Resplendent models generated by indiscernibles

by

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Abstract

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As first proved by Ehrenfeucht and Mostowski [EM], every first-order theory which has infinite models, has models with infinite sets of indiscernibles. Ramsey's Theorem is a crucial component of the proof of this result. If the structure has built-in Skolem functions, taking the Skolem hull of the set of indiscernibles will produce structures generated by a set of indiscernibles.

In this thesis I study the question: Which first-order structures are generated by indiscernibles? J. Schmerl showed that if $\mathcal{L}$ is a finite language, every countable recursively saturated $\mathcal{L}$-structure in which a form of coding of finite functions is available is generated by indiscernibles. Further, he showed that such a structure has arbitrarily large extensions which are generated by a set of indiscernibles, resplendent, and $\mathcal{L}_{\infty,\omega}$-equivalent to the original structure. Proofs of these theorems are complex and use a combinatorial lemma whose proof in Schmerl's paper has an acknowledged gap. I offer a complete proof of a more direct combinatorial lemma from which Schmerl's theorems follow.

The other subject of this thesis is cofinal extensions of linearly ordered structures. It is related to the work of R. Kaye [Kaye3], [Kaye4], who used a weak notion of saturation to give a sufficient condition under which a countable model of PA^- has a proper elementary cofinal extension. I give two different proofs of the fact that every countable recursively saturated linearly ordered structure with no last element has a proper cofinal elementary extension.

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Contents

Introduction

Peano Arithmetic, PA, is the first order theory over a finite language $\mathcal{L}_{\text{PA}} = \{0, 1, +, \cdot, \leq\}$ whose intended model is the structure of natural numbers. Its nonstandard models exist by Completeness Theorem. To understand what kind of nonstandard models exist and what they look like, many aspects of models of PA are studied, including extensions of models. There are two important types of extensions: cofinal extensions and end extensions. A cofinal extension is an extension of whch each element is below an element of the ground model; an end extension is an extension which has the original model as its initial segment. There are two fundamental theorems regarding these two types of extensions. Gaifman's Splitting Theorem [Gaifman1] states that every extension $\mathcal{M} \subseteq \mathcal{N}$ of models of PA has an intermediate model $\mathcal{K}$ between them, so that $\mathcal{K}$ is a cofinal elementary extension of $\mathcal{M}$ while $\mathcal{N}$ is an end extension of $\mathcal{K}$. Gaifman's Splitting Theorem and Compactness Theorem imply that every nonstandard model of PA has a proper elementary cofinal extension. The other theorem, by MacDowell and Specker [MS], states that every model of PA has a proper elementary end extension. Many aspects of elementary end extensions and elementary cofinal extensions are studied. Gaifman showed that every model $\mathcal{M}$ of PA has an elementary minimal end extension $\mathcal{N}$; that is, if $\mathcal{M} \prec \mathcal{K} \prec \mathcal{N}$, then either $\mathcal{K} = \mathcal{M}$ or $\mathcal{K} = \mathcal{N}$ [Gaifman2]. One of the immediate questions would be which models of PA^- have such extensions and which models don't. Lessan showed that for $\mathcal{M} \models \mathrm{PA}$ and $n > 0$, if $K^n(\mathcal{M}) \neq \mathbb{N}$ where $K^n(\mathcal{M})$ is the collection of Σ_n-definable elements of $\mathcal{M}$, then $K^n(\mathcal{M})$ has no proper elementary end or cofinal extensions [Lessan]. R. Kaye gave a sufficient condition under which a countable model of PA^- has a proper cofinal elementary extension and showed that there are some having no such extensions in [Kaye3] and [Kaye4][1]. Other important results on elementary end and cofinal extensions were obtained by Kotlarski [Kotlarski], Smoryński [Smoryński2], Stavi , [SS], and J. Schmerl [Schmerl1], [Schmerl2].

In addition to elementary extensions, I focus on the concepts of recursive saturation, resplendence, and models generated by indiscernibles. Recursive saturation and resplendence are concepts about expandability of a structure. An $\mathcal{L}$-structure $\mathcal{M}$ is recursively saturated if for each tuple $\bar{b} \in M$, every

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[1] R. Kaye's work will be introduced in Chapter 5

recursive type $p(x,\bar{b})$ consistent with $Th(M,\bar{b})$ is realized in $\mathcal{M}$. So, $\mathcal{M}$ is expandable to an $\mathcal{L}\cup\{\bar{c}\}$-structure $(\mathcal{M},\bar{c})$ with $\bar{c}$ being new constant symbols so that $(\mathcal{M},\bar{c})\vDash p(c,\bar{b})$. An $\mathcal{L}$-structure $\mathcal{M}$ is resplendent if for each tuple $\bar{b}\in M$ and each $\mathcal{L}\cup\{X\}$-sentence $\psi(X,\bar{b})$, with X being a new relational symbol whenever $\mathrm{Th}(\mathcal{M},\bar{b})+\psi(X,\bar{b})$ is consistent, $\mathcal{M}$ can be expanded to an $\mathcal{L}\cup\{X\}$-structure $(\mathcal{M},X)$ so that $(\mathcal{M},X)\vDash\psi(X,\bar{b})$. These concepts were first introduced and studied by Barwise and Schlipf [BS] and Ressayre [Ressayre]. An $\mathcal{L}$-structure $\mathcal{M}$ is chronically resplendent if it is both resplendent and, for each $\varphi(X,\bar{b})$ consistent with $\mathrm{Th}(\mathcal{M},\bar{b})$, $\mathcal{M}$ can be expanded to $(\mathcal{M},X)$ so that $(\mathcal{M},X)\vDash\psi(X,\bar{b})$ and $(\mathcal{M},X)$ is again resplendent. A final concept of expandability, total resplendence, was introduced by J. Schmerl [Schmerl2]. An $\mathcal{L}$-structure $\mathcal{M}$ is totally resplendent if there are countably many relations $X_0,X_1,X_2,\ldots$ on M such that every expansion $(\mathcal{M},X_0,X_1,\ldots,X_{n-1})$ is resplendent and if

$$(\mathcal{M},X_0,X_1,X_2,\ldots)\prec(\mathcal{M}',X_0',X_1',X_2',\ldots)\vDash\exists X\varphi(\bar{b},R),$$

then $(\mathcal{M},X_0,X_1,X_2,\ldots,X)\vDash\varphi(\bar{b},X)$ for some X parametrically definable in $(\mathcal{M},X_0,X_1,X_2,\ldots)$. Chapter 4 presents these four notions as equivalent among countable structures over a finite language. However, it is not known yet whether chronic resplendence is strictly stronger than resplendence, or whether total resplendence is strictly stronger than chronic resplendence [Schmerl2]. Many basic facts about recursive saturation can be found in [Smoryński1].

Indiscernibility also plays an important role in this thesis. Roughly speaking, a set of indiscernibles of a structure is a set of elements of the structure that have the same truth values over all formulas. Models generated by a set of indiscernibles are called Ehrenfeucht-Mostowski models. Ehrenfeucht and Mostowski showed that every theory with infinite models has an Ehrenfeucht-Mostowski model. The proof involves Ramsey's Theorem. Chapter 2 discusses definitions in and basic facts about indiscernibles and Ramsey's Theorem.

The two concepts of saturation and generation by a set of indiscernibles can be considered opposites in some ways. Saturated models are "large," while models generated by indiscernibles seem "small." However, both classes of models are in fact closely related. Chapter 3 introduces a theorem by J. Schmerl combining these two notions (Theorem 3.1.1) which states that every countable recursively saturated model of **CFF** is generated by a set of indiscernibles. Here, **CFF** is a theory whose models can code all finite functions. Its models have a binary function β that does the coding. In the proof of the theorem, Schmerl constructs a decreasing sequence of subsets of the model satisfying certain properties. To achieve the construction, a combinatorial result was needed; for this, Schmerl applies Nešetřil-Rödl Theorem (Theorem 3.1.2), a generalization of Ramsey's Theorem. He also introduces his own alternative combinatorial lemma (Lemma 4.1.2) to replace the rather heavy machinery of Nešetřil-Rödl Theorem. This combinatorial lemma better fits his argument and needs no additional combinatorial definitions, as Theorem Nešetřil-Rödl Theorem would require. However, Schmerl's combinatorial lemma has an error in the original proof [Schmerl1]. In chapter 4, I provide a complete proof of the lemma. Schmerl generalized Theorem 3.1.1 further, proving that for every countable recursively saturated model $\mathfrak{A}$ of **CFF** there is an indiscernible-type Σ such that, if $\mathfrak{B}$ is a model of Th($\mathfrak{A}$) generated by a set I realizing Σ, then $\mathfrak{B}$ is totally resplendent and $\mathfrak{A} \equiv_{\infty,\omega} \mathfrak{B}$. The proof of this theorem in [Schmerl2] is very terse and has many unexplained claims. Moreover, the combinatorial lemma used in Theorem 3.1.1 is applied in this proof as well. Therefore, I also provide an expository proof of Theorem 4.2.6 in Chapter 4.

Meanwhile, Chapter 3 introduces another approach to the problem of combining recursive saturation and generation by indiscernibles. In an unpublished result, R. Kossak showed that every completion of PA has countable recursively saturated models generated by indiscernibles. This result is weaker than Theorem 3.1.1, but the argument is much simpler. The proof is done within recursive saturation and the standard form of Ramsey's Theorem, and does not use advanced combinatorics.

The final chapter starts with a review of R. Kaye's work on fragments of PA. Kaye gave a sufficient condition for countable models of PA^- that have a proper elementary cofinal extension in terms of 'tallness'. He investigated further and show that every countable model of $B\Sigma_n + \exp + \neg I\Sigma_n$ has a proper elementary cofinal extension. The chapter provides then in a generalized setting two different proofs of the theorem that every countable recursively saturated linearly ordered structure with no last element has an elementary cofinal extension. The first proof by Schmerl uses the concept of resplendence and a set theoretic argument; my proof is much more elementary, basically a back-and-forth argument which can be formalized to show that the result is a theorem of ACA_0. It is not clear if Schmerl's proof can be formalized in ACA_0, due to its set theoretic argument.

Chapter 1

Preliminaries

Notation

If $\mathcal{M}$ is an $\mathcal{L}$-structure, M denotes the universe of $\mathcal{M}$. Let $(A, <)$ be an ordered set and $B \subseteq A$. If for all $x \in A$ there exists $y \in B$ such that $x \leq y$, then B is cofinal in A and denoted by $B \subseteq_{\text{cof}} A$. If $\mathcal{M} \prec \mathcal{N}$ and $M \subseteq_{\text{cof}} N$, then we will write $\mathcal{M} \prec_{\text{cof}} \mathcal{N}$. For a linearly ordered set $(X, <)$ and $n < \omega$, X^n is the collection of all n-tuples of elements of X, and $[X]^n$ is the collection of all increasing n-tuples of elements of X. Here, 'increasing' always means 'strictly increasing'. So, $\langle a_0, a_1, \ldots, a_{n-1} \rangle \in [X]^n$ means that for all $i < n$, $a_i \in X$ and $a_0 < a_1 < \cdots < a_{n-1}$. The notation $[X]^n$ will often occur without specifying the ordering; in those cases, the ordering used will be clear in the context. The notations $[A]^{>n}$ and $[A]^{<n}$ denote $\bigcup_{n<i}[A]^i$ and $\bigcup_{i<n}[A]^i$ respectively.

Peano Arithmetic

The language $\mathcal{L}_{\mathrm{PA}}$ is the first order language $\{0, 1, +, \cdot, \leq\}$. The basic arithmetic theory, denoted by PA^- is the $\mathcal{L}_{\mathrm{PA}}$-theory of nonnegative parts of discretely ordered rings whose axioms are the following formulas:

Ax1: $\forall x, y, z((x + y) + z = x + (y + z))$

Ax2: $\forall x, y(x + y = y + x)$

Ax3: $\forall x, y, z((x \cdot y) \cdot z = x \cdot (y \cdot z))$

Ax4: $\forall x, y(x \cdot y = y \cdot x)$

Ax5: $\forall x, y, z(x \cdot (y + z) = x \cdot y + x \cdot z)$

Ax6: $\forall x(x + 0 = x) \wedge (x \cdot 0 = 0))$

Ax7: $\forall x(x \cdot 1 = x)$

Ax8: $\forall x, y, z((x < y \wedge y < z) \rightarrow x < z)$

Ax9: $\forall x \neg(x < x)$

Ax10: $\forall x, y(x < y \vee x = y \vee y < x)$

Ax11: $\forall x, y, z(x < y \rightarrow x + z < y + z)$

Ax12: $\forall x, y, z(0 < z \wedge x < y \rightarrow x \cdot z < y \cdot z)$

Ax13: $\forall x, y(x < y \rightarrow \exists z(x + z = y))$

Ax14: $0 < 1 \wedge \forall x(x > 0 \rightarrow x \geq 1)$

Ax15: $\forall x(x \geq 0)$

The axioms of Peano Arithmetic(PA) consist of PA^- and the axiom scheme of the inductions of all $\mathcal{L}_{\mathrm{PA}}$-formulas; for all $\mathcal{L}_{\mathrm{PA}}$-formulas $\varphi(x, \bar{y})$,

$$I_x\varphi := \forall\bar{y}(\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \rightarrow \varphi(x+1, \bar{y})) \rightarrow \forall x\varphi(x, \bar{y}).$$

The arithmetic hierarchy

There is a hierarchy of $\mathcal{L}_{\mathrm{PA}}$-formulas. An $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(\bar{x})$ is $\Delta_0(= \Sigma_0 = \Pi_0)$ iff every quantifier of φ is bounded, i.e., of the form $\forall y < t$ or $\exists x < t$ for some term t. For $n \in \mathbb{N}$, an $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(\bar{x})$ is Σ_{n+1} iff it is of the form $\exists\bar{y}\varphi(\bar{x}, \bar{y})$ with $\varphi(\bar{x}, \bar{y}) \in \Pi_n$; it is Π_{n+1} iff it is of the form $\forall\bar{y}\varphi(\bar{x}, \bar{y})$ with $\varphi(\bar{x}, \bar{y}) \in \Sigma_n$. $\varphi(\bar{x})$ is Σ_n or Π_n if it is equivalent to a Σ_n or a Π_n-formula in the theory or the model being considered. An $\mathcal{L}_{\mathrm{PA}}$-formula is Δ_n iff it is equivalent to both a Σ_n-formula and a Π_n-formula in the theory or the model being considered. Since every formula is equivalent in the predicate calculus to a formula in prenex normal form, it belongs to some Σ_n or Π_n. These classes of formulas form the arithmetic hierarchy by inclusions; for all $n \in \mathbb{N}$,

$$\Sigma_n \subseteq \Delta_{n+1} \subseteq \Sigma_{n+1} \text{ and } \Pi_n \subseteq \Delta_{n+1} \subseteq \Pi_{n+1}.$$

For a class of formulas Γ, $I\Gamma$ denotes the theory axiomatized by PA^- together with all induction axioms for all $\varphi \in \Gamma$. Typical examples are $I\Sigma_n$ and $I\Pi_n$. It is easy to see that PA is equivalent to $I\Sigma_1 + I\Sigma_2 + I\Sigma_3 + \cdots$ and $I\Pi_1 + I\Pi_2 + I\Pi_3 + \cdots$. For an $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x, \bar{y}, \bar{z})$, the collection axiom $B_{x,\bar{y}}\varphi$ is,

$$\forall \bar{z}, t(\forall x < t \exists \bar{y} \varphi(x, \bar{y}, \bar{z}) \to \exists s \forall x < t \exists \bar{y} < s \varphi(x, \bar{y}, \bar{z})).$$

PA is equivalent to the theory axiomatized by $I\Delta_0$ together with all collection axioms for all $\mathcal{L}_{\mathrm{PA}}$-formulas. $B\Sigma_n$ denotes the theory axiomatized by $I\Delta_0$ together with all collection axioms for all Σ_n $\mathcal{L}_{\mathrm{PA}}$-formulas.

Gödel numbering

When a language $\mathcal{L}$ is countable, a *Gödel numbering* is assumed. The Gödel number assigned to φ is denoted by $\ulcorner \varphi \urcorner$, and often the formula φ is dealt with as if it is the natural number itself. A language $\mathcal{L}$ is recursive if each set of Gödel numbers of constant symbols, function symbols, and relation symbols is recursive. A benefit of a language $\mathcal{L}$ being recursive is that algorithms can be easily constructed to determine if a given natural number is a Gödel number of an $\mathcal{L}$-term, an $\mathcal{L}$-formula, an $\mathcal{L}$-formula with one free variable, etc. For precise definitions, refer to Chapter 11 of [Kaye1].

Coding

For simple coding tasks, we use *Cantor's pairing function*:

$$\prec x, y \succ = \frac{1}{2}[(x+y)^2 + 3x + y].$$

For every model $\mathcal{M}$ of PA, Cantor's pairing function establishes a one-to-one correspondence between M^2 and M. Cantor's pairing function can be extended to a one-to-one correspondence between M^n and M by defining $\prec x_0, x_1, \ldots, x_{n-1} \succ = \prec x_0, \prec x_1, \ldots, x_{n-1} \succ\succ$ for each $n \geq 2$. The n-ary function $\prec, \succ$ can be used to code n-tuples of elements of M. However, a way of coding uniform in the length of the sequence is also necessary. Gödel's β-function $(x)_y$ (see [Kaye1] for the definition) as defined by using the Chinese Remainder Theorem suffices:

Lemma 1.1. PA *proves*

(1) $\forall x \exists y (y)_0 = x$

(2) $\forall x, y, z \exists w (\forall i < z((w)_i = (y)_i) \wedge (w)_z = x)$

(3) $\forall x, y \ (x)_y \leq x$

The above lemma allows us to code bounded definable sequences in a uniform way: Let $\mathcal{M}$ be a nonstandard model of PA, $\bar{a} \in M$, and $\varphi(x, \bar{z})$ be an $\mathcal{L}_{\mathrm{PA}}$-formula. Then

$$\mathcal{M} \models \forall z \exists w \forall x < z(\varphi(x, \bar{a}) \leftrightarrow (w)_x \neq 0).$$

The above statement can be proved using induction on z.

Recursive saturation

Let T be an $\mathcal{L}$-theory and $p(\bar{x})$ be a collection of $\mathcal{L}$-formulas in variables $x_0, \ldots, x_{n-1}$ for $n \in \mathbb{N}$. $p(\bar{x})$ is an *n-type* of T if $T + p(\bar{c})$ is consistent where $\bar{c}$ is a tuple of new constants. $p(\bar{x})$ is *recursive* if the set of all Gödel numbers of the formulas $\varphi(\bar{x})$ in $p(\bar{x})$ is recursive. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and Γ be a collection of $\mathcal{L}$-formulas. Let $\bar{a} \in M^n$ be a finite tuple and $p(\bar{x}, \bar{y})$ be a collection of formulas in Γ in the variables $x_0, \cdots, x_{k-1}$ and $y_0, \ldots, y_{n-1}$. Then, $p(\bar{x}, \bar{a}) = \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in p(\bar{x}, \bar{y})\}$ is a Γ-*type over* $\mathcal{M}$ with parameters $a_0, \ldots, a_{n-1}$ if it is a type of the complete $\mathcal{L} \cup \{\bar{a}\}$-theory of the expanded structure $(\mathcal{M}, \bar{a})$. Further if $p(\bar{x}, \bar{y})$ is recursive then $p(\bar{x}, \bar{a})$ is a *recursive* Γ-*type*. $\mathcal{M}$ is Γ-*recursively saturated* if every recursive Γ-type over $\mathcal{M}$ with a finite number of parameters from M is realized in $\mathcal{M}$. $\mathcal{M}$ is *recursively saturated* if it is Λ-recursively saturated where Λ is the collection of all $\mathcal{L}$-formulas.

Let Γ be a class of $\mathcal{L}_{\mathrm{PA}}$-formulas. A *satisfaction relation* for Γ is an $\mathcal{L}_{\mathrm{PA}}$-formula $\mathrm{Sat}_{\Gamma}(u, v)$ such that for each $\varphi(x_0, \dots, x_{n-1}) \in \Gamma$,

$$\mathrm{PA} \vdash \forall x(\varphi((x)_0, \dots, (x)_{n-1}) \leftrightarrow \mathrm{Sat}_{\Gamma}(\ulcorner \varphi(v_0, \dots, v_{n-1}) \urcorner, x))$$

It is well known that there are satisfaction relations $\mathrm{Sat}_{\Sigma_n}(u, v)$ and $\mathrm{Sat}_{\Pi_n}(u, v)$ for each $n < \omega$. This fact and the discussion below Lemma 1.1 lead to the following theorem.

Theorem 1.2. *Let $\mathcal{M} \models \mathrm{PA}$ be nonstandard. Then $\mathcal{M}$ is Σ_n-recursively saturated for all $n < \omega$.*

Infinitary logic

Let $\mathcal{L}$ be a language. In the infinitary logic $\mathcal{L}_{\infty,\omega}$, the formulas are defined inductively as follows:

1. Every atomic formula of $\mathcal{L}$ is a formula of $\mathcal{L}_{\infty,\omega}$
2. If X is any set of formulas of $\mathcal{L}_{\infty,\omega}$ with free variables from the same finite set, then

   $$\bigwedge_{\varphi \in X} \varphi \quad \text{and} \quad \bigvee_{\varphi \in X} \varphi$$

   are formulas in $\mathcal{L}_{\infty,\omega}$.

3. If φ is a formula in $\mathcal{L}_{\infty,\omega}$ and x is a variable, then so are $\neg\varphi$, $\forall x\varphi$ and $\exists x\varphi$.

An $\mathcal{L}_{\infty,\omega}$-sentence is an $\mathcal{L}_{\infty,\omega}$-formula with no free variable. Satisfaction for $\mathcal{L}_{\infty,\omega}$-formulas is defined similarly as in the first order logic.

Two $\mathcal{L}$ -structures $\mathcal{M}$ and $\mathcal{N}$ are *equivalent* in $\mathcal{L}_{\infty,\omega}$; $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ if for all $\mathcal{L}_{\infty,\omega}$-sentences φ, $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$.

A back-and-forth system from $\mathcal{M}$ to $\mathcal{N}$ is a set I of pairs of $(\bar{a}, \bar{b})$ of tuples with $\bar{a} \in M^n$ and $\bar{b} \in N^n$ for some $n < \omega$ such that

1. if $(\bar{a}, \bar{b}) \in I$, then $(\mathcal{M}, \bar{a}) \equiv_0 (\mathcal{N}, \bar{b})$, i.e., the two structures are equivalent with respect to the atomic sentences,

2. I is not empty, and

3. if $(\bar{a}, \bar{b}) \in I$, for every $c \in M$ there is an element $d \in N$ such that $(\bar{a}c, \bar{b}d) \in I$ and for every $d \in N$ there is an element $c \in M$ such that $(\bar{a}c, \bar{b}d) \in I$.

One of the central results of infinitary logic is Karp's Theorem.

Theorem 1.3. (Karp's Theorem)*Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be* $\mathcal{L}$*-structures. There is a back-and-forth system from* $\mathcal{M}$ *to* $\mathcal{N}$ *iff* $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$.

For the proof, see [Hodges].

The following is immediate.

Theorem 1.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be recursively saturated structures. If $\mathcal{M} \equiv \mathcal{N}$ and they realize the same types, then there is a back-and-forth system from $\mathcal{M}$ to $\mathcal{N}$, and thus $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$.*

Cofinal and end extensions

Cofinal extensions and end extensions of models have played an important role in the study of PA. There are two fundamental theorems regarding elementary extensions of models of arithmetic.

Theorem 1.5. (Gaifman's Splitting Theorem [Gaifman1]) *If $\mathcal{M} \subseteq \mathcal{N}$ are models of* PA *then there is a unique $\mathcal{K} \models \mathrm{PA}$ such that $\mathcal{M} \subseteq_{\mathrm{cof}} \mathcal{K} \subseteq_{\mathrm{end}} \mathcal{N}$; and moreover $\mathcal{M} \prec \mathcal{K}$.*

This theorem together with Compactness Theorem establishes the existence of proper elementary cofinal extensions, and also that every cofinal extension of a model of PA to another model of PA is elementary.

Theorem 1.6. (MacDowell and Specker [MS]). *Every model $\mathcal{M} \models \mathrm{PA}$ has a proper elementary end-extension.*

Second order arithmetic

We define Z_2, the *formal system of second order arithmetic.* The *language of second order arithmetic*, $\mathcal{L}_2$ is a two-sorted language. The variables of the first sort are called number variables and denoted by lower case letters. The variables of the second sort are called set variables and denoted by capital letters. Numerical terms are the same as those of $\mathcal{L}_{\mathrm{PA}}$. Atomic formulas are $t_1 = t_2, t_1 < t_2$, and $t_1 \in X$ where t_1 and t_2 are numerical terms and X is any set variable. Formulas are built from atomic formulas by means of propositional connectives, $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, number quantifiers $\forall n, \exists n$, and set quantifiers $\forall X, \exists X$.

A *structure for* $\mathcal{L}_2$, is an ordered 7-tuple

$$\mathcal{M} = (M, S_{\mathcal{M}}, +_{\mathcal{M}}, \cdot_{\mathcal{M}}, 0_{\mathcal{M}}, 1_{\mathcal{M}}, <_{\mathcal{M}})$$

where $S_{\mathcal{M}}$ is a subset of $\mathcal{P}(M)$ and serves as the range of the set variables.

The axioms of the formal system of second order arithmetic are the following $\mathcal{L}_2$-formulas together with the basic axioms of PA:

1. Induction axiom:

$$(0 \in X \wedge \forall x(x \in X \rightarrow x + 1 \in X)), \rightarrow \forall x(x \in X)$$

2. Comprehension scheme:

$$\exists X \forall x (x \in X \leftrightarrow \varphi(x))$$

where $\varphi(x)$ is any formula of $\mathcal{L}_2$ in which X does not occur freely.

There are five important subsystems of Z_2 classified by their capability to define sets. This thesis introduces ACA_0 and refers to [Simpson] for the other subsystems. An $\mathcal{L}_2$-formula with no set quantifiers an *arithmetical formula*. The *arithmetical comprehension scheme* is the restriction of the comprehension scheme above to arithmetical formulas $\varphi(x)$. ACA_0 is the subsystem of Z_2 whose axioms are the arithmetical comprehension scheme, the induction axiom, and the basic axioms PA.

Chapter 2

Ramsey's Theorem and Ehrenfeucht-Mostowski Models

This chapter provides basic definitions and facts about Ramsey's Theorem and the concept of being generated by indiscernibles. Let $\mathfrak{A}$ be a structure over a language $\mathcal{L}$ and X be a linearly ordered subset of A. If every pair of tuples $\langle \bar{a} \rangle, \langle \bar{b} \rangle \in [X]^n$ satisfies the same first order formulas in $\mathfrak{A}$, we say that X is *an indiscernible sequence*. Ramsey's Theorem proves that every theory with infinite models has a model with an infinite indiscernible sequence. The models generated by an indiscernible sequence are called Ehrenfeucht-Mostowski models. The definitions and the theorems in this chapter are mostly taken from [Marker] and [Hodges].

2.1 Ramsey's Theorem

Let κ, η, and λ be cardinals, and n an integer. Let $f : [X]^n \to \lambda$ be a function. $Y \subseteq X$ is *homogeneous* for f if f is constant on $[Y]^n$. $\kappa \to (\eta)^n_\lambda$ if for all $f : [X]^n \to \lambda$ with $|X| \geq \kappa$, there is $Y \subseteq X$ such that $|Y| \geq \eta$ and Y is homogeneous for f.

Theorem 2.1.1 (Ramsey's Theorem)**.** *If $k, n < \omega$, then $\aleph_0 \to (\aleph_0)^n_k$.*

The finite version follows from Ramsey's Theorem by using König lemma.

Definition 2.1.2. *A finite branching tree* is a partial order $(T, <)$ such that:

i) there is $r \in T$ such that $r \leq x$ for all $x \in T$;

ii) if $x \in T$, then $\{y : y < x\}$ is finite and liniearly ordered by $<$;

iii) if $x \in T$, then there is a finite (possibly empty) set $\{y_1, \ldots, y_m\}$ of incomparable elements such that each $y_i > x$ and, if $z > x$, then $z \geq y_i$ for some i.

A *path* through T is a function $f : \omega \to T$ such that $f(n) < f(n+1)$.

Lemma 2.1.3 (König's Lemma)**.** *If T is an infinite finite branching tree, then there is a path through T.*

Theorem 2.1.4 (Finite Ramsey Theorem)**.** *For all $k, n, m < \omega$, there is $l < \omega$ such that $l \to (m)^n_k$.*

Finite Ramsey's Theorem can be proved in PA by formalizing one of its direct proofs. Ramsey's Theorem is not formalizable in PA but can be formalized and proved in ACA_0. For the proof, see Theorem 2.2.16 in [KS].

2.2 Indiscernibles

Ehrenfeucht and Mostowski introduced the notion of indiscernibles.

Definition 2.2.1. Let $\mathcal{L}$ be a language and $\mathcal{M}$ be an $\mathcal{L}$-structure. Let X be a subset of M linearly ordered by some ordering $<$. Let Φ be a set of $\mathcal{L}$-formulas. X is called a Φ-*indiscernible sequence* if for all $n < \omega$, all increasing n-tuples $\langle a_0, \ldots, a_{n-1}\rangle, \langle b_0, \ldots, b_{n-1}\rangle$ in $[X]^n$, and all formulas $\phi(x_0, \ldots, x_{n-1}) \in \Phi$, $\mathcal{M} \vDash \phi(a_0, \ldots, a_{n-1}) \leftrightarrow \phi(b_0, \ldots, b_{n-1})$. If Φ is the set of all $\mathcal{L}$-formulas with n-free variables for some $n > 0$, then X is also called an n-*indiscernible sequence*. If Φ is the set of all $\mathcal{L}$-formulas, Φ is omitted, and X is simply called an *indiscernible sequence*.

For example, if A is a vector space and X is a basis of A, then X is clearly an indiscernible sequence for any ordering of X. The question of how common a phenomenon it is that a structure has an infinite indiscernible sequence can be answered by the theorem below; the proof of the theorem is taken from [Marker], and repeated here to provide taste of how Ramsey's Theorem is used to find an indiscernible sequence and why Ramsey's Theorem is essential for it.

Theorem 2.2.2 (Ehrenfeucht-Mostowski)**.** *Let T be an $\mathcal{L}$-theory with infinite models. For any infinite linear order $(I, <)$, there is $\mathcal{M} \vDash T$ containing an indiscernible sequence $(c_i : i \in I)$.*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_i : i \in I\}$. Let Γ be the union of T and

- $c_i \neq c_j$ for $i, j \in I$ with $i \neq j$;
- $\phi(c_{i_0}, \ldots, c_{i_{n-1}}) \leftrightarrow \phi(c_{j_0}, \ldots, c_{j_{n-1}})$ for all $\mathcal{L}$-formulas $\phi(x_0, \ldots, x_{n-1})$, where $n < \omega$ and $\langle i_0, \ldots, i_{n-1} \rangle, \langle j_0, \ldots, j_{n-1} \rangle \in [I]^n$.

Let $\mathcal{N}$ be an infinite model of T. We claim that $\mathcal{N}$ is a model of every finite subset of Γ. By compactness, Γ has a model $\mathcal{M}$ which contains the indiscernible sequence $(c_i : i \in I)$. Let Δ be a finite subset of Γ. Let I_0 be the finite subset of I such that if c_i occurs in Δ, then $i \in I$. Let $\{\phi_i : i < m\}$ be the set of the formulas such that Δ asserts indiscernibility with respect to the formulas ϕ_i, $i \leq m$. Let $x_0, \ldots, x_{k-1}$ be the free variables from $\phi_0, \ldots, \phi_{m-1}$. To apply Ramsey's Theorem, we fix a linear ordering $<_{\mathcal{N}}$ on $\mathcal{N}$ and define a function $f : [(N, <_{\mathcal{N}})]^k \to \mathcal{P}(\{0, 1, \ldots, m-1\})$ so that $f(a_0, \ldots, a_{k-1}) = \{i : \mathcal{N} \vDash \phi_i(a_0, \ldots, a_{k-1})\}$. Ramsey's Theorem holds that there is an infinite subset Y of N which is homogeneous for f. Thus, for each ϕ_i, $i < m$, every k-tuple from $[Y]^k$ has the same truth value, producing infinitely many choices for interpretations of c_i's, $i \in I_0$. $\square$

When the ordering $<$ considered is clear, an indiscernible sequence will be called an indiscernible set or a set of indiscernibles.

Definition 2.2.3. Let Σ be a complete set of $\mathcal{L}$-formulas. Let $(I, <)$ be a countable ordered set. Let $\mathcal{L}' = \mathcal{L} \cup \{i : i \in I\}$ where every element of I is a new constant symbol not in $\mathcal{L}$. Consider the set of $\mathcal{L}'$ sentences

$$T = \{\varphi(\bar{i}) : n < \omega, \langle \bar{i} \rangle \in [I]^n, \varphi(\bar{x}) \text{ is an } n\text{-ary } \mathcal{L}\text{-formula in } \Sigma\}.$$

If T is consistent, then Σ is called an *indiscernible type.*

If Σ is an indiscernible type, then for an arbitrary infinite ordered set $(I, <)$, T can be defined as above. If $I \subseteq A$ and $\mathfrak{A} \models T$, then I is an indiscernible sequence of $\mathfrak{A}$.

Note that if J is an infinite indiscernible sequence of $\mathfrak{A}$, then $\mathrm{tp}(J) = \{\varphi(\bar{x}) : n < \omega, \langle \bar{j} \rangle \in [J]^n, \varphi(\bar{x}) \text{ is an } n\text{-ary formula}, \mathfrak{A} \models \varphi(\bar{j})\}$ is an indiscernible type. We call $\mathrm{tp}(J)$ *the indiscernible type of* J.

Indiscernible type defined above should not be confused with the definition of indiscernible type in [KS]. The latter is a 1-type.

2.3 Ehrenfeucht-Mostowski models

Let T be an $\mathcal{L}$-theory. We say that T has *built-in Skolem functions* if for all $\mathcal{L}$-formulas $\varphi(\bar{x}, y)$, there exists a function symbol f such that

$$T \vDash \forall \bar{x}(\exists y \varphi(\bar{x}, y) \to \varphi(\bar{x}, f(\bar{x})).$$

If T has neither built-in Skolem functions nor definable Skolem functions, we still can find an expanded language $\mathcal{L}^*$ of $\mathcal{L}$ and an $\mathcal{L}^*$-theory $T^* \supseteq T$ having built-in Skolem functions such that if $\mathcal{M}$ is any model of T, there is an expansion $\mathcal{M}^*$ to $\mathcal{L}^*$ such that $\mathcal{M}^* \vDash \mathrm{T}^*$. T^* is called a *skolemization* of T. For the details, see [Marker].

Let T be a theory with built-in Skolem functions. Let $X \subseteq \mathcal{M} \vDash T$. Then, there exists the *Skolem hull* $\mathcal{H}_{\mathcal{M}}(X)$, the unique $\mathcal{L}$-substructure of $\mathcal{M}$ generated by X. Every element of $\mathcal{H}_{\mathcal{M}}(X)$ is generated as $t(\bar{c})$ for some term t and $\bar{c} \in X$. By the Tarski-Vaught Test, $\mathcal{H}_{\mathcal{M}}(X)$ is an elementary substructure of $\mathcal{M}$. In particular, models built as Skolem hulls of indiscernible sequences are called *Ehrenfeucht-Mostowski* models. By Theorem 2.2.2, every theory having built-in Skolem functions has an Ehrenfeucht-Mostowski model of every cardinality.

PA does not have built-in Skolem functions. However, for each formula

$\varphi(\bar{x}, y)$, there is the *Skolem term* t_φ defined as

$$t_\varphi(\bar{x}) = \begin{cases} \min\{y : \varphi(\bar{x}, y)\}, & \text{if } \exists y \varphi(\bar{x}, y) \\ 0 & \text{otherwise.} \end{cases}$$

t_φ is well defined in PA because of its least number principle. So, PA can be regarded as having built-in Skolem functions. For $\mathcal{M} \models \mathrm{PA}$, the Skolem hull of X in $\mathcal{M}$ is called the Skolem closure of X in $\mathcal{M}$ and denoted by $\mathrm{Scl}^{\mathcal{M}}(X)$. If there is no confusion, this is written as $\mathrm{Scl}(X)$.

The following are some basic features of Ehrenfeucht-Mostowski models taken from [Marker].

Lemma 2.3.1. *Let T be an $\mathcal{L}$-theory with built-in Skolem functions. Let $\mathcal{M} \models T$. Let $I \subseteq \mathcal{M}$ be an infinite indiscernible sequence. Suppose that $\gamma : I \to I$ is an order-preserving permutation. Then there is an automorphism $\sigma : \mathcal{H}^{\mathcal{M}}(I) \to \mathcal{H}^{\mathcal{M}}(I)$ extending γ.*

Proof. Define an automorphism by $\sigma(t(\bar{c})) = t(\gamma(\bar{c}))$ for terms t and $\bar{c} \in I$. This extends γ. □

Let I be an indiscernible sequence in $\mathcal{M}$. Let

$$\mathrm{tp}(I) = \{\varphi(\bar{x}) : \mathcal{M} \models \varphi(\bar{i}), \langle \bar{i} \rangle \in [I]^n, n < \omega\}.$$

be the indiscernible type of I. If a sequence J of elements of any structure

$\mathcal{N}$ over the same language as $\mathcal{M}$ realizes $\mathrm{tp}(I)$, then J is an indiscernible sequence of $\mathcal{N}$.

Lemma 2.3.2. *Let T be an $\mathcal{L}$-theory with built-in Skolem functions. Suppose that I is an infinite indiscernible sequence in $\mathcal{M} \models T$. If $(J, <)$ is any infinite ordered set, there is $\mathcal{N} \models T$ containing J as its indiscernible sequence and $\mathrm{tp}(I) = \mathrm{tp}(J)$.*

The proof is a straightforward application of the Compactness Theorem.

Lemma 2.3.3. *Let T be an $\mathcal{L}$-theory with built-in Skolem functions. If I is an indiscernible sequence in $\mathcal{M} \models T$ and J is an indiscernible sequence in $\mathcal{N} \models T$ with $\mathrm{tp}(I) = \mathrm{tp}(J)$, then any order-preserving map $\tau : I \to J$ extends to an elementary embedding $\sigma : \mathcal{H}^{\mathcal{M}}(I) \to \mathcal{H}^{\mathcal{N}}(J)$.*

Proof. As in lemma 2.3.1, the embedding is defined by $\sigma(t(\bar{c})) = t(\tau(\bar{c}))$ for terms t and $\bar{c} \in I$. □

Chapter 3

Recursive saturation and Ehrenfeucht-Mostowski models

The main question of this thesis is what structures are generated by an indiscernible sequence, in particular what models of Peano Arithmetic are generated by an indiscernible sequence. D. Marker and S. Smith raised the question: Is there a recursively saturated model of Peano Arithmetic generated by a set of indiscernibles? Schmerl wrote that this question was raised by A. Macintyre [Schmerl1]; however, R. Kossak affirmed that Marker and Smith raised the question through a personal communication with Marker. We discuss these questions and give some answers.

3.1 Countable recursively saturated models

F. Abramson and J. Knight, answering the question raised by Marker and Smith, showed that every consistent extension of PA has a countable, recursively saturated model generated by a set of indiscernibles. The proof of this theorem was in a personal letter to Macintyre [Schmerl1] and thus unavailable. Instead, a proof is offered by Kossak. This positive answer was unexpected because recursively saturated models are in some sense large, whereas models generated by indiscernibles are small [Schmerl1]. More surprisingly, Schmerl showed that there is a stronger relation between the two concepts by showing that every countable recursively saturated structure capable of coding any finite function is an Ehrenfeucht-Mostowski model.

Theorem 3.1.1 (Schmerl, [Schmerl1]). *Suppose T is a theory in a recursive language which includes the binary function symbol β such that for each $n < \omega$ the sentence*

$$\forall x_0, \ldots, x_{n-1} \forall y_0, \ldots, y_{n-1} \exists x \left[\bigwedge_{i<j<n} x_i \neq x_j \rightarrow \bigwedge_{i<n} \beta(x_i, x) = y_i \right]$$

is a consequence of T. Then every countable, recursively saturated model of T is generated by a set of indiscernibles. Such a theory is called **CFF***(for "codes finite functions").*

The proof of this theorem will not be provided here because a stronger theorem, Theorem 4.2.6, will be proved in the next chapter. Instead, we will discuss the involvement of some combinatorics in the proof of the theorem. The combinatorial theorems are described using definitions taken from [KS].

For $n < \omega$, let $\{V\}^n$ be the set of all n-element subsets of V. Then $H = (V, E)$ is an *n-uniform hypergraph* if $E \subseteq \{V\}^n$. An n-uniform hypergraph (V, E) is *sparse* if there is no $K \in \{V\}^{n+1}$ such that $\{K\}^n \subseteq E$. If $H = (V, E)$ and $H' = (V', E')$ are n-uniform hypergraphs, then H is a *subhypergraph* of H' iff $V \subseteq V'$ and $E = E' \cap \{V\}^n$.

Theorem 3.1.2 (Nešetřil-Rödl Theorem)**.** *Suppose $1 \leq n < \omega$ and $H = (V, E)$ is a finite, sparse $(n+1)$-uniform hypergraph. Then there is a finite sparse $(n+1)$-uniform hypergraph $H' = (V', E')$ such that whenever $\{V'\}^{n+1} = F_1 \cup F_2$, then (V', E') has a subhypergraph (W, F) which is isomorphic to (V, E) and such that either $F \subseteq F_1$ or $F \subseteq F_2$.*

Schmerl used this theorem in the proof of Theorem 3.1.1 However, he thought that Nešetřil-Rödl Theorem was rather heavy machinery for what was actually needed, providing his own alternative combinatorial theorem (Lemma 4.1.2) while claiming that the proof of his alternative theorem was a nearly direct transcription of one of the standard proofs of Ramsey's Theorem. He explained that Lemma 4.1.2 was weaker than Nešetřil-Rödl Theorem because it had an infinite version, which Nešetřil-Rödl did not. However, the proof he gave for this was incomplete, as recognized by R. Kaye [Kaye2]. This combinatorial theorem and a full proof of it will appear in the next chapter to prove Theorem 4.2.6, a generalization of Theorem 3.1.1. This will confirm that Schmerl's claim about the proof of his combinatorial theorem that the proof is a nearly direct translation of one of the standard proofs of Ramsey's Theorem.

The original proof of Theorem 3.1.1 was very terse and hard to understand. Kaye [Kaye2] and Schmerl [KS] provided expository proofs of the theorem. In this expository proof, Schmerl used a somewhat weakened variation of the Nešetřil-Rödl Theorem which was proved independently by Abramson and Harrington [AH]. The following definition and theorem are taken from [KS].

Definition 3.1.3. Let $\mathcal{L}$ be a finite language consisting only of relation symbols among which is the binary relation symbol $<$ for ordering. Let $\mathfrak{A} = (A, <, \ldots)$ be a finite ordered $\mathcal{L}$-structure and f be a function on $[A]^{<\omega}$. f is *homogeneous* on $\mathfrak{A}$ if, whenever $X, Y \subseteq A$ and $\mathfrak{A} \restriction X \cong \mathfrak{A} \restriction Y$, then $f(X) = f(Y)$.

Theorem 3.1.4 (AH/NR Theorem). *Suppose* $\mathfrak{A} = (A, <, ...)$ *is a finite ordered* $\mathcal{L}$*-structure. Then there is a finite ordered* $\mathcal{L}$*-structure* $\mathfrak{B} = (B, <, ...)$ *such that whenever* $f : [B]^{<\omega} \to \{0, 1\}$*, then there is* $\mathfrak{A}' \subseteq \mathfrak{B}$ *such that* $\mathfrak{A}' \cong \mathfrak{A}$ *and* f *is homogeneous on* $\mathfrak{A}'$*.*

It is known that this theorem is provable in PA [AH].

3.2 Kossak's proof

For proof of the theorem by Abramson and Knight mentioned in the previous section, Kossak suggested building a recursively saturated model of PA by constructing an indiscernible sequence of which each element codes an ω-sequence. This section provides the results. To do so, some lemmas are necessary. The following is a weaker version of Gaifman's Splitting Theorem. The proof is well-known and so omitted.

Lemma 3.2.1. *If* $\mathcal{M} \prec \mathcal{N} \models \mathrm{PA}$ *and* $\mathcal{K} = \sup(\mathcal{M})$ *in* $\mathcal{N}$*, then* $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K} \prec_{\mathrm{end}} \mathcal{N}$.

Lemma 3.2.2. *Let* $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$ *be nonstandard models of* PA*, and suppose for some* $a \in \mathcal{N}$*,* $\mathcal{M} = \sup\{(a)_n : n < \omega\}$*, and* $\mathrm{Scl}((a)_i) < (a)_{i+1}$ *for all* $i < \omega$*. If* $\varphi(x, \bar{y})$ *is a formula and* $\bar{b} \in M$ *with* $\bar{b} < (a)_m$ *for some* $m < \omega$*, then* $\mathcal{M} \models \exists x \varphi(x, \bar{b})$ *iff* $\mathcal{M} \models \exists x < (a)_{m+1} \varphi(x, \bar{b})$*, and* $\mathcal{M} \models \forall x \varphi(x, \bar{b})$ *iff* $\mathcal{M} \models \forall x < (a)_{m+1} \varphi(x, \bar{b})$.

Proof. Let $\mathcal{M}_m = \sup(\mathrm{Scl}((a)_m))$. By the previous lemma, $\mathcal{M}_m \prec \mathcal{M}$. Suppose $\mathcal{M} \models \exists x \varphi(x, \bar{b})$. Then, $\mathcal{M}_m \models \exists x \varphi(x, \bar{b})$. Since $\mathcal{M}_m < (a)_{m+1}$, $\mathcal{M} \models \exists x < (a)_{m+1} \varphi(x, \bar{b})$. The other direction is clear.

For the second equivalence, suppose $\mathcal{M} \vDash \forall x < (a)_{m+1}\varphi(x, \bar{b})$. Then, $\mathcal{M}_m \vDash \forall x\varphi(x, \bar{b})$. So, $\mathcal{M} \vDash \forall x\varphi(x, \bar{b})$. The other direction is clear. □

Corollary 3.2.3. *Let $\mathcal{M}, \mathcal{N}$, and $a \in N$ satisfy the same conditions in the above lemma. If $\varphi(\bar{y})$ is a formula and $b \in M$, then there exists a Δ_0-formula $\varphi^*(\bar{y}, x)$ such that for all $\bar{c} < b$, $\mathcal{M} \vDash \varphi(\bar{c})$ iff $\mathcal{N} \vDash \varphi^*(\bar{c}, a)$. Moreover, $\varphi^*(\bar{y}, x)$ can be found in an effective way.*

Proof. A formula in prenex normal form equivalent to $\varphi(\bar{y})$ can be found in an effective way. Let $Q_1x_1, Q_2x_2, \ldots, Q_nx_n\psi(\bar{x}, \bar{y})$ be the formula in prenex normal form. Here, $\psi(\bar{x}, \bar{y})$ is quantifier-free and Q_i's are $\exists$ or $\forall$. Let $b < (a)_m$ for some $m < \omega$. By the previous lemma, the first quantifier Q_1 of $Q_1x_1, Q_2x_2, \ldots, Q_nx_n\psi(\bar{x}, \bar{c})$ can be replaced with a bounded quantifier so that

$$\begin{gathered}\mathcal{M} \vDash Q_1x_1, Q_2x_2, \ldots, Q_nx_n\psi(\bar{x}, \bar{c}) \\ \text{if and only if } \mathcal{M} \vDash Q_1x_1 < (a)_{m+1}, Q_2x_2, \ldots, Q_nx_n\psi(\bar{x}, \bar{c}).\end{gathered}$$

Then, Q_2 is replaced by a bounded quantifier. If $d_1 < (a)_{m+1}$, then again by the previous lemma,

$$\begin{gathered}\mathcal{M} \vDash Q_2x_2, \ldots, Q_nx_n\psi(d_1, x_2, \ldots, x_n, \bar{c}) \\ \text{if and only if } \mathcal{M} \vDash Q_2x_2 < (a)_{m+2}, Q_3x_3, \ldots, Q_nx_n\psi(d_1, x_2, \ldots, x_n, \bar{c}).\end{gathered}$$

This produces:

$$\mathcal{M} \models Q_1 x_1, Q_2 x_2, \ldots, Q_n x_n \psi(\bar{x}, \bar{c})$$
$$\text{if and only if } \mathcal{M} \models Q_1 x_1 < (a)_{m+1} Q_2 x_2 < (a)_{m+2}, Q_3 x_3, \ldots, Q_n x_n \psi(\bar{x}, \bar{c}).$$

Each of the remaining unbounded quantifiers is replaced with a bounded quantifier in this way until the unbounded quantifiers are all exhausted. Then, the resulting formula $\varphi^*(\bar{y}, a)$ will be

$$Q_1 x_1 < (a)_{m+1} Q_2 x_2 < (a)_{m+2}, \ldots, Q_n x_n < (a)_{m+n} \psi(\bar{x}, \bar{y})$$

Since $\mathcal{M} \prec \mathcal{N}$, φ^* has the desired property. $\square$

Lemma 3.2.4. *Let $\mathcal{M}, \mathcal{N}$, and $a \in N$ satisfy the same conditions in the previous lemma. Then, $\mathcal{M}$ is recursively saturated.*

Proof. Let $\tau(x, \bar{b})$ be a recursive type over $\mathcal{M}$ for some $\bar{b} \in M$. Then there exists some $m < \omega$ such that $\bar{b} < (a)_m$. Then, by Lemma 3.2.2, $\tau(x, \bar{b}) \cup \{x < (a)_{m+1}\}$ is also a recursive type over $\mathcal{M}$. Let $\tau^*(x, \bar{b}, a) = \{\varphi^*(x, \bar{b}, a) : \varphi(x, \bar{b}) \in \tau(x, \bar{b})\} \cup \{x < (a)_{m+1}\}$ where φ^* is the Δ_0-formula obtained by using the above corollary so that for all $c < (a)_{m+1}$, $\mathcal{M} \models \varphi(c, \bar{b})$ iff $\mathcal{N} \models \varphi^*(c, \bar{b}, a)$. Then, τ^* is a Δ_0-recursive type over $\mathcal{N}$, and so by Theorem 1.2 it is realized in $\mathcal{N}$. Therefore, $\tau(x, \bar{b})$ is realized in $\mathcal{M}$. $\square$

Theorem 3.2.5 (R. Kossak (unpublished)). *Every consistent extension of* PA *has a countable, recursively saturated model which is generated by a set of indiscernibles.*

Proof. Let S be a consistent complete extension of PA. Let $\mathcal{L} = \mathcal{L}_{\mathrm{PA}} \cup \{c\}$ and T be the $\mathcal{L}$-theory consisting of S and the following:

$$\begin{aligned}&\{\varphi((c)_{m_1}, (c)_{m_2}, \ldots, (c)_{m_k}) \leftrightarrow \varphi((c)_{n_1}, (c)_{n_2}, \ldots, (c)_{n_k}) : \\ &\qquad k < \omega, \langle \bar{m} \rangle, \langle \bar{n} \rangle \in [\omega]^k, \varphi \text{ is an } \mathcal{L}_{\mathrm{PA}}\text{-formula}\} \\ &\cup \{(c)_n < (c)_{n+1} : n < \omega\} \\ &\cup \{(c)_n < ((c)_{n+1})_0 : n < \omega\} \\ &\cup \{t(((c)_n)_i) < ((c)_n)_{i+1} : n, i < \omega, t \text{ is an } \mathcal{L}_{\mathrm{PA}}\text{-term}\}\end{aligned}$$

Suppose T is consistent. Then there exists a model $\mathcal{N}$ of T. Let $\mathcal{M} = \mathrm{Scl}((c)_n : n < \omega)$, and $\mathcal{M}_n = \sup(\mathrm{Scl}(((c)_n)_i : i < \omega))$ in $\mathcal{M}$ for each $n < \omega$. Since $\mathcal{M} \prec \mathcal{N}$, $\mathcal{M}$ a model of T and is generated by the set of indiscernibles $\{(c)_n : n < \omega\}$. For each $n < \omega$, since $\mathrm{Scl}(((c)_n)_i : i < \omega) \prec \mathcal{M}$, by Lemma 3.2.1, $\mathcal{M}_n \prec_{\mathrm{end}} \mathcal{M}$. The last set of sentences of T implies that $\mathrm{Scl}(((c)_n)_i) < ((c)_n)_{i+1}$ for all $n, i < \omega$. So by Lemma 3.2.4, $\mathcal{M}_n$ is recursively saturated. Since $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}_n$, $\mathcal{M}$ is recursively saturated.

So, it suffices to demonstrate that T is consistent, by showing that every finite fragment of T is consistent. Let $t_0, t_1, \ldots$ be an enumeration of $\mathcal{L}_{\mathrm{PA}}$-

terms, and $\varphi_0, \varphi_1, \ldots$ an enumeration of all $\mathcal{L}_{\mathrm{PA}}$-formulas such that each φ_j has at most j free variables. Let $k < \omega$. Consider the following fragment T_k of T:

$$\begin{aligned}
&\mathrm{PA} \cup \{\varphi_l((c)_{m_1}, (c)_{m_2}, \ldots, (c)_{m_{k_l}}) \leftrightarrow \varphi_l((c)_{n_1}, (c)_{n_2}, \ldots, (c)_{n_{k_l}}) : \\
&\qquad\qquad \langle \bar{m} \rangle, \langle \bar{n} \rangle \in [k]^{k_l}, l < k\} \\
&\cup \{(c)_n < (c)_{n+1} : n < k-1\} \\
&\cup \{(c)_n < ((c)_{n+1})_0 : n < k-1\} \\
&\cup \{t_l(((c)_n)_i) < ((c)_n)_{i+1} : n, l < k, i < k-1\}
\end{aligned}$$

So, T_k is a subtheory of T whose sentences involves only the indices less than k. Let $\mathfrak{A}$ be any model of S. Consider the set of formulas Ψ consisting of $t_0(x) < y, t_1(x) < y, \ldots, t_{k-1}(x) < y$. Ramsey's Theorem provides an infinite sequence of elements of $\mathfrak{A}$ indiscernible over Ψ. Choose k elements among them, say $b_0^0 < b_1^0 < \cdots < b_{k-1}^0$. Take an element a_0 in $\mathfrak{A}$ coding $b_0^0, b_1^0, \ldots, b_{k-1}^0$, i.e., $(a_0)_0 = b_0^0, \ldots, (a_0)_{k-1} = b_{k-1}^0$. Then, apply Ramsey's Theorem again to the set $A_{>a_0} = \{x \in A : x > a_0\}$ which is still infinite, to find a finite sequence $b_0^1, b_1^1, \ldots, b_{k-1}^1$ in $A_{>a_0}$ indiscernible over Ψ. Take an element $a_1 \in A$ coding $b_0^1, b_1^1, \ldots, b_{k-1}^1$; repeating this process produces an infinite increasing sequence $a_0, a_1, a_2, \ldots$. Apply Ramsey's Theorem to this sequence to find a subsequence of size k indiscernible over φ_l's for $l < k$. Let

$d \in A$ code the subsequence. If d interpret c, then, clearly, $(\mathfrak{A}, d)$ satisfies T_k. $\square$

Chapter 4

Schmerl's Theorem

Schmerl generalized his Theorem 3.1.1 further. This chapter is dedicated to a proof of Schmerl's theorem that every countable, recursively saturated structure $\mathcal{M}$ capable of coding all finite functions has arbitrarily large recursively saturated structures $\mathcal{N} \equiv_{\infty,\omega} \mathcal{M}$ generated by a set of indiscernibles [Schmerl2]. Its original proof in [Schmerl2] is very terse and contains many claims that lack explanation; furthermore, as mentioned in the previous chapter, the proof of Lemma 4.1.2 used in the proof of his theorems has an error. Here, I fix the proof of the lemma and give an expository proof of Schmerl's theorem.

4.1 Combinatorial preliminaries

Definition 4.1.1. Let $(A, <)$ be an infinite linearly ordered set. Let $G = \langle g_n : [A]^n \to A \rangle_{1<n<\omega}$ be a sequence of functions. Let Y be a finite subset of A and $a \in A$. Let $r \in \mathbb{N}$. For a function $f : [Y]^{\geq r} \to A$, we say that f is *coded by a via G* if for all $s \geq r$ and for all $\langle b_0, \ldots, b_{s-1} \rangle \in [Y]^s$,

$$g_{s+1}(b_0, b_1, \ldots, b_{s-1}, a) = f(b_0, b_1, \ldots, b_{s-1}).$$

We also say that a subset $I \subseteq A$ is *(G, r)-free* if it satisfies the following: if Y is a finite subset of I and $f : [Y]^{\geq r} \to A$ is a function, then for each $a \in I$ there is $b \in I$ with $b > a$ that codes f via G. Note that for all $r < s < \omega$ if I is (G, r)-free, then it is (G, s)-free, and if I is $(G, 1)$-free, then it is $(G, 0)$-free.

Let $0 < m < \omega$ and $\langle b_0, b_1, \ldots, b_{m-1} \rangle, \langle b'_0, b'_1, \ldots, b'_{m-1} \rangle \in [A]^m$. We write $\langle b_0, b_1, \ldots, b_{m-1} \rangle \sim_{(G,r)} \langle b'_0, b'_1, \ldots, b'_{m-1} \rangle$ if $g_s(b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}) = g_s(b'_{i_0}, b'_{i_1}, \ldots, b'_{i_{s-1}})$ whenever $r < s \leq m$ and $\langle i_0, i_1, \ldots, i_{s-1} \rangle \in [\{0, 1, \ldots, m-1\}]^s$.

Let $\langle b_0, b_1, b_2, \ldots \rangle, \langle b'_0, b'_1, b'_2, \ldots \rangle \in [A]^\omega$. We write $\langle b_0, b_1, b_2 \ldots \rangle \sim_{(G,r)} \langle b'_0, b'_1, b'_2 \ldots \rangle$ if $\langle b_0, b_1, \ldots, b_{m-1} \rangle \sim_{(G,r)} \langle b'_0, b'_1, \ldots, b'_{m-1} \rangle$ for all $m < \omega$ with $r < m$.

If $I \subseteq A$ is (G, r)-free, then every function defined on a finite subset of $[I]^{\geq r}$ can be coded by an element of I via the functions from G.

Observations: Let $r < \omega$. Let I be a (G, r)-free subset of ω with $G = \langle g_n : [\omega]^n \to \omega \rangle_{1<n<\omega}$. Then,

1. I is infinite with no last element.

2. If B is a finite subset of I, then $I \setminus B$ is also (G, r)-free. In particular, for each $n < \omega$, the set $\{i \in I : i > n\}$ is also (G, r)-free.

3. If $m \leq r$, then for any $\langle b_0, b_1, \ldots, b_{m-1}\rangle, \langle b'_0, {b'}_1, \ldots, {b'}_{m-1}\rangle \in [\omega]^m$, $\langle b_0, b_1, \ldots, b_{m-1}\rangle \sim_{(G,r)} \langle {b'}_0, {b'}_1, \ldots, {b'}_{m-1}\rangle$.

4. Let $I = \{b_0, b_1, b_2, \ldots\}$ with $b_0 < b_1 < b_2 < \cdots$. Let $J = \{c_0, c_1, c_2, \ldots\}$ be another subset of ω with $c_0 < c_1 < c_2 < \cdots$. If $\langle b_0, b_1, b_2, \ldots\rangle \sim_{(G,r)} \langle c_0, c_1, c_2, \ldots, \rangle$, then J is also (G, r)-free.

5. Let $0 < n < \omega$, $\langle b_0, b_1, \ldots, b_{n-1}\rangle \in [I]^n$, $\langle c_0, c_1, \ldots, c_{n-1}\rangle \in [\omega]^n$, and $\langle b_0, b_1, \ldots, b_{n-1}\rangle \sim_{(G,r)} \langle c_0, c_1, \ldots, c_{n-1}\rangle$. Then for each $c \in \omega$ with $c > c_{n-1}$, there exists $b \in I$ such that $\langle b_0, b_1, \ldots, b_{n-1}, b\rangle \sim_{(G,r)} \langle c_0, c_1, \ldots, c_{n-1}, c\rangle$. In fact, if $n \geq r$ and $f : [\{b_0, b_1, \ldots, b_{n-1}\}]^{\geq r} \to \omega$ is defined by $f(b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}) = g_{s+1}(c_{i_0}, c_{i_1}, \ldots, c_{i_{s-1}}, c)$ for $s \geq r$ and $\langle b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}\rangle \in [\{b_0, b_1, \ldots, b_{n-1}\}]^s$, then by (G, r)-freeness of I there exists $b \in I$ that codes f:

$$\begin{aligned} g_{s+1}(b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}, b) &= f(b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}) \\ &= g_{s+1}(c_{i_0}, c_{i_1}, \ldots, c_{i_{s-1}}, c) \end{aligned}$$

   for all $s \geq r$ and $\langle b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}\rangle \in [\{b_0, b_1, \ldots, b_{n-1}\}]^s$.

The following lemma will be the main combinatorial tool used in the proof of Theorem 4.2.6 in the next section. The proof is technical, but uses only elementary arguments.

Lemma 4.1.2. *Let* $0 < r < \omega$, $G = \{g_n : [\omega]^n \to \omega : 1 < n < \omega\}$ *and* $I \subseteq \omega$ *be* (G, r)*-free. Let* $F : [I]^r \to \{0, 1\}$*. Then there is a* (G, r)*-free subset* J *of* I *such that* F *is constant on* $[J]^r$*.*

Proof. Let $n_0, n_1, n_2, \ldots$ be the list of elements of I with $n_0 < n_1 < n_2 < \cdots$. We prove an equivalent statement, that if $0 < k \leq r$ and $F : [I]^k \to 2$, then there exists a (G, r)-free subset J of I on which F is constant. Proceeding inductively on k, for each $k < r$ we assume that both $F^{-1}(0)$ and $F^{-1}(1)$ are infinite, because if $F^{-1}(i)$ is finite, then $I \setminus \bigcup F^{-1}(i)$ will be (G, r)-free by Observation 2.

Suppose $k = 1$. We try to construct a sequence $c_0 < c_1 < c_2 < \cdots$ of elements of I so that for each $i < \omega$, $\langle c_0, c_1, \ldots, c_{i-1}\rangle \sim_{(G,r)} \langle n_0, n_1, \ldots, n_{i-1}\rangle$ and $F(c_i) = 0$. If we succeed, then the set $\{c_0, c_1, c_2, \ldots\}$ will be (G, r)-free and F will be constant on it. By Observation 3 above, there exist r-elements $c_0, c_1, \ldots, c_{r-1}$ such that $\langle c_0, c_1, \ldots, c_{r-1}\rangle \sim_{(G,r)} \langle n_0, n_1, \ldots, n_{r-1}\rangle$ and $F(c_j) = 0$ for all $j < r$. Suppose we fail to proceed at the $(p+1)$-st step for some $p \geq r$ and have a finite sequence $\langle c_0, c_1, \ldots, c_{p-1}\rangle \in [I]^p$ such that

$\langle c_0, c_1, \ldots, c_{p-1}\rangle \sim_{(G,r)} \langle n_0, n_1, \ldots, n_{p-1}\rangle$, $F(c_j) = 0$ for all $j < p$, and for no $c \in I$ with $F(c) = 0$ $\langle c_0, c_1, \ldots, c_{p-1}, c\rangle \sim_{(G,r)} \langle n_0, n_1, \ldots, n_{p-1}, n_p\rangle$. Then, inductively build an increasing sequence $b_0, b_1, b_2, \ldots$ such that $c_{p-1} < b_0 < b_1 < \cdots$ and for each $i < \omega$ $\langle c_0, c_1, \ldots, c_{p-1}, b_i\rangle \sim_{(G,r)} \langle n_0, n_1, \ldots, n_{p-1}, n_p\rangle$ and $\langle b_0, b_1, \ldots, b_i\rangle \sim_{(G,r)} \langle n_0, n_1, \ldots, n_i\rangle$. Choose $b_0 \in I$ such that $c_{p-1} < b_0$ and $\langle c_0, c_1, \ldots, c_{p-1}, b_0\rangle \sim_{(G,r)} \langle n_0, n_1, \ldots, n_{p-1}, n_p\rangle$. Such b_0 exists by Observation 5 above. Note that $F(b_0) = 1$ because there is no such b_0 with $F(b_0) = 0$. For each $0 < i < \omega$, to find b_i consider the function $f : [\{c_0, c_1, \ldots, c_{p-1}\} \cup \{b_0, b_1, \ldots, b_{i-1}\}]^{\geq r} \to \omega$ defined by

$$\begin{aligned} f(c_{i_0}, c_{i_1}, \ldots, c_{i_{s-1}}) &= g_{s+1}(n_{i_0}, n_{i_1}, \ldots, n_{i_{s-1}}, n_p) \\ &\qquad \text{if } r \leq s \text{ and } \langle c_{i_0}, c_{i_1}, \ldots, c_{i_{s-1}}\rangle \in [\{c_0, c_1, \ldots, c_{p-1}\}]^s, \\ f(b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}) &= g_{s+1}(n_{i_0}, n_{i_1}, \ldots, n_{i_{s-1}}, n_i) \\ &\qquad \text{if } r \leq s \text{ and } \langle b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}\rangle \in [\{b_0, b_1, \ldots, b_{i-1}\}]^s, \\ f = n_0 &\qquad \text{otherwise.} \end{aligned}$$

There exists such b_i that codes f via G by (G, r)-freeness of I. Then, for each $i < \omega$, $F(b_i) = 1$, and $J = \{b_0, b_1, b_2, \ldots\}$ is the desired subset of I.

Suppose $0 < k < r$, and for any function from $[I]^k$ to $\{0, 1\}$ there exists a (G, r)-free subset of I on which the function is constant. Let $F : [I]^{k+1} \to$

$\{0,1\}$ be an arbitrary function. Inductively construct a sequence $c_0 < c_1 < c_2 < \cdots$ of elements of I and a sequence $I = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ so that for each $i < \omega$

(1) $c_i = \min X_i < c_{i+1}$,

(2) $\{c_0, c_1, \ldots, c_i\} \cup X_{i+1} \sim_{(G,r)} I,$ and

(3) whenever $x, y \in X_{i+1}$ and $\langle i_0, i_1, \ldots, i_{k-1}\rangle \in [i+1]^k,$ then

$$F(c_{i_0}, c_{i_1}, \ldots, c_{i_{k-1}}, x) = F(c_{i_0}, c_{i_1}, \ldots, c_{i_{k-1}}, y).$$

Let $X_0 = I$ and $c_0 = \min X_0$. Let $i < \omega$ and suppose we have constructed $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_i$ and $c_0 < c_1 < c_2 \cdots < c_i = \min X_i$ satisfying the conditions. We want to find $X_{i+1} \subseteq X_i$ such that $\langle c_0, c_1, \ldots, c_i\rangle \cup X_{i+1} \sim_{(G,r)} I$, and for each element $\langle c_{i_0}, c_{i_1}, \ldots, c_{i_{k-1}}\rangle$ of $[c_0, c_1, \ldots, c_i]^k$ the unary function $F(c_{i_0}, c_{i_1}, \ldots, c_{i_{k-1}}, x)$ is constant on X_{i+1}. Note that $[c_0, c_1, \ldots, c_i]^k$ is finite and for each element $\langle c_{i_0}, c_{i_1}, \ldots, c_{i_{k-1}}\rangle$ of $[c_0, c_1, \ldots, c_{i-1}]^k$ the property that the unary function $F(c_{i_0}, c_{i_1}, \ldots, c_{i_{k-1}}, x)$ is constant on X_i is already established. So, we list all the elements of $[c_0, c_1, \ldots, c_i]^k \setminus [c_0, c_1, \ldots, c_{i-1}]^k$, say $\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_{t-1}$ where $t = |[c_0, c_1, \ldots, c_i]^k \setminus [c_0, c_1, \ldots, c_{i-1}]^k|$, and construct a finite sequence $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{t-1}$ such that $X_i \supseteq Y_0$, $c_i < \min Y_0$, and for each $n < t$

$$\langle c_0, c_1, \ldots, c_i\rangle \cup Y_n \sim_{(G,r)} I; \text{ and}$$

$$\text{the function } F_n : Y_n \to \{0,1\} \text{ defined by}$$

$$F_n(x) = F(\bar{c}_n, x) \text{ is constant on } Y_n.$$

Y_n's can be found similarly to the case of $k = 1$. We let $X_{i+1} = Y_{t-1}$. Then, X_{i+1} and $c_{i+1} = \min X_{i+1}$ will have the desired property.

Let $X = \{c_0, c_1, c_2, \ldots\}$. Then, clearly, $X \sim_{(G,r)} I$, and so X is (G,r)-free. Moreover, for each finite sequence $\langle c_{i_0}, c_{i_1}, \cdots, c_{i_{k-1}}\rangle \in [X]^k$, the unary function $F(c_{i_0}, c_{i_1}, \ldots, c_{i_{k-1}}, x)$ is constant on the set $\{x \in X : x > c_{i_{k-1}}\}$. Define $H : [X]^k \to 2$ so that $H(x_0, x_1, \cdots, x_{k-1}) = F(x_0, x_1, \cdots, x_{k-1}, c)$ for some (or every) $c \in X$ with $c > x_{k-1}$. Then, by the induction hypothesis we have a (G,r)-free subset J of X on which H is constant. Then for all $\langle x_{i_0}, x_{i_1}, \cdots, x_{i_k}\rangle, \langle x'_{i_0}, x'_{i_1}, \cdots, x'_{i_k}\rangle \in [J]^{k+1}$, we have

$$\begin{aligned}&F(x_{i_0}, x_{i_1}, \cdots, x_{i_k}) = H(x_{i_0}, x_{i_1}, \cdots, x_{i_{k-1}})\\ &= H(x'_{i_0}, x'_{i_1}, \cdots, x'_{i_{k-1}}) = F(x'_{i_0}, x'_{i_1}, \cdots, x'_{i_k}).\end{aligned}$$

So, J is as desired. □

4.2 Schmerl's Theorem

Various concepts of expandability of a structure are required to formulate Schmerl's theorem in its full generality; this section begins with those concepts.

Definition 4.2.1. Let $\mathcal{L}$ be a first order language and $\mathfrak{A}$ be an $\mathcal{L}$-structure. Let R be a relation symbol not in $\mathcal{L}$ and let $\varphi(R, \bar{x})$ be a formula of the expanded language $\mathcal{L} \cup \{R\}$. For $\bar{a} \in A$, we write $\mathfrak{A} \models \exists R \varphi(R, \bar{a})$ to indicate that there is a relation R on $\mathfrak{A}$ such that $(\mathfrak{A}, R) \models \varphi(R, \bar{a})$.

An $\mathcal{L}$-structure $\mathfrak{A}$ is *resplendent* if whenever $a_0, a_1, \ldots, a_{n-1} \in A$ and $\varphi(\bar{x}, R)$ is an n-ary $(L \cup \{R\})$-formula such that $\mathfrak{B} \models \exists R \varphi(\bar{a}, R)$ for some $\mathfrak{B} \succ \mathfrak{A}$, then $\mathfrak{A} \models \exists R \varphi(\bar{a}, R)$. $\mathfrak{A}$ is *chronically resplendent* if it is both resplendent and, whenever $\mathfrak{A} \models \exists R \varphi(\bar{a}, R)$, there exists a relation R on $\mathfrak{A}$ such that $(\mathfrak{A}, R)$ is resplendent and $(\mathfrak{A}, R) \models \varphi(\bar{a}, R)$. $\mathfrak{A}$ is *totally resplendent* if there are countably many relations $R_0, R_1, R_2, \ldots$ on A such that each expansion $(\mathfrak{A}, R_0, R_1, \ldots, R_{n-1})$ is resplendent; moreover, if $(\mathfrak{A}, R_0, R_1, \ldots) \models \exists R \varphi(\bar{a}, R)$, then there exists a relation R parametrically definable in $(\mathfrak{A}, R_0, R_1, \ldots)$ such that $(\mathfrak{A}, R_0, R_1, \ldots, R) \models \varphi(\bar{a}, R)$.

Resplendence was defined by J. Barwise and J. Schlipf [BS] and J. P. Ressayre [Ressayre]; total resplendence was defined by Schmerl [Schmerl2].

Clearly totally resplendent structures are chronically resplendent and chronically resplendent structures are resplendent. The theorems below show that the notions of recursive saturation, resplendence, chronic resplendence, and total resplendence coincide in countable structures over a finite language. The formulation of the following theorem is taken from [Kaye1]

Theorem 4.2.2 (Barwise, Schlipf, Ressayre)**.** *Let $\mathcal{L}$ be a recursive language and $\mathcal{M}$ be a countable recursively saturated $\mathcal{L}$-structure. Let $\bar{a} \in M$. Let $\mathcal{L}'$ be a recursive extension of $\mathcal{L} \cup \{\bar{a}\}$ and T a recursively axiomatized $\mathcal{L}'$-theory. Then, if* $\mathrm{Th}(\mathcal{M}, \bar{a}) + T$ *is consistent, there is an expansion of $(\mathcal{M}, \bar{a})$ to $\mathcal{L}'$ satisfying T that is recursively saturated as an $\mathcal{L}'$-structure.*

The above theorem shows that every countable recursively saturated structure over a recursive language is chronically resplendent and is again recursively saturated. If starting with a countable recursively saturated structure, this allows for repeated expansion of a structure with respect to a sequence of formulas of a expanded language. This is one of the main tools used in the proof of Schmerl's theorem.

Corollary 4.2.3. *Let $\mathcal{L}$ be a countable language and $\mathcal{M}$ be a countable recursively saturated $\mathcal{L}$-structure. Then $\mathcal{M}$ is totally resplendent.*

Proof. For simplicity, we start by assuming that $\mathcal{M}$ has a pairing function $\prec -, - \succ$: $M^2 \to M$ which is one-to-one and onto. This is possible by Theorem 4.2.2 because the bijective condition of $\prec -, - \succ$ can be expressed by a formula and such a function can be found in $\mathcal{M}$. Let $\mathcal{L}(\mathcal{M})$ be a language $\mathcal{L}$ together with a new constant symbol a for each $a \in M$. Let $\mathcal{L}' = \mathcal{L}(\mathcal{M}) \cup \{R_1, R_2, \ldots\} \cup \{R\}$ where R_i and R are new unary relation symbols. Let $\varphi_1(R), \varphi_2(R), \varphi_3(R), \ldots$ be an enumeration of all $\mathcal{L}'$-sentences containing R where $\varphi_i(R)$ is a sentence over $\mathcal{L}(\mathcal{M}) \cup \{R_1, \ldots, R_{i-1}\} \cup \{R\}$. For each i, if $\varphi_i(R)$ is consistent with $(\mathcal{M}, R_1^{\mathcal{M}}, \ldots, R_{i-1}^{\mathcal{M}})$, then by Theorem 4.2.2 there is a relation $R_i^{\mathcal{M}}$ interpreting R_i such that $(\mathcal{M}, R_1^{\mathcal{M}}, \ldots, R_i^{\mathcal{M}})$ is recursively saturated. This produces a sequence of relations $R_1^{\mathcal{M}}, R_2^{\mathcal{M}}, \ldots$. We claim that the sequence of relations witnesses total resplendence of $\mathcal{M}$. Suppose for $\bar{a} \in M$, $(\mathcal{M}, R_0^{\mathcal{M}}, R_1^{\mathcal{M}}, \ldots, S^{\mathcal{M}}) \vDash \varphi(\bar{a}, S)$ where S is an n-ary new relation symbol for some $n \in \mathbb{N}$. Define a formula $\varphi'(\bar{a}, R)$ from $\varphi(\bar{a}, S)$ by replacing every occurrence of the subformulas of the form $\bar{x} \in S$ by $\prec \bar{x} \succ \in R$. Then by the construction of the sequence $R_1^{\mathcal{M}}, R_2^{\mathcal{M}}, \ldots$, there is $R_l^{\mathcal{M}}$ for some $l \in \mathbb{N}$ interpreting R such that $(\mathcal{M}, R_1^{\mathcal{M}}, \ldots, R_l^{\mathcal{M}}) \vDash \varphi'(\bar{a}, R)$. So, the relation S' defined by the formula $\prec \bar{x} \succ \in R_l$ in $\mathcal{M}$ realizes $\varphi(\bar{a}, S)$. □

The following theorem and corollary show that the converse of 4.2.2 is also true; their formulations are taken from [Kaye1].

Theorem 4.2.4 (Kleene). *Let $\mathcal{L}$ be a finite language. Let $\{\theta_i(\bar{x}) : i < \omega\}$ be a recursive set of $\mathcal{L}$-formulas involving finitely many free variables. Then there is a formula $\Theta(X, \bar{x})$ of the expanded language $\mathcal{L} \cup \{X\}$ with X a new* *relation symbol such that in any infinite $\mathcal{L}$-structure $\mathcal{M}$, for all $\bar{a} \in M$,*

$$\mathcal{M} \vDash \exists X \Theta(X, \bar{a}) \textit{ iff } \mathcal{M} \vDash \theta_i(\bar{a}) \quad \textit{for all } i < \omega.$$

Corollary 4.2.5. *Let $\mathcal{L}$ be a finite language. If an $\mathcal{L}$-structure $\mathcal{M}$ is resplendent, it is recursively saturated.*

Let **CFF** be the $\mathcal{L}$-theory as in the 3.1.1. For the rest of this section, we fix a finite language $\mathcal{L}$ with the binary function symbol β and a countable recursively saturated $\mathcal{L}$-structure $\mathfrak{A}$ which is a model of **CFF**. A term $t(\bar{x})$ is a *β-term* if it contains no function symbols other than β, and for $I \subseteq A$, a subset B of A is the *β-closure* of I if it is the smallest subset of A which contains I and closed under β. Note that if B is the β-closure of I, then every element of B is generated as $t(i_0, i_1, \ldots, i_{n-1})$ for some β-term t and some $(i_0, i_1, \ldots, i_{n-1}) \in I^n$. In addition, $\mathfrak{A}$ is *β-generated by I* if A itself is the β-closure of $I \subseteq A$.

Theorem 4.2.6. (J. Schmerl, [Schmerl2]) *Let $\mathfrak{A}$ be a countable recursively saturated model of **CFF**. Then there is an indiscernible type Σ in the language $\mathcal{L}$ such that if I is a linearly ordered set with no last element and $T = \{\varphi(i_0.\ldots, i_{n-1}) : n \in \mathbb{N},\ \varphi(x_0, \ldots, x_{n-1}) \in \Sigma,$ and $\langle i_0, \ldots, i_{n-1}\rangle \in [I]^n\}$, then every model $\mathfrak{C}$ of T has the elementary substructure $\mathfrak{B}$ which* *is β-generated by I and is totally resplendent, and such that $\mathfrak{B} \equiv_{\infty,\omega} \mathfrak{A}$ as $\mathcal{L}$-structures.*

Proof. We start by expanding $\mathfrak{A}$ to a recursively saturated linearly ordered structure $(\mathfrak{A}, <)$ satisfying the sentences:

$$\forall x_0, \ldots, x_{n-1} \forall y_0, \ldots, y_{n-1} \forall z \exists x > z \left[\bigwedge_{i<j<n} x_i \neq x_j \rightarrow \bigwedge_{i<n} \beta(x_i, x) = y_i \right]$$

for $n < \omega$. Here, $<$ is a new relation symbol not in $\mathcal{L}$. If A is linearly ordered by $<$ of the order type ω, then the above sentences are clearly satisfied. Since the set of above sentences is recursive, $\mathfrak{A}$ can be expanded by Theorem 4.2.2 to a linearly ordered structure $(\mathfrak{A}, <)$ which satisfies the above sentences and is recursively saturated.

$\mathfrak{A}$ is also assumed to have a bijection $\prec -, - \succ: A^2 \rightarrow A$ (pairing function), again by using Theorem 4.2.2.

The following lemma is necessary to define a sequence of functions for freeness of $\mathfrak{A}$.

Lemma 4.2.7. *There are distinct $a_0, a_1, a_2, \ldots \in A$ such that whenever $0 < n < \omega$, then $\beta(a_n, a_0) = a_{n+1}$.*

Proof. This requires two elements $a_0, a_1 \in A$ realizing the set of formulas consisting of

$$x \neq y$$

$$\beta(y,x) \neq x \wedge \beta(y,x) \neq y$$

$$\beta(\beta(y,x),x) \neq x \wedge \beta(\beta(y,x),x) \neq y \wedge \beta(\beta(y,x),x) \neq \beta(y,x)$$

$$\vdots$$

where $\beta(y,x), \beta(\beta(y,x),x), \beta(\beta(\beta(y,x),x),x), \ldots$ are intended to represent $a_2, a_3, a_4, \ldots$. Since this set is recursive, by recursive saturation it suffices to show that for each $n < \omega$ there are distinct $a_0, a_1, a_2, \ldots, a_{n+2} \in A$ such that $\beta(a_{i+1}, a_0) = a_{i+2}$ for $i \leq n$. Choose any distinct $a_1, a_2, \ldots, a_{n+2}$ from A. Then since $\mathfrak{A} \models$ **CFF**, there are infinitely many x for which $\bigwedge_{i \leq n+1} \beta(a_{i+1}, x) = a_{i+2}$. Choose a_0 to be such an x not in $\{a_1, a_2, \ldots, a_{n+2}\}$.

□

Fix some notation; let $a_0, a_1, a_2, \ldots \in A$ be the sequence as in the above lemma throughout the proof. For $n \geq 2$, set $\beta(x_0, x_1, \ldots, x_n) = \beta(\beta(x_0, x_1, \ldots, x_{n-1}), x_n)$. For each $n \in \mathbb{N}$ with $n > 0$, there is an $(n+1)$-ary function $\beta(a_{n+1}, -)$ from $[A]^{n+1}$ to A. To discuss freeness in $\mathfrak{A}$, let $G = \{\beta(a_{n+1}, -) : 0 < n < \omega\}$. Let $d_0, d_1, d_2, \ldots$ be an enumeration of the elements of A.

To get a desired indiscernible type Σ, we will construct a sequence of recursively saturated expansions $\mathfrak{A}_n$ of $\mathfrak{A}$, where $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_{n+1} = (\mathfrak{A}_n, I_n, R_n, d_n) = (\mathfrak{A}, I_0, I_1, \dots, I_n, R_0, R_1, \dots, R_n, d_0, d_1, \dots, d_n)$ with I_n's and R_n's being new unary relations on $\mathfrak{A}_{n+1}$ and d_n's being new constants. The new constant symbols d_n's are interpreted by d_n's in $\mathfrak{A}$. Let $\mathfrak{A}_\omega = \bigcup_{n<\omega} \mathfrak{A}_n$, and for $n \leq \omega$ let $\mathcal{L}_n$ be the language appropriate for $\mathfrak{A}_n$. Let $\langle \varphi_n(x_0, \dots, x_{n-1}, y) : 0 < n < \omega \rangle$ be a list of $\mathcal{L}_\omega$-formulas such that φ_n is an $(n+1)$-ary $\mathcal{L}_n$-formula and each $\mathcal{L}_\omega$-formula with free variables among $y, x_0, x_1, x_2, \dots$ is equivalent to one in the list. Let $\langle \psi_n(R) : n < \omega \rangle$ be a list of the $(\mathcal{L}_\omega \cup \{R\})$-sentences with R being a new unary relation symbol such that $\psi_n(R)$ is an $\mathcal{L}_n \cup \{R\}$-formula. We will construct the sequence so that $\mathfrak{A}_0 = \mathfrak{A}$ and for each $n < \omega$

(1.1) $\mathfrak{A}_{n+1} = (\mathfrak{A}_n, d_n, I_n, R_n)$ is recursively saturated,

(1.2) $I_0 \subseteq A$ and if $n > 0$, $I_n \subseteq I_{n-1}$,

(1.3) for all $\langle b_0, b_1 \rangle \in [I_n]^2, \beta(b_0, b_1) = a_0$ and $\beta(a_0, b_0) = a_1$.

(1.4) I_n is an n-indiscernible sequence in $\mathfrak{A}_n$,

(1.5) If $n > 1$, $\langle b_0, \dots, b_{n-1} \rangle \in [I_n]^n$, and $\mathfrak{A}_n \models \exists y \varphi_{n-1}(b_0, b_1, \dots, b_{n-2}, y)$, then

$$\mathfrak{A}_n \models \varphi_{n-1}(b_0, b_1, \dots, b_{n-2}, \beta(a_n, b_0, b_1, \dots, b_{n-2}, b_{n-1})), \text{ and}$$

(1.6) If $\mathfrak{A}_n \models \exists R \psi_n(R)$, then $\mathfrak{A}_n \models \psi_n(R_n)$.

Recall that (1.4) means that I_n is indiscernible over all $\mathcal{L}_n$-formulas with n free variables, i.e., for all $\mathcal{L}_n$-formulas $\varphi(x_0, x_1, \ldots, x_{n-1})$ with n free variables and all $\langle \bar{b} \rangle, \langle \bar{c} \rangle \in [I_n]^n$, $\mathfrak{A}_n \vDash \varphi(\bar{b}) \leftrightarrow \varphi(\bar{c})$.

Before constructing the sequence of $\mathfrak{A}_n$'s, we show how it works. Suppose that the sequence has been constructed. Let Σ_ω be the set of all $\mathcal{L}_\omega$-formulas $\varphi(x_0, x_1, \ldots, x_{n-1})$ such that for all sufficiently large r (i.e., there exists some $s < \omega$ such that for all $r > s$), whenever $\langle b_0, b_1, \ldots, b_{n-1} \rangle \in [I_r]^n$, then $\mathfrak{A}_\omega \vDash \varphi(\bar{b})$. Note that $\Sigma_\omega \vdash \mathrm{Th}_A(\mathfrak{A}_\omega)$ where $\mathrm{Th}_A(\mathfrak{A}_\omega)$ is the theory of the structure $(\mathfrak{A}, \{a : a \in A\})$ with a new constant a for each $a \in A$. We claim that Σ_ω is complete. Let $\psi(x_0, \ldots, x_{n-1})$ be an $\mathcal{L}_\omega$-formula. Then, there is some $r > n$ such that $\psi(x_0, \ldots, x_{n-1})$ is an $\mathcal{L}_r$-formula, and so for all $s \geq r$, I_s is $\{\psi(x_0, \ldots, x_{n-1})\}$-indiscernible. Hence, $\psi(x_0, \ldots, x_{n-1})$ or $\neg\psi(x_0, \ldots, x_{n-1})$ is in Σ_ω.

Let I be a linearly ordered set with no last element and let $T_\omega = \{\varphi(i_0, \ldots, i_{n-1}) : n < \omega, \langle i_0, \cdots, i_{n-1} \rangle \in [I]^n, \varphi(\bar{x}) \in \Sigma_\omega\}$. By the construction of Σ_ω, T_ω is consistent and in every model of T_ω, Σ_ω is the indiscernible type of I.

Lemma 4.2.8. *Let I and T_ω be as above. Let $\mathfrak{C}_\omega$ be a model of T_ω and B_ω be the β-closure of I in $\mathfrak{C}_\omega$. Then B_ω is an elementary substructure of $\mathfrak{C}_\omega$,* *and so it is a model of* T_ω β*-generated by* I *having indiscernible type* Σ_ω.

Proof. Tarski-Vaught Test will be used to show that B_ω is an elementary substructure of $\mathfrak{C}_\omega$. Recall that every $\mathcal{L}_\omega$-formula is equivalent to some formula in the list $\langle \varphi_n(x_0, x_1, \ldots, x_{n-1}, y) \ : \ 0 < n < \omega \rangle$. So, suppose $0 < n < \omega$, $(c_0, c_1, \ldots, c_{n-1}) \in B_\omega^n$, and $\mathfrak{C}_\omega \models \exists y \varphi_n(\bar{c}, y)$. Then, there exist some $\langle i_0, i_1, \ldots, i_{k-1} \rangle \in [I]^k$ for some $k < \omega$ and β-terms $t_0, t_1, \ldots, t_{n-1}$ such that $t_j(i_0, i_1, \ldots, i_{k-1}) = c_j$ for all $j < n$. So, we can assume without loss of generality that $\langle c_0, c_1, \ldots, c_{n-1} \rangle \in [I]^n$. Since Σ_ω is complete, $\exists y \varphi_n(\bar{x}, y) \in \Sigma_\omega$. So, by the construction of Σ_ω, for all sufficiently large r, $\mathfrak{A}_\omega \models \forall \langle \bar{x} \rangle \in [I_r]^n \exists y \varphi_n(\bar{x}, y)$ and, by (1.5), whenever $\langle b_0, b_1, \ldots, b_n \rangle \in [I_r]^{n+1}$, then $\mathfrak{A}_\omega \models \varphi_n(b_0, b_1, \ldots, b_{n-1}, \beta(a_{n+1}, b_0, b_1, \ldots, b_n))$. Since by (1.3), a_{n+1} is in the β-closure of any two elements of I_{n+1} in $\mathfrak{A}$, there exists some β-term $s(x_0, x_1, \ldots, x_n)$ such that whenever $\langle b_0, b_1, \ldots, b_n \rangle \in [I_r]^{n+1}$, then $\mathfrak{A}_\omega \models \varphi_n(b_0, b_1, \ldots, b_{n-1}, s(b_0, b_1, \ldots, b_n))$. So, the formula $\varphi_n(x_0, x_1, \ldots, x_{n-1}, s(x_0, x_1, \ldots, x_n))$ is in Σ_ω and $\mathfrak{C}_\omega \models \varphi_n(c_0, c_1, \ldots, c_{n-1}, s(c_0, c_1, \ldots, c_n))$ for all $c_n \in I$ greater than c_{n-1} in I if there is any. Since I is a linearly ordered set with no last element, there are plenty of such c_n's in I. Since $s(c_0, c_1, \ldots, c_n)$ is in B_ω, $\mathfrak{B}_\omega \prec \mathfrak{C}_\omega$. □

Let $\Sigma = \Sigma_\omega \upharpoonright_{\mathcal{L}}$ be the reduct of Σ_ω to $\mathcal{L}$ and $T =$

$\{\varphi(i_0, \ldots, i_{n-1}) : n < \omega, \langle i_0, \cdots, i_{n-1}\rangle \in [I]^n, \varphi(\bar{x}) \in \Sigma\}$. Let $\mathfrak{C}$ be a model of T and B be the β-closure of I in $\mathfrak{C}$. Then, the same argument as in the above proof shows that B is the elementary substructure of $\mathfrak{C}$ β-generated by I having indiscernible type Σ. Let $\mathfrak{C}_\omega$ be a model of T_ω and $\mathfrak{B}_\omega$ the β-closure of I in $\mathfrak{C}_\omega$ as in the above lemma. Then, by Lemma 2.3.3, $\mathfrak{B}$ is isomorphic to the reduct of $\mathfrak{B}_\omega$ to $\mathcal{L}$. So, $\mathfrak{B}_\omega$ can be considered as an expansion of $\mathfrak{B}$ to $\mathcal{L}_\omega$ β-generated by I having indiscernible type Σ_ω.

Lemma 4.2.9. *Let $\mathfrak{B}$ and $\mathfrak{B}_\omega$ be as above. Then, $\mathfrak{A}$ and $\mathfrak{B}$ realize the same $\mathcal{L}$-types.*

Proof. Let $p(\bar{x})$ be a type of $\mathfrak{B}$ realized by $(c_0, c_1, \ldots, c_{k-1}) \in B^k$. Since $\mathfrak{B}$ is β-generated by I, there are β-terms, $t_0, t_1, \ldots, t_{k-1}$ and $\langle i_0, i_1, \cdots, i_{m-1}\rangle \in [I]^m$ for some $m < \omega$ such that $c_j = t_j(i_0, i_1, \ldots, i_{m-1})$ for all $j < k$. For each $\varphi(\bar{x}) \in p(\bar{x})$, let $\varphi'(\bar{y}) = \varphi(t_0(y_0, \ldots, y_{m-1}), \ldots, t_{k-1}(y_0, \ldots, y_{m-1}))$. Then, the type $p'(\bar{y}) = \{\varphi'(\bar{y}) : \varphi(\bar{x}) \in p(\bar{x})\}$ is realized by $\langle i_0, i_1, \ldots, i_{m-1}\rangle \in [I]^m$ in $\mathfrak{B}$. Since Σ is complete, each formula $\varphi'(\bar{y})$ in $p'(\bar{y})$ is in Σ and is realized in $\mathfrak{A}$ by every increasing m-sequence of elements of $I_r \subseteq A$ for all sufficiently large r's. Moreover, since if $r > m$ then $I_r \subseteq I_m$ and I_m is m-indiscernible, $p'(\bar{y})$ is realized in $\mathfrak{A}$ by every increasing m-sequence of I_m. So, if $\langle b_0, b_1, \ldots, b_{m-1}\rangle \in [I_m]^m$ and $t_i(b_0, b_1, \ldots, b_{m-1}) = e_i$ ($\in A$) for each

$i < m$, then $\bar{e}$ realizes $p(\bar{x})$ in $\mathfrak{A}$. Conversely, suppose $q(\bar{x})$ is a type of $\mathfrak{A}$ and is realized by some $(e_0, e_1, \ldots, e_{k-1}) \in A^k$. Since $T_\omega \supseteq \mathrm{Th}_A(\mathfrak{A}_\omega)$ and e_j are already in B_ω by the condition (1.1), $q(\bar{x})$ is realized in $\mathfrak{B}_\omega$ and thus in $\mathfrak{B}$. $\square$

It only remains to show that $\mathfrak{B}$ is totally resplendent because, then since $\mathfrak{A}$ and $\mathfrak{B}$ are recursively saturated, realize the same types, and $\mathfrak{A} \equiv \mathfrak{B}$, by Theorem 1.4 we have $\mathfrak{A} \equiv_{\infty,\omega} \mathfrak{B}$.

Lemma 4.2.10. *$\mathfrak{B}$ is totally resplendent.*

Proof. The relations $R_0, R_1, \ldots$ of $\mathfrak{B}_\omega$ will witness that $\mathfrak{B}_\omega$ is totally resplendent and thus so is $\mathfrak{B}$. Let $b_0, b_1, \ldots, b_{m-1} \in B$ and $\varphi(\bar{x}, R)$ be an $\mathcal{L}_\omega \cup \{R\}$-formula. Suppose $\exists R \varphi(\bar{b}, R)$ is satisfied in some elementary extension of $\mathfrak{B}_\omega$. Since $\mathfrak{B}_\omega$ is β-generated by I, we can, without loss of generality, assume $\langle b_0, b_1, \ldots, b_{m-1} \rangle \in [I]^m$. Let $k < \omega$ be such that $m < (k+1)$ and $\varphi(\bar{x}, R)$ be an $\mathcal{L}_{k+1} \cup \{R\}$-formula, and let $\mathrm{Th}_{k+1}(\mathfrak{B}_\omega, \bar{b})$ be the theory of the structure $(\mathfrak{B}_\omega, \bar{b})$ restricted to $\mathcal{L}_{k+1} \cup \{\bar{b}\}$. Then, $\mathrm{Th}_{k+1}(\mathfrak{B}_\omega, \bar{b}) + \varphi(\bar{b}, R)$ is consistent. Note that if $\psi(x_0, x_1, \ldots, x_m)$ is an $\mathcal{L}_{k+1}$-formula, then $\mathfrak{B}_\omega \models \psi(\bar{b})$ iff $\psi(\bar{x}) \in \Sigma_\omega$ iff $\mathfrak{A}_{k+1} \models \forall \langle \bar{x} \rangle \in [I_k]^m \psi(\bar{x})$ by the construction of Σ_ω and the conditions (1.2) and (1.4). So, if $\langle \bar{i} \rangle \in [I_k]^m$, then $\mathrm{Th}_{k+1}(\mathfrak{A}_\omega, \bar{i}) + \varphi(\bar{i}, R)$ is consistent, and so, by resplendence of $\mathfrak{A}_{k+1}$, there exists $R_{\bar{i}} \subseteq A$ such that

$(\mathfrak{A}_{k+1}, R_{\bar{i}}) \models \varphi(\bar{i}, R_{\bar{i}})$. Let $S = \{\prec \bar{y}, \bar{i} \succ \in A \mid \bar{y} \in R_{\bar{i}} \text{ and } \langle \bar{i} \rangle \in [I_k]^n\}$. Then, $(\mathfrak{A}_{k+1}, S) \models \forall \langle \bar{x} \rangle \in [I_k]^m \varphi'(\bar{x}, S)$ where $\varphi'(\bar{x}, S)$ is the formula obtained from $\varphi(\bar{x}, R)$ by replacing every subformula of the form $\bar{y} \in R$ by $\prec \bar{y}, \bar{x} \succ \in S$. By the construction of the sequence $R_0, R_1, R_2, \ldots$, there exists $l < \omega$ such that $\mathfrak{A}_\omega \models \forall \langle \bar{x} \rangle \in [I_k]^m \varphi'(\bar{x}, R_l)$, and so $\varphi'(\bar{x}, R_l) \in \Sigma_\omega$. Thus, $\mathfrak{B}_\omega \models \varphi'(\bar{b}, R_l)$. If R is the relation defined in $\mathfrak{B}_\omega$ by the formula $\prec \bar{y}, \bar{b} \succ \in R_l$, then $(\mathfrak{B}_\omega, R) \models \varphi(\bar{b}, R)$. Therefore $\mathfrak{B}_\omega$ is totally resplendent and so is $\mathfrak{B}$. □

Now we construct the sequence of $\mathfrak{A}_n$'s. Adding d_n and R_n to satisfy conditions (1.1) and (1.6) is easy due to Theorem 4.2.2. So, discussing adding d_n's and R_n's is unnecessary for the rest of the construction. The only concern is to find I_n satisfying (1.2),(1.3),(1.4), and (1.5); that makes $\mathfrak{A}_{n+1}$ recursively saturated. To facilitate the construction, we add one more condition on I_n:

(1.7) I_n is (G, n)-free.

We want to use recursive saturation of $\mathfrak{A}_n$ and Theorem 4.2.2 to get $\mathfrak{A}_{n+1}$ satisfying (1.1). So, we first show that for each $n < \omega$, (1.2) + (1.3) + (1.4) + (1.5) + (1.7) can be expressed by a recursive set of $\mathcal{L}_{n+1}$-formulas with parameters from a finite subset of A. Expressing (1.2) + (1.3) + (1.5) is straightforward. Condition (1.4) can be expressed by the following recursive set of L_{n+1}-sentences: for each $\mathcal{L}_n$ formula $\varphi(x_0, x_1, \ldots, x_{n-1})$

$$\forall\langle\bar{x}\rangle \in [I_n]^n, \forall\langle\bar{y}\rangle \in [I_n]^n(\varphi(\bar{x}) \leftrightarrow \varphi(\bar{y})).$$

Condition (1.7) requires that for each $k \geq n$, if $x_0 < x_1 < x_2 < \cdots < x_{k-1}$ are elements of I_n and $f : [\{x_0, x_1, \ldots, x_{k-1}\}]^{\geq n} \to A$ is any function, then there are arbitrarily large $y \in I_n$ (i.e., for all $w \in I_n$, there exists $y > w$) such that

$$\beta(a_{s+1}, x_{i_0}, x_{i_1}, x_{i_2}, \ldots, x_{i_{s-1}}, y) = f(x_{i_0}, x_{i_1}, x_{i_2}, \ldots, x_{i_{s-1}})$$

for all $\langle x_{i_0}, x_{i_1}, x_{i_2}, \ldots, x_{i_{s-1}}\rangle \in [\{x_0, x_1, x_2, \ldots, x_{k-1}\}]^{\geq n}$. Let $l_{n,k} < \omega$ denote the cardinality of the set $[\{x_0, x_1, x_2, \ldots, x_{k-1}\}]^{\geq n}$ and let

$$\rho_{n,k} : \{0, 1, 2, \ldots, l_{n,k} - 1\} \to [\{x_0, x_1, x_2, \ldots, x_{k-1}\}]^{\geq n}$$

be an enumeration of $[\{x_0, x_1, x_2, \ldots, x_{k-1}\}]^{\geq n}$. Then, (1.7) is expressed by the following recursive set of $\mathcal{L}_{n+1}$-sentences: for each k with $n \leq k < \omega$

$$\forall\langle x_0, x_1, \ldots, x_{k-1}\rangle \in [I_n]^k \ \forall v_0, v_1, \ldots, v_{l_{n,k-1}} \ \forall z \in I_n,\ \exists y \in I_n \left(y > z \wedge \bigwedge_{i<l_{n,k}} \beta(a_{\mathrm{len}(\rho_{n,k}(i))+1}, \rho_{n,k}(i), y) = v_i \right)$$

where $\mathrm{len}(\rho_{n,k}(i))$ is the length of the sequence $\rho_{n,k}(i)$. Therefore, (1.2) + (1.3) + (1.4) + (1.5) + (1.7) is recursively axiomatized.

For each $n < \omega$ we will show that (1.2) + (1.3) + (1.4) + (1.5) + (1.7) is realized in $\mathfrak{A}_n$. Then, by Theorem 4.2.2 there exists a subset $I_n \subseteq A$ that makes $(\mathfrak{A}_n, I_n)$ recursively saturated.

The following lemma will allow us to obtain I_0.

Lemma 4.2.11. *Let $\mathfrak{A}$ be the structure of Theorem 4.2.6. There exists a $(G, 0)$-free (equivalently $(G, 1)$-free) sequence $b_0 < b_1 < b_2, \cdots$ of elements of A such that for all $\langle i, j\rangle \in [\omega]^2$, $\beta(b_i, b_j) = a_0$ and $\beta(a_0, b_i) = a_1$.*

Proof. We will construct inductively an increasing sequence $b_0, b_1, b_2, \ldots$ so that for each $m < \omega$, $\beta(b_i, b_m) = a_0$ for all $i < m$, $\beta(a_0, b_m) = a_1$, and whenever $f : [\{b_0, b_1, \ldots, b_{m-1}\}]^{>0} \to \{d_0, d_1, \ldots, d_{m-1}\}$ is an arbitrary function, then there exists some $k < \omega$ with $k > m - 1$ such that b_k codes f via the functions in G.

At the 0-th step, find $b_0 \in A$ such that $\beta(a_0, b_0) = a_1$ and $b_0 \neq a_i$ for each $i < \omega$. The condition that $b_0 \neq a_i$ for each $i < \omega$, can be expressed by a recursive set of formulas with parameters a_0 and a_1: $x \neq a_0$, $x \neq a_1$, $x \neq \beta(a_1, a_0)$, $x \neq \beta(\beta(a_1, a_0), a_0), \ldots$. By recursive saturation and β-coding capability of $\mathfrak{A}$, there exists an element b_0 satisfying the conditions. The condition $b_0 \neq a_i$ is required; otherwise, for example, if $b_0 = a_3$, then $\beta(a_3, b_6) = \beta(a_3, b_7) = a_0$ and so any function f with $f(b_6, b_8) \neq f(b_7, b_8)$

can not be coded. By the same reason, we will also require $b_n \neq a_i$ for all $n, i < \omega$.

At the $(m+1)$-st step for $m < \omega$, we start with a finite sequence $b_0 < b_1 < \cdots < b_m < \cdots < b_{t_m - 1}$ from the previous step and add a finite number of elements to the sequence to obtain a sequence $b_0 < b_1 < \cdots < b_{t_m-1} < \cdots < b_{t_{m+1}-1}$ for some $t_{m+1} < \omega$. (So, we will have a sequence of natural numbers $1 = t_0, t_1, t_2, \ldots, t_m, \ldots$ indicating the length of the sequence resulting from each step). For this, we consider all the functions from $[\{b_0, b_1, b_2, \ldots, b_m\}]^{>0}$ to $\{d_0, d_1, \ldots, d_m\}$. There are a finite number of such functions. For each such function, we will add an element coding the function via G to the sequence. So, the number of elements newly added at the $(m+1)$-st step, which is $t_{m+1} - t_m$, will be the same as the number of functions from $[\{b_0, b_1, \ldots, b_m\}]^{>0}$ to $\{d_0, d_1, \ldots, d_m\}$. However, to proceed in this way requires avoiding some bad situations. For example, if $\beta(a_4, b_2, b_3) = \beta(a_5, b_1, b_2, b_4)$, then coding a function f with $f(b_2, b_3, b_5) \neq f(b_1, b_2, b_4, b_5)$ will be impossible. So, we have to require that whenever $r, s < \omega$, $\langle \bar{b} \rangle \in [\{b_0, b_1, b_2, \ldots, \}]^{<r}$, $\langle \bar{b}' \rangle \in [\{b_0, b_1, b_2, \ldots, \}]^{<s}$, and $(a_{r+1}, \bar{b})$ and $(a_{s+1}, \bar{b}')$ are distinct, then

$$\beta(a_{r+1}, \bar{b}) \neq \beta(a_{s+1}, \bar{b}').$$

Let $f : [\{b_0, b_1, \ldots, b_m\}]^{>0} \to \{d_0, d_1, \ldots, d_m\}$ be the function under consideration. Suppose we have constructed $b_0, b_1, \ldots, b_{k-1}$ for $k \geq t_m$ and want to find an element b_k of A that codes f. We list all the properties that we want b_k to have:

(2.1) $b_{k-1} < x$.

(2.2) whenever $i < k$, $\beta(b_i, x) = a_0$, and $\beta(a_0, x) = a_1$.

(2.3) for all $r, s < \omega$, if $\langle \bar{b} \rangle \in [\{b_0, \ldots, b_{k-1}\}]^{<r}$ and $\langle \bar{b}' \rangle \in [\{b_0, \ldots, b_{k-1}\}]^{<s}$, then $\beta(a_r, \bar{b}) \neq \beta(a_{s+1}, \bar{b}', x)$, and if $\langle a_{r+1}, \bar{b} \rangle$ and $\langle a_{s+1}, \bar{b}' \rangle$ are different, then $\beta(a_{r+1}, \bar{b}, x) \neq \beta(a_{s+1}, \bar{b}', x)$.

(2.4) $x \neq a_i$ for each $i < \omega$.

(2.5) whenever $0 < i \leq m+1$ and $\langle \bar{b} \rangle \in [\{b_0, b_1, \ldots, b_m\}]^i$, then $\beta(a_{i+1}, \bar{b}, x) = f(\bar{b})$.

Note that every condition except for (2.4) can be expressed by a formula and (2.4) can be expressed by a recursive set of formulas as in the 0-th step. So, the set of properties above can be written as a recursive set of formulas with parameters $a_0, a_1, b_0, b_1, \ldots, b_{k-1}$. By β-coding capability of $\mathfrak{A}$, the set of formulas is finitely realizable. So by recursive saturation of $\mathfrak{A}$, there exists $b_k \in A$ realizing all the formulas. The sequence $b_0, b_1, b_2, \ldots$ constructed in this way is clearly $(G, 0)$-free. $\square$

Note that if $n = 0$, then the conditions (1.4) and (1.5) are automatically satisfied. So, the above lemma provides a subset of A realizing I_0 of (1.2) + (1.3) + (1.4) + (1.5) + (1.7) for $n = 0$. By Theorem 4.2.2, there exists such a subset I_0 of A that makes $(\mathfrak{A}_0, I_0)$ recursively saturated.

For I_1, (1.4) needs to be considered as well but (1.5) is again automatically satisfied. Note that (1.3) is satisfied automatically for all $n > 0$ because $I_n \subseteq I_0$. Similarly to the proof of Lemma 4.2.11, we can find a $(G, 1)$-free subset I_0' of I_0 whose order type is ω. Then, by Lemma 4.1.2, for each $\mathcal{L}_1$-formula $\varphi(x)$, we can find a $(G, 1)$-free subset I_1 of I_0' which is indiscernible over $\varphi(x)$. So, by Theorem 4.2.2, a desired I_1 that makes $(\mathfrak{A}_1, I_1)$ recursively saturated can be obtained.

Suppose $I_0, I_1, \ldots, I_{n-1}$ for $n > 1$ have been constructed and $\mathfrak{A}_n = (\mathfrak{A}, d_0, \ldots, d_{n-1},\ I_0, \ldots, I_{n-1}, R_0, \ldots, R_{n-1})$ satisfies the conditions (1.1) + $\cdots$ + (1.7). To find I_n, (1.5) needs to be considered. First we find an intermediate subset Z of I_{n-1} such that $(\mathfrak{A}_n, Z)$ is recursively saturated and Z realizes I_n of (1.5) + (1.7); that is, Z is (G, n)-free and, if $\langle b_0, \ldots, b_{n-1}\rangle \in [Z]^n$ and $\mathfrak{A}_n \models \exists y \varphi_{n-1}(b_0, b_1, \ldots, b_{n-2}, y)$, then

$$\mathfrak{A}_n \models \varphi_{n-1}(b_0, b_1, \ldots, b_{n-2}, \beta(a_n, b_0, b_1, \ldots, b_{n-2}, b_{n-1})).$$

This Z can be obtained by using recursive saturation of $\mathfrak{A}_n$ and Theorem 4.2.2

if (1.5)+(1.7) is shown to be consistent by constructing a subset of elements of I_{n-1} which realizes I_n of (1.5) + (1.7). We construct inductively an increasing sequence $b_0, b_1, \ldots$ realizing I_n of (1.5) + (1.7). Similarly again to the proof of Lemma 4.2.11, there exists a $(G, n-1)$-free sequence $\langle g_0, g_1, g_2, \ldots\rangle$ of elements of I_{n-1}. This sequence offers a model for (G, n)-freeness of the sequence $b_0, b_1, \ldots$. We start by choosing an arbitrary increasing sequence $b_0, \ldots, b_{n-2}$ from I_{n-1}. Then, by Observation 3 in section 4.1, $\langle b_0, \ldots, b_{n-2}\rangle \sim_{(G,n)} \langle g_0, \ldots, g_{n-2}\rangle$. Suppose that we have constructed $b_0, b_1, \ldots, b_{k-1}$ for some $k \geq n-1$ such that $\langle b_0, b_1, \ldots, b_{k-1}\rangle \sim_{(G,n)} \langle g_0, g_1, \ldots, g_{k-1}\rangle$, and for each $\langle b_{i_0}, b_{i_1}, \ldots, b_{i_{n-1}}\rangle \in [\{b_0, b_1, \ldots, b_{k-1}\}]^n$ if $\mathfrak{A}_n \vDash \exists y \varphi_{n-1}(b_{i_0}, \ldots, b_{i_{n-2}}, y)$, then $\mathfrak{A}_n \vDash \varphi_{n-1}(b_{i_0}, \ldots, b_{i_{n-2}}, \beta(a_n, b_{i_0}, \ldots, b_{i_{n-1}}))$. We want to find $b_k \in I_{n-1}$ such that $b_{k-1} < b_k$, $\langle b_0, b_1, \ldots, b_k\rangle \sim_{(G,n)} \langle g_0, g_1, \ldots, g_k\rangle$, and for each $\langle \bar{b}\rangle \in [\{b_0, b_1, \ldots, b_{k-1}\}]^{n-1}$, if $\mathfrak{A}_n \vDash \exists y \varphi_{n-1}(\bar{b}, y)$, then $\mathfrak{A}_n \vDash \varphi_{n-1}(\bar{b}, \beta(a_n, \bar{b}, b_k))$. Consider the function $f : [\{b_0, \ldots b_{k-1}\}]^{\geq n-1} \rightarrow A$ defined as follows:

if $s \geq n$ and $\langle b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}\rangle \in [\{b_0, b_1, \ldots, b_{k-1}\}]^s$,

$$\text{then } f(b_{i_0}, b_{i_1}, \ldots, b_{i_{s-1}}) = \beta(a_{s+1}, g_{i_0}, g_{i_1}, \ldots, g_{i_{s-1}}, g_k),$$

if $\bar{b} \in [\{b_0, b_1, \ldots, b_{k-1}\}]^{n-1}$ and $\mathfrak{A}_{n-1} \vDash \exists y \varphi_{n-1}(\bar{b}, y)$,

$$\text{then } \mathfrak{A}_n \vDash \varphi_{n-1}(\bar{b}, f(\bar{b})),$$

otherwise, $f = g_0$.

Since I_{n-1} is $(G, n-1)$-free, there exists an element $b_k \in I_{n-1}$ such that $b_{k-1} < b_k$, and which codes f via G. The resulting sequence $b_0, b_1, b_2, \ldots$ clearly realizes I_n of $(1.5) + (1.7)$. Here, we got this sequence by consuming one degree of freedom of I_{n-1} ($(G, n-1)$-free to (G, n)-free) to get an additional property (1.7). By recursive saturation of $\mathfrak{A}_n$, there exists a subset Z of I_{n-1} such that $(\mathfrak{A}_n, Z)$ is recursively saturated and Z realizes I_n of $(1.5) + (1.7)$.

Now, we will find $I_n \subseteq Z$ which is n-indiscernible and (G, n)-free. Similarly again to the proof of Lemma 4.2.11, we can find a (G, n)-free subset of Z of ω-type. This set still realizes I_n of (1.7). Then, by Lemma 4.1.2, for each formula $\varphi(\bar{x})$ with n free variables there exists a (G, n)-free subset of Z which is indiscernible over $\varphi(\bar{x})$. So, by recursive saturation of $(\mathfrak{A}_n, Z)$ and Theorem 4.2.2, this produces a desired subset I_n of I_{n-1} that makes $(\mathfrak{A}_n, I_n)$ recursively saturated. This completes the proof of Theorem 4.2.6. $\square$

If a countable linearly ordered set with no last element is taken for I in the above theorem, the structure $\mathfrak{B}$ generated by I is countable. Then, by Karp's Theorem, there is a back-and-forth system between $\mathfrak{A}$ and $\mathfrak{B}$, and hence $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. So, the above theorem generalizes Theorem 3.1.1.

The coding capability of a theory as in the **CFF** has been explored in a notable paper by Emil Jeřábek [Jeřábek]. A theory T is *sequential* if it contains Robinson's arithmetic Q relativized to some formula $N(x)$ and there is a formula $\beta(x, i, w)$ such that T proves

$$\forall w, x, k \exists w' \forall i, y[(N(k) \wedge i \leq k) \rightarrow$$

$$[\beta(y, i, w') \leftrightarrow ((i < k \wedge \beta(y, i, w)) \vee (i = k) \wedge (y = k))]]$$

So, the formula $\beta(x, i, w)$ can code sequences in T as the Gödel β-function does in PA. Jeřábek found some formulas $N(x)$ and $\beta(x, i, w)$ to show that the induction-free PA$^-$ is a sequential theory; this result makes Theorems 3.1.1 and 4.2.6 applicable to PA^-.

Chapter 5

Elementary cofinal extensions

This chapter explores structures that have a proper elementary cofinal extension. By Gaifman's Splitting Theorem and Compactness Theorem, every nonstandard model of PA is such a structure. Regarding fragments of PA, Kaye showed that every countable model of $B\Sigma_n + \exp + \neg I\Sigma_n$ has a proper elementary cofinal extension [Kaye4]. He also showed that $I\Sigma_n + \neg B\Sigma_{n+1}$ has both models having elementary cofinal extensions and models having no such extensions [Kaye3]. We discuss the same problem in a more general setting. Let $\mathcal{L}$ be a finite language with a symbol $<$ for ordering and $\mathcal{M}$ be a countable linearly ordered $\mathcal{L}$-structure with no last element. Schmerl proved that if $\mathcal{M}$ is countable and recursively saturated, then it has a proper elementary cofinal extension. In his proof Schmerl applied a set theoretic argument which bothered him; he wanted to know if there was a more elementary proof without using any set theoretic argument. So, we offer another proof of it, carrying out a direct construction of a proper isomorphic embedding of $\mathcal{M}$ to itself.

5.1 R. Kaye's work

This section reviews R. Kaye's work on models of fragments of PA which have proper elementary cofinal extensions. C. Parsons [Parsons], J. Paris, L. Kirby [PK], and H. Lessan [Lessan] have shown that the following implications hold and do not reverse; for all $n \in \mathbb{N}$,

$$I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n.$$

So, there are models $\mathcal{M}$ of $I\Sigma_n + \neg B\Sigma_{n+1}$; that is, $\mathcal{M} \models I\Sigma_n$ but $\mathcal{M} \not\models B\Sigma_{n+1}$. Similarly, there are models of $B\Sigma_{n+1} + \neg I\Sigma_{n+1}$. The question was raised by Kossak whether every countable model of $I\Sigma_n + \neg B\Sigma_{n+1}$ has proper elementary cofinal extensions, or if any such models exist [Kaye3]. Kaye gave a complete answer to this question by offering a sufficient condition for countable models of PA^- which have a proper elementary cofinal extension and providing a countable model of $I\Sigma_n + \neg B\Sigma_{n+1}$ which has no proper elementary cofinal extensions.

Definition 5.1.1. (Kaye [Kaye3]) If $\mathcal{M} \models \mathrm{PA}^-$ is an $\mathcal{L}$-structure, where $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{PA}}$ is a recursive first-order language and Γ is a recursive class of $\mathcal{L}$-

formulas, $\mathcal{M}$ is Γ-*tall* iff for any recursive sequence of formulas $(\phi_n(x,\bar{y}))_{n\in\mathbb{N}}$ from Γ and any $\bar{a}\in M$, if

$$\mathcal{M}\models Q^*x\phi_n(x,\bar{a})$$

and

$$\mathcal{M}\models\forall x(\phi_{n+1}(x,\bar{a})\rightarrow\phi_n(x,\bar{a}))$$

for all $n\in\mathbb{N}$ then $\{\phi_n(x,\bar{a}):n\in\mathbb{N}\}$ is realized in $\mathcal{M}$. Here, Q^* is the quantifier such that $Q^*x\phi(x,\bar{y})$ is $\exists z\forall x(x>z\rightarrow\phi(x,\bar{y}))$.

The sequence of the definable sets of $\phi_n(x,\bar{a})$ forms a descending chain. Tallness (Σ_n-tall for all $n\in\mathbb{N}$) is a weaker notion than recursive saturation. Indeed, by the observation known as Craig's trick (see [Kaye1]) that every recursively enumerable set of formulas is equivalent in the predicate calculus to a recursive set of formulas, every Γ-recursive saturated structure is Γ-tall. Hence, every nonstandard model of PA is Σ_n-tall for all $n\in\mathbb{N}$. However, there are nonstandard models that are not recursively saturated. For example, for any model $\mathcal{M}\models$ PA, its Skolem closure $\mathrm{Scl}^{\mathcal{M}}(\emptyset)$ is not recursively saturated because the recursive type $\{x>t : t \text{ is a Skolem term}\}$ is not realized in $\mathcal{M}$. In the following theorem Kaye gives a sufficient condition under which a countable model of PA^- has a proper elementary cofinal extension.

Theorem 5.1.2 (Kaye [Kaye3]). *Let* $\mathcal{M} \models \mathrm{PA}^-$ *be countable and* Σ_n*-tall for all* $n \in \mathbb{N}$*. Then* $\mathcal{M}$ *has a proper elementary cofinal extension.*

This provides an affirmative answer to Kossak's question because there are countable recursively saturated models of $I\Sigma_n + \neg B\Sigma_{n+1}$. Let exp denote a formula stating that the exponential function x^y is total. Kaye also gave a negative answer;

Theorem 5.1.3 (Kaye [Kaye3]). *For each* $n \in \mathbb{N}$ *there is a countable model of* $I\Sigma_n + \exp + \neg B\Sigma_{n+1}$ *with no proper elementary cofinal extensions.*

Kaye also proved the following theorem.

Theorem 5.1.4 (Kaye [Kaye4]). *Every countable model* $\mathcal{M}$ *of* $B\Sigma_n + \exp + \neg I\Sigma_n$ *has a proper elementary cofinal extension.*

5.2 Schmerl's proof

This section investigates countable models which have proper elementary cofinal extensions in a more general setting. With Schmerl's kind permission, I presents this unpublished proof of the theorem that every countable recursively saturated linearly ordered structure with no last element has a proper elementary cofinal extension.

Lemma 5.2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be countable $\mathcal{L}$-structures. If $\mathcal{N} \prec \mathcal{M}$, and $(\mathcal{M}, \mathcal{N})$ is recursively saturated, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. Clearly, $\mathcal{M}$ is recursively saturated. If $\bar{b} \in N$ and $p(\bar{b}, \bar{x})$ is a recursive type of $\mathcal{N}$, then $p'(\bar{b}, \bar{x}) = \{\varphi(\bar{b}, \bar{x}) \wedge \bar{x} \in \mathcal{N} : \varphi(\bar{b}, \bar{x}) \in p(\bar{b}, \bar{x})\}$ is a recursive type of $\mathcal{M}$. So, $\mathcal{N}$ is also recursively saturated. By Theorem 1.4, if $\mathcal{M}$ and $\mathcal{N}$ realize the same types, then there exists a back-and-forth system from $\mathcal{M}$ to $\mathcal{N}$, and so an isomorphism from $\mathcal{M}$ to $\mathcal{N}$ can easily be built. So, it remains only to show that $\mathcal{M}$ and $\mathcal{N}$ realize the same types. Since $\mathcal{N} \prec \mathcal{M}$, every type realized in $\mathcal{N}$ is realized in $\mathcal{M}$. Suppose $p(\bar{x})$ is a type of $\mathrm{Th}(\mathcal{M})$ and $\bar{a} \in M$ realizes the type. Consider a set of formulas, $q(\bar{a}, \bar{x}) = \{\varphi(\bar{a}) \leftrightarrow \bar{x} \in N \wedge \varphi(\bar{x}) : \varphi(\bar{x})$ is an $\mathcal{L}$- formula$\}$. $q(\bar{a}, \bar{x})$ is consistent with $\mathrm{Th}(\mathcal{M})$ because $\mathcal{M} \equiv \mathcal{N}$ and clearly is a recursive type. Thus, there is $\bar{b} \in N$ realizing $q(\bar{a}, \bar{x})$ in $(\mathcal{M}, \mathcal{N})$. Then, $\bar{b}$ realizes $p(\bar{x})$ in $\mathcal{N}$. $\square$

Theorem 5.2.2 (Schmerl)**.** *Let $\mathcal{L}$ be a finite language. Let $(\mathcal{M}, <)$ be a countable, recursively saturated $\mathcal{L}$-structure which is linearly ordered and has no last element. Then, there exists a proper elementary cofinal extension of $\mathcal{M}$ which is again recursively saturated.*

Proof. Let T be a theory consisting of $\exists x (x \notin X)$, $\forall y \exists x \in X (y \leq x)$ and the following formulas

$$\forall \bar{z} \in X\{\exists u\varphi(u, \bar{z}) \to \exists x \in X\varphi(x, \bar{z})\}$$

for all $\mathcal{L}$-formulas φ. Then, T expresses that X is a proper cofinal subset and X satisfies Tarski-Vaught Test. Thus, if $(\mathcal{M}, X) \vDash T$, then X is a proper elementary cofinal substructure of $\mathcal{M}$. Clearly, T is recursive. So, if T is consistent, by Theorem 4.2.2 and the recursive saturation of $\mathcal{M}$ there exists a relation X of $\mathcal{M}$ such that $(\mathcal{M}, X) \vDash T$ and is recursively saturated. Then, by the above lemma $X \cong \mathcal{M}$. If X viewed as $\mathcal{M}$ and $\mathcal{M}$ as an extension of $\mathcal{M}$, the work is complete.

For the consistency of T, we construct an elementary chain $\mathcal{M}_0 = \mathcal{M} \prec \mathcal{M}_1, \ldots \prec \mathcal{M}_{\aleph_1} \prec \mathcal{M}_{\aleph_1+1} \prec \ldots \prec \mathcal{M}_{\aleph_1+\omega}$ such that for all $\alpha < \aleph_1 + \omega$, $\mathcal{M}_{\alpha+1}$ contains an element $c_\alpha > \mathcal{M}_\alpha$. For all limit ordinal β take $\mathcal{M}_\beta = \cup_{\alpha<\beta}\mathcal{M}_\alpha$. Use Compactness Theorem and the fact that $\mathcal{M}_\alpha$ has no last element to construct this chain. Let $\mathcal{N} = \mathcal{M}_{\aleph_1+\omega}$ and $C = \{c_{\aleph_1+n} : n < \omega\}$. Then, $C \subseteq_{\text{cof}} \mathcal{M}_{\aleph_1+\omega}$. By Löwenheim-Skolem Theorem, there is a countable elementary substructure $\mathcal{N}'$ of $\mathcal{N}$ containing C. Since C is cofinal in $\mathcal{N}$, so is $\mathcal{N}'$. Since $|\mathcal{N}| \geq \aleph_1$, $\mathcal{N}' \prec \mathcal{N}$ is proper. Therefore $(\mathcal{M}, \mathcal{N}) \vDash T$. $\square$

5.3 A more elementary proof

The following is another proof of the theorem of the previous section which does not use uncountable cardinals and the notion of resplendence. Instead, we construct inductively an elementary embedding of $\mathcal{M}$ to itself whose image is proper and cofinal in $\mathcal{M}$.

Definition 5.3.1. Let A be a subset of an $\mathcal{L}$-structure $\mathcal{M}$. $b \in M$ is algebraic over A if there is a formula $\varphi(x, \bar{a})$ with $\bar{a} \in A$ such that $\varphi(\mathcal{M}, \bar{a})$ is finite and $\mathcal{M} \models \varphi(b, \bar{a})$. We denote the set of all elements algebraic over A by $\mathrm{acl}(A)$. When A is finite, e.g., $A = \{a_1, a_2, \ldots, a_n\}$, we also use the notation $\mathrm{acl}(a_1, a_2, \ldots, a_n)$.

Proof. First we show that $M \setminus \mathrm{acl}(\emptyset) \neq \emptyset$. Consider the recursive set of formulas $p(x) = \{\exists^{=k} y \psi(y) \to \neg\psi(x) : k < \omega, \psi \text{ an } \mathcal{L}\text{-formula}\}$. Since $\mathcal{M}$ is infinite, any finite subset of $p(x)$ is realized in $\mathcal{M}$. So, $p(x)$ is a recursive type of $\mathcal{M}$ and is realized in $\mathcal{M}$. Fix an element a of $\mathcal{M}$ realizing $p(x)$. Then $a \notin \mathrm{acl}(\emptyset)$.

Let $a_1, a_2, a_3 \ldots$ be an enumeration of $\mathcal{M}$. We will construct, by induction, a sequence of elements, $b_1, b_2, b_3 \ldots$ exhausting all the elements of $\mathcal{M}$ and a sequence of partial elementary maps $f_n : \{b_1, b_2, b_3 \ldots, b_n\} \to \mathcal{M}$ such that $\emptyset = f_0 \subseteq f_1 \subseteq f_2 \subseteq f_3 \subseteq \cdots$, $a \notin \mathrm{acl}(f_n(b_1), f_n(b_2), \ldots, f_n(b_n))$ for each $n \in \mathbb{N}$, and $f_n(b_n) > a_{i+1}$ whenever $n = 2(i+1)$ and $i \in \mathbb{N}$. Then, $f = \bigcup_{n<\omega} f_n$ will be an embedding of $\mathcal{M}$ into itself whose image is a proper cofinal subset of $\mathcal{M}$. Since the image is isomorphic to $\mathcal{M}$, we are done.

For each $n \in \mathbb{N}$, we will let $\varphi_0(\bar{w}), \varphi_1(\bar{w}), \varphi_2(\bar{w}), \ldots$ be a recursive enumeration of all the $\mathcal{L}$-formulas with $n+1$ free variables and $\psi_0(\bar{v}), \psi_1(\bar{v}), \psi_2(\bar{v}), \ldots$ be a recursive enumeration of all the $\mathcal{L}$-formulas with $n+2$ free variables. At each n-th stage of the construction, we will let $\bar{b}$ denote the n-tuple $(b_1, b_2, \ldots, b_n)$ and$f_n(\bar{b})$ the n-tuple $(f_n(b_1), f_n(b_2), \ldots, f_n(b_n))$.

We carry out a back-and-forth construction. Let $n < \omega$ and suppose we have constructed $b_1, b_2, \ldots, b_n$ and $\emptyset = f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots \subseteq f_n$. We want to find b_{n+1} and extend f_n to f_{n+1}.

Case $n = 2i$ for some $i < \omega$.

Let $b_{n+1} = a_{i+1}$. We want to find $c \in M$ such that $\mathrm{tp}^{\mathcal{M}}(\bar{b}, b_{n+1}) = \mathrm{tp}^{\mathcal{M}}(f_n(\bar{b}), c)$ and $a \notin \mathrm{acl}(f_n(\bar{b}), c)$. So, c can be any element realizing the set of the formulas $q(x) =$

$$\{\varphi_j(\bar{b}, b_{n+1}) \leftrightarrow \varphi_j(f_n(\bar{b}), x) : j \in \mathbb{N}\} \cup$$

$$\{\exists^{=k} y \psi_j(f_n(\bar{b}), x, y) \rightarrow \neg\psi_j(f_n(\bar{b}), x, a) : k, j \in \mathbb{N}\}.$$

Clearly, $q(x)$ is recursive. Suppose $q(x)$ is not finitely realizable in $\mathcal{M}$. Then, there exists $l < \omega$ such that

$$\begin{aligned}\mathcal{M} \vDash &\forall x\Big[\bigwedge_{j<l}\{\varphi_j(\bar{b}, b_{n+1}) \leftrightarrow \varphi_j(f_n(\bar{b}), x)\} \to \\ &\bigvee_{k,j<l}\{\exists^{=k} y\psi_j(f_n(\bar{b}), x, y) \wedge \psi_j(f_n(\bar{b}), x, a)\}\Big].\end{aligned}$$

For each $j < l$, let $\varphi'_j(\bar{w}) = \varphi_j(\bar{w})$ if $\mathcal{M} \vDash \varphi_j(\bar{b}, b_{n+1})$ and $\varphi'_j(\bar{w}) = \neg\varphi_j(\bar{w})$ otherwise. Let

$$\begin{aligned}\theta(f_n(\bar{b}), z) = \forall x&\Big[\bigwedge_{i<l}\varphi'_i(f_n(\bar{b}), x) \to \\ &\bigvee_{k,j<l}\{\exists^{=k} y\psi_j(f_n(\bar{b}), x, y) \wedge \psi_j(f_n(\bar{b}), x, z)\}\Big].\end{aligned}$$

Then, $\mathcal{M} \vDash \theta(f_n(\bar{b}), a)$. Since f_n is a partial elementary map, we get $\mathcal{M} \vDash \exists x \bigwedge_{i<l} \varphi'_i(f_n(\bar{b}), x)$. In addition, for any $e \in M$, $\bigvee_{k,j<l}\{\exists^{=k} y\psi_j(f_n(\bar{b}), e, y) \wedge \psi_j(f_n(\bar{b}), e, z)\}$ defines a finite subset of M. Thus, $\theta(f_n(\bar{b}), z)$ defines a finite subset of M containing a. This contradicts the induction hypothesis that $a \notin \operatorname{acl}(f_n(\bar{b}))$. Hence, $q(x)$ is a recursive type of $\mathcal{M}$ and, by recursive saturation of $\mathcal{M}$, is realized by some $c \in M$. Let $f_{n+1} = f_n \cup \{(b_{n+1}, c)\}$. Then, f_{n+1} is a partial elementary map, and we have $a \notin \operatorname{acl}(f_{n+1}(b_1), f_{n+1}(b_2), \ldots, f_{n+1}(b_{n+1}))$.

Case $n = 2i + 1$ for some $i < \omega$.

We want to extend f_n to f_{n+1} so that f_{n+1} is a partial elementary map, $a \notin \operatorname{acl}(f_{n+1}(\bar{b}))$, and $f_{n+1}(b_{n+1}) > a_{i+1}$. First, we will find $d \in M$ such that

$d > a_{i+1}$ and $a \notin \mathrm{acl}(f_n(\bar{b}), d)$. Let

$$\begin{aligned} r(x) = &\{\exists^{=k} y \psi_j(f_n(\bar{b}), x, y) \to \neg \psi_j(f_n(\bar{b}), x, a) : j, k < \omega\} \\ &\cup \{x > a_{i+1}\}. \end{aligned}$$

Clearly, $r(x)$ is recursive. Suppose that $r(x)$ is not finitely realizable in $\mathcal{M}$. Then, there exists $l \in \mathbb{N}$ such that

$$\mathcal{M} \vDash \forall x > a_{i+1} \bigvee_{j,k<l} \{\exists^{=k} y \psi_j(f_n(\bar{b}), x, y) \wedge \psi_j(f_n(\bar{b}), x, a)\}.$$

Let

$$\eta(f_n(\bar{b}), z) = \exists w \forall x > w \bigvee_{j,k<l} \{\exists^{=k} y \psi_j(f_n(\bar{b}), x, y) \wedge \psi_j(f_n(\bar{b}), x, z)\}.$$

Then, $\mathcal{M} \vDash \eta(f_n(\bar{b}), a)$. To see that $\eta(f_n(\bar{b}), z)$ defines a finite subset of $\mathcal{M}$, we let $\eta''(f_n(\bar{b}), x, z) := \bigvee_{j,k<l} \{\exists^{=k} y \psi_j(f_n(\bar{b}), x, y) \wedge \psi_j(f_n(\bar{b}), x, z)\}$ and $\eta'(f_n(\bar{b}), w, z) := \forall x > w\ n''(f_n(\bar{b}), x, z)$. So, $\eta(f_n(\bar{b}), z)$ is $\exists w\ \eta'(f_n(\bar{b}), w, z)$. Then, for every element $e \in M$, the size of $\eta''(f_n(\bar{b}), e, \mathcal{M})$ is less than l^2, and so is the size of $\eta'(f_n(\bar{b}, e, \mathcal{M})$ because $\eta'(f_n(\bar{b}, e, \mathcal{M}) = \bigcap_{x>e} \eta''(f_n(\bar{b}, x, \mathcal{M})$. Note also that if $w_1 < w_2$, then $\eta'(f_n(\bar{b}), w_1, \mathcal{M}) \subseteq \eta'(f_n(\bar{b}, w_2, \mathcal{M})$. Since $\eta(f_n(\bar{b}), \mathcal{M}) = \bigcup_{w \in M} \eta'(f_n(\bar{b}), w, \mathcal{M})$, the size of $\eta(f_n(\bar{b}), \mathcal{M})$ is less than l^2. Thus, $\eta(f_n(\bar{b}), z)$ defines a finite subset of $\mathcal{M}$ containing a. It contradicts the induction hypothesis that $a \notin \mathrm{acl}(f_n(\bar{b}))$. Hence, $r(x)$ is a recursive type of

$\mathcal{M}$ and is realized by some $d \in M$. Then, by using recursive saturation of $\mathcal{M}$, we can find an element $b_{n+1} \in M$ such that

$$\mathrm{tp}^{\mathcal{M}}(\bar{b}, b_{n+1}) = \mathrm{tp}^{\mathcal{M}}(f_n(\bar{b}), d).$$

Let $f_{n+1} = f_n \cup \{(b_{n+1}, d)\}$. Then, f_{n+1} satisfies all the required properties.

□

Corollary 5.3.2. *Let $(\mathcal{M}, <)$ be a countable, recursively saturated $\mathcal{L}$-structure which is linearly ordered and has no last element. Then Then $(\mathcal{M}, <)$ has a proper elementary cofinal extension of cardinality $\aleph_1$.*

Proof. By Theorem 5.2.2, we can construct an elementary chain of countable recursively saturated structures of length ω_1 with each extension being proper:

$$\mathcal{M} = \mathcal{M}_0 \prec_{\mathrm{cof}} \mathcal{M}_1 \prec_{\mathrm{cof}} \cdots \mathcal{M}_\alpha \prec_{\mathrm{cof}} \cdots (\alpha < \omega_1)$$

Then $\mathcal{N} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$ is an elementary cofinal extension of $\mathcal{M}$ of cardinality ω_1. □

Schmerl's proof of Theorem 5.2.2 can not be formalized in second order arithmetic because of the use of uncountable cardinals. The proof presented in this section does formalize. In fact, we can prove the following theorem.

Theorem 5.3.3. *Theorem 5.2.2 is provable in* ACA_0*.*

Proof. (Sketch) Check if every step of the above proof can be carried out in ACA_0. The first part of the proof is to find a such that $a \notin \mathrm{acl}(\emptyset)$. This has no set existence argument involved, and so can be clearly carried out within ACA_0. Then, enumerate $a_1, a_1, \ldots$ of $|\mathcal{M}|$. This can be done even in RCA_0, which is weaker than ACA_0. See Lemma II.3.7 of [Simpson]. To find the function $f : |\mathcal{M}| \to |\mathcal{M}|$ and an enumeration of $|\mathcal{M}|, b_1, b_2, \ldots$ as in the proof, one thing to consider is that when the function is constructed, the choices of b_n and $f(b_n)$ are not unique. With the original argument there is no guarantee that there is such a function. This problem can be resolved by choosing the smallest b_n and $f(b_n)$ at each stage of the construction. $\square$

Chapter 6

Questions

There are many open questions related to the topics covered in this thesis. I will list only three.

Question 1. *Is there a simpler proof of Theorem 4.2.6? What is the weakest subsystem of second order arithmetic in which Theorem 4.2.6 theorem is provable?*

Chapter 4 discussed the question proposed by Marker and Smith, whether there is a recursively saturated structure generated by indiscernibles. Schmerl proved that every countable recursively saturated model $\mathfrak{A}$ of **CFF** has an indiscernible-type Σ such that whenever $\mathfrak{B}$ is generated by a set I of indiscernibles having indiscernible-type Σ and having no last element, then $\mathfrak{B}$ is β-generated by I, is totally resplendent and $\mathfrak{B} \equiv {}_{\infty,\omega}\mathfrak{A}$. The theorem answers the question affirmatively and reveals that the two seemingly opposite notions are closely related. However, its proof is very complicated, especially when compared with that of Theorem 3.2.5, which is weaker but gives another affirmative answer to the same question. So, questions remain as to whether there is a simpler proof of Theorem 4.2.6, and what is the weakest subsystem of second order arithmetic in which Theorem 4.2.6 theorem is provable.

Question 2. *Let $\mathfrak{A}$ be a countable model of **CFF**. If $\mathfrak{A}$ has generating indiscernible sequences of different order types, is it recursively saturated? Or, what conditions make the converse of Theorem 4.2.6 hold?*

Theorem 4.2.6 implies that if $\mathfrak{A}$ is a countable recursively saturated model of **CFF**, then for every countable order type $\mathfrak{A}$ is generated by an indiscernible sequence of the order type. Does the condition that $\mathfrak{A}$ is generated by two indiscernible sequences of different order types allow the converse of Theorem 4.2.6 hold? More generally, this raises issues concerning characterizing countable models of **CFF** generated by indiscernibles.

Question 3. *Let $\mathfrak{A}$ be a countable linearly ordered structure having no last element. If $\mathfrak{A}$ is recursively saturated, does it have a cofinal elementary extension of every infinite cardinality?*

The result of Theorem 5.2.2 could be extended to cardinality ω_1 by constructing an elementary chain of length ω_1 in Corollary 5.3.2. However, this method does not allow for reaching a higher cardinality because Theorem 5.2.2 applies only to countable structures. It would be interesting to explore whether $\mathfrak{A}$ has an elementary cofinal extension of every higher cardinality.

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