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TOWARDS A THEORY OF ATOLL DECOMPOSITIONS

by

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I. Introduction

All graphs considered throughout this work are finite, undirected, and without loops or multiple lines. The terminology and notation employed here will for the most part follow that of Harary [1], unless otherwise specified. First the definition of an island decomposition, which is a concept that is fundamental to this work, and then the definition of an atoll decomposition, which is the concept that is the central theme of this work, will be set forth.

An island decomposition of a graph G is an acyclic spanning subgraph of G in which every point has degree $\leqq 2$. Thus it follows that an island decomposition is actually a collection of point disjoint paths that cover the points of G. An isolated point can be a member of such a collection. The point disjoint paths that together make up the collection of paths of an island decomposition are called the islands of the island decomposition. Throughout this work, by a path is meant a path as is defined in [1], by the length of a path is meant the number of lines in the path, and by an open path is meant either a point K_1, or a path K_2, or a path having length $\geqq 2$ and whose two endpoints are nonadjacent. Obviously, every graph has an island decomposition, e.g., the set of all the points of a graph is an island decomposition of the graph. The definition of an island decomposition just given above is the one composed by Goodman and Hedetniemi [2] in a paper in which they introduce the term island decomposition, and in which they study the problem of determining the Hamiltonian completion number hc(G) of a graph G, i.e., hc(G) is the minimum number of lines that need to be added to G in order to make it Hamiltonian. Chronologically, however, the notion of an island decomposition, though not the name island decomposition itself, was, apparently, first introduced by Ore [3] in a classic paper in which he calls an island decomposition an arc covering, and in which he derives a sufficient condition for a graph to be Hamiltonian in terms of the number of lines of the graph, viz., if G is a graph with $p \geq 3$ points and $q \geq (p^2-3p+6)/2$ lines, then G is Hamiltonian. Also, the problem of finding an island decomposition of a graph, s.t. this island decomposition contains a minimum number of islands, has been considered by Boesch, Chen, and McHugh [4]. In their paper they do not actually use the term island decomposition, but instead they let $\zeta(G)$ denote the minimum number of point disjoint paths that cover the points of the graph G , and they call $\zeta(G)$ the path-to-point covering number of G . In this work $\zeta(G)$ will denote the minimum number of islands in an island decomposition of the graph G , i.e., the symbol $\zeta(G)$ will denote precisely the same thing as it does in [4]. Now at this juncture, the concept of an atoll decomposition, a concept which I have formed myself, will be introduced for the first time.

An atoll decomposition of a graph G is an acyclic spanning subgraph A of G , s.t. the degree of every point of A is ≤ 2 and s.t. A is linewise maximal with respect to this degree constraint, i.e., if any line of $G \setminus A$ is added to A , then at least one talon (i.e., $K_{1,3}$) is created. Therefore it follows that A is essentially a collection of point disjoint open paths that cover the points of G , s.t. no endpoint of any path of A is adjacent to an endpoint of any other path of A . The point disjoint open paths that together make up the collection of paths of an atoll decomposition are called the atolls of the atoll decomposition. It ought to be stated and emphasized here that throughout this work, whenever any reference or mention is made to paths of a certain atoll decomposition, it is understood that all of these paths under consideration are atolls of this particular atoll decomposition. It immediately follows that every atoll decomposition is an island decomposition, but not conversely. Note that not every graph has an atoll decomposition, e.g., no clique (i.e., complete graph) which contains at least three points has an atoll decomposition. Unfortunately, no useful or elegant characterization of those graphs which have an atoll decomposition is known at the present time. Accordingly, let $\mathcal{A}$ denote the as yet uncharacterized class of all those graphs which have an atoll decomposition. Also, from here on, the term atoll decomposition will, when convenient, be abbreviated by a.d.

Thus, let G be a graph, s.t. $G \in \mathcal{A}$, i.e., s.t. A_G is nonempty, where $A_G = \{A_1, A_2, \ldots, A_\psi\}$ is the set of all atoll decompositions of G. If $\forall\, i,\ 1 \le i \le \psi$, each A_i has $\varkappa_i$ components (i.e., atolls) consisting of exactly λ_i^0 paths of length 0 (i.e., points K_1), plus λ_i^1 paths of length 1 (i.e., paths K_2), plus $\lambda_i^{\ge 2}$ (open) paths of length ≥ 2, then, of course, $\varkappa_i = \lambda_i^0 + \lambda_i^1 + \lambda_i^{\ge 2}$. Let $\beta(G) = \min_{1 \le i \le \psi} \varkappa_i$, and let $\Omega(G) = \max_{1 \le i \le \psi} \varkappa_i$. Hence $\beta(G)$ $(\Omega(G))$ denotes the minimum (maximum) number of atolls in an atoll decomposition of G, and it follows at once that if a graph G has an atoll decomposition, then $\zeta(G) \le \beta(G)$. Further, $\forall\, i,\ 1 \le i \le \psi$, let $\rho(A_i) = \lambda_i^0 + \lambda_i^1 + 2\lambda_i^{\ge 2}$. Let $\mu(G) = \min_{1 \le i \le \psi} \rho(A_i)$, and let $\tau(G) = \max_{1 \le i \le \psi} \rho(A_i)$. As a general rule, the symbol λ_j^{ℓ} is reserved throughout this work to signify the number of point disjoint open paths of length ℓ in a certain collection of point disjoint open paths that is specified by ∂ .

A stable set of a graph G is a subset L of the set V of points of G , s.t. the points of L are pairwise nonadjacent. For any graph G , let $\alpha(G)$ denote the largest number of points in a stable set of G . The value $\alpha(G)$ is called the stability number of G . It follows directly from the definition of an atoll decomposition, that if a graph G has an atoll decomposition and that if $\Pi'(A)$ denotes the set of all the endpoints of all the paths (i.e., atolls) which together comprise the collection of paths of the a.d. A of G , less exactly one endpoint from each path of A consisting of a path K_2 , then $\Pi'(A)$ is a stable set of G . The stable set $\Pi'(A)$ of G is called a terminal stable set of G (i.e., the points of $\Pi'(A)$ are terminal points (i.e., endpoints) of the paths comprising A) and A is said to conform to $\Pi'(A)$ in G . Therefore it follows that ρ and consequently both μ and τ each measure the cardinality of certain terminal stable sets of G . The value $\tau(G)$ is called the terminal stability number of G . The following two observations may now be made: First, note that if λ_j^1 denotes the number of paths of length 1 (i.e., again, paths K_2) of the a.d. A_j of the graph $G \in \mathcal{A}$, then it follows that A_j conforms to precisely $2^{\lambda_j^1}$ terminal stable sets of G . And second, note that if L is a stable set of a graph $G \in \mathcal{A}$, then either L is not a terminal stable set of G , or else L is a terminal stable set of G and it follows that $\exists$ one or more atoll decompositions of G , each of which conforms to L in G .
A concise but comprehensive outline will now be presented, an outline which sets forth all the essential concepts that have not yet been

given above, and which includes all the main results of this work.

In Part II, atoll decomposition theory is applied to the topic of triangulated graphs. A triangulated graph is a graph G_Δ, s.t. every cycle of G_Δ having length >3 contains a diagonal (i.e., a line joining two nonconsecutive points of the cycle). The concept of a triangulated graph is originally due to Hajnal and Surányi [5]. For any subset U of the set V of points of a graph G, let $\langle U \rangle$ denote the subgraph of G induced by the point set U. And if u is any point of G, then let $G(u)$ denote the set of all those points of G that are adjacent to u. The set $G(u)$ is called the neighborhood of u. A simplicial point of a graph G is a point s of G, s.t. $\langle G(s) \rangle = K_n$, where $n \geq 0$, i.e., the neighborhood $G(s)$ of a simplicial point s of a graph G induces a clique of G. Dirac [6] showed that every triangulated graph contains at least one simplicial point, so let $\sigma(G_\Delta)$ denote the maximum cardinality of a stable set of simplicial points of the triangulated graph G_Δ, where a stable set of simplicial points is, as expected, a set of simplicial points that are pairwise ndnadjacent. The value $\sigma(G_\Delta)$ is called the simplicity number of G_Δ, and it follows at once that $\sigma(G_\Delta) \leq \alpha(G_\Delta)$. The chromatic number $\chi(G)$ of a graph G is the minimum order of a partition of the set V of points of G into stable subsets. The complement $\bar{G}$ of a graph G has the same set V of points as has G, but two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G. It is shown that if G is a graph, s.t. $G \in \mathcal{A}$, then $\tau(G) \leq \chi(\bar{G})$. The principal result of Part II may now be stated: If G_Δ is a triangulated graph, s.t. $G_\Delta \in \mathcal{A}$ and $\sigma(G_\Delta) = \alpha(G_\Delta)$, then $\tau(G_\Delta) = \chi(\bar{G}_\Delta)$.

In Part III, atoll decomposition theory is applied to the subject of trees. A tree is a connected acyclic graph, and so it immediately follows that a tree is a triangulated graph. For any nontrivial tree T (i.e., $T \neq K_1$), let $\epsilon(T)$ denote the number of endpoints (i.e., points of T having degree one) of T, and let $\epsilon(K_1) = 1$. It is shown that every tree T has an atoll decomposition and that if $T \neq K_2$, then $\sigma(T) = \epsilon(T)$. Thus as a corollary of the principal result of Part II, it is proved that if T is a tree, s.t. $T \neq K_2$, and if $\epsilon(T) = \alpha(T)$, then $\tau(T) = \chi(\bar{T})$ ($\tau(K_2) = \chi(\bar{K}_2)$, nonetheless). A pair of lines is said to be nonadjacent if they do not share a common point, an independent set of lines or a matching is a set of lines that are pairwise nonadjacent, a maximum matching of a graph G contains the largest number of lines in a matching of G, and a partial matching of G is nonempty but contains fewer lines than does a maximum matching of G. A line cover of a graph G is a subset Y of the set E of lines of G, s.t. each point of G is a point of some line of Y, and a minimum line cover of G contains the smallest number of lines in a line cover of G. Let φ_E denote the empty set of lines. If X is a set of lines of a graph G, s.t. $X = X_1 \cup X_2 \cup \ldots \cup X_\Gamma$, where $\forall i$, $1 \leq i \leq \Gamma$, it holds that $X_i \neq \varphi_E$, and where $\forall i,j$, $1 \leq i < j \leq \Gamma$, it holds that $X_i \cap X_j = \varphi_E$, then X is said to be the direct sum of its subsets $X_1, X_2, \ldots, X_\Gamma$. This direct sum decomposition of X may be written as $X = \oplus_{i=1}^{\Gamma} X_i$, or as $X = \oplus_{i \in \{I\}} X_i$, where $\{I\}$ is a finite index set, s.t. $|\{I\}| = \Gamma$. It is shown that if Y is a minimum line cover of a nontrivial tree T, then Y has a minimum direct sum decomposition into matchings of T:

$$Y = \left(\oplus_{k=1}^{\omega} M_k \right) \oplus \left(\oplus_{j \in \{J\}} P_j \right),$$

where $\omega \geq 1$ and $\forall k, 1 \leq k \leq \omega$, each M_k is a maximum matching of T, and, where $|\{J\}| = \omega' \geq 0$ and $\forall j \in \{J\}$, each P_j is a partial matching of T; furthermore, the value $\omega + \omega'$ is the smallest number of matchings of T in a direct sum decomposition of Y into matchings of T, and the value $\omega = \omega(Y)$ is the greatest number of pairwise line disjoint maximum matchings of T that are contained in Y. If $\{Y_\ell\}_{\ell=1}^{\Lambda}$ is the set of all minimum line covers of a nontrivial tree T, then let $\omega(T) = \max_{1 \leq \ell \leq \Lambda} \omega(Y_\ell)$, and let $\omega(K_1) = 0$. The chief result of Part III can now be stated: If T is a tree, s.t. $\omega(T) \geq 2$, then $\tau(T) = \chi(\bar{T})$.

An example is provided that shows that this last result is not implied by the principal result of Part II. If $\pi_j = (u_1, u_2, \ldots, u_{k+1}, u_{k+2}, \ldots, u_{k+r}, \ldots, u_c)$ is any path of a graph G, where $1 \leq k \leq c$ and $0 \leq r \leq c-1$, then the subpath $(u_k, u_{k+1}, u_{k+2}, \ldots, u_{k+r})$ of π_j is called a segment of π_j of length r. Thus it follows that a path is a segment of itself. Another new concept will now be introduced, viz., that of a maximal island decomposition. A maximal island decomposition of a graph G is an island decomposition I_a of G, s.t. if every island of I_a is a segment of an island of an island decomposition I_b of G, then $I_b = I_a$. It follows that every minimum island decomposition (i.e., a minimum island decomposition is an island decomposition which contains the smallest number of islands in an island decomposition) is a maximal island decomposition. A significant result of Part III is the following characterization: An island decomposition I_a of a tree T is a maximal island decomposition of T if and only if I_a is an atoll decomposition of T. Also in Part III it is proved that $\forall$ trees T, it holds that $\zeta(T) = \beta(T)$.

In Part IV, atoll decomposition theory is applied to the topic of Hamiltonian graphs. A Hamiltonian graph is a graph which contains a spanning cycle. A singular atoll of an a.d. A of a graph G is an atoll $\pi_s = (u_1, u_2, \ldots, u_d)$ of A having length ≥ 3 (i.e., $d \geq 4$) and containing a certain distinguished line $x = u_a u_{a+1}$, where $2 \leq a < a+1 \leq d-1$, s.t. both the lines $u_1 u_{a+1}$ and $u_a u_d$ belong to $G \setminus A$. A singular atoll decomposition of a graph G is an a.d. A_s of G, s.t. A_s consists of precisely one atoll π_s and this atoll π_s is a singular atoll. It follows directly from the definition of an atoll decomposition that a graph G has no atoll decomposition if and only if $\forall$ island decompositions I_e of G, either it holds that $\exists$ two distinct islands π_x and π_y of I_e, s.t. an endpoint of π_x is adjacent to an endpoint of π_y, or it holds that $\exists$ an island π_z of I_e having length ≥ 2, s.t. the two endpoints of π_z are adjacent (i.e., either the first condition holds or the second condition holds or both conditions hold simultaneously). Therefore, since clearly every atoll decomposition is also a maximal island decomposition, then it follows that $G \notin \mathcal{A}$ if and only if $\forall$ maximal island decompositions I_a of G, it holds that $\exists$ an island π_w of I_a having length ≥ 2, s.t. the two endpoints of π_w are adjacent. The primary result of Part IV is the following characterization of Hamiltonian graphs in terms of atoll decompositions: A graph G is Hamiltonian if and only if exactly one of the following two conditions holds: (1) G has no atoll decomposition and $\zeta(G) = 1$, (2) G has a singular atoll decomposition. Also in Part IV it is shown as a corollary of this last result, that if H is a Hamiltonian graph, s.t. $H \in \mathcal{A}$, then $\zeta(H) = \beta(H)$. A graph is said to be planar if it can be drawn in the plane so that no two of its lines intersect, i.e., so that its lines intersect only at their points, if they intersect at all. An interesting consequence of the material presented in Part IV is the next statement: If H_π is a Hamiltonian planar graph with $p \geq 3$ points and which is neither C_p nor K_4, then $H_\pi \in \mathcal{A}$, and it follows that $\chi(H_\pi) \leq p-\tau(H_\pi) + 1$.

In Part V, atoll decomposition theory is applied to the theory of domination. A dominating set of a graph G is a subset D of the set V of points of G, s.t. every point of G that is not in D is adjacent to at least one point of D, and a minimal dominating set of G is a dominating set of G that does not properly contain another dominating set of G. Let $\delta(G)$ denote the smallest number of points in a minimal dominating set of G. The value $\delta(G)$ is called the domination number of G, and a dominating set of G that contains exactly $\delta(G)$ points is called a minimum dominating set of G. A maximal stable set of a graph G is a stable set of G that is not properly contained in any other stable set of G, and a key decomposition (as in Florida Keys) of a graph G is an atoll decomposition KA of G, s.t. KA conforms to a maximal stable set of G. Let $\mathcal{KA}$ denote the class of all those graphs which have a key decomposition. Thus if G is a graph, s.t. $G \in \mathcal{KA}$, then KA_G is nonempty, where $KA_G = \{KA_1, KA_2, \ldots, KA_{\psi_k}\}$ is the set of all key decompositions of G. Let $\mu_k(G) = \min_{1 \leq e \leq \psi_k} \rho(KA_e)$. A ternary atoll decomposition of a graph G is an atoll decomposition TA of G, s.t. every atoll of TA has length ≤ 3, and $\mathcal{TA}$ denotes the class of all those graphs which have a ternary atoll decomposition. It follows that $\mathcal{TA} \subseteq \mathcal{KA} \subseteq \mathcal{A}$. An elementary contraction of a graph G is the identification of two adjacent points u_1 and u_2 of G, i.e., both u_1 and u_2 are removed from G and they are replaced in G by a new point u_3, s.t. u_3 is adjacent to all those points to which at least one of u_1 and u_2 was adjacent. A contraction is a composition of finitely many elementary contractions, and a graph G^* is said to be a contraction of a graph G if G^* can be produced from G by means of a contraction of G. Let $\{r\}$ denote the smallest positive nonzero integer not less than the real number r, so that $\{0\} = 1$, and let $\{r\}'$ denote the smallest integer not less than r, so that $\{0\}' = 0$. Lastly, if A is an atoll decomposition of a graph G, then let $\eta(A)$ denote the length of the longest atoll of A.

The single result of Part V may now be given: If G is a graph, s.t. $G \in \mathcal{A}$, then

(1) if A_m is an atoll decomposition of G, then $\delta(G) \le \sum_{\ell=0}^{\eta(A_m)} \left[\left\{\frac{\ell+1}{3}\right\} \lambda_m^\ell\right]$,

where $\forall \ell$, $0 \le \ell \le \eta(A_m)$, each λ_m^ℓ denotes the number of atolls of A_m having length ℓ,

(2) if $G \in \mathcal{TA}$, then $\delta(G) \le \mu_k(G)$, and

(3) if $G \notin \mathcal{TA}$ and if A_m is an atoll decomposition of G, then $\exists$ a contraction G^* of G, s.t. $G^* \in \mathcal{TA}$ and $2\delta(G^*) - \sum_{\ell=0}^{\eta(A_m)} \left[\left\{\frac{2\ell}{\ell+1}\right\} \lambda_m^\ell\right] \le \delta(G)$, where $\forall \ell$, $0 \le \ell \le \eta(A_m)$, each λ_m^ℓ denotes the number of atolls of A_m having length ℓ.

In Part VI, atoll decomposition theory is applied to the topic of matchings. Let $\gamma(G)$ denote the number of lines in a maximum matching of the graph G. An elementary expansion of a graph G is the removal of a point v_1 of G and its replacement in G by two new points v_2 and v_3, s.t. v_2 is adjacent to v_3 and, moreover, each of v_2 and

v_3 is adjacent to all those points to which v_1 was adjacent. An expansion is a composition of finitely many elementary expansions, and a a graph G^- is said to be an expansion of a graph G if G^- can be produced from G by means of an expansion of G. An odd atoll decomposition of a graph G is an atoll decomposition OA of G, s.t. every atoll of OA has odd length, and $\mathcal{OA}$ denotes the class of all those graphs which have an odd atoll decomposition. An atoll of length 0 (i.e., again, a point K_1) is said to have even length. The single result of Part VI is the following assertion, which represents both in form and in substance a counterpart or twin of the result of Part V, that was just given before: If G is a graph, s.t. $G \in \mathcal{A}$, then

(1) if A_n is an atoll decomposition of G, then

$\sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{\ell}{2} \right\}' \lambda_n^{\ell} \right] \leq \gamma(G)$, where $\forall \ell,\ 0 \leq \ell \leq \eta(A_n)$, each λ_n^{ℓ} denotes the number of atolls of A_n having length ℓ,

(2) if $G \in \mathcal{OA}$ and if OA_d is an odd atoll decomposition of G, then

$\gamma(G) = \sum_{\ell=0}^{\eta(OA_d)} \left[\left\{ \frac{\ell}{2} \right\} \lambda_d^{\ell} \right]$, where $\forall \ell,\ 0 \leq \ell \leq \eta(OA_d)$, each λ_d^{ℓ} denotes the number of atolls of OA_d having length ℓ, and

(3) if $G \notin \mathcal{OA}$ and if A_n is an atoll decomposition of G, then $\exists$ an expansion G^- of G, s.t. $G^- \in \mathcal{OA}$ and

$\gamma(G) \leq 2\gamma(G^-) - \sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda_n^{\ell} \right]$, where $\forall \ell,\ 0 \leq \ell \leq \eta(A_n)$, each λ_n^{ℓ} denotes the number of atolls of A_n having length ℓ.

II. Atoll Decompositions and Triangulated Graphs

To recapitulate once more for purposes of review and clarification, an atoll decomposition of a graph G is an acyclic spanning subgraph A of G, s.t. the degree of every point of A is ≤ 2 and s.t. A is linewise maximal with respect to this degree constraint, i.e., if any line of $G\setminus A$ is added to A, then at least one talon (i.e., $K_{1,3}$) is created. Therefore if by a path is meant a path as is defined in [1], and if by an open path is meant either a point K_1, or a path K_2, or a path having length ≥ 2 and whose two endpoints are nonadjacent, then it follows that A is actually a collection of point disjoint open paths (which are called the atolls of A) that cover the points of G, s.t. no endpoint of any path (i.e., atoll) of A is adjacent to an endpoint of any other path (i.e., again, atoll) of A. The term atoll decomposition will, when convenient, be abbreviated by a.d.

Thus if $\mathcal{A}$ denotes the class of all those graphs which have an atoll decomposition and if G is a graph, s.t. $G \in \mathcal{A}$, then A_G is nonempty, where $A_G = \{A_1, A_2, \ldots, A_\psi\}$ is the set of all atoll decompositions of G. Further, if $\forall i,\ 1 \leq i \leq \psi$, each A_i has $\varkappa_i$ components (i.e., atolls) consisting of exactly λ_i^o paths of length 0 (i.e., points K_1), plus λ_i^1 paths of length 1 (i.e., paths K_2), plus $\lambda_i^{\geq 2}$ (open) paths of length ≥ 2, then $\forall i,\ 1 \leq i \leq \psi$, let $\rho(A_i) = \lambda_i^o + \lambda_i^1 + 2\lambda_i^{\geq 2}$, and let $\tau(G) = \max\limits_{1 \leq i \leq \psi} \rho(A_i)$.

A stable set of a graph G is a subset L of the set V of points of G, s.t. the points of L are pairwise nonadjacent, and $\alpha(G)$ denotes the greatest number of points in a stable set of G.

The value $\alpha(G)$ is called the stability number of G. It now follows directly from the definition of an atoll decomposition, that if G is a graph, s.t. $G \in \mathcal{A}$, and that if $\Pi'(A)$ denotes the set of all the endpoints of all the paths (i.e., atolls) which together comprise the collection of paths of the a.d. A of G, less exactly one endpoint from each path of A consisting of a path K_2, then $\Pi'(A)$ is a stable set of G. The stable set $\Pi'(A)$ is called a terminal stable set of G and A is said to conform to $\Pi'(A)$ in G. Therefore it follows that ρ and consequently τ each measure the cardinality of certain terminal stable sets of G. The value $\tau(G)$ is called the terminal stability number of G. Note that if L is a stable set of a graph $G \in \mathcal{A}$, then either L is not a terminal stable set of G, or else L is a terminal stable set of G and it follows that $\exists$ one or more atoll decompositions of G, each of which conforms to L in G.

A triangulated graph is a graph G_Δ, s.t. every cycle of G_Δ having length > 3 contains a diagonal (i.e., a line joining two nonconsecutive points of the cycle). For any subset U of the set V of points of a graph G, let $\langle U \rangle$ denote the subgraph of G induced by the point set U, and for any point u of G, let $G(u)$ denote the set of all those points of G that are adjacent to u. The set $G(u)$ is called the neighborhood of u in G. A simplicial point of a graph G is a point s of G, s.t. $\langle G(s) \rangle = K_n$, where $n \geq 0$, i.e., the neighborhood $G(s)$ of s in G induces a clique (i.e., complete graph) of G. Although obviously not every graph contains a simplicial point, Dirac [6] proved the next statement which treats triangulated graphs:

Proposition II.1. (Dirac [6]) If G_Δ is a triangulated graph, then G_Δ contains at least one simplicial point. (cf. [19], p. 851).

Thus let $\sigma(G_\Delta)$ denote the maximum cardinality of a stable set of simplicial points of the triangulated graph G_Δ, where a stable set of simplicial points is a set of simplicial points that are pairwise nonadjacent. The value $\sigma(G_\Delta)$ is called the simplicity number of G_Δ. The proof of the next proposition is obvious:

Proposition II.2. If G_Δ is a triangulated graph, then $\sigma(G_\Delta) \leq \alpha(G_\Delta)$.

The partition number $\theta(G)$ of a graph G is the minimum order of a partition of the set V of points of G into subsets, each of which induces a clique of G. The proof of the next proposition follows at once from the fact that no two points of a stable set can both be points of a clique:

Proposition II.3. If G is a graph, then $\alpha(G) \leq \theta(G)$.

More than the content of this last proposition is true with regard to triangulated graphs:

Proposition II.4. (Hajnal and Surányi [5]). If G_Δ is a triangulated graph, then $\alpha(G_\Delta) = \theta(G_\Delta)$.

The chromatic number $\chi(G)$ of a graph G is the minimum order of a partition of the set V of points of G into stable subsets. The complement $\bar{G}$ of a graph G has the same set V of points as has G, but two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G. The proof of the next proposition follows directly from the fact that a set U of points of a graph G induces a clique (is a stable set) of G if and only if U is a stable set (induces a clique) of $\bar{G}$:

Proposition II.5. If G is a graph, then $\theta(G) = \chi(\bar{G})$.

The proof of the next proposition is obvious:

Proposition II.6. If G is a graph, s.t. $G \in \mathcal{A}$, then $\tau(G) \leq \alpha(G)$.

The next proposition now follows at once by Propositions II.3, II.5., and II.6.:

Proposition II.7. If G is a graph, s.t. $G \in \mathcal{A}$, then $\tau(G) \leq \chi(\bar{G})$.

Proposition II.8. If G_Δ is a triangulated graph, s.t. $G_\Delta \in \mathcal{A}$, and if $\exists$ a stable set L of G_Δ , s.t. L is a terminal stable set of G_Δ and $|L| = \alpha(G_\Delta)$, then $\tau(G_\Delta) = \chi(\bar{G}_\Delta)$.

Proof: Since L is a terminal stable set of G_Δ , then $\exists$ an a.d. A of G_Δ that conforms to L in G_Δ , and thus it holds that $|L| = \rho(A)$. Hence by Propositions II.4 and II.5, it follows that $\chi(\bar{G}_\Delta) = \theta(G_\Delta) = \alpha(G_\Delta) = |L| = \rho(A) \leq \tau(G_\Delta)$, i.e., $\chi(\bar{G}_\Delta) \leq \tau(G_\Delta)$. Therefore, by Proposition II.7., it results that $\tau(G_\Delta) = \chi(\bar{G}_\Delta)$.
If A is an atoll decomposition of a graph $G \in \mathcal{A}$, then let $\Pi(A)$ denote the set of all the endpoints of all the paths (i.e., again, atolls) which together comprise the collection of paths of A .

Proposition II.9. If G is a graph, s.t. $G \in \mathcal{A}$ and G contains at least one simplicial point, and if A_ι is an atoll decomposition of G and s is a simplicial point of G , s.t. $s \notin \Pi(A_\iota)$, then $\exists$ an a.d. A_ξ of G , s.t. $s \in \Pi(A_\xi)$ and $\rho(A_\xi) \geq \rho(A_\iota)$.

Proof: A maximal clique of a graph G is a clique of G whose set of points is not properly contained in the set of points of any other clique of G . Thus, since s is a simplicial point of G , then it follows that $\exists$ a maximal clique Q of G , s.t. s is a point of Q and $\langle G(s)\rangle = Q-s$, where $Q-s$ is a clique of G . Let V_Q denote the set of points of Q, so that $G(s) = V_Q \backslash s$. Since A_l is a spanning subgraph of G that consists entirely of point disjoint open paths and since by hypothesis $s \notin \Pi(A_l)$, then it follows that s must be an interior point (i.e., nonendpoint) of a path π_t of A_l. Let $\pi_t = (u_1, u_2, \ldots, u_{k-1}, s, u_{k+1}, \ldots, u_e)$ where $s = u_k$. Thus s is adjacent to exactly two points of π_t (viz., u_{k-1} and u_{k+1}), and therefore it follows that s is adjacent to at least two points of G, i.e., $|G(s)| = |V_Q \backslash s| \geq 2$, and so $|V_Q| \geq 3$.

Further, it is required that the paths of A_l cover the points of the clique Q (since A_l is a spanning subgraph of G) in such a way that no two endpoints of any path of A_l having length ≥ 2 are both points of Q, and that no two endpoints of any two distinct paths of A_l (i.e., one endpoint from each of the two paths) are both points of Q (since A_l is an atoll decomposition of G, i.e., since every path of A_l is an open path and since no endpoint of any path of A_l is adjacent to an endpoint of any other path of A_l). Thus it follows that at most two points of $\Pi(A_l)$ are points of Q, and that if two points of $\Pi(A_l)$ are both points of Q, then these two points are the two endpoints of a path K_2 of A_l. Moreover, by hypothesis it holds that $s \notin \Pi(A_l)$, i.e., s is an interior point of a path π_t of A_l. Hence, in view of all the requirements and considerations that have been set forth above, it directly follows that $\exists$ just four ways in which the paths of A_l can (and must) cover the points of Q, and so therefore only the following four cases may arise:

(a) every point of $Q-s$ is an interior point of a path of A_l, the two endpoints u_1 and u_e of π_t are both points of $G \backslash Q$, and s is an interior point of π_t ,

(b) two points v_1 and v_2 of Q-s are the endpoints of a path π_r of A_ι having length 1 (i.e., a path K_2) , every other point of Q-s is an interior point of a path of A_ι , both u_1 and u_e are points of $G\backslash Q$, and s is an interior point of π_t ,

(c) one point w_1 of Q-s is the endpoint of a path π_p of A_ι , every other point of Q-s is an interior point of a path of A_ι , both u_1 and u_e are points of $G\backslash Q$, s is an interior point of π_t , and if $\pi_p \neq K_1$, then the other endpoint w_f of π_p is a point of $G\backslash Q$,

(d) u_1 is a point of $G\backslash Q$, u_e is a point of Q-s , every other point of Q-s is an interior point of a path of A_ι , and s is an interior point of π_t .

Diagram 2.1 illustrates the four cases just described, and consists of four separate figures, each of which represents one of the four cases and each of which focuses in and around that local part of G centered on Q . For the sake of visual clarity, each figure represents the case in question only in its essentials.

If A is an atoll decomposition of a graph $G \in \mathcal{A}$, then let $\Sigma(A)$ denote the collection of all the paths (i.e., again, atolls) of A . A path transformation of an a.d. A of a graph $G \in \mathcal{A}$ is a finite sequence of additions and/or deletions of points and/or lines executed on the paths of a subcollection of $\Sigma(A)$, s.t. a new a.d. A_ξ of G is obtained from A by means of this sequence of operations. Since Q is a clique of G , then every two points of Q are joined by a line, and therefore it follows from this fact that each of the

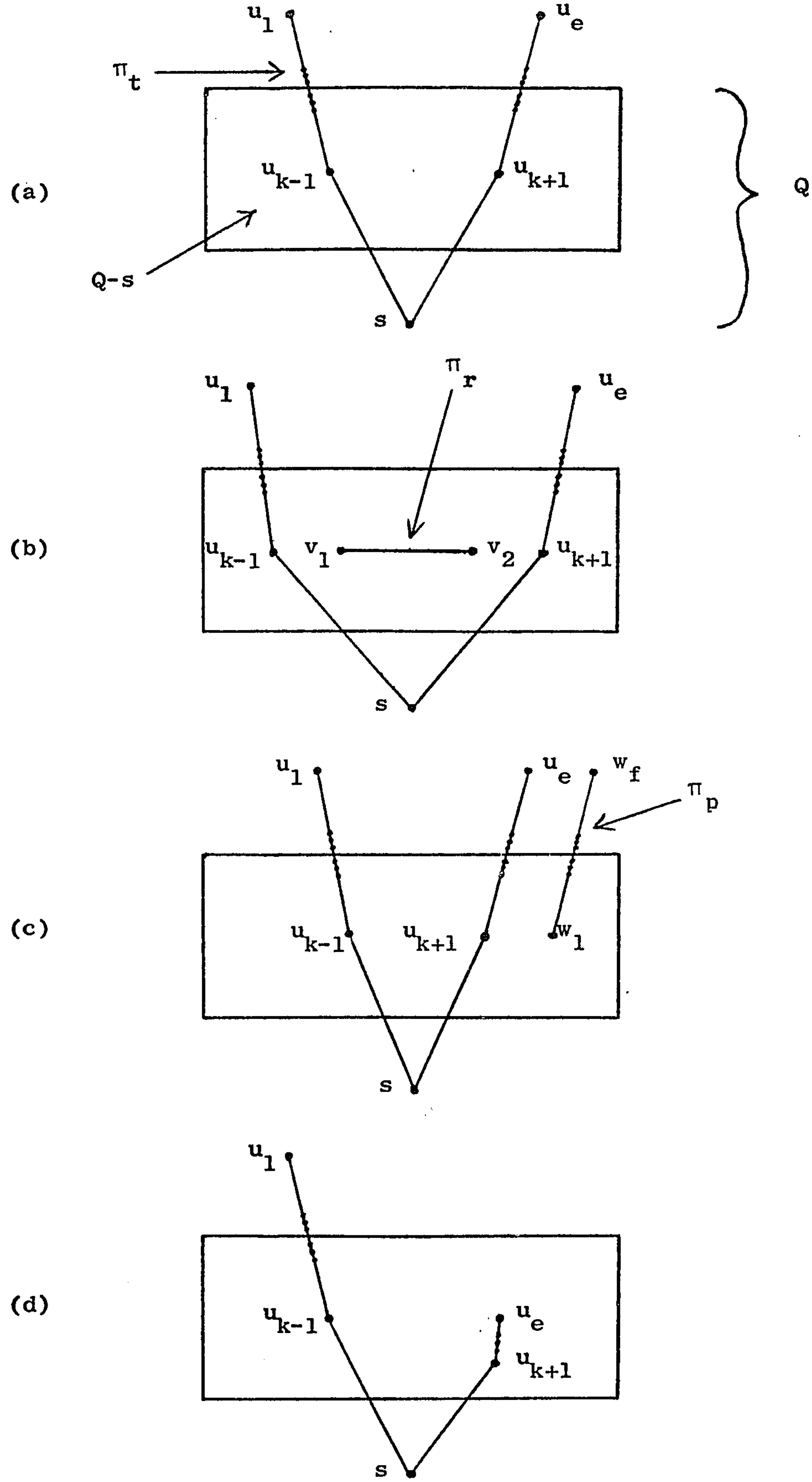


Diagram 2.1. The four ways in which A_l can cover Q .

following four path transformations of A_ι may be carried out:

$\xi(a)$: in case (a), first delete the point s from π_t, and then add the line $u_{k-1}u_{k+1}$ to $\pi_t - s$, thereby forming two new paths π'_t and s of an a.d. A_ξ of G, for which $\Sigma(A_\xi) = ((\Sigma(A_\iota))\setminus\pi_t) \cup \{\pi'_t, s\}$;

$\xi(b)$: in case (b), first delete the point s from π_t, then add the path π_r to $\pi_t - s$, and then add the lines $u_{k-1}v_1$ and v_2u_{k+1} to $(\pi_t - s) \cup \pi_r$, thereby forming two new paths π''_t and s of an a.d. A_ξ of G, for which $\Sigma(A_\xi) = ((\Sigma(A_\iota))\setminus\{\pi_r, \pi_t\}) \cup \{\pi''_t, s\}$;

$\xi(c)$: in case (c), first delete the point s from π_t, then add the line $u_{k-1}u_{k+1}$ to $\pi_t - s$, and then first add the point s to π_p and then add the line $w_1 s$ to $\pi_p \cup s$, thereby forming two new paths π'''_t and π'_p of an a.d. A_ξ of G, for which $\Sigma(A_\xi) = ((\Sigma(A_\iota))\setminus\{\pi_p, \pi_t\}) \cup \{\pi'''_t, \pi'_p\}$;

$\xi(d)$: in case (d), first delete the line $u_{k-1}s$ from π_t, and then add the line $u_{k-1}u_e$ to $\pi_t - u_{k-1}s$, thereby forming a new path π''''_t of an a.d. A_ξ of G, for which $\Sigma(A_\xi) = ((\Sigma(A_\iota))\setminus\pi_t) \cup \pi''''_t$.

Diagram 2.2 illustrates the four path transformations just described, and consists of four separate figures, each of which represents the result of one of the four path transformations and each of which focuses in and around that local part of G centered on Q .

It is clear that each of the four sequences of operations, viz., $\xi(a), \xi(b), \xi(c)$ and $\xi(d)$, are valid path transformations, and that each of them yields an a.d. A_ξ of G, s.t. $s \in \Pi(A_\xi)$. Further, it

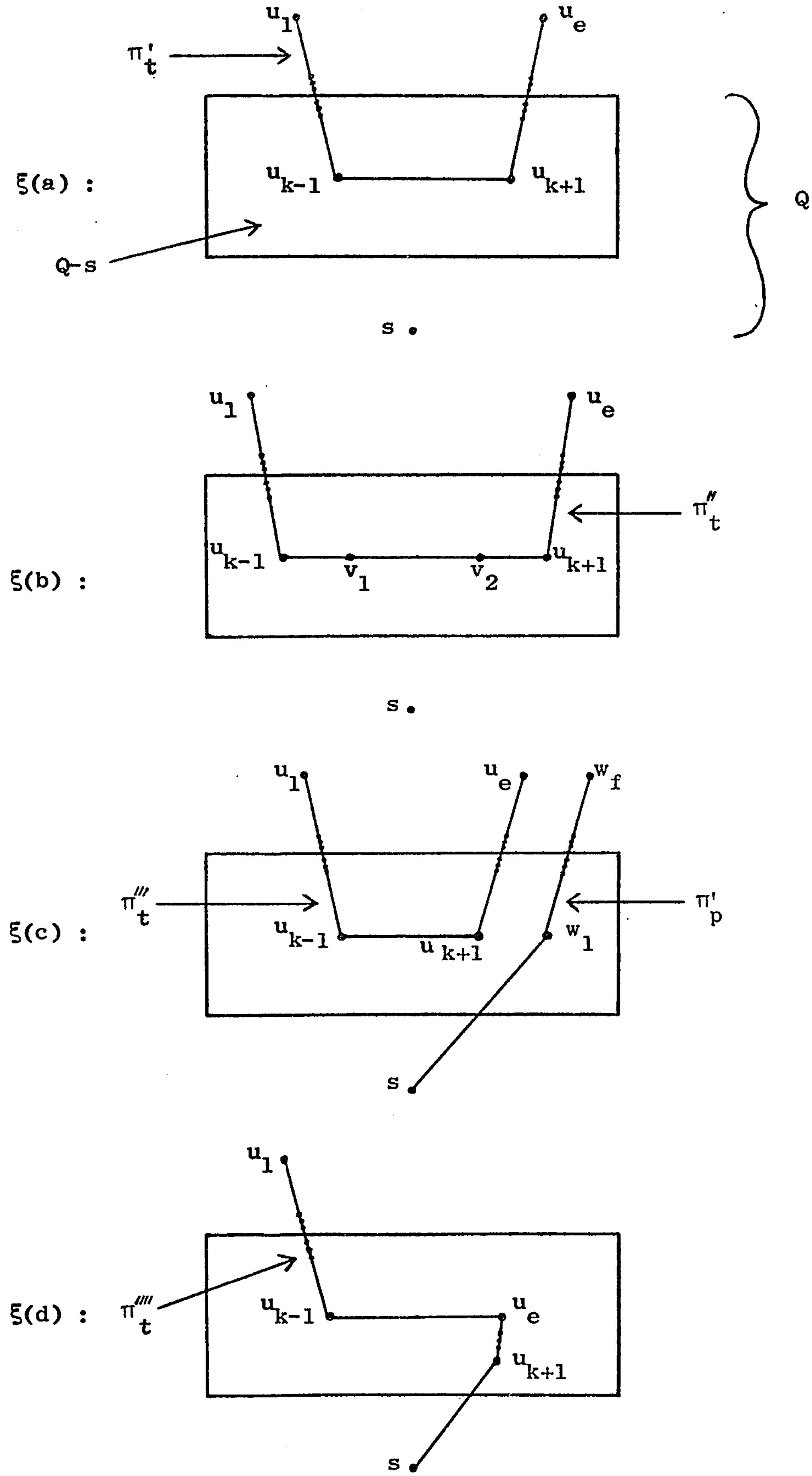


Diagram 2.2. The four path transformations of A_l .

follows that path transformation $\xi(a)$ yields an a.d. A_ξ of G for which $\rho(A_\xi) = \rho(A_\iota) + 1$, that path transformation $\xi(b)$ yields an a.d. A_ξ of G for which $\rho(A_\xi) = \rho(A_\iota)$, that path transformation $\xi(c)$ yields an a.d. A_ξ of G for which $\rho(A_\xi) = \rho(A_\iota)$ if π_p is a path K_1 or if π_p is a path having length ≥ 2, and for which $\rho(A_\xi) = \rho(A_\iota) + 1$ if π_p is a path K_2, and that path transformation $\xi(d)$ yields an a.d. A_ξ of G for which $\rho(A_\xi) = \rho(A_\iota)$. Therefore $\forall$ four path transformations, it holds that $\rho(A_\xi) \geq \rho(A_\iota)$. Q.E.D.

Proposition II.10. If G is a graph, s.t. $G \in \mathcal{Q}$ and G contains at least one simplicial point, and if A_ι is an atoll decomposition of G, S_0 is a stable set of simplicial points of G and s' is a point of S_0, s.t. $|S_0| \geq 2$, $s' \notin \Pi(A_\iota)$, and $S_0 \setminus s' \subseteq \Pi(A_\iota)$, then $\exists$ an a.d. A_ξ of G, s.t. $S_0 \subseteq \Pi(A_\xi)$.

Proof: Let s^* be any point of $S_0 \setminus s'$ and let Q^* be the maximal clique of G, s.t. s^* is a point of Q^* and $\langle G(s^*) \rangle = Q^* - s^*$, and let Q' be the maximal clique of G, s.t. s' is a point of Q' and $\langle G(s') \rangle = Q' - s'$. Since by hypothesis s^* is not adjacent to s', then it follows both that s^* is not a point of Q' and that s' is not a point of Q^*. Moreover by hypothesis it holds both that $s^* \in \Pi(A_\iota)$ and that $s' \notin \Pi(A_\iota)$, i.e., s^* is a terminal point (i.e., an endpoint) of a path (i.e., atoll) of A_ι and s' is an interior point of a path of A_ι. If now an appropriate path transformation is carried out with respect to s', then by Proposition II.9 it follows that $\exists$ an a.d. A_ξ of G, s.t. $s' \in \Pi(A_\xi)$. It is obvious from an examination of each of the four path transformations which are described in the proof of Proposition II.9 that the only new terminal point of a path (of A_ξ) that is created after a path transformation is carried out with respect to s' is s'. Thus since s' is not a point of Q^*, then it follows that no new terminal point of a path (of A_ξ) is created as a point of Q^*. It is also obvious by examination that since s^* is not a point of Q', then it follows that s^* is not transformed into an interior point of a path (of A_ξ) after a path transformation is carried out with respect to s', i.e., s^* remains an endpoint (i.e., again, a terminal point) of a path (of A_ξ). Hence, since no new terminal point of a path (of A_ξ) is created as a point of Q^* and since s^* remains a terminal point of a path (of A_ξ), then it follows that $s^* \in \Pi(A_\xi)$, and therefore $S_0 \subseteq \Pi(A_\xi)$.

Proposition II.11. If G is a graph, s.t. $G \in \mathcal{A}$ and G contains at least one simplicial point, and if S is a stable set of simplicial points of G, then $\exists$ an a.d. A_ξ of G, s.t. $\forall s \in S$, it holds that $s \in \Pi(A_\xi)$.

Proof: The proof of this proposition follows directly by Propositions II.9 and II.10. If A_τ is any atoll decomposition of G, then either $\forall s \in S$, it holds that $s \in \Pi(A_\tau)$, and in this case it follows that $A_\xi = A_\tau$, or else it holds that S' is nonempty, where $S' = \{s\colon s \in S$ and $s \notin \Pi(A_\tau)\}$. If the latter case holds, then repeatedly apply Proposition II.9 with respect to each point of S' (i.e., one point after the other, in any order whatsoever, so that every point of S' has been taken on). Proposition II.10 guarantees that this repeated application of Proposition II.9 with respect to each point of S' will finally yield an a.d. A_ξ of G, s.t. $S \subseteq \Pi(A_\xi)$.

Proposition II.12. If G_Δ is a triangulated graph, s.t. $G_\Delta \in \mathcal{A}$ and $\sigma(G_\Delta) = \alpha(G_\Delta)$, then $\tau(G_\Delta) = \chi(\bar{G}_\Delta)$.

Proof: Let S be a stable set of simplicial points of G_Δ, s.t. $|S| = \sigma(G_\Delta)$. Therefore, since by hypothesis G_Δ is a triangulated graph, s.t. $G_\Delta \in \mathcal{A}$, then by Propositions II.1 and II.11, it follows that $\exists$ an a.d. A_ξ of G_Δ, s.t. $\forall s \in S$, it holds that $s \in \Pi(A_\xi)$. Hence, since $\forall s \in S$ it holds that $s \in \Pi(A_\xi)$, then it follows that the stable set S of G_Δ must be contained in a terminal stable set L_ξ of G_Δ, s.t. A_ξ conforms to L_ξ in G_Δ, and thus it follows that $\rho(A_\xi) = |L_\xi| \geq |S|$, i.e., $\rho(A_\xi) \geq |S|$. But $\alpha(G_\Delta) \geq \tau(G_\Delta)$ by Proposition II.6, and $\tau(G_\Delta) \geq \rho(A_\xi)$ by definition. Therefore it holds that $\alpha(G_\Delta) \geq \tau(G_\Delta) \geq \rho(A_\xi) \geq |S| = \sigma(G_\Delta)$, and since $\sigma(G_\Delta) = \alpha(G_\Delta)$ by hypothesis, then it follows that $\alpha(G_\Delta) = |S| = \rho(A_\xi)$. Hence, since $\forall s \in S$, it holds that $s \in \Pi(A_\xi)$, then in view of $|S| = \rho(A_\xi)$, it follows that A_ξ conforms to S in G_Δ, and thus it holds that S is a terminal stable set of G_Δ, s.t. $|S| = \alpha(G_\Delta)$. Therefore by Proposition II.8, it follows that $\tau(G_\Delta) = \chi(\bar{G}_\Delta)$.

A dominating set of a graph G is a subset D of the set V of points of G, s.t. every point of G that is not in D is adjacent to at least one point of D (see Part V below). The proof of the next proposition is straightforward:

<u>Proposition II.13.</u> If G is a graph which contains at least one simplicial point, then $\sigma(G) = \alpha(G)$ if and only if G contains at least one set of simplicial points which is a dominating set of G.

Therefore the next proposition follows at once by Propositions II.1, II.12, and II.13 :

<u>Proposition II.14.</u> If G_Δ is a triangulated graph, s.t. $G_\Delta \in \mathcal{A}$ and G_Δ contains at least one set of simplicial points which is a dominating set of G_Δ, then $\tau(G_\Delta) = \chi(\bar{G}_\Delta)$.

Propositions II.12 and II.14 can be employed for a triangulated graph G_Δ only if $G_\Delta \in \mathcal{A}$. Diagram 4.1 (see below) depicts the graph G_o which is a triangulated graph but which does not have an atoll decomposition so that Propositions II.12 and II.14 are not applicable here (moreover, with regard to Proposition II.12, it holds that $\sigma(G_o) = 3 \neq 4 = \alpha(G_o)$) . Further, if G_q is a clique which contains at least three points, then G_q is a triangulated graph and $\sigma(G_q) = 1 = \alpha(G_q)$, but $G_q \notin \mathcal{A}$, so that Proposition II.12 is not applicable in this case too; and since $G_q \notin \mathcal{A}$, then Proposition II.14 is also not applicable in this case (even though every point of G_q constitutes a set of simplicial points of G_q which is a dominating set of G_q).

III. Atoll Decompositions and Trees

Again, an island decomposition of a graph G is an acyclic spanning subgraph of G in which every point has degree ≤ 2, i.e., an island decomposition is a collection of point disjoint paths (which are called the islands of the island decomposition) that cover the points of G. An isolated point can be a member of such a collection. Clearly every graph has an island decomposition, e.g., the set of all the points of a graph is an island decomposition of the graph. A minimum island decomposition is an island decomposition which contains the smallest number of islands in an island decomposition. Obviously every graph has a minimum island decomposition. In general, an endpoint of an atoll of an atoll decomposition will also be referred to as a terminal point of the atoll, whereas an endpoint of an island of an island decomposition will not have another name.

A tree is a connected acyclic graph.

Proposition III.1. If T is a tree, then T is a triangulated graph and $T \in \mathcal{A}$.

Proof: Since a tree contains no cycles, then T is a priori a triangulated graph. Let I_ζ be a minimum island decomposition of T. It follows by the definition of a minimum island decomposition that no two endpoints of any two distinct islands of I_ζ (i.e., one endpoint from each of the two islands) are adjacent (since otherwise an island decomposition of T could be formed containing fewer islands than I_ζ contains, in contradiction to the fact that I_ζ is a minimum island decomposition of T). Moreover, since T contains no cycles, then it follows that no two endpoints of any island of I_ζ having length ≥ 2

are adjacent (since otherwise T would contain a cycle), i.e., every island of I_ζ is an open path. Thus it follows that I_ζ is a collection of point disjoint open paths (i.e., the islands of I_ζ) that cover the points of T, s.t. no endpoint of any path of I_ζ is adjacent to an endpoint of any other path of I_ζ, and so therefore it follows that I_ζ is an atoll decomposition of T.

If T is a nontrivial tree (i.e., $T \neq K_1$), then let $\epsilon(T)$ denote the number of endpoints of T (i.e., those points of T having degree one), and let $\epsilon(K_1) = 1$, so that $\sigma(K_1) = \epsilon(K_1)$.

<u>Proposition III.2.</u> If T is a tree, s.t. $T \neq K_2$, then $\sigma(T) = \epsilon(T)$.

Proof: Since T is a triangulated graph by Proposition III.1, then T contains at least one simplicial point by Proposition II.1, i.e., $\sigma(T)$ is meaningful. If s is a simplicial point of T $(T \neq K_1)$, then $\langle T(s) \rangle = K_n$, where $n \geq 1$. But T contains no cycles and so it follows that n must be equal to 1, i.e., s is an endpoint of T. Conversely, if e is an endpoint of T, then $\langle T(e) \rangle = K_1$, i.e., e is a simplicial point of T. Hence the set of all simplicial points of T is precisely the set of all endpoints of T. Furthermore, if $T \neq K_2$, then plainly any two endpoints of T are nonadjacent (since otherwise T would contain a cycle), and thus any two simplicial points of T are nonadjacent, i.e., the set of all simplicial points of T is a stable set of T. Hence it follows that the set of all simplicial points of T contains exactly $\sigma(T)$ points, and therefore it follows that $\sigma(T) = \epsilon(T)$.

The next proposition now follows at once by Propositions II.12, III.1, and III.2 :

<u>Proposition III.3.</u> If T is a tree, s.t. $T \neq K_2$, and if $\epsilon(T) = \alpha(T)$,

then $\tau(T) = \chi(\bar{T})$.

A bipartite graph is a graph B whose set V_B of points can be partitioned into two stable subsets V_B^1 and V_B^2 , i.e., if x is any line of B , then x has one of its points in V_B^1 and its other point in V_B^2 .

Proposition III.4. (König [7]) A graph is bipartite if and only if all its cycles have even length.

The proof of the next proposition is obvious (since a tree is a connected acyclic graph) and follows immediately by Proposition III.4 :

Proposition III.5. A tree is a connected bipartite graph.

A pair of lines is said to be nonadjacent if they do not share a common point, an independent set of lines or a matching is a set of lines that are pairwise nonadjacent, a maximum matching of a graph G contains the largest number of lines in a matching of G , and a partial matching of G is nonempty but contains fewer lines than does a maximum matching of G . Let $\gamma(G)$ denote the number of lines in a maximum matching of the graph G . A point cover for G is a subset W of the set V of points of G , s.t. every line of G has at least one of its points in W , and the smallest number of points in any point cover for G is called its point covering number and is denoted by $\nu(G)$. Analogously, a line cover of G is a subset Z of the set E of lines of G , s.t. every point of G is a point of some line of Z , and the smallest number of lines in any line cover of G is called its line covering number and is denoted by $\nu'(G)$. A minimum line cover of G is a line cover of G which contains $\nu'(G)$ lines.

Proposition III.6. (König [8]) If B is a bipartite graph, then

$\gamma(B) = \nu(B)$.

Proposition III.7. (Gallai [9]) If G is a nontrivial connected graph with p points, then $\alpha(G) + \nu(G) = p = \gamma(G) + \nu'(G)$.

The next proposition now follows immediately by Propositions III.6 and III.7 :

Proposition III.8. If B is a nontrivial connected bipartite graph, then $\alpha(B) = \nu'(B)$.

Thus the next proposition follows at once by Propositions III.5 and III.8 :

Proposition III.9. If T is a nontrivial tree, then $\alpha(T) = \nu'(T)$.

Let B be a bipartite graph with bipartition $\{V_B^1, V_B^2\}$ of its point set V_B , where $|V_B^1| = m$ and $|V_B^2| = n$. If B contains every line joining each point of V_B^1 to each point of V_B^2 , then B is called a complete bipartite graph and is denoted by $K_{m,n}$. Hence a talon is a complete bipartite graph $K_{1,3}$, and by a star is meant either a complete bipartite graph $K_{1,n}$, where $n \geq 1$, or a point K_1 (and thus a talon is a star which contains exactly three lines). A forest is an acyclic graph.

Let Y be a minimum line cover of the nontrivial tree T , so that $|Y| = \nu'(T)$. Since every point of T is a point of some line of Y and since T is acyclic, then it follows that Y induces a spanning forest R of T . Moreover, since Y is a minimum line cover of T , then it follows that $\nexists$ a line $z \in Y$, s.t. one point of z is a point of a line $y_1 \in Y$ and the other point of z is a point of a line $y_2 \in Y$, where $y_1 \neq y_2$ (because if Y contained such a line z , then $Y' = Y \setminus z$ would be a line cover of T with $|Y'| < |Y|$, in contradiction to the fact that Y is a minimum line cover of T). Hence it results that R contains no path having length ≥ 3 , and therefore it follows that R is the point disjoint union of complete bipartite subgraphs K_{1,n_o} of T (which are called the component stars of R), s.t. $n_o \geq 1$, i.e., every component star of R contains a line. Further, if L is any stable set of T , s.t. $|L| = \alpha(T)$, then by Proposition III.9 $\exists$ a bijection $b: L \to Y$ between the points of L and the lines of Y . Therefore, since each point of L is a point of a line of Y (because Y is a line cover of T) and no two points of L are the two points of a line of Y (because L is a stable set of T), and since $|L| = |Y|$ (by Proposition III.9), then it results that b may be defined as follows: if $u \in L$, then $b(u) = y \in Y$, where u is a point of y . Hence, in view of these considerations, it follows that L furnishes the set of endpoints (i.e., again, those points having degree one) of the component stars K_{1,n_o} of R , for $n_o \geq 2$, and that each component star $K_{1,1}$ of R contains exactly one point of L as one of its endpoints. Therefore, if V_T denotes the set of points of T , then it follows that $V_T \backslash L$ furnishes the set of centerpoints (i.e., nonendpoints) or foci of the component stars K_{1,n_o} of R , for $n_o \geq 2$, and that each component star $K_{1,1}$ of R contains exactly one point of $V_T \backslash L$ as one of its endpoints, i.e., $|V_T \backslash L|$ is the number of component stars of R (i.e., $|V_T \backslash L|$ is the number of components of R). Thus, since by Proposition III.7 it holds that $|V_T \backslash L| = |V_T| - |L| = |V_T| - \alpha(T) = \nu(T)$, then it follows that R contains exactly $\nu(T)$ component stars. Therefore, in view of the fact that R is the point disjoint union of $\nu(T)$ stars K_{1,n_o} , where $n_o \geq 1$ (i.e., again, every component star of R contains a line), it now follows that if precisely one line is chosen from each component star of R and that if M_r is the set of lines so selected, then M_r is a matching of R . Furthermore it is clear that M_r is in fact a maximum matching of R , and thus, since M_r contains exactly $\nu(T)$ lines, then it follows that $\gamma(R) = \nu(T)$. And it is also clear that every maximum matching of R can be obtained by means of the selection process just described, i.e., again, one line is chosen from each component star of R . The proof of the next proposition is essentially due to Norman and Rabin (cf. [10], p.319) :

<u>Proposition III.10.</u> (Norman and Rabin [10]) M_t is a maximum matching of a tree T if and only if M_t is a maximum matching of a spanning forest of T that is induced by a minimum line cover of T .

Let φ_E denote the empty set of lines. If X is a set of lines of a graph G , s.t. $X = X_1 \cup X_2 \cup \ldots \cup X_\Gamma$, where $\forall i,\ 1 \leq i \leq \Gamma$, it holds that $X_i \neq \varphi_E$, and where $\forall i,j\ ,\ 1 \leq i < j \leq \Gamma$, it holds that $X_i \cap X_j = \varphi_E$, then X is said to be the direct sum of its subsets $X_1, X_2, \ldots, X_\Gamma$ (i.e., $\{X_1, X_2, \ldots, X_\Gamma\}$ is a partition of X). This direct sum decomposition of X may be written as $X = \oplus_{i=1}^{\Gamma} X_i$, or as $X = \oplus_{i \in \{I\}} X_i$,where $\{I\}$ is a finite index set, s.t. $|\{I\}| = \Gamma$. A minimum direct sum decomposition of X has the minimum order of a direct sum decomposition of X .

<u>Proposition III.11.</u> If Y is a minimum line cover of a nontrivial tree T , then Y has a minimum direct sum decomposition into matchings of T :

$$Y = \left(\oplus_{k=1}^{\omega} M_k \right) \oplus \left(\oplus_{j \in \{J\}} P_j \right) ,$$

where $\omega \geq 1$ and $\forall k\ ,\ 1 \leq k \leq \omega$, each M_k is a maximum matching of T , and, where $|\{j\}| = \omega' \geq 0$ and $\forall j \in \{j\}$, each P_j is a partial matching of T ; furthermore, the value $\omega+\omega'$ is the smallest number of matchings of T in a direct sum decomposition of Y into matchings of T, and the value $\omega = \omega(Y)$ is the greatest number of pairwise line disjoint maximum matchings of T that are contained in Y.

Proof: As was shown above, Y induces a spanning forest R of T, s.t. R is the point disjoint union of stars K_{1,n_0} of T, where $\forall$ such component stars K_{1,n_0} of R, it holds that $n_0 \geq 1$, i.e., each component star of R contains a line. It was also shown above that if $c(R)$ denotes the number of component stars of R, then $c(R) = \gamma(R)$. But $\gamma(R) = \gamma(T)$ by Proposition III.10, and so it follows that $c(R) = \gamma(T)$. Clearly the following "greedy" type of algorithm yields a minimum direct sum decomposition of Y into matchings of T of the kind described in the statement of this proposition: Choose exactly one line from each component star K_{1,n_0} of R, so that if N_1 is the set of lines so selected, then N_1 is a maximum matching M_1 of T (since $c(R) = \gamma(T)$). Next choose exactly one line from each component star K_{1,n_1} of $R-N_1$, s.t., $n_1 \geq 1$ (i.e., s.t. K_{1,n_1} contains at least one line of $R-N_1$, where $R-N_1$ is the spanning forest of T obtained from R by the deletion of the set N_1 of lines from R, so that $R-N_1$ is the point disjoint union of stars K_{1,n_1} of T), so that if N_2 is the set of lines so selected, then N_2 is a matching of T. Specifically, if every component star of $R-N_1$ contains a line, then N_2 is a maximum matching M_2 of T (since in this case, $c(R-N_1) = c(R) = \gamma(T)$); if at least one component star of $R-N_1$ contains a line, but fewer than $c(R)$ component stars of $R-N_1$ contain a line, then N_2 is obviously a partial matching P_1 of T ; and if no component star of $R-N_1$ contains a line (i.e., $R-N_1$ consists entirely of points K_1), then $N_2 = \varphi_E$, and stop. In general, choose exactly one line from each component star K_{1,n_h} of $R - (N_1 \cup N_2 \cup \ldots \cup N_h)$, s.t. $n_h \geq 1$, so that if N_{h+1} is the set of lines so selected, then N_{h+1} is a matching of T. Specifically, if every component star of $R - (N_1 \cup N_2 \cup \ldots \cup N_h)$ contains a line, then N_{h+1} is a maximum matching M_{h+1} of T; if at least one component star of $R - (N_1 \cup N_2 \cup \ldots \cup N_h)$ contains a line, but fewer than $c(R)$ component stars of $R - (N_1 \cup N_2 \cup \ldots \cup N_h)$ contain a line, then N_{h+1} is a partial matching P_h of T; and if no component star of $R - (N_1 \cup N_2 \cup \ldots \cup N_h)$ contains a line, then $N_{h+1} = \varphi_E$, and stop. Note that $|V_T| \geq c(R-(N_1 \cup N_2 \cup \ldots \cup N_h)) \geq c(R-(N_1 \cup N_2 \cup \ldots \cup N_{h-1})) \geq \ldots \geq c(R-N_1) \geq c(R)$, and this algorithm terminates when that spanning forest $R - (N_1 \cup N_2 \cup \ldots \cup N_g)$ of T is obtained for which $c(R - (N_1 \cup N_2 \cup \ldots \cup N_g)) = |V_T|$.

If $\{Y_\ell\}_{\ell=1}^{\Lambda}$ is the set of all minimum line covers of a nontrivial tree T, then let $\omega(T) = \max_{1 \leq \ell \leq \Lambda} \omega(Y_\ell)$, and let $\omega(K_1) = 0$.

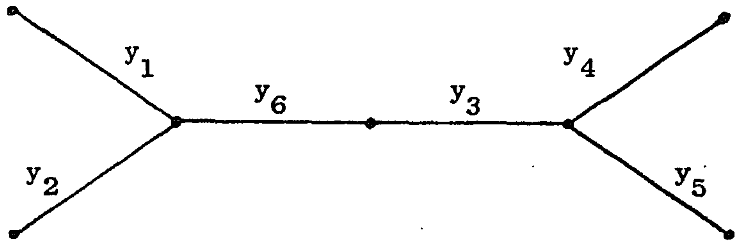
<u>Proposition III.12.</u> If T is a tree, s.t. $\omega(T) \geq 2$, then $\tau(T) = \chi(\bar{T})$.

Proof: Let Y^* be a minimum line cover of T, s.t. $\omega(Y^*) = \omega(T) \geq 2$, and let R^* be the spanning forest of T induced by Y^*, so that R^* is the point disjoint union of stars K_{1,n^*} of T. Since $\omega(Y^*) \geq 2$, then it follows by Proposition III.11 that $\forall$ component stars K_{1,n^*} of R^*, it holds that $n^* \geq 2$, i.e., no component star of R^* is either a point K_1 or a path K_2, and every component star of R^* contains at least two lines. Further, it was shown above that if L is a stable set of T, s.t. $|L| = \alpha(T)$, then L constitutes the set of endpoints of the component stars of R^* . If all but two lines of each component star of R^* which contains more than two lines are now deleted from each component star of R^* which contains more than two lines, then plainly a spanning forest A^* of T is obtained, s.t. A^* contains no talons (i.e., again, $K_{1,3}$' s), i.e., the degree of every point of A^* is ≤ 2 . It is obvious that A^* consists entirely of points K_1 and of paths $K_{1,2}$, and that the $\alpha(T)$ points of L now constitute the set of end-points of all the paths which together comprise A^* (i.e., A^* has at least one path $K_{1,2}$, and A^* may or may not have points K_1 as its paths). Hence it follows that the points of $V_T \backslash L$ constitute the set of foci of the component stars $K_{1,2}$ (i.e., paths $K_{1,2}$) of A^* . It is now an easy matter to verify that A^* is in fact an atoll decomposition of T , s.t. $\Pi(A^*) = L$, and thus it follows that L is a terminal stable set of T , s.t. $|L| = \alpha(T)$. Therefore it follows by Propositions II.8 and III.1 that $\tau(T) = \chi(\bar{T})$.

Diagram 3.1 depicts the tree T_o which shows that Proposition III.12 is not implied by Proposition III.3.

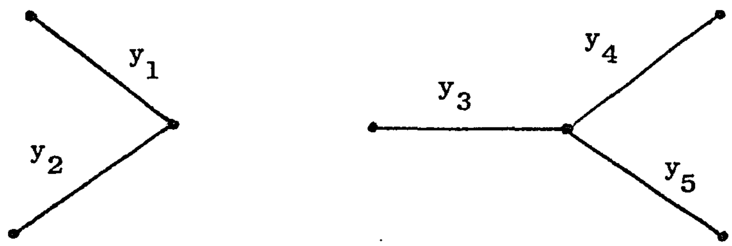
If $\pi_j = (u_1, u_2, \ldots, u_k, u_{k+1}, u_{k+2}, \ldots, u_{k+r}, \ldots, u_c)$ is any path of a graph G , where $1 \leq k \leq c$ and $0 \leq r \leq c-1$, then the subpath $(u_k, u_{k+1}, u_{k+2}, \ldots, u_{k+r})$ of π_j is called a segment of π_j of length r . Thus it follows that a path is a segment of itself. A maximal island decomposition of a graph G is an island decomposition I_a of G , s.t. if every island of I_a is a segment of an island of an island decomposition I_b of G , then $I_b = I_a$. It follows that every minimum island decomposition is a maximal island decomposition, and so therefore, since every graph has a minimum island decomposition, then every graph has a maximal island decomposition.

T_o



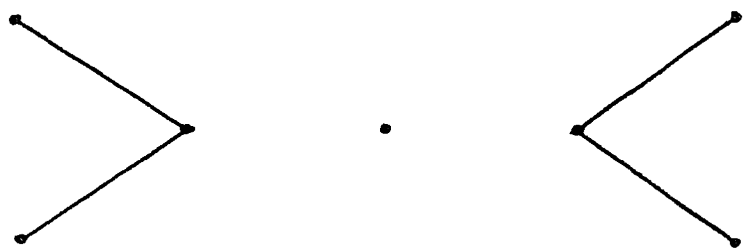
Proposition III.3.: $T_o \neq K_2$, $\epsilon(T_o) = 4 \neq \alpha(T_o) = 5$.

R^*



Proposition III.12.: $Y^* = \{y_1, y_2, y_3, y_4, y_5\}$, Y^* induces R^* , $\omega(Y^*) = 2 = \omega(T_o)$, i.e., $\omega(T_o) \geq 2$.

A^*



Proposition III.12.: Since T_o is a tree, s.t. $\omega(T_o) \geq 2$, then $\tau(T_o) = \chi(\bar{T}_o)$, where $\tau(T_o) = \rho(A^*) = 5$.

Diagram 3.1. An example which shows that Proposition III.12 is not implied by Proposition III.3.

Proposition III.13. An island decomposition I_a of a tree T is a maximal island decomposition of T if and only if I_a is an atoll decomposition of T.

Proof: If A_t is any atoll decomposition of T, then by the definition of an atoll decomposition it follows both that A_t is an island decomposition of T and that no two endpoints of any two distinct paths (i.e., atolls) of A_t are adjacent (i.e., one endpoint from each of the two paths). Assume that A_t is not a maximal island decomposition of T. Hence it follows that $\exists$ an island decomposition I_b of T, s.t. every path (i.e., again, atoll) of A_t is a segment of a path (i.e., island) of I_b, and that $I_b \neq A_t$. Therefore it follows that $\exists$ two distinct paths $\pi_{t'}$ and $\pi_{t''}$ of A_t, s.t. an endpoint v' of $\pi_{t'}$ is adjacent to an interior point v'' of $\pi_{t''}$ (since every path of A_t is a segment of a path of I_b, and since no two endpoints of any two distinct paths of A_t are adjacent). And therefore it follows that the path $(\pi_{t'}, v'')$ (which is not an atoll of A_t) must be a segment of an island of I_b. But since v'' is an interior point of $\pi_{t''}$, then this last assertion implies that $\pi_{t''}$ cannot be a segment of any island of I_b, which is a contradiction. Therefore the original assumption is false, and so A_t is a maximal island decomposition of T. Conversely, if I_a is any maximal island decomposition of T, then I_a is a collection of point disjoint paths that cover the points of T (since I_a is an island decomposition of T), s.t. no two endpoints of any path of I_a having length ≥ 2 are adjacent (since T is acyclic) and s.t. no two endpoints of any two distinct paths of I_a are adjacent (since I_a is a maximal island decomposition of T), i.e., the islands of I_a are open paths, and it follows at once that I_a is an atoll decomposition of T .

For any graph G , $\zeta(G)$ denotes the smallest number of islands in an island decomposition of G , and if in addition $G \in \mathcal{A}$, then $\beta(G)$ denotes the smallest number of atolls in an atoll decomposition of G . Since every atoll decomposition is an island decomposition, then the proof of the next proposition is immediate:

<u>Proposition III.14</u>. If G is a graph, s.t. $G \in \mathcal{A}$, then $\zeta(G) \leq \beta(G)$.

<u>Proposition III.15</u>. If T is a tree, then $\zeta(T) = \beta(T)$.

Proof: By Proposition III.1, it holds that $T \in \mathcal{A}$, and so by Proposition III.14, it follows that $\zeta(T) \leq \beta(T)$. On the other hand, let I_ζ be any minimum island decomposition of T , so that I_ζ contains exactly $\zeta(T)$ islands. It follows from the proof of Proposition III.1 that I_ζ is an atoll decomposition of T , and so it follows that $\beta(T) \leq \zeta(T)$. Therefore it results that $\zeta(T) = \beta(T)$.

<u>Proposition III.16</u>. (Boesch, Chen, and McHugh [4]). If G is a connected graph and if $\{T_1, T_2, \ldots, T_\upsilon\}$ is the set of all spanning trees of G , then

$$\zeta(G) = \min_{1 \leq k \leq \upsilon} \zeta(T_k) .$$

The next proposition follows at once by Propositions III.14, III.15, and III.16 :

<u>Proposition III.17</u>. If G is a connected graph, s.t. $G \in \mathcal{A}$, and if $\{T_1, T_2, \ldots, T_\upsilon\}$ is the set of all spanning trees of G , then

$$\beta(G) \geq \min_{1 \leq k \leq \upsilon} \beta(T_k) .$$

If G is a graph, s.t. $G \in \mathcal{A}$, then $\Omega(G)$ denotes the greatest number of atolls in an atoll decomposition of G .

<u>Proposition III.18</u>. If G is a connected graph, s.t. $G \in \mathcal{A}$, and if $\{T_1, T_2, \ldots, T_\upsilon\}$ is the set of all spanning trees of G , then

$$\Omega(G) \leq \max_{1 \leq k \leq \upsilon} \Omega(T_k) .$$

Proof: Let A_h be an atoll decomposition of G, s.t. A_h contains exactly $\Omega(G)$ atolls. Since A_h is an atoll decomposition of G, then it follows that A_h is a spanning forest of G. Thus if each line of A_h is assigned a length equal to one and if each line of $G \backslash A_h$ is assigned a length greater than one, and then if Kruskal's algorithm for obtaining a minimum length spanning tree of a connected graph (cf. [11]) is applied with respect to G, then a spanning tree T_g of G is obtained (where $1 \leq g \leq \upsilon$), s.t. the lines of A_h are contained in T_g and A_h is a spanning forest of T_g. Moreover, since A_h is an atoll decomposition of G, then it is clear that A_h is also an atoll decomposition of T_g which contains exactly $\Omega(G)$ atolls. Hence it follows that $\Omega(T_g) \geq \Omega(G)$, and so therefore it follows that $\max_{1 \leq k \leq \upsilon} \Omega(T_k) \geq \Omega(G)$.

IV. Atoll Decompositions and Hamiltonian Graphs.

A Hamiltonian graph is a graph which contains a spanning cycle. A graph G is called randomly Hamiltonian if G has the property that the following procedure (when applied to G) goes forward unblocked and terminates successfully (in the sense implied below): Begin at any point u of G and then proceed to any point adjacent to u. In general, on arriving at a point v, first select any point w which is both adjacent to v and has not previously been encountered, and then proceed to w. When, after a finite number of repetitions of this last step just described, no new points remain, then a line exists which joins the final point chosen and u. The procedure now terminates and every point of G has been encountered exactly once, i.e., a Hamiltonian cycle of G has been obtained (and thus G is Hamiltonian).

Proposition IV.1. (Chartrand and Kronk [12]). A graph G with $p \geq 3$ points is randomly Hamiltonian if and only if G is one of the graphs C_p, K_p, or $K_{n,n}$ with $p = 2n$ (i.e., or p is even and $G = K_{p/2,p/2}$).

Again, if $\pi_j = (u_1, u_2, \ldots, u_k, u_{k+1}, u_{k+2}, \ldots, u_{k+r}, \ldots, u_c)$ is any path of a graph G, where $1 \leq k \leq c$ and $0 \leq r \leq c-1$, then the subpath $(u_k, u_{k+1}, u_{k+2}, \ldots, u_{k+r})$ of π_j is called a segment of π_j of length r (to reiterate, throughout this work by a path is meant a path as is defined in [1]).

Proposition IV.2. (Dirac and Thomassen [13]). If G is a finite connected graph with $p \geq 3$ points, then the following statements are equivalent: (1) G is randomly Hamiltonian, (2) G is one of the graphs C_p, K_p, or $K_{n,n}$ with $p = 2n$, (3) Every path of G is contained in a cycle of G, (4) If the points of any path of G are deleted, then the remaining graph is connected, (5) G contains a Hamiltonian (spanning) path and every Hamiltonian path of G is contained in a Hamiltonian cycle of G (i.e., in other words, the two endpoints of every Hamiltonian path of G are adjacent).

It follows directly from the definition of an atoll decomposition that a graph G has no atoll decomposition if and only if $\forall$ island decompositions I_e of G, either it holds that $\exists$ two distinct islands π_x and π_y of I_e, s.t. an endpoint of π_x is adjacent to an endpoint of π_y, or it holds that $\exists$ an island π_z of I_e having length ≥ 2, s.t. the two endpoints of π_z are adjacent (i.e., either the first condition holds or the second condition holds or both conditions hold simultaneously). Again, a maximal island decomposition of a graph G is an island decomposition I_a of G, s.t. if every island of I_a is a segment of an island of an island decomposition I_b of G, then $I_b = I_a$. Therefore, since clearly every atoll decomposition is a maximal island decomposition, then it follows that $G \notin \mathcal{A}$ if and only if $\forall$ maximal island decompositions I_a of G, it holds that $\exists$ an island π_w of I_a having length ≥ 2, s.t. the two endpoints of π_w are adjacent. It is only a routine matter to verify that each of the graphs C_p, K_p, and $K_{n,n}$ with $p = 2n$ has no atoll decomposition. Furthermore, it is obvious that $\zeta(C_p) = \zeta(K_p) = \zeta(K_{n,n}) = 1$ with $p = 2n$. Hence, in view of Proposition IV.1, the next proposition now follows:

<u>Proposition IV.3.</u> If a graph G is randomly Hamiltonian, then G has no atoll decomposition and $\zeta(G) = 1$.

<u>Proposition IV.4.</u> If G is a graph, s.t. G has no atoll decomposition and $\zeta(G) = 1$, then G is Hamiltonian.

Proof: Since $\zeta(G) = 1$ by hypothesis, then it immediately follows that G contains a Hamiltonian path $\pi_h = (v_1, v_2, \ldots, v_p)$, i.e., the set $\{v_1, v_2, \ldots, v_p\}$ of points of π_h is precisely the set V of points of G with the points of V labeled $v_1, v_2, \ldots, v_p$ and with v_i adjacent to v_{i+1} for $i = 1,2,\ldots,p-1$, where $p = |V|$ (i.e., π_h orders V). Assume that v_1 is not adjacent to v_p . Since G has no atoll decomposition, then it follows that $p \geq 3$, and therefore π_h is an open path of G (since v_1 is not adjacent to v_p) that covers the points of G . But this means that π_h is an atoll decomposition of G , in contradiction to the hypothesis that G has no atoll decomposition. Hence the original assumption is false, and thus v_1 is adjacent to v_p . Therefore it follows that G contains a Hamiltonian cycle C_h which contains π_h , i.e., $C_h = [v_1, v_2, \ldots, v_p]$, where v_i is adjacent to v_{i+1} , for $i = 1,2,\ldots,p-1$, and v_1 is adjacent to v_p (i.e., C_h orders V).

The proof of the next proposition follows directly from the proof of Proposition IV.4:

Proposition IV.5. If G is a graph, s.t. G has no atoll decomposition and $\zeta(G) = 1$, then G contains a Hamiltonian path and every Hamiltonian path of G is contained in a Hamiltonian cycle of G .

If G is a (finite) graph, s.t. G has no atoll decomposition, then, again, it follows that $p \geq 3$; and if, in addition, it holds that $\zeta(G) = 1$, then obviously G is also connected. Hence the next proposition follows at once by Propositions IV.2 (1),(5), and IV.5:

Proposition IV.6. If G is a graph, s.t. G has no atoll decomposition and $\zeta(G) = 1$, then G is randomly Hamiltonian.

Therefore the next proposition follows by Propositions IV.3 and

IV.6, and yields a new characterization of randomly Hamiltonian graphs in terms of the concept of an atoll decomposition and in terms of the parameter ζ :

Proposition IV.7. A graph G is randomly Hamiltonian if and only if G has no atoll decomposition and $\zeta(G) = 1$.

Diagram 4.1 depicts the graph G_o which has no atoll decomposition, and for which it holds that $\zeta(G_o) = 2$; moreover, G_o is neither randomly Hamiltonian nor Hamiltonian (since $\zeta(G_o) = 2$).

A singular atoll of an atoll decomposition A of a graph $G \in \mathcal{A}$ is an atoll $\pi_s = (u_1, u_2, \ldots, u_d)$ of A having length ≥ 3 (i.e., $d \geq 4$) and containing a certain distinguished line $x = u_a u_{a+1}$, where $2 \leq a < a+1 \leq d-1$, s.t. both the lines $u_1 u_{a+1}$ and $u_a u_d$ belong to $G \setminus A$. A singular atoll decomposition of a graph G is an a.d. A_s of G , s.t. A_s consists of precisely one atoll π_s and this atoll π_s is a singular atoll. Therefore it follows that a singular atoll decomposition is a Hamiltonian path whose two endpoints are non-adjacent, i.e., a Hamiltonian path which is an open path (however, not every Hamiltonian path which is an open path is a singular atoll decomposition). The proof of the next proposition follows immediately from the definitions of a singular atoll decomposition and a singular atoll (i.e., since $A_s = \pi_s$ is a spanning open path of G , s.t. $u_1 u_{a+1}$ and $u_a u_d$ belong to G):

Proposition IV.8. If a graph G has a singular atoll decomposition, then G is Hamiltonian.

Let G be a Hamiltonian graph with point set $V = \{v_1, v_2, \ldots, v_p\}$ and let $C_h = [v_1, v_2, \ldots, v_p]$ be a Hamiltonian cycle of G , where $p \geq 3$, v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, p-1$, and v_1 is

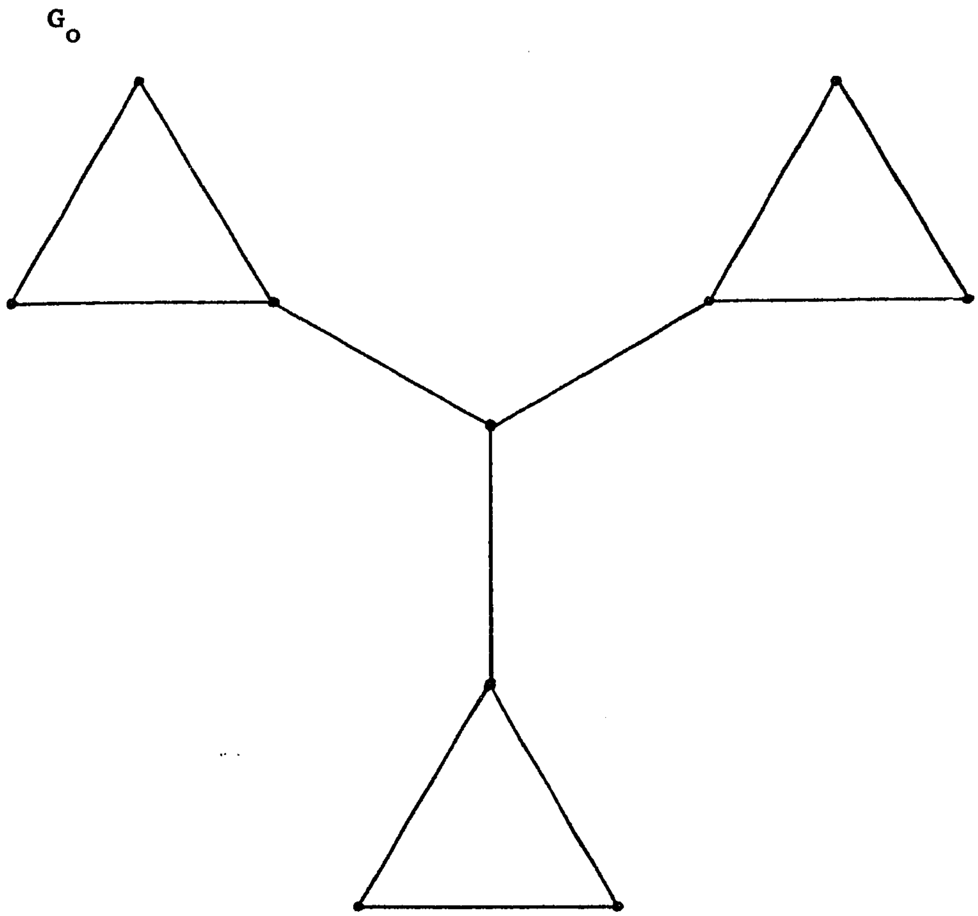


Diagram 4.1. The graph $G_o \notin \mathcal{A}$, for which $\zeta(G_o) = 2$.

adjacent to v_p . If $G \neq C_h$ and if y is a line of $G \setminus C_h$, then y joins two nonconsecutive points of C_h , and y is called a diagonal of G with respect to C_h . The proof of the next proposition is essentially due to Chartrand and Kronk (cf. [12]):

Proposition IV.9. (Chartrand and Kronk [12]). If G is a Hamiltonian graph with $p \geq 3$ points, then G is one of the graphs C_p, K_p, or $K_{n,n}$ with $p = 2n$ if and only if for every diagonal $v_j v_k$ of G with respect to a Hamiltonian cycle $C_h = [v_1, v_2, \ldots, v_p]$ of G , where $1 \leq j < k \leq p$, G also contains both the diagonal $v_{j+1} v_{k+1}$ and the diagonal $v_{j-1} v_{k-1}$, where the numbers $j+1$, $k+1$, $j-1$, and $k-1$ are expressed modulo p .

The next proposition follows directly by Propositions IV.1 and IV.9:

Proposition IV.10. If G is a Hamiltonian graph with $p \geq 3$ points, then G is not randomly Hamiltonian only if $\exists$ a diagonal $v_j v_k$ of G with respect to a Hamiltonian cycle $C_h = [v_1, v_2, \ldots, v_p]$ of G , s.t. either G does not contain the line $v_{j+1} v_{k+1}$, or G does not contain the line $v_{j-1} v_{k-1}$, or both the lines $v_{j+1} v_{k+1}$ and $v_{j-1} v_{k-1}$ do not belong to G , where $1 \leq j < k \leq p$ and where $j+1, k+1, j-1$, and $k-1$ are expressed modulo p .

The next proposition follows at once by Proposition IV.10 and the definitions of a singular atoll and a singular atoll decomposition:

Proposition IV.11. If G is a Hamiltonian graph, then G is not randomly Hamiltonian only if G has a singular atoll decomposition.

And the next proposition follows at once by Proposition IV.2 (1), (5), and the definitions of a singular atoll and a singular atoll decomposition:

Proposition IV.12. If G is a randomly Hamiltonian graph, then G has no singular atoll decomposition.

Therefore the next proposition follows by Propositions IV.7, IV.8, IV.11, and IV.12 , and yields a characterization of Hamiltonian graphs in terms of the concept of an atoll decomposition and in terms of the parameter ζ :

Proposition IV.13. A graph G is Hamiltonian if and only if exactly one of the following two conditions holds:

(1) G has no atoll decomposition and $\zeta(G) = 1$,

(2) G has a singular atoll decomposition.

Proposition IV.14. If H is a Hamiltonian graph, s.t. $H \in \mathcal{A}$, then $\zeta(H) = \beta(H)$.

Proof: Since $H \in \mathcal{A}$, then $\zeta(H) \leq \beta(H)$ by Proposition III.14. On the other hand, since H is a Hamiltonian graph, s.t. $H \in \mathcal{A}$, then H has a singular atoll decomposition A_s by Proposition IV.13. Hence, since $\zeta(H) = 1$ and A_s is a Hamiltonian path of H , then it follows that H has an atoll decomposition (i.e., A_s) which contains exactly $\zeta(H)$ atolls. Thus it follows that $\beta(H) \leq \zeta(H)$, and so therefore $\zeta(H) = \beta(H)$.

Proposition IV.15. (Gaddum and Nordhaus [14]). If G is a graph with p points, then $\chi(G) + \chi(\bar{G}) \leq p+1$.

Proposition IV.16. If H is a Hamiltonian graph with $p \geq 3$ points and which is neither C_p nor K_p nor $K_{n,n}$ with $p = 2n$, then $H \in \mathcal{A}$, and it follows that $\chi(H) \leq p - \tau(H)+1$.

Proof: Since H is a Hamiltonian graph which is neither C_p nor K_p nor $K_{n,n}$ with $p = 2n$, then by Propositions IV.1 and IV.7, it follows that $H \in \mathcal{A}$, and thus it holds that $\tau(H) \leq \chi(\bar{H})$ by Proposition II.7 , i.e., $-\chi(\bar{H}) \leq -\tau(H)$. Therefore, since $\chi(H) \leq p - \chi(\bar{H}) + 1$ by Proposition IV.15, then it follows that $\chi(H) \leq p - \tau(H) + 1$.

A graph is said to be planar if it can be drawn in the plane so that no two of its lines intersect, i.e., so that its lines intersect only at their points, if they intersect at all.

Proposition IV.17. (Whitney [15]). $\forall$ planar graphs G_π, it holds that $\chi(G_\pi) \leq 4$ if and only if $\forall$ Hamiltonian planar graphs H_π, it holds that $\chi(H_\pi) \leq 4$.

Clearly, if $p \geq 3$, then $\chi(C_p) = 2$ if p is even, $\chi(C_p) = 3$ if p is odd, $\chi(K_p) = p \leq 4$ if K_p is planar (i.e., if $p \leq 4$), and $\chi(K_{p/2,p/2}) = 2$ if $K_{p/2,p/2}$ is planar (i.e., if $p \leq 4$). Therefore, in view of these last observations, Proposition IV.16 yields Proposition IV.18, which is of interest, in view of Proposition IV.17:

Proposition IV.18. If H_π is a Hamiltonian planar graph with $p \geq 3$ points and which is neither C_p nor K_4, then $H_\pi \in \mathcal{A}$, and it follows that $\chi(H_\pi) \leq p - \tau(H_\pi) + 1$.

In view of Propositions IV.1, IV.2, and IV.7, it is now clear that Chartrand and Kronk [12] and Dirac and Thomassen [13] have actually characterized those graphs G, s.t. $G \notin \mathcal{A}$ and $\zeta(G) = 1$. It appears that the next problem is to find a characterization of those graphs G, s.t. $G \notin \mathcal{A}$ and $\zeta(G) = 2$. Accordingly, I believe that the following conjecture yields in fact a characterization of those graphs which do not have an atoll decomposition and which can be covered by two islands, but which cannot be covered by one island (it is simple to prove the sufficiency of the condition given in the statement of the conjecture); let $c(F)$ denote the number of components of the graph F.

<u>Conjecture</u>. If G is a graph, then $G \notin \mathcal{A}$ and $\zeta(G) = 2$ if and only if either

(1) $c(G) = 2$ and one component of G contains a Hamiltonian path and the other component of G is randomly Hamiltonian, or else

(2) G is connected and G contains a cutpoint v_ρ, s.t. $c(G-v_\rho) = 3$ and each component of $G-v_\rho$ is randomly Hamiltonian.

V. Atoll Decompositions and Minimum Dominating Sets.

A dominating set of a graph G is a subset D of the set V of points of G , s.t. every point of G that is not in D is adjacent to at least one point of D . A stable dominating set of G is a dominating set of G that is also a stable set of G, and a minimal dominating set of G is a dominating set of G that does not properly contain another dominating set of G . Let $\delta(G)$ denote the smallest number of points in a minimal dominating set of G . The value $\delta(G)$ is called the domination number of G , and a dominating set of G that contains exactly $\delta(G)$ points is called a minimum dominating set of G . A maximal stable set of G is a stable set of G that is not properly contained in any other stable set of G .

Proposition V.1. (Berge [16]). (1) D is a stable dominating set of a graph G if and only if D is a maximal stable set of G , and (2) if D is a stable dominating set of G , then D is a minimal dominating set of G .

Proposition V.2. (Vizing [17]). If G is a graph with p points and q lines, then $\delta(G) \leq p+1 - \sqrt{1+2q}$.

If a graph $G \in \mathcal{A}$ contains a stable set L , s.t. L is both a maximal stable set of G and a terminal stable set of G , then G is said to have a key decomposition, i.e., explicitly, a key decomposition (as in Florida Keys) of a graph G is an atoll decomposition KA of G , s.t. KA conforms to a maximal stable set of G . Let $\mathcal{K}\mathcal{A}$ denote the class of all those graphs which have a key decomposition. Thus if G is a graph, s.t. $G \in \mathcal{K}\mathcal{A}$, then KA_G is nonempty, where $KA_G = \{KA_1, KA_2, \ldots, KA_{\psi_k}\}$ is the set of all key decompositions of G . Let

$\mu_k(G) = \min_{1 \leq e \leq \psi_k} \rho(KA_e)$. A ternary atoll decomposition of a graph G is an atoll decomposition TA of G , s.t. every atoll of TA has length ≤ 3 , and $\mathcal{T}\mathcal{A}$ denotes the class of all those graphs which have a ternary atoll decomposition. An elementary contraction of a graph G is the identification of two adjacent points u_1 and u_2 of G , i.e., both u_1 and u_2 are removed from G and they are replaced in G by a new point u_3 , s.t. u_3 is adjacent to all those points to which at least one of u_1 and u_2 was adjacent. A contraction is a composition of finitely many elementary contractions, and a graph $G^{\bullet}$ is said to be a contraction of a graph G if $G^{\bullet}$ can be produced from G by means of a contraction of G . Let $\{r\}$ denote the smallest positive nonzero integer not less than the real number r , so that $\{0\} = 1$, and let $\{r\}'$ denote the smallest integer not less than r , so that $\{0\}' = 0$. And if A is an atoll decomposition of a graph G , then let $\eta(A)$ denote the length of the longest atoll of A (so that if TA is a ternary atoll decomposition of a graph $G \in \mathcal{T}\mathcal{A}$, then $\eta(TA) \leq 3$).

<u>Proposition V.3.</u> If G is a graph, s.t. $G \in \mathcal{A}$, then

(1) if A_m is an atoll decomposition of G , then

$$\delta(G) \leq \sum_{\ell=0}^{\eta(A_m)} \left[\left\{ \frac{\ell+1}{3} \right\} \lambda_m^{\ell} \right] ,$$

where $\forall \ell,\ 0 \leq \ell \leq \eta(A_m)$, each λ_m^{ℓ} denotes the number of atolls of A_m having length ℓ ,

(2) if $G \in \mathcal{T}\mathcal{A}$, then $\delta(G) \leq \mu_k(G)$, and

(3) if $G \notin \mathcal{T}\mathcal{A}$ and if A_m is an atoll decomposition of G , then $\exists$ a contraction $G^{\bullet}$ of G , s.t. $G^{\bullet} \in \mathcal{T}\mathcal{A}$ and

$$2\delta(G') - \sum_{\ell=0}^{\eta(A_m)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda_m^{\ell} \right] \leq \delta(G) ,$$

where $\forall \ell,\ 0 \leq \ell \leq \eta(A_m)$, each λ_m^{ℓ} denotes the number of atolls of A_m having length ℓ .

Proof: In order to prove statement (1), assume that $G \in \mathcal{A}$, let A_m be an atoll decomposition of G , and let $\Sigma(A_m) = \left[\pi_{m_1}^{\ell_1}, \pi_{m_2}^{\ell_2}, \ldots, \pi_{m_{\varkappa_m}}^{\ell_{\varkappa_m}} \right]$ be the collection of all the paths (i.e., atolls) of A_m , where $\forall j$, $1 \leq j \leq \varkappa_m$, the length of $\pi_{m_j}^{\ell_j}$ is equal to ℓ_j . Thus $\Sigma(A_m)$ constitutes a spanning forest of G (since A_m is an atoll decomposition of G) , and so it follows that every point of G is a point of a path of A_m . Further, if (as was specified above) $\eta(A_m)$ denotes the length of the longest path of A_m , and if π_m^{ℓ} is any path of A_m of length ℓ , where $0 \leq \ell \leq \eta(A_m)$, then it is a simple matter to verify the following sequence of values of $\delta(\pi_m^{\ell})$:

$$\delta(\pi_m^0) = 1,\ \delta(\pi_m^1) = 1,\ \delta(\pi_m^2) = 1,\ \delta(\pi_m^3) = 2 ,$$

$$\delta(\pi_m^4) = 2,\ \delta(\pi_m^5) = 2,\ \delta(\pi_m^6) = 3,\ \delta(\pi_m^7) = 3 ,$$

$$\delta(\pi_m^8) = 3,\ \delta(\pi_m^9) = 4,\ \ldots,\ \delta(\pi_m^{\eta(A_m)}) = \left\{ \frac{\eta(A_m) + 1}{3} \right\} .$$

Thus if now exactly one minimum dominating set of each path of A_m is selected from each path of A_m , and if the union of all these minimum dominating sets is formed, then plainly this union constitutes a dominating set D_m of G (since every point of G is a point of a path of A_m) . Moreover, in view of the above sequence of values of $\delta(\pi_m^{\ell})$, it follows that if $\forall \ell$, $0 \leq \ell \leq \eta(A_m)$, the symbol λ_m^{ℓ} denotes the number of paths of A_m having length ℓ , then $|D_m| = 1\lambda_m^0 + 1\lambda_m^1 + 1\lambda_m^2 + 2\lambda_m^3 + 2\lambda_m^4 +$

$2\lambda_m^5 + 3\lambda_m^6 + 3\lambda_m^7 + 3\lambda_m^8 + 4\lambda_m^9 + \ldots + \left\{ \frac{\eta(A_m)+1}{3} \right\} \lambda_m^{\eta(A_m)}$. (since A_m is a collection of point disjoint open paths), i.e., $|D_m| = \sum_{\ell=0}^{\eta(A_m)} \left[\left\{ \frac{\ell+1}{3} \right\} \lambda_m^{\ell} \right]$. Therefore it follows that $\delta(G) \leq \sum_{\ell=0}^{\eta(A_m)} \left[\left\{ \frac{\ell+1}{3} \right\} \lambda_m^{\ell} \right]$ (since every dominating set contains a minimal dominating set), and so an upper bound for $\delta(G)$ is obtained and statement (1) holds.

In order to prove statement (2), assume that $G \in \mathcal{TA}$, and let TA be a ternary atoll decomposition of G . Thus every atoll of TA has length ≤ 3 (by the definition of a ternary atoll decomposition) and TA spans G , i.e., TA is a collection of point disjoint open paths (i.e., again, atolls) that cover the points of G , s.t. no endpoint of any path of TA is adjacent to an endpoint of any other path of TA (by the definition of an atoll decomposition). Hence it follows that if $\Pi'(TA)$ denotes the set of all the endpoints of all the paths which together comprise the collection of paths of TA , less exactly one endpoint from each path of TA consisting of a path K_2 , and that if v is a point of V (again, where V is the set of points of G), s.t. $v \in V \setminus \Pi'(TA)$, then v is a point of a path of TA (since TA spans G) and v is adjacent to at least one point of $\Pi'(TA)$ (since every path of TA has length ≤ 3), i.e., $\Pi'(TA)$ is a dominating set of G . But since TA is an atoll decomposition of G that conforms to $\Pi'(TA)$ in G , then $\Pi'(TA)$ is a (terminal) stable set of G . Thus $\Pi'(TA)$ is a stable dominating set of G , and so it follows by Proposition V.1 (1) that $\Pi'(TA)$ is a maximal stable set of G . Therefore, since TA conforms to $\Pi'(TA)$ in G , then TA is a key decomposition of G (and thus it has been shown that if a graph has a ternary atoll decomposition, then this graph also has a key decomposition, i.e., $\mathcal{TA} \subseteq \mathcal{KA} \subseteq \mathcal{A}$), and so

KA_G is nonempty, where, again, $KA_G = \{KA_1, KA_2, \ldots, KA_{\psi_k}\}$ is the set of all key decompositions of G. Hence, if $\mu_k(G) = \min_{1 \leq e \leq \psi_k} \rho(KA_e)$, then $\mu_k(G)$ is the cardinality of a certain maximal stable set of G, and thus it follows by Proposition V.1. (1). (2) that $\mu_k(G)$ is the cardinality of a certain minimal dominating set of G. Therefore it follows that $\delta(G) \leq \mu_k(G)$, and so statement (2) holds.

In order to prove statement (3), assume that $G \in (\mathcal{A} \setminus \mathcal{JA})$, and let A_m be an atoll decomposition of G. Since $G \notin \mathcal{JA}$, then it follows that A_m contains at least one path (i.e., again, atoll) of length ≥ 4 so let π_f be a path of A_m of length ≥ 4, and let w_1 and w_2 be two adjacent interior points of π_f (i.e., neither w_1 nor w_2 are endpoints of π_f). Identify w_1 and w_2 into a new point w_3, so that a new path $\pi^{\bullet}_{f3}$ is produced which is a contraction of π_f, and also so that, simultaneously, a new graph $G^{\bullet}_{\iota}$ is obtained which is a contraction of G (and which is called an intermediate contraction of G with respect to A_m). Since w_1 and w_2 are adjacent interior points of π_f and since π_f has length ≥ 4, then it follows directly from the definition of an atoll decomposition that $G^{\bullet}_{\iota} \in \mathcal{A}$ and that $((\Sigma(A_m) \setminus \pi_f) \cup \pi^{\bullet}_{f3})$ constitutes the collection of paths of an atoll decomposition $A^{\bullet}_{m\iota}$ of $G^{\bullet}_{\iota}$ (i.e., $\Sigma(A^{\bullet}_{m\iota}) = ((\Sigma(A_m) \setminus \pi_f) \cup \pi^{\bullet}_{f3})$); in addition, it is clear that $\rho(A^{\bullet}_{m\iota}) = \rho(A_m)$. If $\pi^{\bullet}_{f3}$ has length ≥ 4, then (just as with π_f) contract two adjacent interior points of $\pi^{\bullet}_{f3}$ so that a new path $\pi^{\bullet}_{f4}$ is produced which is a contraction of π_f. And if $\pi^{\bullet}_{f3}$ has length 3, then do not contract $\pi^{\bullet}_{f3}$. Specifically, apply the following path contraction procedure to π_f: Continue contracting adjacent interior points of successive paths $\pi^{\bullet}_{fi}$ (where $\forall i,\ 4 \leq i \leq b$, it holds that $\pi^{\bullet}_{fi}$ is a contraction of $\pi^{\bullet}_{f(i-1)}$, and $\pi^{\bullet}_{f3}$ is a contraction of π_f, so that $\pi^{\bullet}_{fi}$ is a contraction of π_f) until finally a path $\pi^{\bullet}_{fb}$ is obtained, s.t. $\pi^{\bullet}_{fb}$ is a contraction of π_f and $\pi^{\bullet}_{fb}$ has length 3. Do not contract $\pi^{\bullet}_{fb}$ (i.e., do not contract π_f any further).

Moreover and generally, apply the path contraction procedure just described to every path of A_m having length ≥ 4 (i.e., in other words, contract the interior of every path of A_m having length ≥ 4 so as to obtain a path of length 3 whose two endpoints are the same two endpoints of the original uncontracted path, and do not contract at all any path having length ≤ 3; it results that every path of A_m having length ≥ 4 is contracted to a path of length 3, and every path of A_m having length ≤ 3 remains uncontracted). It follows that the collection of paths which are ultimately obtained when the path contraction procedure (applied to $\Sigma(A_m)$) terminates constitutes the collection of paths of a ternary atoll decomposition TA_Ψ of a graph $G^{\bullet}$, s.t. TA_Ψ is a contraction of A_m, and $G^{\bullet}$ is a contraction of G (in fact, $G^{\bullet}$ is the last intermediate contraction in the sequence of intermediate contractions which is produced as a consequence of the path contraction procedure applied to $\Sigma(A_m)$); in addition, it follows that $\rho(TA_\Psi) = \rho(A_m)$ (since $\rho(K_1) = \rho(K_2) = 1$, ρ (path of length ≥ 2) $= 2$, and no path of length ≤ 3 is ever contracted). But as was shown above in the proof of statement (2), it holds that TA_Ψ conforms to a stable set L_t of $G^{\bullet}$ which is a minimal dominating set of $G^{\bullet}$. Hence it follows that $\delta(G^{\bullet}) \leq |L_t| = \rho(TA_\Psi) = \rho(A_m)$. However, $\rho(A_m) = \lambda^{o}_{m} + \lambda^{1}_{m} + 2\lambda^{\geq 2}_{m} = \lambda^{o}_{m} + \lambda^{1}_{m} + 2\lambda^{2}_{m} + 2\lambda^{3}_{m} + 2\lambda^{4}_{m} + \ldots + 2\lambda^{\eta(A_m)}_{m} =$

$$\sum_{\ell=0}^{\eta(A_m)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda^{\ell}_{m} \right],$$

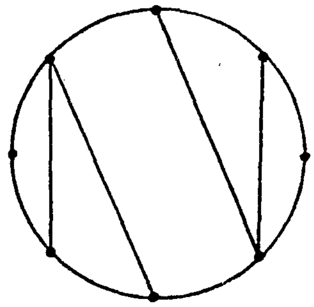
and thus it holds that

$$\delta(G') \leq \sum_{\ell=0}^{\eta(A_m)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda_m^{\ell} \right].$$

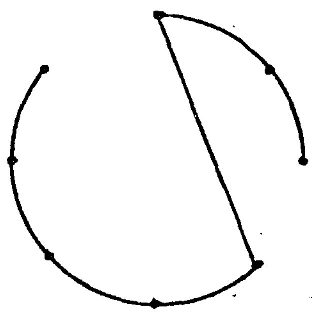
It is a straightforward exercise to prove that if F' is any contraction of a graph F , then $\delta(F') \leq \delta(F)$, and so $\delta(G') \leq \delta(G)$. Therefore it follows that $2\delta(G') - \sum_{\ell=0}^{\eta(A_m)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda_m^{\ell} \right] \leq \delta(G)$, and so a lower bound for $\delta(G)$ is obtained and statement (3) holds. Q.E.D.

Diagram 5.1 depicts the Hamiltonian graph H_0 (where $H_0 \in \mathcal{A}$ by Propositions IV.1. and IV.7.; in fact, it is clear that $H_0 \in \mathcal{JA}$), this graph being a case in which Proposition V.3. yields a tighter bound than the bound provided by Proposition V.2.

H_o



KA_o



Proposition V.2.: $\delta(H_o) = 2 \leq 4 = p_o + 1 - \sqrt{1+2q_o}$.

Proposition V.3.: $H_o \in \mathcal{JA}$; $\rho(KA_o) = \mu_k(H_o)$, and

$$\delta(H_o) = 2 = \mu_k(H_o) .$$

Diagram 5.1. A case in which Proposition V.3. yields a tighter bound than does Proposition V.2.

VI. Atoll Decompositions and Maximum Matchings.

Once again, a pair of lines is said to be nonadjacent if they do not share a common point, an independent set of lines or a matching is a set of lines that are pairwise nonadjacent, and a maximum matching of a graph G contains the largest number of lines in a matching of G. Let $\gamma(G)$ denote the number of lines in a maximum matching of the graph G. A matching N of a graph G saturates a point u of G, and u is said to be N-saturated, if u is a point of some line of N; otherwise, if u is a point of no line of N, then u is N-unsaturated. If every point of G is N-saturated, the matching N is said to be a perfect matching; clearly, a perfect matching is a maximum matching. Let $\{r\}^*$ denote the largest integer not greater than the real number r.

Proposition VI.1. (Erdös and Posá [18]). If G is a graph with p points and minumum degree d_μ, then $\gamma(G) \geq \min\left(d_\mu, \left\{\frac{p}{2}\right\}^*\right)$.

An elementary expansion of a graph G is the removal of a point v_1 of G and its replacement in G by two new points v_2 and v_3, s.t. v_2 is adjacent to v_3 and, moreover, each of v_2 and v_3 is adjacent to all those points to which v_1 was adjacent. An expansion is a composition of finitely many elementary expansions, and a graph G^- is said to be an expansion of a graph G if G^- can be produced from G by means of an expansion of G. Thus an elementary expansion of a point v_1 of a graph G out into two adjacent points v_2 and v_3, followed by an elementary contraction of v_2 and v_3, yields the original graph G. An odd atoll decomposition of a graph G is an atoll decomposition OA of G, s.t. every atoll of OA has odd length (an atoll of length 0 is said to have even length), and $\mathcal{O}\mathcal{A}$ denotes the class of all those graphs which have an odd atoll decomposition. Again, let $\{r\}$ denote the smallest positive nonzero integer not less than the real number r, let $\{r\}'$ denote the smallest integer not less than r, and if A is an atoll decomposition of a graph G, then let $\eta(A)$ denote the length of the longest atoll of A.

<u>Proposition VI.2.</u> If G is a graph, s.t. $G \in \mathcal{A}$, then

(1) if A_n is an atoll decomposition of G, then

$$\sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{\ell}{2} \right\}' \lambda_n^{\ell} \right] \le \gamma(G)$$, where $\forall \ell$, $0 \le \ell \le \eta(A_n)$, each λ_n^{ℓ} denotes the number of atolls of A_n having length ℓ,

(2) if $G \in \mathcal{O}\mathcal{A}$ and if OA_d is an odd atoll decomposition of G, then $\gamma(G) = \sum_{\ell=0}^{\eta(OA_d)} \left[\left\{ \frac{\ell}{2} \right\} \lambda_d^{\ell} \right]$, where $\forall \ell$, $0 \le \ell \le \eta(OA_d)$, each λ_d^{ℓ} denotes the number of atolls of OA_d having length ℓ, and

(3) if $G \notin \mathcal{O}\mathcal{A}$ and if A_n is an atoll decomposition of G, then $\exists$ an expansion G^- of G, s.t. $G^- \in \mathcal{O}\mathcal{A}$ and $\gamma(G) \le 2\gamma(G^-) -$ $\sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda_n^{\ell} \right]$, where $\forall \ell$, $0 \le \ell \le \eta(A_n)$, each λ_n^{ℓ} denotes the number of atolls of A_n having length ℓ.

Proof: In order to prove statement (1), assume that $G \in \mathcal{A}$, let A_n be an atoll decomposition of G, and let $\Sigma(A_n) = \left[\pi_{n_1}^{\ell_1}, \pi_{n_2}^{\ell_2}, \ldots, \pi_{n_{\varkappa_n}}^{\ell_{\varkappa_n}} \right]$ be the collection of all the paths (i.e., again, atolls) of A_n, where $\forall j$, $1 \le j \le \varkappa_n$, the length of $\pi_{n_j}^{\ell_j}$ is equal to ℓ_j. Thus $\Sigma(A_n)$ constitutes a spanning forest of G which consists of point disjoint open paths (since A_n is an atoll decomposition of G), and so it follows that every point of G is a point of exactly one path of A_n. Further, if $\eta(A_n)$ denotes the length of the longest path of A_n, and if π_n^{ℓ} is any path of A_n of length ℓ, where $0 \leq \ell \leq \eta(A_n)$, then it is a simple matter to verify the following sequence of values of $\gamma(\pi_n^{\ell})$:

$$\gamma(\pi_n^{o}) = 0 \ , \ \gamma(\pi_n^{1}) = 1 \ , \ \gamma(\pi_n^{2}) = 1 \ ,$$

$$\gamma(\pi_n^{3}) = 2 \ , \ \gamma(\pi_n^{4}) = 2 \ , \ \gamma(\pi_n^{5}) = 3 \ ,$$

$$\gamma(\pi_n^{6}) = 3 \ , \ \gamma(\pi_n^{7}) = 4 \ , \ \ldots \ , \ \gamma(\pi_n^{\eta(A_n)}) = \left\{ \frac{\eta(A_n)}{2} \right\}' .$$

Thus, if now exactly one maximum matching of each path of A_n is selected from each path of A_n, and if the union of all these maximum matchings is formed, then plainly this union constitutes a matching N_n of G (since A_n is a collection of point disjoint open paths that spans G). Moreover, in view of the above sequence of values of $\gamma(\pi_n^{\ell})$, it follows that if $\forall \ell$, $0 \leq \ell \leq \eta(A_n)$, the symbol λ_n^{ℓ} denotes the number of paths of A_n having length ℓ, then $|N_n| = 0\lambda_n^{o} + 1\lambda_n^{1} + 1\lambda_n^{2} + 2\lambda_n^{3} + 2\lambda_n^{4} + 3\lambda_n^{5} + 3\lambda_n^{6} + 4\lambda_n^{7} + \ldots + \left\{ \frac{\eta(A_n)}{2} \right\}' \lambda_n^{\eta(A_n)}$ (since A_n is a collection of point disjoint open paths), i.e., $|N_n| = \sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{\ell}{2} \right\}' \lambda_n^{\ell} \right]$.

Therefore it follows that $\sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{\ell}{2} \right\}' \lambda_n^{\ell} \right] \leq \gamma(G)$ (since a maximum matching contains the largest number of lines in a matching), and so a lower bound for $\gamma(G)$ is obtained and statement (1) holds.

In order to prove statement (2), assume that $G \in \mathcal{OA}$, and let OA_d be an odd atoll decomposition of G. Thus every atoll of OA_d has odd length (by the definition of an odd atoll decomposition) and OA_d spans G, i.e., OA_d is a collection of point disjoint open paths that cover the points of G, s.t. no endpoint of any path of OA_d is adjacent to an endpoint of any other path of OA_d (by the definition of an atoll decomposition). Hence, if $\eta(OA_d)$ denotes the length of the longest path of OA_d (as was specified above), and if π_d^{ℓ} is any path of OA_d of length $\ell = 2g+1$, where $g \geq 0$ (since every path of OA_d has odd length) and $0 \leq \ell \leq \eta(OA_d)$, then it is obvious that π_d^{ℓ} contains exactly one perfect matching and it is not difficult to verify that $\gamma(\pi_d^{\ell}) = \{\frac{\ell}{2}\}$. Thus, if now the unique perfect matching of each path of OA_d is taken from each path of OA_d, and if the union of all these perfect matchings is formed, then clearly this union constitutes a perfect matching M_d of G (since every point of G is a point of exactly one path of OA_d). Moreover, in view of $\gamma(\pi_d^{\ell}) = \{\frac{\ell}{2}\}$, it follows that if $\forall \ell$, $0 \leq \ell \leq \eta(OA_d)$, the symbol λ_d^{ℓ} denotes the number of paths of OA_d having length ℓ, then $|M_d| = \sum_{\ell=0}^{\eta(OA_d)} \left[\left\{\frac{\ell}{2}\right\} \lambda_d^{\ell}\right]$ (since OA_d is a collection of point disjoint open paths). Therefore it follows that $\gamma(G) = \sum_{\ell=0}^{\eta(OA_d)} \left[\left\{\frac{\ell}{2}\right\} \lambda_d^{\ell}\right]$ (since a perfect matching is a maximum matching), and so statement (2) holds.

In order to prove statement (3), assume that $G \in (\mathcal{A} \setminus \mathcal{OA})$, and let A_n be an atoll decomposition of G. Since $G \notin \mathcal{OA}$, then it follows that A_n contains at least one path of even length. Select precisely one endpoint w_{Ξ} from each path $\pi_{n\Xi}^{\ell}$ of A_n of even length, and then carry out a series of elementary expansions, s.t. each elementary expansion in this series is executed with respect to exactly one such w_{Ξ} and s.t. every w_{Ξ} is accounted for in this series (i.e., every w_{Ξ} is expanded). It follows that the collection of paths which are obtained as a result of this series of elementary expansions constitutes the collection of paths of an odd atoll decomposition OA_Y of a graph G^-, s.t. OA_Y is an expansion of A_n, and G^- is an expansion of G ; in addition, it follows that $\rho(OA_Y) = \rho(A_n)$ (since $\rho(K_1) = \rho(K_2) = 1$, ρ (path of length ≥ 2) $= 2$, and no path of A_n of length 1 is expanded to a path (of OA_Y) of length 2 when forming OA_Y).

As was indicated above, it is obvious that an odd atoll decomposition yields a perfect matching, and so therefore it directly follows that $\gamma(G^-) \geq \rho(OA_Y)$. Hence it follows that $\gamma(G^-) \geq \rho(OA_Y) = \rho(A_n) = \lambda_n^o + \lambda_n^1 + 2\lambda_n^{\geq 2} = \lambda_n^o + \lambda_n^1 + 2\lambda_n^2 + 2\lambda_n^3 + 2\lambda_n^4 + \dots + 2\lambda_n^{\eta(A_n)} =$

$\sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda_n^{\ell}\right]$, and thus it holds that $\gamma(G^-) \geq \sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda_n^{\ell}\right]$.

It follows immediately by the definition of a matching and by the definition of an expansion that every matching of a graph F is also a matching of any expansion F^- of F , and that thus every maximum matching of F is also a matching of F^- . Hence $\gamma(G) \leq \gamma(G^-)$. Therefore it follows that $\gamma(G) \leq 2\gamma(G^-) - \sum_{\ell=0}^{\eta(A_n)} \left[\left\{ \frac{2\ell}{\ell+1} \right\} \lambda_n^{\ell}\right]$, and so an upper bound for $\gamma(G)$ is obtained and statement (3) holds. **Q.E.D.**

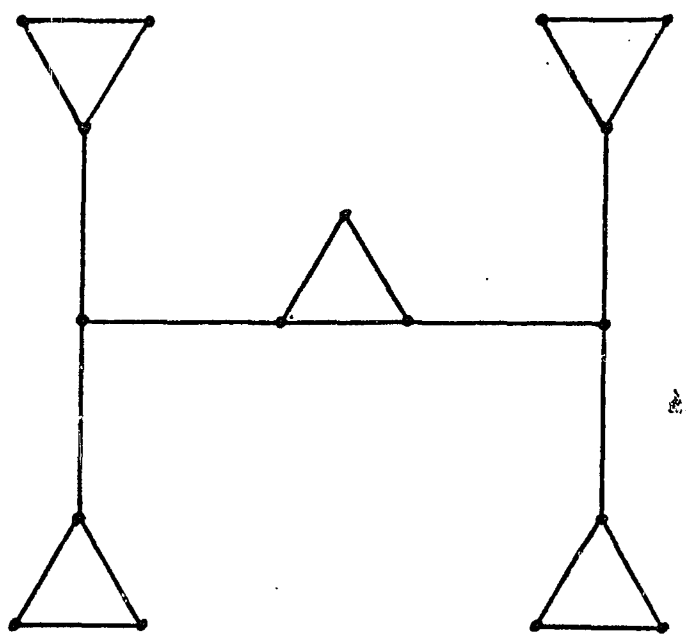
Diagram 6.1 depicts the tree T_1 (where $T_1 \in \mathcal{A}$ by Proposition III.1), this graph being a case in which Proposition VI.2 yields a tighter bound than the bound provided by Proposition VI.1.

The next consideration ends this work on atoll decompositions. A large number of examples leads me to believe that the following conjecture yields in fact a characterization of those graphs which do not have an atoll decomposition (and so therefore yields a characterization of those graphs which do have an atoll decomposition; it is straightforward to prove the sufficiency of the condition given in the statement of the conjecture); let $c(F)$ denote the number of components of the graph F.

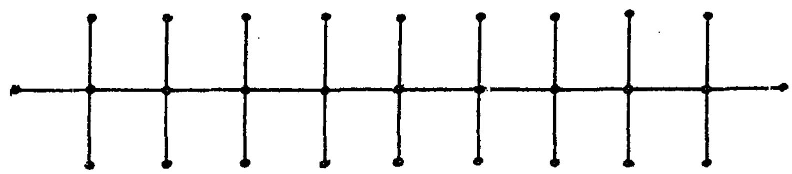
<u>Conjecture</u>. If G is a connected graph, then $G \notin \mathcal{A}$ if and only if either (1) G is randomly Hamiltonian, or else (2) G contains a cutpoint u_ρ, s.t. $c(G-u_\rho) \geq 3$ and $G - u_\rho$ contains three components, each of which is randomly Hamiltonian.

Note: The necessity of the conjecture's condition has since been proven false by the author (though the sufficiency of the condition is still true), e.g., the graph $G_1 \notin \mathcal{A}$:

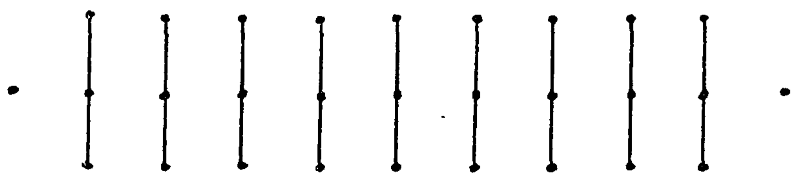
G_1



T_1



A_1



Proposition VI.1.: $\gamma(T_1) = 9 \geq 1 = \min\ (d_{\mu_1}, \left\{\frac{p_1}{2}\right\}^*)$.

Proposition VI.2.: $T_1 \in \mathcal{A}$; if for $\ell = 0,1,2 = \eta(A_1)$, the symbol λ_1^{ℓ} denotes the number of atolls of A_1 having length ℓ , then

$$\sum_{\ell=0}^{\eta(A_1)} \left[\left\{\frac{\ell}{2}\right\}' \lambda_1^{\ell}\right] = 9 = \gamma(T_1) .$$

Diagram 6.1. A case in which Proposition VI.2. yields a tighter bound than does Proposition VI.1.

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