The Differentiability of Renormalized Triple Intersection Local Times

by

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Abstract

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The evolution of the theory of triple intersection times over the past, approximately, two decades has centered primarily on two dimensional Brownian Motion and planar symmetric stable processes. The one dimensional cases have gone largely unstudied. In this thesis, we examine the differentiability of renormalized triple intersection local times for the two aforementioned Markov processes in $\mathbf{R}^1$. In more detail, we prove that the single partial derivative with respect to each spatial variable exists and show that each partial derivative is, in fact, jointly continuous in both space and time variables. During the course of our analysis, we discover that these results hold for the class of symmetric stable process for which $3/2 < \beta < 2$.

Acknowledgments

I begin by dedicating this work and my degree to the memory of my late father, Devinder Singh Dhamoon. The unconditional love and respect he bestowed to me will live forever in my heart.

This achievement would not have been possible without the support of my mother, Minal, sister, Amrita, and wife, Manmeet. I express my deepest gratitude to each of you for everything you did for me during the time I pursued my doctorate.

I would like to acknowledge some feedback I received from Jay Rosen on some preliminary drafts and highlight the contributions of Michael Marcus and Elena Kosygina to the probability research group.

In the course of my academic life, I have encountered several teachers who have had a great impact on my education and inspired me to pursue this degree. I would like to acknowledge and, sincerely, thank Sharon Crumpton, Greg Faulk, and Ed Lewis, from Belmont University, and Leslie Sibner, from Polytechnic University, for their contributions.

In the course of reflecting on my time as a student at the Graduate Center, I have arrived at two firm conclusions. First, I am extremely proud to have been associated with the City University of New York. The exposure to the values and ideology of CUNY and the GC has made a profound and permanent impression on me. Without any doubt, I view the world differently today as opposed to my outlook several years ago when I started at CUNY.

Second, I know that I will remember my time at the GC as some of the happiest times of my life and the people I interacted with on a daily basis are the direct reason I feel this way. I am forever indebted to Rob Landsman for all of his support. I have yet to meet anyone at the Graduate Center who does so much for students. I will never forget a dinner I had with Larry Santoro in the spring of 2008 where we discussed our families and exchanged many stories about our lives. You are a true friend, Larry. To my thesis brother from another mother, Michael Carlisle, I simply say, it is time to let the healing begin.

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Considering the many days, if not months, I spent on the second floor of the Mina Reese Library, I would be at great fault if I did not acknowledge some of the great people who work there. I thank Michael Adams, Curtis Mathews, Nancy Sada, and Carline Dennery for always allowing me to talk and visit with them.

None of this success or prosperity would have ever been possible without the grace of God.

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Chapter 1

Introduction

1.1 Notation

We will denote by $\mathbf{R}$ the set of real numbers and $\mathbf{R}_+ = [0, \infty)$. The set of integers will be indicated by $\mathbf{Z}$, where $\mathbf{Z}_+ = [0, 1, ...)$. For $x, y \in \mathbf{R}^n$, (x, y) will denote the ordinary inner product in $\mathbf{R}^n$ and $|x| = (x, x)^{1/2}$ the standard Euclidean norm. The σ-field of Borel subsets of n-dimensional Euclidean space will be written as $\mathscr{B}(\mathbf{R}^n)$. The symbol " $\doteq$ " means 'defined to equal'.

The Dirac delta function concentrated at x will be expressed by δ_x. The function f_ϵ will be an approximate delta function, which means that for any $\epsilon > 0$, $f_\epsilon(x) = \frac{1}{\epsilon} f(x/\epsilon)$ with f a positive, C^1, even function of x supported on $[-1, 1]$ with $\int f_\epsilon(x)\, \mathrm{d}x = 1$.

Each stochastic process, X_t, will be defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with state space $(\mathbf{R}^n, \mathscr{B}(\mathbf{R}^n))$. The notation BM^d is reserved for d-dimensional Brownian motion.

1.2 Occupation Measures and Local Times

The inception of the theory of local times can be accredited to Lévy and his work with Brownian local times [16]. As Revuz-Yor [21], p. 277, note, the advancement of this theory has since branched into roughly three directions. The first precept formulizes local times for Markov processes through the utilization of potential theory and the theory of additive functionals. This was initiated by Blumenthal-Getoor [2] and fostered through the Isomorphism Theorems of Marcus-Rosen [17], which connect the law of local time processes to Gaussian processes. The second is motivated by Itô's formula and the theory of semimartingales. The progenitors of this approach include Tanaka [31], Millar [19], and Meyer [18].

The third school of thought is grounded in the notion of occupation densities, which is the methodology we adopt for our work and develop in detail below. The survey of Geman-Horowitz [10] delineates the roots of this branch of local time theory.

Definition 1.2.1. *Let $X : \boldsymbol{R}^N \to \boldsymbol{R}^d$ $(t \mapsto X_t)$ be a Borel measurable function and denote the Lebesgue measure on $\boldsymbol{R}^N$ by λ_N. For each $A \subset \mathscr{B}(\boldsymbol{R}^N)$ we define the* ***occupation measure*** *μ_A on $\boldsymbol{R}^d$ by*

$$\mu_A(B) = \lambda_N(A \cap X^{-1}(B))$$

with $B \subset \mathscr{B}(\boldsymbol{R}^d)$.

If we set $\mathbf{R}^N = \mathbf{R}_+$, we can consider $\mathbf{R}_+$ as a time set and interpret the occupation measure as the amount of time spent by X_t in B during time period A.

Definition 1.2.2. *Let μ_A be an occupation measure and λ_d the Lebesgue measure in $\boldsymbol{R}^d$. If $\mu_A \ll \lambda_d$, the Radon-Nikodym theorem asserts the existence of a unique measurable* *function, denoted by* $\alpha(x, A)$*, such that for each* $B \subset \mathscr{B}(\boldsymbol{R}^d)$*,*

$$\lambda_N(t \in A \mid X_t \in B) = \int_B \alpha(x, A) \, \mathrm{d}\lambda_d$$

and the Radon-Nikodym derivative

$$\alpha(x, A) = \frac{\mathrm{d}\mu_A}{\mathrm{d}\lambda_d}(x) \tag{1.1}$$

is defined as the ***occupation density*** *on* A *and we say that* X *has a local time on* A*.*

The uniqueness of $\alpha(x, A)$ refers to the fact that if $\tilde{\alpha}(\cdot, \cdot)$ is any measurable function with the same property, then $\alpha(x, A) = \tilde{\alpha}(x, A)$ almost everywhere $\mathrm{d}\lambda_d$.

Considering, again, $\mathbf{R}^n = \mathbf{R}_+$ and setting $d = 1$, Gurevich-Shilov [12], p. 207, indicates the right hand side of (1.1) can be written as

$$\lim_{\epsilon \to 0} \frac{\lambda_N \left(t \in A \mid X_t \in (x, x + \epsilon]\right)}{\epsilon}, \qquad \text{for a.e. } x.$$

and the local time $\alpha(x, A)$ can intuitively be thought of as the time spent at x during the time period A.

To this point, the concepts of occupation measure and local time have been construed as real variable statements. We would now like to extend these definitions to a probabilistic framework.

Definition 1.2.3. *Let* X_t *be a stochastic process defined on a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$*, such that* $X : \Omega \times \boldsymbol{R}_+ \to \boldsymbol{R}^d$*. We say that* X *has a* ***local time*** *on* A *if* $X(\omega, \cdot)$ *has a local time on* A *for almost all* ω*.*

1.3 Multiple Points and Intersection Local Times.

Having introduced the notion of a local time in the previous section, we now expand this concept and develop the key ideas of intersection local times and renormalized intersection local times. Our focus will primarily be on BM^2, BM^3 and planar symmetric stable processes and in the next chapter we discuss the one dimensional case. In the course of expounding these random measures, we trace the origins of this idea and present a brief survey concerning the critical results. To work towards this goal, we start with the path properties of multi-dimensional Brownian motion and symmetric stable processes. These features will motivate the definitions and theorems to follow.

1.3.1 k-Multiple Points

Definition 1.3.1. *Suppose X_t is a stochastic process with state space $(\boldsymbol{R}^n, \mathscr{B}(\boldsymbol{R}^n))$. We call a point $x \in \boldsymbol{R}^n$ a **double point** if there exist time instances $t_1 < t_2$ such that $X_{t_1} = X_{t_2} = x$. More generally, x is called a k-**multiple point** if there exist distinct time elements $t_1 < \cdots < t_k$ such that $X_{t_1} = \cdots = X_{t_k} = x$.*

Simply stated, a point $x \in \mathbf{R}^d$ is a k-multiple point if the sample path of X_t visits x at least k times. When $d \geq 2$, such an x can be visualized as a point in $\mathbf{R}^d$ in which the path of the process intersects itself. Figure 1.1 above assists with this notion for double points in $\mathbf{R}^2$.

The historical origins of our problem can be traced back to the mid-twentieth century and the inquiry into the existence of k-multiple points for BM^d. In 1942, Levy [15] initiated

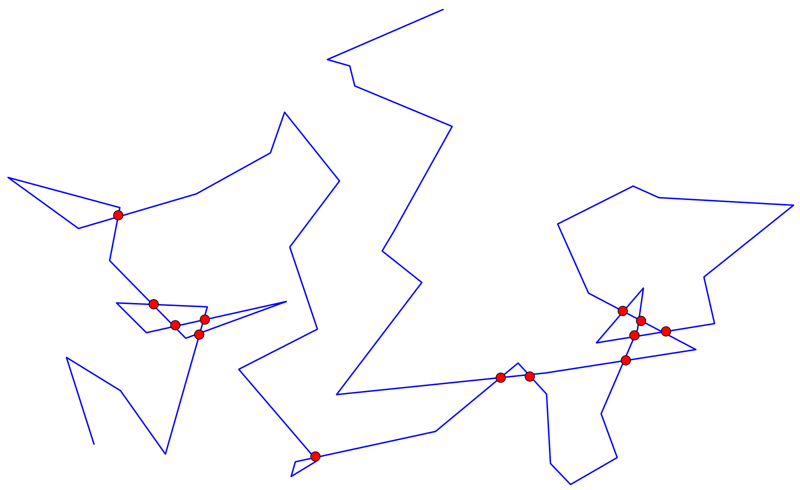

Figure 1.1: Double Points and Self-Intersections.

the study of the self-intersection of Brownian paths when he proved the existence of double points for planar Brownian motion. Two years later Kakutani [13] investigated the Brownian path in $\mathbf{R}^n$, for $n \geq 5$, and determined that the sample path of the process will not intersect itself. The case for $n = 3, 4$ was resolved by Dvoretzky, Erdös, and Kakutani [5]. They established the existence of double points for the Brownian path in three dimensions and the non-existence of double points for BM^4. These three authors confirmed the existence of recurrent points for the planar Brownian path in 1954 [6] and, together with Taylor [7], they showed that three dimensional Brownian motion has no triple points. A simulation of BM^2 in Figure 1.2 above illustrates the notion of multiple points.

The path properties of n-dimensional symmetric stable processes were studied by several authors in the 1950's and 1960's. Taylor summarized these finding in [32] and we list them in Table 1.1 below. For each value of n and β, k represents the maximum multiplicity for the sample path.

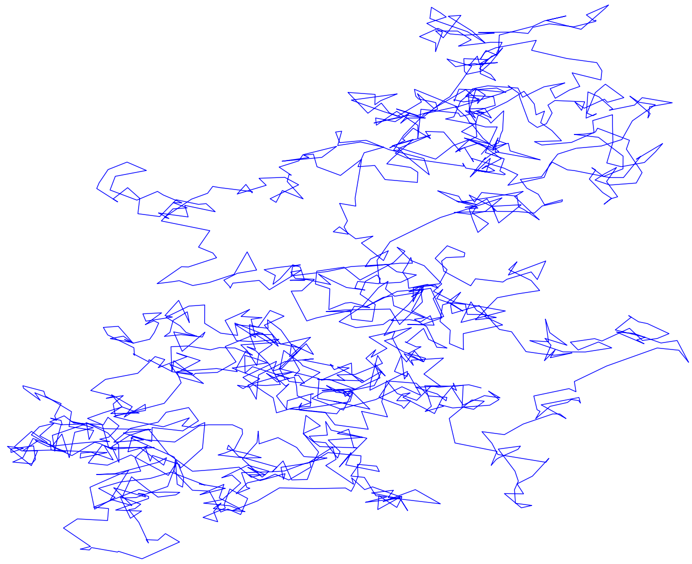

Figure 1.2: Simulation of BM^2 Illustrating k-multiple Points.

1.3.2 Intersection Local Times and Renormalization

The motivation for intersection local times, for a stochastic process X_t, is to define a random measure on $\mathbf{R}_+^k$ supported on

$$\left\{(t_1, ..., t_k) \in \mathbf{R}_+^k \mid X_{t_1} = \cdots = X_{t_k}\right\} \tag{1.2}$$

which, heuristically speaking, measures the amount of time a process spends intersecting itself at points in $\mathbf{R}^n$. Our discussion above, regarding the existence of k-multiple points for n-dimensional symmetric stable processes, leads us to conclude (1.2) may be empty for certain dimensions in $\mathbf{R}^n$ and parameters β. Hence, we must exercise some diligence and seek to define an intersection local time supported on sets which are not empty. To

$\mathbf{R}^n$	k-multiple/ recurrent points	Parameter
$n=1$	$k=1$ $k=\ell,\ \ell=2,3,...$ recurrent	$0<\beta\le\frac{1}{2}$ $\dfrac{\ell-1}{\ell}<\beta\le\dfrac{\ell}{\ell+1}$ $1\le\beta\le 2$
$n=2$	$k=1$ $k=\ell,\ \ell=2,3,...$ recurrent	$0<\beta\le 1$ $\dfrac{2\ell-2}{\ell}<\beta\le\dfrac{2\ell}{\ell+1}$ $\beta=2$
$n=3$	k=1 k=2	$0<\beta\le\frac{3}{2}$ $\frac{3}{2}<\beta\le 2$
$n\ge 4$	k=1	$0<\beta\le 2$

Table 1.1: Summary of k-multiple points for symmetric stable processes in $\mathbf{R}^n$.

foster this notion, we begin by considering processes which visit points in $\mathbf{R}^n$ at least twice. The existence of k-multiple points for the sample path of BM^1 and BM^2, along with the occurrence of double points for BM^3, make these processes valid candidates. We can also contemplate one and two dimensional symmetric stable processes of order $1/2<\beta<2$ and $1<\beta<2$, respectively. In fact, if we let X_t denote any of these processes, then the set

$$\big\{(t_1,t_2)\in\mathbf{R}_+^2 \mid X_{t_1}=X_{t_2}\big\} \tag{1.3}$$

is not empty with probability 1. This leads to our first formulation of intersection local times.

Let X_t be a one or two dimensional symmetric stable process of order $1/2<\beta<2$ or $1<\beta<2$, respectively, or a BM^3. The *intersection local time* for X_t is formally given by

$$\alpha(x,B)\doteq\iint\limits_B \delta_x(X_t-X_s)\ \mathrm{d}s\ \mathrm{d}t;\quad B\subset\mathscr{B}(\mathbf{R}_+^2), \tag{1.4}$$

where δ_x is the Dirac delta function concentrated at x.

The intersection local time can be analyzed as a functional of X_t which increases only when the difference between X_t and X_s is equal to x. When $x = 0$, $\alpha(0, B)$ increases when the path of the process intersects itself and can be inferred as the amount of time spent by the process at double points. For small values of x, (1.4) measures the amount of time of "near intersections".

Setting

$$\alpha_\epsilon(x, B) \doteq \iint_B f_\epsilon(X_t - X_s - x)\ \mathrm{d}s\ \mathrm{d}t; \quad B \subset \mathscr{B}(\mathbf{R}_+^2), \tag{1.5}$$

the expression appearing in (1.4) can be defined as

$$\alpha(x, B) = \lim_{\epsilon \to 0} \alpha_\epsilon(x, B). \tag{1.6}$$

With respect to BM^2, expression (1.4) initially appeared in the physics and chemistry literature in connection with Symanzik's work in Euclidean quantum field theory [20] and Edward's and Westwater's study of polymer chains [9],[33]. It was in the appendix of Symanzik's aforementioned work in which Varadhan first studied $\alpha(x, B)$ as a random variable and introduced the construction of a renormalized intersection local time. First, by applying

$$g_k(x) = \frac{k}{2\pi} \exp\left(\frac{-k|x|^2}{2}\right)$$

as a sequence of approximate delta functions converging weakly to δ_0 and setting W_t to a two dimensional Brownian motion, he proved

$$\lim_{k \to \infty} \int_0^1 \int_0^1 g_k(W_t - W_s)\ \mathrm{d}s\ \mathrm{d}t \longrightarrow \infty, \quad \mathbb{P} - \text{a.s.} \tag{1.7}$$

However, if the sequence of integrals above were renormalized, in the sense that an "infinite term" was subtracted from the integrals, Varadhan concluded the resulting series would converge to a random variable. Specifically, he proved we can select a sequence of constants $\{c_k \mid \lim_{k\to\infty} c_k = \infty\}$, independent of the path of W_t, such that

$$\lim_{k\to\infty}\left(\int_0^1\int_0^1 g_k(W_t - W_s)\ \mathrm{d}s\ \mathrm{d}t - c_k\right) \tag{1.8}$$

converges a.s. in $L^2(\mathrm{d}\mathbb{P})$ to a well defined random variable.

Whereas the renormalization introduced by Varadhan was opportune for Symanzik's work, Rosen [23] argued that the main obstacle in expanding the theory of quantum fields lies in the local behavior of $\alpha(x, B)$. Considering both the two and three dimensional Brownian path, he deduced that the infinite result of (1.7) was attributable to global and local properties. First, through the use of Fourier analysis, Rosen discovered the divergent outcome was a direct consequence of the fact that the image of BM^2 and BM^3 has Lebesgue measure zero. His second approach centered on the local analysis of the "set variable" B and he proved the elements on the diagonal of $\mathbf{R}^2_+$ contribute to the absence of a limit. Nonetheless, his main conclusions revealed that for certain $B \subset \mathscr{B}(\mathbf{R}^2_+)$, particularly those elements off the diagonal, $\alpha(x, B)$ can be identified as a local time which is continuous in space and time. Assigning

$$\mathbf{R}^2_+(\epsilon) = \{(s,t) \mid 0 \le s,t, |s-t| \ge \epsilon, \epsilon > 0\},$$
$$Q_{S,T,\epsilon} = \{(s,t) \mid 0 \le s \le S, 0 \le t \le T, |s-t| \ge \epsilon, \epsilon > 0\},$$

the following existence and continuity properties were obtained.

Theorem 1.3.2 ([23]). *Let W_t be a BM^2 or BM^3 and define $\widetilde{W}_t = W_t - W_s$.*

i) If B is a bounded Borel set in $\boldsymbol{R}^2_+$, then $\widetilde{W}_t$ has an occupation density $\alpha(x, B)$ on B.

ii) *For any bounded $B \subset \mathscr{B}(\boldsymbol{R}^2_+(\epsilon))$, $\alpha(x, B)$ is locally Hölder continuous of order $\alpha < 1$ if $d = 2$ and $\alpha < 1/2$ if $d = 3$. Specifically, for any compact subset $K \subset \boldsymbol{R}^d$ we can choose a version of $\alpha(x, B)$ such that*

$$\sup_{x,y \in K} \frac{|\alpha(x, B) - \alpha(y, B)|}{|x - y|^{\alpha}} < \infty$$

with $\alpha < 1$ and $\alpha < 1/2$ when $d = 2$ and $d = 3$, respectively.

iii) *We can choose of version of $\alpha(x, Q_{S,T,\epsilon})$ which is a measure supported by*

$$\{(s, t) \mid W_t - W_s = x\}$$

and continuous in (x, S, T).

Notice, $Q_{S,T}$ simply bounds the time variables in the domain of integration of $\alpha(x, B)$ and has allowed us to index our local time by $(S, T) \in \mathbf{R}^2_+$ (below, we will apply a single time index as opposed to the two time parameters appearing here). Accordingly, we can change our frame of reference and consider our local time as a stochastic process such that if W_t is BM^2,

$$\alpha(\omega, x, Q_{S,T}) : \Omega \times \mathbf{R}^2 \times \mathbf{R}^2_+ \longrightarrow \mathbf{R}$$

(resp. $x \in \mathbf{R}^3$ if we consider BM^3). A version of the local time $\alpha(x, Q_{S,T})$ indicates the existence of a stochastic process $\hat{\alpha}(\cdot, \cdot)$ which is itself a local time for $\widetilde{W_t}$ at x, jointly continuous on $\mathbf{R}^2 \times \mathbf{R}^2_+$ ($\mathbf{R}^3 \times \mathbf{R}^2_+$) such that for all $x \in \mathbf{R}^2$ ($x \in \mathbf{R}^3$) and $(S, T) \in \mathbf{R}^2_+$

$$\alpha(x, Q_{S,T}) = \hat{\alpha}(x, Q_{S,T}), \quad \mathbb{P} - \text{a.s.}$$

The existence of a continuous version addresses a classical problem in the theory of the local times of Markov processes, in which we refer the reader to [17] for an in depth discussion.

Theorem 1.3.2 which provides the existence of a local time for the process $\widetilde{W}_t$, sheds light on the local behavior of the Brownian functional $\alpha(x, B)$, and serves as precursor of developments to come. Namely, a primary objective in the study of the intersection local times of Markov processes is to devise a renormalization, i.e. a subtraction of "infinite parts", for arbitrary bounded Borel sets in $\mathbf{R}_+^2$ which result in a version, continuous in both time and space.

The renormalization introduced by Varadhan of (1.8) has since led to a large body of literature in which alternate representation have been constructed and [8] includes a brief survey. We discuss two renormalizations relevant to our study and begin with a formulation introduced by Le Gall [14] for planar Brownian motion.

Seeking applications of intersection local times to the analysis of the Weiner sausage, Le Gall considered the ensuing quantity

$$\tilde{\gamma}(x, B) \doteq \alpha(x, B) - \mathbb{E}\left[\alpha(x, B)\right], \tag{1.9}$$

where B is any Borel set in the triangle

$$T = \{(s, t) \mid 0 \leq s \leq t \leq 1\}.$$

Results concerning (1.9) are captured in the following

Theorem 1.3.3 ([14]). *Consider a planar Brownian motion W_t and let B be any Borel set of T. Then*

i) for any $x \in \boldsymbol{R}^2/\{0\}$, there exists a finite random measure $\alpha(x, B)$, supported on

$$\{(s, t) \in \boldsymbol{R}_+^2 \mid W_t - W_s = x\}$$

such that $x \mapsto \alpha(x, \cdot)$ is weakly continuous in x;

ii) We can select a version of $\tilde{\gamma}(x, B)$ which is continuous for all $x \in \boldsymbol{R}^2$.

From the symmetry of T, we can extend the above conclusions to any Borel set in $\mathbf{R}_+^2$.

In comparison to the results of the local analysis in Theorem 1.3.2, Le Gall proved the existence of $\alpha(x, B)$ for arbitrary Borel sets of T which included elements on the diagonal, i.e. $s = t$. Statements (i) and (ii) accentuate the motivation for a renormalization. Alone, $\alpha(x, B)$ admits weak continuity for $x \neq 0$. However, upon subtracting the renormalized term $\mathbb{E}[\alpha(x, B)]$, we gain the stronger conclusion of continuity for $\tilde{\gamma}(x, B)$ for all $x \in \mathbf{R}^2$. Such functionals are often referred to in the literatures as *renormalized intersection local times.*

The techniques employed by Le Gall in obtaining Theorem 1.3.3 (ii) will be duplicated to derive our results and we briefly mention the methodology here. Recall, the multidimensional version of Kolmogorov's continuity criterion [21], p. 26, provides necessary conditions to establish the existence of a version of a Banach valued process which is locally Hölder continuous of order α. By setting

$$\tilde{\gamma}_\epsilon(x, B) \doteq \alpha_\epsilon(x, B) - \mathbb{E}[\alpha_\epsilon(x, B)], \tag{1.10}$$

with $\alpha_\epsilon(x, B)$ defined in (1.5), we observe that $\gamma_\epsilon(x, B)$ is continuous in x. By satisfying the hypothesis of Kolmogorov's criterion

$$\mathbb{E}\left[(\tilde{\gamma}_\epsilon(x, B) - \tilde{\gamma}_{\epsilon'}(x', B))^p\right] \leq c_p |(\epsilon, x) - (\epsilon', x')|^{\delta p}; \quad \delta > 0, \tag{1.11}$$

the subsequent local properties hold

$$|\tilde{\gamma}_\epsilon(x, B) - \tilde{\gamma}_{\epsilon'}(x', B)| \leq c_p |(\epsilon, x) - (\epsilon', x')|^{\xi}; \quad \xi > 0.$$

The net effect of the above inequality reveals the subsequent limit exists

$$\lim_{\epsilon \to 0} \tilde{\gamma}_\epsilon(x, B) = \tilde{\gamma}(x, B)$$

and since $\tilde{\gamma}_\epsilon(x, B)$ is continuous uniformly in ϵ, the limit is continuous in x. In essence, to derive existence and continuity results concerning renormalized intersection local times, one approach is to arrive at a bound similar to (1.11) and then apply Kolmogorov's continuity criterion.

The second renormalization we review is a product of the utilization of stochastic integrals to the study of intersection local times by Rosen-Yor [29], in which the "infinite term" subtracted from $\alpha(x, B)$ involves the potential of our process and a time parameter. Let W_t be a BM^2 and denote the density of this process by

$$p_s(x) = \frac{1}{2\pi s} \exp\left(-\frac{|x|^2}{2s}\right)$$

and let

$$u_\epsilon(x) \doteq \int_\epsilon^1 p_s(x)\ \mathrm{d}s, \quad u(x) \doteq u_0(x). \tag{1.12}$$

With

$$\overline{\Gamma}_T = \{0 \leq s \leq t \leq T\}, \quad \alpha(x, \overline{\Gamma}_T) \doteq \alpha(x, T),$$

Rosen-Yor concluded the next result.

Theorem 1.3.4 ([29])**.** *Consider the planar Brownian motion W_t, with potential $u(x)$. For any $x \in \boldsymbol{R}^2/\{0\}$, the following limit exists*

$$\alpha(x, T) = \lim_{\epsilon \to 0} \int_0^T \int_0^t p_\epsilon\left(W_t - W_s - x\right)\ \mathrm{d}s\ \mathrm{d}t$$

Additionally, the process defined by

$$\tilde{\alpha}(x,T) \doteq \alpha(x,T) - Tu(x), \quad x \neq 0 \tag{1.13}$$

admits a bicontinuous extension to all $x \in \boldsymbol{R}^2$ and $T \in \boldsymbol{R}_+$.

A consequence to the above theorem can be deduced as follows. Since

$$v(x) = u(x) - \frac{1}{\pi}\log\frac{1}{|x|}; \quad x \neq 0, \tag{1.14}$$

can be extended by continuity to all of $\mathbf{R}^2$, the process defined by

$$\tilde{\tilde{\alpha}}(x,T) \doteq \alpha(x,T) - \frac{T}{\pi}\log\frac{1}{|x|}; \quad x \neq 0, \tag{1.15}$$

can also be extended by continuity to all of $\mathbf{R}^2$ and be characterized as isolating the singularities of the intersection local time.

Having assessed various renormalizations of (1.4) with respect to Brownian motion, we shift focus and now address intersection local times for two dimensional symmetric stable processes of order $1 < \beta < 2$. Along the way, we will learn that the analytic properties regarding the weak continuity of $\alpha(x,B)$ in the space variable for the Brownian case extends to the larger class of Markov processes now under consideration. Hence, our objective will be to identify a renormalization which leads to a bicontinuous version in the space and time variable, $\{(x,T) \mid x \in \mathbf{R}^2, T \in \mathbf{R}_+\}$. We will outline a single renormalized intersection local time parallel to Theorem 1.3.4 where the potential of our process and a time parameter appear as our renormalized term.

For a planar symmetric stable process X_t of index $0 < \beta \leq 2$, the density is equal to

$$p_t(x) = \frac{1}{(2\pi)^2} \int \exp\left(iq \cdot x\right) \exp\left(-t|q|^\beta\right) \, \mathrm{d}^2 q; \quad q, x \in \mathbf{R}^2,$$

where $\mathrm{d}^2 q$ is the Lebesgue measure in $\mathbf{R}^2$. The potential for X_t is then

$$U(x) \doteq \int_0^\infty p_t(x) \, \mathrm{d}t = \frac{c(\beta)}{|x|^{2-\beta}} \tag{1.16}$$

with

$$c(\beta) = (2^\beta \pi)^{-1} \frac{\Gamma\left(\frac{2-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}. \tag{1.17}$$

Applying the similar analytical techniques introduced by Le Gall in the study of (1.10), Rosen [25] concluded the subsequent results.

Theorem 1.3.5 ([25])**.** *Let X_t be a two-dimensional symmetric stable process order $4/3 < \beta < 2$ and set $\widetilde{X}_t = X_t - X_s$. Then there exists a local time $\alpha(x, \cdot)$ for the process $\widetilde{X}_t$ which is weakly continuous in $x \neq 0$. Furthermore, the renormalized intersection local time*

$$\hat{\alpha}(x, T) \doteq \alpha\left(x, [0, T]^2\right) - 2T \frac{c(\beta)}{|x|^{2-\beta}} \tag{1.18}$$

has a version which is continuous for all $x \in \boldsymbol{R}^2$.

1.3.3 Renormalized Triple Intersection Local Times.

Having defined a measure for the set (1.3), it is only natural to expand the concept of a local time supported by the set

$$\left\{(t_1, t_2, t_3) \in \mathbf{R}_+^3 \mid X_{t_1} = X_{t_2} = X_{t_3}\right\}, \tag{1.19}$$

which measures the amount of time spent by a process intersecting itself three times at a point $x \in \mathbf{R}^d$. To identify the processes we can consider for such a measure, Table 1.1 reveals BM^3 must be excluded, since the path of this process will intersect itself at any point $x \in \mathbf{R}^3$ at most two times. However, the existence k-multiple points, with $k \geq 3$, for BM^1 and BM^2 and for one and two dimensional symmetric stable processes of order $2/3 < \beta < 2$ and $4/3 < \beta < 2$, respectively, indicates (1.19) is not empty with probability 1 for these processes. This directs us to the following functional.

Let X_t be a one dimensional symmetric stable process of order $2/3 < \beta \leq 2$ or a two dimensional symmetric stable process of order $4/3 < \beta \leq 2$. For any $x, y \in \mathbf{R}^2$ the triple intersection local time is formally given by

$$\alpha(x, y, B) \doteq \iiint\limits_B \delta_x (X_s - X_r)\, \delta_y (X_t - X_s) \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t; \quad B \subset \mathscr{B}(\mathbf{R}^3_+), \tag{1.20}$$

with

$$\mathbf{R}^3_+ = \{(r, s, t) \mid 0 \leq r, s, t\}.$$

Notice that we have included BM^1 and BM^2 in the above definition by allowing $\beta = 2$. Observe that $\alpha(x, y, B)$ is parametrized by the two variables x and y, compared to just the single variable x of the intersection local time. The interpretation accompanying $\alpha(0, 0, B)$ is the time spent by the process at triple points. For small values of x, y, (1.20) measures the time spent at "near intersections".

Setting f_ϵ as an approximate delta function, we can let

$$\alpha_\epsilon(x, y, B) \doteq \iint\limits_B f_\epsilon (X_s - X_r - x)\, f_\epsilon (X_t - X_s - y) \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t$$

and define (1.20) as the almost sure limit

$$\alpha(x,y,B) = \lim_{\epsilon \to 0} \alpha_\epsilon(x,y,B).$$

In the course of our investigation of (1.20), we will encounter characteristics parallel to intersection local times. Namely, the weak continuity of $\alpha(x,B)$ with respect to x will extend to the two spatial variables of $\alpha(x,y,B)$. Therefore, we aspire to ascertain renormalizations which produce versions continuous for all $x, y \in \mathbf{R}^2$ *and* an appropriate time parameter $T \in \mathbf{R}_+$. To work towards this goal, we start by establishing the existence of a local time and examine the local behavior of $\alpha(x,y,B)$. We move forward and then introduce renormalizations for BM^2 and planar symmetric stable processes.

As with the case of intersection local times, the local analysis of $\alpha(x,y,B)$ begins with the study of the time variables (r,s,t) off the diagonal, i.e. $r \neq s$ and $s \neq t$, which inclines us to define the following sets,

$$\begin{aligned} \mathbf{R}^3_{\leq} &= \{(r,s,t) \mid 0 \leq r \leq s \leq t\} \\ \mathbf{R}^3_{\leq}(\epsilon) &= \{(r,s,t) \mid 0 \leq r \leq s \leq t, |s-r| \geq \epsilon, |t-s| \geq \epsilon, \epsilon > 0\} \end{aligned}$$

With the process defined by

$$\widetilde{X}(r,s,t) = (X_s - X_r, X_t - X_s),$$

an extension of Theorem 1.3.2 (i) and (ii) is captured in the following.

Theorem 1.3.6 ([27])**.** *Let X_t be a planar symmetric stable process of order $4/3 < \beta \leq 2$. Then*

i) $\widetilde{X}(r,s,t)$ has a occupation density $\alpha(x,y,B)$ for any bounded $B \subset \mathscr{B}(\boldsymbol{R}^3_{\leq})$.

ii) If B is a bounded Borel set in $\boldsymbol{R}^3_{\leq}(\epsilon)$, then for all $x, y \in \boldsymbol{R}^2$, there exists a version of $\alpha(x, y, B)$ which is a measure supported on

$$\{(r, s, t) \in \boldsymbol{R}^3_{\leq} \mid X_s - X_r = x, X_t - X_s = y\}$$

and weakly continuous in x, y.

Although the set (1.19) is not empty for planar symmetric stable processes of order $1 < \beta$, the above reveals the existence of a local time for $\widetilde{X}(r, s, t)$ only if the process X_t is of order $4/3 < \beta$. For values off the diagonal, the support of $\alpha(x, y, \cdot)$ exists for all $x, y \in \mathbf{R}^2$. The aim of the next result, akin to Theorem 1.3.3 (i), is to dissect our local time for arbitrary bounded Borel sets of $\mathbf{R}^3_{\leq}$.

Theorem 1.3.7 ([27]). *Let X_t be a planar symmetric stable process of order $8/5 < \beta \leq 2$ and consider any bounded $B \subset \mathscr{B}(\boldsymbol{R}^3_{\leq})$. With probability one, for any $x, y \in \boldsymbol{R}^2/\{0\}$, $\alpha(x, y, B)$ is a measure, weakly continuous in x, y and supported on the set*

$$\{(r, s, t) \in \boldsymbol{R}^3_{\leq} \mid X_s - X_r = x, X_t - X_s = y\}.$$

As we enlarge the analysis of our random measure from elements (r, s, t) off the diagonal to arbitrary bounded sets in $\mathbf{R}^3_{\leq}$ in the pursuit of existence and continuity results for x and y, the class of processes shrinks from order $4/3 < \beta$ to $8/5 < \beta$. The weak continuity of $\alpha(x, y, B)$ motivates the discussion of a renormalized triple intersection local time. The specific renormalizations we consider not only involve the potentials of BM^2 and planar symmetric stable processes, as defined in (1.12) and (1.16), respectively, but also the renormalized intersection local times!

The work of Rosen-Yor which yielded the renormalization for $\alpha(x, B)$ in terms of $u(x)$ and a time index in Theorem 1.3.4 was in fact a consequence of their study of stochastic integrals and triple intersection local times. To introduce a time index, set

$$\Gamma_T = \{(r,s,t) \mid 0 \le r \le s \le t \le T\}, \quad \alpha(x,y,\Gamma_T) \doteq \alpha(x,y,T),$$

so that we can consider $\alpha(x,y,T)$ as a stochastic process. For the Brownian case, the subsequent renormalization was shown to be continuous in x, y, and T.

Theorem 1.3.8 ([29])**.** *Let W_t be a planar Brownian motion with $u(x)$ as in (1.12). Consider the renormalized intersection local time $\tilde{\alpha}(x,T)$ of equation* (1.13)*. Then the following process*

$$\tilde{\gamma}(x,y,T) \doteq \alpha(x,y,T) - Tu(x)u(y) - u(x)\tilde{\alpha}(y,T) - u(y)\tilde{\alpha}(x,T), \qquad x,y \neq 0 \tag{1.21}$$

has a jointly continuous extension to all x, y, T.

The capacity to isolate the singularities of $\tilde{\alpha}(x,T)$ via (1.15) through the continuity of (1.14) can be replicated to $\tilde{\gamma}(x,y,T)$, so that the process given by

$$\begin{aligned}\tilde{\tilde{\gamma}}(x,y,T) \doteq & \alpha(x,y,T) - \frac{T}{\pi^2}\log\left(\frac{1}{|x|}\right)\log\left(\frac{1}{|y|}\right) \\ & - \frac{1}{\pi}\left(\log\frac{1}{|x|}\right)\tilde{\alpha}(y,T) - \frac{1}{\pi}\left(\log\frac{1}{|y|}\right)\tilde{\alpha}(x,T), \qquad x,y \neq 0\end{aligned} \tag{1.22}$$

has a jointly continuous extension to all x, y, T. The above expression carries the similar interpretation of isolating the singularities of $\tilde{\gamma}(x,y,T)$. A striking feature of both (1.21) and (1.22) is the symmetry of the renormalizations in the x and y variables.

Moving forward, we turn our attention to planar symmetric stable processes and examine a renormalization formulated by Rosen [27] similar to (1.21).

Theorem 1.3.9 ([27])**.** *Let X_t be a planar symmetric stable process of order $8/5 < \beta < 2$ with potential density $U(x)$ and set $\hat{\alpha}(x,T)$ as the renormalized intersection local time of* *equation* (1.18) . *The following process given below*

$$\hat{\gamma}(x, y, T) \doteq \alpha(x, y, T) - TU(x)U(y) - U(x)\hat{\alpha}(y, T) - U(y)\hat{\alpha}(x, T), \qquad x, y \neq 0$$

has a jointly continuous extension to all x, y, T.

1.4 Summary

Having defined and studied local times supported by the set (1.3) and (1.19), the existence of recurrent points for two dimensional stable process of order $1 \leq \beta \leq 2$, naturally, leads to the notion of an n-fold intersection local time supported on the set

$$\{(t_1, ..., t_n) \in \mathbf{R}_+^n \mid X_{t_1} = \cdots = X_{t_n}\}.$$

This random measure has in fact been studied in by Geman, Horowitz, and Rosen in [11],[24] and more recently by Bass-Khoshnevisan [1]. Our primary focus will be on triple intersection local times for one-dimensional processes, which we discuss in detail in the next chapter.

We have concentrated our discussion on the existence and continuity properties of renormalized intersection local times. From a probabilistic perspective, one may inquire about the convergence in distribution of these processes. This has, in fact, been addressed in [26],[34].

Finally, the initial study of intersection local times was motivated by the fields of quantum field theory and chemistry. In the four decades since the introduction of these random measures, their applications have advanced in these fields of origin and expanded to other disciplines. We refer the reader to [3] for further investigation of the utilization of renormalized intersection local times in probability theory and in the physical sciences.

Chapter 2

Derivatives of Intersection Local Times

In the previous chapter, we presented a brief survey of intersection local times. A keen observer will have noticed that our discussion has been restricted to, BM^2, BM^3, and planar symmetric stable processes and all mention of intersection local times for the one-dimensional case has been excluded up to this point. As we will discuss below, intersection local times for $\mathbf{R}^1$ valued processes were not considered particularly interesting Markovian functionals since they could be reformulated in terms of the local time L_t^y. However, in recent years, these random measures have been revisited and it has been discovered that these processes indeed bear profound characteristics.

In the first section of this chapter, we motivate the interest of local times in $\mathbf{R}^1$ and extend the results concerning existence, continuity, and renormalization from the case of two and three dimensional intersection local times to the one dimensional case. After addressing these issues, we delve deeper into the analytic properties of this random measure and prove that the distributional derivatives of this process exists. Finally, we will ascertain various characteristics of this "differentiable" process.

The objective of the second section is to present the principal results of this thesis and introduce the stochastic functionals which will be examined in great detail in later chapters.

2.1 Derivatives of Intersection Local Times

Motivated by the work of Rogers-Walsh [22] on the intrinsic Brownian local time sheet, Rosen [28] was compelled to examine the quantity formally given by

$$\gamma_t' = -\int_0^t \int_0^s \delta'(X_s - X_r)\ \mathrm{d}s\ \mathrm{d}t, \tag{2.1}$$

with X_t a one-dimensional symmetric stable process, and δ' the derivative of the Dirac delta function concentrated at zero. Noting the comparison of γ_t' with the intersection local time

$$\alpha_t(y) = \int_0^t \int_0^s \delta(X_s - X_r - y)\ \mathrm{d}r\ \mathrm{d}s,$$

naturally leads to the notion of the distributional derivative of intersection local times with respect to the spatial variable x. As Rosen writes the process $\alpha_t(y)$ has not been studied much in one-dimension since it can be expressed in terms of the local time L_t^y of the process X_t: $\alpha_t(y) = \int \int_0^t L_s^{x-y}\, d_s L_s^x\, dx$ and $\alpha_t(0) = \frac{1}{2}\int (L_t^y)^2\, dy$.

Hence, even before advancing the concept of differentiability, we must first constitute the existence, continuity, and renormalization of self intersection local times in one-dimension. To work towards this goal, we define our nascent delta function in terms of an inverse Fourier transform

$$f_\epsilon(y) = \frac{1}{2\pi} \int_{\mathbf{R}} \exp(iyq)\hat{f}(\epsilon q)\ \mathrm{d}q. \tag{2.2}$$

and set

$$\alpha_{t,\epsilon}(y) \doteq \int_0^t \int_0^s f_\epsilon(X_s - X_r - y) \, \mathrm{d}r \, \mathrm{d}s, \quad \gamma_{t,\epsilon}(y) \doteq \alpha_{t,\epsilon}(y) - \mathbb{E}\left[\alpha_{t,\epsilon}(y)\right]$$

Within this framework, Rosen derived the preliminary results.

Theorem 2.1.1 ([28]). *Let X_t be a one-dimensional symmetric stable process of order $3/2 < \beta \leq 2$. Then for any $y \in \boldsymbol{R}$ and $t \in \boldsymbol{R}_+$:*

i) the following two limits exist,

$$\alpha_t(y) \doteq \lim_{\epsilon \to 0} \alpha_{t,\epsilon}(y), \quad \gamma_t(y) \doteq \lim_{\epsilon \to 0} \gamma_{t,\epsilon}(y), \tag{2.3}$$

with convergence of $\alpha_t(y)$ holding a.s. and in all $L^p(\mathrm{d}\mathbb{P})$, $p > 0$, spaces ;

ii) $\gamma_t(y)$ is continuous in y;

iii) for any continuous function $g(y)$,

$$\int_0^t \int_0^s g(X_s - X_r) \, \mathrm{d}r \, \mathrm{d}s = \int_{\boldsymbol{R}} g(y)\alpha_t(y) \, \mathrm{d}y. \tag{2.4}$$

Since $\alpha_t(y)$ satisfies the Occupation Times Formula in (iii) we can conclude it is itself a local time for the process $X_t - X_s$.

With the existence of one-dimensional intersection local times settled by the above results, we can now undertake the core question at hand. We can formally differentiate $f_\epsilon(y)$ of equation (2.2) with respect to y and write

$$\frac{\mathrm{d}}{\mathrm{d}y} f_\epsilon(y) \doteq f'_\epsilon(y) = \frac{1}{2\pi} \int \exp(iyq) iq \hat{f}(\epsilon q) \, \mathrm{d}q.$$

Furthermore, let

$$c(\beta) = \begin{cases} -\pi^{-1}\Gamma(2-\beta)\cos\big((1-\beta)\pi/2\big) & \text{for } 1<\beta<2 \\ 0 & \text{for } \beta = 2 \end{cases}$$

and

$$h_\beta(x) = \begin{cases} c(\beta)\mathrm{sgn}(x)|x|^{\beta-2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Fixing

$$\alpha'_{t,\epsilon}(y) \doteq \int_0^t \int_0^s f'_\epsilon(X_s - X_r - y)\ \mathrm{d}r\ \mathrm{d}s, \quad \gamma'_{t,\epsilon}(y) \doteq \alpha'_{t,\epsilon}(y) - \mathbb{E}\left[\alpha'_{t,\epsilon}(y)\right]$$

we can duplicate the results of the previous theorem for the above "differentiated" stochastic functionals.

Theorem 2.1.2 ([28]). *Let X_t be a one-dimensional symmetric stable process of order $3/2 < \beta \leq 2$ and consider any two points $(t, y) \in \boldsymbol{R}_+ \times \boldsymbol{R}$.*

i) The following two limits exist,

$$\alpha'_t(y) \doteq \lim_{\epsilon\to 0} \alpha'_{t,\epsilon}(y), \qquad \gamma'_t(y) \doteq \lim_{\epsilon\to 0} \gamma'_{t,\epsilon}(y), \tag{2.5}$$

where $\alpha'_t(y)$ converges a.s. and in all $L^p(\mathrm{d}\mathbb{P})$, $p > 0$, spaces ;

ii) for any C^1 function g,

$$\int_0^t \int_0^s g'(X_s - X_r)\ \mathrm{d}r\ \mathrm{d}s = \int_{\boldsymbol{R}} g(y)\alpha'_t(y)\ \mathrm{d}y; \tag{2.6}$$

iii) $\alpha'_t(y) - h_\beta(y)t$ is continuous in y.

iv)

$$\frac{\mathrm{d}}{\mathrm{d}y}\gamma_t(y) = \gamma_t'(y), \tag{2.7}$$

with $\gamma_t'(y)$ *continuous in* y*;*

v) $\alpha_t'(0) = \gamma_t'(0)$.

Connecting Theorems 2.1.1 and 2.1.2 leads us to answer questions inspired by (2.1). At first glance, it would appear $\alpha_t'(y)$ satisfies the Occupation Times Formula in (ii). However, this is not the case since $f_\epsilon'(X_s - X_r - y)$ is not an approximate delta function and we can conclude $\alpha_t'(y)$ is not a local time. However, (2.4) and (2.6) do allow us to identify $\alpha_t'(y)$ as the distributional derivative of $\alpha_t(y)$. In (iii), we identify a renormalization of $\alpha_t'(y)$ which renders continuity in y. However, since $h_\beta(y)$ is not continuous at $y = 0$, this renormalization prevents us from identifying $\alpha_t'(0)$. The continuity of $\gamma_t(t)$ opens the possibility of differentiating the renormalized intersection local time and (2.7) characterizes the derivative exactly.

Our goal in the next section is to extend the idea of differentiability to the triple intersection local times of one-dimensional symmetric stable processes.

2.2 The Differentiability of Triple Intersection Local Times

With X_t a symmetric stable process in $\mathbf{R}^1$, the stochastic functionals which motivate our work are the "near intersection" local times for X_t, formally defined as

$$\alpha(x,y,T) = \int_{B_T} \delta(X_s - X_r - x)\delta(X_t - X_s - y) \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t$$
$$\gamma(x,y,T) = \int_{B_T} \{\delta(X_s - X_r - x)\}_0\{\delta(X_t - X_s - y)\}_0 \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t,$$

where

$$B_T \doteq \{(r,s,t) \mid 0 \le r \le s \le t \le T\}.$$

and $\{Y\}_0 \doteq Y - \mathbb{E}[Y]$, for any random variable Y. When $x = y = 0$, the above processes are then coined the triple intersection local time and renormalized triple intersection local time, respectively. The principal results concern the existence and continuity of the derivatives in the space variables, (x,y), for the renormalized process

$$\gamma_x(x,y,T) = -\int_{B_T} \{\delta'(X_s - X_r - x)\}_0\{\delta(X_t - X_s - y)\}_0 \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t,$$
$$\gamma_y(x,y,T) = -\int_{B_T} \{\delta(X_s - X_r - x)\}_0\{\delta'(X_t - X_s - y)\}_0 \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t.$$

The structure which facilitates the study is as follow. The inverse Fourier transform of the probability density function of X_t is

$$p_t(y) = \frac{1}{2\pi}\int_{\mathbf{R}} \exp(iqy)\exp(-t|q|^{\beta}) \; \mathrm{d}q, \tag{2.8}$$

which we can differentiate with respect to y and write

$$p'_t(y) \doteq \frac{\partial}{\partial y} p_t(y) = \frac{1}{2\pi} \int_{\mathbf{R}} iq \exp(iqy) \exp(-t|q|^\beta) \, \mathrm{d}q. \tag{2.9}$$

When $\beta = 2$, X_t reduces to Brownian motion. With the two expressions above, set

$$\alpha(\epsilon, x, y, T) \doteq \int_{B_T} p_\epsilon(X_s - X_r - x) p_\epsilon(X_t - X_s - y) \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t, \tag{2.10}$$

$$\gamma(\epsilon, x, y, T) \doteq \int_{B_T} \{p_\epsilon(X_s - X_r - x)\}_0 \{p_\epsilon(X_t - X_s - y)\}_0 \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t, \tag{2.11}$$

$$\gamma_x(\epsilon, x, y, T) \doteq \frac{\partial}{\partial x}\gamma(\epsilon, x, y, T) = \int_{B_T} \{p'_\epsilon(X_s - X_r - x)\}_0 \{p_\epsilon(X_t - X_s - y)\}_0 \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t, \tag{2.12}$$

$$\gamma_y(\epsilon, x, y, T) \doteq \frac{\partial}{\partial y}\gamma(\epsilon, x, y, T) = \int_{B_T} \{p_\epsilon(X_s - X_r - x)\}_0 \{p'_\epsilon(X_t - X_s - y)\}_0 \; \mathrm{d}r \; \mathrm{d}s \; \mathrm{d}t. \tag{2.13}$$

With the above definitions, the main result settles the issue of differentiability.

Theorem 2.2.1. *Let X_t be a one dimensional symmetric stable process of order $3/2 < \beta \leq 2$ and select arbitrary $(x, y, T) \in \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}_+$.*

1. *The following limits exist*

$$\lim_{\epsilon \to 0} \gamma_\epsilon(x, y, T) \doteq \gamma(x, y, T) \tag{2.14}$$

$$\lim_{\epsilon \to 0} \gamma_x(\epsilon, x, y, T) \doteq \gamma_x(x, y, T) \tag{2.15}$$

$$\lim_{\epsilon \to 0} \gamma_y(\epsilon, x, y, T) \doteq \gamma_y(x, y, T) \tag{2.16}$$

with convergence holding a.s. and in all $L^p(\mathrm{d}\mathbb{P})$ spaces.

2. *With probability 1, $\gamma(x,y,T)$ is differentiable in (x,y) with*

$$\frac{\partial}{\partial x}\gamma(x,y,T) = \gamma_x(x,y,T) \quad \textit{and} \quad \frac{\partial}{\partial y}\gamma(x,y,T) = \gamma_y(x,y,T). \tag{2.17}$$

   The differentiated processes, $\gamma_x(x,y,T)$, and $\gamma_y(x,y,T)$, are jointly continuous in (x,y,T).

3. *Considered as a mapping, $\gamma(x,y,T) : \boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}_+ \to \boldsymbol{R}$, $\gamma(\cdot)$ is a $C^{1,0,0}$ and $C^{0,1,0}$ function, almost surely*

4. *The following limit exist*

$$\lim_{\epsilon\to 0}\alpha_\epsilon(x,y,T) \doteq \alpha(x,y,T), \tag{2.18}$$

   with convergence holding a.s. and in all $L^p(\mathrm{d}\mathbb{P})$ spaces. Furthermore, there exists a version of the process, $\alpha(x,y,T)$, which is a local time for $(X_t - X_s, X_s - X_r)$. Specifically, for any bounded Borel function $h(x,y) : \boldsymbol{R}^2 \to \boldsymbol{R}$, there exists a function $\alpha(x,y,T)$ such that

$$\int_{\boldsymbol{R}^2} h(x,y)\alpha(x,y,T)\ \mathrm{d}x\ \mathrm{d}y = \int_{B_T} h(X_s - X_r, X_t - X_s)\ \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t.$$

Our methods show that in fact the differentiated processes, $\gamma_x(x,y,T)$, and $\gamma_y(x,y,T)$, are jointly Holder continuous in (x,y,T).

The Brownian motion part of this Theorem has been established in [30] using very different techniques.

With B is an arbitrary bounded Borel set in $\mathbf{R}^3_+ = \{(r,s,t) \mid 0 \le r,s,t\}$, we expand the integrand of $\gamma(\epsilon, x, y, B)$ and write

$$\gamma(\epsilon, x, y, B) = \phi(\epsilon, x, y, B) - \kappa(\epsilon, x, y, B) \tag{2.19}$$

where

$$\phi(\epsilon, x, y, B) \doteq \iiint\limits_B p_\epsilon(X_s - X_r - x)\{p_\epsilon(X_t - X_s - y)\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \tag{2.20}$$

$$\kappa(\epsilon, x, y, B) \doteq \iiint\limits_B \mathbb{E}\left[p_\epsilon(X_s - X_r - x)\right]\{p_\epsilon(X_t - X_s - y)\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t. \tag{2.21}$$

A similar expansion for $\gamma_y(\epsilon, x, y, B)$ leads to

$$\gamma_y(\epsilon, x, y, B) = \phi_y(\epsilon, x, y, B) - \kappa_y(\epsilon, x, y, B) \tag{2.22}$$

with

$$\phi_y(\epsilon, x, y, B) \doteq \iiint\limits_B p_\epsilon(X_s - X_r - x)\{p'_\epsilon(X_t - X_s - y)\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \tag{2.23}$$

$$\kappa_y(\epsilon, x, y, B) \doteq \iiint\limits_B \mathbb{E}\left[p_\epsilon(X_s - X_r - x)\right]\{p'_\epsilon(X_t - X_s - y)\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \tag{2.24}$$

and

$$\gamma_x(\epsilon, x, y, B) = J_x(\epsilon, x, y, B) - \kappa_x(\epsilon, x, y, B). \tag{2.25}$$

In the expression above, $J_x(\epsilon, x, y, B)$ and $\kappa_x(\epsilon, x, y, B)$ are defined analogous with (2.23) and (2.24).

To prove Theorem 2.2.1, we apply Kolmogorov's Continuity Criterion. Specifically, establishing the continuity of each process in the space parameters assures us of a limit which is continuous in (x, y). However, the analyses of $\gamma(\epsilon, x, y, B)$ and $\gamma_y(\epsilon, x, y, B)$ is simplified if we focus on the stochastic functionals $\phi(\epsilon, x, y, B)$ and $\phi_y(\epsilon, x, y, B)$, reasoning techniques applied to these quantities can be can then be extended to the second summands appearing in (2.19) and (2.22). We focus solely on the y derivative, since methods developed for this case extend naturally to $J_x(\epsilon, x, y, B)$. This last fact can be deduced from the symmetry of the integrands of (2.12) and (2.13). Along the way, it will become apparent that proofs involving the existence and continuity of $\alpha(\epsilon, x, y, B)$ are naturally acquired in the study of the aforementioned functions and we outline the ideas at the appropriate locations.

Establishing the Hölder continuity for the previously cited functionals will consume a large portion of our analysis. To accomplish this objective, consider the following dyadic decomposition

$$A(k, n, 2) \doteq \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right]^2_{\leq} \tag{2.26}$$

$$D(2) \doteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} A(k, n, 2) \tag{2.27}$$

so that

$$A(1, 1, 2) \doteq [0, 1/2] \times [1/2, 1]^2_{\leq}. \tag{2.28}$$

Trivially, $A(1, 1, 2) \subset D(2)$. Furthermore, let

$$A(k, n, 1) \doteq \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right]^2_{\leq} \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right] \tag{2.29}$$

$$D(1) \doteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} A(k, n, 1) \tag{2.30}$$

where

$$A(1, 1, 1) \doteq [0, 1/2]^2_{\leq} \times [1/2, 1], \tag{2.31}$$

k	n	$\left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right]$	$\left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right]$
1	1	[0,1/2]	[1/2,1]
1	2	[0,1/4]	[1/4,2/4]
2	2	[2/4,3/4]	[3/4,1]
1	3	[0,1/8]	[1/8,2/8]
2	3	[2/8,3/8]	[3/8,4/8]
3	3	[4/8,5/8]	[5/8,6/8]
4	3	[6/8,7/8]	[7/8,1]
1	4	[0,1/16]	[1/16,2/16]
2	4	[2/16,3/16]	[3/16,4/16]
3	4	[4/16,5/16]	[5/16,6/16]
4	4	[6/16,7/16]	[7/16,8/16]
5	4	[8/16,9/16]	[9/16,10/16]
6	4	[10/16,11/16]	[11/16,12/16]
7	4	[12/16,13/16]	[13/16,14/16]
8	4	[14/16,15/16]	[15/16,1]

Table 2.1: Dyadic decomposition for $n = 1, 2, 3, 4$.

with $A(1,1,1) \subset D(1)$. Set

$$B_1 \doteq \{(r,s,t) \mid 0 \le r \le s \le t \le 1\}.$$

From our definitions above, $D(1) \cap D(2) = \varnothing$ and $B_1 \subset D(1) \cup D(2)$. Table 2.1 above assists in illustrating these sets.

Our motivation to define the above sets is as follows. We start by considering an arbitrary $B \subset D(1) \cup D(2)$. However, the symmetry of $D(1)$ and $D(2)$ indicates we can restrict our attention to $B \subset D(2)$. In Lemmas 2.2.2 and 2.2.3 below we derive both probabilistic and analytical characteristics of $\phi(\epsilon, x, y, B)$ and $\phi_y(\epsilon, x, y, B)$ with $B \subset A(1,1,2)$. Via a scaling argument, we prove through Lemma 2.2.4 that our Markovian functionals are, in fact, Hölder continuous in L^m for arbitrary $B \subset D(2)$.

Lemma 2.2.2. *Consider the random variables $\phi(\epsilon, x, y, B)$ and $\phi_y(\epsilon, x, y, B)$, with $x, y \in$ $\boldsymbol{R}$, $B \subseteq [0, \frac{1}{2}] \times [\frac{1}{2}, 1]^2_{\le}$ and $\epsilon > 0$. For all positive integers m, there exists constants*

$c_1, c_2, \in \boldsymbol{R}$ *such that*

$$\mathbb{E}\left[\left(\phi(\epsilon, x, y, B)\right)^m\right] \leq c_1 \tag{2.32}$$

$$\mathbb{E}\left[\left(\phi_y(\epsilon, x, y, B)\right)^m\right] \leq c_2. \tag{2.33}$$

The procedure to obtain the probabilistic results of (2.32) and (2.33) will be leveraged in our proof below concerning local analytic properties.

Lemma 2.2.3. *Let* $x, x', y, y' \in \boldsymbol{R}$*,* $\epsilon \in [0,1)$*,* $B \subseteq [0, \frac{1}{2}] \times [\frac{1}{2}, 1]^2_{\leq}$*. For any positive integer* m *there exists* $\xi > 0$ *and* $c_m(\xi)$ *such that*

$$\|\phi(\epsilon, x, y, B) - \phi(\epsilon', x', y', B)\|_m \leq c_m(\xi) \, \|(\epsilon, x, y) - (\epsilon', x', y')\|^{\xi} \tag{2.34}$$

and, additionally,

$$\|\phi_y(\epsilon, x, y, B) - \phi_y(\epsilon', x', y', B)\|_m \leq c_m(\xi) \, \|(\epsilon, x, y) - (\epsilon', x', y')\|^{\xi} \tag{2.35}$$

In other words, both functionals are Hölder continuous in L^m*.*

Upon establishing (2.34) and (2.35), the hypothesis of Kolmogorov's Continuity Criterion is satisfied. Therefore, we are certain that the convergence of

$$\lim_{\epsilon \to 0} \phi(\epsilon, x, y, B) = \phi(x, y, B) \quad \text{and} \quad \lim_{\epsilon \to 0} \phi_y(\epsilon, x, y, B) = \phi_y(x, y, B)$$

holds locally uniformly in (x, y), with the limit functions continuous in (x, y). We address the set variable B in Lemma 2.2.5 below.

As we remarked above, a scaling argument will be utilized to attain the following.

Lemma 2.2.4. *For any arbitrary bounded region B of $D(2)$, both $\phi(\epsilon, x, y, B)$ and $\phi_y(\epsilon, x, y, B)$ are locally Hölder continuous in L^m.*

Having addressed the spatial variables x and y, we shift focus to the set variable.

Lemma 2.2.5. *For any $B \subset D(2)$, there exists $\xi, \tilde{\xi} > 0$ and constants $c_m(\xi), c_m(\tilde{\xi})$ independent of B such that*

$$\mathbb{E}\left[\left(\phi(\epsilon, x, y, B)\right)^m\right] \leq c_m(\xi)|B|^{m\xi} \tag{2.36}$$

$$\mathbb{E}\left[\left(\phi_y(\epsilon, x, y, B)\right)^m\right] \leq c_m(\tilde{\xi})|B|^{m\tilde{\xi}}. \tag{2.37}$$

At the end of section 3.4 we will show how to use the above results to prove continuity in T.

Chapter 3

The Case of Brownian Motion

In this chapter, we establish the proofs of our main results of section 2.2 for one-dimensional Brownian motion, which we denote throughout this chapter as W_t. To set the framework for this case, recall the inverse Fourier transforms of equations (2.8) and (2.9),

$$\begin{aligned} p_\epsilon(x) &= \frac{1}{2\pi}\int_{\mathbf{R}} \exp(iqx)\exp(-\epsilon q^2/2)\ \mathrm{d}q \\ p'_\epsilon(y) &= \frac{1}{2\pi}\int_{\mathbf{R}} iq\exp(iqy)\exp(-\epsilon q^2/2)\ \mathrm{d}q. \end{aligned}$$

3.1 Principal estimate of the finite moments

Lemma 2.2.2 involves the m^{th} moment of our random variables. We initiate our study with the y derivative and aim to establish a bound on the expected value

$$\mathbb{E}\left[\left(\phi_y(\epsilon, x, y, B)\right)^m\right], \tag{3.1}$$

uniform in $x, y, B \subseteq [0, \frac{1}{2}] \times [\frac{1}{2}, 1]^2_{\leq}$ and $\epsilon > 0$. Recalling the value of $\phi_y(\epsilon, x, y, B)$ in (2.23),

$$\phi_y(\epsilon, x, y, B) = \iiint\limits_B p_\epsilon(W_s - W_r - x)\{p'_\epsilon(W_t - W_s - y)\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t,$$

we can write (3.1) as

$$\mathbb{E}\left[\left(\phi_y(\epsilon, x, y, B)\right)^m\right] = \frac{i^m}{(2\pi)^{2m}} \iiint\limits_{\mathbf{R}^{2m} \times B^m} \mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_j W(r_j, s_j)\right) \prod_{j=1}^m \left\{\exp\left(iq_j W(s_j, t_j)\right)\right\}_0\right]$$
$$\cdot F(\epsilon, x, y, p, q) \prod_{j=1}^m q_j \, \mathrm{d}p_j \, \mathrm{d}q_j \, \mathrm{d}r_j \, \mathrm{d}s_j \, \mathrm{d}t_j, \tag{3.2}$$

with

$$F(\epsilon, x, y, p, q) \doteq \exp\left(ix\sum_{j=1}^m p_j\right) \exp\left(iy\sum_{j=1}^m q_j\right) \prod_{j=1}^m \hat{f}(\epsilon, p_j) \, \hat{f}(\epsilon, q_j) \tag{3.3}$$

and

$$\hat{f}(\epsilon, a_j) \doteq \exp(-\epsilon a_j^2/2).$$

We simplified notation in (3.2) by setting $W(a_j, b_j) = (W_{b_j} - W_{a_j})$.

Observe, the primary multiple integral of interest can be broken down into the following sum

$$\sum_{\pi^1, \pi^2, \pi^3} \left[\frac{i^m}{(2\pi)^{2m}} \iiint\limits_{\mathbf{R}^{2m} \times B_{m,\pi}} \mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_{\pi_j} W(r_{\pi_j}, s_{\pi_j})\right) \prod_{j=1}^m \left\{\exp\left(iq_{\pi_j} W(s_{\pi_j}, t_{\pi_j})\right)\right\}_0\right]\right.$$
$$\left. \cdot F(\epsilon, x, y, p, q) \prod_{j=1}^m q_j \, \mathrm{d}p_j \, \mathrm{d}q_j \, \mathrm{d}r_j \, \mathrm{d}s_j \, \mathrm{d}t_j\right], \tag{3.4}$$

with

$$B_{m,\pi} = \left\{[0, 1/2, 1/2] < (r_{\pi^1_1}, s_{\pi^2_2}, t_{\pi^3_3}) < \cdots < (r_{\pi^1_m}, s_{\pi^2_m}, t_{\pi^3_m}) < [1/2, 1, 1]\right\}$$

and where π^1, π^2, and π^3 each represent permutations of the set $\{1, ..., m\}$. Therefore, without loss of generality, we can analyze (3.2) in which the time variables are sorted for a fixed order. Understanding the impact of this reconciliation and examining all orderings of the time parameters forms the crux of our first proof. To obtain this insight, we begin by ordering the (r, s, t) terms in an expression which will be utilized in Sections 3.1.1 and 3.1.2,

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, s_j)\right)\exp\left(i\sum_{j=1}^{m} q_j W(s_j, t_j)\right)\right]. \tag{3.5}$$

We end this preliminary discussion by introducing terminology which will be used throughout the analysis of the intersection local times.

The independent increments of Brownian motion allows us to write the above as

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, 1/2)\right)\right]\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(1/2, s_j) + i\sum_{j=1}^{m} q_j W(s_j, t_j)\right)\right]. \tag{3.6}$$

To evaluate (3.6) we reorder the $r, s,$ and t time variables in increasing order. Analyzing first r terms, we are led to

$$\sum_{j=1}^{m} p_j W(r_j, 1/2) = \sum_{k=1}^{m} \bar{p}_k W(\bar{r}_k, \bar{r}_{k+1}), \tag{3.7}$$

where $\{\bar{r}_1, ..., \bar{r}_{m+1}\}$ are the r_j's relabeled so that $\bar{r}_1 \leq \cdots \leq \bar{r}_{m+1} \doteq 1/2$. Expanding both sides of (3.7) and equating the W_{r_j} and $W_{\bar{r}_k}$ terms yields

$$\bar{p}_k = \sum_{\{j: r_j \leq \bar{r}_k\}} p_j. \tag{3.8}$$

Rearranging the terms in the above equation implies that every p_j can be written as a linear combination of the $\bar{p}_k$'s, specifically, in more detail, as

$$p_j = \bar{p}_1, \text{ for } r_j = \bar{r}_1, \quad \text{and } \; p_j = \bar{p}_k - \bar{p}_{k-1}, \text{ for } r_j = \bar{r}_k, \; k > 1.$$

Applying a similar argument and reordering the s and t terms produces,

$$\sum_{j=1}^{m} p_j W(1/2, s_j) + \sum_{j=1}^{m} q_j W(s_j, t_j) = \sum_{k=1}^{2m} u_k W(d_k, d_{k+1}) \tag{3.9}$$

where $\{1/2 \doteq d_1, ..., d_{2m+1}\}$ are the points $\{1/2, s_1, ..., s_m, t_1, ..., t_m\}$ in increasing order. As a result, (3.9) implies that $u_1 = \sum_{j=1}^{m} p_j$ and more generally

$$u_k = \sum_{\{j: s_j > d_k\}} p_j + \sum_{\{\ell: s_\ell \le d_k < t_\ell\}} q_\ell \tag{3.10}$$

$$u_k - u_{k-1} = \begin{cases} q_\ell - p_\ell & \text{if } d_k = s_\ell \\ -q_\ell & \text{if } d_k = t_\ell. \end{cases} \tag{3.11}$$

Equation (3.10), combined with the fact that $u_{2m+1} = 0$, reveals the set $\{u_k\}$ spans $\{p_j\} \cup \{q_\ell\}$.

Inserting equations (3.7) and (3.9) into expression (3.6) gives us

$$\begin{aligned} &\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, s_j)\right)\exp\left(i\sum_{j=1}^{m} q_j W(s_j, t_j)\right)\right] \\ &\quad = \mathbb{E}\left[\exp\left(i\sum_{k=1}^{m} \bar{p}_k W(\bar{r}_k, \bar{r}_{k+1})\right)\right]\mathbb{E}\left[\exp\left(i\sum_{k=1}^{2m} u_k W(d_k, d_{k+1})\right)\right] \\ &\quad = \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{k=1}^{2m} u_k^2(\tilde{d}_k)\right), \end{aligned} \tag{3.12}$$

where, for the last equality, we used the definition of the characteristic function of Brownian motion and substituted the following,

$$\tilde{r}_k \doteq \bar{r}_{k+1} - \bar{r}_k \tag{3.13}$$

$$\tilde{d}_k \doteq d_{k+1} - d_k. \tag{3.14}$$

We now introduce some terminology associated to the reordered s and t terms.

Definition 3.1.1. *If the interval $[s_\ell, t_\ell] = [d_k, d_{k+1}]$, for some k, we will refer to both $[s_\ell, t_\ell]$ and $[d_k, d_{k+1}]$ as **isolated intervals**. Intervals which are not isolated will be referred to as **non-isolated intervals**. In more detail, $[s_\ell, t_\ell]$ is non-isolated if $s_\ell = d_k$ and $t_\ell = d_{k+m}$ for some $m > 1$. Furthermore, we will identify $q_{\ell_{(k)}}$ as the **isolated variable** (**non-isolated variable**) associated with the isolated interval (non-isolated interval) $[s_\ell, t_\ell] = [d_k, d_{k+1}]$.*

Figure 3.1 below provides examples of these definitions.

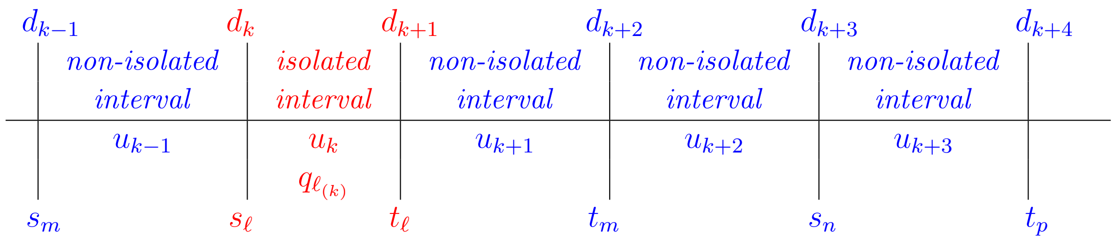

Figure 3.1: Examples of isolated and non-isolated intervals .

We will need to introduce some set notation to identify and index the non-isolated/isolated intervals with respect to both the time variables $\{(s_1, t_1), ..., (s_m, t_m)\}$ and the ordered set $\{d_1, ..., d_{2m+1}\}$. Let

$$I_j \doteq \{j \mid [s_j, t_j] \text{ is isolated}\}, \quad I_j^c \doteq \{j \mid [s_j, t_j] \text{ is non-isolated}\}, \tag{3.15}$$

$$I \doteq \{k \mid [d_k, d_{k+1}] \text{ is isolated}\}, \quad I^c \doteq \{k \mid [d_k, d_{k+1}] \text{ is non-isolated}\}. \tag{3.16}$$

Observe the cardinality of the sets $|I| + |I^c| = 2m$ and $|I_j| + |I_j^c| = m$. The cardinality of $|I| + |I^c| \neq 2m + 1$, since $d_1 \doteq 1/2$ in (3.9). Hence, the interval $[d_1, d_2]$ is neither a non-isolated interval nor an isolated interval. We will let $I_j^{c_1}$ and $I_j^{c_2}$ denote a disjoint union of I_j^c.

Additionally, define the sets

$$S \doteq \{k \mid d_k = s_\ell \text{ for some } \ell\} \tag{3.17}$$

$$T \doteq \{k \mid d_k = t_\ell \text{ for some } \ell\} \tag{3.18}$$

and note $|S \cup T| = 2m$. Introduce the subsequent set notation

$$\{u_k\}_A \doteq \{u_k \mid k \in A\}. \tag{3.19}$$

We will say that the elements of the set $\{u_k\}_S$ are from an s-interval and elements from the set $\{u_k\}_T$ are from a t-interval. Figure 3.2 below illustrates this terminology.

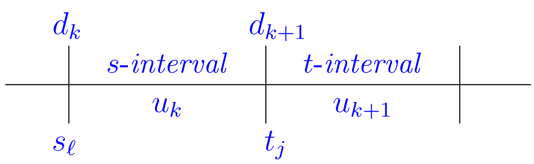

Figure 3.2: Illustration of an s and t interval .

Combining the above nomenclature permits us to identify non-isolated and isolated intervals originating from s and t-intervals.

In some arguments, we will need to select certain elements from the sets $\{1, ..., m\}$ and $\{1, ..., 2m\}$, for some fixed m. To facilitate this, the index sets defined below will appear throughout the work

$$\mathcal{M} \doteq \{1, 2, ..., m\} \tag{3.20}$$

$$\mathcal{M}_2 \doteq \{1, ..., 2m\} \tag{3.21}$$

Having studied (3.5) and introduced the notions of isolated and non-isolated intervals, let us return our attention to (3.2). To bound this quantity, we must consider two cases and the first involves scenarios in which the reordering of the s and t variables in (3.9) returns only non-isolated intervals. On the other hand, the reordering of these terms could potentially yield both isolated and non-isolated intervals and this is the second case which we must analyze. As we will see below in the following sections, the two cases result in separate and interesting analyses.

In the course of our proofs, the letter c will represent a constant which can vary from line to line.

3.1.1 Analysis of $\phi_y(\epsilon, x, y, B)$ for the non-isolated intervals

The proof over the non-isolated intervals begins with expanding the product in the expected value of (3.4). This produces an expression of the form

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_{\pi_j} W(r_{\pi_j}, s_{\pi_j})\right)\exp\left(i\sum_{j=1}^{m} q_{\pi_j} W(s_{\pi_j}, t_{\pi_j})\right)\right] \tag{3.22}$$

$$+\sum\left(\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_{\pi_j} W(r_{\pi_j}, s_{\pi_j})\right)\exp\left(i\sum_{I_j^{c1}} q_{\pi_j} W(s_{\pi_j}, t_{\pi_j})\right)\right]\right.$$

$$\left.\cdot(-1)^{m_2}\prod_{I_j^{c2}}\mathbb{E}\left[\exp\left(iq_{\pi_j} W(s_{\pi_j}, t_{\pi_j})\right)\right]\right), \tag{3.23}$$

where $I_j^{c_1}$ and $I_j^{c_2}$ form a disjoint union of the set $\{1, ..., m\}$ and $|I_j^{c_2}| = m_2$. We first establish techniques which bound the expression, (3.22), and briefly outline how to apply these methods to the remaining summands arising in (3.23).

Inserting (3.12) into (3.2), we begin with the integral

$$\frac{i^m}{(2\pi)^{2m}} \iiint\limits_{\mathbf{R}^{2m}\times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} u_k^2(\tilde{d}_k)\right) \cdot F(\epsilon, x, y, p, q) \prod_{j=1}^{m} q_j \, \mathrm{d}p_j \, \mathrm{d}q_j \, \mathrm{d}r_j \, \mathrm{d}s_j \, \mathrm{d}t_j. \tag{3.24}$$

Observing the trivial inequalities,

$$\left| \frac{i^m}{(2\pi)^{2m}} \right| \leq c^m \tag{3.25}$$

and

$$|F(\epsilon, x, y, p, q)| \leq 1 \tag{3.26}$$

we can bound the integrand in (3.24) by

$$c^m \iiint\limits_{\mathbf{R}^{2m}\times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} u_k^2(\tilde{d}_k)\right) \prod_{j=1}^{m} |q_j| \, \mathrm{d}p_j \, \mathrm{d}q_j \, \mathrm{d}r_j \, \mathrm{d}s_j \, \mathrm{d}t_j. \tag{3.27}$$

To integrate the time variables, r_j, s_j, and t_j, notice that a consequence of (3.13) and (3.14) is that $\{\tilde{r}_k\}$ generates the set $\{r_j\}$ and $\{\tilde{d}_k\}$ generates $\{s_j\} \cup \{t_j\}$. Applying these facts, along with the property that for any $u \in \mathbf{R}$ and $\alpha \in \mathbf{R}_+$,

$$\int_0^1 \exp\left(-x|u|^\alpha\right) \mathrm{d}x = \frac{1 - \exp\left(-|u|^\alpha\right)}{|u|^\alpha} \leq \frac{c}{1 + |u|^\alpha} \tag{3.28}$$

reveals (3.27) is bounded above by

$$c \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1+\bar{p}_k^2\right)^{-1} \prod_{k=1}^{2m} \left(1+u_k^2\right)^{-1} \prod_{j=1}^{m} \, |q_j| \, \mathrm{d}p_j \, \mathrm{d}q_j. \tag{3.29}$$

To prove this multiple integral is finite, we focus on $\prod_j |q_j|$. and recall that equation (3.11) allows us to write each q_j in terms of u_k and u_{k-1}, depending on the value of d_k. Let us reexpress each q_j appearing in (3.29) via the following relationship,

$$u_{k-1} - u_k = q_\ell \quad \text{for } d_k = t_\ell, \tag{3.30}$$

so that the product involving the $|q_j|$ terms can be bounded above by

$$\prod_{j=1}^{m} \, |q_j| = \prod_{k\in T} |u_{k-1} - u_k| \leq \prod_{k\in T} \left(|u_{k-1}| + |u_k|\right). \tag{3.31}$$

We now investigate the polynomial which results from the expansion of the right hand side of the inequality above. In the course of our analysis, we will discover a dependency between the degree of the $|u_k|$ terms of this polynomial and the s-intervals and t-intervals from which they originate. The relationship between these two notions will be critical in establishing our desired bound. To facilitate our analysis and as we will see in the proof of Proposition 3.1.2 below, the polynomial which emerges from the expansion of (3.31) will assume the general form

$$\sum \left(\prod_S |u_k|^{\omega_k} \prod_T |u_k|^{\omega_k} \right).$$

In order to investigate the orders of the above polynomial, define the following sets

$$S^{-} \doteq \{\ k \in S\ \big| k+1 \in T \text{ and } \omega_k = 1\} \tag{3.32}$$

$$T^{+} \doteq \{k \in T \mid k-1 \in S^{-}\} \tag{3.33}$$

$$S^{'} \doteq \{k \in S \mid k \notin S^{-}\} \tag{3.34}$$

$$T^{'} \doteq \{k \in T \mid k \notin T^{+}\}. \tag{3.35}$$

Note, $S^{-} \cup S^{'}$ and $T^{+} \cup T^{'}$ form a disjoint union of the sets S and T, defined in (3.17) and (3.18), respectively. In other words, these sets account for all of the non-isolated s and t-intervals and $|S^{-} \cup S^{'} \cup T^{+} \cup T^{'}| = 2m$. Figure 3.3 below provides an example and the motivations for the introduction of these sets becomes clearer in Proposition 3.1.2 below.

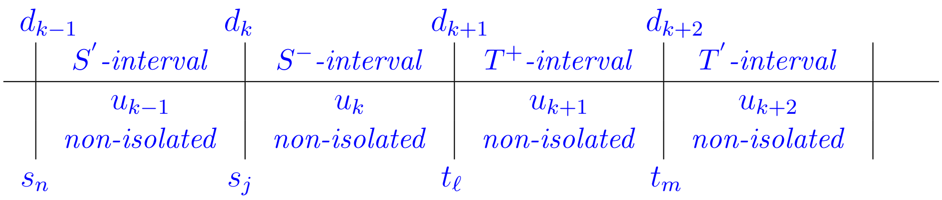

Figure 3.3: Examples of $S^{'}, S^{-}, T^{+}$ and $T^{'}$ intervals .

Proposition 3.1.2. *Expansion of the product*

$$\prod_{k\in T}\left(|u_{k-1}|+|u_k|\right) \tag{3.36}$$

generates a polynomial of the form

$$\mathcal{P}_1(|u_k|) \doteq \prod_{k\in T}(|u_{k-1}|+|u_k|) = \sum \left(\prod_{S'}|u_k|^{\omega_k}\prod_{S^-}|u_k|^{\omega_k}\prod_{T^+}|u_k|^{\omega_k}\prod_{T'}|u_k|^{\omega_k}\right), \tag{3.37}$$

where

$$deg\ |u_k| = \omega_k = \begin{cases} 0, & if\ k \in S' \\ 1, & if\ k \in S^- \\ 0\ \ or\ 1, & if\ k \in T^+ \\ 0, 1,\ \ or\ 2, & if\ k \in T'. \end{cases} \tag{3.38}$$

Proof. We begin the proof by focussing on the orders of the u_k terms as they relate to the sets S and T, defined in (3.17) and (3.18), respectively. Since we are focussing on the non-isolated intervals, it is important to notice equation (3.30) implies that $\deg|u_k| \le 2$, when $k \in T$. This fact stems from cases in which $k \in T$ and $k+1 \in T$. Specifically, if $d_k = t_\ell$ and $d_{k+1} = t_j$, for some ℓ and j, we then have

$$q_\ell = u_{k-1} - u_k \tag{3.39}$$

$$q_j = u_k - u_{k+1}. \tag{3.40}$$

Hence, u_k appears as a summand for both q_ℓ and q_j and the polynomial resulting from the expansion of (3.36) will be of maximum order two. The ensuing function can be written as

$$\mathcal{P}_1(|u_k|) = \sum \left(\prod_S |u_k|^{\omega_k}\prod_T |u_k|^{\omega_k}\right), \tag{3.41}$$

with ω_k taking values in $\{0, 1, 2\}$.

Consider, now, the values of ω_k, for $k \in S$. Begin by noticing elements from $\{u_k\}_S$ appear in (3.41) only if $k-1 \in S$ and $k \in T$. Specifically, for these cases, when $d_k = t_\ell$ (3.30) implies

$$q_\ell = u_{k-1} - u_k. \tag{3.42}$$

The u_{k-1} appearing above is from an s-interval and can not appear as a summand for another value of q. Therefore, upon expanding (3.36), we can conclude the values of $\{\omega_k\}_S$ are at most equal to 1.

We have superficially identified the degree of the $|u_k|$ variables of (3.41) and look to delve deeper into the matter. To do so, consider any product which appears as a summand in this polynomial

$$\prod_S |u_k|^{\omega_k} \prod_T |u_k|^{\omega_k} \tag{3.43}$$

The definitions of the sets S^- and $S^{'}$, combined with our discussion above regarding the values of $\{\omega_k\}_S$, imply the elements of $\{\omega_k\}_{S^-}$ and $\{\omega_k\}_{S'}$ are equal to 1 and 0, respectively. The product appearing in (3.43) can be reexpressed as

$$\prod_S |u_k|^{\omega_k} \prod_T |u_k|^{\omega_k} = \prod_{S^-} |u_k| \prod_{T^+} |u_k|^{\omega_k} \prod_{T'} |u_k|^{\omega_k}. \tag{3.44}$$

To investigate the values of $\{\omega_k\}_{T^+}$, consider (3.30) when $d_k = t_\ell$ for $k \in T^+$. The u_{k-1} and u_k appearing in the summand for q_ℓ are from a s-interval and t-interval, respectively. Upon expanding (3.36), u_{k-1} and u_k can not appear in the same term of the polynomial. However, from our earlier discussion, we know that u_k can appear at most once as a summand for a q. In other words, for those values of k such that $k \in T^+$ and $k+1 \in T^{'}$. From this, we can deduce that ω_k is equal to 0 or 1 for $k \in T^+$.

Take into account the values of $\{\omega_k\}_{T'}$. If $k \in T^{'}$ and $k+1 \in T^{'}$, then (3.30) reveals u_k appears in the summand for two values of q and hence we can conclude $\{\omega_k\}_{T'}$ is at most equal to 2. With the assistance of (3.44), (3.41) can be rewritten in more detail

$$\mathcal{P}_1(|u_k|) = \sum \left(\prod_{S'} |u_k|^{\omega_k} \prod_{S^-} |u_k|^{\omega_k} \prod_{T^+} |u_k|^{\omega_k} \prod_{T'} |u_k|^{\omega_k} \right)$$

and we close the proof. □

Our next step is to apply the above proposition to multiple integral (3.29). Before proceeding with that step, we establish key properties regarding the u variables linked to the sets S', S^-, and T^+ in Proposition 3.2.3 below. Before delving into the proof, it is helpful to recall that equation (3.8) permits us to express each $\bar{p}_k$ in terms of p_j via

$$\bar{p}_k = \sum_{\{j: r_j \leq \bar{r}_k\}} p_j.$$

and any u_k can be written in terms of p_j and q_ℓ by relation (3.10),

$$u_k = \sum_{\{j: s_j > d_k\}} p_j + \sum_{\{\ell: s_\ell \leq d_k < t_\ell\}} q_\ell,$$

Proposition 3.1.3. *Let $\{q_j\}$ represent a set of non-isolated variables associated with the set I_j^c of* (3.15)*. Let $|\{\bar{p}_k\}| = m$ and $|I_j^c| = n$ where $m, n > 0$ and $n \leq m$. With the sets $S^{'}$, T^+, and S^- defined in* (3.32)-(3.34) *, respectively, we have the following;*

i.) the set $\{\bar{p}_k\} \cup \{u_k\}_{S'} \cup \{u_k\}_{S^-}$ forms a basis for $\{p_j\} \cup \{q_j\}$; and

ii.) the set $\{\bar{p}_k\} \cup \{u_k\}_{S'} \cup \{u_k\}_{T^+}$ forms a basis for $\{p_j\} \cup \{q_j\}$.

Proof. i.) Begin by noting $S^{'}$ and S^- form a disjoint union of S. The cardinality of the sets

$$|\{\bar{p}_k\}| = |\{p_j\} = m \quad \text{and} \quad |\{u_k\}_S|| = |\{q_j\}| = n$$

reveals,

$$|\{\bar{p}_k\} \cup \{u_k\}_S| = |\{p_j\} \cup \{q_j\}| = m + n.$$

Rearranging the terms in (3.8) and solving for each p_j entails $\{\bar{p}_k\}$ generates $\{p_j\}$.

Let $\{k_1, k_2, ..., k_n\}$ be the elements of the set S, such that $k_1 < k_2 < \cdots < k_n$. Via (3.10), appearing in the summand of u_{k_1} are those q_j, where j is such that $s_j \leq d_{k_1} < t_j$. Proceeding sequentially through the ordered set $\{u_{k_1}, u_{k_2}, ..., u_{k_n}\}$ and noting the values of q_j which appear as summands for each u_k we see each q_j appears as a summand for a u_k term one at a time. This implies each q_j can be written as a sum involving terms from $\{\bar{p}_k\}$ and $\{u_k\}_S$. Hence, upon rearranging terms and solving for q_j, we can conclude $\{\bar{p}_k\} \cup \{u_k\}_S$ generates $\{p_j\} \cup \{q_j\}$. Since $|\{\bar{p}_k\} \cup \{u_k\}_S|$ is finite, the set forms our required basis.

ii.) We have addressed the sets $\{\bar{p}_k\}$ and $\{p_j\}$ in the proof of *i.)*. By the definition of S^- it follows that $|S^-| = |T^+| = \tilde{n}$, with $\tilde{n} < n$. In turn, this implies

$$|\{\bar{p}_k\} \cup \{u_k\}_{S'} \cup \{u_k\}_{T^+}| = |\{p_j\} \cup \{q_j\}| = m + n.$$

To establish our result, we mimic the technique used in the proof of *i.)* above. Let $\{k_1, k_2, ..., k_n\}$ be the elements of the set $S^{'} \cup T^+$, in their natural order. Proceed sequentially through the ordered set $\{u_{k_1}, u_{k_2}, ..., u_{k_n}\}$ and reexpress each u_k in terms of p_j and q_j through (3.10). Not every q_j will appear as a summand in u_k, $k \in S^{'}$ and we explain how these variables appear as a summand in u_k, $k \in T^+$.

From the definition of the set T^+, the interval preceding a T^+-interval is a S^--interval.

Consider any element $u_k \in \{u_k\}_{T^+}$ with $d_{k-1} = s_v,\ d_k = t_w,\ k-1 \in S^-,\ k \in T^+$ and $[s_v, t_w]$ a non-isolated interval. As a consequence of the non-isolated intervals, appearing as summand for u_k will be a q_j term associated with the preceding S^--interval such that $j = v$. These are the precise q_j's which do not appear as a summand in u_k, $k \in S^{'}$.

Upon reexpressing each term in the set $\{u_{k_1}, u_{k_2}, ..., u_{k_n}\}$, we discover each q appears at least once as a summand for a $\{u_k\}$. As a result, each q_j can be expressed as a sum involving terms from $\{\bar{p}_k\}$ and $\{u_k\}_{S'} \cup \{u_k\}_{T^+}$. Hence $\{\bar{p}_k\} \cup \{u_k\}_{S'} \cup \{u_k\}_{T^+}$ generates $\{p_j\} \cup \{q_j\}$. The fact that $|\{\bar{p}_k\} \cup \{u_k\}_{S'} \cup \{u_k\}_{T^+}|$ is finite, yields our desired result. $\square$

Returning to the primary expression, inserting $\mathcal{P}_1(|u_k|)$ into (3.29) produces the sum

$$\sum \left[\iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} (1+\bar{p}_k^2)^{-1} \prod_{S'} \frac{1}{1+u_k^2} \prod_{S^-} \frac{|u_k|}{1+u_k^2} \prod_{T^+} \frac{|u_k|^{\omega_k}}{1+u_k^2} \prod_{T'} \frac{|u_k|^{\omega_k}}{1+u_k^2} \prod_{j=1}^{m} \mathrm{d}p_j \ \mathrm{d}q_j \right]. \tag{3.45}$$

Since $|u_k|^{\omega_k}(1+u_k^2)^{-1} \leq c$, for all $k \in T^{'}$, any multiple integral appearing as a summand in (3.172) can be bounded above by

$$c \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} (1+\bar{p}_k^2)^{-1} \prod_{S'} \frac{1}{1+u_k^2} \prod_{S^-} \frac{|u_k|}{1+u_k^2} \prod_{T^+} \frac{|u_k|^{\omega_k}}{1+u_k^2} \prod_{j=1}^{m} \mathrm{d}p_j \ \mathrm{d}q_j. \tag{3.46}$$

The emergence of this expression motivates the introduction of Proposition 3.1.4. As we will see, proofs in future sections will be heavily dependent on this critical result. Additionally, we not that, in our current analysis, $|I_j^c| = m$. However, when we begin to study the case for the isolated intervals, the cardinality of I_j^c will assume values less then m. This is the motivation for the introduction of the parameter n in the statement below.

Proposition 3.1.4. *Let each of the two sets*

$$\{\bar{p}_k\} \cup \{u_k\}_{S'} \cup \{u_k\}_{S^-} \quad \textit{and} \quad \{\bar{p}_k\} \cup \{u_k\}_{S'} \cup \{u_k\}_{T^+}$$

form a basis for $\{p_j\} \cup \{q_j\}$, *where the later set is associated with the non-isolated intervals* I_j^c. *Denote the cardinality of the sets,* $|\{\bar{p}_k\}| = m$, $|I_j^c| = n$, *where* $m, n > 0$, *and* $n \leq m$. *Recall the index set* $\mathcal{M}$ *defined in* (3.20). *Consider any parameters* $a, b > 0$ *and* $w_k, x_k, y_k, z_k > 0$. *The* $m + n$ *multidimensional integral*

$$\iint\limits_{\boldsymbol{R}^m \times \boldsymbol{R}^n} \prod_{k=1}^{m} \frac{|\bar{p}_k|^{w_k}}{(1 + |\bar{p}_k|^b)^a} \prod_{S'} \frac{|u_k|^{x_k}}{(1 + |u_k|^b)^a} \prod_{S^-} \frac{|u_k|^{y_k}}{(1 + |u_k|^b)^a} \prod_{T^+} \frac{|u_k|^{z_k}}{(1 + |u_k|^b)^a} \prod_{j=1}^{m} dp_j \prod_{I_j^c} dq_j$$

is finite if each of the following inequalities are satisfied,

$$w_k < ab - 1, \text{ for all } k \in \mathcal{M}, \qquad x_k < ab - 1, \text{ for all } k \in S'$$

$$2y_k < 2ab - 1, \text{ for all } k \in S^-, \qquad 2z_k < 2ab - 1, \text{ for all } k \in T^+.$$

Proof. Simplify notation by setting

$$\mathcal{A} = \prod_{k=1}^{m} \frac{|\bar{p}_k|^w}{(1 + |\bar{p}_k|^b)^a}, \quad \mathcal{B}' = \prod_{S'} \frac{|u_k|^x}{(1 + |u_k|^b)^a}, \quad \mathcal{B}^- = \prod_{S^-} \frac{|u_k|^y}{(1 + |u_k|^b)^a}, \quad \mathcal{C}^+ = \prod_{T^+} \frac{|u_k|^z}{(1 + |u_k|^b)^a}.$$

An application of Cauchy-Schwarz inequality reveals

$$\begin{aligned}
&\iint\limits_{\mathbf{R}^{m \times n}} \mathcal{A}\, \mathcal{B}' \mathcal{B}^- \, \mathcal{C}^+ \prod_{j=1}^{m} \mathrm{d}p_j \prod_{I_j^c} \mathrm{d}q_j \\
&\quad = \iint\limits_{\mathbf{R}^{m \times n}} \left(\mathcal{A}\, \mathcal{B}'\right)^{1/2} \mathcal{B}^- \left(\mathcal{A}\, \mathcal{B}'\right)^{1/2} \mathcal{C}^+ \prod_{j=1}^{m} \mathrm{d}p_j \prod_{I_j^c} \mathrm{d}q_j \\
&\quad \leq \left[\iint\limits_{\mathbf{R}^{m \times n}} \mathcal{A}\, \mathcal{B}' \left(\mathcal{B}^-\right)^2 \prod_{j=1}^{m} \mathrm{d}p_j \; \mathrm{d}q_j\right]^{1/2} \cdot \left[\iint\limits_{\mathbf{R}^{m \times n}} \mathcal{A}\, \mathcal{B}' \left(\mathcal{C}^+\right)^2 \prod_{j=1}^{m} \mathrm{d}p_j \prod_{I_j^c} \mathrm{d}q_j\right]^{1/2}.
\end{aligned} \tag{3.47}$$

Proposition 3.2.3 indicates

$$\prod_{j=1}^{m} \mathrm{d}p_j = \prod_{j=1}^{m} \mathrm{d}\bar{p}_k, \quad \prod_{I_j^c} \mathrm{d}q_j = \prod_{S',S^-} \mathrm{d}u_k, \quad \prod_{I_j^c} \mathrm{d}q_j = \prod_{S',T^+} \mathrm{d}u_k.$$

Upon changing measures, (3.47) is equal to

$$\begin{aligned}
&\left[\iint_{\mathbf{R}^{m\times n}} \mathcal{A}\,\mathcal{B}'\,\left(\mathcal{B}^-\right)^2 \prod_{k=1}^{m} \mathrm{d}\bar{p}_k \prod_{S',S^-} \mathrm{d}u_k \right]^{1/2} \cdot \left[\iint_{\mathbf{R}^{m\times n}} \mathcal{A}\,\mathcal{B}'\,\left(\mathcal{C}^+\right)^2 \prod_{k=1}^{m} \mathrm{d}\bar{p}_k \prod_{S',T^+} \mathrm{d}u_k \right]^{1/2} \\
&= \left[\prod_{k=1}^{m} \int_{\mathbf{R}} \mathcal{A}\,\mathrm{d}\bar{p}_k \prod_{S'} \int_{\mathbf{R}} \mathcal{B}'\,\mathrm{d}u_k \prod_{S^-} \int_{\mathbf{R}} \left(\mathcal{B}^-\right)^2\,\mathrm{d}u_k \right]^{1/2} \\
&\qquad \cdot \left[\prod_{k=1}^{m} \int_{\mathbf{R}} \mathcal{A}\,\mathrm{d}\bar{p}_k \prod_{S'} \int_{\mathbf{R}} \mathcal{B}'\,\mathrm{d}u_k \prod_{T^+} \int_{\mathbf{R}} \left(\mathcal{C}^+\right)^2\,\mathrm{d}u_k \right]^{1/2}.
\end{aligned} \tag{3.48}$$

Now, consider any one dimensional integral taking the general form

$$\int_{\mathbf{R}} \frac{|x|^{\delta}}{(1+|x|^{\beta})^{\alpha}}\,\mathrm{d}x, \quad \text{with } \alpha,\beta > 0, \quad \delta \geq 0,$$

and write

$$\begin{aligned}
\int_{\mathbf{R}} \frac{|x|^{\delta}}{(1+|x|^{\beta})^{\alpha}}\,\mathrm{d}x &= \int_0^1 \frac{|x|^{\delta}}{(1+|x|^{\beta})^{\alpha}}\,\mathrm{d}x + \int_1^{\infty} \frac{|x|^{\delta}}{(1+|x|^{\beta})^{\alpha}}\,\mathrm{d}x \\
&= c + \int_1^{\infty} \frac{|x|^{\delta}}{(1+|x|^{\beta})^{\alpha}}\,\mathrm{d}x \leq c + \int_1^{\infty} \frac{|x|^{\delta}}{|x|^{\beta\alpha}}\,\mathrm{d}x = c + \int_1^{\infty} \frac{1}{|x|^{\beta\alpha-\delta}}\,\mathrm{d}x \leq c,
\end{aligned}$$

where the last integral in the last line over the domain of integration $[1,\infty)$ is finite for $1 < \beta\alpha - \delta$ or equivalently $\delta < \beta\alpha - 1$. Applying this rule to each expression emerging in (3.48), closes the argument.

$\square$

Examination of the powers of the $|\bar{p}_k|$ and $|u_k|$ variables emerging in the numerators of each of the rational functions in (3.46) reveals that the conditions of Proposition 3.1.4 are satisfied to conclude this aforementioned quantity is finite.

We illustrate how to derive an upper bound for the remaining integrals materializing as summands in (3.23),

$$\frac{i^m}{(2\pi)^{2m}} \iiint\limits_{\mathbf{R}^{2m}\times B^m} \mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, s_j)\right)\exp\left(i\sum_{I_j^{c1}} q_j W(s_j, t_j)\right)\right]$$
$$\prod_{I_j^{c2}} \mathbb{E}\left[\exp\left(iq_j W(s_j, t_j)\right)\right] F(\epsilon, x, y, p, q) \prod_{j=1}^{m} q_j \,\mathrm{d}p_j \,\mathrm{d}q_j \,\mathrm{d}r_j \,\mathrm{d}s_j \,\mathrm{d}t_j. \quad (3.49)$$

Start by replicating the notions which equates (3.5) to (3.6),

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, s_j)\right)\exp\left(i\sum_{I_j^{c1}} q_j W(s_j, t_j)\right)\right]$$
$$= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, 1/2)\right)\right]\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(1/2, s_j) + i\sum_{I_j^{c1}} q_j W(s_j, t_j)\right)\right],$$

and reorder the r terms to arrive at (3.8). Arrange the entire set of time variables, $\{1/2, s_1, ..., s_m, t_1, ..., t_m\}$, in increasing order and, as before, denote this new set by $\{1/2 \doteq d_1, ..., d_{2m+1}\}$. By setting

$$\sum_{j=1}^{m} p_j W(1/2, s_j) + \sum_{I_j^{c1}} q_j W(s_j, t_j) = \sum_{k=1}^{2m} \bar{u}_k W(d_k, d_{k+1}), \quad (3.50)$$

and equating the Brownian terms we can conclude

$$\bar{u}_1 = \sum_{j=1}^{m} p_j \ , \quad \bar{u}_k \ = \sum_{\{j:s_j>d_k\}} p_j + \sum_{\{\ell\in I_j^{c1}:s_\ell\le d_k<t_\ell\}} q_\ell \ , \quad \bar{u}_{2m+1} = 0,$$

$$\bar{u}_k - \bar{u}_{k-1} \ = \begin{cases} q_\ell - p_\ell & \text{if } d_k = s_\ell \text{ and } \ell \in I_j^{c_1} \\ -q_\ell & \text{if } d_k = t_\ell \text{ and } \ell \in I_j^{c_1}. \end{cases}$$

The above indicate $\{\bar{u}_k\}$ spans $\{p_j\} \cup \{q_\ell\}$, for those q_ℓ with $\ell \in I_j^{c_1}$.

Gathering these facts and applying (3.25) and (3.26), the primary integral of concern is

$$c^m \iiint_{\mathbf{R}^m\times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} \bar{u}_k^2(\tilde{d}_k)\right) \cdot \exp\left(-\sum_{I_j^{c2}} q_j^2(t_j - s_j)\right) \prod_{j=1}^{m} |q_j| \ \mathrm{d}p_j \ \mathrm{d}q_j \ \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j. \tag{3.51}$$

To advance our argument, we examine in more detail the product

$$\exp\left(-\sum_{k=1}^{2m} \bar{u}_k^2(\tilde{d}_k)\right) \exp\left(-\sum_{I_j^{c2}} q_j^2(t_j - s_j)\right). \tag{3.52}$$

Observe, any interval (s_j, t_j), with $j \in I_j^{c_2}$, is a non-isolated interval which means $s_j = d_k$ and $t_j = d_{k+n}$, for $n \geq 2$. To identify the d_k variables which lie between s_j and t_j, set

$$\widetilde{\mathcal{K}}_j \doteq \{k \mid s_j \leq d_k < t_j\}. \tag{3.53}$$

For each $j \in I_j^{c_2}$, the difference, $t_j - s_j$, can be expressed by means of the $\tilde{d}_k$ variables through

$$t_j - s_j = \sum_{k\in\widetilde{\mathcal{K}}_j} (d_{k+1} - d_k) = \sum_{k\in\widetilde{\mathcal{K}}_j} \tilde{d}_k.$$

Since there is at least one s-interval or t-interval between s_j and t_j, the sum on the right hand side is comprised of at least two $\tilde{d}_k$ terms. Incorporating the above and recalling the index set defined in (3.21), the exponential functions in (3.52) can be written as

$$
\begin{aligned}
&\exp\left(-\sum_{k=1}^{2m} \bar{u}_k^2(\tilde{d}_k)\right)\exp\left(-\sum_{I_j^{c2}} q_j^2(t_j - s_j)\right) \\
&= \exp\left(-\sum_{k=1}^{2m} \bar{u}_k^2(\tilde{d}_k)\right)\exp\left(-\sum_{I_j^{c2}} q_j^2\Big(\sum_{k\in\widetilde{\mathcal{K}}_j} \tilde{d}_k\Big)\right) \\
&= \exp\left(-\sum_{k=1}^{2m} \bar{u}_k^2(\tilde{d}_k)\right)\exp\left(-\sum_{k}\Big(\sum_{j\in I_j^{c2}} q_j^2\Big)(\tilde{d}_k)\right) \\
&= \exp\left(-\sum_{\mathcal{M}_2/\widetilde{\mathcal{K}}_j} \bar{u}_k^2(\tilde{d}_k)\right)\exp\left(-\sum_{k}\Big(\sum_{\{j|k\in\widetilde{\mathcal{K}}_j\}} q_j^2 + \bar{u}_k^2\Big)(\tilde{d}_k)\right). \qquad (3.54)
\end{aligned}
$$

Consider $\beta \geq 1$. If we equate (3.9) with (3.50) , then two important facts follow. For any $k \in \mathcal{M}_2/\widetilde{\mathcal{K}}_j$, $u_k(\tilde{d}_k) = \bar{u}_k(\tilde{d}_k)$ and $|u_k|^\beta = |\bar{u}_k|^\beta$, for any $\beta \in \mathbf{R}$. Second, with $k \in \widetilde{\mathcal{K}}_j$,

$$
u_k = \sum_{\{j|k\in\widetilde{\mathcal{K}}_j\}} q_j + \bar{u}_k \quad \text{and} \quad u_k(\tilde{d}_k) = \Big(\sum_{\{j|k\in\widetilde{\mathcal{K}}_j\}} q_j + \bar{u}_k\Big)(\tilde{d}_k)
$$

With $m_2 = |I_j^{c2}|$ and $\beta > 0$, this implies

$$
|u_k|^\beta = \Big|\sum_{\{j|k\in\widetilde{\mathcal{K}}_j\}} q_j + \bar{u}_k\Big|^\beta \leq c\Big[\ |m_2+1|^{\beta-1}\sum_{\{j|k\in\widetilde{\mathcal{K}}_j\}} |q_j|^\beta + |\bar{u}_k|^\beta\ \Big]
$$

The last upper bound holds since Hölder's inequality implies

$$
\frac{1}{n}\Big|\sum_{j=1}^{n} x_j\Big| \leq \frac{1}{n}\sum_{j=1}^{n}|x_j| \leq \Big(\frac{1}{n}\sum_{j=1}^{n}|x_j|^\beta\Big)^{1/\beta}
$$

which leads to

$$\left|\sum_{j=1}^{n} x_j\right|^{\beta} \le n^{\beta-1}\sum_{j=1}^{n}|x_j|^{\beta}.$$

Collecting these facts, (3.54) can be bounded above by

$$\begin{aligned}
&\exp\left(-\sum_{\mathcal{M}_2/\widetilde{\mathcal{K}}_j}\bar{u}_k^2(\tilde{d}_k)\right)\exp\left(-\sum_{k}\left(\sum_{\{j|k\in\widetilde{\mathcal{K}}_j\}}q_j^2+\bar{u}_k^2\right)(\tilde{d}_k)\right)\\
&\quad\le \exp\left(-\sum_{\mathcal{M}_2/\widetilde{\mathcal{K}}_j}\bar{u}_k^2(\tilde{d}_k)\right)\exp\left(-\sum_{k\in\widetilde{\mathcal{K}}_j}u_k^2(\tilde{d}_k)\right)=\exp\left(-\sum_{k=1}^{2m}u_k^2(\tilde{d}_k)\right)
\end{aligned}$$

Hence, upon applying the above upper bound to (3.51) we are returned to the previous analysis.

3.1.2 Analysis of $\phi_y(\epsilon, x, y, B)$ for the isolated intervals

Having completed the case for the non-isolated intervals, we discuss now a procedure to bound (3.2) over both isolated and non-isolated intervals. To motivate our plan, let $q_{\ell_{(k)}}$ be an isolated variable associated with the isolated interval $[d_k, d_{k+1}]$. The isolated intervals possess the special property that $q_{\ell_{(k)}}$ appears as a summand of u_i if and only if $i = k$. More specifically, from equations (3.11), we note that if $d_k = s_\ell$ and $d_{k+1} = t_\ell$, then

$$\begin{aligned}
u_k &= u_{k+1} + q_{\ell_{(k)}} \\
u_{k-1} &= u_{k+1} + p_{\ell_{(k)}}.
\end{aligned} \tag{3.55}$$

Figure 3.4 below illustrates this unique property and inspires the terminology of an isolated interval.

Notice the expected value appearing in (3.4) can be partitioned over isolated and non-

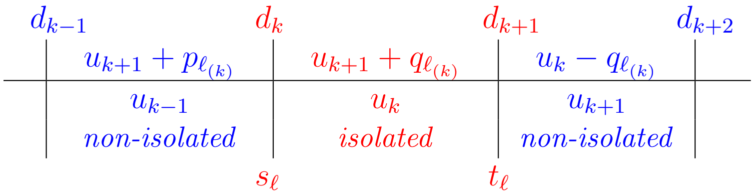

Figure 3.4: Isolated intervals and sums involving $q_{\ell_{(k)}}$.

isolated intervals so that

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_{\pi_j} W(r_{\pi_j}, s_{\pi_j})\right)\prod_{j=1}^{m}\left\{\exp\left(iq_{\pi_j}W(s_{\pi_j}, t_{\pi_j})\right)\right\}_0\right]$$
$$= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_{\pi_j} W(r_{\pi_j}, s_{\pi_j})\right)\prod_{I_j^c}\left\{\exp\left(iq_{\pi_j}W(s_{\pi_j}, t_{\pi_j})\right)\right\}_0\prod_{I_j}\left\{\exp\left(iq_{\pi_j}W(s_{\pi_j}, t_{\pi_j})\right)\right\}_0\right].$$

If we were to expand the above over brackets involving the non-isolated intervals I_j^c and leave brackets involving I_j intact, we would obtain a sum of the form

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_{\pi_j} W(r_{\pi_j}, s_{\pi_j})\right)\exp\left(i\sum_{I_j^c} q_{\pi_j} W(s_{\pi_j}, t_{\pi_j})\right)\prod_{I_j}\left\{\exp\left(iq_{\pi_j}W(s_{\pi_j}, t_{\pi_j})\right)\right\}_0\right] \tag{3.56}$$

$$+\sum\left(\mathbb{E}\Bigg[\exp\left(i\sum_{j=1}^{m} p_{\pi_j} W(r_{\pi_j}, s_{\pi_j})\right)\exp\left(i\sum_{I_j^{c1}} q_{\pi_j} W(s_{\pi_j}, t_{\pi_j})\right)\right.$$
$$\left.\prod_{I_j}\left\{\exp\left(iq_{\pi_j}W(s_{\pi_j}, t_{\pi_j})\right)\right\}_0\Bigg]\prod_{I_j^{c2}}\mathbb{E}\left[\exp\left(iq_{\pi_j}W(s_{\pi_j}, t_{\pi_j})\right)\right]\right), \tag{3.57}$$

where I_j^{c1} and I_j^{c2} form a disjoint union of I_j^c. We first derive a bound for integrals consisting of the first summand above and explain how to address (3.57) at the end of this section.

To simplify (3.56), insert the product of isolated intervals,

$$\prod_{I_j}\left\{\exp\left(iq_jW(s_j, t_j)\right)\right\}_0 = \prod_{I_j}\Big(\exp\left(iq_jW(s_j, t_j)\right) - \mathbb{E}\left[\exp\left(iq_jW(s_j, t_j)\right)\right]\Big),$$

into (3.5) and apply the subsequent analysis of that expression to arrive at

$$\exp\left(-\sum_{k=1}^{m}\bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{I^c}u_k^2(\tilde{d}_k)\right)\prod_{k\in I}\left(\exp\left(-u_k^2(\tilde{d}_k)\right)-\exp\left(-\left(u_{k+1}^2+q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right), \tag{3.58}$$

where we applied the identity $u_k = u_{k+1}+q_{\ell_{(k)}}$. Embedding the above into (3.2), we discover

$$\begin{aligned}\frac{i^m}{(2\pi)^{2m}}\iiint\limits_{\mathbf{R}^{2m}\times B^m}&\exp\left(-\sum_{k=1}^{m}\bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{I^c}u_k^2(\tilde{d}_k)\right)\\&\prod_{k\in I}\left(\exp\left(-u_k^2(\tilde{d}_k)\right)-\exp\left(-\left(u_{k+1}^2+q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right)\\&\cdot F(\epsilon,x,y,p,q)\prod_{j=1}^{m}q_j\ \mathrm{d}p_j\ \mathrm{d}q_j\ \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j.\end{aligned} \tag{3.59}$$

Use (3.25) and (3.26) times the absolute value of (3.58) to bound the above integrand,

$$\begin{aligned}c^m\iiint\limits_{\mathbf{R}^{2m}\times B^m}&\exp\left(-\sum_{k=1}^{m}\bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{I^c}u_k^2(\tilde{d}_k)\right)\\&\cdot\left|\prod_{k\in I}\left(\exp\left(-u_k^2(\tilde{d}_k)\right)-\exp\left(-\left(u_{k+1}^2+q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right)\right|\\&\prod_{j=1}^{m}|q_j|\ \mathrm{d}p_j\ \mathrm{d}q_j\ \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j.\end{aligned} \tag{3.60}$$

The strategy to prove (3.60) is finite is as follows. We first focus on the isolated intervals and the isolated variables. This leads us to focus on the absolute value appearing in the above integrand,

$$\iint\limits_{\mathbf{R}\times B}\left|\exp\left(-u_k^2(\tilde{d}_k)\right)-\exp\left(-\left(u_{k+1}^2+q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right|\ |q_{\ell_{(k)}}|\ \mathrm{d}(\tilde{d}_k)\ \mathrm{d}q_{\ell_{(k)}}. \tag{3.61}$$

Notice we are first integrating the $\tilde{d}_k$ terms and, then, the isolated variables $q_{\ell_{(k)}}$. In more detail, these $\tilde{d}_k$ terms, defined in (3.14) as $(d_{k+1} - d_k)$, are isolated intervals associated to the isolated variables $q_{\ell_{(k)}}$. After integrating these terms, we, naturally, shift focus to the non-isolated variables.

To begin, substitute (3.55) for the u_k terms appearing in (3.61),

$$\begin{aligned}
&\left|\exp\left(-u_k^2(\tilde{d}_k)\right) - \exp\left(-\left(u_{k+1}^2 + q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right| \\
&\qquad = \left|\exp\left(-\left(u_{k+1} + q_{\ell_{(k)}}\right)^2(\tilde{d}_k)\right) - \exp\left(-\left(u_{k+1}^2 + q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right|.
\end{aligned}$$

Simplify notation by setting $u_{k+1} = u$ and $q_{\ell_{(k)}} = q$. By partitioning the real line into the two disjoint sets where $|q| \leq 4|u|$ and $|q| \geq 4|u|$ and recollecting $B \subseteq [0, 1/2] \times [1/2, 1]^2$, we can write (3.61) as

$$\begin{aligned}
&\iint\limits_{\mathbf{R}\times B} \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; |q| \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q \\
&\qquad \leq \int\limits_{|q|\leq 4|u|} \int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; |q| \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q \qquad (3.62) \\
&\qquad\quad + \int\limits_{|q|\geq 4|u|} \int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; |q| \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q. \qquad (3.63)
\end{aligned}$$

Starting with (3.62), we first integrate the $\tilde{d}_k$ terms, which directs us to

$$\begin{aligned}
&\int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; \mathrm{d}(\tilde{d}_k) \\
&\qquad \leq c \int_0^1 \exp\left(-(u+q)^2(\tilde{d}_k)\right) + \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right) \; \mathrm{d}(\tilde{d}_k) \\
&\qquad \leq c\left[\frac{1}{1+(u+q)^2} + \frac{1}{1+(u^2+q^2)}\right]. \qquad (3.64)
\end{aligned}$$

This implies

$$\int\limits_{|q|\leq 4|u|} \int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)\left(\tilde{d}_k\right)\right)\right| \ |q| \ \mathrm{d}(\tilde{d}_k) \ \mathrm{d}q$$
$$\leq c \int\limits_{|q|\leq 4|u|} \left[\frac{1}{1+(u+q)^2} + \frac{1}{1+(u^2+q^2)}\right] \ |q| \ \mathrm{d}q$$
$$\leq cu \int\limits_{|q|\leq 4|u|} \frac{1}{1+(u+q)^2} + \frac{1}{1+(u^2+q^2)} \ \mathrm{d}q$$
$$\leq cu. \tag{3.65}$$

To procure a bound for (3.63), we must consider the two cases when either

$$(u+q)^2 \geq (u^2+q^2) \quad \text{or} \quad (u+q)^2 \leq (u^2+q^2). \tag{3.66}$$

We first work with the integral involving the measure $\mathrm{d}(\tilde{d}_k)$ and enlarge the domain of integration to obtain

$$\int_0^1 \left|\exp\left(-(u+q)^2\left(\tilde{d}_k\right)\right) - \exp\left(-\left(u^2+q^2\right)\left(\tilde{d}_k\right)\right)\right| \ \mathrm{d}(\tilde{d}_k)$$
$$\leq \int_0^\infty \left|\exp\left(-(u+q)^2\left(\tilde{d}_k\right)\right) - \exp\left(-\left(u^2+q^2\right)\left(\tilde{d}_k\right)\right)\right| \ \mathrm{d}(\tilde{d}_k). \tag{3.67}$$

Considering the first of the two cases appearing in (3.66), (3.67) is expressed as

$$\int_0^\infty \exp\left(-\left(u^2+q^2\right)\left(\tilde{d}_k\right)\right) - \exp\left(-(u+q)^2\left(\tilde{d}_k\right)\right) \ \mathrm{d}(\tilde{d}_k) = \frac{1}{u^2+q^2} - \frac{1}{(u+q)^2} \tag{3.68}$$

and the second case reveals

$$\int_0^\infty \exp\left(-(u+q)^2\left(\tilde{d}_k\right)\right) - \exp\left(-\left(u^2+q^2\right)\left(\tilde{d}_k\right)\right) \ \mathrm{d}(\tilde{d}_k) = \frac{1}{(u+q)^2} - \frac{1}{u^2+q^2}. \tag{3.69}$$

Utilizing the previous two computations and recalling the power series expansions for $|x| \leq 1$,

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty}(-1)^k(k+1)x^k = 1 - 2x + 3x^3 - 4x^4 + \cdots \tag{3.70}$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots, \tag{3.71}$$

we obtain the following bounds for (3.63)

$$\begin{aligned}
&\int\limits_{|q|\geq 4|u|} \int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| |q| \, \mathrm{d}(\tilde{d}_k) \, \mathrm{d}q \\
&\quad \leq \int\limits_{|q|\geq 4|u|} \left|\frac{1}{(u+q)^2} - \frac{1}{u^2+q^2}\right| |q| \, \mathrm{d}q \leq \int\limits_{|q|\geq 4|u|} \left|\frac{1}{(1+u/q)^2} - \frac{1}{(1+u^2/q^2)}\right| \frac{|q|}{|q|^2} \, \mathrm{d}q \\
&\quad \leq \int\limits_{|q|\geq 4|u|} \left|\left(1 - 2\left(\frac{u}{q}\right) + 3\left(\frac{u}{q}\right)^2 - \cdots\right) - \left(1 - \left(\frac{u^2}{q^2}\right)^2 + \left(\frac{u^2}{q^2}\right)^4 - \cdots\right)\right| \frac{1}{|q|} \, \mathrm{d}q \\
&\quad \leq c \int\limits_{|q|\geq 4|u|} \left|\left(\frac{u}{q}\right)\right| \frac{1}{|q|} \, \mathrm{d}q \leq c \int\limits_{|q|\geq 4|u|} \frac{|u|}{|q|^2} \, \mathrm{d}q \leq c.
\end{aligned} \tag{3.72}$$

In summary, to establish a bound on (3.60), we first addressed the isolated intervals $\tilde{d}_k$ and their associated isolated variables q_{ℓ_k}. This directed us to resolve multiple integrals of the form (3.61). By partitioning the real line into two disjoint sets, we scrutinized the two integrals (3.62) and (3.63) over the two cases appearing in (3.66). Combining our results (3.65) and (3.72) and resubstituting $u = u_{k+1}$ indicates (3.61) can be bounded by $c + u_{k+1}$, $k \in I$. With this, (3.60) is less than

$$\iiint\limits_{\mathbf{R}^m \times \mathbf{R}^n \times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right)$$
$$\prod_{k\in I} \left[c + |u_{k+1}|\right] \prod_{I_j^c} |q_j| \, \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}r_j \, \mathrm{d}s_j \, \mathrm{d}t_j \, \mathrm{d}p_j \, \mathrm{d}q_j,$$

where q_j are the remaining non-isolated variables, I_j^c is defined by (3.15), and the n appearing in the domain of integration is the cardinality of I_j^c.

We are now in position to shift our attention to the non-isolated intervals. Addressing the first of the two cases in the brackets above leads us to

$$c \iiint\limits_{\mathbf{R}^m \times \mathbf{R}^n \times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right) \prod_{k \in I} \prod_{I_j^c} |q_j| \,\mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}r_j \,\mathrm{d}s_j \,\mathrm{d}t_j \,\mathrm{d}p_j \,\mathrm{d}q_j,$$

and this returns us to the study of (3.27). Turning to the second and the more critical summand in the aforementioned brackets, we look to derive a bound for

$$\iiint\limits_{\mathbf{R}^m \times \mathbf{R}^n \times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right) \prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j| \,\mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}r_j \,\mathrm{d}s_j \,\mathrm{d}t_j \,\mathrm{d}p_j \,\mathrm{d}q_j, \tag{3.73}$$

Integrating the $\tilde{r}_k$ terms and the remaining $\tilde{d}_k$ terms by applying (3.28) unveils (3.73) is bounded by

$$\iint\limits_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{k=1}^{m} \left(1 + \bar{p}_k^2\right)^{-1} \prod_{I^c} \left(1 + u_k^2\right)^{-1} \prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j| \,\mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j. \tag{3.74}$$

By substituting (3.30) for each $|q_j|$ term, the quantity of interest appears in the second line below

$$\prod_{k\in I} |u_{k+1}| \prod_{I_j^c} |q_j| = \prod_{k\in I} |u_{k+1}| \prod_{k\in T} |u_{k-1} - u_k|$$
$$\leq \prod_{k\in I} |u_{k+1}| \prod_{k\in T} (|u_{k-1}| + |u_k|) . \tag{3.75}$$

With regard to the product of u_{k+1} terms, the definition of set I, provided in (3.16), reveals these terms originate from t-intervals and Figure 3.4 assists in our identification. Formally, define this set as

$$T^{I+1} \doteq \{k \in T \mid k-1 \in I\} \tag{3.76}$$

so that (3.75) is synonymous with

$$\prod_{k\in T^{I+1}} |u_k| \prod_{k\in T} (|u_{k-1}| + |u_k|) .$$

We now aim to study the interconnection between the polynomial resulting from the expansion of (3.77) and the remaining non-isolated s-intervals and t-intervals.

Proposition 3.1.5. *The expansion of the products*

$$\mathcal{P}_2(|u_k|) \doteq \prod_{k\in T^{I+1}} |u_k| \prod_{k\in T} (|u_{k-1}| + |u_k|) \tag{3.77}$$

generates a polynomial of the form

$$\mathcal{P}_2(|u_k|) = \sum \left(\prod_{S'} |u_k|^{\omega_k} \prod_{S^-} |u_k|^{\omega_k} \prod_{T^+} |u_k|^{\omega_k} \prod_{T^{I+1}} |u_k|^{\omega_k} \prod_{T''} |u_k|^{\omega_k} \right) , \tag{3.78}$$

with orders given by

$$deg\ |u_k| = \omega_k = \begin{cases} 0, & \text{if } k \in S' \\ 1, & \text{if } k \in S^- \\ 0 \text{ or } 1, & \text{if } k \in T^+ \\ 1 \text{ or } 2, & \text{if } k \in T^{I+1} \\ 0, 1, \text{ or } 2, & \text{if } k \in T''. \end{cases} \tag{3.79}$$

Proof. To begin our analysis, focus first on the set T^{I+1} and consider the following two scenarios. For the first case, let $[s_\ell, t_\ell]$ be an isolated interval followed by a non-isolated t-interval. Specifically, $d_k = s_\ell$, $d_{k+1} = t_\ell$, and $d_{k+2} = t_m$, for some m. Recall, we have already integrated the isolated variable $q_{\ell_{(k)}}$ affiliated with the interval $[s_\ell, t_\ell]$, which implies this q does not appear in the product $\prod_{I_j^c} |q_j|$. However, since $d_{k+2} = t_m$, (3.30) shows $q_m = u_{k+1} - u_{k+2}$. The term u_{k+1} appears twice in (3.77) and elements from the set $\{u_k\}_{T^{I+1}}$ assume a maximum degree of two in the expansion of those products.

For the second event, let $[s_\ell, t_\ell]$ be an isolated interval followed by a s-interval which is either isolated or non-isolated; $d_k = s_\ell$, $d_{k+1} = t_\ell$, and $d_{k+2} = s_n$, for some n with $k+2 \in I$ or $k+2 \in I^c$. In either case, since u_{k+2} is allied to a s-interval, q_n cannot be recomposed via (3.30), which entails u_{k+1} does not appear as summand for q_n. Combining these two cases, we can conclude

$$\deg |u_k| \in \{1, 2\}, \quad k \in T^{I+1}.$$

Having completed the case of the T^{I+1} intervals, we shift focus to the product appearing on the right side of (3.77) and return to the previous section. The study of (3.36) leads to (3.37) and (3.38) and we can incorporate those results to simplify the current analysis. However, a slight modification is required to account for the T^{I+1}-intervals, since these quantities did not appear in the prior case. The set $T^{'}$, declared in (3.35), must be replaced with

$$T^{''} \doteq \{k \in T \mid k \notin T^{I+1} \cup T^{+}\}. \tag{3.80}$$

If $k, k+1 \in T^{''}$, then (3.39) and (3.40) permit us to conclude the maximum degree elements from this set can assume is two and more generally,

$$\deg |u_k| \in \{0, 1, 2\}, \quad k \in T^{''}.$$

Figure 3.5 below highlights the sets under consideration.

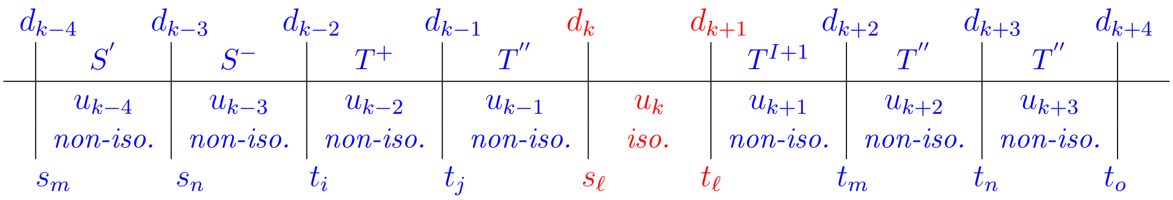

Figure 3.5: Introduction of T^{I+1} and $T^{''}$ -intervals .

All non-isolated intervals emerging in (3.74) have been accounted for and a deeper inspection reveals the sets S^{-}, T^{+}, $S^{'}$, T^{I+1}, and $T^{''}$ form a disjoint union of I^c. Consolidating this information, we can expand (3.77) to formulate polynomial (3.78).

□

Inserting $\mathcal{P}_2(|u_k|)$ into (3.74) produces the sum

$$\sum \Bigg[\iint_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{k=1}^{m} \left(1 + \bar{p}_k^2\right)^{-1} \prod_{I^c} \left(1 + u_k^2\right)^{-1} \prod_{S^{'}} |u_k|^{\omega_k} \prod_{S^{-}} |u_k|^{\omega_k}$$
$$\prod_{T^{+}} |u_k|^{\omega_k} \prod_{T^{I+1}} |u_k|^{\omega_k} \prod_{T^{''}} |u_k|^{\omega_k} \prod_{I_j^c} \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j \Bigg]. \tag{3.81}$$

To prove any multi-variable integral within the brackets above is finite, note, for any $k \in T^{I+1} \cup T^{''}$, $|u_k|^{\omega_k}\left(1+u_k^2\right)^{-1} \leq c$, so that any summand in (3.153) can be bounded by

$$\iint\limits_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{k=1}^{m} \left(1+\bar{p}_k^2\right)^{-1} \prod_{I^c/\{T^{I+1} \cup T^{''}\}} \left(1+u_k^2\right)^{-1} \prod_{S^{'}} |u_k|^{\omega_k} \prod_{S^-} |u_k|^{\omega_k} \prod_{T^+} |u_k|^{\omega_k} \prod_{I_j^c} \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j. \tag{3.82}$$

Proposition 3.1.6 below undertakes spanning issues associated with the introduction of the T^{I+1} and $T^{''}$-intervals. Upon completing the proof of this statement, we can apply Proposition 3.1.4 to conclude the multiple integrals originating from (3.56) are finite.

Proposition 3.1.6. *Consider the non-isolated variables $\{q_j\}$ associated with multiple integral* (3.82) *and let the non-isolated intervals $S^{'}, S^-, T^+, T^{I+1}, T^{''}$ form a disjoint union of I^c. Both $\{\bar{p}_k\} \cup \{u_k\}_{S^{'}} \cup \{u_k\}_{S^-}$ and $\{\bar{p}_k\} \cup \{u_k\}_{S^{'}} \cup \{u_k\}_{T^+}$ form a basis for $\{p_j\} \cup \{q_j\}$.*

Proof. Although we have removed the non-isolated T^{I+1} and $T^{''}$-intervals from I^c, this does not hinder us from duplicating the techniques of Proposition 3.2.3. Hence, we provide a sketch of the proof.

Beginning with the cardinality of the sets, the equivalence in cardinality $|S^-| = |T^+|$ implies $\left|I^c/\{T^{I+1} \cup T^{''} \cup T^+\}\right| = \left|I^c/\{T^{I+1} \cup T^{''} \cup S^-\}\right| = n$ and, therefore,

$$|\{\bar{p}_k\} \cup \{u_k\}_{S^{'}} \cup \{u_k\}_{S^-}| = |\{\bar{p}_k\} \cup \{u_k\}_{S^{'}} \cup \{u_k\}_{T^+}| = m + n.$$

Reordering the elements from $S^{'} \cup S^-$ in their natural order and formulating each element of the ordered set $\{u_{k_1}, u_{k_2}, ..., u_{k_{n_1}}\}$ in terms of p_j and q_j leads to the first desired conclusion. We can close our argument by replicating the proof of part *ii.)* of Proposition 3.2.3 with respect to the remaining set in question. □

Consider the second summand emerging in (3.57),

$$\frac{i^m}{(2\pi)^{2m}} \iiint_{\mathbf{R}^{2m}\times B^m} \mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_j W(r_j,s_j)\right) \exp\left(i\sum_{I_j^{c1}} q_j W(s_j,t_j)\right) \prod_{I_j} \left\{ \exp\left(iq_j W(s_j,t_j)\right)\right\}_0 \right]$$
$$\cdot \prod_{I_j^{c2}} \mathbb{E}\left[\exp\left(iq_j W(s_j,t_j)\right)\right] F(\epsilon,x,y,p,q) \prod_{j=1}^m q_j \,\mathrm{d}p_j\,\mathrm{d}q_j\,\mathrm{d}r_j\,\mathrm{d}s_j\,\mathrm{d}t_j. \tag{3.83}$$

and recall $I_j^{c_1}$ and $I_j^{c_2}$ form a disjoint union of I_j^c. The first step in the study of the above is to replicate the derivation of (3.58) and (3.60) . Apply the independent increments of Brownian motion to write

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_j W(r_j,s_j)\right) \exp\left(i\sum_{I_j^{c1}\cup I_j} q_j W(s_j,t_j)\right)\right]$$
$$= \mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_j W(r_j,1/2)\right)\right] \mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_j W(1/2,s_j) + i\sum_{I_j^{c1}\cup I_j} q_j W(s_j,t_j)\right)\right]$$

and reorder the r terms as in (3.7) to conclude (3.8). Address the time variables s and t in the exact fashion as in (3.9)

$$\sum_{j=1}^m p_j W(1/2,s_j) + \sum_{I_j^{c1}\cup I_j} q_j W(s_j,t_j) = \sum_{k=1}^{2m} \hat{u}_k W(d_k,d_{k+1}). \tag{3.84}$$

After equating the Brownian terms, we come to

$$\hat{u}_1 = \sum_{j=1}^{m} p_j \ , \qquad \hat{u}_k = \sum_{\{j:s_j>d_k\}} p_j + \sum_{\{\ell\in I_j^{c1}\cup I_j: s_\ell\le d_k<t_\ell\}} q_\ell \ , \qquad \hat{u}_{2m+1}=0,$$

$$\hat{u}_k - \hat{u}_{k-1} = \begin{cases} q_\ell - p_\ell, & \text{if } d_k = s_\ell \text{ and } \ell \in I_j^{c_1}\cup I_j \\ -q_\ell, & \text{if } d_k = t_\ell \text{ and } \ell \in I_j^{c_1}\cup I_j. \end{cases}$$

The above indicate $\{\hat{u}_k\}$ spans $\{p_j\}\cup\{q_\ell\}_{I_j^{c_1}\cup I_j}$. Replicating the above study with regard to the product

$$\prod_{I_j} \left\{ \exp\left(iq_{\pi_j} W(s_{\pi_j}, t_{\pi_j})\right)\right\}_0$$

implies (3.83) can be bounded above and rewritten as

$$\begin{aligned} c^m \iiint\limits_{\mathbf{R}^{2m}\times B^m} &\exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} \hat{u}_k^2(\tilde{d}_k)\right) \exp\left(-\sum_{I_j^{c_2}} q_j^2(t_j-s_j)\right) \\ &\cdot \left| \prod_{k\in I} \left(\exp\left(-\hat{u}_k^2(\tilde{d}_k)\right) - \exp\left(-\left(\hat{u}_{k+1}^2 + q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right) \right| \\ &\prod_{j=1}^{m} |q_j| \ \mathrm{d}p_j \ \mathrm{d}q_j \ \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j. \end{aligned} \tag{3.85}$$

The above integral reduces to the one emerging in (3.60). Equating (3.9) with (3.84) yields

$$\begin{aligned} &\left| \prod_{k\in I} \left(\exp\left(-u_k^2(\tilde{d}_k)\right) - \exp\left(-\left(u_{k+1}^2 + q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right) \right| \\ &\quad = \left| \prod_{k\in I} \left(\exp\left(-\hat{u}_k^2(\tilde{d}_k)\right) - \exp\left(-\left(\hat{u}_{k+1}^2 + q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right) \right| \end{aligned}$$

and modifying the analysis of (3.52) yields

$$\exp\left(-\sum_{k=1}^{2m}\hat{u}_k^2(\tilde{d}_k)\right)\exp\left(-\sum_{I_j^{c2}} q_j^2(t_j-s_j)\right)\leq \exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right).$$

This completes the proof of

$$\mathbb{E}\left[\left(\phi_y(\epsilon,x,y,B)\right)^m\right]\leq c,$$

over both class of intervals.

3.1.3 Finite moments of $\phi(\epsilon, x, y, B)$ and $\alpha(\epsilon, x, y, B)$

With the status of the m^{th} moments with respect to the derivative of the spatial variable y clarified, we turn our focus now and look to duplicate our results for (2.32),

$$\mathbb{E}\left[\left(\phi(\epsilon,x,y,B)\right)^m\right]\leq c.$$

This directs us to $\phi(\epsilon, x, y, B)$ of equation (2.20), with expected value expressed as

$$\mathbb{E}\left[\left(\phi(\epsilon,x,y,B)\right)^m\right]=\frac{1}{(2\pi)^{2m}}\iiint\limits_{\mathbf{R}^{2m}\times B^m}\mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_jW(r_j,s_j)\right)\prod_{j=1}^m\left\{\exp\left(iq_jW(s_j,t_j)\right)\right\}_0\right]$$
$$\cdot F(\epsilon,x,y,p,q)\prod_{j=1}^m \mathrm{d}p_j\ \mathrm{d}q_j\ \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j. \tag{3.86}$$

In distinguishing the above equation with the that of (3.2), we begin by noticing the two expressions are very similar except (3.86) does not include a product of $|q_j|$ terms. Equation (2.9) allows us to deduce that these variables are an after-effect of differentiating the inverse Fourier transform of the probability density function of W_t. The absence of these terms will simplify our inquiry of $\mathbb{E}\left[(\phi(\epsilon, x, y, B))^m\right]$. In more detail, after expanding the expression

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, s_j)\right)\prod_{j=1}^{m}\left\{\exp\left(iq_j W(s_j, t_j)\right)\right\}_0\right]$$

we are directed to an expression of the form

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, s_j)\right)\exp\left(i\sum_{j=1}^{m} q_j W(s_j, t_j)\right)\right]$$
$$\pm\sum\left(\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j, s_j)\right)\exp\left(i\sum_{\pi'} q_j W(s_j, t_j)\right)\right.\right.$$
$$\left.\left.\prod_{\pi''}\mathbb{E}\left[\exp\left(i\sum_{\pi_1} q_j W(s_j, t_j)\right)\right]\right]\right),$$

with $\pi^{'}$ and $\pi^{''}$ forming a disjoint union of $\mathcal{M}$. Hence, it suffices to address the first quantity appearing in the sum above, since the techniques for this case can be, naturally, applied to the remaining summands. Therefore, we begin with a study of an expression parallel to (3.24),

$$\frac{1}{(2\pi)^{2m}}\iiint\limits_{\mathbf{R}^{2m}\times B^m}\exp\left(-\sum_{k=1}^{m}\bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{k=1}^{2m}u_k^2(\tilde{d}_k)\right)$$
$$\cdot F(\epsilon, x, y, p, q)\prod_{j=1}^{m}\mathrm{d}p_j\ \mathrm{d}q_j\ \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j,. \tag{3.87}$$

Applying (3.25), (3.26) and bounding the $\mathrm{d}r_j, \mathrm{d}s_j$, and $\mathrm{d}t_j$ integrals with (3.28) produces

$$c^m\iint\limits_{\mathbf{R}^{2m}}\prod_{k=1}^{m}\left(1+\bar{p}_k^2\right)^{-1}\prod_{k=1}^{2m}\left(1+u_k^2\right)^{-1}\ \mathrm{d}p_j\ \mathrm{d}q_j. \tag{3.88}$$

We simply recall the facts following (3.8) and (3.10) to justify the change of variables in

(3.88) to arrive at the product of finite multiple integrals

$$\prod_{k=1}^{m} \int_{\mathbf{R}} \left(1+\bar{p}_k^2\right)^{-1} \, \mathrm{d}\bar{p}_k \prod_{k=1}^{2m} \int_{\mathbf{R}} \left(1+u_k^2\right)^{-1} \, \mathrm{d}u_k. \tag{3.89}$$

The construction of the proof for

$$\mathbb{E}\left[\left(\alpha(\epsilon,x,y,B)\right)^m\right] \leq c$$

follows naturally from the techniques derived in the study of $\phi(\epsilon,x,y,B)$, since the expected value emerging in the integrand of $\mathbb{E}\left[\left(\alpha(\epsilon,x,y,B)\right)^m\right]$ is of the form

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j W(r_j,s_j)\right)\exp\left(i\sum_{j=1}^{m} q_j W(s_j,t_j)\right)\right].$$

3.2 Hölder continuity over $A(1,1,2)$

We turn to Lemma 2.2.3 and look to secure equation (2.35)

$$\left\|\phi_y(\epsilon,x,y,B)-\phi_y(\epsilon',x',y',B)\right\|_m \leq c_m(\xi)\left\|(\epsilon,x,y)-(\epsilon',x',y')\right\|^{m\xi}. \tag{3.90}$$

By writing,

$$\begin{aligned}\left\|\phi_y(\epsilon,x,y,B)-\phi_y(\epsilon',x',y',B)\right\|_m \leq{}& \left\|\phi_y(\epsilon,x,y,B)-\phi_y(\epsilon,x',y,B)\right\|_m \\ &+\left\|\phi_y(\epsilon,x',y,B)-\phi_y(\epsilon,x',y',B)\right\|_m \\ &+\left\|\phi_y(\epsilon,x',y',B)-\phi_y(\epsilon',x',y',B)\right\|_m \end{aligned} \tag{3.91}$$

our goal is to establish a bound on each term above. The net effect of varying (x, y, ϵ) will impact the function $F(\epsilon, x, y, p, q)$, defined in (3.3) as

$$F(\epsilon, x, y, p, q) \doteq \exp\left(ix\sum_{j=1}^{m} p_j\right)\exp\left(iy\sum_{j=1}^{m} q_j\right)\prod_{j=1}^{m}\hat{f}(\epsilon, p_j)\ \hat{f}(\epsilon, q_j),$$

with $\hat{f}(\epsilon, a_j) \doteq \exp(-\epsilon a_j^2)$. Altering this deterministic function will yield an alternate set of multiple integrals which we must prove are finite. Furthermore, we must consider in our analysis the two summands (3.22) and (3.23) over the non-isolated intervals and scenarios (3.56) and (3.57) for the isolated cases.

We first study perturbations in the x, y, and ϵ terms for $\phi_y(\epsilon, x, y, B)$ in Sections 3.2.1 and 3.2.2, and duplicate our result for $\phi(\epsilon, x, y, B)$ and $\alpha(\epsilon, x, y, B)$ in Section 3.2.3. To assist in this analysis, we observe the following bounds. For any $\alpha, \gamma, \gamma', p, q \in \mathbf{R}$ and $\xi \in [0, 1]$,

$$|\exp(-i\alpha\gamma) - \exp(-i\alpha\gamma')| \leq c|\alpha|^{\xi}|\gamma - \gamma'|^{\xi} \tag{3.92}$$

$$|\exp(-\gamma|\alpha|) - \exp(-\gamma'|\alpha|)| \leq c|\alpha|^{\xi}|\gamma - \gamma'|^{\xi} \tag{3.93}$$

$$|\exp(-i\alpha\gamma)\exp(-\xi|p|)\exp(-\xi|q|)| \leq 1 \tag{3.94}$$

$$|\alpha - \gamma|^{\xi} \leq c\left(|\alpha|^{\xi} + |\gamma|^{\xi}\right) \tag{3.95}$$

3.2.1 Local properties and the non-isolated intervals

Starting with the variation in x, (3.3) now becomes

$$F(\epsilon, x, x', y, p, q) = \prod_{j=1}^{m}\Bigg(\exp\left(-ixp_j\right) - \exp\left(-ix'p_j\right)\Bigg)\exp\left(-iy\sum_{j=1}^{m} q_j\right)\prod_{j=1}^{m}\hat{f}(\epsilon, p_j)\ \hat{f}(\epsilon, q_j), \tag{3.96}$$

so that we must prove the upper bound

$$\mathbb{E}\left[\left(\phi_y(\epsilon,x,y,B)-\phi_y(\epsilon,x',y,B)\right)^m\right] \leq c_m(\xi)|x-x'|^{m\xi} \tag{3.97}$$

over both the non-isolated and isolated intervals.

Beginning with (3.22), replace $F(\epsilon,x,y,p,q)$ in (3.24) with (3.96). Integrating the time variables and applying (3.25), (3.92), and (3.94), work over the non-isolated intervals begins with the subsequent expression,

$$c^m|x-x'|^{m\xi}\iint\limits_{\mathbf{R}^{2m}}\prod_{k=1}^{m}\left(1+\bar{p}_k^2\right)^{-1}\prod_{k=1}^{2m}\left(1+u_k^2\right)^{-1}\prod_{j=1}^{m}|q_j|\;|p_j|^{\xi}\;\mathrm{d}p_j\;\mathrm{d}q_j. \tag{3.98}$$

The above integral is comparable to (3.29), except for the product of $|p_j|^{\xi}$ terms. In the course of our analysis, we will discover we can apply the techniques from the study of this prior integral to our current problem, once we have investigated the impact of the $|p_j|^{\xi}$ terms on (3.98). We start our analysis here.

Proposition 3.2.1. *The product* $\prod|p_j|^{\xi}$ *arising in* (3.98) *can be bounded above by*

$$\prod_{j=1}^{m}|p_j|^{\xi}=\prod_R|\bar{p}_{k'}-\bar{p}_{k''}|^{\xi}\leq c\prod_R|\bar{p}_{k'}|^{\xi}+|\bar{p}_{k''}|^{\xi}.$$

The polynomial resulting from the expansion of the upper bound takes the form

$$\mathcal{P}_3(|\bar{p}_k|)\doteq\prod_R|\bar{p}_{k'}|^{\xi}+|\bar{p}_{k''}|^{\xi}=\sum\left(\prod_R|\bar{p}_k|^{\eta_k\xi}\right), \tag{3.99}$$

where

$$\deg|\bar{p}_k|=\eta_k\xi\in\{0,\xi,2\xi\}.$$

Proof. Equation (3.8),

$$\bar{p}_k = \sum_{\{j : r_j \leq \bar{r}_k\}} p_j,$$

allows us to compose each $\bar{p}_k$ as the sum of the p_j terms. Note the index of summation appearing in that expression and set

$$R \doteq \{k \mid \bar{r}_k = r_j\} \quad \text{or, equivalently,} \quad R = \{1, ..., m\}. \tag{3.100}$$

Let $\{k_1, k_2, ..., k_m\}$ be the elements of the set R, such that $k_1 < k_2 < \cdots < k_m$. As we proceed through the ordered set $\{\bar{p}_{k_1}, \bar{p}_{k_2}, ..., \bar{p}_{k_m}\}$, writing each term as a sum of p_j's, we notice the sums are increasing. Upon rearranging these terms and solving for each p_j, two key observations become apparent. We first remark that each p_j term is represented as the difference of at most two $\bar{p}_k$'s,

$$p_j = \bar{p}_{k'} - \bar{p}_{k''} \ , \tag{3.101}$$

with $k^{'}, k^{''} \in R$. The second important fact to note is that each $\bar{p}_k$ appears at most twice as a summand for a p_j and the following recursive relationship emerges. For any ℓ, j, and m, with $\bar{r}_{k_m} = r_\ell$ and $\bar{r}_{k_{m+1}} = r_j$, we discover

$$\begin{aligned} p_\ell =& \bar{p}_{k_m} - \bar{p}_{k_{m-1}} \\ p_j =& \bar{p}_{k_{m+1}} - \bar{p}_{k_m}. \end{aligned} \tag{3.102}$$

Inequality (3.95) allows us to write (3.101) as

$$|p_j|^\xi = |\bar{p}_{k'} - \bar{p}_{k''}|^\xi \leq |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi ,$$

and more generally,

$$\prod_{j=1}^{m} |p_j|^{\xi} = \prod_{R} |\bar{p}_{k'} - \bar{p}_{k''}|^{\xi} \leq c \prod_{R} |\bar{p}_{k'}|^{\xi} + |\bar{p}_{k''}|^{\xi} .$$

We define

$$\mathcal{P}_3(|\bar{p}_k|) \doteq \prod_{R} |\bar{p}_{k'}|^{\xi} + |\bar{p}_{k''}|^{\xi} = \sum \left(\prod_{R} |\bar{p}_k|^{\eta_k \xi} \right),$$

where

$$\deg |\bar{p}_k| = \eta_k \xi \in \{0, \xi, 2\xi\}.$$

The order of the polynomial follows from our second observation above regarding the frequency of the elements $\bar{p}_k$ emerging in (3.101). □

Replacing the product of $|p_j|^{\xi}$ variables with $\mathcal{P}_3(|\bar{p}_k|)$, integral (3.98) can be bounded above by a sum of multiple integrals,

$$c \sum \left[\iint_{\mathbf{R}^{2m}} \prod_{R} |\bar{p}_k|^{\eta_k \xi} \prod_{k=1}^{m} (1 + \bar{p}_k^2)^{-1} \prod_{k=1}^{2m} \left(1 + u_k^2\right)^{-1} \prod_{j=1}^{m} \ |q_j| \ \mathrm{d}p_j \ \mathrm{d}q_j \right] . \tag{3.103}$$

We now turn to our proof of the bound of $\phi_y(\epsilon, x, y, B)$ over the non-isolated intervals. Consider any integral appearing as a summand in (3.103) and proceed with the exact analysis of (3.29), starting with an application of Proposition 3.1.2. Continuing with this line of reasoning, we arrive at integrals parallel to (3.46), with one slight modification regarding rational functions linked to the measure $\mathrm{d}\bar{p}_k$,

$$\iint_{\mathbf{R}^{2m}} \prod_{R} |\bar{p}_k|^{\eta_k \xi} \prod_{k=1}^{m} \frac{1}{1 + \bar{p}_k^2} \prod_{S'} \frac{1}{1 + u_k^2} \prod_{S^-} \frac{|u_k|}{1 + u_k^2} \prod_{T^+} \frac{|u_k|^{\omega_k}}{1 + u_k^2} \prod_{j=1}^{m} \mathrm{d}p_j \ \mathrm{d}q_j . \tag{3.104}$$

The above multiple integral is in the form required to apply Proposition 3.1.4. For any m, we can select ξ sufficiently small such that

$$\eta_k \xi < 1, \quad \text{for all } k \in \mathcal{M}, \tag{3.105}$$

indicating (3.104) can be bounded above by $c(\xi)$. With this, we have established (3.97) over the non-isolated intervals.

To investigate the variation in y, we must validate over both classes of intervals

$$\mathbb{E}\Big[\big(\phi_y(\epsilon, x, y, B) - \phi_y(\epsilon, x, y', B)\big)^m\Big] \leq c_m(\xi)|y - y'|^{m\xi}. \tag{3.106}$$

With the obvious changes, we arrive at an expression parallel to (3.98)

$$c^m |y - y'|^{m\xi} \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1 + \bar{p}_k^2\right)^{-1} \prod_{k=1}^{2m} \left(1 + u_k^2\right)^{-1} \prod_{j=1}^{m} \; |q_j| \; |q_j|^{\xi} \; \mathrm{d}p_j \; \mathrm{d}q_j. \tag{3.107}$$

The above integral is similar to (3.29) and, recall, we began our analysis of this quantity by studying the expansion of the product (3.36). In a similar fashion, we will ascertain our desired bound by first focusing on $\prod \; |q_j| \; |q_j|^{\xi}$.

As before, rewrite each $|q_j|$ in terms of u_k and u_{k-1} via (3.30),

$$u_{k-1} - u_k = q_\ell \quad \text{for } d_k = t_\ell.$$

However, for the $|q_j|^{\xi}$ variables, substitute the first case appearing in (3.11),

$$u_k - u_{k-1} + p_\ell = q_\ell, \quad \text{for } d_k = s_\ell. \tag{3.108}$$

In the process of deriving a bound for the variation in x, it was demonstrated in equation

(3.101) how each p_ℓ term can be expressed as the difference of two $\bar{p}_k$ terms. Applying (3.95) reveals,

$$\begin{aligned}|q_j|^\xi &= |u_k - u_{k-1} + p_\ell|^\xi = |u_k - u_{k-1} + \bar{p}_{k'} - \bar{p}_{k''}|^\xi \\ &\leq \left(|u_k|^\xi + |u_{k-1}|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi\right), \end{aligned} \tag{3.109}$$

with $k \in S$ and $k^{'}, k^{''} \in R$. Sets S and R are defined in (3.17) and (3.100), respectively. Gathering these fact yields the upper bound

$$\prod_{j=1}^{m} |q_j|\ |q_j|^\xi \leq \prod_{k\in T} (|u_{k-1}| + |u_k|) \prod_{\substack{k\in S \\ k^{'},k^{''}\in R}} \left(|u_k|^\xi + |u_{k-1}|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi\right),$$

with T is defined in (3.18).

We would like to expand the products surfacing on the right hand side of the inequality above and understand the impact that the terms, which appear in the resulting polynomial, have on (3.107). The $\bar{p}_k$ terms will play a minor role in deriving our desired bound. However, understanding the relationship between the degree of the $|u_k|$ terms and the s-intervals and t-intervals from which they originate will be essential to obtaining our bound. We now begin exploring the interplay between these notions and start with the $\{u_k\}_T$ and $\{u_k\}_S$ terms. As we will see, the properties of these terms from our previous analysis extend naturally to our current problem and, with the goal of applying former results, we will need to define sets parallel to the ones appearing in (3.32)-(3.35). Specifically, as we will see in the proposition below, multiplying out the right hand side of the inequality above directs us to a polynomial of the form

$$\sum \left(\prod_S |u_k|^{\omega_k + \theta_k \xi} \prod_T |u_k|^{\omega_k + \theta_k \xi} \right).$$

With respect to the above, let

$$S_\xi^- \doteq \{k \in S \mid k \in S, k+1 \in T, \text{ and } \omega_k + \theta_k \xi \geq 1\} \tag{3.110}$$

$$T_\xi^+ \doteq \{k \in T \mid k-1 \in S_\xi^-\} \tag{3.111}$$

$$S_\xi^{'} \doteq \{k \in S \mid k \notin S_\xi^-\} \tag{3.112}$$

$$T_\xi^{'} \doteq \{k \in T \mid k \notin T_\xi^+\}. \tag{3.113}$$

We will make extensive use of the sets defined above in our current and future analysis.

Proposition 3.2.2. *The expansion of the products*

$$\mathcal{P}_4(|\bar{p}_k|, |u_k|) \doteq \prod_{k\in T} (|u_{k-1}| + |u_k|) \prod_{\substack{k\in S \\ k^{'}, k^{''} \in R}} \left(|u_k|^\xi + |u_{k-1}|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi\right) \tag{3.114}$$

generates a polynomial of the form

$$\mathcal{P}_4(|\bar{p}_k|, |u_k|) = \sum \left(\prod_R |\bar{p}_k|^{\eta_k \xi} \prod_{S_\xi^{'}} |u_k|^{\theta_k \xi} \prod_{S_\xi^-} |u_k|^{\omega_k + \theta_k \xi} \prod_{T_\xi^+, T_\xi^{'}} |u_k|^{\omega_k + \theta_k \xi} \right) \tag{3.115}$$

where

$$\deg |\bar{p}_k| = \eta_k \xi = 0, \xi, 2\xi \ \textit{if } k \in R$$

$$\textit{deg } |u_k| = \omega_k + \theta_k \xi \begin{cases} = 0, \xi, \ \textit{or } 2\xi, & \textit{if } k \in S_\xi^{'} \\ = 1 \ \textit{or } 1+\xi, & \textit{if } k \in S_\xi^- \\ = \xi \ \textit{or } 1, & \textit{if } k \in T_\xi^+ \\ \leq 2, & \textit{if } k \in T_\xi^{'}. \end{cases} \tag{3.116}$$

Proof. For reference purposes, let us split the products appearing in (3.114) and label them as

$$\prod_{k\in T}\left(|u_{k-1}|+|u_k|\right) \tag{3.117}$$

$$\prod_{k\in S}\prod_{k',k''\in R}\left(|u_k|^{\xi}+|u_{k-1}|^{\xi}+|\bar{p}_{k'}|^{\xi}+|\bar{p}_{k''}|^{\xi}\right). \tag{3.118}$$

Let us begin by examining the t-intervals and determine the frequency in which an element from $\{u_k\}_T$ can appear in (3.114). We make the general observation that (3.36) and (3.117) yield the same polynomial, which reveals elements from the set $\{u_k\}_T$ appear at most twice in both products. When $k \in T$ and $k+1 \in T$, the u_k term appearing in both (3.39) and (3.40) is proceeded by a t-interval and (3.108) implies this particular u_k does not surface again in (3.118). However, this does not imply elements from the T-intervals can not arise in the aforementioned product. When $d_k = s_\ell$ and $d_{k-1} = t_j$, for some ℓ and j, (3.108) suggests elements from $\{u_k\}_T$ can appear at most once in (3.118). Ensuing the expansion of (3.114), we can conclude terms from $\{u_k\}_T$ will be of maximum order 2.

We shift our attention to the s-intervals. If $k \in S$ and $k-1 \in S$, then elements from $\{u_k\}_S$ can emerge twice in (3.118). In more detail, if $d_k = s_\ell$ and $d_{k-1} = s_j$, for some ℓ and j, we have from (3.108),

$$q_\ell = u_k - u_{k-1} + p_\ell \tag{3.119}$$

$$q_j = u_{k-1} - u_{k-2} + p_j \tag{3.120}$$

and u_{k-1} has appeared as a summand for two values of q. Furthermore, this u_{k-1} term cannot appear in (3.117). Hence, there exists terms from $\{u_k\}_S$ which can have order 2ξ in the expansion of (3.114). It is important to note this is not the maximum degree elements from this set can assume.

From our analysis of (3.41) and the discussion following that polynomial, we can conclude elements from $\{u_k\}_S$ appear in (3.117) when $k \in T$ and $k-1 \in S$. Note the u_{k-1} term appearing in (3.42). Reexpressing the $|q_j|^\xi$ terms through (3.108) implies this u_{k-1} will appear for another value of q and, consequently, in the product (3.118). Upon expanding (3.114), we can conclude elements from $\{u_k\}_S$, specifically those which appear in both (3.117) and (3.118), attain a maximum order of $1+\xi$.

With our analysis of the S-intervals and T-intervals complete, we see the products appearing in (3.114) generate a polynomial of the form

$$\begin{aligned}\mathcal{P}_4(|\bar{p}_k|, |u_k|) &\doteq \prod_{k\in T}(|u_{k-1}| + |u_k|) \prod_{\substack{k\in S\\ k',k''\in R}} \left(|u_k|^\xi + |u_{k-1}|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi\right) \\ &= \sum \left(\prod_R |\bar{p}_k|^{\eta_k \xi} \prod_{S,T} |u_k|^{\omega_k + \theta_k \xi}\right), \end{aligned} \tag{3.121}$$

where η_k, ω_k, and θ_k take values in $\{0,1,2\}$. Recall, the values of η_k follow from our discussion of (3.101). Specifically, if $\bar{r}_{k'} = r_\ell$ and $\bar{r}_{k'+1} = r_j$, for some k', ℓ, and j, then the recursive relationship presented in (3.102) indicates (3.119) and (3.120) can be expressed as

$$\begin{aligned} q_\ell &= u_k - u_{k-1} + p_\ell = u_k - u_{k-1} + \bar{p}_{k'} - \bar{p}_{k'-1} \\ q_j &= u_{k-1} - u_{k-2} + p_j = u_{k-1} - u_{k-2} + \bar{p}_{k'+1} - \bar{p}_{k'}. \end{aligned} \tag{3.122}$$

Hence, $\bar{p}_{k'}$ emerges twice in the above summands and η_k can assume orders less than or equal to 2ξ in polynomial $\mathcal{P}_4$.

The analysis of the s-intervals above leads us to conclude the maximum order of elements from $\{u_k\}_{S'_\xi}$ and $\{u_k\}_{S^-_\xi}$ are 2ξ and $1+\xi$, respectively. With the assistance of these newly defined sets, $\mathcal{P}_4(|\bar{p}_k|,|u_k|)$ is equal to

$$\mathcal{P}_4(|\bar{p}_k|,|u_k|)=\sum\left(\prod_{R}|\bar{p}_k|^{\eta_k\xi}\prod_{S'_\xi}|u_k|^{\theta_k\xi}\prod_{S^-_\xi}|u_k|^{\omega_k+\theta_k\xi}\prod_{T^+_\xi,T'_\xi}|u_k|^{\omega_k+\theta_k\xi}\right).$$

To determine the order of the $\{u_k\}_{T^+_\xi}$ terms appearing above, let us return to the products (3.114)-(3.118). Recognize, elements from $\{u_k\}_{S^-_\xi}$ take values 1 or $1+\xi$ only if u_k, $k\in S$, has appeared in (3.117) or in both (3.117) and (3.118), respectively. Consider two terms, u_{k-1} and u_k, appearing in (3.115), with $k-1\in S^-_\xi$ and $k\in T^+_\xi$. Both terms appear as summands for a q via (3.30) and, therefore, in (3.117). Upon expanding (3.114), these terms can not appear together in polynomial (3.115). However, u_{k+1} will be from either a t-interval or s-interval, which implies u_k will appear once more in (3.117) or (3.118). In either event, upon expanding (3.114), we can conclude when u_{k-1} and u_k appear in (3.115), the order of u_k will be either ξ or 1. Hence, members of $\{u_k\}_{T^+_\xi}$ attain a maximum order of 1.

From our discussion above regarding the t-intervals, elements from $\{u_k\}_{T'_\xi}$ are of maximum degree 2 in (3.115) and we arrive at the order of the $|u_k|$ terms presented in (3.116). □

Before proceeding further, we pause and note that the properties of the s and t-intervals defined in (3.32)-(3.34) are nearly identical to those emerging in the proof of the proposition above, (3.110)-(3.112), with two a slight modifications. The definition of the S'_ξ-intervals has been changed to account for the higher orders that the u_k variables, originating from this interval, can assume in the expansion of (3.114). Additionally, we have also changed notation due to the introduction of the ξ terms in our new analysis. The imperative observation to make is that the techniques derived to validate Proposition 3.2.3 can be replicated to establish a parallel result.

Proposition 3.2.3. *Let $\{q_j\}$ represent a set of non-isolated variables associated with the* *set* I_j^c *of* (3.15)*. Denote the cardinality of the two sets by* $|\{\bar{p}_k\}| = m$ *and* $|I_j^c| = n$ *where* $m, n > 0$ *and* $n \leq m$*. With the sets* S_ξ', T_ξ^+, *and* S_ξ^- *defined in* (3.110)-(3.112) *, respectively, we can conclude that both*

$$\{\bar{p}_k\} \cup \{u_k\}_{S_\xi'} \cup \{u_k\}_{S_\xi^-} \quad \textit{and} \quad \{\bar{p}_k\} \cup \{u_k\}_{S_\xi'} \cup \{u_k\}_{T_\xi^+}$$

form a basis for $\{p_j\} \cup \{q_j\}$

Proof. Replicate the proof for Proposition 3.2.3 □

Returning to our problem at hand, inserting $\mathcal{P}_4(|\bar{p}_k|, |u_k|)$ into (3.107) we arrive at

$$\sum \Bigg[\iint\limits_{\mathbf{R}^{2m}} \prod_R |\bar{p}_k|^{\eta_k \xi} \prod_{k=1}^m \left(1 + \bar{p}_k^2\right)^{-1} \prod_{S,T} (1 + u_k^2)^{-1} \\ \prod_{S_\xi'} |u_k|^{\theta_k \xi} \prod_{S_\xi^-} |u_k|^{\omega_k + \theta_k \xi} \prod_{T_\xi^+, T_\xi'} |u_k|^{\omega_k + \theta_k \xi} \prod_{j=1}^m \mathrm{d}p_j \; \mathrm{d}q_j \Bigg]. \tag{3.123}$$

We can select ξ sufficiently small so that for all $k \in T_\xi'$,

$$\frac{|u_k|^{\omega_k + \theta_k \xi}}{1 + u_k^2} \leq c. \tag{3.124}$$

This permits us to bound any multiple integral appearing as a summand in the above expression by

$$\iint\limits_{\mathbf{R}^{2m}} \prod_R |\bar{p}_k|^{\eta_k \xi} \prod_{k=1}^m \frac{1}{1 + \bar{p}_k^2} \prod_{S_\xi'} \frac{|u_k|^{\theta_k \xi}}{1 + u_k^2} \prod_{S_\xi^-} \frac{|u_k|^{\omega_k + \theta_k \xi}}{1 + u_k^2} \prod_{T_\xi^+} \frac{|u_k|^{\omega_k + \theta_k \xi}}{1 + u_k^2} \prod_{j=1}^m \mathrm{d}p_j \; \mathrm{d}q_j \tag{3.125}$$

leading us to a point to utilize Proposition 3.1.4. Remembering the order of the $|\bar{p}_k|$ and

u_k variables in (3.114), ξ can be selected close to zero such that for any m,

$$\eta_k\xi < 1, \text{ for all } k \in R, \quad \theta_k\xi < 1, \text{ for all } k \in S_\xi', \quad \omega_k + \theta_k\xi < 3/2, \text{ for all } k \in S_\xi^- \cup T_\xi^+.$$

Hence, $c(\xi)$ is an upper bound for (3.125) and we have secured (3.106) over the non-isolated intervals.

Fortunately, to establish the inequality with respect to ϵ over the non-isolated intervals,

$$\mathbb{E}\Big[\big(\phi_y(\epsilon, x', y', B) - \phi_y(\epsilon', x', y', B)\big)^m\Big] \le c_m(\xi)|\epsilon - \epsilon'|^{m\xi}, \tag{3.126}$$

we will be able to leverage much of our analysis in establishing (3.97) and (3.106). The variation in this last variable changes (3.3) to

$$\begin{aligned} &F(\epsilon, \epsilon', x, y, p, q) \\ &\quad = \exp\left(-ix\sum_{j=1}^m p_j\right)\exp\left(-iy\sum_{j=1}^m q_j\right)\prod_{j=1}^m \left(\hat f(\epsilon, p_j)\hat f(\epsilon, q_j) - \hat f(\epsilon', p_j)\hat f(\epsilon', q_j)\right) \end{aligned} \tag{3.127}$$

and (3.93) and (3.95) reveal

$$\begin{aligned} \prod_{j=1}^m \hat f(\epsilon, p_j)\hat f(\epsilon, q_j) - \hat f(\epsilon', p_j)\hat f(\epsilon', q_j) &= \prod_{j=1}^m \exp\left(-\epsilon\left(p_j^2+q_j^2\right)\right) - \exp\left(-\epsilon'\left(p_j^2+q_j^2\right)\right) \\ \le c^m|\epsilon-\epsilon'|^{m\hat\xi}\prod_{j=1}^m\left(|p_j^2+q_j^2|^{\hat\xi}\right) &\le c^m|\epsilon-\epsilon'|^{m\hat\xi}\prod_{j=1}^m\left(|p_j^2|^{\hat\xi}+|q_j^2|^{\hat\xi}\right) \\ \le c^m|\epsilon-\epsilon'|^{m\xi}\prod_{j=1}^m\left(|p_j|^{\xi}+|q_j|^{\xi}\right), \end{aligned} \tag{3.128}$$

where $\hat\xi$ is chosen sufficiently small and $\xi = 2\hat\xi$. With these facts, the multiple integral of interest emerges as

$$c^m|\epsilon-\epsilon'|^{m\xi}\iint_{\mathbf{R}^{2m}}\prod_{k=1}^{m}\left(1+\bar{p}_k^2\right)^{-1}\prod_{k=1}^{2m}\left(1+u_k^2\right)^{-1}\prod_{j=1}^{m}\ |q_j|\ \left(|p_j|^\xi+|q_j|^\xi\right)\ \mathrm{d}p_j\ \mathrm{d}q_j. \tag{3.129}$$

Introduce a disjoint union of the set $\{1,...,m\}$ by means of $\pi^{'}$ and $\pi^{''}$. Upon expanding the crucial product in the integrand, an upper bound can be initiated,

$$\prod_{j=1}^{m}\ |q_j|\ \left(|p_j|^\xi+|q_j|^\xi\right)=\sum\left(\prod_{\pi'}|p_j|^\xi\prod_{j=1}^{m}\ |q_j|\ \prod_{\pi''}|q_j|^\xi\right) \tag{3.130}$$

$$\leq\sum\left(\mathcal{P}_5(|\bar{p}_k|,|u_k|)\right) \tag{3.131}$$

where

$$\mathcal{P}_5(|\bar{p}_k|,|u_k|)\doteq\mathcal{P}_3(|\bar{p}_k|)\mathcal{P}_4(|\bar{p}_k|,|u_k|)$$

We can leverage our previous work to dissect the expansion of the two polynomials which generate $\mathcal{P}_5(|\bar{p}_k|,|u_k|)$. To deduce the relationship between the order of the $|u_k|$ variables and the non-isolated intervals, simply refer to Proposition 3.2.2. The contribution from the $|\bar{p}_k|$ terms can be established by multiplying

$$\prod_R|\bar{p}_k|^{\eta_k\xi}\prod_{\substack{k\in S\\ k^{'},k^{''}\in R}}\left(|u_k|^\xi+|u_{k-1}|^\xi+|\bar{p}_{k'}|^\xi+|\bar{p}_{k''}|^\xi\right).$$

and (3.99) and (3.118) indicate

$$\deg\ |\bar{p}_k|=\hat{\eta}_k\xi\in\{0,\xi,2\xi,3\xi,4\xi\}$$

Therefore, (3.131) is equivalent to

$$\sum\left(\sum\left(\prod_{R}|\bar{p}_k|^{\hat{\eta}_k\xi}\prod_{S'_\xi,S^-_\xi}|u_k|^{\omega_k+\theta_k\xi}\prod_{T^+_\xi,T'_\xi}|u_k|^{\omega_k+\theta_k\xi}\right)\right). \tag{3.132}$$

Insert the above into (3.129) and bound all rational functions from the T'-intervals by a constant. After employing Proposition 3.1.4 to the resulting expression, the study over the non-isolated intervals is complete.

3.2.2 Resolution of the isolated intervals

To establish (3.97),(3.106), and (3.126) over the isolated intervals, it suffices to scrutinize the variation in x,y, and ϵ on the integral appearing in (3.59) and (3.83). As we witnessed with the case of the non-isolated intervals, the variation of these terms with respect to the function (3.3) will have a direct impact on our analysis.

We initiate our study with the x variable. Substitute (3.96) for $F(\epsilon,x,y,p,q)$ appearing in (3.59) and apply (3.25), (3.92), and (3.94) to attain the integral,

$$\begin{aligned} c^m|x-x'|^{m\xi}\iiint\limits_{\mathbf{R}^{2m}\times B^m}\exp\left(-\sum_{k=1}^{m}\bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{I^c}u_k^2(\tilde{d}_k)\right) \\ \cdot\left|\prod_{k\in I}\left(\exp\left(-u_k^2(\tilde{d}_k)\right)-\exp\left(-\left(u_{k+1}^2+q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right)\right| \\ \prod_{j=1}^{m}|q_j|\;|p_j|^{\xi}\;\mathrm{d}p_j\;\mathrm{d}q_j\;\mathrm{d}r_j\;\mathrm{d}s_j\;\mathrm{d}t_j. \end{aligned} \tag{3.133}$$

Proceeding as in our analysis of (3.60), we first integrate the isolated intervals, $\tilde{d}_k$, and then the isolated variables, $q_{\ell_{(k)}}$, from the above expression. Notice, $\prod|p_j|^{\xi}$ has no influence on these terms. The study of (3.61) allows us to bound the integral in (3.133) by

$$\iiint\limits_{\mathbf{R}^m \times \mathbf{R}^n \times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right) \prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j| \, \mathrm{d}q_j \prod_{j=1}^{m} |p_j|^{\xi} \, \mathrm{d}p_j \, \mathrm{d}r_j \, \mathrm{d}s_j \, \mathrm{d}t_j, \tag{3.134}$$

where q_j are the remaining non-isolated variables, I_j^c is defined by (3.15), and the n appearing in the domain of integration is the cardinality of I_j^c.

In the above expression, we will address integrals involving the $\tilde{r}_k$ terms and the remaining non-isolated variables $\tilde{d}_k$ as we did in (3.73), leaving the product of $|p_j|^{\xi}$ terms to analyze. However, in our inquiry of the variation of x over the non-isolated intervals, we derived a bound for this product, specifically, $\mathcal{P}_3(|\bar{p}_k|)$ of (3.99). Combining these assertions, (3.134) can be bounded by the sum

$$\sum \left[\iint\limits_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{R} |\bar{p}_k|^{\eta_k \xi} \prod_{k=1}^{m} \left(1 + \bar{p}_k^2\right)^{-1} \prod_{I^c} \left(1 + u_k^2\right)^{-1} \prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j| \, \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j \right]. \tag{3.135}$$

Consider any multiple integral emerging as a summand in the above expression and notice the quantity

$$\prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j|$$

emerged previously in (3.74). The diagnosis of these products led to the introduction of sets T^{I+1} and T'' of (3.76) and (3.80), respectively, and eventually to polynomial defined in (3.78). Since $\prod |\bar{p}_k|^{\eta_k \xi}$ is independent of the products above, we can insert $\mathcal{P}_2(|u_k|)$ into any integral of (3.135), apply the bound $|u_k|^{\omega_k}(1 + u_k^2)^{-1} \leq c$ for all $k \in T^{I+1} \cup T''$, and reduce our study to integrals of the form

$$\iint_{\mathbf{R}^m\times\mathbf{R}^n} \prod_R |\bar{p}_k|^{\eta_k\xi} \prod_{k=1}^m \left(1+\bar{p}_k^2\right)^{-1} \prod_{I^c/\{T^{I+1}\cup T''\}} \left(1+u_k^2\right)^{-1}$$
$$\prod_{S'} |u_k|^{\omega_k} \prod_{S^-} |u_k|^{\omega_k} \prod_{T^+} |u_k|^{\omega_k} \prod_{I_j^c} \mathrm{d}q_j \prod_{j=1}^m \mathrm{d}p_j.$$

This, however, is the precise integral appearing in (3.104). Employ the methods from that analysis to conclude (3.97) over the isolated intervals.

We now analyze the variation of y on $\phi_y(\epsilon, x, y, B)$ over the isolated intervals. Following the strategy presented to produce (3.133), we come to the following integrals,

$$c^m|y-y'|^{m\xi} \iiint_{\mathbf{R}^{2m}\times B^m} \exp\left(-\sum_{k=1}^m \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right)$$
$$\cdot \left| \prod_{k\in I} \left(\exp\left(-u_k^2(\tilde{d}_k)\right) - \exp\left(-\left(u_{k+1}^2 + q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right) \right|$$
$$\prod_{j=1}^m |q_j|\ |q_j|^\xi \ \mathrm{d}p_j\ \mathrm{d}q_j\ \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j. \tag{3.136}$$

In the course of our examination of integrals similar to the above, for example (3.60), it was customary at this point to proceed with first integrating the isolated intervals and then the associated isolated variables. We will continue with this pattern however, we must now consider the introduction of isolated variables taking the form $|q_j|^\xi$. We initiate our bound of (3.136) by studying an integral similar to (3.61),

$$\iint_{\mathbf{R}\times B} \left| \left(\exp\left(-u_k^2(\tilde{d}_k)\right) - \exp\left(-\left(u_{k+1}^2 + q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\right) \right| \ |q_{\ell_{(k)}}|\ \ |q_{\ell_{(k)}}|^\xi \ \mathrm{d}(\tilde{d}_k)\ \mathrm{d}q_{\ell_{(k)}}. \tag{3.137}$$

To start working with the above, apply the precise methodologies which produced the upper bounds appearing in (3.62) and (3.63),

$$\iint\limits_{\mathbf{R}\times B} \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; |q| \; |q|^{\xi} \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q$$

$$\leq \int\limits_{|q|\leq 4|u|} \int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; |q| \; |q|^{\xi} \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q \quad (3.138)$$

$$+ \int\limits_{|q|\geq 4|u|} \int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; |q| \; |q|^{\xi} \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q. \quad (3.139)$$

Evaluating first the isolated intervals $\tilde{d}_k$ and duplicating (3.64) and (3.65) implies (3.138) can be bounded by

$$\int\limits_{|q|\leq 4|u|} \int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; |q| \; |q|^{\xi} \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q$$

$$\leq c|u|^{1+\xi}. \quad (3.140)$$

Turning focus to (3.139), (3.67)-(3.72) shows

$$\int\limits_{|q|\geq 4|u|} \int_0^1 \left|\exp\left(-(u+q)^2(\tilde{d}_k)\right) - \exp\left(-\left(u^2+q^2\right)(\tilde{d}_k)\right)\right| \; |q| \; |q|^{\xi} \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q$$

$$\leq \int_{|q|\geq 4|u|} \left|\frac{1}{(1+u/q)^2} - \frac{1}{(1+u^2/q^2)}\right| \; \frac{1}{|q|^{1-\xi}} \; \mathrm{d}q$$

$$\leq c \int_{|q|\geq 4|u|} \frac{|u|}{|q|^{2-\xi}} \; \mathrm{d}q$$

$$\leq c|u|^{\xi}. \quad (3.141)$$

The two bounds, (3.140) and (3.141), entails (3.137) is bounded by

$$c|u_{k+1}|^{1+\xi} + c|u_{k+1}|^{\xi} \leq |u_{k+1}|^{1+\xi} + c, \quad k \in I$$

since, $|u_{k+1}|^{\xi} \leq 1$ for small u_{k+1}. This inclines us to bound the integral appearing in (3.136) by

$$\iiint\limits_{\mathbf{R}^m\times\mathbf{R}^n\times B^m} \exp\left(-\sum_{k=1}^{m}\bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{I^c}u_k^2(\tilde{d}_k)\right)$$
$$\prod_{k\in I}\left[c+|u_{k+1}|^{1+\xi}\right]\prod_{I_j^c}|q_j|\;|q_j|^{\xi}\;\mathrm{d}q_j\prod_{k=1}^{m}\mathrm{d}p_j\;\mathrm{d}r_j\;\mathrm{d}s_j\;\mathrm{d}t_j,$$

where q_j are the remaining non-isolated variables, I_j^c is defined by (3.15), and the n appearing in the domain of integration is the cardinality of I_j^c.

Addressing the first summand in the brackets above returns us to the case of the variation in y parameter over the non-isolated interval. Working with the second summand,

$$\iiint\limits_{\mathbf{R}^m\times\mathbf{R}^n\times B^m} \exp\left(-\sum_{k=1}^{m}\bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{I^c}u_k^2(\tilde{d}_k)\right)$$
$$\prod_{k\in I}|u_{k+1}|^{1+\xi}\prod_{I_j^c}|q_j|\;|q_j|^{\xi}\;\mathrm{d}q_j\prod_{k=1}^{m}\mathrm{d}p_j\;\mathrm{d}r_j\;\mathrm{d}s_j\;\mathrm{d}t_j, \tag{3.142}$$

an application of (3.28), with respect to $\tilde{r}_k$ and the remaining non-isolated $\tilde{d}_k$ variables, indicates (3.142) is less than

$$\iint\limits_{\mathbf{R}^m\times\mathbf{R}^n}\prod_{k=1}^{m}\left(1+\bar{p}_k^2\right)^{-1}\prod_{I^c}\left(1+u_k^2\right)^{-1}\prod_{k\in I}|u_{k+1}|^{1+\xi}\prod_{I_j^c}|q_j|\;|q_j|^{\xi}\;\mathrm{d}q_j\prod_{j=1}^{m}\mathrm{d}p_j. \tag{3.143}$$

To prove that the above integral is finite, the first expression we diagnose is

$$\prod_{k\in I}|u_{k+1}|^{1+\xi}\prod_{I_j^c}|q_j|\;|q_j|^{\xi}. \tag{3.144}$$

The product involving the $|u_{k+1}|$ variables was identified in (3.74). We reintroduce the set

T^{I+1} of (3.76). To account for the addition of the ξ term in the exponent, let

$$T_{\xi}^{I+1} \doteq \{k \in T \mid k-1 \in I \text{ and } \deg |u_k| = 1+\xi\} \tag{3.145}$$

and write

$$\prod_{k \in I} |u_{k+1}|^{1+\xi} = \prod_{k \in T_{\xi}^{I+1}} |u_k|^{1+\xi}.$$

The product $\prod_{I_j^c} |q_j| \, |q_j|^{\xi}$ initially arose in (3.107) and the subsequent discussion following those multiple integrals leads us to Proposition 3.2.2. This implies (3.144) can be bounded above by

$$\begin{aligned}
&\prod_{k \in I} |u_{k+1}|^{1+\xi} \prod_{I_j^c} |q_j| \, |q_j|^{\xi} \\
&\quad \leq \prod_{k \in T_{\xi}^{I+1}} |u_k|^{1+\xi} \prod_{k \in T} (|u_{k-1}| + |u_k|) \prod_{\substack{k \in S \\ k', k'' \in R}} \left(|u_k|^{\xi} + |u_{k-1}|^{\xi} + |\bar{p}_{k'}|^{\xi} + |\bar{p}_{k''}|^{\xi}\right).
\end{aligned}$$

Our aim is to evaluate the order of the $|u_k|$ variables emerging from the expansion of the upper bound. To our advantage, we can leverage much of our work from previous propositions.

Proposition 3.2.4. *The polynomial generated by multiplying the products*

$$\mathcal{P}_6(|\bar{p}_k|, |u_k|) \doteq \prod_{k \in T_{\xi}^{I+1}} |u_k|^{1+\xi} \prod_{k \in T} (|u_{k-1}| + |u_k|) \prod_{\substack{k \in S \\ k', k'' \in R}} \left(|u_k|^{\xi} + |u_{k-1}|^{\xi} + |\bar{p}_{k'}|^{\xi} + |\bar{p}_{k''}|^{\xi}\right) \tag{3.146}$$

is of the form

$$\mathcal{P}_6(|\bar{p}_k|,|u_k|) = \sum \left(\prod_R |\bar{p}_k|^{\eta_k \xi} \prod_{S'_\xi} |u_k|^{\theta_k \xi} \prod_{S^-_\xi} |u_k|^{\omega_k + \theta_k \xi} \prod_{T^{I+1}_\xi, T^+_\xi, T''_\xi} |u_k|^{\omega_k + \theta_k \xi} \right), \tag{3.147}$$

where

$$\deg |\bar{p}_k| = \eta_k \xi \in \{0, \xi, 2\xi\}, \qquad \textit{if } k \in R$$

$$deg\ |u_k| = \omega_k + \theta_k \xi \begin{cases} = 0, \xi, \ \textit{or } 2\xi, & \textit{if } k \in S'_\xi \\ = 1 \ \textit{or } 1+\xi, & \textit{if } k \in S^-_\xi \\ = \xi \ \textit{or } 1, & \textit{if } k \in T^+_\xi \\ = 1+\xi, 1+2\xi, \ \textit{or } 2+\xi, & \textit{if } k \in T^{I+1}_\xi \\ \le 2, & \textit{if } k \in T''_\xi. \end{cases} \tag{3.148}$$

Proof. We start with the T^{I+1}_ξ-intervals and isolate the products

$$\prod_{k \in T^{I+1}_\xi} |u_k|^{1+\xi} \prod_{k \in T} (|u_{k-1}| + |u_k|). \tag{3.149}$$

In the analysis of (3.77), we considered the two cases when the isolated interval $[s_\ell, t_\ell]$ is followed by either a non-isolated t-interval or a s-interval which is either isolated or non-isolated. Reapplying this argument, we can conclude deg $\{|u_k|\}_{T^{I+1}_\xi} \in \{1+\xi, 2+\xi\}$ in the expansion of (3.149).

Now, take into account the quantity

$$\prod_{k \in T^{I+1}_\xi} |u_k|^{1+\xi} \prod_{k \in S} \prod_{k',k'' \in R} \left(|u_k|^\xi + |u_{k-1}|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi\right). \tag{3.150}$$

To understand the contribution from the T^{I+1}_ξ-intervals, we need to examine the lone case when a non-isolated s-interval is preceded by a isolated interval $[s_\ell, t_\ell]$; $d_k = s_\ell$, $d_{k+1} = t_\ell$, and $d_{k+2} = s_m$, for some m. Substitution (3.108) shows $q_m = u_{k+2} - u_{k+1} + p_m$ and (3.109) allows us to bound

$$|q_m|^\xi \leq |u_{k+2}|^\xi + |u_{k+1}|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi.$$

The $|u_{k+1}|^\xi$ on the right side of the inequality is from a T_ξ^{I+1}-interval and since this term can emerge twice in (3.150), we can conclude the maximum degree of the elements from $\{|u_k|\}_{T_\xi^{I+1}}$ in this product are equal to $1 + 2\xi$. Combining this result with the conclusion following (3.149), we have

$$\deg |u_k| \in \{1 + \xi, 1 + 2\xi, 2 + \xi\}, \quad k \in T_\xi^{I+1},$$

with respect to the the polynomial resulting from (3.146).

The remaining non-isolated s-intervals and t-intervals to examine originate from

$$\prod_{k \in T} (|u_{k-1}| + |u_k|) \prod_{k \in S} \prod_{k', k'' \in R} \left(|u_k|^\xi + |u_{k-1}|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi \right). \tag{3.151}$$

This provides us with the opportunity to incorporate Proposition 3.2.2. Recollect sets $S_\xi^-, T_\xi^+, S_\xi^{'}$ of (3.110), (3.111), (3.112), respectively, and the summary of the degree of the $|u_k|$ variables in $\mathcal{P}_4(|\bar{p}_k|, |u_k|)$. The $T_\xi^{'}$-intervals of (3.113) must be readjusted to account for the introduction of the product $\prod_{k \in T_\xi^{I+1}} |u_k|^{1+\xi}$ of (3.146). Introduce the non-isolated t-intervals

$$T_\xi^{''} \doteq \{k \in T \mid k \notin T_\xi^{I+1} \cup T_\xi^+\} \tag{3.152}$$

and observe the maximum order these terms can attain in the expansion of (3.151) is of order 2. To prove this fact, we point the reader to the discussion subsequent to (3.118).

With the order of the $|u_k|$ variable settled, (3.146) can be expressed as

$$\mathcal{P}_6(|\bar{p}_k|,|u_k|) = \sum \left(\prod_{R} |\bar{p}_k|^{\eta_k \xi} \prod_{S'_\xi} |u_k|^{\theta_k \xi} \prod_{S^-_\xi} |u_k|^{\omega_k + \theta_k \xi} \prod_{T^{I+1}_\xi, T^+_\xi, T''_\xi} |u_k|^{\omega_k + \theta_k \xi} \right).$$

□

Inserting $\mathcal{P}_6(|\bar{p}_k|,|u_k|)$ into (3.143) yields

$$\sum \Bigg[\iint\limits_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{R} |\bar{p}_k|^{\eta_k \xi} \prod_{k=1}^{m} \left(1+\bar{p}_k^2\right)^{-1} \prod_{S,T} \left(1+u_k^2\right)^{-1} \prod_{S'_\xi} |u_k|^{\theta_k \xi} \prod_{S^-_\xi} |u_k|^{\omega_k + \theta_k \xi}$$
$$\prod_{T^+_\xi} |u_k|^{\omega_k + \theta_k \xi} \prod_{T^{I+1}_\xi} |u_k|^{\omega_k + \theta_k \xi} \prod_{T''_\xi} |u_k|^{\omega_k + \theta_k \xi} \prod_{I^c_j} \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j \Bigg]. \tag{3.153}$$

Our objective is to reduce the above multiple integral to the form required to apply Proposition 3.1.4 . To implement that result, we need to bound the rational functions connected with the sets T^{I+1}_ξ and T''_ξ. The former of the non-isolated t-intervals are problematic since there exists $k \in T^{I+1}_\xi$ with $\deg |u_k| = 2+\xi$ so that

$$\lim_{u_k \to \infty} \frac{|u_k|^{2+\xi}}{1+u_k^2} = \infty. \tag{3.154}$$

To highlight these cases set

$$T^{I+1}_{2+\xi} \doteq \left\{ k \in T^{I+1}_\xi \;\middle|\; \deg |u_k| = 2+\xi \right\}. \tag{3.155}$$

With the assistance of other s and t-intervals, we demonstrate how to resolve this dilemma. Recall, $\deg |u_k| = 2+\xi$, $k \in T^{I+1}_\xi$, in polynomial (3.147) is a consequence of (3.149) and the exact scenarios which generate these cases are ones in which an isolated interval is followed two non-isolated t-intervals. In more precise notation, $[s_\ell, t_\ell]$ is an isolated interval with

$d_k = s_\ell$, $d_{k+1} = t_\ell$, and $d_{k+2} = t_m$. Obviously $k+1 \in T_\xi^{I+1}$ and definition (3.152) reveals $k+2 \in T_\xi^{''}$. Notice, u_{k-1} is associated with a non-isolated interval preceding $[s_\ell, t_\ell]$ and originates from either a $S_\xi^{'}$-interval or any of the t-intervals $T_\xi^{+}, T_\xi^{I+1}, T_\xi^{''}$. To identify these sets let

$$S_\xi^{I-1} \doteq \left\{ k \in S_\xi^{'} \mid k+2 \in T_{2+\xi}^{I+1} \right\} \tag{3.156}$$

$$T_\xi^{I-1} \doteq \left\{ k \in T_\xi^{+} \cup T_\xi^{I+1} \cup T_\xi^{''} \mid k+2 \in T_{2+\xi}^{I+1} \right\}. \tag{3.157}$$

Figure 3.6 below illustrates the scenarios which generate the $T_{2+\xi}^{I+1}$-intervals. The u variables affiliated with the above sets play a vital role in our argument and two properties of these terms are of particular interest and presented in Proposition 3.2.5.

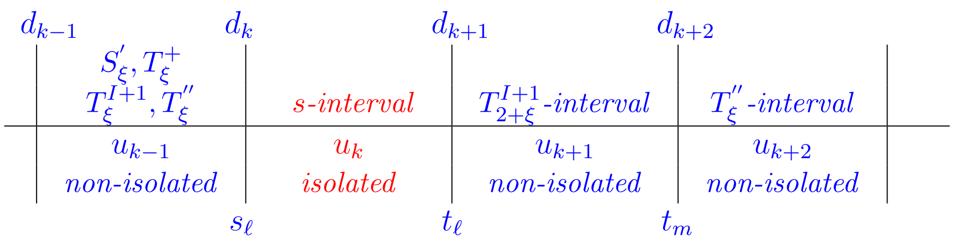

Figure 3.6: Scenarios which generate the $T_{2+\xi}^{I+1}$-intervals .

Proposition 3.2.5. *Let u_k be such that $k \in S_\xi^{I-1} \cup T_\xi^{I-1}$.*

i.) With respect to polynomial $\mathcal{P}_6(|\bar{p}_k|, |u_k|)$ of (3.147),

$$\deg |u_k| \leq \begin{cases} \xi, & \textit{if } k \in S_\xi^{I-1}, \\ 1+\xi, & \textit{if } k \in T_\xi^{I-1}. \end{cases} \tag{3.158}$$

ii.) Any u_k variable can be reexpressed as the sum

$$u_k = u_{k+2} + p_{\ell_{(k+1)}}, \tag{3.159}$$

with $k+1 \in I$ and $k+2 \in T_\xi^{I+1}$.

Proof. i.) Beginning with the s-intervals under analysis, consider a u_k variable such that $k \in S_\xi^{I-1}$. The preceding non-isolated interval is either another $S_\xi^{'}$-interval or linked to any of the t-intervals, T_ξ^{I+1} or $T_\xi^{''}$. In each of these three cases, u_k will appear at most once in the second product on the right hand side of (3.151). Hence, $\deg |u_k| \leq \xi$ with respect to $\mathcal{P}_6(|\bar{p}_k|, |u_k|)$.

Turning to the t-interval, the scenarios presented in (3.148) leave us to resolve two cases. The first involves $k \in T_\xi^{I+1} \cap T_\xi^{I-1}$ and we can conclude $\deg |u_k| = 1 + \xi$ for these specific variables. In more detail, if $k \in T_\xi^{I+1}$ and $k+1$ is an isolated interval, then u_k will appear at most once in (3.149). For the second case, let k be an element from the set $T_\xi^{''} \cap T_\xi^{I-1}$. Since $k+1 \in I$, u_k emerges only once in the product

$$\prod_{k \in T} (|u_{k-1}| + |u_k|)$$

and this proves the first property.

ii.) The second acute characteristic shared by the elements from $\{u_k\}_{S_\xi^{I-1}} \cup \{u_k\}_{T_\xi^{I-1}}$ states that each term can be rewritten so that the variable associated with the T_ξ^{I+1}-interval surfaces as a summand. Indeed, if $k \in S_\xi^{I-1} \cup T_\xi^{I-1}$, Figure 3.4 reveals

$$u_k = u_{k+2} + p_{\ell_{(k+1)}},$$

with $k+1 \in I$ and $k+2 \in T_\xi^{I+1}$.

□

Let us now return to the rational integrands of (3.154), which motivated this discussion, and apply the second result from the above proposition. If $k \in S_\xi^{I-1} \cup T_\xi^{I-1}$ and $k+2 \in T_{2+\xi}^{I+1}$, we can write

$$\frac{|u_{k+2}|^{2+\xi}}{1+u_{k+2}^2} = \frac{|u_{k+2}|^2 |u_{k+2}|^\xi}{1+u_{k+2}^2} = |u_{k+2}|^\xi \le c|u_k - p_{\ell_{(k+1)}}|^\xi \le c\left(|u_k|^\xi + |p_{\ell_{(k+1)}}|^\xi\right) \tag{3.160}$$

and bound the product of rational functions originating from the T_ξ^{I+1}-intervals by

$$\begin{aligned}\prod_{T_\xi^{I+1}} \frac{|u_k|^{\omega_k+\theta_k\xi}}{1+u_k^2} &= \prod_{T_\xi^{I+1}/T_{2+\xi}^{I+1}} \frac{|u_k|^{\omega_k+\theta_k\xi}}{1+u_k^2} \prod_{T_{2+\xi}^{I+1}} \frac{|u_k|^{2+\xi}}{1+u_k^2} \le c \prod_{T_\xi^{I+1}/T_{2+\xi}^{I+1}} \frac{|u_k|^{\omega_k+\theta_k\xi}}{1+u_k^2} \prod_{T_{2+\xi}^{I+1}} |u_k|^\xi \\ &\le c \prod_{T_\xi^{I+1}/T_{2+\xi}^{I+1}} \frac{|u_k|^{\omega_k+\theta_k\xi}}{1+u_k^2} \prod_{\substack{k\in S_\xi^{I-1}\cup T_\xi^{I-1} \\ k+1\in I}} \left(|u_k|^\xi + |p_{\ell_{(k+1)}}|^\xi\right).\end{aligned}$$

If we expand the product with respect to the $|u_k|$ and $|p_{\ell_{(k+1)}}|$ variables and recall property (3.101), we arrive at the subsequent polynomial,

$$\prod_{\substack{k\in S_\xi^{I-1}\cup T_\xi^{I-1} \\ k+1\in I}} \left(|u_k|^\xi + |p_{\ell_{(k+1)}}|^\xi\right) = \prod_{\substack{k\in S_\xi^{I-1}\cup T_\xi^{I-1} \\ k',k''\in R}} \left(|u_k|^\xi + |\bar p_{k'} - \bar p_{k''}|^\xi\right) \le c\mathcal{P}_7(|\bar p_k|, |u_k|)$$

with

$$\mathcal{P}_7(|\bar p_k|, |u_k|) \doteq \prod_{\substack{k\in S_\xi^{I-1}\cup T_\xi^{I-1} \\ k',k''\in R}} \left(|u_k|^\xi + |\bar p_{k'}|^\xi + |\bar p_{k''}|^\xi\right) = \sum \left(\prod_{S_\xi^{I-1}\cup T_\xi^{I-1}} |u_k|^\xi \prod_R |\bar p_k|^{\tilde\eta_k\xi} \right), \tag{3.161}$$

where $\tilde\eta_k\xi$ takes values in the set $\{0, \xi, 2\xi\}$. Hence, we have shown how to express u variables linked to the $T_{2+\xi}^{I-1}$-intervals in terms of other u_k, $k \in S_\xi^{I-1} \cup T_\xi^{I-1}$, and $\bar p_{k'}$, $k' \in R$, variables.

Our next step is to analyze the impact of the above polynomial on (3.153). However, before proceeding in this manner, we first look to reconcile the order of the $|\bar{p}_k|$ variables as they originate from the two polynomials $\mathcal{P}_6(|\bar{p}_k|, |u_k|)$ and $\mathcal{P}_7(|\bar{p}_k|, |u_k|)$.

To set up the next proposition, we recall several key facts. With respect to $\mathcal{P}_6(|\bar{p}_k|, |u_k|)$, contributions to the order of the $|\bar{p}_k|$ terms, denoted as $\eta_k \in \{0, \xi, 2\xi\}$ in (3.148), originate from the p variables associated with the non-isolated s-intervals and, specifically, from the products below

$$\begin{aligned}\prod_{I_j^c} |q_j|^\xi &= \prod_{\substack{k\in S\\ j\in I_j^c}} \left(|u_k - u_{k-1} + p_j|^\xi\right) = \prod_{\substack{k\in S\\ k',k''\in R}} \left(|u_k - u_{k-1} + \bar{p}_{k'} - \bar{p}_{k''}|^\xi\right)\\ &\leq c \prod_{\substack{k\in S\\ k',k''\in R}} \left(|u_k|^\xi + |u_{k-1}|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi\right). \end{aligned} \tag{3.162}$$

Recall, we formulated this upper bound by combining (3.108),

$$u_k - u_{k-1} + p_\ell = q_\ell, \quad \text{for } d_k = s_\ell,$$

with (3.109). Additionally, in the process of deriving (3.153), we addressed the isolated intervals and their associated isolated variables. Hence, if $[s_\ell, t_\ell]$ is an isolated interval, then the p_ℓ variable paired with this interval will not appear in (3.162).

Replicating the above ideas, contributions to the order of the $|\bar{p}_k|$ terms in $\mathcal{P}_7(|\bar{p}_k|, |u_k|)$, listed as $\tilde{\eta}_k \in \{0, \xi, 2\xi\}$, can be traced to the products below and, particularly, to the p terms linked to isolated intervals,

$$\prod_{T_{2+\xi}^{I+1}} |u_k|^\xi \leq \prod_{\substack{k\in S_\xi^{I-1}\cup T_\xi^{I-1}\\ k+1\in I}} \left(|u_k|^\xi + |p_{\ell_{(k+1)}}|^\xi\right) \leq c \prod_{\substack{k\in S_\xi^{I-1}\cup T_\xi^{I-1}\\ k',k''\in R}} \left(|u_k|^\xi + |\bar{p}_{k'}|^\xi + |\bar{p}_{k''}|^\xi\right). \tag{3.163}$$

Recollect, the above was derived using Proposition 3.2.5 (ii). With these clarification, we state and prove the following,

Proposition 3.2.6. *Upon reexpressing the rational functions in* $\mathcal{P}_6(|\bar{p}_k|, |u_k|)$ *that are of the form presented in* (3.154)*, with the assistance of* $\mathcal{P}_7(|\bar{p}_k|, |u_k|)$*, the maximum degree the* $|\bar{p}_k|$ *variables can assume in the resulting polynomial is* 2ξ*.*

Proof. The three cases which we must consider are $(\eta, \tilde{\eta}) \in \{(2\xi, 0), (\xi, \xi), (0, 2\xi)\}$ and the recursive relations amongst the p terms, as described in (3.102), will be critical in our analysis. To set the notation for the the proof, consider the following for any k', m, n, and ℓ;

$$
\begin{aligned}
p_m &= \bar{p}_{k'} - \bar{p}_{k'-1}, \quad &&\text{for } \bar{r}_{k'} = r_m \\
p_n &= \bar{p}_{k'+1} - \bar{p}_{k'}, \quad &&\text{for } \bar{r}_{k'+1} = r_n \\
p_\ell &= \bar{p}_{k'+2} - \bar{p}_{k'+1}, \quad &&\text{for } \bar{r}_{k'+2} = r_\ell.
\end{aligned}
$$

Turing attention to the first case, $(\eta, \tilde{\eta}) = (2\xi, 0)$, let $d_k = s_m$, $k \in S_\xi^{'}$, $d_{k+1} = s_n$, $k+1 \in S_\xi^{'}$. If $k+2$ is a non-isolated s or t-interval, then we do not receive any contribution from the p variable paired with this interval in (3.163). Hence, if $\eta = 2$, then $\tilde{\eta} = 0$. This is the exact case discussed with regard to (3.122).

Now, consider a scenario when $d_k = s_m$, $k \in S_\xi^{'}$, $d_{k+1} = s_n$, $k+1 \in S_\xi^{I-1}$, $d_{k+2} = s_\ell$, $k+2 \in I$, and $d_{k+3} = t_\ell$, $k+3 \in T_{2+\xi}^{I+1}$. To understand the order of the $|\bar{p}_k|$ terms with respect to $\mathcal{P}_6$, write

$$
\begin{aligned}
q_m &= |u_k - u_{k-1} + p_m| = |u_k - u_{k-1} + \bar{p}_{k'} - \bar{p}_{k'-1}| \\
q_n &= |u_{k+1} - u_k + p_n| = |u_{k+1} - u_k + \bar{p}_{k'+1} - \bar{p}_{k'}|
\end{aligned}
$$

and, since q_n can not appear again in (3.162), observe that deg $|\bar{p}_{k'+1}| \leq \xi$. Shifting to $\mathcal{P}_7$, since $k+3 \in T^{I+1}_{2+\xi}$, we must rewrite the u term paired with this t-interval as

$$u_{k+3} = u_{k+1} - p_{k+2} = u_{k+1} - p_\ell = u_{k+1} - \bar{p}_{k'+2} + \bar{p}_{k'+1}.$$

Since p_ℓ is affiliated with an isolated interval, $[s_\ell, t_\ell]$,this term can not arise in (3.162) and, therefore, deg $|\bar{p}_{k'+1}| \leq \xi$. Combining these facts, we see that when $\eta \leq \xi$, then $\tilde{\eta} \leq \xi$.

Finally, to prove $(\eta, \tilde{\eta}) = (0, 2\xi)$ let $[s_n, t_n]$ and $[s_\ell, t_\ell]$ each be isolated intervals which are followed by $T^{I+1}_{2+\xi}$ intervals. To satisfy these conditions, we must introduce a parameter $b \geq 3$ and write, $d_k = s_n$, $k \in I$, $d_{k+1} = t_n$, $k+1 \in T^{I+1}_{2+\xi}$, $d_{k+b} = s_\ell$, $k+b \in I$, and $d_{k+b+1} = t_\ell$, $k+b+1 \in T^{I+1}_{2+\xi}$. The u variables of concern can be reformulated as

$$\begin{aligned} u_{k+1} &= u_{k+1} - p_k = u_{k+1} - p_n = u_{k+1} - \bar{p}_{k'+1} + \bar{p}_{k'} \\ u_{k+b+1} &= u_{k+b-1} - p_{k+b} = u_{k+b-1} - p_\ell = u_{k+b-1} - \bar{p}_{k'+2} + \bar{p}_{k'+1} \end{aligned}$$

and note that $\bar{p}_{k'+1}$ emerges as a summand twice. Since p_n and p_ℓ can not appear again in (3.162), we conclude that when $\tilde{\eta} = 2\xi$, then $\eta = 0$. $\square$

Having completed the above analysis, we return to (3.153) and observe, with the assistance of $\mathcal{P}_7(|\bar{p}_k|, |u_k|)$, the subsequent upper bound

$$\begin{aligned} & c^m \iint_{\mathbf{R}^m \times \mathbf{R}^n} \prod_R |\bar{p}_k|^{(\eta_k + \tilde{\eta}_k)\xi} \prod_{k=1}^m \left(1 + \bar{p}_k^2\right)^{-1} \prod_{S'_\xi} \frac{|u_k|^{\theta_k \xi}}{1 + u_k^2} \prod_{S^{I-1}_\xi} |u_k|^\xi \prod_{S^-_\xi} \frac{|u_k|^{\omega_k + \theta_k \xi}}{1 + u_k^2} \\ & \prod_{T^+_\xi} \frac{|u_k|^{\omega_k + \theta_k \xi}}{1 + u_k^2} \prod_{T^+_\xi \cap T^{I-1}_\xi} |u_k|^\xi \left[\prod_{T^{I+1}_\xi / T^{I+1}_{2+\xi}} \frac{|u_k|^{\omega_k + \theta_k \xi}}{1 + u_k^2} \prod_{T^{I+1}_\xi \cap T^{I-1}_\xi} |u_k|^\xi \right] \\ & \left[\prod_{T''_\xi} \frac{|u_k|^{\omega_k + \theta_k \xi}}{1 + u_k^2} \prod_{T''_\xi \cap T^{I-1}_\xi} |u_k|^\xi \right] \prod_{I^c_j} \mathrm{d}q_j \prod_{j=1}^m \mathrm{d}p_j. \end{aligned} \tag{3.164}$$

To address the product of rational functions over the t-intervals confined in the brackets above, we confront three cases. When $k \in \{T_\xi^{I+1} \cap T_\xi^{I-1}\} \cup \{T_\xi^{''} \cap T_\xi^{I-1}\}$, (3.158) suggests the power of $|u_k|$, $\omega_k + \theta_k \xi$, is less than or equal to $1+\xi$. Therefore, the introduction of the $|u_k|^\xi$ terms for these scenarios does not preclude the bound

$$\frac{|u_k|^{\omega_k+\theta_k\xi}|u_k|^\xi}{1+u_k^2} \leq c, \quad k \in \{T_\xi^{I+1} \cap T_\xi^{I-1}\} \cup \{T_\xi^{''} \cap T_\xi^{I-1}\}$$

For the second and third case, if $k \in \{T_\xi^{I+1}/T_{2+\xi}^{I+1}\}/T_\xi^{I-1}$ or $k \in T_\xi^{''}/T_\xi^{I-1}$, then (3.148) reveals, deg $|u_k| = 1+2\xi$ or deg $|u_k| \leq 2$, respectively, which leads to the bound $|u_k|^{\omega_k+\theta_k\xi}\left(1+u_k^2\right)^{-1} \leq c$. Multiple integral (3.164) can now be bounded by

$$c \iint\limits_{\mathbf{R}^m \times \mathbf{R}^n} \prod_R |\bar{p}_k|^{(\eta_k+\tilde{\eta_k})\xi} \prod_{k=1}^m \left(1+\bar{p}_k^2\right)^{-1} \prod_{S_\xi^{'}} \frac{|u_k|^{\theta_k\xi}}{1+u_k^2} \prod_{S_\xi^{I-1}} |u_k|^\xi \prod_{S_\xi^-} \frac{|u_k|^{\omega_k+\theta_k\xi}}{1+u_k^2}$$
$$\prod_{T_\xi^+} \frac{|u_k|^{\omega_k+\theta_k\xi}}{1+u_k^2} \prod_{T_\xi^+ \cap T_\xi^{I-1}} |u_k|^\xi \prod_{I_j^c} \mathrm{d}q_j \prod_{j=1}^m \mathrm{d}p_j.$$

Proposition 3.1.4 can be successfully applied to the above expression, upon addressing two concerns. The emergence of the product

$$\prod_{S_\xi^{I-1} \cup \{T_\xi^{''} \cap T_\xi^{I-1}\}} |u_k|^\xi$$

indicates the maximum power of the $\{|u_k|\}_{S_\xi^{I-1}}$ and $\{|u_k|\}_{\{T_\xi^{''} \cap T_\xi^{I-1}\}}$ terms are 3ξ and $1+\xi$, respectively. With these facts in mind, the parameter ξ can be chosen close to zero to ensure that the subsequent inequalities are satisfied;

$$(\eta_k + \tilde{\eta_k})\xi < 1, \text{ for all } k \in \mathcal{M}, \qquad \theta_k\xi + \mathbb{1}_{S_\xi^{I-1}}\xi < 1, \text{ for all } k \in S_\xi^{'},$$
$$\omega_k + \theta_k\xi < 3/2, \text{ for all } k \in S_\xi^-, \quad \omega_k + \theta_k\xi + \mathbb{1}_{T_\xi^{I-1}}\xi < 3/2, \text{ for all } k \in T_\xi^+.$$

To procure (3.126) over the isolated intervals, (3.127) and (3.128) concedes

$$c^m|\epsilon-\epsilon'|^{m\xi}\iiint_{\mathbf{R}^{2m}\times B^m}\exp\left(-\sum_{k=1}^{m}\bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{I^c}u_k^2(\tilde{d}_k)\right)$$
$$\cdot\left|\prod_{k\in I}\left(\exp\left(-u_k^2(\tilde{d}_k)\right)-\exp\left(-\left(u_{k+1}^2+q_{\ell(k)}^2\right)(\tilde{d}_k)\right)\right)\right|$$
$$\prod_{j=1}^{m}\left(|p_j|^{\xi}+|q_j|^{\xi}\right)\ |q_j|\ \mathrm{d}p_j\ \mathrm{d}q_j\ \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j. \tag{3.165}$$

Applying equality (3.130) returns us to the study of the x and y parameter. Therefore, we have established (3.90) and the Hölder continuity of $\phi_y(\epsilon,x,y,B)$ over $A(1,1,2)$. Application of Kolmogorov's Continuity Criterion guarantees the local uniform convergence of

$$\lim_{\epsilon\to 0}\phi_y(\epsilon,x,y,B)=\phi_y(x,y,B),$$

with the $\phi_y(x,y,B)$ continuous in (x,y).

3.2.3 Local properties of the intersection local times

With the detailed analysis regarding the continuity of $\phi_y(\epsilon,x,y,B)$ complete, we now look to extend the same properties to $\phi(\epsilon,x,y,B)$. In Section 3.1.3 we derived a bound for the m^{th} moment of the later quantity and saw the proof was simplified by the fact that the product $\prod q_j$ which appears in (3.2) does not appear in (3.86). Our new analysis will also be void of these terms, thus, permitting us to streamline our arguments. Most of the hard work has been completed in the previous §3.2.1 and §3.2.2 and we will leverage many of those conclusions. To begin, we look to establish a bound on each individual term appearing on the right hand side of the inequality below

$$\|\phi(\epsilon,x,y,B)-\phi(\epsilon',x',y',B)\|_m \le \|\phi(\epsilon,x,y,B)-\phi(\epsilon,x',y,B)\|_m \tag{3.166}$$

$$+\|\phi(\epsilon,x',y,B)-\phi(\epsilon,x',y',B)\|_m \tag{3.167}$$

$$+\|\phi(\epsilon,x',y',B)-\phi(\epsilon',x',y',B)\|_m . \tag{3.168}$$

Focussing with the perturbation in x, (3.87) and (3.96) indicates we must derive a bound for the quantity

$$c^m|x-x'|^{m\xi}\iint\limits_{\mathbf{R}^{2m}}\prod_{k=1}^{m}\left(1+\bar{p}_k^2\right)^{-1}\prod_{k=1}^{2m}\left(1+u_k^2\right)^{-1}\prod_{j=1}^{m}|p_j|^\xi\ \mathrm{d}p_j\ \mathrm{d}q_j. \tag{3.169}$$

After arising in (3.98), the product $\prod|p_j|^\xi$ was inspected in great detail and the crucial ideas, captured in Proposition 3.2.1, exposed the order of the $\{|\bar{p}_k|\}_R$ terms emerging in polynomial (3.99),

$$\prod_{j=1}^{m}|p_j|^\xi \le c\mathcal{P}_3(|\bar{p}_k|)=\sum\left(\prod_R|\bar{p}_k|^{\eta_k\xi}\right),$$

were of maximum degree 2ξ ($\eta_k\in\{0,1,2\}$). This conclusion was independent of the fact that the Brownian functional under investigation was $\phi_y(\epsilon,x,y,B)$ and can, therefore, be reapplied here. Inserting $\mathcal{P}_3(|\bar{p}_k|)$ into (3.169), we obtain the upper bound

$$c\sum\left[\iint\limits_{\mathbf{R}^{2m}}\prod_R|\bar{p}_k|^{\eta_k\xi}\prod_{k=1}^{m}(1+\bar{p}_k^2)^{-1}\prod_{k=1}^{2m}\left(1+u_k^2\right)^{-1}\prod_{j=1}^{m}\ \mathrm{d}p_j\ \mathrm{d}q_j\right]. \tag{3.170}$$

After employing the change of variable argument which transformed (3.88) to (3.89), any multiple integral appearing as a summand above can be reexpressed as the product of finte single variable integrals. Utilizing Proposition 3.1.4 and referring to (3.105) allows us to complete (3.166) over the non-isolated intervals.

Turning to the variation in y, we must formulate a bound for

$$c^m|y-y'|^{m\xi}\iint\limits_{\mathbf{R}^{2m}}\prod_{k=1}^{m}\left(1+\bar{p}_k^2\right)^{-1}\prod_{k=1}^{2m}\left(1+u_k^2\right)^{-1}\prod_{j=1}^{m}\,|q_j|^{\xi}\ \mathrm{d}p_j\ \mathrm{d}q_j. \tag{3.171}$$

To our advantage, we can implement the study of $\mathcal{P}_1(|u_k|)$ of (3.41) which permits us to bound the above integral by

$$\sum\left[\iint\limits_{\mathbf{R}^{2m}}\prod_{k=1}^{m}\left(1+\bar{p}_k^2\right)^{-1}\prod_{S',S^-}\frac{|u_k|^{\omega_k\xi}}{1+u_k^2}\prod_{T^+,T'}\frac{|u_k|^{\omega_k\xi}}{1+u_k^2}\prod_{j=1}^{m}\mathrm{d}p_j\ \mathrm{d}q_j\right], \tag{3.172}$$

with ω_k taking values in (3.38). Bounding rational functions associated with the T^+ and T'-intervals and changing variables closes the argument.

The variation in ϵ affords us an opportunity to apply the results of the variation in the spatial variables x and y. In more detail, (3.127) and (3.128) reinforces the examination of,

$$c^m|\epsilon-\epsilon'|^{m\xi}\iint\limits_{\mathbf{R}^{2m}}\prod_{k=1}^{m}(1+\bar{p}_k^2)^{-1}\prod_{k=1}^{2m}\left(1+u_k^2\right)^{-1}\prod_{j=1}^{m}\ \left(|p_j|^{\xi}+|q_j|^{\xi}\right)\ \ \mathrm{d}p_j\ \mathrm{d}q_j. \tag{3.173}$$

Let π' and π'' form a disjoint union of $\{1,...,m\}$, so we can generate the polynomial

$$\begin{aligned}\prod_{j=1}^{m}\left(|p_j|^{\xi}+|q_j|^{\xi}\right)&=\sum\left(\prod_{\pi'}|p_j|^{\xi}\prod_{\pi''}|q_j|^{\xi}\right)\\&\leq\sum\left(\sum\left(\prod_{R}|\bar{p}_k|^{\eta_k\xi}\prod_{S'\cup S^-}|u_k|^{\omega_k\xi}\prod_{T^+\cup T'}|u_k|^{\omega_k\xi}\right)\right)\end{aligned}$$

and this produces our result. Hence, we have resolved the local uniform convergence of

$$\lim_{\epsilon\to 0}\phi(\epsilon,x,y,B)=\phi(x,y,B),$$

with $\phi(x, y, B)$ continuous in (x, y).

We mention, proofs involving the local properties and convergence of

$$\lim_{\epsilon \to 0} \alpha(\epsilon, x, y, B) = \alpha(x, y, B)$$

are developed using the analysis of $\phi(\epsilon, x, y, B)$.

3.3 Scaling and extending Hölder continuity to $D(2)$

To this point, we have established Hölder continuity for our key functionals for arbitrary $B \subset A(1, 1, 2)$. The goal of this section is to extend continuity to arbitrary $B \subset D(2)$, with $D(2)$ defined in (2.27). To establish Lemma 2.2.4, we introduce a trio of scaling identities and a derive a proposition which utilizes these expressions. For $\lambda \in \mathbf{R}_+$,

$$W(\lambda t) \stackrel{\text{dist}}{=} \lambda^{1/2} W(t), \tag{3.174}$$

and the transition densities can be expressed as

$$p_\epsilon(y/k) = k p_{\epsilon k^2}(y) \tag{3.175}$$

$$p'_\epsilon(y/k) = k^2 p'_{\epsilon k^2}(y). \tag{3.176}$$

Proposition 3.3.1. *For standard Brownian motion, the subsequent equalities in distribu-*

tion hold;

$$\phi_y(\epsilon, x, y, B) \overset{\text{dist}}{=} 2^{-3n/2}\phi_y(2^n\epsilon, 2^{n/2}x, 2^{n/2}y, 2^nB) \tag{3.177}$$

$$\phi(\epsilon, x, y, B) \overset{\text{dist}}{=} 2^{-2n}\phi(2^n\epsilon, 2^{n/2}x, 2^{n/2}y, 2^nB) \tag{3.178}$$

$$\alpha(\epsilon, x, y, B) \overset{\text{dist}}{=} 2^{-2n}\alpha(2^n\epsilon, 2^{n/2}x, 2^{n/2}y, 2^nB) \tag{3.179}$$

Proof. With $B \subset A(1, 1, 2)$, application of (3.174) yields

$$\begin{aligned}
&\phi_y(\epsilon, x, y, B) \\
&= \frac{1}{2^{3n}} \int_{2^nB} p_{\epsilon,x}\left(W(2^{-n}s, 2^{-n}r)\right)\left\{p'_{\epsilon,y}\left(W(2^{-n}t, 2^{-n}s)\right)\right\}_0 \; \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t \\
&= \frac{1}{2^{3n}} \int_{2^nB} p_{\epsilon}\left(W(2^{-n}s) - W(2^{-n}r) - x\right)\left\{p'_{\epsilon}\left(W(2^{-n}t) - W(2^{-n}s) - y\right)\right\}_0 \; \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t \\
&\overset{\text{dist}}{=} \frac{1}{2^{3n}} \int_{2^nB} p_{\epsilon}\left(2^{-n/2}W(s) - 2^{-n/2}W(r) - x\right) \\
&\qquad\qquad \cdot \left\{p'_{\epsilon}\left(2^{-n/2}W(t) - 2^{-n/2}W(s) - y\right)\right\}_0 \; \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t \\
&= \frac{1}{2^{3n}} \int_{2^nB} p_{\epsilon}\left(2^{-n/2}\left(W(s) - W(r) - 2^{n/2}x\right)\right) \\
&\qquad\qquad \cdot \left\{p'_{\epsilon}\left(2^{-n/2}\left(W(t) - W(s) - 2^{n/2}y\right)\right)\right\}_0 \; \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t
\end{aligned}$$

and via (3.175) and (3.176) this last expression is equal to

$$\begin{aligned}
&\frac{1}{2^{3n}} \int_{2^nB} 2^{n/2} p_{2^n\epsilon}\left(\left(W(s) - W(r) - 2^{n/2}x\right)\right) \\
&\qquad\qquad \cdot 2^n \left\{p'_{2^n\epsilon}\left(\left(W(t) - W(s) - 2^{n/2}y\right)\right)\right\}_0 \; \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t \\
&= 2^{-3n/2}\phi_y(2^n\epsilon, 2^{n/2}x, 2^{n/2}y, 2^nB)
\end{aligned}$$

Turning to (3.178), we utilize (3.174) and (3.175)

$$
\begin{aligned}
&\phi(\epsilon, x, y, B) \\
&= \frac{1}{2^{3n}} \int_{2^n B} p_\epsilon\left(W(2^{-n}s) - W(2^{-n}r) - x\right)\left\{p_\epsilon\left(W(2^{-n}t) - W(2^{-n}s) - y\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&\stackrel{\text{dist}}{=} \frac{1}{2^{3n}} \int_{2^n B} p_\epsilon\left(2^{-n/2}W(s) - 2^{-n/2}W(r) - x\right) \\
&\qquad\qquad \cdot \left\{p_\epsilon\left(2^{-n/2}W(t) - 2^{-n/2}W(s) - y\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&= \frac{1}{2^{3n}} \int_{2^n B} 2^{n/2} p_{2^n\epsilon}\left(\left(W(s) - W(r) - 2^{n/2}x\right)\right) \\
&\qquad\qquad \cdot 2^{n/2}\left\{p_{2^n\epsilon}\left(\left(W(t) - W(s) - 2^{n/2}y\right)\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&= 2^{-2n}\phi(2^n\epsilon, 2^{n/2}x, 2^{n/2}y, 2^n B).
\end{aligned}
$$

The proof for

$$\alpha(\epsilon, x, y, B) \stackrel{\text{dist}}{=} 2^{-2n}\alpha(2^n\epsilon, 2^{n/2}x, 2^{n/2}y, 2^n B)$$

follows by replicating the above. $\square$

To extend our continuity results for $\phi_y(\cdot)$, consider any $B \subseteq D(2)$. If we define $B(k, n) \doteq B \cap A(k, n, 2)$, then B can be expressed as the union

$$B = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} B(k, n). \tag{3.180}$$

This permits us to write

$$\begin{aligned}
&\|\phi_y(\epsilon,x,y,B)-\phi_y(\epsilon',x',y',B)\|_m && (3.181)\\
&\quad=\left\|\sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}}\phi_y\big(\epsilon,x,y,B(k,n)\big)-\phi_y\big(\epsilon',x',y',B(k,n)\big)\right\|_m\\
&\quad\le\sum_{n=1}^{\infty}\left\|\sum_{k=1}^{2^{n-1}}\phi_y\big(\epsilon,x,y,B(k,n)\big)-\phi_y\big(\epsilon',x',y',B(k,n)\big)\right\|_m. && (3.182)
\end{aligned}$$

For fixed n and arbitrary $k\in[0,2^{n-1}]$, the independent increments of W_t implies

$$\begin{aligned}
&\mathbb{E}\left[\phi_y\left(\epsilon,x,y,B(k,n)\right)\right]=\mathbb{E}\left[\int_{B(k,n)}p_{\epsilon,x}(W(s,r))\{p'_{\epsilon,y}(W(t,s))\}_0\ \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t\right]\\
&=\int_{B(k,n)}\mathbb{E}\left[p_{\epsilon,x}(W(s,r))\right]\mathbb{E}\left[p'_{\epsilon,y}(W(t,s))\right]\ \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t\\
&\quad-\int_{B(k,n)}\mathbb{E}\left[p_{\epsilon,x}(W(s,r))\right]\mathbb{E}\left[p'_{\epsilon,y}(W(t,s))\right]\ \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t\\
&=0
\end{aligned}$$

and using independence we can apply Rosenthal's inequality [4, p. 46] to see that (3.182) can be bounded above by

$$\sum_{n=1}^{\infty}2^{(n-1)/2}\sup_k\left\|\phi_y\big(\epsilon,x,y,B(k,n)\big)-\phi_y\big(\epsilon',x',y',B(k,n)\big)\right\|_m.$$

Replacing n with $n-1$ in equation (3.177), this last expression is equivalent to

$$\begin{aligned}
&\sum_{n=1}^{\infty} 2^{(n-1)/2} \sup_k \left\|\phi_y\big(\epsilon, x, y, B(k,n)\big) - \phi_y\big(\epsilon', x', y', B(k,n)\big)\right\|_m \\
&\quad = \sum_{n=1}^{\infty} 2^{(n-1)/2} 2^{-(3(n-1))/2} \sup_k \Big\|\phi_y\left(2^{n-1}\epsilon, 2^{(n-1)/2}x, 2^{(n-1)/2}y, 2^{n-1}B(k,n)\right) \\
&\qquad\qquad - \phi_y\left(2^{n-1}\epsilon', 2^{(n-1)/2}x', 2^{(n-1)/2}y', 2^{n-1}B(k,n)\right)\Big\|_m .
\end{aligned} \tag{3.183}$$

Since $2^{n-1}B(k,n) \subseteq A(1,1,2)$, Lemma 2.2.3 reveals the supremum over k in the above expression can be bounded by, $c_m(\xi)2^{\xi(n-1)}\|(\epsilon,x,y)-(\epsilon',x',y')\|^{\xi}$. We can select a ξ small so that (3.183) is bounded by

$$\begin{aligned}
&c_m(\xi)\sum_{n=1}^{\infty} 2^{(n-1)/2}2^{-(3(n-1))/2}2^{\xi(n-1)}\|(\epsilon,x,y)-(\epsilon',x',y')\|^{\xi} \\
&= c_m(\xi)\sum_{n=1}^{\infty} 2^{(\xi-1)(n-1)}\|(\epsilon,x,y)-(\epsilon',x',y')\|^{\xi} \\
&\leq c_m(\xi)\|(\epsilon,x,y)-(\epsilon',x',y')\|^{\xi}
\end{aligned}$$

To address $\phi(\cdot)$, we begin by reexpressing any $B \subseteq D(2)$ via (3.180). Replicate the analysis following (3.181) and apply the fact that $\mathbb{E}\left[\phi(\epsilon,x,y,B(k,n))\right]=0$ to arrive at the bound

$$\begin{aligned}
&\|\phi(\epsilon,x,y,B(k,n)) - \phi(\epsilon',x',y',B(k,n))\|_m \\
&\quad \leq \sum_{n=1}^{\infty} 2^{(n-1)/2} \sup_k \left\|\phi\big(\epsilon,x,y,B(k,n)\big) - \phi\big(\epsilon',x',y',B(k,n)\big)\right\|_m .
\end{aligned}$$

Combining (3.178) and Lemma 2.2.3 indicates the above quantity can be bounded by

$$
\begin{aligned}
&\sum_{n=1}^{\infty} 2^{(n-1)/2} 2^{-2(n-1)} \sup_k \Big\| \phi\left(2^{n-1}\epsilon, 2^{(n-1)/2}x, 2^{(n-1)/2}y, 2^{n-1}B(k,n)\right) \\
&\qquad\qquad\qquad - \phi\left(2^{n-1}\epsilon', 2^{(n-1)/2}x', 2^{(n-1)/2}y', 2^{n-1}B(k,n)\right) \Big\|_m \\
&\quad \le c_m(\xi) \sum_{n=1}^{\infty} 2^{(n-1)/2} 2^{-2(n-1)} 2^{\xi(n-1)} \|(\epsilon, x, y) - (\epsilon', x', y')\|^{\xi} \\
&\quad = c_m(\xi) \sum_{n=1}^{\infty} 2^{(-3/2+\xi)(n-1)} \|(\epsilon, x, y) - (\epsilon', x', y')\|^{\xi} \\
&\quad \le c_m(\xi) \|(\epsilon, x, y) - (\epsilon', x', y')\|^{\xi}
\end{aligned}
$$

3.4 Continuity in the set variable B

To prove (2.37) of Lemma 2.2.5,

$$
\mathbb{E}\left[\left(\phi_y(\epsilon, x, y, B)\right)^m\right] \le c_m(\xi)|B|^{m\xi}
$$

we must modify our proof of (2.33) over both classes of intervals. For the first scenario involving the non-isolated case we turn to (3.27),

$$
c^m \iiint\limits_{\mathbf{R}^{2m}\times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} u_k^2(\tilde{d}_k)\right) \prod_{j=1}^{m} |q_j| \; \mathrm{d}p_j \; \mathrm{d}q_j \; \mathrm{d}r_j \; \mathrm{d}s_j \; \mathrm{d}t_j,
$$

and single out the multiple integral with respect to the time variables

$$
\int_{B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} u_k^2(\tilde{d}_k)\right) \prod_{j=1}^{m} \mathrm{d}r_j \; \mathrm{d}s_j \; \mathrm{d}t_j. \tag{3.184}
$$

Let a and a' be conjugate exponents taking specific values

$$1 < a' < 1 + \delta \quad \text{and} \quad a = (1 - 1/a')^{-1}\,, \tag{3.185}$$

with δ very small. In other words, a' is chosen close to one and a is a large value. With $\prod \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j$ as our measure, Hölder's inequality suggests

$$\begin{aligned}
&|B|^{m/a} \left(\int_{B^m} \left[\exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k) \right) \right]^{a'} \left[\exp\left(-\sum_{k=1}^{2m} u_k^2(\tilde{d}_k) \right) \right]^{a'} \prod_{j=1}^{m} \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j \right)^{1/a'} \\
&\qquad \le c|B|^{m/a} \left(\prod_{k=1}^{m} \frac{1}{1 + a'\bar{p}_k^2} \prod_{k=1}^{2m} \frac{1}{1 + a'u_k^2} \right)^{1/a'} \\
&\qquad = c|B|^{m/a} \prod_{k=1}^{m} \left(1 + a'\bar{p}_k^2\right)^{-1/a'} \prod_{k=1}^{2m} \left(1 + a'u_k^2\right)^{-1/a'} \\
&\qquad \le c|B|^{m/a} \prod_{k=1}^{m} \left(1 + \bar{p}_k^2\right)^{-1/a'} \prod_{k=1}^{2m} \left(1 + u_k^2\right)^{-1/a'}
\end{aligned} \tag{3.186}$$

where (3.28) was utilized to procure the bound appearing in the second line. Inserting the last expression into (3.27) yields the upper bound

$$c^m |B|^{m/a} \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1 + \bar{p}_k^2\right)^{-1/a'} \prod_{k=1}^{2m} \left(1 + u_k^2\right)^{-1/a'} \prod_{j=1}^{m} |q_j| \ \mathrm{d}p_j \ \mathrm{d}q_j. \tag{3.187}$$

Set

$$\zeta = 1 - 1/a' \tag{3.188}$$

so that the following equality holds

$$\prod_{j=1}^{m} |q_j| = \prod_{j=1}^{m} |q_j|^{1/a'} \ |q_j|^{\zeta}.$$

Substitute (3.30) and (3.108) for each $|q_j|^{1/a'}$ and $|q_j|^\zeta$ term, respectively, to produce

$$\prod_{j=1}^{m} |q_j| = \prod_{j=1}^{m} |q_j|^{1/a'} \ |q_j|^\zeta \leq \mathcal{P}_8(|\bar{p}_k|, |u_k|), \tag{3.189}$$

where

$$\mathcal{P}_8(|\bar{p}_k|, |u_k|) \doteq \prod_{k\in T} \left(|u_{k-1}|^{1/a'} + |u_k|^{1/a'}\right) \prod_{\substack{k\in S \\ k', k''\in R}} \left(|u_k|^\zeta + |u_{k-1}|^\zeta + |\bar{p}_{k'}|^\zeta + |\bar{p}_{k''}|^\zeta\right). \tag{3.190}$$

In the course of securing Hölder continuity in the y parameter over the non-isolated intervals, a parallel expression arises in (3.114). Upon applying Proposition 3.2.2 and reincorporating the sets defined in (3.110)-(3.113), $\mathcal{P}_8(|\bar{p}_k|, |u_k|)$ expands to the following polynomial, where we make the appropriate modifications to account for the insertion of the $1/a'$ in the exponent;

$$\mathcal{P}_8(|\bar{p}_k|, |u_k|) = \sum \left(\prod_R |\bar{p}_k|^{\eta_k\zeta} \prod_{S'_\xi} |u_k|^{\theta_k\zeta} \prod_{S^-_\xi} |u_k|^{\omega_k(1/a')+\theta_k\zeta} \prod_{T^+_\xi, T'_\xi} |u_k|^{\omega_k(1/a')+\theta_k\zeta} \right),$$

with order

$$\deg \ |\bar{p}_k| = \eta_k\zeta \in \{0, \zeta, 2\zeta\}$$

$$\deg |u_k| = \omega_k(1/a') + \theta_k\zeta \begin{cases} = 0, \zeta, \text{ or } 2\zeta, & \text{if } k \in S'_\xi \\ = 1/a' \text{ or } 1, & \text{if } k \in S^-_\xi \\ = \zeta \text{ or } 1/a', & \text{if } k \in T^+_\xi \\ \leq 2(1/a'), & \text{if } k \in T'_\xi. \end{cases}$$

Upon implanting $\mathcal{P}_8(|\bar{p}_k|, |u_k|)$ into (3.187) and utilizing the bound

$$\frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1 + a'u_k^2)^{1/a'}} \leq c, \quad k \in T'_\xi, \tag{3.191}$$

we are directed to

$$\iint\limits_{\mathbf{R}^{2m}} \prod_{R} |\bar{p}_k|^{\eta_k \zeta} \prod_{k=1}^{m} \frac{1}{(1+\bar{p}_k^2)^{1/a'}} \prod_{S'_\xi} \frac{|u_k|^{\theta_k \zeta}}{(1+u_k^2)^{1/a'}} \prod_{S^-_\xi \cup T^+_\xi} \frac{|u_k|^{\omega_k + \theta_k \zeta}}{(1+u_k^2)^{1/a'}} \prod_{j=1}^{m} \mathrm{d}p_j \ \mathrm{d}q_j.$$

Proposition 3.1.4 reveals the above expression is finite, since the parameter a' can be chosen close to 1 to produce

$$\begin{aligned} \eta_k \zeta &< (2/a') - 1, \quad \text{for all } k \in \mathcal{M} \\ \theta_k \zeta &< (2/a') - 1, \quad \text{for all } k \in S_\xi^{'} \\ 2[\omega_k(1/a') + \theta_k \zeta] &< (4/a') - 1, \quad \text{for all } k \in S^-_\xi \cup T^+_\xi. \end{aligned}$$

Scenarios involving the isolated intervals require a similar modification, but a more subtle proof and we provide the general procedures to be taken. With the assistance of some Fourier analysis, we reduce our starting point, multiple integral (3.192), to the expression emerging in (3.202). After utilizing Hölder 's inequality and some basic real analysis on this last quantity, we are, then, directed to integral (3.204). From that point, we simply reincorporate the methods constructed to derive Hölder continuity in the y variable for the isolated variables to conclude the argument.

We start with (3.59). Apply $(2\pi)^{-m} \leq c^m$ and note that both quantities $|\exp(ixp)|$ and $|\exp(-\epsilon p^2)|$, which appear in $F(\epsilon, x, y, p, q)$, can be bounded by one to arrive at

$$\begin{aligned} (ci)^m \iiint\limits_{\mathbf{R}^{2m} \times B^m} &\exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} u_k^2(\tilde{d}_k)\right) \\ &\cdot \prod_{k \in I} \left(\exp\left(-u_k^2(\tilde{d}_k)\right) - \exp\left(-\left(u_{k+1}^2 + q_{\ell(k)}^2\right)(\tilde{d}_k)\right)\right) \\ &\exp\left(iy \sum_{j=1}^{m} q_j\right) \prod_{j=1}^{m} \hat{f}(\epsilon, q_j)\ q_j \ \mathrm{d}p_j \ \mathrm{d}q_j \ \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j. \end{aligned} \tag{3.192}$$

Before isolating integrals with respect to the time measure, we first address the isolated variables $q_{\ell_{(k)}}$ appearing in the product above and look to acquire a bound for

$$\left|\int_{\mathbf{R}} \exp\left(-u_k^2(\tilde{d}_k)\right) - \exp\left(-\left(u_{k+1}^2 + q_{\ell_{(k)}}^2\right)(\tilde{d}_k)\right)\ iq_{\ell_{(k)}} \exp\left(iyq_{\ell_{(k)}}\right) \exp\left(-\epsilon q_{\ell_{(k)}}^2\right)\ \mathrm{d}q_{\ell_{(k)}}\right|. \tag{3.193}$$

Note the insertion imaginary exponential into the integrand.

Substitute (3.55) for the u_k terms above and simplify notation by setting

$$u_{k+1} = u, \quad q_{\ell_{(k)}} = q, \quad \text{and} \quad \tilde{d}_k = t \tag{3.194}$$

Recalling (2.8) and (2.9) as the inverse Fourier transforms for $p_t(z)$ and $p_t'(z)$, respectively, an application of Parseval's identity and closure under convolution implies

$$\begin{aligned}
&\left|\int_{\mathbf{R}} \left(\exp\left(-t(u+q)^2\right) - \exp\left(-t(u^2+q^2)\right)\right) iq \exp\left(iyq\right) \exp\left(-\epsilon q^2\right)\ \mathrm{d}q\right| \\
&= \left|\int_{\mathbf{R}} \exp\left(-t(u+q)^2\right) \exp\left(iyq\right) iq \exp\left(-\epsilon q^2\right) \mathrm{d}q \right.\\
&\qquad \left. - \int_{\mathbf{R}} \exp\left(-tu^2\right) \exp\left(-tq^2\right) \exp\left(iyq\right) iq \exp\left(-\epsilon q^2\right)\ \mathrm{d}q\right| \\
&= \left|\int_{\mathbf{R}} \exp(iuz) p_t(z) p_\epsilon'(z-y)\ \mathrm{d}z - \int_{\mathbf{R}} \exp(-tu^2) p_t(z) p_\epsilon'(z-y)\ \mathrm{d}z\right| \\
&= \left|\int_{\mathbf{R}} \left(\exp(iuz) - \exp(-tu^2)\right) p_t(z) p_\epsilon'(z-y)\ \mathrm{d}z\right|.
\end{aligned} \tag{3.195}$$

Integration by parts yields

$$\left|\int_{\mathbf{R}} \big(\exp(iuz) - \exp(-tu^2)\big)\, p_t(z) p'_\epsilon(z-y)\ \mathrm{d}z\right|$$
$$\leq \big|\, \big[\big(\exp(iuz) - \exp(-tu^2)\big)\, p_t(z)\big]\, p_\epsilon(z-y)\big|\Big|_{z=-\infty}^{z=\infty} \tag{3.196}$$
$$+ \left|\int_{\mathbf{R}} iu \exp(iuz) p_t(z) p_\epsilon(z-y)\ \mathrm{d}z\right| \tag{3.197}$$
$$+ \left|\int_{\mathbf{R}} \big(\exp(-tu^2) - \exp(iuz)\big)\, p'_t(z) p_\epsilon(z-y)\ \mathrm{d}z\right|. \tag{3.198}$$

Starting work with (3.196), the definition of absolute value leads us to consider two cases. For the first case we must analyze

$$\lim_{z\to\infty} \exp(iuz) p_t(z) - \lim_{z\to\infty} \exp(-tu^2) p_t(z).$$

Since $-cp_t(z) \leq \exp(iuz)p_t(z) \leq cp_t(z)$ and $\lim_{z\to\infty} -cp_t(z) = \lim_{z\to\infty} cp_t(z) = 0$, the Squeeze Theorem leads us to, $\lim_{z\to\infty} \exp(iuz)p_t(z) = 0$. Repeated application of this argument concedes

$$\big|\, \big[\big(\exp(iuz) - \exp(-tu^2)\big)\, p_t(z)\big]\, p_\epsilon(z-y)\big|\Big|_{z=-\infty}^{z=\infty} = 0. \tag{3.199}$$

Focussing on (3.197), an application of the Chapman-Kolmogorov equation produces

$$\left|\int_{\mathbf{R}} iu \exp(iuz) p_t(z) p_\epsilon(z-y)\ \mathrm{d}z\right| \leq |u| \int_{\mathbf{R}} p_t(z) p_\epsilon(z-y)\ \mathrm{d}z$$
$$= |u| p_{t+\epsilon}(y) \leq c|u| p_{t+\epsilon}(0) \leq c\frac{|u|}{t^{1/2}}. \tag{3.200}$$

Before turning attention to (3.198), we mention two points. First, formally differentiate

$p'_t(z)$ to arrive at

$$p'_t(z) = \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{2\pi t}}\exp\left(\frac{-z^2}{2t}\right)\right] = \frac{-z}{t}p_t(z).$$

Second, the subsequent upper bound holds for all t

$$\sup_z\left[\left|\frac{z}{t^{1/2}}\right| |p_t\left(z\right)|\right], \sup_z\left[\left|\frac{z^2}{t}\right| |p_t\left(z\right)|\right] \le \frac{c}{t^{1/2}}.$$

Turning to the integral of interest,

$$\begin{aligned}
&\left|\int_{\mathbf{R}} \left(\exp(-tu^2) - \exp(iuz)\right) p'_t(z)p_\epsilon(z-y)\ \mathrm{d}z\right| \\
&\quad\le \int_{\mathbf{R}} \left(|1-\exp(-tu^2)| + |1-\exp(iuz)|\right)\left|\frac{z}{t}\right| |p_t(z)|p_\epsilon(z-y)\ \mathrm{d}z \\
&\quad\le c\sup_z\left[|1-\exp(-tu^2)| + |1-\exp(iuz)|\right]\left|\frac{z}{t}\right| |p_t(z)| \\
&\quad\le c\sup_z\left[t^{1/2}|u| + |uz|\right]\left|\frac{z}{t}\right| |p_t(z)| \le c|u|\sup_z\left[\left|\frac{z}{t^{1/2}}\right| |p_t(z)| + \left|\frac{z^2}{t}\right| |p_t(z)|\right] \\
&\quad\le c\frac{|u|}{t^{1/2}}.
\end{aligned} \tag{3.201}$$

Combining (3.199), (3.200), and (4.59), any integral of the form (3.193) can be bounded above by $c|u_{k+1}|(\tilde{d}_k)^{-1/2}$, where $k \in I$. With the assistance of this quantity and the observations, $|i\exp(-iyq_j)|, |\hat{f}(\epsilon, q_j)| \le 1$, $j \in I_j^c$, (3.192) reduces to

$$\begin{aligned}
c^m \iiint\limits_{\mathbf{R}^m\times\mathbf{R}^n\times B^m} &\exp\left(-\sum_{k=1}^m \bar{p}_k^2(\tilde{r}_k)\right)\exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right)\prod_{k\in I}\frac{|u_{k+1}|}{\tilde{d}_k^{1/2}} \\
&\prod_{I_j^c}|q_j|\ \mathrm{d}q_j \prod_{j=1}^m \mathrm{d}p_j\ \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j,
\end{aligned} \tag{3.202}$$

with q_j the remaining non-isolated variables and $n = \left|I_j^c\right|$.

Separate the following quantity from the above

$$\int_{B^m} \exp\left(-\sum_{k=1}^m \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right) \prod_{k\in I} \frac{1}{\tilde{d}_k^{1/2}} \prod_{j=1}^m \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j. \tag{3.203}$$

With a and a' selected as in (3.185), Hölder's inequality yields the upper bound

$$|B|^{m/a} \left(\int_{B^m} \left[\exp\left(-\sum_{k=1}^m \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right) \prod_{k\in I} \frac{1}{\tilde{d}_k^{1/2}}\right]^{a'} \prod_{j=1}^m \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j \right)^{1/a'}.$$

A change of variables through (3.13) and (3.14) permits us to reexpress the multiple integral in the parenthesis as

$$\begin{aligned}
&\int_{[0,1/2]^m} \left[\exp\left(-\sum_{k=1}^m \bar{p}_k^2(\tilde{r}_k)\right)\right]^{a'} \prod_{j=1}^m \mathrm{d}r_j \\
&\quad \cdot \int_{[1/2,1]^m} \left[\exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right) \prod_{k\in I} \frac{1}{\tilde{d}_k^{1/2}}\right]^{a'} \prod_{j=1}^m \mathrm{d}s_j\ \mathrm{d}t_j \\
&= \prod_{k=1}^m \int_{[1/2,1]} \left[\exp\left(-\sum_{k=1}^m \bar{p}_k^2(\tilde{r}_k)\right)\right]^{a'} \mathrm{d}(\tilde{r}_k) \\
&\qquad \prod_{I^c} \int_{[0,1/2]} \left[\exp\left(-\sum_{I^c} u_k^2(\tilde{d}_k)\right)\right]^{a'} \mathrm{d}(\tilde{d}_k) \ \prod_{I} \int_{[0,1/2]} \left[\frac{1}{\tilde{d}_k^{1/2}}\right]^{a'} \mathrm{d}(\tilde{d}_k).
\end{aligned}$$

Enlarge the domain of integration and apply (3.28) for measures involving $\mathrm{d}(\tilde{r}_k)$ and $\mathrm{d}(\tilde{d}_k)$, $k \in I^c$. Since integrals concerning $\mathrm{d}(\tilde{d}_k)$, for $k \in I$, are finite, we arrive at the following upper bound for (3.203),

$$\begin{aligned}
c|B|^{m/a} &\left(\prod_{k=1}^m \frac{1}{1+\bar{p}_k^2} \prod_{I^c} \frac{1}{1+u_k^2} \right)^{1/a'} \\
&= c|B|^{m/a} \prod_{k=1}^m \left(1+\bar{p}_k^2\right)^{-1/a'} \prod_{I^c} \left(1+u_k^2\right)^{-1/a'},
\end{aligned}$$

which implies (3.202) is less than or equal to

$$c^m |B|^{m/a} \iint\limits_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{k=1}^{m} \left(1+\bar{p}_k^2\right)^{-1/a'} \prod_{I^c} \left(1+u_k^2\right)^{-1/a'} \prod_{k\in I} |u_{k+1}| \prod_{I_j^c} |q_j| \, \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j. \quad (3.204)$$

Assign ζ as in (3.188) so we can utilize the upper bound presented in (3.189). Recollect the definition of the T^{I+1}-intervals, (3.76), to write

$$\prod_{k\in I} |u_{k+1}| \prod_{I_j^c} |q_j|$$
$$\leq \prod_{k\in T^{I+1}} |u_k| \prod_{k\in T} \left(|u_{k-1}|^{1/a'} + |u_k|^{1/a'}\right) \prod_{\substack{k\in S \\ k',k''\in R}} \left(|u_k|^{\zeta} + |u_{k-1}|^{\zeta} + |\bar{p}_{k'}|^{\zeta} + |\bar{p}_{k''}|^{\zeta}\right).$$

To discern the order of the $|u_k|$ terms materializing in the expansion of the upper bound, we return to a similar polynomial generated by (3.146). After implementing the proper adjustments for the change in notation, application of Proposition 3.2.4 permits us to conclude that the upper bound above is equivalent to

$$\mathcal{P}_9(|\bar{p}_k|, |u_k|)$$
$$\doteq \prod_{k\in T^{I+1}} |u_k| \prod_{k\in T} \left(|u_{k-1}|^{1/a'} + |u_k|^{1/a'}\right) \prod_{\substack{k\in S \\ k',k''\in R}} \left(|u_k|^{\zeta} + |u_{k-1}|^{\zeta} + |\bar{p}_{k'}|^{\zeta} + |\bar{p}_{k''}|^{\zeta}\right)$$
$$= \sum \left(\prod_{R} |\bar{p}_k|^{\eta_k \zeta} \prod_{S_\xi'} |u_k|^{\theta_k \zeta} \prod_{S_\xi^-} |u_k|^{\omega_k(1/a') + \theta_k \zeta} \prod_{T^{I+1}, T_\xi^+, T_\xi''} |u_k|^{\omega_k(1/a') + \theta_k \zeta} \right),$$

where,

$$\deg |\bar{p}_k| = \eta_k \zeta \in \{0, \zeta, 2\zeta\}, \qquad \text{if } k \in R$$

$$\deg|u_k| = \omega_k(1/a') + \theta_k\zeta \begin{cases} = 0, \zeta, \text{ or } 2\zeta, & \text{if } k \in S_\xi^{'} \\ = 1/a' \text{ or } 1, & \text{if } k \in S_\xi^{-} \\ = \zeta \text{ or } 1/a', & \text{if } k \in T_\xi^{+} \\ = 1, 1+\zeta, \text{ or } 1+1/a', & \text{if } k \in T^{I+1} \\ \leq 2(1/a'), & \text{if } k \in T_\xi^{''}. \end{cases}$$

Instil $\mathcal{P}_9(|\bar{p}_k|, |u_k|)$ into (3.204) to arrive at a sum of multiple integrals in which each summand assumes the form

$$\begin{aligned} &\iint_{\mathbf{R}^m\times\mathbf{R}^n} \prod_R |\bar{p}_k|^{\eta_k\zeta} \prod_{k=1}^m \left(1+\bar{p}_k^2\right)^{-1/a'} \prod_{I^c} \left(1+u_k^2\right)^{-1/a'} \prod_{S_\xi^{'}} |u_k|^{\theta_k\zeta} \prod_{S_\xi^{-}} |u_k|^{\omega_k(1/a')+\theta_k\zeta} \\ &\prod_{T_\xi^{+}} |u_k|^{\omega_k(1/a')+\theta_k\zeta} \prod_{T^{I+1}} |u_k|^{\omega_k(1/a')+\theta_k\zeta} \prod_{T_\xi^{''}} |u_k|^{\omega_k(1/a')+\theta_k\zeta} \prod_{I_j^c} \mathrm{d}q_j \prod_{j=1}^m \mathrm{d}p_j. \end{aligned} \tag{3.205}$$

The above integral is nearly in the format required to apply Proposition 3.1.4, except for one problematic case. Parallel with (3.153) and (3.154), the rational functions originating from the T^{I+1}-intervals are of concern since

$$\lim_{u_k\to\infty} \frac{|u_k|^{1+1/a'}}{\left(1+u_k^2\right)^{1/a'}} = \infty. \tag{3.206}$$

After making the appropriate changes in notation, we can replicate the resolution of the previously mentioned multiple integral to resolve this issue. To diagram the proof, identify the following t-intervals

$$T_{1+1/a'}^{I+1} \doteq \left\{k \in T^{I+1} \mid \deg|u_k| = 1 + 1/a'\right\}.$$

Replace (3.156) and (3.157), with

$$S_\xi^{I-1} \doteq \left\{ k \in S_\xi' \mid k+2 \in T_{1+1/a'}^{I+1} \right\}, \quad T_\xi^{I-1} \doteq \left\{ k \in T_\xi^+ \cup T_\xi^{I+1} \cup T_\xi'' \mid k+2 \in T_{1+1/a'}^{I+1} \right\}.$$

The two critical characteristics regarding u_k, $k \in S_\xi^{I-1} \cup T_\xi^{I-1}$, are depicted in Proposition 3.2.5. The point to observe is that the derivation of (3.158) and (3.159) was based solely on the ordering of the s and t variables and can be reapplied to (3.206). With regard to the first property, we discover in the current framework that

$$\deg |u_k| \le \begin{cases} \zeta, & \text{if } k \in S_\xi^{I-1}, \\ 1, & \text{if } k \in T_\xi^{I-1}. \end{cases}$$

To utilize the second characteristic note

$$1 + 1/a' = 2(1/a') + 1 - 1/a' = 2(1/a') + \zeta$$

and, recall, ζ is close to zero. Therefore, for $k \in S_\xi^{I-1} \cup T_\xi^{I-1}$ and $k+2 \in T_{1+1/a'}^{I+1}$,

$$\frac{|u_{k+2}|^{1+1/a'}}{\left(1 + a' u_{k+2}^2\right)^{1/a'}} = \frac{|u_{k+2}|^{2(1/a')} |u_{k+2}|^\zeta}{\left(1 + a' u_{k+2}^2\right)^{1/a'}} \le c|u_{k+2}|^\zeta \le c|u_k - p_{\ell_{(k+1)}}|^\zeta \le c\left(|u_k|^\zeta + |p_{\ell_{(k+1)}}|^\zeta\right),$$

which produces,

$$\begin{aligned}
\prod_{T^{I+1}} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} &= \prod_{T^{I+1}/T^{I+1}_{1+1/a'}} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{T^{I+1}_{1+1/a'}} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \\
&\leq c \prod_{T^{I+1}/T^{I+1}_{1+1/a'}} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{\substack{k\in S_\xi^{I-1}\cup T_\xi^{I-1} \\ k+1\in I}} \left(|u_k|^\zeta + |p_{\ell_{(k+1)}}|^\zeta\right) \\
&= c \prod_{T^{I+1}/T^{I+1}_{1+1/a'}} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \sum \left(\prod_{S_\xi^{I-1}\cup T_\xi^{I-1}} |u_k|^\zeta \prod_R |\bar{p}_k|^{\tilde{\eta}_k\zeta} \right),
\end{aligned} \tag{3.207}$$

where the last inequality was derived in a similar fashion as $\mathcal{P}_7(|\bar{p}_k|, |u_k|)$, of (3.161).

Having resolved our troublesome case emerging in (3.206), (3.205) converts to

$$\begin{aligned}
&c^m \iint_{\mathbf{R}^m\times\mathbf{R}^n} \prod_R |p_{k'}|^{(\eta_k+\tilde{\eta}_k)\zeta} \prod_{k=1}^m \frac{1}{(1+\bar{p}_k^2)^{1/a'}} \prod_{S'_\xi} \frac{|u_k|^{\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{S_\xi^{I-1}} |u_k|^\zeta \prod_{S_\xi^-} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \\
&\prod_{T_\xi^+} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{T_\xi^+\cap T_\xi^{I-1}} |u_k|^\zeta \left[\prod_{T^{I+1}/T^{I+1}_{1+1/a'}} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{T^{I+1}\cap T_\xi^{I-1}} |u_k|^\zeta \right] \\
&\left[\prod_{T''_\xi} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{T''_\xi\cap T_\xi^{I-1}} |u_k|^\zeta \right] \prod_{I_j^c} \mathrm{d}q_j \prod_{j=1}^m \mathrm{d}p_j.
\end{aligned} \tag{3.208}$$

To prove the product of rational functions originating from the $T^{I+1}\cup T''_\xi$-intervals can be bounded by a constant, we reexamine the three instances succeeding (3.164). This directs us to

$$\deg |u_k| = \omega_k(1/a') + \theta_k\zeta \begin{cases} \leq 1 & \text{if } k \in \{T^{I+1}\cap T_\xi^{I-1}\} \cup \{T''_\xi \cap T_\xi^{I-1}\} \\ = 1+\zeta, & \text{if } k \in \{T^{I+1}/T^{I+1}_{1+1/a'}\}/T_\xi^{I-1} \\ \leq 2(1/a')\ , & \text{if } k \in T''_\xi/T_\xi^{I-1}. \end{cases}$$

Therefore,

$$\left[\prod_{T^{I+1}/T^{I+1}_{1+1/a'}} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{T^{I+1}\cap T^{I-1}_\xi} |u_k|^\zeta\right] \cdot \left[\prod_{T''_\xi} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{T''_\xi\cap T^{I-1}_\xi} |u_k|^\zeta\right] \leq c$$

produces the subsequent upper bound for (3.208) ,

$$\begin{aligned} c^m \iint_{\mathbf{R}^m\times\mathbf{R}^n} \prod_R |p_{k'}|^{(\eta_k+\tilde{\eta}_k)\zeta} \prod_{k=1}^m \frac{1}{(1+\bar{p}_k^2)^{1/a'}} \prod_{S'_\xi} \frac{|u_k|^{\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{S^{I-1}_\xi} |u_k|^\zeta \prod_{S^-_\xi} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \\ \prod_{T^+_\xi} \frac{|u_k|^{\omega_k(1/a')+\theta_k\zeta}}{(1+u_k^2)^{1/a'}} \prod_{T^+_\xi\cap T^{I-1}_\xi} |u_k|^\zeta \prod_{I^c_j} \mathrm{d}q_j \prod_{j=1}^m \mathrm{d}p_j. \end{aligned}$$

If a' selected close to one, ζ is slightly larger than zero and any integral above is finite since,

$$\begin{aligned} \eta_k+\tilde{\eta}_k\zeta &< 2/a'-1, \text{ for all } k\in\mathcal{M} \\ \theta_k\zeta+\mathbb{1}_{S^{I-1}_\xi}\zeta &< 2/a'-1, \text{ for all } k\in S'_\xi \\ 2[\omega_k(1/a')+\theta_k\zeta] &< 4/a'-1, \text{ for all } k\in S^-_\xi \\ 2[\omega_k(1/a')+\theta_k\zeta+\mathbb{1}_{T^{I-1}_\xi}\zeta] &< 4/a'-1, \text{ for all } k\in T^+_\xi. \end{aligned}$$

Having established 2.2.5, we now prove continuity in the set variable. Let $B_T \doteq \{0 \leq\leq s \leq t \leq T\}$. If $T, T' \leq M \leq \infty$, then $|B_T - B_{T'}| \leq M|T-T'|$. Since $\mathbb{E}\left[\left(\phi_y(\epsilon,x,y,B)\right)^m\right] \leq c_m(\xi)|B|^{m\xi}$, we have for some constant c,

$$\mathbb{E}\left[\left(\phi_y(\epsilon,x,y,T)-\phi_y(\epsilon,x,y,T')\right)^m\right] \leq c_m(\xi)|T-T'|^{m\xi}.$$

To generate (2.36),

$$\mathbb{E}\left[\left(\phi(\epsilon,x,y,B)\right)^m\right] \leq c_m(\xi)|B|^{m\xi},$$

we look to (3.87) and upon employing (3.25) and (3.26) we are directed to the upper bound

$$c^m \iiint\limits_{\mathbf{R}^{2m}\times B^m} \exp\left(-\sum_{k=1}^{m} \bar{p}_k^2(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} u_k^2(\tilde{d}_k)\right) \prod_{j=1}^{m} \mathrm{d}p_j \ \mathrm{d}q_j \ \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j. \tag{3.209}$$

If we extract the multiple integral involving the measure $\prod \ \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j$, we are led to the exact expression appearing (3.184). Following the application of Hölder's inequality, we can insert (3.186) into (3.209) producing

$$\begin{aligned}
c^m|B|^{m/a} &\iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \frac{1}{(1+a'\bar{p}_k^2)^{1/a'}} \prod_{k=1} \frac{1}{(1+a'u_k^2)^{1/a'}} \prod_{j=1}^{2m} \mathrm{d}p_j \ \mathrm{d}q_j \\
&\leq c^m|B|^{m/a} \prod_{k=1}^{m} \int_{\mathbf{R}} \frac{1}{(1+\bar{p}_k^2)^{1/a'}} \ \mathrm{d}\bar{p}_j \ \prod_{k=1}^{2m} \int_{\mathbf{R}} \frac{1}{(1+u_k^2)^{1/a'}} \ \mathrm{d}u_j \\
&= c^m|B|^{m/a} \prod_{k=1}^{m} \frac{1}{(1+\bar{p}_k^2)^{[1-2(1/a')]/2}} \prod_{k=1}^{2m} \frac{1}{(1+u_k^2)^{[1-2(1/a')]/2}} \\
&\leq c^m|B|^{m/a}
\end{aligned}$$

where the quantity $[1-2(1/a')]/2$ is slightly less than $1/2$.

The bound involving the set variable is proved in a similar fashion for $\alpha(\epsilon, x, y, B)$. To identify the local time for the process defined by $(X_t - X_s, X_s - X_r)$, let $h(x, y) : \mathbf{R}^2 \to \mathbf{R}$, be a continuous function with compact support. By the local uniform convergence,

$\lim_{\epsilon\to 0}\alpha(\epsilon,x,y,T)=\alpha(x,y,T)$,

$$\begin{aligned}
&\int_{\mathbf{R}^2} h(x,y)\alpha(x,y,T)\ \mathrm{d}x\ \mathrm{d}y\\
&\quad=\lim_{\epsilon\to 0}\int_{\mathbf{R}^2} h(x,y)\alpha(\epsilon,x,y,T)\ \mathrm{d}x\ \mathrm{d}y\\
&\quad=\lim_{\epsilon\to 0}\int_{B_T}\left[\int_{\mathbf{R}^2} h(x,y)p_\epsilon(X_s-X_r-x)p_\epsilon(X_t-X_s-y)\ \mathrm{d}x\ \mathrm{d}y\right]\ \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t\\
&\quad=\int_{B_T}\lim_{\epsilon\to 0}p_\epsilon * h(X_s-X_r,X_t-X_s)\ \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t,\\
&\quad=\int_{B_T} h(X_s-X_r,X_t-X_s)\ \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}t.
\end{aligned}$$

where we used the uniform convergence, $\lim_{\epsilon\to 0}p_\epsilon * h(X_s-X_r,X_t-X_s)\to h(X_s-X_r,X_t-X_s)$, to produce the last line.

Finally, to prove $\gamma(x,y,T)$ is differentiable in y, as stated in (2.17), observe that for any $\epsilon>0$,

$$\frac{\partial}{\partial y}\gamma(\epsilon,x,y,T)=\gamma_y(\epsilon,x,y,T).$$

Therefore,

$$\gamma(\epsilon,x,y',T)=\gamma(\epsilon,x,y,T)+\int_y^{y'}\gamma_z(\epsilon,x,z,T)\ \mathrm{d}z.$$

However, the local uniform convergence of $\gamma(\epsilon,x,y,T)$ and $\gamma_y(\epsilon,x,y,T)$ reveals

$$\gamma(x,y',T)=\gamma(x,y,T)+\int_y^{y'}\gamma_z(x,y,T)\ \mathrm{d}z$$

and we conclude

$$\frac{\partial}{\partial y}\gamma(x,y,T)=\gamma_y(x,y,T).$$

Chapter 4

The Case of Symmetric Stable Processes

We now initiate our discussion of the derivatives of triple intersection local times for a class of symmetric stable processes in $\mathbf{R}^1$. In the case for standard Brownian motion, the parameter β appearing in the inverse Fourier transform of the transition density function in (2.8) and (2.9) was set equal to 2 and this value was critical in the study of the multi-dimensional integrals which emerged during the study of this first case. In our new framework, our objective will be to determine the smallest value β can assume in the interval $(1, 2)$ to render our results. Fortunately, we will be able to incorporate much of the analysis and the details of the Brownian case to streamline the proofs for the new Markov processes under scrutiny.

Through out this chapter we let X_t denote a one dimensional symmetric stable process.

4.1 Primary bounds for $\phi_y(\epsilon,x,y,B), \phi(\epsilon,x,y,B)$, and $\alpha(\epsilon,x,y,B)$

We divide the proofs involving the m^{th} moments of the three main functional in two sections.

4.1.1 Finite moments of $\phi_y(\epsilon,x,y,B)$ over both class of intervals

To establish Lemma 2.2.2 we examine

$$\mathbb{E}\left[(\phi_y(\epsilon,x,y,B))^m\right] = \frac{i^m}{(2\pi)^{2m}} \iiint\limits_{\mathbf{R}^{2m}\times B^m} \mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_j X(r_j,s_j)\right)\prod_{j=1}^m \left\{\exp\left(iq_j X(s_j,t_j)\right)\right\}_0\right] \cdot F(\beta,\epsilon,x,y,p,q)\prod_{j=1}^m \ q_j \ \mathrm{d}p_j \ \mathrm{d}q_j \ \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j, \tag{4.1}$$

with

$$F(\beta,\epsilon,x,y,p,q) \doteq \exp\left(ix\sum_{j=1}^m p_j\right)\exp\left(iy\sum_{j=1}^m q_j\right)\prod_{j=1}^m \ \hat{f}(\beta,\epsilon,p_j) \ \hat{f}(\beta,\epsilon,q_j) \tag{4.2}$$

and

$$\hat{f}(\beta,\epsilon,a_j) \doteq \exp(-\epsilon|a_j|^\beta).$$

Replicating the study of (3.2) and (3.4), we focus on

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^m p_j X(r_j,s_j)\right)\exp\left(i\sum_{j=1}^m q_j X(s_j,t_j)\right)\right]$$

and upon reordering the time variables we arrive at a computation analogous to (3.12). Noting the characteristic function of X_t is given by $\mathbb{E}\left[\exp\left(i\omega X_t\right)\right] = \exp\left(-t|\omega|^{\beta}\right)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\sum_{j=1}^{m} p_j X(r_j, s_j)\right)\exp\left(i\sum_{j=1}^{m} q_j X(s_j, t_j)\right)\right] \qquad (4.3)\\
&\quad = \mathbb{E}\left[\exp\left(i\sum_{k=1}^{m} \bar{p}_k X(\bar{r}_k, \bar{r}_{k+1})\right)\right]\mathbb{E}\left[\exp\left(i\sum_{k=1}^{2m} u_k X(d_k, d_{k+1})\right)\right]\\
&\quad = \exp\left(-\sum_{k=1}^{m} |\bar{p_k}|^{\beta}(\tilde{r}_k)\right)\exp\left(-\sum_{k=1}^{2m} |u_k|^{\beta}(\tilde{d}_k)\right).
\end{aligned}$$

Taking the non-isolated intervals into consideration , (3.22), prompts us to establish a bound on the following quantity

$$\begin{aligned}
&\frac{i^m}{(2\pi)^{2m}} \iiint\limits_{\mathbf{R}^{2m}\times B^m} \exp\left(-\sum_{k=1}^{m} |\bar{p}_k|^{\beta}(\tilde{r}_k)\right)\exp\left(-\sum_{k=1}^{2m} |u_k|^{\beta}(\tilde{d}_k)\right)\\
&\qquad\qquad \cdot F(\beta,\epsilon,x,y,p,q)\prod_{j=1}^{m} |q_j|\ \mathrm{d}p_j\ \mathrm{d}q_j\ \mathrm{d}r_j\ \mathrm{d}s_j\ \mathrm{d}t_j. \qquad (4.4)
\end{aligned}$$

Exercising the two upper bounds $|F(\beta,\epsilon,x,y,p,q)| \leq 1$, and $|i^m(2\pi)^{2m}| \leq c^m$ and integrating the time variables with the assistance of (3.28) curtails (4.4) to

$$c^m \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m}\left(1+|\bar{p}_k|^{\beta}\right)^{-1}\prod_{k=1}^{2m}(1+|u_k|^{\beta})^{-1}\prod_{j=1}^{m} |q_j|\ \mathrm{d}p_j\ \mathrm{d}q_j. \qquad (4.5)$$

To begin the study of the above expression, we look to subtract a small, finite power from each of the $|q_j|$ terms in the product above. As we will see in future steps of the analysis below, extracting this small quantity positions us to apply Theorem 3.1.4 to critical multi-dimensional integrals. Define

$$\xi_\beta \doteq 1 - \beta/2$$

and recall, β assumes values between one and two. With this in mind, split the product of

$|q_j|$ terms as follows,

$$\prod_{j=1}^{m} |q_j| = \prod_{j=1}^{m} |q_j|^{\beta/2} \, |q_j|^{\xi_\beta} \tag{4.6}$$

where $1/2 < \beta/2 < 1$ and $0 < \xi_\beta < 1/2$. In more detail, as β takes values close to 2, ξ_β approaches 0. Conversely, when β is close to 1, ξ_β approaches 1/2. Given these parameters, produce the upper bound below

$$\begin{aligned}\prod_{j=1}^{m} |q_j| &= \prod_{j=1}^{m} |q_j|^{\beta/2}|q_j|^{\xi_\beta} \\ &\leq \prod_{k\in T} \left(|u_{k-1}|^{\beta/2} + |u_k|^{\beta/2}\right) \prod_{\substack{k\in S \\ k', k'' \in R}} \left(|u_k|^{\xi_\beta} + |u_{k-1}|^{\xi_\beta} + |\bar{p}_{k'}|^{\xi_\beta} + |\bar{p}_{k''}|^{\xi_\beta}\right). \end{aligned} \tag{4.7}$$

In our usual manner, we analyze the expansion of the right hand side of the inequality above.

Proposition 4.1.1. *The polynomial generated by multiplying the products*

$$\mathcal{Q}_1(|\bar{p}_k|, |u_k|) \doteq \prod_{k\in T} \left(|u_{k-1}|^{\beta/2} + |u_k|^{\beta/2}\right) \prod_{\substack{k\in S \\ k', k'' \in R}} \left(|u_k|^{\xi_\beta} + |u_{k-1}|^{\xi_\beta} + |\bar{p}_{k'}|^{\xi_\beta} + |\bar{p}_{k''}|^{\xi_\beta}\right). \tag{4.8}$$

is of the form

$$\mathcal{Q}_1(|\bar{p}_k|, |u_k|) = \sum \left(\prod_{R} |\bar{p}_k|^{\eta_k \xi_\beta} \prod_{S'_\xi} |u_k|^{\theta_k \xi_\beta} \prod_{S^-_\xi} |u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta} \prod_{T^+_\xi, T'_\xi} |u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta} \right)$$

where

$$deg\ |\bar{p}_k| = \eta_k \xi_\beta = 0, \xi_\beta, \ or\ 2\xi_\beta, \quad if\ k \in R,$$

Measure/Interval	***Primary Inequality***	***Values of*** β **and** ξ_β
$\mathrm{d}\bar{p}_k$	$1+\eta_k\xi_\beta<\beta$	$3/2<\beta<2$ $0<\xi_\beta<1/4$
$S_\xi^{'}$	$1+\theta_k\xi_\beta<\beta$	$3/2<\beta<2$ $0<\xi_\beta<1/4$
S_ξ^{-}	$\frac{1+2[\omega_k(\beta/2)+\theta_k\xi_\beta]}{2}<\beta$	$3/2<\beta<2$ $0<\xi_\beta<1/4$
T_ξ^{+}	$\frac{1+2[\omega_k(\beta/2)+\theta_k\xi_\beta]}{2}<\beta$	$1<\beta<2$ $0<\xi_\beta<1/2$

Table 4.1: Parameters β and ξ_β over the non-isolated intervals.

$$deg\ |u_k| = \omega_k(\beta/2)+\theta_k\xi_\beta \begin{cases} = 0, \xi_\beta,\ \ or\ 2\xi_\beta, & if\ k\in S_\xi^{'} \\ = \beta/2\ \ or\ 1, & if\ k\in S_\xi^{-} \\ = \xi_\beta\ \ or\ \beta/2, & if\ k\in T_\xi^{+} \\ \leq \beta, & if\ k\in T_\xi^{'}. \end{cases} \tag{4.9}$$

Proof. Initiating the analysis of $\mathcal{P}_4(|\bar{p}_k|,|u_k|)$, defined as (3.114) in Proposition 3.2.2, leads to our desired result. $\square$

Note that the maximum degree of $|\bar{p}_k|$ is equal to $2\xi_\beta$ and this order is bounded by 1.

Insert $\mathcal{Q}_1(|\bar{p}_k|,|u_k|)$ into (4.5) and utilize the bound $|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}\left(1+|u_k|^\beta\right)^{-1}\leq c$, for all $k\in T_\xi^{'}$ to arrive at

$$\iint\limits_{\mathbf{R}^{2m}} \prod_R |\bar{p}_k|^{\eta_k\xi_\beta} \prod_{k=1}^m \frac{1}{1+|\bar{p}_k|^\beta} \prod_{S_\xi^{'}} \frac{|u_k|^{\theta_k\xi_\beta}}{1+|u_k|^\beta} \prod_{S_\xi^{-}} \frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta} \prod_{T_\xi^{+}} \frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta} \prod_{j=1}^m \mathrm{d}p_j\ \mathrm{d}q_j. \tag{4.10}$$

Proposition 3.1.4 permits us to conclude the above products are finite for $\beta>3/2$. A summary of this analysis is provided in Table 4.1 above. Looking back to (4.6), if $3/2<\beta<2$, then $0<\xi_\beta<1/4$.

Over the isolated intervals, we aim to derive a bound for the expression

$$\iiint_{\mathbf{R}^{2m}\times B^m} \exp\left(-\sum_{k=1}^{m} |\bar{p}_k|^\beta(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} |u_k|^\beta(\tilde{d}_k)\right)$$
$$\cdot \left| \prod_{k\in I} \left(\exp\left(-|u_k|^\beta(\tilde{d}_k)\right) - \exp\left(-\left(|u_{k+1}|^\beta + |q_{\ell(k)}|^\beta\right)(\tilde{d}_k)\right)\right)\right|$$
$$\prod_{j=1}^{m} |q_j| \ \mathrm{d}p_j \ \mathrm{d}q_j \ \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j. \tag{4.11}$$

Referring to the layout for the Brownian case, we first integrate the isolated intervals and the isolated variables from the above. This endorses the investigation of

$$\iint_{\mathbf{R}\times B} \left|\exp\left(-|u_k|^\beta(\tilde{d}_k)\right) - \exp\left(-\left(|u_{k+1}|^\beta + |q_{\ell(k)}|^\beta\right)(\tilde{d}_k)\right)\right| \ |q_{\ell(k)}| \ \mathrm{d}(\tilde{d}_k) \ \mathrm{d}q_{\ell(k)}. \tag{4.12}$$

Now, borrow portions of our evaluation of (3.61). Substitute (3.55) for the u_k variables and simplify notation by setting $u_{k+1} = u$ and $q_{\ell(k)} = q$. Partition the domain of integration involving $\mathbf{R}$ into disjoints sets

$$\iint_{\mathbf{R}\times B} \left|\exp\left(-\left(|u+q|^\beta\right)(\tilde{d}_k)\right) - \exp\left(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\right)\right| \ |q| \ \mathrm{d}(\tilde{d}_k) \ \mathrm{d}q$$
$$= \int_{|q|\le 4|u|} \int_0^1 \left|\exp\left(-\left(|u+q|^\beta\right)(\tilde{d}_k)\right) - \exp\left(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\right)\right| \ |q| \ \mathrm{d}(\tilde{d}_k) \ \mathrm{d}q \tag{4.13}$$
$$+ \int_{|q|\ge 4|u|} \int_0^1 \left|\exp\left(-\left(|u+q|^\beta\right)(\tilde{d}_k)\right) - \exp\left(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\right)\right| \ |q| \ \mathrm{d}(\tilde{d}_k) \ \mathrm{d}q. \tag{4.14}$$

Working with (4.13), integrate first the isolated intervals

$$\begin{aligned}
&\int\limits_{|q|\le 4|u|} \int_0^1 \left|\exp\left(-\left(|u+q|^\beta\right)(\tilde{d}_k)\right) - \exp\left(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\right)\right| \ |q| \ \mathrm{d}(\tilde{d}_k) \ \mathrm{d}q \\
&\qquad \le c \int\limits_{|q|\le 4|u|} \left[\frac{1}{1+(|u+q|^\beta)} + \frac{1}{1+(|u|^\beta + |q|^\beta)}\right] \ |q| \ \mathrm{d}q \\
&\qquad \le c|u| \left[\ \int\limits_{|q|\le 4|u|} \frac{1}{1+(|u+q|^\beta)} \ \mathrm{d}q \ + \int\limits_{|q|\le 4|u|} \frac{1}{1+(|u|^\beta + |q|^\beta)} \ \mathrm{d}q\right]. \qquad (4.15)
\end{aligned}$$

Looking closely at the first integral in the brackets, the cusp at $q = -u$ directs us to two integrals

$$\int\limits_{|q|\le 4|u|} \frac{1}{1+(|u+q|^\beta)} \ \mathrm{d}q = \int\limits_{-4|u|}^{-u} \frac{1}{1+(|u+q|^\beta)} \ \mathrm{d}q + \int\limits_{-u}^{4|u|} \frac{1}{1+(|u+q|^\beta)} \ \mathrm{d}q \le c.$$

Shifting attention to the second integral appearing in that bracket, the symmetry of the integrand along $q = 0$ implies

$$\int\limits_{|q|\le 4|u|} \frac{1}{1+(|u|^\beta + |q|^\beta)} \ \mathrm{d}q = 2 \int\limits_{0}^{4|u|} \frac{1}{1+(|u|^\beta + q^\beta)} \ \mathrm{d}q \le c$$

and we see that (4.13) can be bounded by

$$\int\limits_{|q|\le 4|u|} \int_0^1 \left|\exp\left(-\left(|u+q|^\beta\right)(\tilde{d}_k)\right) - \exp\left(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\right)\right| \ |q| \ \mathrm{d}(\tilde{d}_k) \ \mathrm{d}q \le c|u|. \qquad (4.16)$$

Turning to (4.14), we first undertake integrals with regard to the measure $\mathrm{d}(\tilde{d}_k)$. By enlarging the domain of integration and considering the two cases when either $\left(|u+q|^\beta\right) \ge$

$\left(|u|^\beta + |q|^\beta\right)$ or $\left(|u+q|^\beta\right) \leq \left(|u|^\beta + |q|^\beta\right)$, we obtain the upper bound

$$\begin{aligned}
&\int_0^1 \left|\exp\left(-\left(|u+q|^\beta\right)(\tilde{d}_k)\right) - \exp\left(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\right)\right| \;\mathrm{d}(\tilde{d}_k) \\
&\qquad \leq \int_0^\infty \left|\exp\left(-\left(|u+q|^\beta\right)(\tilde{d}_k)\right) - \exp\left(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\right)\right| \;\mathrm{d}(\tilde{d}_k) \\
&\qquad \leq c\left|\frac{1}{|u|^\beta + |q|^\beta} - \frac{1}{|u+q|^\beta}\right| \qquad (4.17)
\end{aligned}$$

Before moving to the isolated variables, we bring to mind the binomial series expansion for $|x| < 1$ and $m \notin \mathbf{Z}_+$,

$$(1+x)^m = 1 + \sum_{k=1}^{m} \binom{m}{k} x^k = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$

For values of $1 < \beta < 2$, an application of the above series unravels the following upper bounds,

$$\begin{aligned}
&\int\limits_{|q|\geq 4|u|} \int_0^1 \left|\exp\left(-\left(|u+q|^\beta\right)(\tilde{d}_k)\right) - \exp\left(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\right)\right| \;|q|\;\mathrm{d}(\tilde{d}_k)\;\mathrm{d}q \\
&\qquad \leq c \int\limits_{|q|\geq 4|u|} \left|\frac{1}{|u+q|^\beta} - \frac{1}{|u|^\beta + |q|^\beta}\right| \;|q|\;\mathrm{d}q \\
&\qquad = c \int\limits_{|q|\geq 4|u|} \left|\frac{1}{|1+u/q|^\beta} - \frac{1}{(1+|u|^\beta/|q|^\beta)}\right| \;\frac{|q|}{|q|^\beta}\;\mathrm{d}q \\
&\qquad = c \int\limits_{|q|\geq 4|u|} \left|\left(1 - \beta\frac{|u|}{|q|} + \frac{\beta(\beta-1)}{2!}\left(\frac{|u|}{|q|}\right)^2 - \cdots\right)\right. \\
&\qquad\qquad\qquad \left. - \left(1 - \left(\frac{|u|}{|q|}\right)^\beta + \left(\frac{|u|}{|q|}\right)^{2\beta} - \cdots\right)\right| \;\frac{|q|}{|q|^\beta}\;\mathrm{d}q \\
&\qquad \leq c \int\limits_{|q|\geq 4|u|} \frac{|u|}{|q|}\;\frac{|q|}{|q|^\beta}\;\mathrm{d}q = c \int\limits_{|q|\geq 4|u|} \frac{|u|}{|q|^\beta}\;\mathrm{d}q \leq c|u|^{2-\beta}. \qquad (4.18)
\end{aligned}$$

Adjoining the bounds (4.16) and (4.18) confirms (4.12) is bounded by

$$c|u_{k+1}| + c|u_{k+1}|^{2-\beta} \leq |u_{k+1}|^{1+\xi} + c, \quad k \in I$$

since, $|u_{k+1}|^{2-\beta} \leq 1$ for small u_{k+1}. This analysis, in conjunction with an application of (3.28) for integrals involving the remaining time measures, prompts us to bound (4.11) by

$$\iint_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^\beta\right)^{-1} \prod_{I^c} \left(1 + |u_k|^\beta\right)^{-1} \prod_{k \in I} \left[c + |u_{k+1}|\right] \prod_{I_j^c} |q_j| \, \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j,$$

where q_j are the remaining non-isolated variables, I_j^c is defined by (3.15), and the n appearing in the domain of integration is the cardinality of I_j^c. Working with the first sum in the above brackets returns us to the argument involving the non-isolated intervals. The primary quantity of interest takes form

$$\iint_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^\beta\right)^{-1} \prod_{I^c} \left(1 + |u_k|^\beta\right)^{-1} \prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j| \, \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j. \tag{4.19}$$

Set

$$\begin{aligned} &\mathcal{Q}_2(|\bar{p}_k|, |u_k|) \\ &\doteq \prod_{k \in I} |u_{k+1}| \prod_{k \in T} \left(|u_{k-1}|^{\beta/2} + |u_k|^{\beta/2}\right) \prod_{\substack{k \in S \\ k', k'' \in R}} \left(|u_k|^{\xi_\beta} + |u_{k-1}|^{\xi_\beta} + |\bar{p}_{k'}|^{\xi_\beta} + |\bar{p}_{k''}|^{\xi_\beta}\right) \\ &= \prod_{k \in I} |u_{k+1}| \mathcal{Q}_1(|\bar{p}_k|, |u_k|) \end{aligned} \tag{4.20}$$

so that we may construct the bound for the subsequent products,

$$\prod_{k\in I} |u_{k+1}| \prod_{I_j^c} |q_j| = \prod_{k\in I} |u_{k+1}| \prod_{I_j^c} |q_j|^{\beta/2} \, |q_j|^{\xi_\beta} \leq c\mathcal{Q}_2(|\bar{p}_k|, |u_k|). \tag{4.21}$$

Proposition 4.1.2 characterizes the polynomial emerging in the upper bound.

Proposition 4.1.2. *The expansion of the products*

$$\prod_{k\in I} |u_{k+1}| \prod_{k\in T} \left(|u_{k-1}|^{\beta/2} + |u_k|^{\beta/2}\right) \prod_{\substack{k\in S \\ k^{'},k^{''}\in R}} \left(|u_k|^{\xi_\beta} + |u_{k-1}|^{\xi_\beta} + |\bar{p}_{k'}|^{\xi_\beta} + |\bar{p}_{k''}|^{\xi_\beta}\right)$$

yields a polynomial of the form

$$\mathcal{Q}_2(|\bar{p}_k|, |u_k|) = \sum \Bigg(\prod_R |\bar{p}_k|^{\eta_k \xi_\beta} \prod_{S_\xi^{'}} |u_k|^{\theta_k \xi_\beta} \prod_{S_\xi^-} |u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}$$
$$\prod_{T^{I+1}} |u_k|^{1+\omega_k(\beta/2)+\theta_k\xi_\beta} \prod_{T_\xi^+, T_\xi^{''}} |u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta} \Bigg),$$

with

$$\deg|\bar{p}_k| = \eta_k\xi_\beta = 0, \xi_\beta, \text{ or } 2\xi_\beta, \quad \text{if } k \in R$$

$$\text{deg } |u_k| = \omega_k(\beta/2) + \theta_k\xi_\beta \begin{cases} = 0, \xi_\beta, \text{ or } 2\xi_\beta, & \text{if } k \in S_\xi^{'} \\ = \beta/2 \text{ or } 1, & \text{if } k \in S_\xi^- \\ = \xi_\beta \text{ or } \beta/2, & \text{if } k \in T_\xi^+ \\ \leq \beta, & \text{if } k \in T_\xi^{''}. \end{cases} \tag{4.22}$$

$$\text{deg } |u_k| = 1 + \omega_k(\beta/2) + \theta_k\xi_\beta = 1, 1+\beta/2, \text{ or } 1+\xi_\beta, \quad \text{if } k \in T^{I+1}.$$

Proof. Observe the shared similarities between the products in the above statement and the ones presented in (3.144). After replacing the T_ξ^{I+1}-intervals defined in (3.145) with

$$T^{I+1} \doteq \{k \in T \mid k-1 \in I\},$$

the analysis of (3.146), which culminated in $\mathcal{P}_6(|\bar{p}_k|,|u_k|)$ of (3.147), can be applied to $\mathcal{Q}_2(|\bar{p}_k|,|u_k|)$. Hence, closing the argument. □

Upon implanting the above into (4.19),

$$\sum \Bigg[\iint\limits_{\mathbf{R}^m\times\mathbf{R}^n} \prod_{R} |\bar{p}_k|^{\eta_k\xi_\beta} \prod_{k=1}^{m} \left(1+|\bar{p}_k|^\beta\right)^{-1} \prod_{S,T} \left(1+|u_k|^\beta\right)^{-1} \prod_{S_\xi'} |u_k|^{\theta_k\xi_\beta} \prod_{S_\xi^-} |u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}$$
$$\prod_{T_\xi^+} |u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta} \prod_{T^{I+1}} |u_k|^{1+\omega_k(\beta/2)+\theta_k\xi_\beta} \prod_{T_\xi''} |u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta} \prod_{I_j^c} \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j \Bigg], \tag{4.23}$$

we notice the divergent integrands associated with the T^{I+1}-intervals,

$$\lim_{u_k\to\infty} \frac{|u_k|^{1+\beta/2}}{1+|u_k|^\beta} = \infty. \tag{4.24}$$

To identify these troubled cases set

$$T_{1+\beta/2}^{I+1} \doteq \left\{k \in T^{I+1} \mid \deg|u_k| = 1+\beta/2\right\}.$$

As depicted in (3.154), a similar complication emerged in the process of establishing Hölder continuity in the y variable for the Brownian case. Fortunately, we can leverage and adjust the detailed analysis of the $T_{2+\xi}^{I+1}$-intervals of (3.155) to resolve our current dilemma. To explain how this can be accomplished, begin by replacing (3.156) and (3.157) with

$$S^{I-1}_{\beta/2} \doteq \left\{ k \in S'_\xi \mid k+2 \in T^{I+1}_{1+\beta/2} \right\}, \; T^{I-1}_{\beta/2} \doteq \left\{ k \in T^+_\xi \cup T^{I+1} \cup T''_\xi \mid k+2 \in T^{I+1}_{1+\beta/2} \right\}.$$

As in the Brownian case, the u variables linked to the above intervals are of critical importance and if we revise Proposition 3.2.5 to take into account the variations amongst polynomials $\mathcal{P}_6$ and $\mathcal{Q}_2$, (3.158) can be replaced with

$$\deg |u_k| \leq \begin{cases} \xi_\beta, & \text{if } k \in S^{I-1}_{\beta/2}, \\ 1, & \text{if } k \in T^{I-1}_{\beta/2}. \end{cases}$$

Of course, (3.159) easily carries over to our new analysis since this reexpression is solely dependent on the ordering of the time variables s and t. With these key properties established, our goal is to arrive at a polynomial similar to $\mathcal{P}_7(|\bar{p}_k|, |u_k|)$. Recall $\xi_\beta \doteq 1 - \beta/2$. With $k \in S^{I-1}_{\beta/2} \cup T^{I-1}_{\beta/2}$ and $k+2 \in T^{I+1}_{1+\beta/2}$ write,

$$\frac{|u_{k+2}|^{1+\beta/2}}{1+|u_{k+2}|^\beta} \leq |u_{k+2}|^{\xi_\beta} \leq c\left(|u_k|^{\xi_\beta} + |p_{\ell_{(k+1)}}|^{\xi_\beta} \right).$$

The products linked to the T^{I+1}-intervals can, now, be bounded by

$$\begin{aligned} \prod_{T^{I+1}} \frac{|u_k|^{1+\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta} &\leq c \prod_{T^{I+1}/T^{I+1}_{1+\beta/2}} \frac{|u_k|^{1+\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta} \prod_{\substack{k\in S^{I-1}_{\beta/2}\cup T^{I-1}_{\beta/2} \\ k+1\in I}} \left(|u_k|^{\xi_\beta} + |p_{\ell_{(k+1)}}|^{\xi_\beta} \right) \\ &\leq c \prod_{T^{I+1}/T^{I+1}_{1+\beta/2}} \frac{|u_k|^{1+\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta} \cdot \mathcal{Q}_3(|\bar{p}_k|, |u_k|), \end{aligned}$$

where

$$\mathcal{Q}_3(|\bar{p}_k|,|u_k|) \doteq \prod_{\substack{k\in S^{I-1}_{\beta/2}\cup T^{I-1}_{\beta/2}\\ k',k''\in R}} \left(|u_k|^{\xi_\beta}+|p_{k'}|^{\xi_\beta}+|p_{k''}|^{\xi_\beta}\right)$$
$$=\sum\left(\prod_{S^{I-1}_{\beta/2}\cup T^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\prod_R|\bar{p}_k|^{\tilde{\eta}_k\xi_\beta}\right),$$

Having resolved the problematic rational functions in (4.24), any multi-dimensional integral arising as a summand in (4.23) can be bounded by

$$c^m\iint\limits_{\mathbf{R}^m\times\mathbf{R}^n}\prod_R|\bar{p}_k|^{(\eta_k+\tilde{\eta}_k)\xi_\beta}\prod_{k=1}^m\left(1+|\bar{p}_k|^\beta\right)^{-1}\prod_{S'_\xi}\frac{|u_k|^{\theta_k\xi_\beta}}{1+|u_k|^\beta}\prod_{S^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\prod_{S^-_\xi}\frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta}$$
$$\prod_{T^+_\xi}\frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta}\prod_{T^+_\xi\cap T^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\left[\prod_{T^{I+1}/T^{I+1}_{1+\beta/2}}\frac{|u_k|^{1+\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta}\prod_{T^{I+1}\cap T^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\right]$$
$$\left[\prod_{T''_\xi}\frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta}\prod_{T''_\xi\cap T^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\right]\prod_{I^c_j}\mathrm{d}q_j\prod_{j=1}^m\mathrm{d}p_j. \tag{4.25}$$

To prove the product of rational functions originating from the $T^{I+1}\cup T''_\xi$-intervals can be bounded by a constant, we reexamine the three instances succeeding (3.164). This directs us to

$$\deg|u_k|=\begin{cases}\le 1 & \text{if } k\in\{T^{I+1}\cap T^{I-1}_{\beta/2}\}\cup\{T''_\xi\cap T^{I-1}_{\beta/2}\}\\ =1+\xi_\beta, & \text{if } k\in\{T^{I+1}/T^{I+1}_{1+\beta/2}\}/T^{I-1}_\xi\\ \le\beta\ , & \text{if } k\in T''_\xi/T^{I-1}_{\beta/2}.\end{cases}$$

Measure/Interval	***Primary Inequality***	***Values of*** β **and** ξ_β
$\mathrm{d}\bar{p}_k$	$1+(\eta_k+\tilde{\eta}_k)\xi_\beta<\beta$	$3/2<\beta<2$ $0<\xi_\beta<1/4$
$S'_\xi/S^{I-1}_{\beta/2}$	$1+\theta_k\xi_\beta<\beta$	$3/2<\beta<2$ $0<\xi_\beta<1/4$
$S^{I-1}_{\beta/2}$	$1+\theta_k\xi_\beta+\xi_\beta<\beta$	$3/2<\beta<2$ $0<\xi_\beta<1/4$
S^-_ξ	$\dfrac{1+2[\omega_k(\beta/2)+\theta_k\xi_\beta]}{2}<\beta$	$3/2<\beta<2$ $0<\xi_\beta<1/4$
$T^+_\xi/T^{I-1}_{\beta/2}$	$\dfrac{1+2[\omega_k(\beta/2)+\theta_k\xi_\beta]}{2}<\beta$	$1<\beta<2$ $0<\xi_\beta<1/2$
$T^+_\xi\cap T^{I-1}_{\beta/2}$	$\dfrac{1+2[\omega_k(\beta/2)+\theta_k\xi_\beta+\xi_\beta]}{2}<\beta$	$3/2<\beta<2$ $0<\xi_\beta<1/4$

Table 4.2: Parameters β and ξ_β over the isolated intervals.

Therefore,

$$\left[\prod_{T^{I+1}/T^{I+1}_{1+\beta/2}}\frac{|u_k|^{1+\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta}\prod_{T^{I+1}\cap T^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\right]\cdot\left[\prod_{T''_\xi}\frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta}\prod_{T''_\xi\cap T^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\right]\le c$$

and (4.25) can be bounded by

$$\begin{aligned}c^m\iint_{\mathbf{R}^m\times\mathbf{R}^n}&\prod_R|\bar{p}_k|^{(\eta_k+\tilde{\eta}_k)\xi_\beta}\prod_{k=1}^m\left(1+|\bar{p}_k|^\beta\right)^{-1}\prod_{S'_\xi}\frac{|u_k|^{\theta_k\xi_\beta}}{1+|u_k|^\beta}\prod_{S^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\prod_{S^-_\xi}\frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta}\\&\prod_{T^+_\xi}\frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta}\prod_{T^+_\xi\cap T^{I-1}_{\beta/2}}|u_k|^{\xi_\beta}\prod_{I^c_j}\mathrm{d}q_j\prod_{j=1}^m\mathrm{d}p_j.\end{aligned}$$

A reference to Table 4.2 above, which summarizes the critical details concerning each of the parameters emerging over the various s and t-intervals, permits us to conclude the above multiple integral is finite for $3/2<\beta<2$.

4.1.2 Main estimate for $\phi(\epsilon, x, y, B)$ and $\alpha(\epsilon, x, y, B)$

With the finite moments of $\phi_y(\epsilon, x, y, B)$ settled, we shift attention to the renormalized intersection local time and intersection local time and derive the following,

$$\mathbb{E}\left[\left(\phi(\epsilon, x, y, B)\right)^m\right] \leq c$$
$$\mathbb{E}\left[\left(\alpha(\epsilon, x, y, B)\right)^m\right] \leq c.$$

Proofs concerning these Markovian functionals can be simplified upon leveraging the detailed analysis from the previous section.

The proof for $\phi(\epsilon, x, y, B)$ begins with

$$\frac{i^m}{(2\pi)^{2m}} \iiint\limits_{\mathbf{R}^{2m}\times B^m} \exp\left(-\sum_{k=1}^{m} |\bar{p}_k|^\beta (\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} |u_k|^\beta (\tilde{d}_k)\right)$$
$$\cdot F(\beta, \epsilon, x, y, p, q) \prod_{j=1}^{m} \mathrm{d}p_j \ \mathrm{d}q_j \ \mathrm{d}r_j \ \mathrm{d}s_j \ \mathrm{d}t_j.$$

Bounding $|F(\beta, \epsilon, x, y, p, q) \cdot i^m \cdot (2\pi)^{-2m}| \leq c$ and integrating the time variables yields

$$c^m \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^\beta\right)^{-1} \prod_{k=1}^{2m} (1 + |u_k|^\beta)^{-1} \prod_{j=1}^{m} \mathrm{d}p_j \ \mathrm{d}q_j.$$

The facts following (3.8) and (3.10) justify a change of variables and we can conclude the following product of integrals are finite for $\beta \in (1, 2)$,

$$\prod_{k=1}^{m} \int_{\mathbf{R}} \left(1 + |\bar{p}_k|^\beta\right)^{-1} \ \mathrm{d}\bar{p}_k \prod_{k=1}^{2m} \int_{\mathbf{R}} \left(1 + |u_k|^\beta\right)^{-1} \ \mathrm{d}u_k.$$

We simply mention that the construction of the proof for

$$\mathbb{E}\left[\left(\alpha(\epsilon, x, y, B)\right)^m\right] \leq c$$

follows naturally from the techniques derived in the study of $\phi(\epsilon, x, y, B)$.

4.2 Hölder Continuity for the Three Functionals

The development of the proof

$$\|\phi_y(\epsilon, x, y, B) - \phi_y(\epsilon', x', y', B)\|_m \leq c_m(\bar{\xi}) \left\|(\epsilon, x, y) - (\epsilon', x', y')\right\|^{m\bar{\xi}},$$

with $\bar{\xi} \in (0, 1)$, for symmetric stable processes will proceed in the same manner as in the Brownian case. Namely, the objective is to secure (3.91) over both class of intervals. With respect to the new processes under scrutiny, the deterministic function impacted by changes in the three terms, (x, y, ϵ), is $F(\beta, \epsilon, x, y, p, q)$, defined in (4.2).

Beginning with the variation in x over the first case of the non-isolated intervals, we look to scrutinize

$$c^m |x - x'|^{m\bar{\xi}} \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^{\beta}\right)^{-1} \prod_{k=1}^{2m} \left(1 + |u_k|^{\beta}\right)^{-1} \prod_{j=1}^{m} |q_j| \; |p_j|^{\bar{\xi}} \, \mathrm{d}p_j \, \mathrm{d}q_j, \tag{4.26}$$

where $\bar{\xi} \in (0, 1)$. Separate the product concerning the $|q_j|$ terms as in (4.6) and apply the study of (3.99). This produces the upper bound

$$\prod_{j=1}^{m} |q_j| \; |p_j|^{\bar{\xi}} \leq c\mathcal{Q}_1(|\bar{p}_k|, |u_k|)\mathcal{P}_3(|\bar{p}_k|). \tag{4.27}$$

Upon expanding the right side of the above inequality, we are led to conclude the order of the $|u_k|$ terms are exactly as presented in (4.9). With respect to the $|\bar{p}_k|$ variables, analysis of the expansion of

$$\prod_{\substack{k\in S\\ k',k''\in R}} \left(|u_k|^{\xi_\beta} + |u_{k-1}|^{\xi_\beta} + |\bar{p}_{k'}|^{\xi_\beta} + |\bar{p}_{k''}|^{\xi_\beta}\right) \prod_{k',k''\in R} \left(|\bar{p}_{k'}|^{\bar{\xi}} + |\bar{p}_{k''}|^{\bar{\xi}}\right)$$

reveals

$$\deg\ |\bar{p}_k| = \eta_k \xi_\beta + \bar{\eta}_k \bar{\xi},\ \text{with } \eta_k, \bar{\eta}_k \in \{0,1,2\}. \tag{4.28}$$

The two primary observations to take away from the above is $\bar{\xi}$ can be chosen close to zero and the maximum order the $\bar{p}_k$ variables can assume is $2\xi_\beta + 2\bar{\xi}$. Combining (4.26) with (4.27) and incorporating modifications regarding the rational function affiliated with the measure $\mathrm{d}\bar{p}_k$ directs us to

$$\iint\limits_{\mathbf{R}^{2m}} \prod_{R} |\bar{p}_k|^{\eta_k\xi_\beta + \bar{\eta}_k\bar{\xi}} \prod_{k=1}^{m} \frac{1}{1+|\bar{p}_k|^\beta} \prod_{S'_\xi} \frac{|u_k|^{\theta_k\xi_\beta}}{1+|u_k|^\beta} \prod_{S^-_\xi \cup T^+_\xi} \frac{|u_k|^{\omega_k(\beta/2)+\theta_k\xi_\beta}}{1+|u_k|^\beta} \prod_{j=1}^{m} \mathrm{d}p_j\ \mathrm{d}q_j.$$

We can select $\bar{\xi}$ close to zero so that the above integral is finite for $3/2 < \beta < 2$ and conclude over the non-isolated intervals,

$$\mathbb{E}\left[\left(\phi_y(\epsilon,x,y,B) - \phi_y(\epsilon,x',y,B)\right)^m\right] \le c_m(\bar{\xi})|x-x'|^{m\bar{\xi}} \tag{4.29}$$

Perturbations in the y parameter produce the multiple integral

$$c^m|y-y'|^{m\bar{\xi}} \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1+|\bar{p}_k|^\beta\right)^{-1} \prod_{k=1}^{2m} \left(1+|u_k|^\beta\right)^{-1} \prod_{j=1}^{m}\ |q_j|\ |q_j|^{\bar{\xi}}\ \mathrm{d}p_j\ \mathrm{d}q_j,$$

where $\bar{\xi} \in (0,1)$. Set

$$\bar{\xi}_\beta \doteq \xi_\beta + \bar{\xi}, \quad \text{with } \xi_\beta \doteq 1 - \beta/2,$$

and define the polynomial

$$\mathcal{Q}_4(|\bar{p}_k|, |u_k|) \doteq \prod_{k \in T} \left(|u_{k-1}|^{\beta/2} + |u_k|^{\beta/2}\right) \prod_{\substack{k \in S \\ k', k'' \in R}} \left(|u_k|^{\bar{\xi}_\beta} + |u_{k-1}|^{\bar{\xi}_\beta} + |\bar{p}_{k'}|^{\bar{\xi}_\beta} + |\bar{p}_{k''}|^{\bar{\xi}_\beta}\right) \tag{4.30}$$

to derive the upper bound

$$\prod_{j=1}^{m} |q_j| \; |q_j|^{\bar{\xi}} = \prod_{j=1}^{m} |q_j|^{\beta/2} \; |q_j|^{\xi_\beta + \bar{\xi}} = \prod_{j=1}^{m} |q_j|^{\beta/2} \; |q_j|^{\bar{\xi}_\beta} \leq \mathcal{Q}_4(|\bar{p}_k|, |u_k|). \tag{4.31}$$

Referencing (4.8) and (4.9) leads to the orders of $\mathcal{Q}_4(|\bar{p}_k|, |u_k|)$,

$$\deg |\bar{p}_k| = \bar{\eta}_k \bar{\xi}_\beta = 0, \bar{\xi}_\beta, \text{ or } 2\bar{\xi}_\beta, \quad \text{if } k \in R,$$

$$\deg |u_k| = \omega_k(\beta/2) + \theta_k \bar{\xi}_\beta \begin{cases} = 0, \bar{\xi}_\beta, \text{ or } 2\bar{\xi}_\beta, & \text{if } k \in S'_\xi \\ = \beta/2 \text{ or } 1 + \bar{\xi}, & \text{if } k \in S^-_\xi \\ = \bar{\xi}_\beta \text{ or } \beta/2, & \text{if } k \in T^+_\xi \\ \leq \beta, & \text{if } k \in T'_\xi. \end{cases} \tag{4.32}$$

Bounding the T'_ξ-intervals by a constant and applying Cauchy-Schwarz produces a product of one dimensional integrals similar to (4.10)

$$\iint_{\mathbf{R}^{2m}} \prod_{R} |\bar{p}_k|^{\bar{\eta}_k \bar{\xi}_\beta} \prod_{k=1}^{m} \frac{1}{1 + |\bar{p}_k|^\beta} \prod_{S'_\xi} \frac{|u_k|^{\theta_k \bar{\xi}_\beta}}{1 + |u_k|^\beta} \prod_{S^-_\xi} \frac{|u_k|^{\omega_k(\beta/2) + \theta_k \bar{\xi}_\beta}}{1 + |u_k|^\beta} \prod_{T^+_\xi} \frac{|u_k|^{\omega_k(\beta/2) + \theta_k \bar{\xi}_\beta}}{1 + |u_k|^\beta} \prod_{j=1}^{m} \mathrm{d}p_j \; \mathrm{d}q_j. \tag{4.33}$$

With $\bar{\xi}$ selected close to zero, we can replicate the study which generated Table 4.1 to conclude the above product of one-dimensional integrals are finite for $3/2 < \beta < 2$. Over the non-isolated variables we have shown,

$$\mathbb{E}\left[\left(\phi_y(\epsilon,x,y,B)-\phi_y(\epsilon,x,y',B)\right)^m\right]\le c_m(\bar{\xi})|y-y'|^{m\bar{\xi}}. \tag{4.34}$$

The proof for the case of ϵ begins by replicating inequality (3.128) with respect to $F(\beta,\epsilon,\epsilon',x,y,p,q)$,

$$\begin{aligned}
&\prod_{j=1}^{m}\hat{f}(\beta,\epsilon,p_j)\hat{f}(\beta,\epsilon,q_j)-\hat{f}(\beta,\epsilon',p_j)\hat{f}(\beta,\epsilon',q_j)\\
&\quad=\prod_{j=1}^{m}\exp\left(-\epsilon\left(|p_j|^{\beta}+|q_j|^{\beta}\right)\right)-\exp\left(-\epsilon'\left(|p_j|^{\beta}+|q_j|^{\beta}\right)\right)\\
&\quad\le c^m|\epsilon-\epsilon'|^{m\hat{\xi}}\prod_{j=1}^{m}\left(|p_j|^{\beta\hat{\xi}}+|q_j|^{\beta\hat{\xi}}\right)\le c^m|\epsilon-\epsilon'|^{m\bar{\xi}}\prod_{j=1}^{m}\left(|p_j|^{\bar{\xi}}+|q_j|^{\bar{\xi}}\right),
\end{aligned}$$

where $\hat{\xi}$ can be chosen close to zero and $\bar{\xi}=\beta\hat{\xi}$. This endorses the study of the expression

$$c^m|\epsilon-\epsilon'|^{m\bar{\xi}}\iint_{\mathbf{R}^{2m}}\prod_{k=1}^{m}(1+|\bar{p}_k|^{\beta})^{-1}\prod_{k=1}^{2m}(1+|u_k|^{\beta})^{-1}\prod_{j=1}^{m}\ |q_j|\ \left(|p_j|^{\bar{\xi}}+|q_j|^{\bar{\xi}}\right)\,\mathrm{d}p_j\,\mathrm{d}q_j.$$

Let $\pi^{'}$ and $\pi^{''}$ form a disjoint union of the set $\{1,...,m\}$. The main quantity of interest from the above can be written as

$$\begin{aligned}
\prod_{j=1}^{m}\ |q_j|\ \left(|p_j|^{\bar{\xi}}+|q_j|^{\bar{\xi}}\right)&=\sum\left(\prod_{\pi'}|p_j|^{\bar{\xi}}\prod_{j=1}^{m}\ |q_j|\ \prod_{\pi''}|q_j|^{\bar{\xi}}\right)\\
&=\sum\left(\prod_{\pi'}|p_j|^{\bar{\xi}}\prod_{j=1}^{m}\ |q_j|^{\beta/2}\ |q_j|^{\xi_\beta}\ \prod_{\pi''}|q_j|^{\bar{\xi}}\right).
\end{aligned} \tag{4.35}$$

Polynomials generated by

$$\prod_{\pi'}|p_j|^{\bar{\xi}}\prod_{j=1}^{m}\ |q_j|^{\beta/2}\ |q_j|^{\xi_\beta}\quad\text{and}\quad\prod_{j=1}^{m}\ |q_j|^{\beta/2}\ |q_j|^{\xi_\beta}\ \prod_{\pi''}|q_j|^{\bar{\xi}} \tag{4.36}$$

return us to the case of the x and y parameter, respectively. Note that

$$\prod_{\pi'} |p_j|^{\bar{\xi}} \prod_{j=1}^{m} |q_j|^{\beta/2} \, |q_j|^{\xi_\beta} \prod_{\pi''} |q_j|^{\bar{\xi}} \leq c\mathcal{P}_3(|\bar{p}_k|)\mathcal{Q}_4(|\bar{p}_k|, |u_k|). \tag{4.37}$$

Expansion of the right hand side of the inequality leads us to the orders of the $|u_k|$ terms listed in (4.32) and the $|\bar{p}_k|$ take degrees

$$\deg \ |\bar{p}_k| = \eta_k \bar{\xi} + \bar{\eta}_k \bar{\xi}_\beta, \ \text{ with } \eta_k, \bar{\eta}_k \in \{0, 1, 2\}. \tag{4.38}$$

Applying these facts directs us to products similar to the ones presented in (4.33),

$$\iint_{\mathbf{R}^{2m}} \prod_{R} |\bar{p}_k|^{\eta_k \bar{\xi} + \bar{\eta}_k \bar{\xi}_\beta} \prod_{k=1}^{m} \frac{1}{1 + |\bar{p}_k|^\beta} \prod_{S'_\xi} \frac{|u_k|^{\theta_k \bar{\xi}_\beta}}{1 + |u_k|^\beta} \prod_{S^-_\xi \cup T^+_\xi} \frac{|u_k|^{\omega_k (\beta/2) + \theta_k \bar{\xi}_\beta}}{1 + |u_k|^\beta} \prod_{j=1}^{m} \mathrm{d}p_j \ \mathrm{d}q_j$$

The above expressions are finite for $3/2 < \beta < 2$ and we conclude over the non-isolated intervals.

$$\mathbb{E}\left[\left(\phi_y(\epsilon, x, y, B) - \phi_y(\epsilon', x, y, B)\right)^m\right] \leq c_m(\bar{\xi})|\epsilon - \epsilon'|^{m\bar{\xi}}, \tag{4.39}$$

Shifting attention to the case concerning the isolated intervals, the variation in the x term indicates we must study an expression parallel to (4.19),

$$c^m |x - x'|^{m\bar{\xi}} \iint_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^\beta\right)^{-1} \prod_{I^c} \left(1 + |u_k|^\beta\right)^{-1} \prod_{k \in I} |u_{k+1}| \prod_{I^c_j} |q_j| \ \mathrm{d}q_j \prod_{j=1}^{m} |p_j|^{\bar{\xi}} \ \mathrm{d}p_j. \tag{4.40}$$

The products which generate $\mathcal{P}_3(|\bar{p}_k|)$ and $\mathcal{Q}_2(|\bar{p}_k|, |u_k|)$, defined in (3.99) and (4.20), respectively, permit the formulation of the upper bound

$$\prod_{k\in I}|u_{k+1}|^{2-\beta}\prod_{I_j^c}|q_j|\prod_{j=1}^{m}|p_j|^{\bar{\xi}}\le c\mathcal{Q}_2(|\bar{p}_k|,|u_k|)\mathcal{P}_3(|\bar{p}_k|). \tag{4.41}$$

The degree of the $|u_k|$ variables generated by the right hand side of the inequality above are depicted in (4.22). However, the order of the $\bar{p}_k$ terms over the R-intervals are described in (4.28). Upon incorporating this later change with respect to (4.23), we can reapply the subsequent analysis of that expression to conclude (4.29) over the isolated intervals for $\beta\in(3/2,2)$.

Perturbations in the y parameter direct us to an expression parallel to (4.11) and we can adapt the analysis of this previous multiple integral to study

$$\begin{aligned}c^m|y-y'|^{m\bar{\xi}}\iiint\limits_{\mathbf{R}^{2m}\times B^m}&\exp\left(-\sum_{k=1}^{m}|\bar{p}_k|^{\beta}(\tilde{r}_k)\right)\exp\left(-\sum_{k=1}^{2m}|u_k|^{\beta}(\tilde{d}_k)\right)\\&\cdot\left|\prod_{k\in I}\left(\exp\left(-|u_k|^{\beta}(\tilde{d}_k)\right)-\exp\left(-\left(|u_{k+1}|^{\beta}+|q_{\ell_{(k)}}|^{\beta}\right)(\tilde{d}_k)\right)\right)\right|\\&\prod_{j=1}^{m}|q_j|\;|q_j|^{\bar{\xi}}\;\mathrm{d}p_j\;\mathrm{d}q_j\;\mathrm{d}r_j\;\mathrm{d}s_j\;\mathrm{d}t_j.\end{aligned} \tag{4.42}$$

Integrating the isolated intervals and the isolated variables from the above begins with,

$$\iint\limits_{\mathbf{R}\times B}\left|\exp\left(-|u_k|^{\beta}(\tilde{d}_k)\right)-\exp\left(-\left(|u_{k+1}|^{\beta}+|q_{\ell_{(k)}}|^{\beta}\right)(\tilde{d}_k)\right)\right|\;|q_{\ell_{(k)}}|\;|q_{\ell_{(k)}}|^{\bar{\xi}}\;\mathrm{d}(\tilde{d}_k)\;\mathrm{d}q_{\ell_{(k)}}. \tag{4.43}$$

Upon simplifying notation, $u_{k+1}=u$ and $q_{\ell_{(k)}}=q$, and partitioning the domain of integration over the real line, the above two dimensional integral is equivalent to the sum

$$\int\limits_{|q|\leq 4|u|} \int_0^1 \left|\exp\Big(-\left(|u+q|^\beta\right)(\tilde{d}_k)\Big) - \exp\Big(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\Big)\right| \; |q| \; |q|^{\bar{\xi}} \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q$$
$$+ \int\limits_{|q|\geq 4|u|} \int_0^1 \left|\exp\Big(-\left(|u+q|^\beta\right)(\tilde{d}_k)\Big) - \exp\Big(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\Big)\right| \; |q| \; |q|^{\bar{\xi}} \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q. \tag{4.44}$$

With respect to the first quantity above, observe

$$\begin{aligned}
&\int\limits_{|q|\leq 4|u|} \int_0^1 \left|\exp\Big(-\left(|u+q|^\beta\right)(\tilde{d}_k)\Big) - \exp\Big(-\left(|u|^\beta + |q|^\beta\right)(\tilde{d}_k)\Big)\right| \; |q| \; |q|^{\bar{\xi}} \; \mathrm{d}(\tilde{d}_k) \; \mathrm{d}q \\
&\qquad \leq c|u|^{1+\bar{\xi}} \left[\int\limits_{|q|\leq 4|u|} \frac{1}{1+(|u+q|^\beta)} \, \mathrm{d}q \; + \int\limits_{|q|\leq 4|u|} \frac{1}{1+(|u|^\beta + |q|^\beta)} \, \mathrm{d}q \right] \\
&\qquad \leq c|u|^{1+\bar{\xi}},
\end{aligned}$$

where we applied the study of the integrals emerging in the brackets of (4.15) to obtain the third inequality.

Referencing, (4.17) and the rationale which produced (4.18), the second summand in (4.44) is bounded by

$$\begin{aligned}
&\int\limits_{|q|\geq 4|u|} \int_0^1 \left|\exp\Big(-\big(|u+q|^\beta\big)\,(\tilde{d}_k)\Big) - \exp\Big(-\big(|u|^\beta+|q|^\beta\big)\,(\tilde{d}_k)\Big)\right|\ |q|\ |q|^{\bar{\xi}}\ \mathrm{d}(\tilde{d}_k)\ \mathrm{d}q \\
&\qquad \leq c \int\limits_{|q|\geq 4|u|} \left|\frac{1}{|u+q|^\beta} - \frac{1}{|u|^\beta+|q|^\beta}\right|\ |q|^{1+\bar{\xi}}\ \mathrm{d}q \\
&\qquad = c \int\limits_{|q|\geq 4|u|} \left|\frac{1}{|1+u/q|^\beta} - \frac{1}{(1+|u|^\beta/|q|^\beta)}\right|\ \frac{|q|^{1+\bar{\xi}}}{|q|^\beta}\ \mathrm{d}q \\
&\qquad \leq c \int\limits_{|q|\geq 4|u|} \frac{|u|}{|q|}\ \frac{|q|^{1+\bar{\xi}}}{|q|^\beta}\ \mathrm{d}q = c \int\limits_{|q|\geq 4|u|} \frac{|u|}{|q|^{\beta-\bar{\xi}}}\ \mathrm{d}q \leq c|u|^{2-\beta+\bar{\xi}}.
\end{aligned}$$

Collecting the above bounds for (4.43) and integrating the last time variables from (4.42), induces the study of

$$\iint\limits_{\mathbf{R}^m\times\mathbf{R}^n} \prod_{k=1}^m \big(1+|\bar{p}_k|^\beta\big)^{-1} \prod_{I^c} \big(1+|u_k|^\beta\big)^{-1} \prod_{k\in I} |u_{k+1}|^{1+\bar{\xi}} \prod_{I_j^c} |q_j|\ |q_j|^{\bar{\xi}}\ \mathrm{d}q_j \prod_{j=1}^m \mathrm{d}p_j. \quad (4.45)$$

Now, notice the derivation of the upper bound,

$$\begin{aligned}
\prod_{k\in I} |u_{k+1}|^{1+\bar{\xi}} \prod_{I_j^c} |q_j|\ |q_j|^{\bar{\xi}} &= \prod_{k\in I} |u_{k+1}|^{1+\bar{\xi}} \prod_{I_j^c} |q_j|^{\beta/2}\ |q_j|^{\xi_\beta+\bar{\xi}} \\
&= \prod_{k\in I} |u_{k+1}|^{1+\bar{\xi}} \prod_{I_j^c} |q_j|^{\beta/2}\ |q_j|^{\bar{\xi}_\beta} \qquad (4.46)
\end{aligned}$$

The resemblance of (4.20) with the products on the right side of the inequality above hints that we may re-employ the analysis of this former quantity to our current framework. Pointing the reader to the orders of the $\mathcal{Q}_2(|\bar{p}_k|,|u_k|)$ laid out in (4.22), the primary consequence resulting from the expansion of (4.46) is presented as follows;

$$\deg|\bar{p}_k| = \eta_k\bar{\xi}_\beta = 0, \bar{\xi}_\beta,\ \text{or}\ 2\bar{\xi}_\beta, \qquad \text{if } k\in R$$

$$\deg |u_k| = \omega_k(\beta/2) + \theta_k \bar{\xi}_\beta \begin{cases} = 0, \bar{\xi}_\beta, \text{ or } 2\bar{\xi}_\beta, & \text{if } k \in S'_\xi \\ = \beta/2 \text{ or } 1 + \bar{\xi}, & \text{if } k \in S^-_\xi \\ = \bar{\xi}_\beta \text{ or } \beta/2, & \text{if } k \in T^+_\xi \\ \leq \beta, & \text{if } k \in T''_\xi. \end{cases} \tag{4.47}$$

$$\begin{aligned} \deg |u_k| &= 1 + \bar{\xi} + \omega_k(\beta/2) + \theta_k \bar{\xi}_\beta \\ &= 1 + \bar{\xi}, 1 + \bar{\xi} + \beta/2, \text{ or } 1 + \bar{\xi} + \bar{\xi}_\beta, \quad \text{if } k \in T^{I+1}. \end{aligned}$$

Upon incorporating these details, into (4.45), we can replicate the analysis of (4.19) on the resulting expression to conclude (4.34) renders true over the isolated intervals for $\beta \in (3/2, 2)$.

For the last parameter, we take into account the expression,

$$\begin{aligned} c^m |\epsilon - \epsilon'|^{m\bar{\xi}} \iiint_{\mathbf{R}^{2m} \times B^m} \exp\left(-\sum_{k=1}^m |\bar{p}_k|^\beta (\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} |u_k|^\beta (\tilde{d}_k)\right) \\ \cdot \left| \prod_{k \in I} \left(\exp\left(-|u_k|^\beta (\tilde{d}_k)\right) - \exp\left(-\left(|u_{k+1}|^\beta + |q_{\ell(k)}|^\beta\right)(\tilde{d}_k)\right)\right) \right| \\ \prod_{j=1}^m |q_j| \left(|p_j|^{\bar{\xi}} + |q_j|^{\bar{\xi}}\right) \; \mathrm{d}p_j \; \mathrm{d}q_j \; \mathrm{d}r_j \; \mathrm{d}s_j \; \mathrm{d}t_j. \end{aligned}$$

The details surrounding (4.35) lead us back to the variation of the x and y terms. Finally, (4.39) holds valid for $\beta \in (3/2, 2)$ for both the non-isolated and isolated intervals.

Shifting efforts to the renormalized local time $\phi(\epsilon, x, y, B)$, the variation in the x parameter over the case of the non-isolated intervals directs our attention to

$$c^m |x - x'|^{m\bar{\xi}} \iint_{\mathbf{R}^{2m}} \prod_{k=1}^m \left(1 + |\bar{p}_k|^\beta\right)^{-1} \prod_{k=1}^{2m} (1 + |u_k|^\beta)^{-1} \prod_{j=1}^m |p_j|^{\bar{\xi}} \; \mathrm{d}p_j \; \mathrm{d}q_j.$$

with $\bar{\xi} \in (0,1)$. Simply insert $\sum(\prod_R |\bar{p}_k|^{\eta_k \bar{\xi}})$ into the above integral and apply Proposition 3.1.4 to conclude this first case.

The multiple integral below captures the key changes in the y variable,

$$c^m |y - y'|^{m\bar{\xi}} \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^{\beta}\right)^{-1} \prod_{k=1}^{2m} (1 + |u_k|^{\beta})^{-1} \prod_{j=1}^{m} |q_j|^{\bar{\xi}} \, \mathrm{d}p_j \, \mathrm{d}q_j.$$

Modify polynomial $\mathcal{P}_1(|u_k|)$, characterized in (3.36) , (3.37) , and (3.38) to write,

$$\sum \left[\iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^{\beta}\right)^{-1} \prod_{S'} \frac{1}{1 + |u_k|^{\beta}} \prod_{S^-} \frac{|u_k|^{\bar{\xi}}}{1 + |u_k|^{\beta}} \prod_{T^+} \frac{|u_k|^{\omega_k \bar{\xi}}}{1 + |u_k|^{\beta}} \prod_{T'} \frac{|u_k|^{\omega_k \bar{\xi}}}{1 + |u_k|^{\beta}} \prod_{j=1}^{m} \mathrm{d}p_j \, \mathrm{d}q_j \right]$$

Implement the bound $|u_k|^{\omega_k \bar{\xi}}(1 + |u_k|^{\beta})^{-1} \leq c$, for all $k \in T$. With $\beta \in (0,1)$, changing variables converts any summand above to a finite product of one dimensional integrals. Perturbation in ϵ return us to the previous two cases.

Starting with the variation in x for the isolated intervals, simply insert $\sum(\prod_R |\bar{p}_k|^{\eta_k \bar{\xi}})$ into (4.19) and modify the analysis subsequent that expression to finish this case. For the y parameter, consider the quantity

$$\begin{aligned} &\iiint\limits_{\mathbf{R}^{2m} \times B^m} \exp\left(- \sum_{k=1}^{m} |\bar{p}_k|^{\beta} (\tilde{r}_k) \right) \exp\left(- \sum_{k=1}^{2m} |u_k|^{\beta} (\tilde{d}_k) \right) \\ &\qquad \cdot \left| \prod_{k \in I} \left(\exp\left(-|u_k|^{\beta} (\tilde{d}_k) \right) - \exp\left(- \left(|u_{k+1}|^{\beta} + |q_{\ell(k)}|^{\beta} \right) (\tilde{d}_k) \right) \right) \right| \\ &\qquad \prod_{j=1}^{m} |q_j|^{\bar{\xi}} \, \mathrm{d}p_j \, \mathrm{d}q_j \, \mathrm{d}r_j \, \mathrm{d}s_j \, \mathrm{d}t_j. \end{aligned} \tag{4.48}$$

Recollect the inspection of (4.12) and integrate the isolated terms to write

$$\begin{aligned}
&\int\limits_{\mathbf{R}\times B} \left|\exp\Big(-\left(|u+q|^{\beta}\right)(\tilde{d}_k)\Big) - \exp\Big(-\left(|u|^{\beta}+|q|^{\beta}\right)(\tilde{d}_k)\Big)\right| \; |q|^{\bar{\xi}} \; \mathrm{d}(\tilde{d}_k)\; \mathrm{d}q \\
&\qquad \leq c \int\limits_{\mathbf{R}\times B} \left|\exp\Big(-\left(|u+q|^{\beta}\right)(\tilde{d}_k)\Big) - \exp\Big(-\left(|u|^{\beta}+|q|^{\beta}\right)(\tilde{d}_k)\Big)\right| \; |q| \; \mathrm{d}(\tilde{d}_k)\; \mathrm{d}q \\
&\qquad \leq c|u|^{2-\beta}.
\end{aligned}$$

After addressing the remaining time variables in the regular fashion, (4.48) reduces to

$$\iint\limits_{\mathbf{R}^m\times\mathbf{R}^n} \prod_{k=1}^{m} \left(1+|\bar{p}_k|^{\beta}\right)^{-1} \prod_{I^c} \left(1+|u_k|^{\beta}\right)^{-1} \prod_{k\in I} \; |u_{k+1}|^{2-\beta} \; \prod_{I^c_j} \, \mathrm{d}q_j \; \prod_{j=1}^{m} \, \mathrm{d}p_j.$$

Utilize the bound

$$\prod_{k\in I} \frac{|u_{k+1}|^{2-\beta}}{1+|u_{k+1}|^{\beta}} = \prod_{k\in T^{I+1}} \frac{|u_k|^{2-\beta}}{1+|u_k|^{\beta}} = c$$

and apply Proposition 3.1.4 to conclude the study of the y term. Variation in ϵ returns us to the above two cases.

We mention, proofs involving the local properties and convergence of

$$\lim_{\epsilon\to 0} \alpha(\epsilon, x, y, B) = \alpha(x, y, B)$$

are developed using the analysis of $\phi(\epsilon, x, y, B)$.

4.3 Extension of Local Properties to Arbitrary Sets in $D(2)$

To extend Hölder continuity to any Borel set in $D(2)$, we will mimic the argument presented for the Brownian case in §3.3. The three scaling identities which will advance our argument are

$$\begin{aligned} X(\lambda t) &\stackrel{\text{dist}}{=} \lambda^{1/\beta} X(t) \\ p_\epsilon(y/k) &= k p_{\epsilon|k|^\beta}(y) \\ p'_\epsilon(y/k) &= k^2 p'_{\epsilon|k|^\beta}(y). \end{aligned}$$

and the key proposition is laid out below.

Proposition 4.3.1. *Let X_t be a one dimensional symmetric stable process. The following equalities in distribution hold;*

$$\phi_y(\epsilon, x, y, B) \stackrel{\text{dist}}{=} 2^{3n(1/\beta-1)} \phi_y(2^n\epsilon, 2^{n/\beta}x, 2^{n/\beta}y, 2^n B) \tag{4.49}$$

$$\phi(\epsilon, x, y, B) \stackrel{\text{dist}}{=} 2^{n(2/\beta-3)} \phi(2^n\epsilon, 2^{n/\beta}x, 2^{n/\beta}y, 2^n B) \tag{4.50}$$

$$\alpha(\epsilon, x, y, B) \stackrel{\text{dist}}{=} 2^{n(2/\beta-3)} \alpha(2^n\epsilon, 2^{n/\beta}x, 2^{n/\beta}y, 2^n B) \tag{4.51}$$

Proof. Select an arbitrary $B \subset A(1,1,2)$. The above identities imply

$$
\begin{aligned}
&\phi_y(\epsilon, x, y, B) \\
&= \frac{1}{2^{3n}} \int_{2^n B} p_{\epsilon,x}\left(X(2^{-n}s, 2^{-n}r)\right) \left\{p'_{\epsilon,y}\left(X(2^{-n}t, 2^{-n}s)\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&= \frac{1}{2^{3n}} \int_{2^n B} p_{\epsilon}\left(X(2^{-n}s) - X(2^{-n}r) - x\right) \left\{p'_{\epsilon}\left(X(2^{-n}t) - X(2^{-n}s) - y\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&\overset{\text{dist}}{=} \frac{1}{2^{3n}} \int_{2^n B} p_{\epsilon}\left(2^{-n/\beta} X(s) - 2^{-n/\beta} X(r) - x\right) \\
&\qquad\qquad \cdot \left\{p'_{\epsilon}\left(2^{-n/\beta} X(t) - 2^{-n/\beta} X(s) - y\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&= \frac{1}{2^{3n}} \int_{2^n B} p_{\epsilon}\left(2^{-n/\beta}\left(X(s) - X(r) - 2^{n/\beta} x\right)\right) \\
&\qquad\qquad \cdot \left\{p'_{\epsilon}\left(2^{-n/\beta}\left(X(t) - X(s) - 2^{n/\beta} y\right)\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&= \frac{1}{2^{3n}} \int_{2^n B} 2^{n/\beta} p_{2^n\epsilon}\left(\left(X(s) - X(r) - 2^{n/\beta} x\right)\right) \\
&\qquad\qquad \cdot 2^{2n/\beta} \left\{p'_{2^n\epsilon}\left(\left(X(t) - X(s) - 2^{n/\beta} y\right)\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&= 2^{3n(1/\beta - 1)} \phi_y(2^n\epsilon, 2^{n/\beta} x, 2^{n/\beta} y, 2^n B)
\end{aligned}
$$

Working with the second functional of interest,

$$
\begin{aligned}
&\phi(\epsilon, x, y, B) \\
&= \frac{1}{2^{3n}} \int_{2^n B} p_{\epsilon}\left(X(2^{-n}s) - X(2^{-n}r) - x\right) \left\{p_{\epsilon}\left(X(2^{-n}t) - X(2^{-n}s) - y\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&\overset{\text{dist}}{=} \frac{1}{2^{3n}} \int_{2^n B} p_{\epsilon}\left(2^{-n/\beta} X(s) - 2^{-n/\beta} X(r) - x\right) \\
&\qquad\qquad \cdot \left\{p_{\epsilon}\left(2^{-n/\beta} X(t) - 2^{-n/\beta} X(s) - y\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&= \frac{1}{2^{3n}} \int_{2^n B} 2^{n/\beta} p_{2^n\epsilon}\left(\left(X(s) - X(r) - 2^{n/\beta} x\right)\right) \\
&\qquad\qquad \cdot 2^{n/\beta} \left\{p_{2^n\epsilon}\left(\left(X(t) - X(s) - 2^{n/\beta} y\right)\right)\right\}_0 \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \\
&= 2^{n(2/\beta - 3)} \phi(2^n\epsilon, 2^{n/\beta} x, 2^{n/\beta} y, 2^n B).
\end{aligned}
$$

The proof for

$$\alpha(\epsilon, x, y, B) \stackrel{\text{dist}}{=} 2^{n(2/\beta-3)}\alpha(2^n\epsilon, 2^{n/\beta}x, 2^{n/\beta}y, 2^n B)$$

follows by replicating the techniques presented for $\phi(\epsilon, x, y, B)$.

□

Centering attention on $\phi_y(\cdot)$,

$$\begin{aligned}
&\|\phi_y(\epsilon, x, y, B) - \phi_y(\epsilon', x', y', B)\|_m \\
&\quad \leq \sum_{n=1}^{\infty} \mathbb{E}\left[\left|\sum_{k=1}^{2^{n-1}} \phi_y\big(\epsilon, x, y, B(k,n)\big) - \phi_y\big(\epsilon', x', y', B(k,n)\big)\right|^m\right]^{1/m} \\
&\quad \leq \sum_{n=1}^{\infty} 2^{(n-1)/2} \sup_k \left\|\phi_y\big(\epsilon, x, y, B(k,n)\big) - \phi_y\big(\epsilon', x', y', B(k,n)\big)\right\|_m \\
&\quad = \sum_{n=1}^{\infty} 2^{(n-1)/2} 2^{3(n-1)(1/\beta-1)} \sup_k \Big\|\phi_y\left(2^{n-1}\epsilon, 2^{(n-1)/\beta}x, 2^{(n-1)/\beta}y, 2^{n-1}B(k,n)\right) \\
&\qquad\qquad - \phi_y\left(2^{n-1}\epsilon', 2^{(n-1)/\beta}x', 2^{(n-1)/\beta}y', 2^{n-1}B(k,n)\right)\Big\|_m . \\
&\quad \leq c_m(\xi) \sum_{n=1}^{\infty} 2^{(n-1)/2} 2^{3(n-1)(1/\beta-1)} 2^{(n-1)\xi} \|(\epsilon, x, y) - (\epsilon', x', y')\|^{\xi} \\
&\quad = c_m(\xi) \sum_{n=1}^{\infty} 2^{(n-1)/2} 2^{3(n-1)(1/\beta-1)} 2^{(n-1)\xi} \|(\epsilon, x, y) - (\epsilon', x', y')\|^{\xi} \\
&\quad = c_m(\xi) \sum_{n=1}^{\infty} 2^{(n-1)(-5/2+3/\beta+\xi)} \|(\epsilon, x, y) - (\epsilon', x', y')\|^{\xi}.
\end{aligned}$$

With the parameter ξ chosen close to zero, the inequality below holds true for $\beta \in (6/5, 2)$,

$$\frac{3}{\beta} + \xi < \frac{5}{2}$$

Taking into account the renormalized intersection local time,

$$
\begin{aligned}
&\|\phi(\epsilon,x,y,B)-\phi(\epsilon',x',y',B)\|_m \\
&\quad\le \sum_{n=1}^{\infty} 2^{(n-1)/2}\sup_k \big\|\phi\big(\epsilon,x,y,B(k,n)\big)-\phi\big(\epsilon',x',y',B(k,n)\big)\big\|_m \\
&\quad= \sum_{n=1}^{\infty} 2^{(n-1)/2}2^{(n-1)(2/\beta-3)}\sup_k \Big\|\phi\left(2^{n-1}\epsilon,2^{(n-1)/\beta}x,2^{(n-1)/\beta}y,2^{n-1}B(k,n)\right) \\
&\qquad\qquad\qquad -\phi\left(2^{n-1}\epsilon',2^{(n-1)/\beta}x',2^{(n-1)/\beta}y',2^{n-1}B(k,n)\right)\Big\|_m. \\
&\quad\le c_m(\xi)\sum_{n=1}^{\infty} 2^{(n-1)/2}2^{(n-1)(2/\beta-3)}2^{(n-1)(2/\beta+1)\xi}\|(\epsilon,x,y)-(\epsilon',x',y')\|^{\xi} \\
&\quad= c_m(\xi)\sum_{n=1}^{\infty} 2^{(n-1)/2}2^{(n-1)(2/\beta-3)}2^{(n-1)\xi}\|(\epsilon,x,y)-(\epsilon',x',y')\|^{\xi} \\
&\quad= c_m(\xi)\sum_{n=1}^{\infty} 2^{(n-1)(-5/2+2/\beta+\xi)}\|(\epsilon,x,y)-(\epsilon',x',y')\|^{\xi} \\
&\quad\le c_m(\xi)\|(\epsilon,x,y)-(\epsilon',x',y')\|^{\xi},
\end{aligned}
$$

where the last inequality was derived for $1<\beta<2$.

4.4 Final Continuity Arguments and the Set Parameter B

The derivation of the upper bound consisting of the set variable B,

$$\mathbb{E}\left[\big(\phi_y(\epsilon,x,y,B)\big)^m\right] \le c_m(\xi)|B|^{m\xi},$$

begins with the first case of the non-isolated intervals. An application of $|F(\beta, \epsilon, x, y, p, q)| \leq 1$ and $|i^m (2\pi)^{2m}| \leq c^m$ to (4.4) directs our attention to

$$c^m \iiint\limits_{\mathbf{R}^{2m} \times B^m} \exp\left(-\sum_{k=1}^{m} |\bar{p}_k|^\beta (\tilde{r}_k) \right) \exp\left(-\sum_{k=1}^{2m} |u_k|^\beta (\tilde{d}_k) \right) \prod_{j=1}^{m} |q_j| \; \mathrm{d}p_j \; \mathrm{d}q_j \; \mathrm{d}r_j \; \mathrm{d}s_j \; \mathrm{d}t_j.$$

Select conjugate exponents a and a' as in (3.185) and modify the argument which produced (3.187) to write

$$c^m |B|^{m/a} \iint\limits_{\mathbf{R}^{2m}} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^\beta\right)^{-1/a'} \prod_{k=1}^{2m} \left(1 + |u_k|^\beta\right)^{-1/a'} \prod_{j=1}^{m} |q_j| \; \mathrm{d}p_j \; \mathrm{d}q_j. \tag{4.52}$$

Observe, if we define

$$\zeta_\beta \doteq 1 - \frac{\beta}{2a'} \tag{4.53}$$

then as β approaches 2, ζ_β takes values near zero since $1/a'$ is very close to 1. Conversely, if β is approximately equal to 1, then ζ_β is slightly larger than 1/2. With this parameter in mind, define

$$\mathcal{Q}_5(|\bar{p}_k|, |u_k|) \doteq \prod_{k \in T} \left(|u_{k-1}|^{\beta/(2a')} + |u_k|^{\beta/(2a')} \right) \prod_{\substack{k \in S \\ k', k'' \in R}} \left(|u_k|^{\zeta_\beta} + |u_{k-1}|^{\zeta_\beta} + |\bar{p}_{k'}|^{\zeta_\beta} + |\bar{p}_{k''}|^{\zeta_\beta} \right)$$

so that the subsequent bound maybe formulated

$$\prod_{j=1}^{m} |q_j| = \prod_{j=1}^{m} |q_j|^{\beta/(2a')} \; |q_j|^{\zeta_\beta} \leq \mathcal{Q}_5(|\bar{p}_k|, |u_k|).$$

The similarities between $\mathcal{Q}_1(\bar{p}_k, u_k)$, defined in Proposition 4.1.1, and $\mathcal{Q}_5(\bar{p}_k, u_k)$ hints we can modify the orders presented in (4.9) to conclude this later polynomial assumes degrees

$$\deg |\bar{p}_k| = \eta_k \zeta_\beta = 0, \zeta_\beta, \text{ or } 2\zeta_\beta, \qquad \text{if } k \in R,$$

$$\deg |u_k| = \omega_k[\beta/(2a')] + \theta_k \zeta_\beta \begin{cases} = 0, \zeta_\beta, \text{ or } 2\zeta_\beta, & \text{if } k \in S'_\xi \\ = \beta/(2a') \text{ or } 1, & \text{if } k \in S^-_\xi \\ = \zeta_\beta \text{ or } \beta/(2a'), & \text{if } k \in T^+_\xi \\ \leq \beta/a', & \text{if } k \in T'_\xi. \end{cases}$$

Insert $\mathcal{Q}_5(\bar{p}_k, u_k)$ into (4.52) and employ the bound,

$$\frac{|u_k|^{\omega_k[\beta/(2a')]+\theta_k\zeta_\beta}}{(1+|u_k|^\beta)^{1/a'}}, \qquad \text{for all } k \in T'_\xi,$$

to arrive at

$$\iint\limits_{\mathbf{R}^{2m}} \prod_R |\bar{p}_k|^{\eta_k\zeta_\beta} \prod_{k=1}^m \frac{1}{(1+|\bar{p}_k|^\beta)^{1/a'}} \prod_{S'_\xi} \frac{|u_k|^{\theta_k\zeta_\beta}}{(1+|u_k|^\beta)^{1/a'}} \prod_{S^-_\xi \cup T^+_\xi} \frac{|u_k|^{\omega_k[\beta/(2a')]+\theta_k\zeta_\beta}}{(1+|u_k|^\beta)^{1/a'}} \prod_{j=1}^m \mathrm{d}p_j \; \mathrm{d}q_j.$$

Application of Proposition 3.1.4 indicates the $2m$-dimensional integral above is finite for $\beta \in (3/2, 2)$.

Shifting attention to the isolated intervals, we start with a quantity similar to (3.192)

$$\begin{aligned} (ci)^m \iiint\limits_{\mathbf{R}^{2m}\times B^m} & \exp\left(-\sum_{k=1}^m |\bar{p}_k|^\beta(\tilde{r}_k)\right) \exp\left(-\sum_{k=1}^{2m} |u_k|^\beta(\tilde{d}_k)\right) \\ & \cdot \prod_{k\in I} \left(\exp\left(-|u_k|^\beta(\tilde{d}_k)\right) - \exp\left(-\left(|u_{k+1}|^\beta + |q_{\ell(k)}|^\beta\right)(\tilde{d}_k)\right)\right) \\ & \exp\left(iy\sum_{j=1}^m q_j\right) \prod_{j=1}^m \hat{f}(\beta, \epsilon, q_j)\; q_j \; \mathrm{d}p_j \; \mathrm{d}q_j \; \mathrm{d}r_j \; \mathrm{d}s_j \; \mathrm{d}t_j. \end{aligned} \tag{4.54}$$

and aim to leverage much of the work from the Brownian case. Working towards this goal, we focus first on the isolated variables, $q_{\ell_{(k)}}$, in the one dimensional integral below

$$\left| \int_{\mathbf{R}} \exp\left(-|u_k|^\beta(\tilde{d}_k)\right) - \exp\left(-\left(|u_{k+1}|^\beta + |q_{\ell_{(k)}}|^\beta\right)(\tilde{d}_k)\right) \right.$$
$$\left. \cdot\, iq_{\ell_{(k)}} \exp\left(iyq_{\ell_{(k)}}\right) \exp\left(-\epsilon|q_{\ell_{(k)}}|^\beta\right) \, \mathrm{d}q_{\ell_{(k)}} \right| \tag{4.55}$$

After simplifying notation as in (3.194) and replicating the derivation of (3.196)-(3.198), the above expression can be bounded by the subsequent sum,

$$\begin{aligned}
&\left| \int_{\mathbf{R}} \left(\exp(iuz) - \exp(-t|u|^\beta)\right) p_t(z) p'_\epsilon(z-y) \, \mathrm{d}z \right| \\
&\qquad \leq \left| \left[\left(\exp(iuz) - \exp(-t|u|^\beta)\right) p_t(z)\right] p_\epsilon(z-y) \right| \Bigg|_{z=-\infty}^{z=\infty} \\
&\qquad\quad + \left| \int_{\mathbf{R}} iu \exp(iuz) p_t(z) p_\epsilon(z-y) \, \mathrm{d}z \right| \\
&\qquad\qquad + \left| \int_{\mathbf{R}} \left(\exp(-t|u|^\beta) - \exp(iuz)\right) p'_t(z) p_\epsilon(z-y) \, \mathrm{d}z \right|.
\end{aligned} \tag{4.56}$$

Reapplying the argument which produced (3.199) and adjusting (3.200) to our current problem, we find that

$$\left| \left[\left(\exp(iuz) - \exp(-t|u|^\beta)\right) p_t(z)\right] p_\epsilon(z-y) \right| \Bigg|_{z=-\infty}^{z=\infty} = 0$$

and

$$\begin{aligned}
\left| \int_{\mathbf{R}} iu \exp(iuz) p_t(z) p_\epsilon(z-y) \, \mathrm{d}z \right| &\leq |u| \int_{\mathbf{R}} p_t(z) p_\epsilon(z-y) \, \mathrm{d}z \\
&\leq |u||p_{t+\epsilon}(y)| \leq c|u||p_{t+\epsilon}(0)| \leq c\frac{|u|}{t^{1/\beta}}.
\end{aligned} \tag{4.57}$$

Working with (4.56)

$$\begin{aligned}\left|\int_{\mathbf{R}} \left(\exp(-t|u|^{\beta}) - \exp(iuz)\right) p_t'(z) p_\epsilon(z-y)\ \mathrm{d}z\right| \\ \leq \int_{\mathbf{R}} \left(|1-\exp(-t|u|^{\beta})| + |1-\exp(iuz)|\right) |p_t'(z)| p_\epsilon(z-y)\ \mathrm{d}z.\end{aligned}$$

From (3.92) and (3.93), we can select $\delta, \gamma \in [0,1]$ to bound the above expression by

$$c\int_{\mathbf{R}} \left[t^{\gamma}|u|^{\beta\gamma} + |u|^{\delta}|z|^{\delta}\right] |p_t'(z)| p_\epsilon(z-y)\ \mathrm{d}z \leq c\sup_z \left[t^{\gamma}|u|^{\beta\gamma}|p_t'(z)| + |u|^{\delta}|z|^{\delta}|p_t'(z)|\right]. \quad (4.58)$$

Applying the bound

$$p_t'(z) \leq |p_t'(z)| \leq c\int_{\mathbf{R}} |q| \exp\left(-t|q|^{\beta}\right)\ \mathrm{d}q \leq \frac{c}{t^{2/\beta}}$$

and the scaling identity

$$p_t'(z) = \frac{1}{t^{2/\beta}} p_1'\left(\frac{z}{t^{1/\beta}}\right)$$

the right hand side of the (4.58) can be bounded by

$$\begin{aligned}&\sup_z \left[t^{\gamma}|u|^{\beta\gamma}|p_t'(z)| + |u|^{\delta}|z|^{\delta}|p_t'(z)|\right] \\ &\quad\leq c|u|^{\beta\gamma}\frac{t^{\gamma}}{t^{2/\beta}} + c|u|^{\delta}\sup_z\left[\frac{|z|^{\delta}}{t^{2/\beta}}\left|p_1'\left(\frac{z}{t^{1/\beta}}\right)\right|\right] \\ &\quad= c|u|^{\beta\gamma}\frac{t^{\gamma}}{t^{2/\beta}} + c|u|^{\delta}\left(\frac{t^{\delta/\beta}}{t^{\delta/\beta}}\right)\sup_z\left[\frac{|z|^{\delta}}{t^{2/\beta}}\left|p_1'\left(\frac{z}{t^{1/\beta}}\right)\right|\right] \\ &\quad= c|u|^{\beta\gamma}\frac{1}{t^{2/\beta-\gamma}} + c|u|^{\delta}\frac{1}{t^{2/\beta-\delta/\beta}}\sup_z\left[\frac{|z|^{\delta}}{t^{\delta/\beta}}\left|p_1'\left(\frac{z}{t^{1/\beta}}\right)\right|\right]. \quad (4.59)\end{aligned}$$

Working with the supremum on the right side, write

$$\sup_z \left[|z|\,|p_1'(z)|\right] = \sup_z\left[|z|\left|\frac{1}{2\pi}\int_{\mathbf{R}} iq\exp(iqz)\exp(-|q|^{\beta})\ \mathrm{d}q\right|\right].$$

Partitioning the domain of integration and applying integration by parts reveals,

$$\begin{aligned}
&\sup_z \left[|z| \frac{1}{2\pi} \int_0^\infty iq \exp(iqz) \exp(-q^\beta) \ \mathrm{d}q \right] \\
&\quad = \sup_z \left[\frac{1}{2\pi} |z| \left(\left| -iq \exp(-q^\beta) \frac{\exp(iqz)}{iz} \right| \Big|_0^\infty \right) \right] \\
&\qquad\qquad - \sup_z \left[\frac{1}{2\pi} |z| \left| \int_0^\infty \frac{\exp(iqz)}{iz} \left(-i\beta q^\beta \exp(-q^\beta) + i \exp(-q^\beta) \right) \ \mathrm{d}q \right| \right] \\
&\quad \leq c.
\end{aligned}$$

Since the above argument can be replicated for the domain $(-\infty, 0)$, we can write (4.59) as

$$c|u|^{\beta\gamma} \frac{1}{t^{2/\beta - \gamma}} + c|u|^\delta \frac{1}{t^{2/\beta - \delta/\beta}}.$$

Referencing the time parameter in the denominator on the right side, choose $\delta \in [0,1]$ to satisfy the inequality

$$\frac{2}{\beta} - \frac{\delta}{\beta} = \frac{2-\delta}{\beta} < 1 \text{ , equivalently, } \delta > 2 - \beta.$$

With respect to the time parameter on the left side, select $\gamma \in [0,1]$ such that

$$\frac{2}{\beta} - \gamma = \frac{2 - \beta\gamma}{\beta} < 1 \text{ or, similarly, } \beta\gamma > 2 - \beta.$$

With this formulation in mind, let α be a parameter slightly less than 1 and $\tilde{\epsilon}$ a value very close to zero to produce the upperbound,

$$c|u|^{\beta\gamma} \frac{1}{t^{2/\beta - \gamma}} + c|u|^\delta \frac{1}{t^{2/\beta - \delta/\beta}} \leq c \frac{|u|^{2-\beta+\tilde{\epsilon}}}{t^\alpha}.$$

Upon combining this upper bound with (4.57) and recalling the notation in (3.194), we conclude (4.55) is bounded above by

$$c\left[\frac{|u_{k+1}|}{(\tilde{d}_k)^{1/\beta}}+\frac{|u_{k+1}|^{2-\beta+\tilde{\epsilon}}}{(\tilde{d}_k)^{\alpha}}\right]\le c\left[\frac{|u_{k+1}|}{(\tilde{d}_k)^{1/\beta}}+1\right],\qquad \text{with k } \in I.$$

Note that for small u,

$$\frac{|u_{k+1}|^{2-\beta+\tilde{\epsilon}}}{(\tilde{d}_k)^{1/\beta}}\le 1.$$

Our next step is to investigate the impact of this last upperbound on the primary multiple integral of interest, (4.54). Noting $|i\exp(-iyq_j)|, |\hat{f}(\beta,\epsilon,q_j)|\le 1$, $j\in I_j^c$, we now study

$$c^m\iiint\limits_{\mathbf{R}^m\times\mathbf{R}^n\times B^m}\exp\left(-\sum_{k=1}^m|\bar{p}_k|^\beta(\tilde{r}_k)\right)\exp\left(-\sum_{I_j^c}|u_k|^\beta(\tilde{d}_k)\right)\left[\prod_{k\in I}\frac{|u_{k+1}|}{(\tilde{d}_k)^{1/\beta}}+1\right]$$
$$\prod_{I_j^c}|q_j|\,\mathrm{d}q_j\prod_{j=1}^m\mathrm{d}p_j\;\mathrm{d}r_j\;\mathrm{d}s_j\;\mathrm{d}t_j. \tag{4.60}$$

With respect to the brackets above, we will work individually with each sum and begin with

$$c^m\iiint\limits_{\mathbf{R}^m\times\mathbf{R}^n\times B^m}\exp\left(-\sum_{k=1}^m|\bar{p}_k|^\beta(\tilde{r}_k)\right)\exp\left(-\sum_{I_j^c}|u_k|^\beta(\tilde{d}_k)\right)\prod_{k\in I}\frac{|u_{k+1}|}{(\tilde{d}_k)^{1/\beta}}$$
$$\prod_{I_j^c}|q_j|\,\mathrm{d}q_j\prod_{j=1}^m\mathrm{d}p_j\;\mathrm{d}r_j\;\mathrm{d}s_j\;\mathrm{d}t_j. \tag{4.61}$$

Extract from above the following integral,

$$\int_{B^m}\exp\left(-\sum_{k=1}^m|\bar{p}_k|^\beta(\tilde{r}_k)\right)\exp\left(-\sum_{I_j^c}|u_k|^\beta(\tilde{d}_k)\right)\prod_{k\in I}\frac{1}{(\tilde{d}_k)^{1/\beta}}\prod_{j=1}^m\mathrm{d}r_j\;\mathrm{d}s_j\;\mathrm{d}t_j.$$

Choose a and a' as in (3.185),

$$1 < a' < 1 + \delta \quad \text{and} \quad a = (1 - 1/a')^{-1},$$

and replicate the study of (3.184) to bound (4.61) by

$$c^m |B|^{m/a} \iint_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^{\beta}\right)^{-1/a'} \prod_{I^c} \left(1 + |u_k|^{\beta}\right)^{-1/a'} \prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j| \, \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j. \tag{4.62}$$

Define ζ_β as in (4.53),

$$\zeta_\beta \doteq 1 - \frac{\beta}{2a'},$$

so we may construct the upperbound

$$\prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j| = \prod_{k \in I} |u_{k+1}| \prod_{I_j^c} |q_j|^{\beta/(2a')} |q_j|^{\zeta_\beta} \le c\mathcal{Q}_6(|\bar{p}_k|, |u_k|),$$

where

$$\begin{aligned}
&\mathcal{Q}_6(|\bar{p}_k|, |u_k|) \\
&\doteq \prod_{k \in I} |u_{k+1}| \prod_{k \in T} \left(|u_{k-1}|^{\beta/(2a')} + |u_k|^{\beta/(2a')}\right) \prod_{\substack{k \in S \\ k', k'' \in R}} \left(|u_k|^{\zeta_\beta} + |u_{k-1}|^{\zeta_\beta} + |\bar{p}_{k'}|^{\zeta_\beta} + |\bar{p}_{k''}|^{\zeta_\beta}\right) \\
&= \prod_{k \in I} |u_{k+1}| \mathcal{Q}_5(|\bar{p}_k|, |u_k|)
\end{aligned}$$

Application of Proposition 4.1.2 reveals $\mathcal{Q}_6(|\bar{p}_k|, |u_k|)$ is a polynomial of the form

$$\mathcal{Q}_6(|\bar{p}_k|, |u_k|) = \sum \Bigg(\prod_{R} |\bar{p}_k|^{\eta_k\zeta_\beta} \prod_{S'_\zeta} |u_k|^{\theta_k\zeta_\beta} \prod_{S^-_\zeta} |u_k|^{\omega_k[\beta/(2a')]+\theta_k\zeta_\beta} \prod_{T^{I+1}} |u_k|^{1+\omega_k[\beta/(2a')]+\theta_k\zeta_\beta} \prod_{T^+_\zeta, T''_\zeta} |u_k|^{\omega_k[\beta/(2a')]+\theta_k\zeta_\beta} \Bigg),$$

with orders

$$\deg |\bar{p}_k| = \eta_k\zeta_\beta = 0, \zeta_\beta, \text{ or } 2\zeta_\beta, \quad \text{if } k \in R$$

$$\deg\, |u_k| = \omega_k[\beta/(2a')] + \theta_k\zeta_\beta \begin{cases} = 0, \zeta_\beta, \text{ or } 2\zeta_\beta, & \text{if } k \in S'_\zeta \\ = \beta/(2a') \text{ or } 1, & \text{if } k \in S^-_\zeta \\ = \zeta_\beta \text{ or } \beta/(2a'), & \text{if } k \in T^+_\zeta \\ \leq \beta/a', & \text{if } k \in T''_\zeta. \end{cases}$$

$$\deg\, |u_k| = 1 + \omega_k[\beta/(2a')] + \theta_k\zeta_\beta = 1, 1 + \beta/(2a'), \text{ or } 1 + \zeta_\beta, \quad \text{if } k \in T^{I+1}.$$

Inserting the above into (4.62) yields sums of the form

$$\sum \Bigg[\iint_{\mathbf{R}^m \times \mathbf{R}^n} \prod_{R} |\bar{p}_k|^{\eta_k\zeta_\beta} \prod_{k=1}^{m} \left(1 + |\bar{p}_k|^\beta\right)^{-1} \prod_{S,T} \left(1 + |u_k|^\beta\right)^{-1} \prod_{S'_\zeta} |u_k|^{\theta_k\zeta_\beta} \prod_{S^-_\zeta} |u_k|^{\omega_k[\beta/(2a')]+\theta_k\zeta_\beta}$$
$$\prod_{T^+_\zeta} |u_k|^{\omega_k[\beta/(2a')]+\theta_k\zeta_\beta} \prod_{T^{I+1}} |u_k|^{1+\omega_k[\beta/(2a')]+\theta_k\zeta_\beta}$$
$$\prod_{T''_\zeta} |u_k|^{\omega_k[\beta/(2a')]+\theta_k\zeta_\beta} \prod_{I^c_j} \mathrm{d}q_j \prod_{j=1}^{m} \mathrm{d}p_j \Bigg],$$

We can simply replicate the analysis of (4.23) to conclude any multiple integral emerging as a summand above is finite for $3/2 < \beta < 2$. Additionally, note that working with the second sum in the brackets of (4.60), returns us to the primary bound over of the isolated intervals.

Finally, the proof for

$$\mathbb{E}\left[\left(\phi(\epsilon, x, y, B)\right)^m\right] \leq c_m(\xi)|B|^{m\xi}$$

follows from replicating the analysis for the Brownian case, as presented in section 3.4.

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