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FUCHSIAN GERMS

by

TIMOTHY MOONEY GENDRON

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 1 |
| 2 | Preliminaries | 4 |
| 2.1 | Riemann Surface Laminations | 4 |
| 2.2 | Limits | 8 |
| 2.3 | Inverse Limit Solenoids | 12 |
| 2.4 | Uniform Structures | 14 |
| 2.5 | Topology of $PSL(2, \mathbf{R})$ | 15 |
| 2.6 | Uniform Hausdorff Topology of $PSL(2, \mathbf{R})$ | 18 |
| 3 | Germ of Inverse Limit Solenoids | 21 |
| 3.1 | Group Germs | 21 |
| 3.2 | The Fundamental Germ of an Inverse Limit Solenoid | 24 |

| | |
|--|-----------|
| <i>CONTENTS</i> | iv |
| 4 Fuchsian Nested Sets | 28 |
| 4.1 Pre-Fuchsian Nested Sets | 28 |
| 4.2 Completion | 34 |
| 4.3 Fuchsian Nested Sets | 40 |
| 4.4 The Quotient of $\overline{\mathbf{H}}$ by \mathcal{G} | 44 |
| 4.5 Uniformization via Fuchsian Nested Sets | 51 |
| 5 The Fuchsian Germ | 56 |
| 5.1 Ultrafilters | 57 |
| 5.2 The General Construction | 59 |
| 5.3 The Lockstep Construction | 66 |
| 5.4 The Feigenbaum Germ | 70 |
| 6 Bibliography | 74 |

List of Figures

| | | |
|-----|---|----|
| 2.1 | Holonomy about Geodesic Triangles | 17 |
| 2.2 | Hausdorff Topology vs. Uniform Hausdorff Topology | 20 |
| 4.1 | A Coset | 33 |

Chapter 1

Introduction

We introduce Fuchsian germs to extend the theory of Fuchsian groups to irreducible hyperbolic Riemann surface laminations. Roughly speaking, a Fuchsian germ $[\Gamma]$ is a groupoid consisting of tail equivalence classes of sequences $\{B_i\}$, where $B_i \in \mathcal{G}_i \cdot \mathcal{G}_i^{-1} \subset PSL(2, \mathbf{R})$, and $\mathcal{G} = \{\mathcal{G}_i\}$ is a nested sequence of countable, uniformly discrete subsets of $PSL(2, \mathbf{R})$. (Groupoid multiplication in $[\Gamma]$ being defined componentwise.)

One can define convergence of sequences $\{z_i\}$ in the upper-half plane \mathbf{H} relative to \mathcal{G} and thus define a completion $[\overline{\mathbf{H}}]$ of \mathbf{H} . Under suitable restrictions on \mathcal{G} and $[\Gamma]$, there will be an action of $[\Gamma]$ on $[\overline{\mathbf{H}}]$ in which the quotient $[\overline{\mathbf{H}}]/[\Gamma]$ is a Riemann surface lamination. Conversely, we will show that any

irreducible hyperbolic Riemann surface lamination may be so uniformized.

The organization of the exposition is as follows. We develop the abstract definition of a *Fuchsian nested set* $\mathcal{G} = \{\mathcal{G}_i\}$; then show how such a \mathcal{G} can be used to complete \mathbf{H} to a space $\overline{\mathbf{H}}$ and form a quotient $\overline{\mathbf{H}}_{\mathcal{G}}$ which is an irreducible hyperbolic Riemann surface lamination. Next, we show that from an irreducible hyperbolic Riemann surface lamination L one can construct a Fuchsian directed set \mathcal{G} so that $\overline{\mathbf{H}}_{\mathcal{G}} \cong L$. Finally, from an arbitrary \mathcal{G} we form a groupoid $[\Gamma]$, the *Fuchsian germ*, which acts (as a groupoid) on a direct limit of completions $[\overline{\mathbf{H}}]$ in such a way that $[\overline{\mathbf{H}}]/[\Gamma] \cong \overline{\mathbf{H}}_{\mathcal{G}}$.

One of the motivations for this study came from the perception that the holonomy graph of a foliated space is in a sense an unambitious generalization of fundamental group. Specifically, we suspected that certain surface laminations arising algebraically, such as those occurring as inverse limits of surfaces, should be uniformized by groups rather than by groupoids. Toward this end, we sought to expand the domain and codomain of elements of a putative uniformizing object beyond the singleton sets that are the domains and codomains of elements of the holonomy graph. Following this philosophy, our construction led to uniformizing groups in the case of inverse limits laminations *and* in the case of the Feigenbaum lamination, which is a

quotient of an inverse limit of annuli.

Acknowledgments: There are many to whom I owe debts of gratitude, starting with those who laboured through this document. In the beginning, F. Gardiner and B. Harvey spent many afternoons in the Spring of 1996 critiquing the earliest versions. Later, A. Epstein raised many questions which led to crucial additions to the axiomatization scheme. My advisor, D. Sullivan, willingly spent the waning days of August of this year examining the final version.

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Chapter 2

Preliminaries

For the readers convenience as well as to establish notation, we present in this chapter an introductory discussion of the concepts of Riemann surface lamination, direct and inverse limit, inverse limit solenoid, uniform space, the topology of $PSL(2, \mathbf{R})$ and the uniform Hausdorff topology of $PSL(2, \mathbf{R})$.

2.1 Riemann Surface Laminations

In this section we will give the rudimentary definition and terminology surrounding the objects we intend to uniformize. (Compare [M-Sc], [C].)

In what follows, \mathbf{N} denotes the natural numbers, \mathbf{Z} the integers, \mathbf{R} the

real numbers, I the unit interval $[0, 1]$ in \mathbf{R} , \mathbf{C} the complex numbers, \mathbf{H} the upper half-plane in \mathbf{C} . We will understand \mathbf{H} to be pointed by a fixed z_0 , and will let $\rho_{\mathbf{H}} = |dz|/y$ denote the constant curvature -1 Riemannian metric on \mathbf{H} , and $d_{\mathbf{H}}$ the induced distance function.

Definition 1 *Let L be a separable, metrizable topological space. Suppose that to each $z \in L$ there corresponds a quadruple $(\varphi_z, \Omega_z, T_z, O_z)$, where Ω_z is a domain compactly contained in \mathbf{C} . T_z is a separable, metrizable topological space, O_z is an open set of L containing z and $\varphi_z : \Omega_z \times T_z \rightarrow O_z$ is a homeomorphism. We say that the collection*

$$\mathcal{A} = \{(\varphi_z, \Omega_z, T_z, O_z)\}_{z \in L}$$

defines on L the structure of a Riemann surface lamination if properties

1), 2) and 3) below are satisfied:

1) (Integrity of Leaves) *Suppose that $O_z \cap O_w \neq \emptyset$; let*

$$\psi_{z_w} = \varphi_w^{-1} \circ \varphi_z : \varphi_z^{-1}(O_z \cap O_w) \longrightarrow \varphi_w^{-1}(O_z \cap O_w)$$

denote the implied overlap homeomorphism. Let $\pi_z : \Omega_z \times T_z \rightarrow T_z$ and $\pi_w : \Omega_w \times T_w \rightarrow T_w$ be the projections onto the second factors. Then for each $\zeta \in O_z \cap O_w$, there exists a neighborhood O_ζ of ζ contained in $O_z \cap O_w$ and

a map

$$\phi_\zeta : \pi_z \circ \varphi_z^{-1}(O_\zeta) \longrightarrow T_w,$$

mapping homeomorphically onto its image, such that

$$\pi_w \circ \psi_{zw} = \phi_\zeta \circ \pi_z$$

when restricted to $\varphi_z^{-1}(O_\zeta)$.

2) (Holomorphic Transverse Normality) Whenever $O_z \cap O_w \neq \emptyset$, the collection

$$\left\{ \psi_{zw}^t(\zeta) = \psi_{zw}(\zeta, t) \right\}_{t \in T_{\psi_{zw}}}$$

where $T_{\psi_{zw}} = \pi_z \circ \varphi_z^{-1}(O_z \cap O_w)$, comprises a normal family of holomorphic functions.

3) \mathcal{A} is maximal with respect to properties 1) and 2) i.e. if there exists a quadruple (φ, Ω, T, O) for which the collection $\mathcal{A} \cup (\varphi, \Omega, T, O)$ satisfies properties 1) and 2), then $(\varphi, \Omega, T, O) \in \mathcal{A}$.

A quadruple $(\varphi_z, \Omega_z, T_z, O_z) \in \mathcal{A}$ is called a **chart**; the collection \mathcal{A} is called the **defining atlas** of the Riemann surface lamination structure. The subsets O_z are called **flowboxes**. A subset of L which can be written in the form $P = \varphi(\Omega \times \{t\})$, where $(\varphi, \Omega, T, O) \in \mathcal{A}$ and $t \in T$, is called a **plaque** of the flowbox O .

Definition 2 *If L has the structure of a Riemann surface lamination, then for each $z \in L$, the leaf through z is defined to be the set ℓ_z consisting of points $w \in L$ for which there exists a path $\gamma : (I, 0, 1) \rightarrow (L, z, w)$ and a covering of $\gamma(I)$ by flowboxes O_{z_i} , $i = 1, \dots, n$, such that for each i ,*

$$\gamma(I) \cap O_{z_i} \subset P_i$$

for some plaque P_i of O_{z_i} .

By property 2) in Definition 1, each leaf ℓ_z is a Riemann surface. Thus we have

Definition 3 *If all leaves are hyperbolic Riemann surfaces, L is called hyperbolic; if L has a dense leaf, we say that L is irreducible. A set $T \subset L$ in which $T \cap \ell$ is countable for every leaf ℓ of L is called a transversal. A transversal T is called a flowbox transversal if there exists a chart $(\varphi, \Omega, T, O) \in \mathcal{A}$ and a point $\zeta \in \Omega$ such that $T = \varphi(\{\zeta\} \times T)$. If every flowbox transversal of L is a Cantor set, L is called a solenoid.*

2.2 Limits

In this section we discuss direct and inverse limits. In particular, we introduce an example of each which will lead to, respectively, one construction of Fuchsian germs and a certain class of Riemann surface solenoids.

Recall that a partially ordered set (\mathcal{A}, \leq) can be regarded as a category in which a relation $\alpha \leq \beta$ is regarded as a morphism $\alpha \rightarrow \beta$

Definition 4 A partially ordered set (\mathcal{A}, \leq) is said to be a directed set if for all $\alpha, \beta \in \mathcal{A}$, there exists $\gamma \in \mathcal{A}$ such that $\alpha, \beta \leq \gamma$.

Definition 5 Given a category \mathcal{S} and a directed set \mathcal{A} , an \mathcal{A} -direct (inverse) system is a covariant (contravariant) functor $F : \mathcal{A} \rightarrow \mathcal{S}$.

If $F : \mathcal{A} \rightarrow \mathcal{S}$ is a direct (inverse) system, we will write $X_\alpha = F(\alpha)$ for the image objects and $f_{\alpha\beta} : X_\alpha \rightarrow X_\beta$ ($f^{\alpha\beta} : X_\beta \rightarrow X_\alpha$) for the image morphisms $F(\alpha \rightarrow \beta)$.

Here are two examples which will be of great interest to us:

Example 1 If G_α is any collection of sets/groups/topological spaces/*et cetera* indexed by a directed set \mathcal{A} , let

$$(G_\alpha) = \prod_{\alpha \leq \gamma} G_\gamma.$$

The coordinate forgetting projections $(G_\alpha) \rightarrow (G_\beta)$, $\alpha \leq \beta$, yield an \mathcal{A} -direct system in the category of sets/groups/topological spaces/*et cetera*.

Example 2 Consider a group G and a nested sequence of subgroups $\{G_i\}$. The subgroup relation maps $G_j \subseteq G_i$, $i \leq j$, yield an \mathbf{N} -inverse system.

If $G = \pi_1 X$ for some semi-locally simply connected space X , we may associate to each G_i a covering $s_i : X_i \rightarrow X$ for which $(s_i)_* \pi_1 X_i = G_i$. By covering space theory, for every relation $i < j$, there exists a unique covering map $s^{ij} : X_j \rightarrow X_i$ such that $s_i \circ s^{ij} = s_j$. Thus we obtain an \mathbf{N} -inverse system in the category $Cov(X)$ of coverings of X .

To any system in a category \mathcal{S} , direct or inverse, one may associate a “limiting” set, which may or may not be an object in the ambient category \mathcal{S} .

Definition 6 Let $F : \mathcal{A} \rightarrow \mathcal{S}$ be a direct system. The direct limit of the system is the quotient of the disjoint union $\bigsqcup_{\alpha \in \mathcal{A}} X_\alpha$ by the relation \sim , where $x \sim y$ if and only there exists $\alpha, \beta, \gamma \in \mathcal{A}$ with $x \in X_\alpha$, $y \in X_\beta$ and $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

We denote the direct limit by

$$\varinjlim X_\alpha.$$

Notice that there are canonical maps $p_\alpha : X_\alpha \rightarrow \varinjlim X_\alpha$ induced by the inclusions $X_\alpha \hookrightarrow \coprod X_\alpha$.

Definition 7 Let $G : \mathcal{A} \rightarrow \mathcal{S}$ be an inverse system. The inverse limit is the subset of the direct product $\prod_{\alpha \in \mathcal{A}} X_\alpha$ consisting of those \mathcal{A} -tuples (x_α) for which $x_\alpha = f^{\alpha\beta}(x_\beta)$ whenever $\alpha \leq \beta$.

The notation for inverse limit is

$$\varprojlim X_\alpha.$$

Here there are canonical projections $p^\alpha : \varprojlim X_\alpha \rightarrow X_\alpha$ induced by the projections $\prod X_\alpha \rightarrow X_\alpha$.

When \mathcal{S} is the category of groups or compact Hausdorff spaces, then both the direct and inverse limits may be endowed with group structures resp. topologies so that the limits lie in \mathcal{S} . In this work, we shall always find ourselves in this situation.

Both direct and inverse limits enjoy universality properties which we state here [L].

Proposition 1 *Let $F : \mathcal{A} \rightarrow \mathcal{S}$ be a direct (inverse) system in a category \mathcal{S} for which the limit $\lim_{\overline{F}} X_\alpha$ ($\lim_{\overline{F}} X_\alpha$) lies in \mathcal{S} . Suppose that for some object $X \in \mathcal{S}$ we have a family of morphisms $g_\alpha : X_\alpha \rightarrow X$ ($g^\alpha : X \rightarrow X_\alpha$) that satisfy the condition $g_\alpha = g_\beta \circ f_{\alpha\beta}$ ($g_\alpha = f^{\alpha\beta} \circ g_\beta$) whenever $\alpha \leq \beta$. Then there exists a unique morphism $g : \lim_{\overline{F}} X_\alpha \rightarrow X$ ($g : X \rightarrow \lim_{\overline{F}} X_\alpha$) satisfying the condition $g_\alpha = g \circ p_\alpha$ ($g_\alpha = p^\alpha \circ g$). \square*

Often we will be interested in looking at the limit of certain subsystems of a given system. The condition of *cofinality* will assure isomorphic limits.

Definition 8 *Let \mathcal{A} be a directed set and $\mathcal{B} \subset \mathcal{A}$ a subdirected set of \mathcal{A} . \mathcal{B} is called **cofinal** if for every pair $\alpha, \beta \in \mathcal{A}$, there exists $\gamma \in \mathcal{B}$ such that $\alpha, \beta \leq \gamma$.*

If $F : \mathcal{A} \rightarrow \mathcal{S}$ is a system, direct or inverse, and \mathcal{B} is a cofinal subdirected set of \mathcal{A} , we denote by $F|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{S}$ the system obtained by restriction to \mathcal{B} .

A proof of the following proposition can also be found in [L].

Proposition 2 *If $\mathcal{B} \subset \mathcal{A}$ is cofinal and $F : \mathcal{A} \rightarrow \mathcal{S}$ is a direct (inverse) system, then there exists a canonical isomorphism between $\lim_{\overline{F}} X_\alpha$ and $\lim_{\overline{F|_{\mathcal{B}}}} X_\alpha$ (between $\lim_{\overline{F}} X_\alpha$ and $\lim_{\overline{F|_{\mathcal{B}}}} X_\alpha$). \square*

2.3 Inverse Limit Solenoids

Inverse systems can be used to construct examples of solenoids that will turn out to have very computable Fuchsian germs.

Let S be a closed topological surface of genus > 1 , and consider an \mathbf{N} -inverse system $\mathbf{N} \rightarrow \text{Cov}(S)$ of the type discussed in Example 2, arising from a nested sequence of subgroups of $\pi_1 S$. Let $\hat{S} = \lim_{\leftarrow} S_i$.

Theorem 1 *\hat{S} has the structure of a surface solenoid.*

Proof: When we say simply surface solenoid, we mean that chart overlaps are assumed to be only continuous in the leaf direction, converging in the C^0 -topology in the transverse direction.

Denote by $p : \hat{S} \rightarrow S$ the canonical projection and D the open unit disk in \mathbf{R}^2 . Let $\mathcal{A} = \{(\varphi_z, D, O_z)\}_{z \in S}$ be a manifold atlas for S e.g. O_z is a disk in S containing z and $\varphi_z : D \rightarrow O_z$ is a homeomorphism. We may assume that the disks O_z have been chosen small enough to be evenly covered with respect to any of the coverings $s_i : S_i \rightarrow S$ in the defining inverse system. We will use \mathcal{A} to construct an atlas $\hat{\mathcal{A}}$ for \hat{S} .

Given $(\varphi_z, D, O_z) \in \mathcal{A}$, let $T_z = p^{-1}(z)$, which is homeomorphic to $\prod\{1, \dots, \deg(s^{i+1})\}$, a Cantor set in the Tychonoff topology. Setting $\hat{O}_z =$

$p^{-1}(O_z)$, we now define a chart $\hat{\varphi}_z : D \times T_z \rightarrow \hat{O}_z$. A point $t \in T_z$ is by definition a tuple of points (z, z_1, z_2, \dots) , $z_i \in S_i$, satisfying the condition required to be in the inverse limit. Since O_z is evenly covered by all of the system covers, we may lift φ_z along s_i to $\varphi_{z_i} : D \rightarrow O_{z_i}$ where O_{z_i} is the component of $s_i^{-1}(O_z)$ containing z_i . By the universal property of inverse limits, Proposition 1, there exists a homeomorphism $\hat{\varphi}_z^t : D \rightarrow O_z^t$, where O_z^t is the disk so defined about t in \hat{S} . We thus obtain a homeomorphism $\hat{\varphi}_z : D \times T_z \rightarrow \hat{O}_z$, for each $z \in S$, and an atlas $\hat{\mathcal{A}}$ for \hat{S} .

First, we verify the integrity of leaves, property 1) in Definition 1. If $\hat{O}_z \cap \hat{O}_w \neq \emptyset$, then $O_z \cap O_w \neq \emptyset$ also. If $\hat{\zeta} \in \hat{O}_z \cap \hat{O}_w$, then $\hat{\zeta} \in \hat{O}_\zeta$, where $\zeta = p(\hat{\zeta})$; if O_ζ is selected to be contained in $O_z \cap O_w$, then \hat{O}_ζ will be contained in $\hat{O}_z \cap \hat{O}_w$ also. Now we may define a homeomorphism $\tau_z : T_\zeta \rightarrow T_z$, where $\tau_z(t) = \bar{t}$ if $t \in O_z^{\bar{t}}$. Likewise we define τ_w , and $\phi_\zeta = \tau_w^{-1} \circ \tau_z$ satisfies the requirements of property 1) of Definition 1.

We now demonstrate C^0 -transverse normality, property 2) of Definition 1. But this is trivial in this case. Let $\psi_{zw}^t = (\hat{\varphi}_w^t)^{-1} \circ \hat{\varphi}_z^t$ be the overlap homeomorphism along the intersection of t -indexed plaques, $t \in T_{\psi_{zw}} = T_z$. (Here, we abuse notation in indexing $\hat{\varphi}_w^t$ by $t \in T_z$ also.) Let p_z^t be the

restriction of p to \mathcal{O}_z^t , a homeomorphism onto \mathcal{O}_z . Then

$$\psi_{zw}^t = (\hat{\varphi}_w^t)^{-1} \circ (p_w^t)^{-1} \circ p_z^t \circ \hat{\varphi}_z^t = \varphi_w^{-1} \circ \varphi_z$$

on the domain of ψ_{zw}^t , i.e. the family is constant as one varies t .

Now simply replace $\hat{\mathcal{A}}$ by a maximal compatible atlas, and we are done.

□

2.4 Uniform Structures

We review in this section the definition of a uniform structure on a set X , as well as the topology it induces. Consult [K] for more discussion and proofs.

Let $\Delta_X = \{(x, x) \mid x \in X\}$ be the diagonal in $X \times X$. For $U, V \subset X \times X$, we define

$$U^{-1} = \{(y, x) \mid (x, y) \in U\}$$

and

$$U \circ V = \{(x, z) \mid \exists y \in X \text{ s.t. } (x, y) \in U \text{ and } (y, z) \in V\}.$$

Definition 9 *A collection of sets $\mathcal{U} = \{U_\alpha\}$, $U_\alpha \subset X \times X$, is said to generate a uniform structure on X if the following conditions are satisfied:*

- 1) (reflexivity) $\Delta_X \subset U$ for all $U \in \mathcal{U}$.

2) (openness) *If $U \cap V \neq \emptyset$ for $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \subset U \cap V$.*

3) (symmetry) *If $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \subset U^{-1}$.*

4) (triangle inequality) *If $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.*

The uniform structure on X generated by \mathcal{U} is formed by adding to \mathcal{U} all subsets $W \subset X \times X$ which contain a subset contained in \mathcal{U} .

The following lemma is a sort of iterated triangle inequality.

Lemma 1 *Let \mathcal{U} be a uniform structure. Given any $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \circ V \subset U$. \square*

A uniform structure is a generalization of a metric. Indeed, one can use the collection \mathcal{U} to form a topology on X by taking as a basis sets of the form

$$U(y) = \{x \mid (x, y) \in U\}.$$

(That this collection is a basis follows from the openness condition.)

2.5 Topology of $PSL(2, \mathbf{R})$

Let $SL(2, \mathbf{R})$ be the group of 2×2 matrices of determinant 1 with real entries; the quotient $PSL(2, \mathbf{R}) = SL(2, \mathbf{R}) / \langle I, -I \rangle$, where I is the identity matrix,

is called the *real projective special linear group*. $PSL(2, \mathbf{R})$ is isomorphic as a Lie group to the isometry group of \mathbf{H} via the association

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto Az = \frac{az + b}{cz + d}.$$

Thus, one can understand the topology of $PSL(2, \mathbf{R})$ to be that of compact convergence of isometries of \mathbf{H} . Here and throughout the text we will let $A_* : T\mathbf{H} \rightarrow T\mathbf{H}$ denote the extension of the action of $A \in PSL(2, \mathbf{R})$ to the tangent space.

Now let $T^1\mathbf{H}$ be the unit tangent bundle over \mathbf{H} , which is clearly homeomorphic to $\mathbf{H} \times S^1$. Consider, for any $\epsilon > 0$, sets of the form

$$U_\epsilon = \{((w, \vec{u}), (z, \vec{v})) \in T^1\mathbf{H} \times T^1\mathbf{H} \mid d_{\mathbf{H}}(w, z) < \epsilon \text{ and } \|\vec{u} - \vec{v}_w\|_w < \epsilon\},$$

where $\|\cdot\|_w$ is the norm on $T_w\mathbf{H}$ induced by $\rho_{\mathbf{H}}$ and \vec{v}_w is the parallel translate of \vec{v} from z to w along the unique geodesic connecting z to w .

Proposition 3 *The collection $\mathcal{U} = \{U_\epsilon\}_{\epsilon>0}$ generates a uniform structure on $T^1\mathbf{H}$.*

Proof: Conditions 1) and 2) of Definition 9 are trivially satisfied. Note that

$$\|\vec{u} - \vec{v}_w\|_w = \|\vec{v} - \vec{u}_z\|_z$$

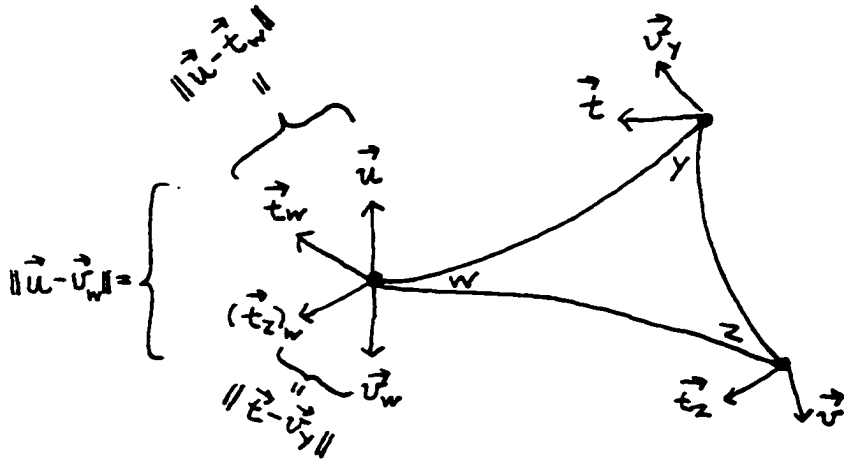


Figure 2.1: Holonomy about Geodesic Triangles

for $(w, \vec{u}), (z, \vec{v}) \in T^1\mathbf{H}$. Thus we have that for all $\epsilon > 0$, $U_\epsilon^{-1} = U_\epsilon$ and so condition 3) is also satisfied. It remains to demonstrate the triangle inequality, condition 4). Since \mathbf{H} is a homogeneous Riemannian manifold, it is *uniformly* almost Euclidean in disks of small radius δ , and there is uniformly small error when one parallel translates vectors about geodesic triangles in such δ -disks. Hence it follows that for any $\epsilon > 0$, we may find δ small enough so that $U_\delta \circ U_\delta \subset U_\epsilon$. More precisely, there exists a function $C(\delta)$, where $C(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, such that whenever $((w, \vec{u}), (y, \vec{t})), ((y, \vec{t}), (z, \vec{v})) \in U_\delta$, we have

$$|\|\vec{u} - \vec{v}_w\|_w - C(\delta)| \leq \|\vec{u} - \vec{t}_w\|_w + \|\vec{t} - \vec{v}_y\|_y.$$

See Figure 2.1. If we take δ such that $C(\delta) + 2\delta < \epsilon$, we have $((w, \vec{u}), (z, \vec{v})) \in$

U_ϵ , since $d_{\mathbf{H}}(w, z) \leq d_{\mathbf{H}}(w, y) + d_{\mathbf{H}}(y, z) < 2\delta < \epsilon$, and $\|\vec{u} - \vec{v}_w\|_w \leq C(\delta) + 2\delta < \epsilon$. This proves the triangle inequality. \square

The sets

$$U_\epsilon^{(z, \vec{v})} = \{(w, \vec{u}) \mid ((w, \vec{u}), (z, \vec{v})) \in U_\epsilon\}$$

generate the topology of $T^1\mathbf{H}$.

If we fix a basepoint (z_0, \vec{v}_0) of $T^1\mathbf{H}$, the correspondence $A \mapsto (Az_0, A_*\vec{v}_0)$ defines a homeomorphism from $PSL(2, \mathbf{R})$ onto $T^1\mathbf{H}$. In particular, a neighborhood basis about A is provided by the sets

$$U_\epsilon(A) = \{B \in PSL(2, \mathbf{R}) \mid (Bz_0, B_*\vec{v}_0) \in U_\epsilon^{(Az_0, A_*\vec{v}_0)}\}.$$

This is how we will express the topology of $PSL(2, \mathbf{R})$.

2.6 Uniform Hausdorff Topology of $PSL(2, \mathbf{R})$

Let $\mathcal{P} = 2^{PSL(2, \mathbf{R})}$ = the set of subsets of $PSL(2, \mathbf{R})$ and define

$$\mathcal{U}_\epsilon^C = \{(A, \mathcal{B}) \in \mathcal{P} \times \mathcal{P} \mid \forall A \in \mathcal{A}, \exists B \in \mathcal{B} \text{ s.t. } A \in U_\epsilon(B)\}$$

$$\mathcal{U}_\epsilon^D = \{(A, \mathcal{B}) \in \mathcal{P} \times \mathcal{P} \mid \forall B \in \mathcal{B}, \exists A \in \mathcal{A} \text{ s.t. } B \in U_\epsilon(A)\}.$$

Proposition 4 *The collection $2^U \equiv \{\mathcal{U}_\epsilon = \mathcal{U}_\epsilon^C \cap \mathcal{U}_\epsilon^D\}$ generates a uniform structure on \mathcal{P} .*

Proof: The first three conditions of Definition 9 are apparent consequences of the fact that U generates a uniform structure on $PSL(2, \mathbf{R})$. Now choose δ so that $U_\delta \circ U_\delta \subset U_\epsilon$. We claim that $\mathcal{U}_\delta \circ \mathcal{U}_\delta \subset \mathcal{U}_\epsilon$ also. Suppose that $(A, B), (B, C) \in \mathcal{U}_\delta$ (so that $(A, C) \in U_\delta \circ U_\delta$). Now for all $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}$ such that $(A, B) \in U_\delta$. Likewise, there exists $C \in \mathcal{C}$ such that $(B, C) \in U_\delta$, so $(A, C) \in U_\delta$ and $A \in U_\delta(C)$. Thus $(A, C) \in \mathcal{U}_\epsilon^C$. In like fashion, $(A, C) \in \mathcal{U}_\epsilon^D$ and $(A, C) \in \mathcal{U}_\epsilon$. \square

Definition 10 *The topology associated to the uniform structure generated by 2^U is called the uniform Hausdorff topology.*

(Here the word *uniform* is used to emphasize uniform proximity, not that this structure comes from a uniform structure.)

Let $\mathcal{U}_\epsilon^C(\mathcal{B}) = \{A \in \mathcal{P} \mid (A, B) \in \mathcal{U}_\epsilon^C\}$. Write $A \subset_\epsilon B$ if $A \in \mathcal{U}_\epsilon^C(\mathcal{B})$, and $A =_\epsilon B$ if $B \subset_\epsilon A$ also.

Definition 11 *A sequence $\{A_i\}$, $A_i \in \mathcal{P}$ for all i , is called an asymptotic subset of \mathcal{A} if for every $\epsilon > 0$, there exists a whole number N such that $A_j \subset_\epsilon \mathcal{A}$ whenever $j \geq N$. We will then write*

$$\varinjlim_{i \rightarrow \infty} A_i \subset \mathcal{A}.$$

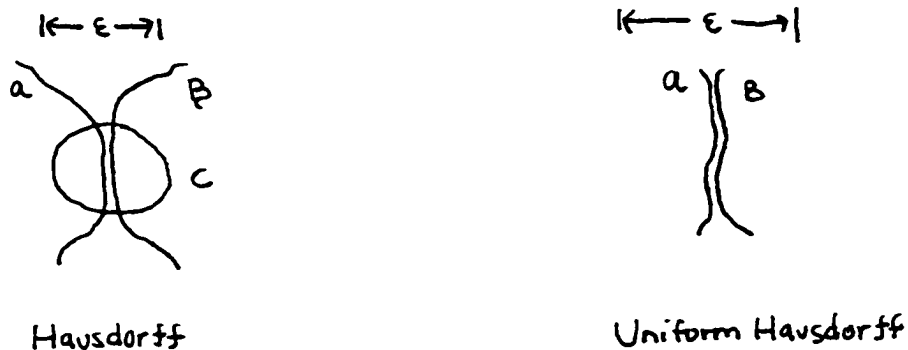


Figure 2.2: Hausdorff Topology vs. Uniform Hausdorff Topology

N.B. Since $PSL(2, \mathbf{R})$ is not compact, the uniform Hausdorff topology on \mathcal{P} is neither compact nor paracompact. It is customary to define a uniform structure given rather by sets $\mathcal{U}_{\epsilon, C}$ which measure proximity on a fixed compact $C \subset PSL(2, \mathbf{R})$. Since we will not be interested in an intrinsic study of \mathcal{P} , and will require a notion of uniform proximity, we will not adopt these provisions. See Figure 2.2.

Chapter 3

Germ of Inverse Limit

Solenoids

Before we begin the task of elaborating the definition of the Fuchsian germ of an arbitrary Riemann surface lamination, we introduce here a motivating example, which will serve as a guide to the forthcoming abstractions.

3.1 Group Germs

Let G be a group and $\mathcal{C} = \{H_i\}$, $i = 1, 2, \dots$, be a nested sequence of normal subgroups of G .

Definition 12 *The pro-completion of G with respect to \mathcal{C} is the group*

$$\widehat{G} = \varprojlim G/H_i.$$

Let \widehat{G}_i denote the limit

$$\varprojlim_{j \geq i} G/H_j;$$

by cofinality (Proposition 2), \widehat{G}_i is canonically isomorphic to \widehat{G} .

Definition 13 *A sequence $\{g_i\}$, $g_i \in G$, is called convergent with respect to \mathcal{C} if for every i , $g_i g_j^{-1} \in H_i$ whenever $j \geq i$.*

Proposition 5 *The set \overline{G} of convergent sequences forms a group with respect to componentwise multiplication.*

Proof: Indeed, given $\{g_i\}, \{h_i\} \in \overline{G}$, we have that

$$g_i h_i (g_j h_j)^{-1} \in g_i H_i g_j^{-1} = g_i H_i g_i^{-1} g_i g_j^{-1} = H_i g_i g_j^{-1} \subset H_i$$

whenever $j \geq i$, so that $\{g_i\}\{h_i\} \in \overline{G}$. In a similar fashion it can be seen that $\{g_i\}^{-1} \in \overline{G}$ also, and since \overline{G} is a subset of the group of all sequences in G , it follows that \overline{G} is a group. \square

Denote by \overline{G}_i the subgroup of \overline{G} consisting of the i th-tails of elements of \overline{G} i.e. sequences of the form $\{g_k\}_{k \geq i}$ where $\{g_k\}_{k \geq 1}$ is an element of \overline{G} .

(Alternatively, an element $\{g_j\} \in \overline{G}_i$ is a sequence which satisfies the stronger condition $g_j g_k^{-1} \in H_{j+i}$ whenever $k \geq j$.) Thus the \overline{G}_i form a direct system of groups with respect to the coordinate forgetting projections $\overline{G}_i \rightarrow \overline{G}_j$, $\{g_k\}_{k \geq i} \mapsto \{g_k\}_{k \geq j}$ (for $j \geq i$).

Definition 14 *The completion of G relative to \mathcal{C} is the group*

$$[\overline{G}] = [\overline{G}]_{\mathcal{C}} = \varprojlim \overline{G}_i.$$

The germ of G with respect to \mathcal{C} is the group

$$[G] = [G]_{\mathcal{C}} = \varprojlim \overline{H}_i.$$

We denote elements of $[\overline{G}]$ using the notation $[g_i]$.

Proposition 6 *$[G]$ is a normal subgroup of $[\overline{G}]$.*

Proof: For each i , the direct product $\overline{H}_i = \prod_{k \geq i} H_k$ is a normal subgroup of \overline{G}_i . Indeed, for $\{g_j\} \in \overline{G}_i$, we have

$$\{g_j\} \overline{H}_i \{g_j\}^{-1} = \prod_{j=1}^{\infty} g_j H_{j+i} g_j^{-1} = \prod_{j=1}^{\infty} H_{j+i} = \overline{H}_i.$$

The statement follows upon passing to direct limits. \square

Proposition 7 *$[\overline{G}]/[G]$ is isomorphic to \widehat{G} .*

Proof: Consider the map $\overline{G}_i \rightarrow \prod G/H_i$ defined $\{g_i\} \mapsto (g_i H_i)$. By definition of convergence, $g_j H_j$ maps onto $g_i H_i$ by the projection $G/H_j \rightarrow G/H_i$, whenever $j \geq i$. Thus we obtain a homomorphism $\varphi : \overline{G} \rightarrow \widehat{G}$, with kernel $\overline{H} = \prod H_i$. In a similar fashion, for each j , we deduce a projection $\varphi_j : \overline{G}_j \rightarrow \widehat{G}_j$ with kernel \overline{H}_j . So we obtain a direct system of exact sequences such that, whenever $j \leq k$, we have a commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \overline{H}_j & \longrightarrow & \overline{G}_j & \xrightarrow{\varphi_j} & \widehat{G}_j \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 1 & \longrightarrow & \overline{H}_k & \longrightarrow & \overline{G}_k & \xrightarrow{\varphi_k} & \widehat{G}_k \longrightarrow 1 \end{array}$$

The limit sequence

$$[G] \longrightarrow [\overline{G}] \xrightarrow{[\varphi_i]} \widehat{G},$$

where $[\varphi_i] = \lim_{\leftarrow} \varphi_i$, is easily seen to be exact: if $[\varphi_i][g_i] = 1$, then $\varphi_i(g_i) = 1$ for all i so that $[g_i] \in [G]$. \square

3.2 The Fundamental Germ of an Inverse Limit

Solenoid

Let S be a surface and let $\mathcal{C} = \{H_i\}$ be a nested sequence of normal subgroups of $\pi_1 S$. Let \widehat{S} denote the surface solenoid associated to an inverse system of

covers over S as in §2-2, Example 2.

Definition 15 *The fundamental germ of \widehat{S} is the group $[\pi_1]\widehat{S} = [\pi_1S]$.*

We construct now a space $[\overline{S}]$ on which $[\pi_1]\widehat{S}$ acts with quotient \widehat{S} . Let $p : \widetilde{S} \rightarrow S$ be a universal cover of S . We will assume that universal covers $p_i : \widetilde{S} \rightarrow S_i$ have been selected for each of the surfaces appearing in the defining system, so that $s^{ij} \circ p_j = p_i$ whenever $i < j$.

Let \overline{S} be the set of all sequences of the form $\{g_i z\}$, where $\{g_i\} \in \overline{\pi_1 S} =$ the completion of $\pi_1 S$ with respect to \mathcal{C} , $z \in \widetilde{S}$ and $g_i z$ is defined by the deck action of $\pi_1 S$ on \widetilde{S} . Likewise, we let \overline{S}_i be the set of i th tails of sequences in \overline{S} ; the \overline{S}_i form a direct system (in the same manner that the \overline{G}_i do).

Definition 16 *The completion of \widetilde{S} with respect to \mathcal{C} is $[\overline{S}] = \lim_{\leftarrow} \overline{S}_i$.*

Elements of $[\overline{S}]$ will be denoted $[g_i z]$.

We construct a projection $[\bar{p}] : [\overline{S}] \rightarrow \widehat{S}$ as follows. Given $\{g_i z\} \in \overline{S}$ we define an element of \widehat{S} by the tuple $(p_i(g_i z))$. That this tuple defines an element of \widehat{S} follows from the fact that $\{g_i\}$ is convergent with respect to \mathcal{C} . Indeed, if $i \leq j$, then $g_i g_j^{-1} \in H_i =$ deck group of p_i , hence $s^{ij} \circ p_j(g_j z) = p_i(g_i z)$. This defines a projection $\bar{p} : \overline{S} \rightarrow \widehat{S}$.

We similarly define projections $\bar{p}_i : \bar{S}_i \rightarrow \hat{S}$ that are compatible with direct limit maps, and hence deduce the projection $[\bar{p}] : [\bar{S}] \rightarrow \hat{S}$.

We equip $[\bar{S}]$ with the pull-back topology (with respect to which the projection becomes an open map).

N.B. The pull-back topology is *not* Hausdorff. The problem of finding a Hausdorff “covering space” topology for $[\bar{S}]$ is related to an algebraic version of the Nielsen realization problem. We will not require a qualitative study of $[\bar{S}]$ *per se*, so the pull-back topology will suffice for our purposes.

There is a natural action of $[\pi_1]\hat{S}$ on $[\bar{S}]$, defined $[h_i] \cdot [g_i z] = [h_i g_i z]$. This action identifies precisely those points identified by $[\bar{p}]$:

Theorem 2 $[\bar{S}]/[\pi_1]\hat{S}$ is homeomorphic to \hat{S} .

Proof: $[g_i z]$ and $[g'_i z']$ project via $[\bar{p}]$ to the same point in \hat{S} if and only if there exists $[h_i] \in [\pi_1]\hat{S}$ such that $[h_i][g_i z] = [g'_i z']$. If $[\mathcal{O}] = [\bar{p}]^{-1}(\mathcal{O})$ is an open, then $[\pi_1]\hat{S} \cdot [\mathcal{O}] = [\mathcal{O}]$, and so we deduce a continuous bijection $[\bar{S}]/[\pi_1]\hat{S} \rightarrow \hat{S}$. Since $[\bar{S}] \rightarrow \hat{S}$ is open by virtue of the pull-back topology, the correspondence is actually homeomorphic. \square

Fuchsian Representations

Suppose that \hat{S} is equipped with the structure of a Riemann surface lamination, pulled back from a Riemann surface structure defined on S_i . Let Γ_i be a Fuchsian group uniformizing S_i , and Γ_j the Fuchsian group uniformizing the Riemann surface structure on S_j pulled back from S_i ($j \geq i$). Let $[\bar{\Gamma}_i]$ be the completion of Γ_i with respect to the nested sequence $\{\Gamma_j\}_{j \geq i}$, $[\Gamma_i]$ the associated germ. In this case, we may identify \tilde{S} with \mathbf{H} to obtain $[\bar{\mathbf{H}}]$ and a conformal version of Theorem 2:

Theorem 3 $[\bar{\mathbf{H}}]/[\Gamma_i]$ is a Riemann surface lamination isomorphic to \hat{S} . \square

When the Riemann surface lamination structure of \hat{S} is *not* pulled back from a conformal structure below, the fiber transversals in \hat{S} no longer enjoy “isometrically” a group structure, and so a more general discussion is required to represent $[\pi_1]\hat{S}$ as a group of tails of Fuchsian sequences. This is the content of the next two chapters.

Chapter 4

Fuchsian Nested Sets

4.1 Pre-Fuchsian Nested Sets

Definition 17 A (decreasing) nested set in $PSL(2, \mathbf{R})$ is a sequence $\mathcal{G} = \{\mathcal{G}_i\}$, $i = 0, 1, 2, \dots$, where $\mathcal{G}_i \in \mathcal{P}$ for all i and $\mathcal{G}_j \subset \mathcal{G}_i$ whenever $j \geq i$.

We will sometimes write $\mathcal{G} = \mathcal{G}_0$ and let $\mathcal{G}_\infty = \bigcap \mathcal{G}_i$.

Definition 18 Two nested sets $\mathcal{G}, \mathcal{G}'$ are called **germ equivalent**, written $\mathcal{G} \sim \mathcal{G}'$, if for each j ,

$$\varprojlim_{i \rightarrow \infty} \mathcal{G}_i \subset \mathcal{G}'_j$$

and vice versa.

Proposition 8 *Germ equivalence is an equivalence relation.*

Proof: Reflexivity and symmetry are immediate. Suppose now that $\mathcal{G} \sim \mathcal{G}'$ and $\mathcal{G}' \sim \mathcal{G}''$. Fix $j \in \mathbf{N}$ and $\epsilon > 0$, and choose $\delta > 0$ such that $U_\delta \circ U_\delta \subset U_\epsilon$, where $U = \{U_\epsilon\}$ is the uniform structure on $PSL(2, \mathbf{R})$ defined in §2-5. By the hypothesis, there exists $k \in \mathbf{N}$ such that $\mathcal{G}'_k \subset_\delta \mathcal{G}''_j$, and there exists $M \in \mathbf{N}$ such that $\mathcal{G}_l \subset_\delta \mathcal{G}'_k$ for all $l \geq M$. Hence, for $l \geq M$ and any $A \in \mathcal{G}_l$, there exists $A' \in \mathcal{G}'_k$ such that $(A, A') \in U_\delta$. But there exists $A'' \in \mathcal{G}''_j$ such that $(A', A'') \in U_\delta$ also. Thus, by the triangle inequality for uniform structures, $(A, A'') \in U_\epsilon$ and we have shown that $\mathcal{G}_l \subset_\epsilon \mathcal{G}''_j$ for all $l \geq M$ i.e. $\lim_{i \rightarrow \infty} \mathcal{G}_i \subset \mathcal{G}''_j$, for any $j \in \mathbf{N}$. This argument is symmetric and it follows that $\mathcal{G} \sim \mathcal{G}''$. \square

Definition 19 *The germ equivalence class of a nested set \mathcal{G} is called the germ of \mathcal{G} , and is denoted $[\mathcal{G}]$.*

Proposition 9 *For any nested set \mathcal{G} , \mathcal{G}_∞ depends only on $[\mathcal{G}]$.*

Proof: $\mathcal{G}_\infty \subset_\epsilon \mathcal{G}'_j$ for any j and any $\epsilon > 0$, hence $\mathcal{G}_\infty \subset \mathcal{G}'_\infty$, and vice versa.

\square

Example 3 If \mathcal{G} is a nested set, then \mathcal{G} is germ equivalent to any cofinal subnested set of itself.

Definition 20 Given nested sets \mathcal{G} and \mathcal{G}' , we say that \mathcal{G} and \mathcal{G}' are orbit equivalent, written $\mathcal{G} \sim_o \mathcal{G}'$, if there exists $M \in PSL(2, \mathbf{R})$ such that $[\mathcal{G}] = [\mathcal{G}'M]$.

(Clearly, \sim_o is also an equivalence relation.)

Definition 21 A nested set \mathcal{G} is called flow if

- i) There exists j such that \mathcal{G}_j acts properly discontinuously on \mathbf{H} .
- ii) \mathcal{G}_∞ is a Fuchsian group.
- iii) \mathcal{G}_∞ acts on the left of \mathcal{G} , preserving its structure of nested set i.e. for all i ,

$$\mathcal{G}_\infty \cdot \mathcal{G}_i = \mathcal{G}_i.$$

Example 4 Let $\mathcal{G}_i = \Gamma$ for all i , where Γ is a Fuchsian group. Then \mathcal{G} is trivially a flow nested set.

Example 5 Let $\mathcal{G}_i = \Gamma_i$, a nested sequence of normal subgroups of a fixed Fuchsian group Γ . Then \mathcal{G} is a flow nested set.

We will assume for convenience that $j = 0$ in condition i) of Definition 21, for any nested set we discuss, since this can always be arranged by omitting the first $j - 1$ sets and re-indexing.

Condition *i*) in Definition 21 then means that, for each $z \in \mathbf{H}$, there exists a constant δ satisfying the property that

$$\mathcal{O} = \mathcal{O}_{z,\delta} = \bigcup_{A \in \mathcal{G}} D_\delta(Az)$$

consists of a disjoint union of hyperbolic disks (here $D_\delta(w)$ signifies the hyperbolic disk of radius δ centered at w).

Definition 22 For $z \in \mathbf{H}$, the set $T_z = \bigcup_{A \in \mathcal{G}} Az$ ($T_z^i = \bigcup_{A \in \mathcal{G}_i} Az$) will be referred to as the \mathcal{G} -orbit (\mathcal{G}_i -orbit) through z . The collection of disks \mathcal{O} appearing in the previous paragraph is called a **preflowbox** and δ the radius of \mathcal{O} . \mathcal{O}^i will denote an *i*-preflowbox, which consists of those disks in \mathcal{O} centered at points in the \mathcal{G}_i -orbit of z . For each $z \in \mathbf{H}$, we define the injectivity radius of \mathcal{G} at z to be the supremum $\bar{\delta}(z)$ of constants δ for which the disks comprising $\mathcal{O}_{z,\delta}$ are disjoint.

Proposition 10 Any flow nested set is countable.

Proof: This is an immediate consequence of property *i*) in Definition 21. \square

Our interest is in irreducible Riemann surface laminations, hence we must include axioms insuring that the germ of \mathcal{G} be sufficient to educe a basis for a completion topology on its orbits.

Definition 23 A flow nested set \mathcal{G} is **self-similar** if for each $j \in \mathbf{N}$ and $A \in \mathcal{G}_j$,

$$\varinjlim \mathcal{G}_i A \subset \mathcal{G}_j.$$

Proposition 11 For any $A \in \mathcal{G}$ and $0 < \delta < \bar{\delta}(z_0)$, there is a whole number N such that, whenever $k \geq N$, there exists a unique set $\mathcal{H}_k^A \subset \mathcal{G}$ for which $\mathcal{H}_k^A =_\delta \mathcal{G}_k A$.

Proof: By self-similarity, there exists j such that $\mathcal{G}_j A \subset_\delta \mathcal{G}$. Since δ is less than the injectivity radius at the basepoint z_0 of \mathbf{H} , it follows that for each $BA \in \mathcal{G}_j A$, there is a unique $C \in \mathcal{G}$ for which $BA(z_0) \in D_\delta(C(z_0))$; hence C is the unique element of \mathcal{G} for which $(BA, C) \in U_\delta$. \square

Definition 24 The \mathcal{H}_k^A are called A (right) **cosets**.

See Figure 3.1.

Clearly, the nested sets of Example 4 and Example 5 are self-similar, since the cosets are precisely cosets in the algebraic sense i.e. $\mathcal{H}_k^A = \mathcal{G}_k A$ for all k and $A \in \mathcal{G}$.

Note: For fixed $A \in \mathcal{G}$, the cosets about A form themselves a nested set $\mathcal{H}^A \in [\mathcal{G}A]$. It follows that $\mathcal{H}_\infty^A = \mathcal{G}_\infty A$.

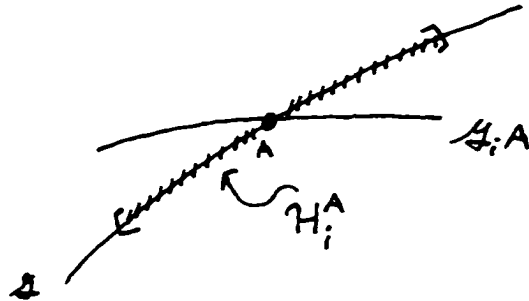


Figure 4.1: A Coset

Consider the collection of all cosets of \mathcal{G} of the form $\mathcal{H}_j^A =_\delta \mathcal{G}_j.A$, where δ satisfies

$$U_\delta \circ U_\delta \circ U_\delta \subset U_{\delta(z_0)}.$$

(That there exists some δ for which the above is true follows from the iterated version of the triangle inequality, Lemma 1 (§2-4).)

Proposition 12 *The above collection of cosets defines a basis for a topology on \mathcal{G} .*

Proof: Clearly every element A of \mathcal{G} is contained in the A -cosets. Suppose now that $C \in \mathcal{H}_j^A \cap \mathcal{H}_k^B$, where $\mathcal{H}_j^A =_\delta \mathcal{G}_j.A$ and $\mathcal{H}_k^B =_\delta \mathcal{G}_k.B$. Then there exists a unique $C' \in \mathcal{G}_j$ such that $(C, C'.A) \in U_\delta$. Now by self-similarity, there exists an index l such that $\mathcal{G}_l.C'.A \subset_\delta \mathcal{G}_j.A$. It follows then that

$$\mathcal{G}_l C \subset_\delta \mathcal{G}_l C' A \subset_\delta \mathcal{G}_j A =_\delta \mathcal{H}_j^A.$$

Thus, by the iterated triangle inequality, we have $\mathcal{G}_l C \subset_{\delta(z_0)} \mathcal{H}_j^A$. By the same sort of argument, we can choose l so that $\mathcal{G}_l C \subset_{\delta(z_0)} \mathcal{H}_k^B$ also. Hence, $\mathcal{H}_l^C \subset \mathcal{H}_j^A \cap \mathcal{H}_k^B$. \square

4.2 Completion

Throughout this section, \mathcal{G} is assumed to be a flow and self-similar nested set.

Definition 25 Let $\mathcal{H} = \{\mathcal{H}_{n_i}^{A_i}\}$, where $\mathcal{H}_{n_i}^{A_i} =_\delta \mathcal{G}_{n_i, A_i}$ is a collection of cosets. We say that \mathcal{H} is a **convergor** if

- 1) \mathcal{H} is a decreasing nested set,
- 2) the sequence n_1, n_2, \dots is a monotone increasing sequence in \mathbf{N} and
- 3) $\delta_i \rightarrow 0$ i.e.

$$\varprojlim_{i \rightarrow \infty} \mathcal{G}_{n_i, A_i} \subset \mathcal{G}.$$

The elements A_i are called the **centers** of the convergor \mathcal{H} , and the sequence $\{n_i\}$ is called the **signature** of the convergor.

Notation: We will henceforward write $\mathcal{H}_i = \mathcal{H}_n^{A_i}$, where it will be understood that the i indices do not in general correspond to the signature of \mathcal{H} .

Example 6 \mathcal{G} is itself a convergor. Also, for fixed $A \in \mathcal{G}$, the coset nested set $\mathcal{H}^A = \{\mathcal{H}_i^A\}$ is a convergor.

Definition 26 A sequence $\mathbf{M} = \{M_i\}$, $M_i \in \mathcal{G}$ for all i , is called **convergent** if there exists a convergor \mathcal{H} such that for every $j \in \mathbf{N}$, there exists a natural number N such that $M_k \in \mathcal{H}_j$ whenever $k \geq N$. If $\mathcal{V} \subset \mathcal{G}$, we say that \mathbf{M} **converges in \mathcal{V}** , and write $\mathbf{M} \subset \mathcal{V}$, if eventually $M_i \in \mathcal{V}$ for all i . A sequence $\mathbf{z} = \{z_i\}$, $z_i \in \mathbf{H}$, **converges** if there exists $z \in \mathbf{H}$ and a convergent sequence \mathbf{M} in \mathcal{G} such that

$$\lim_{i \rightarrow \infty} d_{\mathbf{H}}(z_i, M_i z) = 0.$$

Two convergent sequences \mathbf{z}, \mathbf{w} are called **asymptotic**, written $\mathbf{z} \asymp \mathbf{w}$, if $d_{\mathbf{H}}(z_i, w_i) \rightarrow 0$. Two convergent sequences \mathbf{M} and \mathbf{M}' are called **asymptotic**, written $\mathbf{M} \asymp \mathbf{M}'$, if $\mathbf{M}z \asymp \mathbf{M}'z$ for all $z \in \mathbf{H}$.

Thus, every convergent sequence is asymptotic to $\mathbf{M}z = \{M_i z\}$ for some convergent \mathbf{M} and $z \in \mathbf{H}$.

Proposition 13 *If \mathbf{M} and \mathbf{M}' are convergent, then $\mathbf{M} \asymp \mathbf{M}'$ if and only if $M_i = M'_i$ eventually.*

Proof: This is an immediate consequence of the fact that \mathcal{G} acts properly discontinuously on \mathbf{H} . \square

Definition 27 *Given $\mathcal{V} \subset \mathcal{G}$, we define, respectively, the completion of \mathbf{H} of \mathcal{G} and of \mathcal{V} by*

$$\begin{aligned}\bar{\mathbf{H}} &= \{\text{convergent } \mathbf{z}\} / \asymp \\ \bar{\mathcal{G}} &= \{\text{convergent } \mathbf{M}\} / \asymp \\ \bar{\mathcal{V}} &= \{\text{convergent } \mathbf{M} \subset \mathcal{V}\} / \asymp.\end{aligned}$$

From this point on, when we refer to a convergent sequence \mathbf{z} or \mathbf{M} , we will mean the \asymp -classes, unless we indicate otherwise.

Definition 28 *Given \mathcal{H} a convergor \mathcal{G} , we define, respectively, the target of \mathcal{H} and the orbit of \mathcal{H} by*

$$\begin{aligned}\circledast \mathcal{H} &= \{\mathbf{M} \in \bar{\mathcal{G}} \mid \mathbf{M} \text{ converges by } \mathcal{H}\} \\ \text{Orb}(\mathcal{H}) &= \{\mathbf{M}w \in \bar{\mathbf{H}} \mid w \in \mathbf{H} \text{ and } \mathbf{M} \in \circledast \mathcal{H}\}.\end{aligned}$$

The completions of cosets, taken as closed sets, subgenerate a topology on $\overline{\mathcal{G}}$. In order that this topology yield metrizable, we will need to impose a regularity condition. In what follows, we will use the notation ${}^\circ\mathcal{V}$ to denote the interior of a completion set $\overline{\mathcal{V}}$ in $\overline{\mathcal{G}}$.

Definition 29 *A convergor $\mathcal{H} = \{\mathcal{H}_i\}$ is called **regular** if $\overline{\mathcal{H}}_j \subset {}^\circ\mathcal{H}_i$ whenever $i < j$. A flow nested set \mathcal{G} is called **regular** if \mathcal{G} is regular as a convergor, and if for every ${}^\circ\mathcal{V} \subset \overline{\mathcal{G}}$ and $\mathbf{M} \in {}^\circ\mathcal{V}$, there exists a regular convergor \mathcal{H} such that $\mathbf{M} \in {}^\circ\mathcal{H}$ and $\mathcal{H} \subset \mathcal{V}$.*

Note that if \mathcal{G} is regular, then a coset convergor $\{\mathcal{H}_k^A\}$ is necessarily regular, since it is an asymptotic translate of $\{\mathcal{G}_k\}$ which is regular when viewed as a convergor.

Proposition 14 *i) If \mathcal{H}^A is a coset of a regular \mathcal{G} , then $\mathcal{H}^A \subset {}^\circ\mathcal{H}^A$.*

ii) For any ${}^\circ\mathcal{V}$ and $\mathbf{M} \in {}^\circ\mathcal{V}$, then there exists a coset \mathcal{H}^A of \mathcal{G} such that $\mathbf{M} \in {}^\circ\mathcal{H}^A \subset {}^\circ\mathcal{V}$.

Proof: *i):* By self-similarity, given $B \in \mathcal{H}^A$, there exists a whole number N such that $\mathcal{H}_l^B \subset \mathcal{H}^A$ whenever $l \geq N$. Thus $B \in \overline{\mathcal{H}}_{k+1}^B$ by definition of completion, and

$$\overline{\mathcal{H}}_{k+1}^B \subset {}^\circ\mathcal{H}_k^B \subset {}^\circ\mathcal{H}^A.$$

ii): This is an immediate consequence of the fact that \mathcal{V} contains a regular convergor converging \mathbf{M} . \square

Thus the subspace topology of \mathcal{G} coincides with the topology defined by the cosets of \mathcal{G} , and the coset interiors ${}^{\circ}\mathcal{H}^A$ generate the topology of $\overline{\mathcal{G}}$.

Assumption: From now on, all convergors will be considered to be regular.

Definition 30 *If \mathbf{M} is a fixed convergent sequence in $\overline{\mathcal{G}}$, we define the leaf through \mathbf{M} to be the set*

$$\ell_{\mathbf{M}} = \{\mathbf{M}w \in \overline{\mathbf{H}} \mid w \in \mathbf{H}\}.$$

If $z \in \mathbf{H}$, the completion of the the \mathcal{G}^i -orbit T_z^i is the set

$$\overline{T}_z^i = \{\mathbf{M}z \mid \mathbf{M} \in \overline{\mathcal{G}}_i\}$$

The tangent space of $\ell_{\mathbf{M}}$, $T\ell_{\mathbf{M}}$, is defined to be the collection $\{\mathbf{M}_\bullet \vec{v} \mid \vec{v} \in T\mathbf{H}\}$, where $\mathbf{M}_\bullet \vec{v} = \{(M_i)_\bullet \vec{v}\}$ (modulo \asymp).

A constant curvature -1 Riemannian metric $\rho_{\mathbf{M}}$ is obtained by pushing forward $\rho_{\mathbf{H}}$ along the canonical identification $\mathbf{H} \rightarrow \ell_{\mathbf{M}}$, $w \mapsto \mathbf{M}w$. The induced distance function will be denoted $d_{\mathbf{M}}$. We will let $Isom(\ell_{\mathbf{M}})$ denote the isometry group of $\ell_{\mathbf{M}}$.

The unit tangent bundle $T^1\ell_{\mathbf{M}}$ can be defined to be the set $\{\mathbf{M}_*\vec{v} \mid \vec{v} \in T^1\mathbf{H}\}$. Note that $\mathbf{M}_*\vec{v} \in T^1\ell_{\mathbf{M}}$ if and only if $\rho_{\mathbf{M}}(\mathbf{M}_*\vec{v}, \mathbf{M}_*\vec{v}) = 1$.

Proposition 15 *If $\mathcal{H} \sim_o \mathcal{H}'$, then $Orb(\mathcal{H}) = Orb(\mathcal{H}')$.*

Proof: Indeed, if there exists $M \in PSL(2, \mathbf{R})$ such that $[\mathcal{H}] = [\mathcal{H}'M]$ and if $M'z \in Orb(\mathcal{H}')$, then $M'M \asymp \mathbf{M}$, for some \mathbf{M} converging by \mathcal{H} , so $M'z \asymp \mathbf{M}M^{-1}z \in Orb(\mathcal{H})$. \square

Definition 31 *Let \mathcal{G} be regular. We say that \mathcal{G} is normal if whenever $Orb(\mathcal{H}) \cap Orb(\mathcal{H}') \neq \emptyset$, then $\mathcal{H} \sim_o \mathcal{H}'$.*

As the term suggests, normality will be used to show that preflowboxes with non-trivial intersection intersect in a normal fashion (c.f. the condition of holomorphic transverse normality in Definition 1).

Definition 32 *A flow, self-similar, regular and normal nested set will be called preFuchsian.*

Lemma 2 *Let \mathcal{G} be a preFuchsian nested set and suppose that for $\mathbf{M}, \mathbf{M}' \in \overline{\mathcal{G}}$, there exist points $z, z' \in \mathbf{H}$ such that $\mathbf{M}z \asymp \mathbf{M}'z'$. Then $\ell_{\mathbf{M}} = \ell_{\mathbf{M}'}$.*

Proof: Since \mathcal{G} is normal, if say $\mathbf{M} \in \circlearrowleft\mathcal{H}$ and $\mathbf{M}' \in \circlearrowleft\mathcal{H}'$, then $\mathcal{H} \sim_{\circlearrowleft} \mathcal{H}'$ i.e. there exists $M \in PSL(2, \mathbf{R})$ so that $\mathbf{M}w \asymp \mathbf{M}'Mw$ for all $w \in \mathbf{H}$. The conclusion of the lemma follows immediately. \square

We can now define without ambiguity the leaf through $\mathbf{z} \in \overline{\mathbf{H}}$ to be $\ell_{\mathbf{z}} = \ell_{\mathbf{M}}$ if \mathbf{z} converges via \mathbf{M} . Notice that the Riemannian metric $\rho_{\mathbf{z}} = \rho_{\mathbf{M}}$ is also well-defined. A decomposition of $\overline{\mathbf{H}}$ into leaves isomorphic to \mathbf{H} is now understood.

Note: In the spirit of the note in §3-2, we will not regard $\overline{\mathbf{H}}$ so much as a topological space, but rather as a family of Riemann surfaces, each isomorphic to \mathbf{H} .

4.3 Fuchsian Nested Sets

Let $(z_0, \vec{v}_0) \in T^1\mathbf{H}$ be the basepoint of the unit tangent bundle. Given $\mathbf{M} \in \overline{\mathcal{G}}$, we may basepoint the unit tangent bundle $T^1\ell_{\mathbf{M}}$ of the leaf through \mathbf{M} using $\mathbf{M}(z_0, \vec{v}_0) = (\mathbf{M}z_0, \mathbf{M}_*\vec{v}_0)$. Let $\overline{\mathcal{G}}(z_0, \vec{v}_0)$ denote the totality of such basepoints.

Definition 33 *If \mathbf{M} converges via \mathcal{H} then we define the nested set asso-*

ciated to \mathbf{M} and \mathcal{H} by

$$\mathcal{G}^{\mathbf{M}, \mathcal{H}} = \{\mathcal{A} \in \text{Isom}(\ell_{\mathbf{M}}) \mid \mathcal{A}\mathbf{M}(z_0, \vec{v}_0) \in \overline{\mathcal{G}}(z_0, \vec{v}_0)\}$$

with

$$\mathcal{G}_i^{\mathbf{M}, \mathcal{H}} = \{\mathcal{A} \in \mathcal{G}^{\mathbf{M}, \mathcal{H}} \mid \mathcal{A}\mathbf{M}(z_0, \vec{v}_0) \in \overline{\mathcal{H}}_i(z_0, \vec{v}_0)\}.$$

As earlier, we let

$$\mathcal{G}_{\infty}^{\mathbf{M}, \mathcal{H}} = \bigcap \mathcal{G}_i^{\mathbf{M}, \mathcal{H}}.$$

Proposition 16 *If \mathbf{M} converges via \mathcal{H}' also, then $[\mathcal{G}^{\mathbf{M}, \mathcal{H}}] = [\mathcal{G}^{\mathbf{M}, \mathcal{H}'}]$.*

Proof: If \mathcal{H} and \mathcal{H}' converge \mathbf{M} in \mathcal{G} , it must be that $[\mathcal{H}] = [\mathcal{H}']$. \square

Thus we will drop the superscript \mathcal{H} unless it becomes relevant.

Definition 34 \mathcal{G} is called **uniform** if $\mathcal{G}^{\mathbf{M}}$ is *preFuchsian* for each $\mathbf{M} \in \overline{\mathcal{G}}$.

N.B.: Since \mathcal{G} acts properly discontinuously it follows that for $z \in \mathbf{H}$ the points of \overline{T}_z are uniformly disjoint *i.e.* there exists $\epsilon > 0$ such that for any leaf $\ell \subset \overline{\mathbf{H}}$, $d_{\ell}(\mathbf{w}, \mathbf{w}') > \epsilon$ whenever $\mathbf{w}, \mathbf{w}' \in \overline{T}_z \cap \ell$. Indeed, if it were true that for every $\epsilon > 0$, there exists a leaf ℓ and $\mathbf{w}, \mathbf{w}' \in \overline{T}_z \cap \ell$ with $d_{\ell}(\mathbf{w}, \mathbf{w}') < \epsilon$, then, writing $\mathbf{w} = \mathbf{M}z$ and $\mathbf{w}' = \mathbf{M}'z$, we have $d_{\ell}(\mathbf{M}z, \mathbf{M}'z) = \lim_{i \rightarrow \infty} d_{\mathbf{H}}(M_i z, M'_i z) < \epsilon$, contradicting the proper discontinuity of \mathcal{G} acting on \mathbf{H} . Thus, *a fortiori*, $\mathcal{G}^{\mathbf{M}}$ acts properly discontinuously on $\ell_{\mathbf{M}}$.

Example 7 Take $\mathcal{G} \{ \mathcal{G}_i = \Gamma_i \}$ to be the nested set associated to an inverse limit solenoid. Given $\mathbf{M} = \{ M_i \} \in \overline{\mathcal{G}}$ (converging according to Definition 13 of Chapter 3), we have $\mathbf{M} \in {}^\circ\mathcal{H}$, where $\mathcal{H} = \{ M_i \Gamma_i = \Gamma_i M_i \}$. We claim that

$$\mathcal{G}_i^{\mathbf{M}, \mathcal{H}} = \{ \mathbf{M} A \mathbf{M}^{-1} \mid A \in \mathcal{G}_i \}$$

for all i . Note first that for any $A \in \mathcal{G}_i$

$$\mathbf{M} A \mathbf{M}^{-1}(\mathbf{M}(z_0, \vec{v}_0)) = \mathbf{M} A(z_0, \vec{v}_0) \in \overline{\mathcal{G}}(z_0, \vec{v}_0)$$

since $\mathbf{M} A$ is also a convergent sequence. Moreover, $\mathbf{M} A(z_0, \vec{v}_0) \in \overline{\mathcal{H}}_i(z_0, \vec{v}_0)$ also, since $M_j \Gamma_j \subset \mathcal{H}_i$ whenever $j > i$ (\mathbf{M} is convergent) and $A \in \Gamma_i$.

Conversely, $\mathcal{A} \mathbf{M}(z_0, \vec{v}_0) = \mathbf{N}(z_0, \vec{v}_0) \in \overline{\mathcal{H}}_i(z_0, \vec{v}_0)$ for $\mathcal{A} \in \text{Isom}(\ell_{\mathbf{M}} = \ell_{\mathbf{N}})$ implies that there exists $A \in \Gamma_i$ such that $\mathbf{N} = \mathbf{M} A$, hence \mathcal{A} must be of the form $\mathbf{M} A \mathbf{M}^{-1}$.

Since the properties of being flow, self-similar, regular and normal are stable with respect to conjugation by \mathbf{M} , it follows that $\mathcal{G}^{\mathbf{M}}$ is preFuchsian. Hence \mathcal{G} is uniform.

Assume now that \mathcal{G} is uniform. Since $\mathcal{G}^{\mathbf{M}}$ is then preFuchsian, its cosets $\mathcal{H}^{\mathcal{A}}$, $\mathcal{A} \in \mathcal{G}^{\mathbf{M}}$, form the basis of a topology on $\mathcal{G}^{\mathbf{M}}$; on the other hand, one can define a topology on $\mathcal{G}^{\mathbf{M}}$ coming from \mathcal{G} by taking as a basis sets of the

form

$$\mathcal{H}_M^A = \{A \in \mathcal{G}^M \mid \mathcal{A}M(z_0, \vec{v}_0) \in \overline{\mathcal{H}^A}(z_0, \vec{v}_0)\}$$

where $A \in \mathcal{G}$ and \mathcal{H}^A is any coset of \mathcal{G} .

Definition 35 *The topology on \mathcal{G}^M generated by the \mathcal{H}^A is called the intrinsic topology; the topology on \mathcal{G}^M generated by the \mathcal{H}^A is called the \mathcal{G} -topology.*

Definition 36 *\mathcal{G} is called unified if for each $M \in \overline{\mathcal{G}}$, the \mathcal{G} -topology and the intrinsic topology coincide on \mathcal{G}^M .*

Clearly, in Example 7, \mathcal{G} is unified since $\mathcal{H}_M^A = \mathcal{H}_M^A$ for $\mathcal{A} = M \cdot AM^{-1}$.

Definition 37 *A preFuchsian nested set that is uniform and unified is called Fuchsian.*

Proposition 17 *Let \mathcal{G} be a Fuchsian nested set, and suppose that for $M, M' \in \overline{\mathcal{G}}$ we have $\ell_M = \ell_{M'}$. Then $\mathcal{G}_\infty^M = \mathcal{G}_\infty^{M'}$.*

Proof: This follows from the fact that \mathcal{G} is unified. First note that since $\ell_M = \ell_{M'}$, it follows that $\mathcal{G}^M = \mathcal{G}^{M'}$ (as sets, not as nested sets). In particular, there exists $A \in \mathcal{G}^M$ such that $\mathcal{A}M(z_0, \vec{v}_0) = M'(z_0, \vec{v}_0)$; thus,

$$\mathcal{G}_\infty^{M'}(z_0, \vec{v}_0) = \mathcal{G}_\infty^M \mathcal{A}M(z_0, \vec{v}_0) \subset \overline{\mathcal{G}}(z_0, \vec{v}_0)$$

by self-similarity of $\mathcal{G}^{\mathbf{M}}$.

Moreover, since the intrinsic topology on $\mathcal{G}^{\mathbf{M}}$ coincides with the \mathcal{G} -topology (on $\mathcal{G}^{\mathbf{M}}$), we have that for every coset \mathcal{H}^A of \mathcal{G} for which $\overline{\mathcal{H}^A}$ contains \mathcal{A} ,

$$\mathcal{G}_{\infty}^{\mathbf{M}} \mathcal{A} \subset \overline{\mathcal{H}_{\mathbf{M}}^A}.$$

Hence,

$$\mathcal{G}_{\infty}^{\mathbf{M}} \mathbf{M}'(z_0, \bar{v}_0) \subset \overline{\mathcal{H}_{\mathbf{M}}^A} \mathbf{M}(z_0, \bar{v}_0) = \overline{\mathcal{H}^A}(z_0, \bar{v}_0) = \overline{\mathcal{H}_{\mathbf{M}'}^A} \mathbf{M}'(z_0, \bar{v}_0).$$

But the $\overline{\mathcal{H}_{\mathbf{M}'}^A}$ generate also the intrinsic topology of $\mathcal{G}^{\mathbf{M}'}$, so it follows that $\mathcal{G}_{\infty}^{\mathbf{M}} \subset \mathcal{G}_{\infty}^{\mathbf{M}'}$ for all i , implying that $\mathcal{G}_{\infty}^{\mathbf{M}} \subset \mathcal{G}_{\infty}^{\mathbf{M}'}$. Since the argument is symmetric, the proposition follows. \square

Thus, one can speak of the Fuchsian group \mathcal{G}_{∞}^z associated to the leaf ℓ_z without danger of confusion.

N.B.: In fact, the argument employed in Proposition 17 shows more: that $[\mathcal{G}^{\mathbf{M}}] = [\mathcal{G}^{\mathbf{M}'}]$.

4.4 The Quotient of $\overline{\mathbf{H}}$ by \mathcal{G}

Now define a relation $\sim_{\mathcal{G}}$ on $\overline{\mathbf{H}}$ by $z \sim_{\mathcal{G}} z'$ if and only if there exists $z \in \mathbf{H}$, $\mathbf{M}, \mathbf{M}' \in \overline{\mathcal{G}}$ such that $z = \mathbf{M}z$, $z' = \mathbf{M}'z$ and \mathbf{M}, \mathbf{M}' converge by a common

convergor.

Proposition 18 $\sim_{\mathcal{G}}$ is an equivalence relation.

Proof: Symmetry and reflexivity are apparent. Suppose that $z = Mz \sim_{\mathcal{G}} M'z = z'$ and $z' = N'w \sim_{\mathcal{G}} N''w = z''$, where $M, M' \in \circlearrowleft\mathcal{H}$ and $N', N'' \in \circlearrowleft\mathcal{H}'$. Then $Orb(\mathcal{H}) \cap Orb(\mathcal{H}') \neq \emptyset$, so by normality there exists $A \in PSL(2, \mathbf{R})$ such that $[\mathcal{H}] = [\mathcal{H}'A]$. This implies that $w = Az$ and $N'A \asymp M'$. Moreover, there must exist $M'' \in \circlearrowleft\mathcal{H}$ for which $N''A \asymp M''$. Thus $M''z \asymp N''Az = N''w$, in other words, $z \sim_{\mathcal{G}} z''$. \square

Let $L_{\mathcal{G}}$ denote the quotient of $\overline{\mathbf{H}}$ by $\sim_{\mathcal{G}}$ and $\varrho : \overline{\mathbf{H}} \rightarrow L_{\mathcal{G}}$ the projection.

Proposition 19 Let \mathcal{G} be a Fuchsian nested set, and suppose that $\varrho(z) = \varrho(z')$. Then $\varrho(\ell_z) = \varrho(\ell_{z'})$ and the identification is isometric.

Proof: Clearly the images of the leaves are identical: indeed, if $z = Mz \sim_{\mathcal{G}} M'z = z'$ then $Mw \sim_{\mathcal{G}} M'w$ for all $w \in \mathbf{H}$. Now if $d_{\ell_z}(Mz, Mw) = r$ for $Mz, Mw \in \ell_z$, then by definition of d_{ℓ_z} , $d_{\mathbf{H}}(z, w) = r$ also, and then $d_{\ell_{z'}}(M'z, M'w) = r$. Thus the identification is isometric. \square

Notice that Proposition 19 implies that \mathcal{G}_{∞}^z is isomorphic to $\mathcal{G}_{\infty}^{z'}$. Thus we can make without ambiguity the following

Definition 38 Given \mathcal{G} a Fuchsian nested set and $\hat{\mathbf{z}} = \varrho(\mathbf{z}) \in L_{\mathcal{G}}$, we define the leaf through $\hat{\mathbf{z}}$ to be the Riemann surface $\ell_{\hat{\mathbf{z}}} = \varrho(\ell_{\mathbf{z}})$.

If $\mathbf{z} = z \in \mathbf{H}$, we will simply write ℓ .

We now define a topology on $L_{\mathcal{G}}$. For each $z \in \mathbf{H}$, consider preflowboxes $\mathcal{O} = \mathcal{O}_{z,\delta}^i$, where $\delta < \bar{\delta}(z)$. Let $T = T_z$; recall that $\bar{T} = \bar{\mathcal{G}}z$ is topologized using as subbasis the completions of cosets. Let

$${}^\circ\mathcal{O} = \bigcup_{w \in D_\delta(z)} {}^\circ T_w,$$

$\hat{\mathcal{O}} = \varrho({}^\circ\mathcal{O})$ and $\hat{T} = \varrho({}^\circ T)$. Also, let $\hat{\mathcal{O}}_\ell = \hat{\mathcal{O}} \cap \ell$ and $\hat{T}_\ell = \hat{T} \cap \ell$. Note that $\hat{\mathcal{O}} \cong D_\delta(z) \times \hat{T}$ (as a ‘‘laminated set’’).

Proposition 20 The flowboxes $\hat{\mathcal{O}}$ form a basis for a topology on $L_{\mathcal{G}}$.

Proof: Indeed, if an overlap $\hat{\mathcal{O}} \cap \hat{\mathcal{O}}'$ contains a point $\varrho(w)$, $w \in \mathbf{H}$, then the fact that \mathcal{G} is self-similar implies that there exists a preflowbox \mathcal{O}'' centered at w contained in the intersection $\mathcal{O} \cap \mathcal{O}'$, whereby we have ${}^\circ\mathcal{O}'' \subset {}^\circ\mathcal{O} \cap {}^\circ\mathcal{O}'$ and $\hat{\mathcal{O}}'' \subset \hat{\mathcal{O}} \cap \hat{\mathcal{O}}'$. On the other hand, suppose that the overlap contains a ‘‘completion point’’ $\hat{\mathbf{w}} = \varrho(\mathbf{w})$, where say $\mathbf{w} = \mathbf{M}w$ in ${}^\circ T_w \subset {}^\circ\mathcal{O}$ and $\mathbf{w} = \mathbf{M}'w'$ in ${}^\circ T_{w'} \subset {}^\circ\mathcal{O}'$. Then, by regularity of \mathcal{G} , there are regular convergors $\mathcal{H} = \{\mathcal{H}^{A_i}\}$ and $\mathcal{H}' = \{\mathcal{H}^{A'_i}\}$ with $\mathbf{M} \in {}^\circ\mathcal{H}$ and $\mathbf{M}' \in {}^\circ\mathcal{H}'$ as well as

$\mathcal{H}w \subset T_w$ and $\mathcal{H}'w' \subset T_{w'}$. By normality $\mathcal{H} \sim_o \mathcal{H}'$, so we may find δ'' and an index j for which $\mathcal{O}'' = \mathcal{O}_{A,w,\delta''}^j \subset \mathcal{O} \cap \mathcal{O}'$. Hence, $\hat{w} \subset \hat{\mathcal{O}}'' \subset \hat{\mathcal{O}} \cap \hat{\mathcal{O}}'$. \square

Proposition 21 *$L_{\mathcal{G}}$ is separable and metrizable with respect to the topology described above.*

Proof: Clearly, $L_{\mathcal{G}}$ has a countable topology (using flowboxes $\hat{\mathcal{O}}$ centered at rational points of \mathbf{H} , say). We now show that this topology is regular. Let $\hat{z} = \varrho(z) \in L_{\mathcal{G}}$ be contained in $\hat{\mathcal{O}} = \varrho({}^\circ\mathcal{O})$, where $\mathcal{O} = \mathcal{O}_{z,\delta}^i$. Write $z = \mathbf{M}z$ for $\mathbf{M} \in {}^\circ\mathcal{H}$, $\mathcal{H} = \{\mathcal{H}^{A_i}\}$ a regular convergor, which we may assume is contained in \mathcal{G}_j since $z \in {}^\circ\mathcal{O}$ and \mathcal{G} is regular. Then we may find an index k and $\epsilon > 0$ such that $z \in {}^\circ\mathcal{O}' = {}^\circ\mathcal{O}_{A_k z, \epsilon}^k$ and $\overline{\mathcal{O}'} \subset {}^\circ\mathcal{O}$, i.e. the closure of $\hat{\mathcal{O}}'$ is contained in $\hat{\mathcal{O}}$.

Since the topology on $L_{\mathcal{G}}$ is separable and regular, it follows by Urysohn's theorem that $L_{\mathcal{G}}$ is metrizable. \square

We now define a Riemann surface lamination atlas consisting of charts whose images are flowboxes. For $z \in \mathbf{H}$, $\hat{\mathcal{O}}$ centered at $\hat{z} = \varrho(z)$, we define a chart as follows. Let D be the open unit ball in \mathbf{C} centered at 0, and let $\varphi_I : D \rightarrow D_\delta(z)$ be a conformal homeomorphism, where δ is the radius of $\hat{\mathcal{O}}$. For any $A \in \mathcal{G}$, we define $\varphi_A : D \rightarrow D_\delta(Az)$ by $\varphi_A = A \circ \varphi_I$ and

$\varphi : D \times T \rightarrow \mathcal{O}$ by $\varphi(w, A) = \varphi_A(w)$.

Now define a bijective, leafwise conformal map $\hat{\varphi}_\ell : D \times \hat{T}_\ell \rightarrow \hat{\mathcal{O}}_\ell$ by $\hat{\varphi}_\ell = \varrho \circ \varphi$. This is well defined since ϱ is invariant with respect to precomposition with elements of \mathcal{G}_∞ , that is, $\varrho(Az) = \varrho(z)$ for all $A \in \mathcal{G}_\infty$ and $z \in \mathbf{H}$.

To complete $\hat{\varphi}_\ell$ to map onto $\hat{\mathcal{O}}$, we define, for any completion point $\mathbf{z} = \mathbf{M}z$ in ${}^\circ T \subset {}^\circ \mathcal{O}$ a map $\varphi_{\mathbf{M}} : D \rightarrow D_\delta(\mathbf{z})$ by $w \mapsto \mathbf{M}\varphi_I(w)$. The collection of $\varphi_{\mathbf{z}}$ descend via postcomposition with ϱ to a collection of well-defined conformal homeomorphisms onto the disks comprising $\hat{\mathcal{O}}$ centered at the $\rho(\mathbf{z})$. In this way, we complete $\hat{\varphi}_\ell$ to a homeomorphism $\hat{\varphi} : D \times \hat{T} \rightarrow \hat{\mathcal{O}}$. This concludes our definition of the chart centered at $\hat{z} \in \ell$. (If \mathbf{z} is a completion point, one simply finds a flowbox centered at some $z \in \mathbf{H}$ containing \mathbf{z} , and uses the $\hat{\varphi}$ defined above.)

Lemma 3 *Every non-empty overlap $\hat{\varphi}^{-1} \circ \hat{\varphi}'$ is Riemann surface laminar i.e. holomorphic in the leaf direction, converging uniformly on compacta in the transverse direction.*

Proof: By definition, $\hat{\varphi}^{-1} \circ \hat{\varphi}'$ converges pointwise in the transverse direction, and is holomorphic when restricted to $(\hat{\varphi}')^{-1}(\hat{\mathcal{O}}_\ell \cap \hat{\mathcal{O}}'_\ell)$. Thus, to see that $\hat{\varphi}^{-1} \circ \hat{\varphi}'$ is holomorphic on all of $\hat{\varphi}^{-1}(\hat{\mathcal{O}} \cap \hat{\mathcal{O}}')$, it suffices to demonstrate transversal

equicontinuity (on compacta). This follows from the fact that \mathcal{G} is normal: let $z \in {}^\circ\mathcal{O} \cap {}^\circ\mathcal{O}'$ where, say, $z = \mathbf{M}z$ in ${}^\circ\mathcal{O}$ and $z = \mathbf{M}'z'$ in ${}^\circ\mathcal{O}'$, $\mathbf{M} \in {}^\circ\mathcal{H}$ and $\mathbf{M}' \in {}^\circ\mathcal{H}'$. Then $Orb(\mathcal{H}) \cap Orb(\mathcal{H}') \neq \emptyset$ which implies by normality that $\mathcal{H} \sim_o \mathcal{H}'$, so there exists $M \in PSL(2, \mathbf{R})$ such that $[\mathcal{H}] = [\mathcal{H}'M]$. Hence, for any index i and $\epsilon > 0$, and j large enough, we have $\mathcal{H}_j \subset_\epsilon \mathcal{H}'_i M$. Since \mathcal{G} acts properly discontinuously, if ϵ is chosen small enough, it follows that for each $A \in \mathcal{H}_j$, there exists a unique $A' \in \mathcal{H}'_i$ such that $A \in U_\epsilon(A'M)$, the ϵ -open about A' defined by the uniform structure on $PSL(2, \mathbf{R})$ (see §2-5). Writing $A' = s(A)$, we see that the family $\{A^{-1} \circ s(A) \mid A \in \mathcal{H}_j\}$ is equicontinuous on compact subsets of \mathbf{H} , hence $\hat{\varphi}^{-1} \circ \hat{\varphi}'$ is equicontinuous when restricted to the plaques of $D \times T'$ containing $(\hat{\varphi}')^{-1}(M'_i z')$. It follows that $\hat{\varphi}^{-1} \circ \hat{\varphi}'$ converges on compacta in the transverse direction. By Weierstrass' Theorem, $\hat{\varphi}^{-1} \circ \hat{\varphi}'$ is holomorphic when restricted to any plaque of $(\hat{\varphi}')^{-1}(\hat{\mathcal{O}} \cap \hat{\mathcal{O}}')$. \square

Theorem 4 *$L_{\mathcal{G}}$ has the structure of an irreducible hyperbolic Riemann surface lamination.*

Proof: $L_{\mathcal{G}}$ is separable and metrizable by Proposition 21. The flowboxes constructed above clearly respect leaves, hence property 1) of Definition 1 is satisfied; by Lemma 3, property 2) of Definition 1 is satisfied. Now take any

maximal atlas containing the atlas described above. \square

Our next concern will be the dependence of $L_{\mathcal{G}}$ on the germ $[\mathcal{G}]$. As might be expected, we need a notion of equivalence which is more discriminating than germ equivalence, that takes into account the behavior of \mathcal{G} with respect to the process of completion.

Definition 39 *Two Fuchsian nested sets \mathcal{G} and \mathcal{G}' are called uniformly germ equivalent, written $\mathcal{G} \sim_u \mathcal{G}'$, if $\mathcal{G} \sim \mathcal{G}'$ and if for every regular convergor \mathcal{H} in \mathcal{G} there exists a regular convergor \mathcal{H}' in \mathcal{G}' such that $\mathcal{H} \sim_o \mathcal{H}'$ and vice versa. We denote the associated class $[\mathcal{G}]^u$.*

Proposition 22 *Let \mathcal{G} be a Fuchsian nested set. Then the associated Riemann surface lamination $L_{\mathcal{G}}$ depends only on $[\mathcal{G}]^u$.*

Proof: Clearly we have an equality $\overline{\mathbf{H}} = \overline{\mathbf{H}'}$ between the respective completions of \mathbf{H} and $\mathbf{z} \sim_{\mathcal{G}} \mathbf{w}$ if and only if $\mathbf{z} \sim_{\mathcal{G}'} \mathbf{w}$. Thus $L_{\mathcal{G}} = L_{\mathcal{G}'}$ as point sets; the leaf structures are identical since the orbit equivalent correspondence between respective convergors implies that a leaf with respect to \mathcal{G} is also a leaf with respect to \mathcal{G}' , and vice versa. The corresponding Riemann surface lamination structures are compatible, once again, since the relations

$\mathcal{H} \sim_o \mathcal{H}'$ insure that corresponding flowboxes overlap normally in the sense of clause 2) of Definition 1. \square

4.5 Uniformization via Fuchsian Nested Sets

Let L be an irreducible hyperbolic Riemann surface lamination. We fix the following data:

- A point $\hat{z} \in L$.
- A flowbox transversal \hat{T} containing \hat{z} , and a *regular* nested neighborhood basis $\{\hat{T}^i\}$ in \hat{T} about \hat{z} i.e. one for which the closure of \hat{T}^j is contained in \hat{T}^i whenever $i < j$.
- A continuous section

$$V : \hat{T} \rightarrow T_*^1 L|_{\hat{T}},$$

where $T_*^1 L|_{\hat{T}}$ is the restriction of the leafwise unit tangent bundle of L to \hat{T} .

Let $\ell = \ell_{\hat{z}}$ be the leaf containing \hat{z} and $p : (\mathbf{H}, z_0) \rightarrow (\ell, \hat{z})$ the universal cover map (which is uniquely determined by $V(\hat{z})$). The universal cover map

p in turn determines a Fuchsian group which acts on \mathbf{H} with quotient ℓ . Call this Fuchsian group \mathcal{G}_∞ .

Given $\hat{w} \in \hat{T} \cap \ell$, we may lift $V(\hat{w})$ to vectors V_{w_k} lying in the tangent planes $T_{w_k} \mathbf{H}$, where $w_k \in p^{-1}(\hat{w})$ (this because p is a covering). In particular, each $w_k \in p^{-1}(\hat{w})$ determines an element $A_{w_k} \in PSL(2, \mathbf{R})$, characterized by the property that

$$A_{w_k} z_0 = w_k \quad \text{and} \quad (A_{w_k})_* V_{z_0} = V_{w_k}.$$

Notice that if $w_j, w_k \in p^{-1}(\hat{w})$, then A_{w_j} differs on the left from A_{w_k} by an element of \mathcal{G}_∞ . We define a nested set \mathcal{G} by

$$\mathcal{G} = \{A \in PSL(2, \mathbf{R}) \mid A = A_{w_k} \text{ for some } w_k \in p^{-1}(\hat{w}) \text{ and some } \hat{w} \in \hat{T} \cap \ell\}$$

with

$$\mathcal{G}_i = \{A = A_w \in \mathcal{G} \mid p(w) \in \hat{T}^i\}.$$

We set out to show that \mathcal{G} is Fuchsian and that $L_{\mathcal{G}} \cong L$.

Proposition 23 *\mathcal{G} is a flow nested set.*

Proof: By construction, $\bigcap \mathcal{G}_i = \mathcal{G}_\infty$, which is a Fuchsian group. \mathcal{G} acts properly discontinuously on \mathbf{H} since \hat{T} indexes a flow box in L (e.g. there

exists an $\epsilon > 0$ such that $d_{\mathbf{H}}(w, w') > \epsilon$ uniformly for all $w, w' \in p^{-1}(\hat{T} \cap \ell)$. That \mathcal{G}_∞ acts on the left of \mathcal{G} as a nested set follows from the construction of \mathcal{G} . \square

Given $A \in \mathcal{G}$, let $|A| = \sup_{A \in \mathcal{G}_i} i$.

Proposition 24 \mathcal{G} is self-similar.

Let $T^i = \bigcup_{A \in \mathcal{G}_i} Az_0$ be the pre \mathcal{G}_i -orbit through z_0 , and let $w \in T^i$. The geodesic η in \mathbf{H} with $\eta(0) = z_0$, $\eta(1) = w$ projects to a (geodesic) path $\hat{\eta}$ on ℓ in which $\hat{\eta}(0) = \hat{z}$ and $\hat{\eta}(1) = \hat{w}$, where $p(w) = \hat{w}$.

From the general theory of laminated spaces, one knows that for some natural number j , $\hat{\eta}$ induces a homeomorphism

$$\eta_T : \hat{T}^j \longrightarrow \hat{N} \subset \hat{T},$$

where \hat{N} is some open neighborhood of \hat{w} in \hat{T} . We will interpret this homeomorphism to show that \mathcal{G} is self-similar.

Specifically, $\hat{\eta}_T$ is obtained by covering $\hat{\eta}$ by a finite collection of flowboxes \hat{O}_k , $k = 1, \dots, n$, then sliding some small enough neighborhood \hat{T}^j of \hat{z} along $\hat{\eta}$, through each flow box, until one arrives at \hat{N} .

Let $\frac{d}{dt}\eta(0) = \vec{v}$. For each $A \in \mathcal{G}$, let η_A be the unique geodesic in \mathbf{H} prescribed by the initial conditions

$$\begin{aligned}\eta_A(0) &= Az_0 \\ \frac{d}{dt}\eta_A(0) &= A_*\vec{v}.\end{aligned}$$

Notice that $A \cdot \eta = \eta_A$ and in particular $\eta_A(1) = Aw$. As before, let $\hat{\eta}_A$ denote the projection of η_A to ℓ . Observe that the family of vectors $\frac{d}{dt}\hat{\eta}_A(0) = p_*A_*\vec{v}$ is a continuous vector field on \hat{T} , which converges to $p_*\vec{v}$ as $|A| \rightarrow \infty$. Therefore, in each flowbox \hat{O}_k , we have that the \mathcal{G} family of paths $\hat{\eta}_A(t)$ converges uniformly on compacta in the t variable as $|A| \rightarrow \infty$. In particular, the convergence is true in the final flowbox. This implies that \mathcal{G} is self-similar. \square

The quotient $L_{\mathcal{G}}$ of $\overline{\mathbf{H}}$ by \mathcal{G} is in bijection with L , since $\sim_{\mathcal{G}}$ identifies precisely those convergent sequences identified by the topology of L . Note that the chart φ about \hat{z} , defined as in the previous section by taking any conformal map $\varphi_I : D_{\delta}(0) \rightarrow D_{\delta}(\hat{z})$ and translating by \mathcal{G} , is compatible with the atlas defining the lamination structure of L , since \mathcal{G} was defined using the continuous vector field V . Thus, *a fortiori*, \mathcal{G} is normal.

Since the neighborhood basis $\{\hat{T}^i\}$ about \hat{z} is regular, and since \hat{T} is itself a regular topological space, it follows that any point in an open set $\hat{U} \subset \hat{T}$ can be converged by a regular convergor contained in $\hat{U} \cap \ell$. Thus \mathcal{G} is regular (and so preFuchsian).

Since the \mathcal{G}^M are just the nested sets defined by restricting \hat{T} and V to the various other leaves of L , \mathcal{G} is uniform, and unified since the continuity of V implies all of their topologies are mutually compatible. We have thus proven the

Theorem 5 *For any irreducible, hyperbolic Riemann surface lamination L , there exists a Fuchsian directed set \mathcal{G} such that $L_{\mathcal{G}}$ is isomorphic to L . \square*

Chapter 5

The Fuchsian Germ

In this chapter, we manufacture a groupoid $[\Gamma]$ from a given Fuchsian nested set \mathcal{G} , called the *Fuchsian germ*, generalizing the construction in Chapter 3; and show that the Fuchsian germ is functorial with respect to finite covering maps. Next, we consider the *lockstep* condition: when a Fuchsian nested set satisfies this condition, we may reduce our construction to the simpler version appearing in Chapter 3. We end the chapter by calculating the Fuchsian germ of the *Feigenbaum solenoid*, a solenoid which arises in the study of the dynamics of quadratic mappings of \mathbf{C} .

5.1 Ultrafilters

In order to obtain functoriality of the Fuchsian groupoid with respect to finite covers of surface laminations, it will be necessary to amplify the direct product indexing schemes which figure in the definition of the group germ of §3-1. We do this using the notion of an ultrafilter. A more detailed discussion of this concept can be found in [R].

Let \mathcal{A} be an infinite set.

Definition 40 *A filter on \mathcal{A} is a subset $\mathcal{F} \subset 2^{\mathcal{A}}$ satisfying the following conditions:*

- $\mathcal{A} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
- If $X \in \mathcal{F}$ and $Y \supset X$, then $Y \in \mathcal{F}$.
- If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.

*A filter \mathcal{F} is called **principle** if it contains a finite set and is called an **ultrafilter** if it is maximal i.e. properly contained in no other filter.*

Note that any filter \mathcal{F} may be regarded as a directed set with respect to the ordering $X \leq Y$ if and only if $X \supseteq Y$.

From here on, all filters discussed are assumed to be *non-principle*.

Proposition 25 *Let \mathcal{U} be an ultrafilter on an infinite set \mathcal{A} . Then for any $X \in 2^{\mathcal{A}}$, either $X \in \mathcal{U}$ or $\mathcal{A} \setminus X \in \mathcal{U}$.*

Proof: Suppose that $X \in 2^{\mathcal{A}}$ and $X \notin \mathcal{U}$. Then $\mathcal{A} \setminus X$ intersects every element $Y \in \mathcal{U}$ non-trivially (otherwise $Y \subset X$ which would imply that $X \in \mathcal{U}$). Form a new filter by adding to \mathcal{U} $\mathcal{A} \setminus X$, as well as all intersections of $\mathcal{A} \setminus X$ with elements of \mathcal{U} and all elements of $2^{\mathcal{A}}$ containing $\mathcal{A} \setminus X$. The resulting filter properly contains \mathcal{U} if $\mathcal{A} \setminus X$ is not in \mathcal{U} , hence the contrary must be the case. \square

Corollary 1 *Let \mathcal{U} be an ultrafilter on an infinite set \mathcal{A} . If $\mathcal{A} = \bigcup_{i=1}^n U_i$ a disjoint union of subsets, then $U_i \in \mathcal{U}$ for some index i .*

Proof: Suppose $X_i \notin \mathcal{U}$ for all i . Then $(\mathcal{A} \setminus X_i) \in \mathcal{U}$ for all i , so that $\bigcap (\mathcal{A} \setminus X_i) = \emptyset \in \mathcal{U}$, contradiction. \square .

Now suppose that \mathcal{S} is any category in which the direct product construction is defined (*e.g.* the categories of topological spaces and of groups). Let $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of objects in \mathcal{S} indexed by \mathcal{A} . Then given a filter \mathcal{F} , we may form a direct system $F : \mathcal{F} \rightarrow \mathcal{S}$, where

$$F(X) = \prod_{\alpha \in X} S_\alpha = S_X$$

and $F(\alpha \rightarrow \beta)$ is the associated Cartesian projection.

Definition 41 *The reduced product of the $\{S_\alpha\}$ by \mathcal{F} is*

$$\prod S_\alpha / \mathcal{F} = \lim_{\mathcal{F}} S_X.$$

*If \mathcal{F} is an ultrafilter, then $\prod S_\alpha / \mathcal{F}$ is called an **ultraproduct**. If $S_\alpha = S$ for all α , we call the ultraproduct an **ultrapower** and denote it by *S .*

Example 8 Let \mathcal{U} be an ultrafilter on \mathbb{N} . Then the associated ultrapower of \mathbb{R} , ${}^*\mathbb{R}$, has the structure of a totally ordered, non-Archimedean field $[\mathbb{R}]$; it is called a **nonstandard \mathbb{R}** .

N.B. There are many ultrafilters on a given infinite set \mathcal{A} . It can be shown, for example, that the different nonstandard \mathbb{R} s are all isomorphic, but only if one assumes the Continuum Hypothesis $[\mathbb{R}]$.

5.2 The General Construction

Let \mathcal{G} be a Fuchsian nested set, $\overline{\mathbf{H}}$ the associated completion of \mathbf{H} . Fix an ultrafilter \mathcal{U} on \mathbb{N} .

Let $\mathbf{z} = \{z_i\}$ be an actual convergent sequence corresponding to some \asymp -class in $\overline{\mathbf{H}}$. Given $X = \{n_1, n_2, \dots\} \in \mathcal{U}$, let $P_X(\mathbf{z}) = \{z_{n_i}\}$, which is also

a convergent sequence. Note that if $\mathbf{z} \asymp \mathbf{w}$, then $P_X(\mathbf{z}) \asymp P_X(\mathbf{w})$. Thus P_X descends to a well-defined map $P_X : \overline{\mathbf{H}} \rightarrow \overline{\mathbf{H}}$. It is easy to see that P_X is surjective and takes leaves to leaves. What is more, the collection $\{P_X\}_{X \in \mathcal{U}}$ forms a direct system.

Definition 42 *The germ of $\overline{\mathbf{H}}$ with respect to \mathcal{U} is*

$$[\overline{\mathbf{H}}] = [\overline{\mathbf{H}}]_{\mathcal{U}} = \varinjlim_{P_X, X \in \mathcal{U}} \overline{\mathbf{H}}.$$

We will denote elements of $[\overline{\mathbf{H}}]$ by $[\mathbf{z}]$ or $[z_i]$. When we write $\mathbf{z} \in [\mathbf{z}]$, we mean that \mathbf{z} is an element of $\overline{\mathbf{H}}$ that maps onto $[\mathbf{z}]$ via the natural surjection $\overline{\mathbf{H}} \rightarrow [\overline{\mathbf{H}}]$. If $\mathbf{z}, \mathbf{z}' \in [\mathbf{z}]$ then the leaves $\ell_{\mathbf{z}}$ and $\ell_{\mathbf{z}'}$ are identified isometrically in $[\overline{\mathbf{H}}]$. Thus one also regards $[\overline{\mathbf{H}}]$ as a family of upper-half planes; we write $\ell_{[\mathbf{z}]}$ for the leaf through $[\mathbf{z}]$.

Observe that the relation $\sim_{\mathcal{G}}$ descends to $[\overline{\mathbf{H}}]$. Indeed, if $\mathbf{z} \sim_{\mathcal{G}} \mathbf{w}$ then $\mathbf{z} \sim_{\mathcal{G}} \mathbf{w}'$ for all $\mathbf{w}' \in [\mathbf{w}]$. Thus we can write $[\mathbf{z}] \sim_{\mathcal{G}} [\mathbf{w}]$. In particular, there

is a commutative diagram

$$\begin{array}{ccc}
 \mathbf{H} & & \\
 & \searrow & \\
 \sim_{\mathcal{G}} \downarrow & & [\mathbf{H}] \\
 & \swarrow \sim_{\mathcal{G}} & \\
 L_{\mathcal{G}} & &
 \end{array}$$

We can thus regard $L_{\mathcal{G}}$ as the quotient of the germ $[\mathbf{H}]$ by $\sim_{\mathcal{G}}$; as in §4-4, we will write $\varrho : [\mathbf{H}] \rightarrow L_{\mathcal{G}}$ for the projection.

In like fashion, for each $X \in \mathcal{U}$, we can also define maps $P_X : {}^{\circ}\mathcal{G} \rightarrow {}^{\circ}\mathcal{G}$, where ${}^{\circ}\mathcal{G}$ is the set of convergent sequences in \mathcal{G} that converge via \mathcal{G} (regarded as a convergor).

Let ${}^{\circ}\mathcal{G} \cdot {}^{\circ}\mathcal{G}^{-1}$ be the set of coordinate-wise products of ${}^{\circ}\mathcal{G}$ with ${}^{\circ}\mathcal{G}^{-1}$. In other words, elements of ${}^{\circ}\mathcal{G} \cdot {}^{\circ}\mathcal{G}^{-1}$ are \asymp -classes of sequences of the form $\mathbf{EF}^{-1} = \{E_i F_i^{-1}\}$, where $\mathbf{E}, \mathbf{F} \in {}^{\circ}\mathcal{G}$. We may extend P_X to ${}^{\circ}\mathcal{G} \cdot {}^{\circ}\mathcal{G}^{-1}$ via the instruction $P_X(\mathbf{EF}^{-1}) = P_X(\mathbf{E})P_X(\mathbf{F})^{-1}$. Once again, we obtain a direct system.

Definition 43 *The Fuchsian germ associated to \mathcal{G} is*

$$[\Gamma] = [\Gamma]_{\mathcal{U}} = \varinjlim_{P_X X \in \mathcal{U}} ({}^{\circ}\mathcal{G} \cdot {}^{\circ}\mathcal{G}^{-1}).$$

We will denote elements of $[\Gamma]$ by $[\mathbf{B}]$ or $[B_i]$; if we wish to stress the origin in \mathcal{G} we will write $[\mathbf{E}\mathbf{F}^{-1}]$ or $[E_i F_i^{-1}]$. The notation $\mathbf{B} \in [\mathbf{B}]$ means that there exists $\mathbf{B} \in {}^{\circ}\mathcal{G} \cdot {}^{\circ}\mathcal{G}^{-1}$ such that \mathbf{B} projects onto $[\mathbf{B}]$ via the natural projection onto $[\Gamma]$.

Recall that a *groupoid* is a category in which every morphism has an inverse. In order to regard $[\Gamma]$ as a groupoid, we must first define the domain and range of each element $[\mathbf{B}] \in [\Gamma]$.

Definition 44 Given $[\mathbf{B}] \in [\Gamma]$ and $[z] \in [\overline{\mathbf{H}}]$, we say that $[\mathbf{B}]$ acts on $[z]$ if there exists $\mathbf{B} \in [\mathbf{B}]$ and $\mathbf{z} \in [z]$ such that the sequence $\mathbf{B}\mathbf{z} = \{B_i z_i\}$ is convergent and $\sim_{\mathcal{G}}$ to \mathbf{z} . We will denote its class in $[\Gamma]$ by $[\mathbf{B}][z]$. The sets $[\overline{\mathbf{H}}]_{[\mathbf{B}]} = \{[z] \mid [\mathbf{B}][z] \text{ is defined}\}$ and $[\mathbf{B}][\overline{\mathbf{H}}]_{[\mathbf{B}]}$ are called, respectively, the domain and range of $[\mathbf{B}]$.

Note: The defining condition in Definition 44 is equivalent to requiring that for every $\mathbf{z} \in [z]$, there exists $\mathbf{B} \in [\mathbf{B}]$ such that $\mathbf{B}\mathbf{z} \sim_{\mathcal{G}} \mathbf{z}$. Also, observe that $[\overline{\mathbf{H}}]_{[\mathbf{B}]}$ has a leaf structure inherited from $[\overline{\mathbf{H}}]$: $[z] \in [\overline{\mathbf{H}}]_{[\mathbf{B}]}$ if and only if $\ell_{[z]} \subset [\overline{\mathbf{H}}]_{[\mathbf{B}]}$.

Proposition 26 $[z] \sim_{\mathcal{G}} [w]$ if and only if there exists $[\mathbf{B}] \in [\Gamma]$ such that $[w] = [\mathbf{B}][z]$.

Proof: Indeed, if $z = Mz$ and $w = M'z$ converge via a common convergor with center sequence $\{A_i\}$, then for each i there exists $E_i, F_i \in \mathcal{G}_i$ such that $\{E_i A_i z\} \asymp \{M_i z\}$ and $\{F_i A_i z\} \asymp \{M'_i z\}$, whereby $\{E_i F_i^{-1}\}$ yields the required element $[\mathbf{B}]$ of $[\Gamma]$. \square

Thus we may alternatively regard $L_{\mathcal{G}}$ as the quotient $[\overline{\mathbf{H}}]/[\Gamma]$.

Definition 45 We say that the composition $[\mathbf{B}][\mathbf{C}]$ is defined if $[\overline{\mathbf{H}}]_{[\mathbf{B}]} = [\mathbf{C}][\overline{\mathbf{H}}]_{[\mathbf{C}]}$ and there exists $[\mathbf{D}] \in [\Gamma]$ such that $[\overline{\mathbf{H}}]_{[\mathbf{C}]} = [\overline{\mathbf{H}}]_{[\mathbf{D}]}$ and $[\mathbf{B}](([\mathbf{C}][z]) = [\mathbf{D}][z]$ for all $[z] \in [\overline{\mathbf{H}}]_{[\mathbf{D}]}$.

With these definitions, it is clear that $[\Gamma]$ is a category and we may further state the

Theorem 6 $[\Gamma]$ is a groupoid.

Proof: The inverse of $[\mathbf{B}] = [\mathbf{E}\mathbf{F}^{-1}]$ is $[\mathbf{F}\mathbf{E}^{-1}]$, which is in $[\Gamma]$ by definition.

\square

We now discuss the functoriality of $[\Gamma]$ with respect to finite covering maps of surface laminations.

Definition 46 Let L, L' be two irreducible surface laminations. A covering $f : L \rightarrow L'$ is a continuous, leaf-preserving map in which every point $\tilde{z}' \in L'$

is contained in a neighborhood $\hat{\mathcal{O}}'$ evenly covered by f . The covering is called **finite** if for all $\hat{z}' \in L'$, $\hat{\mathcal{O}}'$ may be selected so that f^{-1} may be written as a disjoint union of a finite number of open sets, each of which map by f homeomorphically onto $\hat{\mathcal{O}}'$.

Recall that a groupoid homomorphism is simply a functor.

Theorem 7 *Let (L, ℓ) , (L', ℓ') be irreducible, hyperbolic Riemann surface laminations (with “base” leaves ℓ and ℓ'), uniformized by Fuchsian germs $[\Gamma]$ and $[\Gamma']$, respectively. A finite covering map $f : (L, \ell) \rightarrow (L', \ell')$ induces a groupoid homomorphism $f_* : [\Gamma] \rightarrow [\Gamma']$.*

Proof: Let $\tilde{f} : \mathbf{H} \rightarrow \mathbf{H}$ be the lift of the restriction of f to ℓ , which is a homeomorphism since f is a covering. \tilde{f} in turn extends to a map $[\tilde{f}] : [\overline{\mathbf{H}}] \rightarrow [\overline{\mathbf{H}}']$ of respective completions, via the instruction $[\tilde{f}](\mathbf{z}) = [\tilde{f}(z_i)]$, where $\{z_i\}$ is any sequence in the class of \mathbf{z} .

We first show that $[\tilde{f}]$ is surjective. Given $\{w'_i\} \in [\mathbf{w}'] \in [\overline{\mathbf{H}}']$, consider the sequence $\{z_i\} = \{\tilde{f}^{-1}(w'_i)\}$. Let $\hat{\mathcal{O}}'$ be a neighborhood of $\varrho'([\mathbf{w}']) = \hat{\mathbf{w}}$ in L' which is evenly covered by neighborhoods $\hat{\mathcal{O}}_1, \dots, \hat{\mathcal{O}}_r$ in L . We may assume that for all i , $\varrho'(w'_i) \in \hat{\mathcal{O}}'$, so it follows then that

$$\{\varrho(z_i)\} \subset \bigcup_{n=1}^r \hat{\mathcal{O}}_n.$$

Let $X_1 = \{n_1, n_2, \dots\} \subset 2^{\mathbf{N}}$ be such that $\varrho(z_{n_i}) \in \hat{\mathcal{O}}_1$ for all i . Define X_j in like fashion, $j = 1, \dots, r$. Then there exists j such that $X_j \in \mathcal{U}$ by Corollary 1. In particular, the sequence $\{\varrho(z_i)\}_{i \in X}$, must be a convergent sequence in L since f is a homeomorphism when restricted to $\hat{\mathcal{O}}_j$ and $\{f(\varrho(z_i))\}$ is convergent in L' . It follows that $\{z_i\}$ defines an element $\mathbf{z} \in [\overline{\mathbf{H}}]$ such that $[\overline{f}](\mathbf{z}) = [\mathbf{w}']$. Thus, $[\overline{f}]$ is surjective.

Now for each $[\mathbf{B}] \in [\Gamma]$, we define $f_*([\mathbf{B}])$ to be the element of $[\Gamma']$ asymptotic to $\tilde{f}[\mathbf{B}]\tilde{f}^{-1}$. Specifically, let $\{B_i\} \in [\mathbf{B}]$, where $B_i = E_i F_i^{-1}$. For any $z \in \mathbf{H}$, the sequence $\{\tilde{f}E_i\tilde{f}^{-1}(\tilde{f}z)\} = \{\tilde{f}(E_i z)\}$ projects to a sequence converging in L' to the projection of $\tilde{f}(z)$ to L' . Hence $\{\tilde{f}E_i\tilde{f}^{-1}\}$ determines a convergent sequence $\{E'_i\} \in {}^{\circ}\mathcal{G}'$. The same is true of $\{\tilde{f}F_i^{-1}\tilde{f}^{-1}\}$, and so it follows that the sequence $\{\tilde{f}B_i\tilde{f}^{-1}\}$ determines an element $f_*([\mathbf{B}])$ of $[\Gamma']$ which is independent of the choice of $\{B_i\} \in [\mathbf{B}]$.

We must show that f_* is homomorphic. Suppose that $[\mathbf{B}][\mathbf{C}]$ is defined in $[\Gamma]$ and is equal to $[\mathbf{D}]$. We must show that $f_*([\mathbf{B}])f_*([\mathbf{C}])$ is defined and equal to $f_*([\mathbf{D}])$. Notice first that

$$[\overline{\mathbf{H}}]_{f_*([\mathbf{B}])} \supset \overline{f}([\overline{\mathbf{H}}]_{[\mathbf{B}]})$$

Indeed, $f_*([\mathbf{B}])\overline{f}(\mathbf{z}) = \overline{f}([\mathbf{B}]\mathbf{z})$ for all $\mathbf{z} \in [\overline{\mathbf{H}}]_{[\mathbf{B}]}$; and $\overline{f}([\mathbf{B}]\mathbf{z}) \sim_{\mathcal{G}'}$ $\overline{f}(\mathbf{z})$.

On the otherhand, suppose $[\mathbf{w}'] \in [\overline{\mathbf{H}}']_{f_*([\mathbf{B}])}$ and write $[\mathbf{w}'] = \overline{f}([\mathbf{z}])$ for some $[\mathbf{z}] \in [\overline{\mathbf{H}}]$. Then $f_*[\mathbf{B}][\mathbf{w}']$ is defined *i.e.* there exists a sequence $\{B_i\} \in [\mathbf{B}]$ and $\{z_i\} \in [\mathbf{z}]$ such that $\{\tilde{f}(B_i z_i)\}$ is convergent and $\sim_{\mathcal{G}'}$ to $[\mathbf{w}']$. The ultrafilter argument used to show that $[\overline{f}]$ is surjective may be used to show that the sequence $\{B_i z_i\}$ is convergent. Hence $[\mathbf{z}] \in [\overline{\mathbf{H}}]_{[\mathbf{B}]}$ and

$$[\overline{\mathbf{H}}']_{f_*([\mathbf{B}])} \subset \overline{f}([\overline{\mathbf{H}}]_{[\mathbf{B}]}).$$

By a similar argument, we have

$$f_*([\mathbf{C}])[\overline{\mathbf{H}}']_{f_*([\mathbf{C}])} = \overline{f}([\mathbf{C}][\overline{\mathbf{H}}]_{[\mathbf{C}]}).$$

It follows then that the composition $f_*([\mathbf{B}])f_*([\mathbf{C}])$ is defined and equal to $f_*([\mathbf{D}])$. \square

5.3 The Lockstep Construction

Let \mathcal{G} be a Fuchsian nested set, \mathcal{U} an ultrafilter on \mathbf{N} . If \mathcal{G} satisfies certain additional properties, we may utilize the simpler construction found in Chapter 3 to uniformize $L_{\mathcal{G}}$.

$$\text{Let } \Gamma_i = \mathcal{G}_i \cdot \mathcal{G}_i^{-1}.$$

Definition 47 *The lockstep Fuchsian germ associated to \mathcal{G} is the ultra-product*

$$[\Gamma_\star] = \prod \Gamma_i / \mathcal{U}.$$

Notice that there exists a natural inclusion $[\Gamma_\star] \rightarrow [\Gamma]$ (as sets; we have not yet imposed a groupoid structure on $[\Gamma_\star]$).

Definition 48 *A convergor \mathcal{H} is called lockstep if the signature of \mathcal{H} is $\{1, 2, \dots\}$. A sequence $\mathbf{M} \in \overline{\mathcal{G}}$ is called lockstep if there exists a lockstep regular convergor \mathcal{H} with $\mathbf{M} \in {}^\circ\mathcal{H}$, and $\{M_i\}$ in the \asymp -class of \mathbf{M} such that $M_i \in \mathcal{H}_i$ for all i . The lockstep completion of \mathbf{H} is the set*

$$\overline{\mathbf{H}}_\star = \{z \in \overline{\mathbf{H}} \mid z = \mathbf{M}z, \mathbf{M} \text{ lockstep}\}.$$

Notice that the maps P_X of the previous section fix $\overline{\mathbf{H}}_\star$ set-wise. We again obtain a direct system and have the

Definition 49 *The germ of $\overline{\mathbf{H}}_\star$ is*

$$[\overline{\mathbf{H}}_\star] = \varinjlim \overline{\mathbf{H}}_\star.$$

Following the prescription of the previous section, we may define a groupoid structure on $[\Gamma]_\star$ relative to $[\overline{\mathbf{H}}_\star]$ e.g. defining domains $[\overline{\mathbf{H}}_\star]_{[\mathbf{B}]} \subset [\overline{\mathbf{H}}_\star]$ for each $[\mathbf{B}] \in [\overline{\mathbf{H}}_\star]$.

Observe that for any Fuchsian nested set, the properties of being regular and of being normal still make sense and hold if we replace the completion $\overline{\mathbf{H}}$ by $\overline{\mathbf{H}}_*$. We can give $\overline{\mathbf{H}}_*$ the subspace topology *e.g.* we may define the open flowboxes to be of the form ${}^\circ\mathcal{O}_* = {}^\circ\mathcal{O} \cap \overline{\mathbf{H}}_*$, for ${}^\circ\mathcal{O}$ an open flowbox of $\overline{\mathbf{H}}$.

Notice now that the inclusion $i : [\Gamma_*] \rightarrow [\Gamma]$ extends to objects via the inclusions $[\overline{\mathbf{H}}_*]_{[\mathbf{B}]} \mapsto [\overline{\mathbf{H}}]_{[\mathbf{B}]}$.

Definition 50 *A Fuchsian nested set \mathcal{G} is lockstep if the inclusion i is an injective groupoid homomorphism which induces an isomorphism of Riemann surface laminations*

$$\hat{i} : [\overline{\mathbf{H}}_*]_{[\Gamma_*]} \longrightarrow [\overline{\mathbf{H}}]_{[\Gamma]}.$$

Note: If \mathcal{G} is lockstep, then the fact that \hat{i} is injective means that $\sim_{\mathcal{G}}$ restricts to an equivalence relation on $\overline{\mathbf{H}}_*$.

Example 9 If $\widehat{\mathcal{S}}$ is an inverse limit Riemann surface solenoid, then $\widehat{\mathcal{S}}$ may be uniformized by a lockstep Fuchsian nested set \mathcal{G} : indeed, if the defining limit is holomorphic, then this is certainly true. If not, then $\widehat{\mathcal{S}}$ is homeomorphic to $\widehat{\mathcal{S}}_0$ a solenoid occurring as a holomorphic limit, and the induced isomorphism of germs (Theorem 7) assures that $\widehat{\mathcal{S}}$ is also lockstep.

Now it is not the case that any finite covering map $f : (L, \ell) \rightarrow (L', \ell')$ induces a *lockstep* homomorphism $f_* : [\Gamma_*] \rightarrow [\Gamma'_*]$. Nevertheless, there is a covering situation in which the lockstep construction pushes forward.

Theorem 8 *Let (L, ℓ) be an irreducible hyperbolic Riemann surface lamination uniformized by a lockstep Fuchsian nested set \mathcal{G} . Suppose that G is a group of isometries of L that acts properly discontinuously and fixes ℓ set-wise, such that $G|_\ell \cong G$ and is fixed point free. Then $(L', \ell') = (L/G, \ell/G)$ may be uniformized by $\mathcal{G}' = G\mathcal{G} = \{G_i\mathcal{G}_i\}$, where G_ℓ is the lift of $G|_\ell$ to \mathbf{H} . Furthermore, \mathcal{G}' is also lockstep, and $[\Gamma'_*] = {}^*G[\Gamma_*]{}^*G$ (where *G is the ultrapower of G).*

Proof: Let $f : L \rightarrow L'$ denote the projection. Suppose that \mathcal{G} is obtained via the datum $(\hat{\varepsilon}, \hat{T} = \{\hat{T}^i\}, V)$ per the construction in §4-5. We may assume further that $\hat{\varepsilon}$ and \hat{T} are contained in an open set mapping homeomorphically by f into L' . Thus the uniformizing datum of \mathcal{G} may be pushed down isometrically to L' . It follows that this pushed down datum will yield the Fuchsian nested set $\mathcal{G}' = G_\ell\mathcal{G}$ for L' . Notice that $\mathcal{G}_i \subset \mathcal{G}'_i$ for all i and that $[\Gamma'_*] = {}^*G[\Gamma_*]{}^*G$.

In fact, it is easy to see that $[\overline{\mathbf{H}}'] = {}^*G_\ell[\overline{\mathbf{H}}]$, $[\overline{\mathbf{H}}'_*] = {}^*G_\ell[\overline{\mathbf{H}}_*]$ and that

the map $i' : [\Gamma'_*] \rightarrow [\Gamma']$ is just the extension of i by ${}^*G_\ell$. Thus i' is also an injective groupoid homomorphism. Moreover, since G_ℓ is an isometry group and i is an isomorphism, it follows that i' is an isomorphism. \square

5.4 The Feigenbaum Germ

Consider the Baumslag-Solitar group $G = \langle x, f : fxf^{-1} = x^2 \rangle$ and the nested set

$$\mathcal{G}_i = \langle f \rangle \langle x^{2^i} \rangle,$$

where $\langle y \rangle$ means the cyclic group generated by y . Notice that $\mathcal{G}_\infty = \langle f \rangle$, but that the \mathcal{G}_i are not themselves subgroups of G . Indeed, $f^{-1}x^{2^i} \in \mathcal{G}_i$, yet its inverse $x^{-2^i}f = fx^{-2^{i-1}} \notin \mathcal{G}_i$.

Let $H_i = \mathcal{G}_i \cdot \mathcal{G}_i^{-1}$ and $[G] = \prod H_i / \mathcal{U}$, for some ultrafilter \mathcal{U} on \mathbb{N} .

Lemma 4 H_i is a group for all i .

Proof: Note first that for $r, m \in \mathbb{Z}$, we have the following generalization of the relation $fx = x^2f$:

$$f^m x^{r2^i} = x^{r2^{i+m}} f^m \quad (*).$$

Now every element $g \in H_i$ may be written in the form

$$g = f^m x^{r2^i} f^n,$$

$m, n, r \in \mathbf{Z}$. Thus an element of the form gh^{-1} , $g, h \in H_i$, may be written in the form

$$gh^{-1} = f^m x^{r2^i} f^n x^{s2^i} f^p,$$

$m, n, p, r, s \in \mathbf{Z}$. By (*), we have then that

$$gh^{-1} = \begin{cases} f^m x^{(r+s2^n)2^i} f^{n+p} \in H_i & \text{if } n > 0 \\ f^{m+n} x^{(r2^n+s)2^i} f^p \in H_i & \text{if } n \leq 0 \end{cases} \quad \square$$

If we let $[\langle x \rangle]$ denote the (lockstep) germ associated to $\mathcal{C} = \{\langle x^{2^i} \rangle\}$, then we have the

Corollary 2 $[G]$ is a group, isomorphic to ${}^* \langle f \rangle [\langle x \rangle] {}^* \langle f \rangle$. \square

Definition 51 We call $[G]$ the **Feigenbaum germ**.

We explain now the terminology of Definition 51. Let A be a hyperbolic annulus and $d : A \rightarrow A$ a degree 2 self-covering map. d is not conformal; let A_1 be the hyperbolic annulus obtained by pulling back the hyperbolic structure of A along d . Thus $d : A_1 \rightarrow A$ is holomorphic. Likewise, we let

A_i be the annulus whose hyperbolic structure is pulled back from d_i . In this way we obtain a dyadic inverse system of holomorphic covers $d_i : A_i \rightarrow A$ i.e. $\deg(d_i) = 2^i$. Let $\hat{A} = \varprojlim A_i$.

Let $B \in PSL(2, \mathbf{R})$ uniformize A . Then B^{2^i} uniformizes A_i . We form a Fuchsian nested set $\mathcal{G} = \{\langle B^{2^i} \rangle\}$. Clearly the associated germ $[\Gamma]$ is isomorphic to the germ $[\langle x \rangle]$ considered above.

The map $d : A \rightarrow A$ lifts to a quasiconformal map $D : \mathbf{H} \rightarrow \mathbf{H}$, which satisfies the Baumslag-Solitar relation in conjunction with B : $DB = B^2D$ (since D is the lift of a degree 2 cover of A). It follows that the (non-Fuchsian) nested set $\{\langle D \rangle \langle B^{2^i} \rangle\}$ has germ isomorphic to the Feigenbaum germ $[G]$.

In [S1], [S2], it is shown that there exists a Riemann surface solenoid \hat{A}' and a quasiconformal homeomorphism $h : \hat{A} \rightarrow \hat{A}'$, such that $D' = hDh^{-1}$ is holomorphic and acts properly discontinuously on \hat{A}' .

Definition 52 *The Feigenbaum solenoid is the Riemann surface solenoid $\hat{A}^{\text{Feig}} = \hat{A}' / \langle D' \rangle$.*

Theorem 9 *\hat{A}^{Feig} may be uniformized by a lockstep Fuchsian nested set such that $[\Gamma_*] \cong [G]$.*

Proof: This is an immediate corollary of Theorem 8. Indeed, if \mathcal{G} is the lock-

step Fuchsian nested set for \hat{A} , then $\mathcal{G}' = h\mathcal{G}h^{-1}$ defines a lockstep Fuchsian nested set for \hat{A}' . By Theorem 8, $\mathcal{G}^{\text{Feig}} = \langle D' \rangle \mathcal{G}' = h\langle D \rangle \mathcal{G}h^{-1}$ uniformizes \hat{A}^{Feig} . Since the germ of $\langle D \rangle \mathcal{G}$ is isomorphic to the Feigenbaum germ, the result follows. \square

Chapter 6

Bibliography

- [C] A. Candel, *Uniformization of surface laminations*, Annales Scientifiques de l'Ecole Normale Supérieure, Quatrième Serie **26** (1993), no. 4, 489-516.
- [K] J.L. Kelly, *General Topology*, Van Nostrand-Reinhold Co., New York, 1965.
- [L] S. Lang, *Algebra*, 3rd edition, Addison-Wesley, New York, 1993.
- [M-Sc] C. Moore and C. Schochet, *Global Analysis on Foliated Spaces*, MSRI Publications **9**, Springer-Verlag, New York-Berlin, 1988.
- [R] A. Robinson, *Non-standard Analysis*, North Holland, Amsterdam, 1966.
- [S1] D. Sullivan, *Bounds, quadratic differentials and renormalization conjectures*, in "Mathematics into the Twenty-First Century", Vol. II, AMS, Providence, 1992.
- [S2] D. Sullivan, *Linking the universalities of Milnor-Thurston, Feigenbaum,*

and Ahlfors-Bers, in “Topological Methods in Modern Mathematics”, Publish or Perish, Houston, 1993.