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ON SOME PROBLEMS CONCERNING ANTI-WICK OPERATORS

BY

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

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Abstract

ON SOME PROBLEMS CONCERNING ANTI-WICK OPERATORS

by

Ernest Gilde

Adviser: Professor Stanley Kaplan

We discuss questions of continuity and integrability of vector valued functions used to define operators with anti-Wick symbols. We also give a necessary and sufficient condition for a pseudodifferential operator of a certain class on $\mathbb{R}^n$ to be a trace class operator.

I wish to thank Professor Kaplan for his unfailing patience and help which made this thesis possible.

I also dedicate this thesis to my wife whose idea all this was.

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1. Introduction.

In this thesis we attempt to answer certain questions raised implicitly in the discussion by Shubin [4] of operators with anti-Wick symbol. In Chapter IV Shubin introduces a special class of pseudodifferential operators in $\mathbb{R}^n$ by means of the following definitions:

1. Definition. For $m \in \mathbb{R}$ and $0 < \rho \leq 1$, $\Gamma^m_\rho(\mathbb{R}^N)$ consists of the functions $a \in C^\infty(\mathbb{R}^n)$ satisfying

$$|\partial^\alpha a(z)| \leq c_\alpha(1+|z|)^{m-\rho|\alpha|}$$

for every multi-index α and $z \in \mathbb{R}^N$.

The operator A with symbol $a(x,\xi) \in \Gamma^m_\rho(\mathbb{R}^n \times \mathbb{R}^n)$ then is defined by the following formula:

2. Definition. $Au(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<x-y,\xi>}a(x,\xi)u(y)dyd\xi$ where $x,y,\xi \in \mathbb{R}^n$ so that $(x,\xi) = z$ in Definition 1 with $N = 2n$ and where $<x-y,\xi> = \sum_{j=1}^{n}(x_j-y_j)\xi_j$. We then say that $A \in G^m_\rho(\mathbb{R}^n)$.

Shubin shows that $A: S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$ continuously (p.170). He then proves that such an operator A can be written in a variety of different ways (Theorem 23.2, p.172):

3. Theorem. Let $A \in G^m_\rho$. Then, for any $\tau \in \mathbb{R}$, A can be written uniquely as $Au(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<x-y,\xi>}b_\tau((1-\tau)x+\tau y,\xi)u(y)dyd\xi$ where $b_\tau \in \Gamma^m_\rho(\mathbb{R}^{2n})$.

4. Definition. The function b_τ in Theorem 3 is called the τ symbol of A. $b_{\frac{1}{2}}$ is called the Weyl symbol of A.

As an aid in proving some theorems about pseudodifferential operators, Shubin defines operators with anti-Wick symbol $a \in \Gamma^m_\rho(\mathbb{R}^{2n})$ by the formula

(1) $$A = \int_{\mathbb{R}^{2n}} a(z)P_z dz$$

where P_z is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the subspace generated by the unit vector $\Phi_z \in L^2(\mathbb{R}^n)$ defined by

$$(2) \qquad \Phi_z(y) = \pi^{-\frac{n}{4}} e^{i<y,\xi>} e^{-\frac{|x-y|^2}{2}}, \ z = (x,\xi)$$

Any pseudodifferential operator can be written as a sum of an operator with anti-Wick symbol and an operator with kernel in $S(\mathbb{R}^{2n})$ (Shubin, Theorem 24.2, p.182) and the Weyl symbol of an operator with anti-Wick symbol $a \in \Gamma^m_\rho(\mathbb{R}^{2n})$ is given by $b(z) = \pi^{-n}\int_{\mathbb{R}^{2n}} e^{-|w-z|^2} a(w)dw$. Furthermore $b \in \Gamma^m_\rho(\mathbb{R}^{2n})$ and $b - a \in \Gamma^{m-2\rho}_\rho(\mathbb{R}^{2n})$. Using this, Shubin obtains necessary and sufficient conditions on the Weyl symbol which guarantee that the operator is bounded on $L^2(\mathbb{R}^n)$ or that the operator is compact (Shubin, Problems 24.8 and 24.9, p.184).

We wish to study (1) using more general operators than the projection operator P_z and more general "symbols" a.

We will use the following notation: If H is any Hilbert space then $\mathcal{B}(H)$ will denote the algebra of bounded linear operators on H. We define M_ξ: $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ by $M_\xi f(y) = e^{i<y,\xi>}f(y)$ where $<y,\xi> = \sum_{j=1}^{n} y_j\xi_j$, T_x: $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ by $T_x f(y) = f(y-x)$, and $U(z) = M_\xi T_x$ for $z = (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Both M_ξ and T_x are unitary operators and hence so is U(z). Thus for any $L \in \mathcal{B}(L^2(\mathbb{R}^n))$ we can define $L_z = U(z)LU(z)^{-1}$. Clearly $L_z \in \mathcal{B}(L^2(\mathbb{R}^n))$ $\forall z \in \mathbb{R}^{2n}$.

With this notation we can rewrite formula (2) as $\Phi_z = U(z)\phi$ where $\phi(y) = \pi^{-\frac{n}{4}} e^{-\frac{|y|}{2}}$. Then the projection P_z used in (1) is given by $P_z f = (f, U(z)\phi)_{L^2(\mathbb{R}^n)} U(z)\phi = U(z)[(U(z)^{-1}f, \phi)\phi] = U(z)PU(z)^{-1}f$ where P is the projection of $L^2(\mathbb{R}^n)$ onto the subspace generated by ϕ (Shubin, p.179).

Since $\forall A,B \in \mathcal{B}(L^2(\mathbb{R}^n))$, $(A+B)_z = U(z)(A+B)U(z)^{-1} = U(z)AU(z)^{-1} + U(z)BU(z)^{-1} = A_z + B_z$, $(AB)_z = U(z)ABU(z)^{-1} = U(z)AU(z)U(z)^{-1}BU(z)^{-1} = A_zB_z$, and $(B^*)_z = U(z)B^*U(z)^{-1} = U(z)B^*U(z)^* = [U(z)BU(z)^*]^* = (B_z)^*$, $L \longmapsto L_z$ is a $*$ homomorphism. Since $L \longmapsto U(z)^{-1}LU(z)$ clearly is the inverse $*$ homomorphism, $L \longmapsto L_z$ is a $*$ automorphism of $\mathscr{B}(L^2(\mathbb{R}^n))$. Denote this $*$ automorphism by $\Phi(z)$.

The maps $\xi \longmapsto M_\xi$ and $x \longmapsto T_x$: $\mathbb{R}^n \longrightarrow \mathscr{B}(L^2(\mathbb{R}^n))$ are group homomorphisms, but $z \longmapsto U(z)$: $\mathbb{R}^{2n} \longrightarrow \mathscr{B}(L^2(\mathbb{R}^n))$ is not since $M_\xi T_x \neq T_x M_\xi$ (see Lemma 5 below). Furthermore, if $\mathscr{F}$ is the Fourier transform, then $M_x = \mathscr{F}^{-1}T_x\mathscr{F}$: for $\phi \in S(\mathbb{R}^n)$, $T_x\mathscr{F}\phi(\xi) = \mathscr{F}\phi(\xi-x) = \int_{\mathbb{R}^n} e^{-i<y,\xi-x>}\phi(y)dy = \int_{\mathbb{R}^n} e^{-i<y,\xi>}M_x\phi(y)dy = \mathscr{F}M_x\phi(\xi)$. Since $S(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and $\mathscr{F}$ and M_x are continuous on $L^2(\mathbb{R}^n)$, $T_x\mathscr{F} = \mathscr{F}M_x$ and thus $\mathscr{F}^{-1}T_x\mathscr{F} = \mathscr{F}^{-1}\mathscr{F}M_x = M_x$.

2. Theorems on Continuity.

We now will study the continuity properties of the map $z \longmapsto L_z$ from $\mathbb{R}^{2n}$ to several vector spaces with various topologies.

5. Lemma. $T_x M_\xi = e^{-i<x,\xi>}M_\xi T_x$.

Proof. $T_xM_\xi f(y) = T_x[e^{i<y,\xi>}f(y)] = e^{i<y-x,\xi>}f(y-x) = e^{-i<x,\xi>}e^{i<y,\xi>}T_xf(y) = e^{-i<x,\xi>}M_\xi T_x f(y)$.

Remark. $z \longmapsto \Phi(z)$ is a group homomorphism of $(\mathbb{R}^{2n},+)$ into the group of $*$ automorphisms of $\mathscr{B}(L^2(\mathbb{R}^n))$.

Proof. $\Phi(z_1+z_2)L = U(z_1+z_2)LU(z_1+z_2)^{-1} = M_{\xi_1+\xi_2}T_{x_1+x_2}LT_{-x_2-x_1}M_{-\xi_2-\xi_1} =$
$M_{\xi_1}M_{\xi_2}T_{x_1}T_{x_2}LT_{-x_2}T_{-x_1}M_{-\xi_2}M_{-\xi_1} =$
$M_{\xi_1}e^{i<x_1,\xi_2>}T_{x_1}M_{\xi_2}T_{x_2}LT_{-x_2}e^{-i<x_1,\xi_2>}M_{-\xi_2}T_{-x_1}M_{-\xi_1} = M_{\xi_1}T_{x_1}U(z_2)LU(z_2)^{-1}T_{-x_1}M_{\xi_1}$
$= U(z_1)L_{z_2}U(z_1)^{-1} = (L_{z_2})_{z_1} = \Phi(z_1)\Phi(z_2)L$.

6. Lemma. $U(z) = R(z,z_0) + e^{i<x_0,\xi-\xi_0>}\, U(z_0)$ where $R(z,z_0) \longrightarrow 0$ in the strong operator topology (SOT) uniformly in z_0 as $z \longrightarrow z_0$.

Proof. $U(z) = M_\xi T_x = M_\xi T_x - M_\xi T_{x_0} + M_\xi T_{x_0} = M_\xi T_{x_0}(T_{x-x_0} - I) +$

$M_\xi T_{x_0} - e^{i<x_0,\xi-\xi_0>} M_{\xi_0} T_{x_0} + e^{i<x_0,\xi-\xi_0>} U(z_0)$. Let $R(z,z_0) = M_\xi T_{x_0}(T_{x-x_0} - I) + M_\xi T_{x_0} - e^{i<x_0,\xi-\xi_0>} M_{\xi_0} T_{x_0}$. Using Lemma 5, we obtain $R(z,z_0) = M_\xi T_{x_0}(T_{x-x_0} - I) + e^{i<x_0,\xi-\xi_0>} M_{\xi_0} T_{x_0}(e^{-i<x_0,\xi-\xi_0>} T_{-x_0} M_{-\xi_0} M_\xi T_{x_0} - I) = M_\xi T_{x_0}(T_{x-x_0} - I) + e^{i<x_0,\xi-\xi_0>} M_{\xi_0} T_{x_0}(T_{-x_0} e^{-i<x_0,\xi-\xi_0>} M_{\xi-\xi_0} T_{x_0} - I) = M_\xi T_{x_0}(T_{x-x_0} - I) + e^{i<x_0,\xi-\xi_0>} M_{\xi_0} T_{x_0}(T_{-x_0} T_{x_0} M_{\xi-\xi_0} - I) = M_\xi T_{x_0}(T_{x-x_0} - I) + e^{i<x_0,\xi-\xi_0>} M_{\xi_0} T_{x_0}(M_{\xi-\xi_0} - I)$. Thus for each $f \in L^2(\mathbb{R}^n)$, $\|R(z,z_0)f\|_{L^2(\mathbb{R}^n)} \le \|T_{x-x_0}f - f\|_{L^2(\mathbb{R}^n)} + \|M_{\xi-\xi_0}f - f\|_{L^2(\mathbb{R}^n)} = \|T_{x-x_0}f - f\|_{L^2(\mathbb{R}^n)} + \|\mathcal{F}^{-1}(T_{\xi-\xi_0} - I)\mathcal{F}f\|_{L^2(\mathbb{R}^n)} = \|T_{x-x_0}f - f\|_{L^2(\mathbb{R}^n)} + (2\pi)^{-\frac{n}{2}}\|T_{\xi-\xi_0}\hat{f} - \hat{f}\|_{L^2(\mathbb{R}^n)} \longrightarrow 0$ uniformly in z_0 as $z = (x,\xi) \longrightarrow (x_0,\xi_0) = z_0$.

7. Lemma. $U(z)^{-1} = \tilde{R}(z,z_0) + e^{-i<x-x_0,\xi_0>}U(z_0)^{-1}$ where $\tilde{R}(z,z_0) \longrightarrow 0$ in SOT uniformly in z_0 as $z \longrightarrow z_0$.

Proof. $U(z)^{-1} = T_{-x}M_{-\xi} - T_{-x}M_{-\xi_0} + T_{-x}M_{-\xi_0} - e^{-i<x-x_0,\xi_0>}T_{-x_0}M_{-\xi_0} + e^{-i<x-x_0,\xi_0>}U(z_0)^{-1}$. Let $\tilde{R}(z,z_0) = T_{-x}M_{-\xi} - T_{-x}M_{-\xi_0} + T_{-x}M_{-\xi_0} - e^{-i<x-x_0,\xi_0>}T_{-x_0}M_{-\xi_0} = T_{-x}M_{-\xi}(I - M_{\xi-\xi_0}) + e^{-i<x-x_0,\xi_0>}T_{-x_0}(e^{i<x-x_0,\xi_0>}T_{x_0-x}M_{-\xi_0} - M_{-\xi_0})$. By Lemma 5, we obtain $\tilde{R}(z,z_0) = T_{-x}M_{-\xi}(I - M_{\xi-\xi_0}) + e^{-i<x-x_0,\xi_0>}T_{-x_0}(M_{-\xi_0}T_{x_0-x} - M_{-\xi_0}) = T_{-x}M_{-\xi}(I - M_{\xi-\xi_0}) + e^{-i<x-x_0,\xi_0>}T_{-x_0}M_{-\xi_0}(T_{x_0-x} - I)$. So $\forall f \in L^2(\mathbb{R}^n)$, $\|\tilde{R}(z,z_0)f\|_{L^2(\mathbb{R}^n)} \le \|f - M_{\xi-\xi_0}f\|_{L^2(\mathbb{R}^n)} + \|T_{x_0-x}f - f\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(I - T_{\xi-\xi_0})\mathcal{F}f\|_{L^2(\mathbb{R}^n)} + \|T_{x_0-x}f - f\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-\frac{n}{2}}\|\hat{f} - T_{\xi-\xi_0}\hat{f}\|_{L^2(\mathbb{R}^n)} + \|T_{x_0-x}f - f\|_{L^2(\mathbb{R}^n)} \longrightarrow 0$ uniformly in z_0 as $z = (x,\xi) \longrightarrow (x_0,\xi_0) = z_0$.

8. Proposition. The maps $z \longmapsto U(z)$ and $z \longmapsto U(z)^{-1}: \mathbb{R}^{2n} \longrightarrow (\mathscr{B}(L^2(\mathbb{R}^n)), \mathrm{SOT})$ are continuous but not uniformly continuous.

Proof. By Lemma 6, $U(z) - U(z_0) = R(z,z_0) + (e^{i<x_0,\xi-\xi_0>} - 1)\, U(z_0)$ and therefore, by Lemma 6, $U(z) - U(z_0) \longrightarrow 0$ in SOT as $z \longrightarrow z_0$. Suppose $z \longmapsto U(z)$ is uniformly continuous, i.e., $U(z) - U(z_0) \longrightarrow 0$ in SOT uniformly in z_0 as $z \longrightarrow z_0$. Then by Lemma 6, $U(z) - U(z_0) - R(z,z_0) \longrightarrow 0$ in SOT uniformly in z_0 as $z \longrightarrow z_0$. Thus $\forall f \in L^2(\mathbb{R}^n)$, $(e^{i<x_0,\xi-\xi_0>} - 1)U(z_0)f \longrightarrow 0$ uniformly in z_0 as $z \longrightarrow z_0$. But this is false for $f \neq 0$: For any $\xi - \xi_0 \neq 0$ we can choose x_0 and hence $z_0 = (x_0,\xi_0)$ such that $e^{i<x_0,\xi-\xi_0>} = -1$. Then for $z = (x,\xi) = (x_0,\xi)$, we have $|z - z_0| = |\xi - \xi_0|$ which can be chosen arbitrarily small but $(e^{i<x_0,\xi-\xi_0>} - 1)U(z_0)f = -2U(z_0)f \neq 0$ for all such z. Therefore $z \longmapsto U(z)$ is not uniformly continuous. The statements for $z \longmapsto U(z)^{-1}$ follow analogously from Lemma 7.

9. Corollary. The map $z \longmapsto L_z: \mathbb{R}^{2n} \longrightarrow (\mathscr{B}(L^2(\mathbb{R}^n)), \mathrm{SOT})$ is continuous.

Proof. Since the set $\mathscr{U}$ of unitary operators is bounded in $\mathscr{B}(L^2(\mathbb{R}^n))$, the map $(\mathscr{U}\times\{L\}\times\mathscr{U}, \mathrm{SOT}\times\mathrm{SOT}\times\mathrm{SOT}) \longrightarrow (\mathscr{B}(L^2(\mathbb{R}^n)), \mathrm{SOT})$: $(U,L,V) \longmapsto ULV$ is continuous. Since $L_z = U(z)LU(z)^{-1}$, the corollary follows from Proposition 8.

10. Proposition. If $L \in \mathscr{B}(L^2(\mathbb{R}^n))$ and $\mathscr{B}(L^2(\mathbb{R}^n))$ has the weak operator topology (WOT) then the map $z \longmapsto L_z: \mathbb{R}^{2n} \longrightarrow \mathscr{B}(L^2(\mathbb{R}^n))$ is uniformly continuous.

Proof. By Lemma 7, $(L_zf, g)_{L^2(\mathbb{R}^n)} = (U(z)LU(z)^{-1}f, g) = (LU(z)^{-1}f, U(z)^{-1}g) = (L[\tilde{R}(z,z_0)f + e^{-i<x-x_0,\xi_0>}U(z_0)^{-1}f], \tilde{R}(z,z_0)g + e^{-i<x-x_0,\xi_0>}U(z_0)^{-1}g) = (L\tilde{R}(z,z_0)f, \tilde{R}(z,z_0)g) + e^{-i<x-x_0,\xi_0>}(LU(z_0)^{-1}f, \tilde{R}(z,z_0)g) + e^{i<x-x_0,\xi_0>}(L\tilde{R}(z,z_0)f, U(z_0)^{-1}g) + e^{-i<x-x_0,\xi_0>}e^{i<x-x_0,\xi_0>}(LU(z_0)^{-1}f, U(z_0)^{-1}g) = (L\tilde{R}(z,z_0)f, \tilde{R}(z,z_0)g) +$

$e^{-i<x-x_0,\xi_0>}(LU(z_0)^{-1}f, \tilde{R}(z,z_0)g) + e^{i<x-x_0,\xi_0>}(L\tilde{R}(z,z_0)f, U(z_0)^{-1}g) +$
$(L_{z_0}f, g)$. Thus $|(L_z f, g) - (L_{z_0} f, g)| \le$
$\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|\tilde{R}(z,z_0)f\|_{L^2(\mathbb{R}^n)}\|\tilde{R}(z,z_0)g\|_{L^2(\mathbb{R}^n)} +$
$\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|U(z_0)^{-1}f\|_{L^2(\mathbb{R}^n)}\|\tilde{R}(z,z_0)g\|_{L^2(\mathbb{R}^n)} +$
$\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|\tilde{R}(z,z_0)f\|_{L^2(\mathbb{R}^n)}\|U(z_0)^{-1}g)\|_{L^2(\mathbb{R}^n)} =$
$\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|\tilde{R}(z,z_0)f\|_{L^2(\mathbb{R}^n)}\|\tilde{R}(z,z_0)g\|_{L^2(\mathbb{R}^n)} +$
$\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|f\|_{L^2(\mathbb{R}^n)}\|\tilde{R}(z,z_0)g\|_{L^2(\mathbb{R}^n)} + \|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|\tilde{R}(z,z_0)f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}$
$\longrightarrow 0$ uniformly in z_0 as $z \longrightarrow z_0$ by Lemma 7.

11. Proposition. $\{L \in \mathcal{B}(L^2(\mathbb{R}^n)) \mid z \longmapsto L_z: \mathbb{R}^{2n} \longrightarrow (\mathcal{B}(L^2(\mathbb{R}^n))$, norm topology) is continuous$\}$ is a norm closed subspace of $\mathcal{B}(L^2(\mathbb{R}^n))$.

Proof. Suppose $\{L_m\}$ is a sequence in $\mathcal{B}(L^2(\mathbb{R}^n))$ which converges to L in the norm topology. Suppose also that $z \longmapsto (L_m)_z: \mathbb{R}^{2n} \longrightarrow (\mathcal{B}(L^2(\mathbb{R}^n))$, norm topology) is continuous for all m. Since $\forall z$ $\|(L_m)_z - L_z\|_{\mathcal{B}(L^2(\mathbb{R}^n))} =$ $\|U(z)(L_m - L)U(z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|L_m - L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}$, $(L_m)_z \longrightarrow L_z$ uniformly in z. Thus $z \longmapsto L_z$ is continuous, i.e., $L \in \{L \in \mathcal{B}(L^2(\mathbb{R}^n)) \mid z \longmapsto L_z: \mathbb{R}^{2n} \longrightarrow$ $(\mathcal{B}(L^2(\mathbb{R}^n))$, norm topology) is continuous$\}$.

12. Lemma. Let g be a non-negative measurable function on $\mathbb{R}^n$. Then ess sup $g(x) = \sup \{\int g(x)f(x)dx: f \in L^1(\mathbb{R}^n), f \ge 0$, and $\int f(x)dx = 1\}$.

Proof. Suppose ess sup $g(x) = \infty$. Let $E_k = \{x: g(x) \ge k\}$. Then $m(E_k) > 0$. Choose $F_k \subseteq E_k$ such that $0 < m(F_k) < \infty$ and let $f_k = \frac{1}{m(F_k)}\chi_{F_k}$. Then $f_k \ge 0$, $\int f_k(x)dx = 1$, and $\int g(x)f_k(x)dx \ge \int kf_k(x)dx = k$. Thus $\sup \{\int g(x)f(x)dx$: $f \ge 0$ and $\int f(x)dx = 1\} = \infty$. Suppose ess sup $g(x) < \infty$. $\forall\varepsilon > 0$ let $E_\varepsilon = \{x$: $g(x) \ge$ ess sup $g(x) - \varepsilon\}$. Then $m(E_\varepsilon) > 0$. Choose $F_\varepsilon \subseteq E_\varepsilon$ such that $0 < m(F_\varepsilon)$ $< \infty$ and let $f_\varepsilon = \frac{1}{m(F_\varepsilon)}\chi_{F_\varepsilon}$. Then $\forall\varepsilon > 0$, $f_\varepsilon \ge 0$, $\int f_\varepsilon(x)dx = 1$, and $\int g(x)f_\varepsilon(x)dx$ $\ge \int[$ess sup $g(x) - \varepsilon]f_\varepsilon(x)dx =$ ess sup $g(x) - \varepsilon$. Thus $\sup \{\int g(x)f(x)dx: f \ge 0$

and $\int f(x)dx = 1\} \geq \operatorname{ess\,sup} g(x)$. Since clearly $\int f(x)g(x)dx \leq \operatorname{ess\,sup} g(x) \int f(x)dx$ when f and g are non-negative measurable functions, we have the required equality.

13. Proposition. Let $\phi \in L^\infty(\mathbb{R}^n)$ and let $\mathcal{M}(\phi)$ be the multiplication operator $f \longmapsto \phi f$: $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$. Then the map $z \longmapsto \mathcal{M}(\phi)_z$: $\mathbb{R}^{2n} \longrightarrow (\mathcal{B}(L^2(\mathbb{R}^n))$, norm topology) is continuous iff $\exists$ a uniformly continuous function ϕ_0 such that $\phi_0 = \phi$ a.e.. If the map $z \longmapsto \mathcal{M}(\phi)_z$ is continuous then, it is uniformly continuous.

Proof. $\exists\theta$ such that θ is Borel measurable and such that $\theta = \phi$ a.e.. $\mathcal{M}(\theta)_z = U(z)\mathcal{M}(\theta)U(z)^{-1} = M_\xi T_x \mathcal{M}(\theta)T_{-x}M_{-\xi}$ and $[T_x\mathcal{M}(\theta)f](y) = [\mathcal{M}(\theta)f](y-x) = [\theta f](y-x) = \theta(y-x)f(y-x) = [T_x\theta](y)[T_xf](y) = [\mathcal{M}(T_x\theta)T_xf](y)$. Thus $T_x\mathcal{M}(\theta) = \mathcal{M}(T_x\theta)T_x$ and so $T_x\mathcal{M}(\theta)T_{-x} = \mathcal{M}(T_x\theta)T_xT_{-x} = \mathcal{M}(T_x\theta)$. Since both $\mathcal{M}(T_x\theta)$ and M_ξ are multiplication operators, they commute. Therefore $\mathcal{M}(\theta)_z = M_\xi T_x\mathcal{M}(\theta)T_{-x}M_{-\xi} = M_\xi\mathcal{M}(T_x\theta)M_{-\xi} = \mathcal{M}(T_x\theta)$. Hence $\|\mathcal{M}(\theta)_z - \mathcal{M}(\theta)_{z_0}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|\mathcal{M}(T_x\theta) - \mathcal{M}(T_{x_0}\theta)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|\mathcal{M}(T_x\theta - T_{x_0}\theta)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|T_x\theta - T_{x_0}\theta\|_{L^\infty(\mathbb{R}^n)} = \operatorname{ess\,sup}\{|\theta(y-x) - \theta(y-x_0)| : y \in \mathbb{R}^n\} = \operatorname{ess\,sup}\{|\theta(u) - \theta(u+x-x_0)|: u \in \mathbb{R}^n\}$. Clearly $z \longmapsto \mathcal{M}(\phi)_z$ is continuous iff $\operatorname{ess\,sup}\{|\theta(u) - \theta(u+x-x_0)|: u \in \mathbb{R}^n\} \longrightarrow 0$ as $x \longrightarrow x_0$. But this limit is uniform in z_0. $z \longmapsto \mathcal{M}(\theta)_z$ is continuous if and only if $\lim_{y\to 0}[\operatorname{ess\,sup}_{y \in \mathbb{R}^n} |\theta(x-y) - \theta(x)|] = 0$. Since θ is Borel measurable and $(x,y) \longmapsto x - y$ is continuous, $\theta(x-y)$ is measurable on $\mathbb{R}^n\times\mathbb{R}^n$. So $\theta(x-y) - \theta(x)$ is measurable on $\mathbb{R}^n\times\mathbb{R}^n$. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ have support in the unit ball $B_1(0)$ and satisfy $\rho \geq 0$ and $\int\rho(x)dx = 1$. $\forall\delta > 0$ define ρ_δ by $\rho_\delta(x) = \delta^{-n}\rho(\delta^{-1}x)$. Then $\rho_\delta \geq 0$, ρ_δ has support in the ball $B_\delta(0)$, $\int\rho_\delta(x)dx = 1$, and $\theta*\rho_\delta \in C^\infty(\mathbb{R}^n)$. By Lemma 12, $\|\theta*\rho_\delta - \theta\|_{L^\infty(\mathbb{R}^n)} = \operatorname{ess\,sup}\{|\int[\theta(x-y) - \theta(x)]\rho_\delta(y)dy|: x \in \mathbb{R}^n\} \leq$

ess sup $\{\int|\theta(x-y) - \theta(x)|\rho_\delta(y)dy\colon x \in \mathbb{R}^n\} = \sup\{\int\int|\theta(x-y - \theta(x)|\rho_\delta(y)dyf(x)dx\colon f \geq 0$ and $\int f(x)dx = 1\} = \sup\{\int\int|\theta(x-y) - \theta(x)|f(x)\rho_\delta(y)dxdy\colon f \geq 0$ and $\int f(x)dx = 1\}$. Now suppose that $z \longmapsto \mathfrak{M}(\phi)_z$ is continuous. Then $\forall\varepsilon > 0\ \exists\delta > 0$ such that $|y| < \delta \Longrightarrow$ ess sup $\{|\theta(x-y) - \theta(x)|\colon x \in \mathbb{R}^n\} < \varepsilon$, i.e., since supp $\rho_\delta \subseteq B_\delta(0)$, for each $y \in \mathbb{R}^n$, $|\theta(x-y) - \theta(x)|f(x)\rho_\delta(y) \leq \varepsilon f(x)\rho_\delta(y)$ a.e. in x. So $\forall y \in \mathbb{R}^n$ $\int|\theta(x-y) - \theta(x)|f(x)\rho_\delta(y)dx \leq \int\varepsilon f(x)\rho_\delta(y)dx = \varepsilon\rho_\delta(y)\int f(x)dx = \varepsilon\rho_\delta(y)$. Therefore $\sup\{\int\int|\theta(x-y) - \theta(x)|f(x)\rho_\delta(y)dxdy\colon f \geq 0$ and $\int f(x)dx = 1\} \leq \varepsilon\int\rho_\delta(y)dy = \varepsilon$. Thus $\lim\limits_{\delta\to 0} \|\theta*\rho_\delta - \theta\|_{L^\infty(\mathbb{R}^n)} = 0$. Let δ_k be a sequence of positive numbers such that $\delta_k \longrightarrow 0$. Then $\theta*\rho_{\delta_k} \longrightarrow \theta$ in $L^\infty(\mathbb{R}^n)$. Thus $\{\theta*\rho_{\delta_k}\}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^n)$. Since $\forall k$, $\theta*\rho_{\delta_k}$ is continuous, $\{\theta*\rho_{\delta_k}\}$ is a Cauchy sequence in the uniform norm. Hence $\exists\phi_0 \in C(\mathbb{R}^n)$ such that $\theta*\rho_{\delta_k} \longrightarrow \phi_0$ uniformly. But then ϕ_0 is continuous and $\phi_0 = \theta = \phi$ a.e.. Furthermore $\forall y \in \mathbb{R}^n$, $\phi_0(x-y)$ is continuous as a function of x. Therefore $\sup\{|\phi_0(x-y) - \phi_0(x)|\colon x \in \mathbb{R}^n\} =$ ess sup $\{|\phi_0(x-y) - \phi_0(x)|\colon x \in \mathbb{R}^n\} =$ ess sup $\{|\theta(x-y) - \theta(x)|\colon x \in \mathbb{R}^n\} \longrightarrow 0$ as $y \longrightarrow 0$, i.e., ϕ_0 is uniformly continuous. The reverse implication is clear.

Let H be any Hilbert space. For $\phi,\theta \in H$, we define $\phi\otimes\bar{\theta}\colon H \longrightarrow H$ by $(\phi\otimes\bar{\theta})f = (f, \theta)_H\phi$. Then $\phi\otimes\bar{\theta}$ is a bounded linear operator with norm $\|\phi\|_H\|\theta\|_H$. $\phi\otimes\bar{\theta}$ is linear in ϕ and conjugate linear in θ. If L is any compact operator on H then $L = \sum\limits_i \lambda_i\phi_i\otimes\bar{\theta}_i$ where $\{\phi_i\}$ and $\{\theta_i\}$ are orthonormal sequences in H. The λ_i's are the positive eigenvalues of $|L| = \sqrt{L^*L}$ and each eigenvalue appears the number of times equal to its multiplicity (Schatten [3], Theorem 7, p.18). Thus $\lambda_i > 0\ \forall i$ and $\{\lambda_i\}$ is finite or $\lambda_i \longrightarrow 0$. Following Dunford & Schwartz [1] (Vol. II, Chapter XI, ¶ 9) we define $\|L\|_p = [\sum\limits_i \lambda_i^p]^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|L\|_\infty$

$= \sup_i \lambda_i$. Let $C_p = \{L \in \mathcal{B}(H)$: L is compact and $\|L\|_p < \infty\}$. Then $\|\ \|_p$ is a norm on C_p and C_p is a Banach space under this norm. C_1 is the space of trace class operators and $\|\ \|_1$ is the trace class norm, C_2 is the space of Hilbert Schmidt operators and $\|\ \|_2$ is the Hilbert Schmidt norm, and C_∞ is the space of compact operators and $\|\ \|_\infty = \|\ \|_{\mathcal{B}(H)}$. Furthermore, if $A \in \mathcal{B}(H)$ and $L \in C_p$ then $\|AL\|_p \le \|A\|_{\mathcal{B}(H)}\|L\|_p$ and $\|LA\|_p \le \|L\|_p\|A\|_{\mathcal{B}(H)}$. If $L \in C_p$, $L = \sum \lambda_i\phi_i\otimes\bar{\theta}_i$. Define $L_k = \sum_{i=1}^{k} \lambda_i\phi\otimes\bar{\theta}_i$ if the representation of L has k terms or more and $L_k = L$ it has fewer than k terms. Then $L_k \longrightarrow L$ in C_p. Clearly $\|\phi\otimes\bar{\theta}\|_p = \|\phi\|_H\|\theta\|_H$ $\forall p$. From now on let $H = L^2(\mathbb{R}^n)$.

14. Lemma. The map $z \longmapsto (\phi\otimes\bar{\theta})_z: \mathbb{R}^{2n} \longrightarrow C_p$ is uniformly continuous.

Proof. $(\phi\otimes\bar{\theta})_z f = U(z)(\phi\otimes\bar{\theta})U(z)^{-1}f = (U(z)^{-1}f, \theta)U(z)\phi = (f, U(z)\theta)U(z)\phi = (U(z)\phi)\otimes\overline{U(z)\theta}f$. So by Lemma 6, $(\phi\otimes\bar{\theta})_z = (U(z)\phi)\otimes\overline{U(z)\theta} =$ $(R(z,z_0)\phi)\otimes\overline{R(z,z_0)\theta} + e^{i<x_0,\xi-\xi_0>}(U(z_0)\phi)\otimes\overline{R(z,z_0)\theta} +$ $e^{-i<x_0,\xi-\xi_0>}(R(z,z_0)\phi)\otimes\overline{U(z_0)\theta} + e^{i<x_0,\xi-\xi_0>}e^{-i<x_0,\xi-\xi_0>}(U(z_0)\phi)\otimes\overline{U(z_0)\theta} =$ $(R(z,z_0)\phi)\otimes\overline{R(z,z_0)\theta} + e^{i<x_0,\xi-\xi_0>}(U(z_0)\phi)\otimes\overline{R(z,z_0)\theta} +$ $e^{-i<x_0,\xi-\xi_0>}(R(z,z_0)\phi)\otimes\overline{U(z_0)\theta} + (\phi\otimes\bar{\theta})_{z_0}$. Thus $\|(\phi\otimes\bar{\theta})_z - (\phi\otimes\bar{\theta})_{z_0}\|_p \le$ $\|(R(z,z_0)\phi)\otimes\overline{R(z,z_0)\theta}\|_p + \|U(z_0)\phi)\otimes\overline{R(z,z_0)\theta}\|_p + \|(R(z,z_0)\phi)\otimes\overline{U(z_0)\theta}\|_p =$ $\|R(z,z_0)\phi\|_{L^2(\mathbb{R}^n)}\|R(z,z_0)\theta\|_{L^2(\mathbb{R}^n)} + \|U(z_0)\phi\|_{L^2(\mathbb{R}^n)}\|R(z,z_0)\theta\|_{L^2(\mathbb{R}^n)} +$ $\|R(z,z_0)\phi\|_{L^2(\mathbb{R}^n)}\|U(z_0)\theta\|_{L^2(\mathbb{R}^n)} = \|R(z,z_0)\phi\|_{L^2(\mathbb{R}^n)}\|R(z,z_0)\theta\|_{L^2(\mathbb{R}^n)} +$ $\|\phi\|_{L^2(\mathbb{R}^n)}\|R(z,z_0)\theta\|_{L^2(\mathbb{R}^n)} + \|R(z,z_0)\phi\|_{L^2(\mathbb{R}^n)}\|\theta\|_{L^2(\mathbb{R}^n)} \longrightarrow 0$ uniformly in z_0 as $z \longrightarrow z_0$.

15. Proposition. If $L \in C_p$ then $z \longmapsto L_z: \mathbb{R}^{2n} \longrightarrow C_p$ is uniformly continuous.

Proof. Let $L \in C_p$ and define L_k as before. By Lemma 14, for each k, $z \longmapsto (L_k)_z: \mathbb{R}^{2n} \longrightarrow C_p$ is uniformly continuous. Thus $\forall\varepsilon > 0$ we can choose k such that $\|L_k - L\|_p < \frac{\varepsilon}{3}$ and we can choose $\delta > 0$ independent of z_0 such that

$\|(L_k)_z - (L_k)_{z_0}\|_p < \frac{\varepsilon}{3}$ when $|z - z_0| < \delta$. Then $|z - z_0| < \delta \longrightarrow \|L_z - L_{z_0}\|_p \le \|L_z - (L_k)_z\|_p + \|(L_k)_z - (L_k)_{z_0}\|_p + \|(L_k)_{z_0} - L_{z_0}\|_p < \|U(z)(L - L_k)U(z)^{-1}\|_p + \frac{\varepsilon}{3} + \|U(z_0)(L - L_k)U(z_0)^{-1}\|_p \le \|U(z)\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|L_k - L\|_p\|U(z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} + \frac{\varepsilon}{3} + \|U(z_0)\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|L_k - L\|_p\|U(z_0)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

Remark. By Proposition 11, $S = \{L \in \mathcal{B}(L^2(\mathbb{R}^n)) \mid z \longmapsto L_z : \mathbb{R}^{2n} \longrightarrow (\mathcal{B}(L^2(\mathbb{R}^n)), \text{norm topology}) \text{ is continuous}\}$ is a closed subspace of $\mathcal{B}(L^2(\mathbb{R}^n))$. By Proposition 13, S contains the operators $\mathcal{M}(\phi)$ when ϕ is uniformly continuous on $\mathbb{R}^n$ and $S \neq \mathcal{B}(L^2(\mathbb{R}^n))$. By Proposition 15 with $p = \infty$, S contains the compact operators.

16. Proposition. Let $\phi \in S(\mathbb{R}^n)$ and $\theta \in L^2(\mathbb{R}^n)$. Then $\forall z \in \mathbb{R}^{2n}$, $(\phi\otimes\bar{\theta})_z : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$ is continuous and $\forall f \in L^2(\mathbb{R}^n)$, $z \longmapsto (\phi\otimes\bar{\theta})_z f : \mathbb{R}^{2n} \longrightarrow S(\mathbb{R}^n)$ is continuous.

Proof. The topology on $S(\mathbb{R}^n)$ is given by the seminorms defined for any non-negative integer k and for any multi-index α by $\rho_{k,\alpha}(\phi) = \sup\{(1+|y|)^k|\partial^\alpha\phi(y)| : y \in \mathbb{R}^n\}$.

a) $M_\xi : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$ is continuous. In fact, $\rho_{k,\alpha}(M_\xi\phi) \le$

$\sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} |\xi^{\alpha'}| \rho_{k,\alpha-\alpha'}(\phi)$. Proof: $\rho_{k,\alpha}(M_\xi\phi) = \sup_{y\in\mathbb{R}^n}(1+|y|)^k|\partial^\alpha[e^{i<y,\xi>}\phi(y)]| =$

$\sup_{y\in\mathbb{R}^n}(1+|y|)^k|\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'} i^{|\alpha'|}\xi^{\alpha'}e^{i<y,\xi>}\partial^{\alpha-\alpha'}\phi(y)| \le \sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}|\xi^{\alpha'}|\rho_{k,\alpha-\alpha'}(\phi)$.

b) $T_x : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$ is continuous. In fact, $\rho_{k,\alpha}(T_x\phi) \le (1+|x|)^k\rho_{k,\alpha}(\phi)$. Proof: $\rho_{k,\alpha}(T_x\phi) = \sup_{y\in\mathbb{R}^n}(1+|y|)^k|\partial_y^\alpha\phi(y-x)| =$

$\sup_{z\in\mathbb{R}^n}(1+|z+x|)^k|\partial^\alpha\phi(z)| \le \sup_{z\in\mathbb{R}^n}(1+|x|)^k(1+|z|)^k|\partial^\alpha\phi(z)| = (1+|x|)^k\rho_{k,\alpha}(\phi)$.

c) $(\phi\otimes\bar{\theta})|S(\mathbb{R}^n) : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$ is continuous: Since $S(\mathbb{R}^n) \xhookrightarrow{i} L^2(\mathbb{R}^n)$ is continuous, it suffices to show that $\phi\otimes\bar{\theta} : L^2(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$ is continuous.

Let $f \in L^2(\mathbb{R}^n)$. Then $\rho_{k,\alpha}((\phi\otimes\bar{\theta})f) = \rho_{k,\alpha}[(f,\theta)\phi] = |(f,\theta)|\rho_{k,\alpha}(\phi) \le \|\theta\|_{L^2(\mathbb{R}^n)}\rho_{k,\alpha}(\phi)\|f\|_{L^2(\mathbb{R}^n)}$.

d) $(\phi\otimes\bar{\theta})_z f\colon \mathbb{R}^{2n} \longrightarrow S(\mathbb{R}^n)$ is continuous by a), b), and c) since $(\phi\otimes\bar{\theta})_z = M_\xi T_x(\phi\otimes\bar{\theta})T_{-x}M_{-\xi}$.

e) $\rho_{k,\alpha}(T_x\phi-\phi) \le n|x|(1+|x|)^k \sum_{j=1}^{n} \rho_{k,\alpha+e_j}(\phi)$ where $\alpha + e_j$ is the multi-index such that $(\alpha + e_j)_i = \alpha_i$ if $i \ne j$ and $(\alpha + e_j)_j = \alpha_j + 1$: $\rho_{k,\alpha}(T_x\phi-\phi) =$

$$\sup_{y\in\mathbb{R}^n} (1+|y|)^k|\partial_y^\alpha[\phi(y-x)-\phi(y)]| = \sup_{y\in\mathbb{R}^n} (1+|y|)^k\Big|\int_0^1 \sum_{j=1}^{n} (-x_j)\partial^{\alpha+e_j}\phi(y-tx)dt\Big| \le$$

$$\sum_{j=1}^{n} |x_j|\int_0^1 \sup_{y\in\mathbb{R}^n} (1+|y|)^k|\partial^{\alpha+e_j}\phi(y-tx)|dt = \sum_{j=1}^{n} |x_j|\int_0^1 \sup_{z\in\mathbb{R}^n} (1+|z+tx|)^k|\partial^{\alpha+e_j}\phi(z)|dt$$

$$\le \sum_{j=1}^{n} |x_j|\int_0^1 (1+|tx|)^k \sup_{z\in\mathbb{R}^n} (1+|z|)^k|\partial^{\alpha+e_j}\phi(z)|dt \le \sum_{j=1}^{n} |x_j|(1+|x|)^k\rho_{k,\alpha+e_j}(\phi) \le$$

$$n|x|(1+|x|)^k \sum_{j=1}^{n} \rho_{k,\alpha+e_j}(\phi).$$

f) $\rho_{k,\alpha}(M_\xi\phi-\phi) \le |\xi|\rho_{k+1,\alpha}(\phi) + \sum_{0<\alpha'\le\alpha} \binom{\alpha}{\alpha'}|\xi^{\alpha'}|\rho_{k,\alpha-\alpha'}(\phi)$:

$$\rho_{k,\alpha}(M_\xi\phi-\phi) = \sup_{y\in\mathbb{R}^n} (1+|y|)^k|\partial_y^\alpha[e^{i<y,\xi>}\phi(y)-\phi(y)]| =$$

$$\sup_{y\in\mathbb{R}^n} (1+|y|)^k\Big|\sum_{\alpha'\le\alpha} \binom{\alpha}{\alpha'}\partial^{\alpha'}(e^{i<y,\xi>}-1)\partial^{\alpha-\alpha'}\phi(y)\Big| \le$$

$$\sup_{y\in\mathbb{R}^n} (1+|y|)^k|e^{i<y,\xi>}-1||\partial^\alpha\phi(y)| +$$

$$\sum_{0<\alpha'\le\alpha} \binom{\alpha}{\alpha'}\sup_{y\in\mathbb{R}^n} (1+|y|)^k|i^{|\alpha'|}\xi^{\alpha'}e^{i<y,\xi>}||\partial^{\alpha-\alpha'}\phi(y)| \le$$

$$\sup_{y\in\mathbb{R}^n} (1+|y|)^k|<y,\xi>||\partial^\alpha\phi(y)| + \sum_{0<\alpha'\le\alpha} \binom{\alpha}{\alpha'}\sup_{y\in\mathbb{R}^n} (1+|y|)^k|i^{|\alpha'|}||\xi^{\alpha'}||\partial^{\alpha-\alpha'}\phi(y)|$$

$$\le \sup_{y\in\mathbb{R}^n} (1+|y|)^{k+1}|\xi||\partial^\alpha\phi(y)| + \sum_{0<\alpha'\le\alpha} \binom{\alpha}{\alpha'}|\xi^{\alpha'}|\rho_{k,\alpha-\alpha'}(\phi) = |\xi|\rho_{k+1,\alpha}(\phi) +$$

$$\sum_{0<\alpha'\le\alpha} \begin{bmatrix}\alpha\\ \alpha'\end{bmatrix} |\xi^{\alpha'}| \rho_{k,\alpha-\alpha'}(\phi).$$

g) $z \longmapsto U(z)\phi$ is continuous: $U(z) - U(z_0) = M_\xi T_x - M_{\xi_0} T_{x_0} =$ $M_\xi T_x - M_\xi T_{x_0} + M_\xi T_{x_0} - M_{\xi_0} T_{x_0} = M_\xi T_{x_0}(T_{x-x_0} - I) + M_{\xi_0}(M_{\xi-\xi_0} T_{x_0} - T_{x_0})$. So $\rho_{k,\alpha}(U(z)\phi - U(z_0)\phi) \le \rho_{k,\alpha}(M_\xi T_{x_0}(T_{x-x_0}\phi - \phi)) + \rho_{k,\alpha}(M_{\xi_0}(M_{\xi-\xi_0} T_{x_0}\phi - T_{x_0}\phi))$. Since we wish to estimate the limit as $(x,\xi) \longrightarrow (x_0,\xi_0)$, we can assume that ξ satisfies $|\xi - \xi_0| < 1$. Then $|\xi| < |\xi_0| + 1$. By a), $\rho_{k,\alpha}(M_\xi T_{x_0}(T_{x-x_0}\phi - \phi))$ is bounded by a finite sum of terms of the form $c_\beta |\xi^\beta| \rho_{k,\alpha-\beta}(T_{x_0}(T_{x-x_0}\phi - \phi)) \le$ $c_\beta (1 + |\xi_0|)^{|\beta|} \rho_{k,\alpha-\beta}(T_{x_0}(T_{x-x_0}\phi - \phi))$. By b), $\rho_{k,\alpha-\beta}(T_{x_0}(T_{x-x_0}\phi - \phi)) \le$ $(1 + |x_0|)^k \rho_{k,\alpha-\beta}(T_{x-x_0}\phi - \phi)$. By e), $\rho_{k,\alpha-\beta}(T_{x-x_0}\phi - \phi) \le$

$$n|x - x_0|(1 + |x - x_0|)^k \sum_{j=1}^{n} \rho_{\alpha-\beta+e_j}(\phi) \longrightarrow 0 \text{ as } x \longrightarrow x_0.$$

Thus $\rho_{k,\alpha}(M_\xi T_{x_0}(T_{x-x_0}\phi - \phi)) \longrightarrow 0$ as $(x,\xi) \longrightarrow (x_0,\xi_0)$. By a), $\rho_{k,\alpha}(M_{\xi_0}(M_{\xi-\xi_0} T_{x_0}\phi - T_{x_0}\phi))$ is bounded by a finite sum of terms of the form $c_\beta |\xi_0^\beta| \rho_{k,\alpha-\beta}(M_{\xi-\xi_0} T_{x_0}\phi - T_{x_0}\phi)$. By f), $\rho_{k,\alpha-\beta}(M_{\xi-\xi_0} T_{x_0}\phi - T_{x_0}\phi)$ is bounded by a sum whose terms consist of $|\xi - \xi_0| \rho_{k+1,\alpha-\beta}(T_{x_0}\phi)$ and a finite number of terms of the form $c_{\beta\gamma} |(\xi - \xi_0)^\gamma| \rho_{k,\alpha-\beta-\gamma}(T_{x_0}\phi)$ where $\gamma > 0$. All these terms clearly go to 0 as $\xi \longrightarrow \xi_0$. Therefore $\rho_{k,\alpha}(M_{\xi_0}(M_{\xi-\xi_0} T_{x_0}\phi - T_{x_0}\phi)) \longrightarrow 0$ as $(x,\xi) \longrightarrow (x_0,\xi_0)$. Hence $\rho_{k,\alpha}(U(z)\phi - U(z_0)\phi) \longrightarrow 0$ as $(x,\xi) \longrightarrow (x_0,\xi_0)$.

h) $z \longmapsto (\phi \otimes \bar\theta)_z f \colon \mathbb{R}^{2n} \longrightarrow S(\mathbb{R}^n)$ is continuous: $(\phi \otimes \bar\theta)_z f =$ $(U(z)^{-1} f, \theta) U(z)\phi = (f, U(z)\theta) U(z)\phi = (f, U(z)\theta - U(z_0)\theta) U(z)\phi +$ $(f, U(z_0)\theta)(U(z)\phi - U(z_0)\phi) + (f, U(z_0)\theta) U(z_0)\phi = (f, U(z)\theta - U(z_0)\theta) U(z)\phi +$ $(f, U(z_0)\theta)(U(z)\phi - U(z_0)\phi) + (\phi \otimes \bar\theta)_{z_0} f$. So $\rho_{k,\alpha}((\phi \otimes \bar\theta)_z f - (\phi \otimes \bar\theta)_{z_0} f) \le$

$|(f, U(z)\theta - U(z_0)\theta)|\rho_{k,\alpha}(U(z)\phi) + |(f, U(z_0)\theta)|\rho_{k,\alpha}(U(z)\phi - U(z_0)\phi) \longrightarrow 0$ by g) and by Proposition 8.

3. Theorems on Integrability.

The following definitions and theorems on the integration of Banach space valued functions are taken from Hille & Phillips [2]. Hille & Phillips use general measures, but we will only use Lebesgue measure on $\mathbb{R}^{2n}$. B will represent a Banach space with norm $\| \ \|$ and B* will be its dual.

17. Definition. Let $f: \mathbb{R}^{2n} \longrightarrow B$ be a function. Then

a) f is countably valued if it assumes a countable set of values in B, assuming each value on a measurable subset of $\mathbb{R}^{2n}$.

b) f is almost separably valued if $\exists$ a null set E such that $f(\mathbb{R}^{2n}\sim E)$ is contained in a separable subspace of B.

c) f is weakly measurable if $g^*f: \mathbb{R}^{2n} \longrightarrow \mathbb{C}$ is measurable $\forall g^* \in B^*$.

d) f is strongly measurable if $\exists$ a sequence of countably valued functions converging almost everywhere to f.

18. Theorem. A Banach space valued function on $\mathbb{R}^{2n}$ is strongly measurable iff it is weakly measurable and almost separably valued.

19. Corollary. A continuous Banach space valued function on $\mathbb{R}^{2n}$ is strongly measurable.

20. Definition. Let $f: \mathbb{R}^{2n} \longrightarrow B$ be a function. Then

a) if f is countably valued then f is Bochner integrable if $\|f\|: \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is integrable. Let $\{x_k\}$ be the set of values of f. We define $\int_{\mathbb{R}^{2n}} f(z)dz = \sum_{k=1}^{\infty} x_k m(E_k)$.

b) f is Bochner integrable if $\exists$ a sequence of countably valued integrable functions f_j converging almost everywhere to f and such that $\lim_{j\to\infty} \int_{\mathbb{R}^{2n}} \|f(z)-$

$f_j(z)\|dz = 0$. We define $\int_{\mathbb{R}^{2n}} f(z)dz = \lim_{j\to\infty}\int_{\mathbb{R}^{2n}} f_j(z)dz$.

21. Theorem. $f\colon \mathbb{R}^{2n} \longrightarrow B$ is Bochner integrable if and only if f is strongly measurable and $\int_{\mathbb{R}^{2n}} \|f(z)\|dz < \infty$.

22. Theorem. $\|\int_{\mathbb{R}^{2n}} f(z)dz\| \le \int_{\mathbb{R}^{2n}} \|f(z)\|dz$.

23. Theorem. Let $T\colon B \longrightarrow B'$ be a bounded linear map of Banach spaces. If $f\colon \mathbb{R}^{2n} \longrightarrow B$ is Bochner integrable then so is Tf and $T\int_{\mathbb{R}^{2n}} f(z)dz = \int_{\mathbb{R}^{2n}} T[f(z)]dz$.

If $B = \mathcal{B}(L^2(\mathbb{R}^n))$, then we now can consider three integrals.

24. Definition. Let $L\colon \mathbb{R}^{2n} \longrightarrow \mathcal{B}(L^2(\mathbb{R}^n))$ be a function. Then

a) if L is Bochner integrable, we will say that L is uniformly integrable. In this case clearly $\int_{\mathbb{R}^{2n}} L(z)dz \in \mathcal{B}(L^2(\mathbb{R}^n))$.

b) if, for each $f \in L^2(\mathbb{R}^n)$, $z \longmapsto L(z)f\colon \mathbb{R}^{2n} \longrightarrow L^2(\mathbb{R}^n)$ is Bochner integrable, then $Af = \int_{\mathbb{R}^{2n}} L(z)f\, dz$ defines an operator A on all of $L^2(\mathbb{R}^n)$. We then will say that L is strongly integrable and $A = \int_{\mathbb{R}^{2n}} L(z)dz$.

c) if, $\forall f,g \in L^2(\mathbb{R}^n)$, $z \longmapsto (L(z)f, g)\colon\ \mathbb{R}^{2n} \longrightarrow \mathbb{C}$ is Lebesgue integrable then $\int_{\mathbb{R}^{2n}} (L(z)f,g)dz$ is a sesquilinear form on $L^2(\mathbb{R}^n)$ and so $(Af,g) = \int_{\mathbb{R}^{2n}} (L(z)f,g)dz$ defines an operator A on $L^2(\mathbb{R}^n)$. If the form is bounded then so is the operator. In this case we will say that L is weakly integrable and $A = \int_{\mathbb{R}^{2n}} L(z)dz$.

By Theorem 23, it is clear that uniformly integrable $\longrightarrow$ strongly integrable $\longrightarrow$ weakly integrable and if any two of these integrals exist they define the same operator.

25. Proposition. If $L \in \mathcal{B}(L^2(\mathbb{R}^n))$ is such that $z \longmapsto L_z\colon \mathbb{R}^{2n} \longrightarrow (\mathcal{B}(L^2(\mathbb{R}^n))$, norm topology) is continuous, and if $a \in L^1(\mathbb{R}^{2n})$, then $a(z)L_z$ is uniformly integrable and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \|a\|_{L^1(\mathbb{R}^{2n})}\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}$.

Proof. Since $z \longmapsto L_z$ is continuous, this map is strongly measurable and so $a(z)L_z$ is strongly measurable. $\int_{\mathbb{R}^{2n}} \|a(z)L_z\|_{\mathcal{B}(L^2(\mathbb{R}^n))}dz =$ $\int_{\mathbb{R}^{2n}} |a(z)|\,\|L_z\|_{\mathcal{B}(L^2(\mathbb{R}^n))}dz = \int_{\mathbb{R}^{2n}} |a(z)|\,\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}dz =$ $\|a\|_{L^1(\mathbb{R}^{2n})}\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}$. So $z \longmapsto a(z)L_z$ is uniformly integrable and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\| \le \int_{\mathbb{R}^{2n}} \|a(z)L_z\|_{\mathcal{B}(L^2(\mathbb{R}^n))}dz = \|a\|_{L^1(\mathbb{R}^{2n})}\|L\|_{\mathcal{B}(L^2(\mathbb{R}^n))}$.

The following lemma essentially is Problem 24.3, p.183 in Shubin.

26. Lemma. If $f,\phi \in L^2(\mathbb{R}^n)$, then $\int_{\mathbb{R}^{2n}} |(f, U(z)\phi)|^2 dz =$ $(2\pi)^n \|f\|^2_{L^2(\mathbb{R}^n)}\|\phi\|^2_{L^2(\mathbb{R}^n)}$.

Proof: $(f, U(z)\phi) = \int_{\mathbb{R}^n} f(y)\overline{e^{i\langle y,\xi\rangle}T_x\phi(y)}dy = \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} f(y)\overline{T_x\phi(y)}dy =$ $[f\overline{T_x\phi(y)}]^\wedge(\xi)$. $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)\overline{T_x\phi(y)}|^2 dydx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)|^2|\overline{T_x\phi(y)}|^2 dxdy =$ $\int_{\mathbb{R}^n} |f(y)|^2 \int_{\mathbb{R}^n} |\phi(y-x)|^2 dxdy = \int_{\mathbb{R}^n} |f(y)|^2 \int_{\mathbb{R}^n} |\phi(x)|^2 dxdy = \|f\|^2_{L^2(\mathbb{R}^n)}\|\phi\|^2_{L^2(\mathbb{R}^n)}$. So $\int_{\mathbb{R}^n} |f(y)\overline{T_x\phi(y)}|^2 dy < \infty$ a.e. dx, i.e., $f\overline{T_x\phi} \in L^2(\mathbb{R}^n)$ a.e. dx. Thus by the Plancherel theorem, $\|[f\overline{T_x\phi}]^\wedge\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}}\|f\overline{T_x\phi}\|^2_{L^2(\mathbb{R}^n)}$ a.e. dx. Therefore $\int_{\mathbb{R}^{2n}} |(f, U(z)\phi)|^2 dz = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |[f\overline{T_x\phi(y)}]^\wedge(\xi)|^2 d\xi dx = \int_{\mathbb{R}^n} \|[f\overline{T_x\phi}]^\wedge\|^2_{L^2(\mathbb{R}^n)}dx =$ $(2\pi)^n \int_{\mathbb{R}^n} \|f\overline{T_x\phi}\|^2_{L^2(\mathbb{R}^n)}dx = (2\pi)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)\overline{T_x\phi(y)}|^2 dydx =$ $(2\pi)^n \|f\|^2_{L^2(\mathbb{R}^n)}\|\phi\|^2_{L^2(\mathbb{R}^n)}$.

27. Lemma. Let $f,g,\phi,\theta \in L^2(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^{2n}} (f,U(z)\theta)(U(z)\phi,g)dz =$ $(2\pi)^n(f,g)(\phi,\theta)$.

Proof. First we will show that $\forall f \in L^2(\mathbb{R}^n)$, $\int_{\mathbb{R}^{2n}} (f, U(z)\theta)(U(z)\phi, f)dz =$ $(2\pi)^n(f,f)(\phi,\theta)$. This follows by polarization since both sides are sesquilinear forms in ϕ and θ and since the quadratic forms obtained by setting

$\phi = \theta$ are equal by Lemma 26. Now fix ϕ and θ. Both sides of the equation in the conclusion of the lemma now are sesquilinear forms in f and g whose quadratic forms we have shown to agree. The conclusion, therefore, also follows by polarization.

The following lemma is very close to Problem 27.4 in Shubin.

28. Lemma. If $L \in C_1$ then $\int_{\mathbb{R}^{2n}}(L_z f, g)dz = (2\pi)^n(f, g)\mathrm{tr}L$ where trL is the trace of L.

Proof. Since $L \in C_1$, it can be represented by $L = \sum \lambda_j \phi_j \otimes \bar{\theta}_j$ where $\lambda_j \geq 0$ $\forall j$, $\sum \lambda_j < \infty$, and $\{\phi_j\}$ and $\{\theta_j\}$ are orthonormal sequences. Then trL = $\sum \lambda_j(\phi_j, \theta_j)$ (Schatten, Lemma 9, p.41). $\int_{\mathbb{R}^{2n}}((\phi_j \otimes \bar{\theta}_j)_z f, g)dz =$ $\int_{\mathbb{R}^{2n}}(U(z)(\phi_j \otimes \bar{\theta}_j)U(z)^{-1}f, g)dz = \int_{\mathbb{R}^{2n}}((U(z)^{-1}f, \theta_j)\phi_j, U(z)^{-1}g)dz =$ $\int_{\mathbb{R}^{2n}}(U(z)^{-1}f, \theta_j)(\phi_j, U(z)^{-1}g)dz = \int_{\mathbb{R}^{2n}}(f, U(z)\theta_j)(U(z)\phi_j, g) =$ $(2\pi)^n(f, g)(\phi_j, \theta_j)$ by Lemma 27. Therefore $|\int_{\mathbb{R}^{2n}}((\phi_j \otimes \bar{\theta}_j)_z f, g)dz| \leq$ $(2\pi)^n|(f, g)|$ So the series $\sum \lambda_j \int_{\mathbb{R}^{2n}}((\phi_j \otimes \bar{\theta}_j)_z f, g)dz$ converges absolutely and therefore $\int_{\mathbb{R}^{2n}} \sum \lambda_j((\phi_j \otimes \bar{\theta}_j)_z f, g)dz = \sum \lambda_j (2\pi)^n(f, g)(\phi_j, \theta_j) = (2\pi)^n(f, g)\mathrm{tr}L$. Let $L, M \in C_1$. Then $\|L_z - M_z\|_1 = \|U(z)(L-M)U(z)^{-1}\|_1 \leq$ $\|U(z)\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|L-M\|_1\|U(z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|L-M\|_1$. Thus $L \longmapsto L_z: C_1 \longrightarrow C_1$ is continuous. So $L_z = \sum \lambda_j(\phi_j \otimes \bar{\theta}_j)_z$ with convergence in C_1. Since convergence in $C_1 \longrightarrow$ convergence in WOT, $(L_z f, g) = \sum \lambda_j((\phi_j \otimes \bar{\theta}_j)_z f, g)$. Therefore $\int_{\mathbb{R}^{2n}}(L_z f, g)dz = \int_{\mathbb{R}^{2n}} \sum \lambda_j((\phi_j \otimes \bar{\theta}_j)_z f, g)dz = (2\pi)^n(f, g)\mathrm{tr}L$.

29. Proposition. If $L \in C_2$ and $a \in L^2(\mathbb{R}^{2n})$, then $z \longmapsto a(z)L_z: \mathbb{R}^{2n} \longrightarrow \mathcal{B}(L^2(\mathbb{R}^n))$ is strongly integrable and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq$ $(2\pi)^{\frac{n}{2}}\|a\|_{L^2(\mathbb{R}^{2n})}\|L\|_2$.

Proof. $L = \sum \lambda_j \phi_j \otimes \bar{\theta}_j$ where the λ_j's are the nonzero eigenvalues of $\sqrt{L^* L}$.

Thus the nonzero eigenvalues of L^*L are the λ_j^2 's. So $\|L\|_2^2 = \mathrm{tr}L^*L$. $\forall f \in L^2(\mathbb{R}^n)$, $\|\int_{\mathbb{R}^{2n}} a(z)L_z f dz\|_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^{2n}} \|a(z)L_z f\|_{L^2(\mathbb{R}^n)} dz = \int_{\mathbb{R}^{2n}} |a(z)| \, \|L_z f\|_{L^2(\mathbb{R}^n)} dz$ $\leq \sqrt{\int_{\mathbb{R}^{2n}} |a(z)|^2 dz}\sqrt{\int_{\mathbb{R}^{2n}} \|L_z f\|^2_{L^2(\mathbb{R}^n)} dz} = \|a\|_{L^2(\mathbb{R}^{2n})}\sqrt{\int_{\mathbb{R}^{2n}} (L_z f, L_z f) dz} =$ $\|a\|_{L^2(\mathbb{R}^{2n})}\sqrt{\int_{\mathbb{R}^{2n}} ((L_z)^* L_z f, f) dz} = \|a\|_{L^2(\mathbb{R}^{2n})}\sqrt{\int_{\mathbb{R}^{2n}} ((L^*)_z L_z f, f) dz} =$ $\|a\|_{L^2(\mathbb{R}^{2n})}\sqrt{\int_{\mathbb{R}^{2n}} ((L^*L)_z f, f) dz} = \|a\|_{L^2(\mathbb{R}^{2n})}\sqrt{(2\pi)^n \|f\|^2_{L^2(\mathbb{R}^n)} \mathrm{tr}L^*L} =$ $(2\pi)^{\frac{n}{2}} \|a\|_{L^2(\mathbb{R}^{2n})} \|L\|_2 \|f\|_{L^2(\mathbb{R}^n)}$. Thus $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq$ $(2\pi)^{\frac{n}{2}} \|a\|_{L^2(\mathbb{R}^{2n})} \|L\|_2$.

The following proposition is an extension of Problem 27.2, p.198 in Shubin.

30. Proposition. if $a \in L^\infty(\mathbb{R}^{2n})$ and $L \in C_1$ then $a(z)L_z$ is weakly integrable and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq (2\pi)^n \|a\|_{L^\infty(\mathbb{R}^{2n})} \|L\|_1$.

Proof. $L = \sum \lambda_j \phi_j \otimes \bar{\theta}_j$ with convergence in C_1. Since $L \longmapsto L_z$: $C_1 \longrightarrow C_1$ is continuous and convergence in $C_1 \longrightarrow$ convergence in WOT, $(L_z f, g) = (L_z f, g)$ $= \sum \lambda_j ((\phi_j \otimes \bar{\theta}_j)_z f, g) = \sum \lambda_j ((U(z)^{-1} f, \theta_j) U(z)\phi_j, g) =$ $\sum \lambda_j (f, U(z)\theta_j)(U(z)\phi_j, g)$. So by Lemma 26, $\forall f,g \in L^2(\mathbb{R}^n)$, $|\int_{\mathbb{R}^{2n}} a(z)(L_z f, g) dz| \leq \int_{\mathbb{R}^{2n}} |a(z)| \, |(L_z f, g)| dz \leq \|a\|_{L^\infty(\mathbb{R}^{2n})} \int_{\mathbb{R}^{2n}} |(L_z f, g)| dz \leq$ $\|a\|_{L^\infty(\mathbb{R}^{2n})} \sum \lambda_j \int_{\mathbb{R}^{2n}} |(f, U(z)\theta_j)| \, |(U(z)\phi_j, g)| dz \leq$ $\|a\|_{L^\infty(\mathbb{R}^{2n})} \sum \lambda_j \sqrt{\int_{\mathbb{R}^{2n}} |(f, U(z)\theta_j)|^2 dz}\sqrt{\int_{\mathbb{R}^2} |(U(z)\phi_j, g)|^2 dz} \leq$ $\|a\|_{L^\infty(\mathbb{R}^{2n})} \sum \lambda_j (2\pi)^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)} \|\theta_j\|_{L^2(\mathbb{R}^n)} (2\pi)^{\frac{n}{2}} \|\phi_j\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} =$ $\|a\|_{L^\infty(\mathbb{R}^{2n})} (2\pi)^n \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \sum \lambda_j = (2\pi)^n \|a\|_{L^\infty(\mathbb{R}^{2n})} \|L\|_1 \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$. So $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq (2\pi)^n \|a\|_{L^\infty(\mathbb{R}^{2n})} \|L\|_1$.

31. Proposition. If $a \in L^1(\mathbb{R}^{2n})$ and $L \in C_p$ then $z \longmapsto a(z)L_z: \mathbb{R}^{2n} \longrightarrow C_p$ is Bochner integrable and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_p \le \|a\|_{L^1(\mathbb{R}^{2n})}\|L\|_p$.

Proof. Since $z \longmapsto L_z: \mathbb{R}^{2n} \longrightarrow C_p$ is continuous by Proposition 15, $z \longmapsto a(z)L_z: \mathbb{R}^{2n} \longrightarrow C_p$ is strongly measurable and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_p \le \int_{\mathbb{R}^{2n}} \|a(z)L_z\|_p dz = \int_{\mathbb{R}^{2n}} |a(z)|\,\|L_z\|_p dz \le \|a\|_{L^1(\mathbb{R}^{2n})}\|L\|_p$ since $\|L_z\|_p = \|U(z)LU(z)^{-1}\|_p \le \|U(z)\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|L\|_p\|U(z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|L\|_p$.

Note. Since for $q \ge p$, $C_p \hookrightarrow C_q$ is continuous, $L \in C_p$ and $a \in L^1(\mathbb{R}^{2n})$ $\Longrightarrow z \longmapsto a(z)L_z: \mathbb{R}^{2n} \longrightarrow C_q$ is Bochner integrable $\forall q \ge p$ and all these integrals coincide.

32. Proposition. Let a be measurable on $\mathbb{R}^{2n}$ and let $L \in C_1$. Suppose $\forall \varepsilon > 0$ $\exists$ $a_\varepsilon \in L^1(\mathbb{R}^{2n})$ and $b_\varepsilon \in L^\infty(\mathbb{R}^{2n})$ satisfying $\|b_\varepsilon\|_{L^\infty(\mathbb{R}^{2n})} \le \varepsilon$ such that $a = a_\varepsilon + b_\varepsilon$. Then $z \longmapsto a(z)L_z$ is weakly integrable and $\int_{\mathbb{R}^{2n}} a(z)L_z dz \in C_\infty$.

Proof. By Proposition 30, $z \longmapsto b_\varepsilon(z)L_z$ is weakly integrable and $\|\int_{\mathbb{R}^{2n}} b_\varepsilon(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \|b_\varepsilon\|_{L^\infty(\mathbb{R}^{2n})} \le \varepsilon$. By Proposition 31, $\int_{\mathbb{R}^{2n}} a_\varepsilon(z)L_z dz \in C_1 \subseteq C_\infty$ and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz - \int_{\mathbb{R}^{2n}} a_\varepsilon(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|\int_{\mathbb{R}^{2n}} b_\varepsilon(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \varepsilon$. So $\int_{\mathbb{R}^{2n}} a_\varepsilon(z)L_z dz \in C_\infty$ and $\int_{\mathbb{R}^{2n}} a_\varepsilon(z)L_z dz \longrightarrow \int_{\mathbb{R}^{2n}} a(z)L_z dz$ in $(\mathcal{B}(L^2(\mathbb{R}^n))$, norm topology). Thus $\int_{\mathbb{R}^{2n}} a(z)L_z dz \in C_\infty$.

33. Corollary. If $a \in L^p(\mathbb{R}^{2n})$, $1 \le p < \infty$ and $L \in C_1$ then $z \longmapsto a(z)L_z$ is weakly integrable and $\int_{\mathbb{R}^{2n}} a(z)L_z dz \in C_\infty$.

Proof. Let $a_\varepsilon(z) = \begin{cases} 0 \text{ if } |a(z)| < \varepsilon \\ a(z) \text{ if } |a(z)| \ge \varepsilon \end{cases}$ and $b_\varepsilon(z) = \begin{cases} a(z) \text{ if } |a(z)| < \varepsilon \\ 0 \text{ if } |a(z)| \ge \varepsilon \end{cases}$.

Then $a = a_\varepsilon + b_\varepsilon$. $\frac{|a_\varepsilon|}{\varepsilon} \ge 1$. So $\left|\frac{a_\varepsilon}{\varepsilon}\right| \le \left|\frac{a_\varepsilon}{\varepsilon}\right|^p \in L^1(\mathbb{R}^{2n})$. Thus $a_\varepsilon \in L^1(\mathbb{R}^{2n})$.

By the definition of b_ε, $\|b_\varepsilon\|_{L^\infty(\mathbb{R}^{2n})} \le \varepsilon$.

34. Corollary. If $L \in C_1$ and $a \in L^1_{loc}(\mathbb{R}^{2n})$ satisfies $\lim_{R\to\infty} \operatorname{ess\,sup}_{z>R} |a(z)| = 0$, then $z \longmapsto a(z)L_z$ is weakly integrable and $\int_{\mathbb{R}^{2n}} a(z)L_z dz \in C_\infty$.

Proof. By hypothesis $\forall\varepsilon > 0\ \exists R_\varepsilon > 0$ such that $\operatorname{ess\,sup}_{z>R_\varepsilon} |a(z)| < \varepsilon$. Let χ_ε be the characteristic function of the ball in $\mathbb{R}^{2n}$ with center 0 and radius R_ε. Then $a_\varepsilon = \chi_\varepsilon a \in L^1(\mathbb{R}^{2n})$ and $b_\varepsilon = (1-\chi_\varepsilon)a$ satisfies $\|b_\varepsilon\|_{L^\infty(\mathbb{R}^{2n})} < \varepsilon$. Clearly $a = a_\varepsilon + b_\varepsilon$.

35. Corollary. Let $L \in C_1$ and let a be an element of the closure in $L^\infty(\mathbb{R}^{2n})$ of $L^1(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$. Then $z \longmapsto a(z)L_z$ is weakly integrable and $\int_{\mathbb{R}^{2n}} a(z)L_z dz \in C_\infty$.

Proof. $\forall\varepsilon > 0\ \exists a_\varepsilon \in L^1(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ such that $b_\varepsilon = a - a_\varepsilon$ satisfies $\|b_\varepsilon\|_{L^\infty(\mathbb{R}^{2n})} < \varepsilon$.

36. Proposition. If $a \in L^2(\mathbb{R}^{2n})$ and $L \in C_2$ then $z \longmapsto a(z)L_z$: $\mathbb{R}^{2n} \longrightarrow \mathcal{B}(L^2(\mathbb{R}^n))$ is strongly integrable, $\int_{\mathbb{R}^{2n}} a(z)L_z dz \in C_\infty$, and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_\infty \le (2\pi)^{\frac{n}{2}} \|a\|_{L^2(\mathbb{R}^{2n})} \|L\|_2$.

Proof. Let χ_R be the characteristic function of the ball in $\mathbb{R}^{2n}$ with center 0 and radius R. Then $a = \chi_R a + (1-\chi_R)a$. By Proposition 31, $z \longmapsto \chi_R(z)a(z)L_z$: $\mathbb{R}^{2n} \longrightarrow C_2$ is Bochner integrable. So $\int_{\mathbb{R}^{2n}} \chi_R(z)a(z)L_z dz \in C_2 \subseteq C_\infty$ and $z \longmapsto \chi_R(z)a(z)L_z$ is strongly integrable. By Proposition 29, $z \longmapsto (1-\chi_R(z))a(z)L_z$: $\mathbb{R}^{2n} \longrightarrow \mathcal{B}(L^2(\mathbb{R}^n))$ is strongly integrable and $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz - \int_{\mathbb{R}^{2n}} \chi_R(z)a(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|\int_{\mathbb{R}^{2n}} (1-\chi_R(z))a(z)L_z dz\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le (2\pi)^{\frac{n}{2}} \|(1-\chi_R)a\|_{L^2(\mathbb{R}^{2n})} \|L\|_2 \longrightarrow 0$ as $R \longrightarrow \infty$. Therefore $\int_{\mathbb{R}^{2n}} a(z)L_z dz \in C_\infty$. By Propositiion 29, $\|\int_{\mathbb{R}^{2n}} a(z)L_z dz\|_\infty \le (2\pi)^{\frac{n}{2}} \|a\|_{L^2(\mathbb{R}^{2n})} \|L\|_2$.

4. Miscellaneous Results.

37. Proposition. If B is a pseudodiffferential operator with Weyl symbol b (see ¶ 1) then B_z is a pseudodifferential operator with Weyl symbol $T_z b$.

Proof. Let $\phi \in S(\mathbb{R}^n)$. Then by Lemma 5, $B_z\phi(w) = U(z)BU(z)^{-1}\phi(w) =$ $M_\xi T_x B T_{-x} M_{-\xi}\phi(w) = M_\xi T_x B e^{-i<-x,-\xi>} M_{-\xi} T_{-x}\phi(w) = e^{-i<x,\xi>} M_\xi T_x B M_{-\xi} T_{-x}\phi(w) =$ $e^{-i<x,\xi>} e^{i<w,\xi>} B M_{-\xi} T_{-x}\phi(w-x) =$

$$e^{i<w-x,\xi>}(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<w-x-y,\eta>} b\left(\frac{w-x+y}{2},\eta\right) e^{i<y,-\xi>}\phi(y+x)\,dy\,d\eta =$$

$$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<w-x-y,\xi>} e^{i<w-x-y,\eta>} b\left(\frac{w-x+y}{2},\eta\right) e^{i<y,-\xi>}\phi(y+x)\,dy\,d\eta =$$

$$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<w-y',\eta'>} b\left(\frac{w+y'-2x}{2},\eta'-\xi\right)\phi(y')\,dy'\,d\eta' =$$

$$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<w-y,\eta>} b\left(\frac{w+y}{2}-x,\eta-\xi\right)\phi(y)\,dy\,d\eta.$$ Since $b\left(\frac{w+y}{2}-x,\eta-\xi\right) = T_z b\left(\frac{w+y}{2},\eta\right)$ if $z = (x,\xi)$, B_z has Weyl symbol $T_z b$.

The following lemma provides a commonly used method for converting some integrals which exist only as iterated integrals to absolutely convergent integrals. We will use the following notation: for $\xi \in \mathbb{R}^n$ we define $<\xi> = \sqrt{1+|\xi|^2}$ and we define $<D_y> = \sqrt{1+D_{y_1}^2+D_{y_2}^2+\ldots+D_{y_n}^2}$ where $D_{y_j} = \frac{1}{i}\frac{\partial}{\partial y_j}$. We will only consider $<D_y>^{2p}$ where p is a positive integer so that $<D_y>^{2p}$ is a differential operator. For each p and multi-index α with $|\alpha| \leq p$ $\exists$ constants $c_\alpha \geq 0$ such that $<\xi>^{2p} = \sum_{|\alpha|\leq p} c_\alpha \xi^{2\alpha}$ and $<D_y>^{2p} = \sum_{|\alpha|\leq p} c_\alpha D_y^{2\alpha}$. We then have $<D_y>^{2p} e^{-i<y,\xi>} = \sum_{|\alpha|\leq p} c_\alpha \xi^{2\alpha} e^{-i<y,\xi>} = <\xi>^{2p} e^{-i<y,\xi>}$ so that $e^{-i<y,\xi>} =$ $<\xi>^{-2p}<D_y>^{2p}e^{-i<y,\xi>}$.

38. Lemma. Let $g \in C^\infty(\mathbb{R}^n)$. Suppose that for each multi-index α $\exists C_\alpha$ such that $\forall y \in \mathbb{R}^n$, $|D^\alpha g(y)| \le C_\alpha(1+|y|)^j$ where $j < -n$. Then $\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} g(y)dy =$ $\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle}\langle\xi\rangle^{-2p}\langle D_y\rangle^{2p} g(y)dy$.

Proof. $\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} g(y)dy = \int_{\mathbb{R}^n}\langle\xi\rangle^{-2p}[\langle D_y\rangle^{2p} e^{-i\langle y,\xi\rangle}]g(y)dy$. We now integrate by parts over a box. As the sides of the box go to ∞, the boundary terms go to 0 due to the inequalities satisfied by $D^\alpha g$. This gives our result.

39. Lemma. Suppose $g(y,\xi,u)$ is measurable on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^N$ and is C^∞ in y for each fixed ξ and u. Suppose that for each multi-index α, g satisfies: $\forall(y,\xi,u) \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^N$ $|D_y^\alpha g(y,\xi,u)| \le (1+|y|)^j(1+|\xi|)^k g_\alpha(u)$ where $j < -n, g_\alpha \ge 0$, and $g_\alpha \in L^1(\mathbb{R}^N)$. Then $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{-i\langle y,\xi\rangle} g(y,\xi,u)dyd\xi du =$ $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{-i\langle y,\xi\rangle} g(y,\xi,u)dudyd\xi$.

Proof. By Lemma 38, $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{-i\langle y,\xi\rangle} g(y,\xi,u)dyd\xi du =$ $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{-i\langle y,\xi\rangle}\langle\xi\rangle^{-2p}\langle D_y\rangle^{2p} g(y,\xi,u)dyd\xi du$. $|\langle D_y\rangle^{2p} g(y,\xi,u)| =$

$$|\sum_{|\alpha|\le p} c_\alpha D_y^{2\alpha} g(y,\xi,u)| \le \sum_{|\alpha|\le p} c_\alpha(1+|y|)^j(1+|\xi|)^k g_{2\alpha}(u) = (1+|y|)^j(1+|\xi|)^k G_p(u)$$

where $G_p = \sum_{|\alpha|\le p} g_\alpha \in L^1(\mathbb{R}^N)$. If we choose p such that $j - 2p < -n$, then the last integral converges absolutely. By Fubini's Theorem, $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{-i\langle y,\xi\rangle}\langle\xi\rangle^{-2p}\langle D_y\rangle^{2p} g(y,\xi,u)dyd\xi du =$ $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{-i\langle y,\xi\rangle}\langle\xi\rangle^{-2p}\langle D_y\rangle^{2p} g(y,\xi,u)dudyd\xi$. Since $\forall\alpha$, $|D_y^\alpha g(y,\xi,u)| \le$ $c_\alpha(1+|\xi|)^k g_\alpha(u)$, $\forall\alpha$, $D_y^\alpha\int_{\mathbb{R}^N} g(y,\xi,u)du = \int_{\mathbb{R}^N} D_y^\alpha g(y,\xi,u)du$. Therefore $\int_{\mathbb{R}^N}\langle D_y\rangle^{2p} g(y,\xi,u)du = \langle D_y\rangle^{2p}\int_{\mathbb{R}^N} g(y,\xi,u)du$. So $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{-i\langle y,\xi\rangle}\langle\xi\rangle^{-2p}\langle D_y\rangle^{2p} g(y,\xi,u)dudyd\xi =$ $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle}\langle\xi\rangle^{-2p}\langle D_y\rangle^{2p}\int_{\mathbb{R}^N} g(y,\xi,u)dudyd\xi$. Since $|D^\alpha\int_{\mathbb{R}^N} g(y,\xi,u)du| =$

$|\int_{\mathbb{R}^N} D^\alpha g(y,\xi,u)du| \le \int_{\mathbb{R}^N} |D^\alpha g(y,\xi,u)|du \le \int_{\mathbb{R}^N}(1+|y|)^j(1+|\xi|)^k g_\alpha(u)du =$ $\|g_\alpha\|_{L^1(\mathbb{R}^N)}(1+|\xi|)^k(1+|y|)^j$, For each ξ, asd a function of y, $\int_{\mathbb{R}^N} g(y,\xi,u)du$ satisfies the conditions of Lemma 38 and therefore $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i<y,\xi>}<\xi>^{-2p}<D_y>^{2p}\int_{\mathbb{R}^N} g(y,\xi,u)dudyd\xi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i<y,\xi>}\int_{\mathbb{R}^N} g(y,\xi,u)dudyd\xi =$ $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{-i<y,\xi>}g(y,\xi,u)dudyd\xi$.

The following is a slightly generalized version of Theorem 24.1, p.180 in Shubin.

40. Proposition. Let $a \in \Gamma_\rho^m(\mathbb{R}^{2n})$ and let B be a pseudodifferential operator with Weyl symbol $b \in S(\mathbb{R}^{2n})$. Then $\int_{\mathbb{R}^{2n}} a(z)B_z dz$ is a pseudodifferential operator with Weyl symbol $a*b \in \Gamma_\rho^m(\mathbb{R}^{2n})$. Furthermore, for each multi-index α, $\exists c_\alpha$ depending on b only such that $\forall N \in \mathbb{N}$, $a*b - \sum_{|\alpha|<N} c_a \partial^\alpha a \in \Gamma_\rho^{m-\rho N}(\mathbb{R}^{2n})$ where $c_0 = (2\pi)^n \mathrm{tr}B$.

Proof. For $z = (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $|z| = |(x,\xi)| = \sqrt{|x|^2+|\xi|^2}$. Let $\phi \in S(\mathbb{R}^n)$. Then by Proposition 37, $\{[\int_{\mathbb{R}^{2n}} a(z)B_z dz]\phi\}(w) = \int_{\mathbb{R}^{2n}} a(z)B_z\phi(w)dz =$

$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi)(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<w-y,\eta>}b\left(\frac{w+y}{2} - x,\eta-\xi\right)\phi(y)dyd\eta dxd\xi =$

$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi)e^{i<w-y,\eta>}b\left(\frac{w+y}{2} - x,\eta-\xi\right)\phi(y)dyd\eta dxd\xi$. Let $g(y,\eta,x,\xi) =$

$e^{i<w,\eta>}a(x,\xi)b\left(\frac{w+y}{2} - x,\eta-\xi\right)\phi(y)$. Then $|D_y^\alpha g(y,\eta,x,\xi)| \le$

$|a(x,\xi)| \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} |D_y^{\alpha'} b\left(\frac{w+y}{2} - x,\eta-\xi\right)| |D_y^{\alpha-\alpha'}\phi(y)| \le$

$c_0(1+|(x,\xi)|)^m \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} \frac{1}{2^{|\alpha'|}}(1+|(\frac{w+y}{2} - x,\eta-\xi)|)^{-2k} c_{p,\alpha-\alpha'}(1+|y|)^{-p} =$

$c_{\alpha,k,p}(1+|(x,\xi)|)^m(1+|(\frac{w+y}{2} - x,\eta-\xi)|)^{-2k}(1+|y|)^{-p}$ where $k > 0$ and $p > 0$ are arbitrary. Clearly $(1+|(x,\xi)|)^m \le (1+|x|)^{|m|}(1+|\xi|)^{|m|}$.
$(1+|(\frac{w+y}{2} - x,\eta-\xi)|)^{-2k} \le (1+|\frac{w+y}{2} - x|)^{-k}(1+|\eta-\xi|)^{-k} \le$
$(1+|\frac{w+y}{2}|)^k(1+|x|)^{-k}(1+|\eta|)^k(1+|\xi|)^{-k} \le (1+|w+y|)^k(1+|x|)^{-k}(1+|\eta|)^k(1+|\xi|)^{-k} \le$
$(1+|w|)^k(1+|y|)^k(1+|x|)^{-k}(1+|\eta|)^k(1+|\xi|)^{-k}$. So $|D_y^\alpha g(y,\eta,x,\xi)| \le$
$c_{\alpha,k,p}(1+|w|)^k(1+|y|)^{k-p}(1+|x|)^{|m|-k}(1+|\eta|)^k(1+|\xi|)^{|m|-k}$. Now we can choose k so that $|m| - k < -n$ and p so that $k - p < -n$. Then g satisfies the conditions of Lemma 39 with $\xi = \eta$ and $u = (x,\xi)$. Therefore $\int_{\mathbb{R}^{2n}} a(z)B_z dz\phi(w)$

$=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi)e^{i<w-y,\eta>}b\left[\frac{w+y}{2} - x,\eta-\xi\right]\phi(y)dxd\xi dyd\eta =$

$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<w-y,\eta>}a*b\left[\frac{w+y}{2},\eta\right]\phi(y)dyd\eta$. By Taylor's Theorem, $a(z') =$

$\sum_{|\alpha|<N} \frac{1}{\alpha!}(\partial^\alpha a(z))(z'-z)^\alpha + r_N(z',z)$ where $r_N(z',z) =$

$\sum_{|\alpha|=N} c_\alpha'(z'-z)^\alpha\int_0^1\partial^\alpha a(z+t(z-z'))(1-t)^{N-1}dt$. So $a*b(z) =$

$\sum_{|\alpha|<N} \frac{1}{\alpha!}\int_{\mathbb{R}^{2n}}(z'-z)^\alpha b(z-z')dz'\,\partial^\alpha a(z) + R_N(z)$ where $R_N(z) =$
$\int_{\mathbb{R}^{2n}} b(z-z')\, r_N(z',z)dz'$. Let $c_\alpha = \frac{1}{\alpha!}\int_{\mathbb{R}^{2n}}(z'-z)^\alpha b(z-z')dz' =$
$\frac{(-1)^{|\alpha|}}{\alpha!}\int_{\mathbb{R}^{2n}} z^\alpha b(z)dz$. Since $b \in S(\mathbb{R}^{2n})$, these integrals exist and $c_0 =$
$\int_{\mathbb{R}^{2n}} b(z)dz = (2\pi)^n \mathrm{tr}B$ by Shubin, Proposition 27.2, p.195.

$R_N(z) = \int_{\mathbb{R}^{2n}} b(z-z') \sum_{|\alpha|=N} c_\alpha'(z'-z)^\alpha\int_0^1\partial^\alpha a(z+t(z-z'))(1-t)^{N-1}dtdz' =$

$\sum_{|\alpha|=N} c_\alpha'\int_{\mathbb{R}^{2n}}\int_0^1 w^\alpha b(-w)\partial^\alpha a(z+tw)(1-t)^{N-1}dtdw$. For each multi-index γ and for arbitrary k > 0, $|\partial_z^{\gamma} w^{\alpha} b(-w)\partial^{\alpha} a(z+tw)(1-t)^{N-1}| = |w^{\alpha} b(-w)\partial^{\alpha+\gamma} a(z+tw)(1-t)^{N-1}| \leq$ $|w|^{|\alpha|} c_k(1+|w|)^{-k} c_{\alpha+\gamma}(1+|z+tw|)^{m-\rho|\alpha+\gamma|} \leq$ $c_k c_{\alpha+\gamma}(1+|w|)^{N}(1+|w|)^{-k}(1+|tw|)^{|m-\rho|\alpha|-\rho|\gamma||}(1+|z|)^{m-\rho|\alpha|-\rho|\gamma|} \leq$ $c_k c_{\alpha+\gamma}(1+|w|)^{N-k+|m-\rho|\alpha|-\rho|\gamma||}(1+|z|)^{m-\rho|\alpha|-\rho|\gamma|}$. Since k was arbitrary, we can choose k so that $N-k+|m-\rho|\alpha|-\rho|\gamma|| < -2n$. Then $|\partial^{\gamma} R_N(z)| =$

$$\Big| \sum_{|\alpha|=N} c_{\alpha}' \int_{\mathbb{R}^{2n}} \int_0^1 w^{\alpha} b(-w)\partial^{\alpha+\gamma} a(z+tw)(1-t)^{N-1} dt dw \Big| \leq$$

$$\sum_{|\alpha|=N} c_{\alpha}' c_k c_{\alpha+\gamma} \int_{\mathbb{R}^{2n}} (1+|w|)^{N-k+|m-\rho|\alpha|-\rho|\gamma||} dw (1+|z|)^{m-\rho|\alpha|-\rho|\gamma|} =$$

$c''_{N,\gamma}(1+|z|)^{m-\rho|\alpha|-\rho|\gamma|}$. So $a*b - \sum_{|\alpha|<N} c_a \partial^{\alpha} a \in \Gamma_{\rho}^{m-\rho N}(\mathbb{R}^{2n})$. In particular $a*b \in \Gamma_{\rho}^{m}(\mathbb{R}^{2n})$. Thus $a*b$ is the Weyl symbol of $\int_{\mathbb{R}^{2n}} a(z)B_z dz$.

The next proposition essentially is Exercise 24.5, p.183 in Shubin.

41. Proposition. Let B be a pseudodifferential operator with Weyl symbol $b \in \Gamma_{\rho}^{m}(\mathbb{R}^{2n})$ and let $\phi,\theta \in S(\mathbb{R}^n)$. Then $(B_{-z}\phi,\theta) = b*\gamma(z)$ where $\gamma(x,\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i<\eta,\xi>}\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}d\eta$.

Proof. It suffices to show that $(B\phi,\theta) = \int_{\mathbb{R}^{2n}} b(-z')\gamma(z')dz'$ since by Proposition 36, B_{-z} has Weyl symbol $T_{-z}b$. $(B\phi,\theta) =$ $(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i<x-y,\xi>} b(\frac{x+y}{2},\xi)\phi(y)dyd\xi\overline{\theta(x)}dx =$ $(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i<x-y,\xi>} b(\frac{x+y}{2},\xi)\phi(y)\overline{\theta(x)}dyd\xi dx$. Let $g(y,\xi,x) =$ $e^{-i<x,\xi>} b(\frac{x+y}{2},\xi)\phi(y)\overline{\theta(x)}$. Then, for arbitrary k > 0 and p > 0, $|D_y^{\alpha} g(y,\xi,x)| \leq$

$$\sum_{\alpha' \leq \alpha} \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} |D_y^{\alpha'} b(\tfrac{x+y}{2},\xi)|\,|D_y^{\alpha-\alpha'} \phi(y)|\,|\theta(x)| \leq$$

$$\sum_{\alpha' \leq \alpha} \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} \frac{1}{2^{|\alpha'|}} c_{\alpha'}(1+|(\tfrac{x+y}{2},\xi)|)^{m-\rho|\alpha'|} c_{k,\alpha-\alpha'}(1+|y|)^{-k} c_{p,0}(1+|x|)^{-p} \leq$$

$\sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} \frac{1}{2^{|\alpha'|}} c_{\alpha'} c_{k,\alpha-\alpha'} c_{p,0} (1+|(\frac{x+y}{2},\xi)|)^{|m|}(1+|y|)^{-k}(1+|x|)^{-p} \le$
$c_{\alpha,k,p}(1+|\frac{x+y}{2}|)^{|m|}(1+|\xi|)^{|m|}(1+|y|)^{-k}(1+|x|)^{-p} \le$
$c_{\alpha,k,p}(1+|x+y|)^{|m|}(1+|\xi|)^{|m|}(1+|y|)^{-k}(1+|x|)^{-p} \le$
$c_{\alpha,k,p}(1+|x|)^{|m|}(1+|y|)^{|m|}(1+|\xi|)^{|m|}(1+|y|)^{-k}(1+|x|)^{-p} =$
$c_{\alpha,k,p}(1+|y|)^{|m|-k}(1+|\xi|)^{|m|}(1+|x|)^{|m|-p}$. Since k and p are arbitrary, we can choose them so that $|m| - k < -n$ and $|m| - p < -n$. Hence g satisfies the conditions of Lemma 39 and therefore $(B\phi,\theta) =$
$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i<x-y,\xi>} b(\frac{x+y}{2},\xi)\phi(y)\overline{\theta(x)}dxdyd\xi$. Furthermore,
$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i<x-y,\xi>} b(\frac{x+y}{2},\xi)\phi(y)\overline{\theta(x)}dxdy$ converges absolutely. Let $x = x' + \frac{\eta}{2}$ and $y = x' - \frac{\eta}{2}$. Then $x - y = \eta$, $\frac{x+y}{2} = -x'$, $\frac{\partial x_i}{\partial x'_j} = -\delta_{ij}$, $\frac{\partial x_i}{\partial \eta'_j} = \frac{1}{2}\,\delta_{ij}$, $\frac{\partial y_i}{\partial x'_j} = -\delta_{ij}$, and $\frac{\partial y_i}{\partial \eta'_j} = \frac{1}{2}\,\delta_{ij}$. Thus the linear transformation given by these formulas has Jacobian determinant $\begin{vmatrix} -I_n & \frac{1}{2}I_n \\ -I_n & -\frac{1}{2}I_n \end{vmatrix} = \begin{vmatrix} -2I_n & 0 \\ -I_n & -\frac{1}{2}I_n \end{vmatrix} = 1$ where I_n is the n × n identity matrix. Therefore $(B\phi,\theta) =$
$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i<\eta,\xi>} b(-x',\xi)\phi(-x'-\frac{\eta}{2})\overline{\theta(-x'+\frac{\eta}{2})}\; d\eta dx' d\xi =$
$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i<\eta,\xi>} b(-x',-\xi)\phi(-x'-\frac{\eta}{2})\overline{\theta(-x'+\frac{\eta}{2})}\; d\eta dx' d\xi =$
$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(-x,-\xi)\gamma(x,\xi)dxd\xi$.

42. Proposition. Let $\phi,\theta \in S(\mathbb{R}^n)$ and $\gamma(x,\xi) =$
$(2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i<\eta,\xi>}\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}d\eta$. Then $\gamma \in S(\mathbb{R}^{2n})$ and γ is the Weyl symbol of the operator $(2\pi)^{-n}\tilde{\phi}\otimes\overline{\tilde{\theta}}$ where $\tilde{\phi}(x) = \phi(-x)$ and $\tilde{\theta}(x) = \theta(-x)$.

Proof. For arbitrary k > 0 and p > 0, $|\partial_\xi^\alpha\partial_x^\beta[e^{-i\langle\eta,\xi\rangle}\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}]| =$

$|\eta^\alpha| \sum_{\beta'\le\beta}\binom{\beta}{\beta'}\partial_x^{\beta'}\phi(-x-\frac{\eta}{2})\partial_x^{\beta-\beta'}\overline{\theta(-x+\frac{\eta}{2})}| \le$

$|\eta|^{|\alpha|}\sum_{\beta'\le\beta}\binom{\beta}{\beta'}c_{\beta',k}(1+|-x-\frac{\eta}{2}|)^{-k}c_{\beta-\beta',p}(1+|-x+\frac{\eta}{2}|)^{-p} \le$
$c_{\alpha,\beta,k,p}(1+|\eta|)^{|\alpha|}(1+|x|)^{k}(1+|\frac{\eta}{2}|)^{-k}(1+|x|)^{p}(1+|\frac{\eta}{2}|)^{-p} \le$
$c_{\alpha,\beta,k,p}(1+|\eta|)^{|\alpha|}(1+|x|)^{k+p}(1+|\frac{\eta}{2}|)^{-k-p} \le$
$c_{\alpha,\beta,k,p}(1+|\eta|)^{|\alpha|}(1+|x|)^{k+p}(\frac{1}{2}+|\frac{\eta}{2}|)^{-k-p} =$
$c_{\alpha,\beta,k,p}2^{k+p}(1+|\eta|)^{|\alpha|}(1+|x|)^{k+p}(1+|\eta|)^{-k-p} =$
$c_{\alpha,\beta,k,p}2^{k+p}(1+|x|)^{k+p}(1+|\eta|)^{|\alpha|-k-p}$. Now we can choose k and p so that $|\alpha| - k - p < -n$. Therefore we can differentiate under the integral sign. So $\gamma \in$

$C^\infty(\mathbb{R}^{2n})$ and $\partial_\xi^\alpha\partial_x^\beta\gamma(x,\xi) = (2\pi)^{-n}\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_{\mathbb{R}^n}e^{-i\langle\eta,\xi\rangle}\eta^\alpha\partial_x^{\beta'}\phi(-x-\frac{\eta}{2})\partial_x^{\beta-\beta'}\overline{\theta(-x+\frac{\eta}{2})}d\eta.$

Since for any k > 0, $(1+|(x,\xi)|)^k \le (1+|x|+|\xi|)^k \le (1+|x|)^k(1+|\xi|)^k$, we will know that $\gamma \in S(\mathbb{R}^{2n})$ if we can show: $\forall k > 0$ and for each multi-index α,β,and γ $\exists c_{k,\alpha,\beta,\gamma}$ such that $\forall x,\ \xi \in \mathbb{R}^n$,

$(1+|x|)^k(1+|\xi|)^k|\int_{\mathbb{R}^n}e^{-i\langle\eta,\xi\rangle}\eta^\alpha\partial^\beta\phi(-x-\frac{\eta}{2})\partial^\gamma\overline{\theta(-x+\frac{\eta}{2})}d\eta| \le c_{k,\alpha,\beta,\gamma}$ Since $\phi,\ \theta \in S(\mathbb{R}^n) \longrightarrow \partial^\beta\phi,\ \partial^\gamma\theta \in S(\mathbb{R}^n)$, it suffices to show that $\forall k,\ \alpha,\ \exists c_{k,\alpha}$ such that $\forall x,$

$\xi \in \mathbb{R}^n$, $(1+|x|)^k(1+|\xi|)^k|\int_{\mathbb{R}^n}e^{-i\langle\eta,\xi\rangle}\eta^\alpha\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}d\eta| \le c_{k,\alpha}$.

$D_\eta^\beta[\eta^\alpha\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}]$ consists of terms of the form $cD_\eta^\gamma\eta^\alpha D_\eta^\delta\phi(-x-\frac{\eta}{2})D_\eta^\varepsilon\overline{\theta(-x+\frac{\eta}{2})}$.
Clearly $|D_\eta^\gamma\eta^\alpha| \le c_{\gamma\alpha}(1+|\eta|)^{|\alpha|}$. Therefore for arbitrary k > 0 and p > 0 we have $|D_\eta^\gamma\eta^\alpha D_\eta^\delta\phi(-x-\frac{\eta}{2})D_\eta^\varepsilon\overline{\theta(-x+\frac{\eta}{2})}| \le c_{\gamma\alpha}(1+|\eta|)^{|\alpha|}c_{\delta k}(1+|-x-\frac{\eta}{2}|)^{-k}c_{\varepsilon p}(1+|-x+\frac{\eta}{2}|)^{-p} \le$
$c_{\gamma\alpha\delta k\varepsilon p}(1+|x|)^{k+p}(1+|\eta|)^{|\alpha|-k-p}$ by estimates we made previously. Therefore we

can apply Lemma 38. Hence $(1+|x|)^k(1+|\xi|)^k|\int_{\mathbb{R}^n} e^{-i<\eta,\xi>}\eta^\alpha\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}d\eta| =$
$(1+|x|)^k(1+|\xi|)^k<\xi>^{-2p}|\int_{\mathbb{R}^n} e^{-i<\eta,\xi>}<D_\eta>^{2p}\eta^\alpha\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}d\eta|$ ($p > 0$ arbitrary).

Since $<D_\eta>^{2p} = \sum_{|\alpha|\leq p} c_\alpha D_\eta^{2\alpha}$, it suffices to estimate
$(1+|x|)^k(1+|\xi|)^k<\xi>^{-2p}|\int_{\mathbb{R}^n} e^{-i<\eta,\xi>}D_\eta^\beta\eta^\alpha D_\eta^\mu\phi(-x-\frac{\eta}{2})\overline{D_\eta^\nu\theta(-x+\frac{\eta}{2})}d\eta|$. For arbitrary $p > 0$ $\exists c_p > 0$ such that $<\xi>^{-2p} \leq c(1+|\xi|)^{-2p}$, $|D_\eta^\beta\eta^\alpha| \leq c_{\alpha,\beta}(1+|\eta|)^{|\alpha|}$, $D^\mu\phi \in S(\mathbb{R}^n)$, and $D^\nu\theta \in S(\mathbb{R}^n)$. Therefore it suffices to show that
$(1+|x|)^k\int_{\mathbb{R}^n}(1+|\eta|)^{|\alpha|}|\phi(-x-\frac{\eta}{2})||\theta(-x+\frac{\eta}{2})|d\eta$ is bounded. Let $\omega = -x - \frac{\eta}{2}$. Then
$-x + \frac{\eta}{2} = -2x - \omega$, $\eta = -2x - 2\omega$, and $d\eta = 2^n d\omega$. We now have, for arbitrary
$p > 0$ and $q > 0$, $(1+|x|)^k\int_{\mathbb{R}^n}(1+|\eta|)^{|\alpha|}|\phi(-x-\frac{\eta}{2})||\theta(-x+\frac{\eta}{2})|d\eta =$
$2^n(1+|x|)^k\int_{\mathbb{R}^n}(1+|-2x-2\omega|)^{|\alpha|}|\phi(\omega)||\theta(-2x-\omega)|d\omega \leq$
$2^n(1+|x|)^k\int_{\mathbb{R}^n}(1+|2x|)^{|\alpha|}(1+|2\omega|)^{|\alpha|}c_p(1+|\omega|)^{-p}c_q(1+|-2x-\omega|)^{-q}d\omega \leq$
$2^n c_p c_q(1+|x|)^k\int_{\mathbb{R}^n}(2+|2x|)^{|\alpha|}(2+|2\omega|)^{|\alpha|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1+|\omega|)^{-p}(1+|2x|)^{-q}(1+|\omega|)^q d\omega \leq$
$2^{n+2|\alpha|}(1+|x|)^{k+|\alpha|-q}\int_{\mathbb{R}^n}(1+|\omega|)^{|\alpha|-p+q}d\omega$. Now we can choose q such that $k + |\alpha| - q < 0$ and p such that $|\alpha| - p + q < -n$. This gives the required bound.
So we have shown that $\gamma \in S(\mathbb{R}^{2n})$. Let $f \in S(\mathbb{R}^n)$. Let B be the operator with Weyl symbol γ. Then $Bf(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i<x-y,\xi>}\gamma(\frac{x+y}{2},\xi)f(y)dyd\xi$. This integral converges absolutely. So $Bf(x) =$
$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i<x-y,\xi>}\gamma(\frac{x+y}{2},\xi)f(y)d\xi dy =$
$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i<x-y,\xi>}(2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i<\eta,\xi>}\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}d\eta f(y)d\xi dy =$
$(2\pi)^{-n}\int_{\mathbb{R}^n}\phi(-x-\frac{x-y}{2})\overline{\theta(-x+\frac{x-y}{2})}f(y)dy$ since, as function of η, $\phi(-x-\frac{\eta}{2})\overline{\theta(-x+\frac{\eta}{2})}$ is in $S(\mathbb{R}^n)$. So $Bf(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\phi(-x)\overline{\theta(-y)}f(y)dy = (2\pi)^{-n}\tilde{\phi}\otimes\bar{\tilde{\theta}}f$. Since $S(\mathbb{R}^n)$ is

dense in $L^2(\mathbb{R}^n)$, $B = (2\pi)^{-n}\tilde{\phi}\otimes\bar{\tilde{\theta}}$.

43. Proposition. Suppose that $A \in G^m_\rho(\mathbb{R}^n)$ and let a be the Weyl symbol of A. Then $A \in C_1$ iff $a \in L^1(\mathbb{R}^{2n})$.

Proof. Suppose that $A \in C_1$. Let $C = (2\pi)^{-n}\tilde{\phi}\otimes\bar{\tilde{\theta}}$ with the notation of Proposition 41. Choose ϕ and θ so that $(2\pi)^n \mathrm{tr} C = (\tilde{\phi},\tilde{\theta}) = 1$. By induction we construct a sequence of operators B_1, B_2, ... of the form $B_j = \int_{\mathbb{R}^{2n}} a_j(z)C_z dz$ which have Weyl symbols b_1, b_2, ... respectively with $b_j \in L^1(\mathbb{R}^{2n})$ for all j satisfying $a_{k+1} = a - \sum_{j=1}^{k} b_j \in \Gamma^{m-\rho k}_\rho(\mathbb{R}^{2n})$: Let $a_1 = a$ and $B_1 = \int_{\mathbb{R}^{2n}} a_1(z)C_z dz$. Then by Proposition 40, $B_1 \in G^m_\rho(\mathbb{R}^n)$, has Weyl symbol $b_1 = a*\gamma \in \Gamma^m_\rho(\mathbb{R}^{2n})$, and $a - b_1 \in \Gamma^{m-\rho}_\rho(\mathbb{R}^{2n})$. By Proposition 41, $a*\gamma = (A_{-z}\phi,\theta)$ and by Proposition 30, $(A_{-z}\phi,\theta) \in L^1(\mathbb{R}^{2n})$. So $b_1 \in L^1(\mathbb{R}^{2n})$. Suppose B_1, ..., B_k, a_1, ..., a_k, and b_1, ..., b_k have been chosen. Let $a_{k+1} = a - \sum_{j=1}^{k} b_j$. By the induction hypothesis, $a_{k+1} \in \Gamma^{m-\rho k}_\rho(\mathbb{R}^{2n})$. Let $B_{k+1} = \int_{\mathbb{R}^{2n}} a_{k+1}(z)C_z dz$. Then B_{k+1} has Weyl symbol $b_{k+1} = a_{k+1}*\gamma \in \Gamma^{m-\rho k}_\rho(\mathbb{R}^{2n})$and $a_{k+1} - b_{k+1} \in \Gamma^{m-\rho(k+1)}_\rho(\mathbb{R}^{2n})$. Therefore $a - \sum_{j=1}^{k+1} b_j = a - \sum_{j=1}^{k} b_j - b_{k+1} = a_{k+1} - b_{k+1} \in \Gamma^{m-\rho(k+1)}_\rho(\mathbb{R}^{2n})$. $b_{k+1} = a_{k+1}*\gamma = (a - \sum_{j=1}^{k} b_j)*\gamma = a*\gamma - \sum_{j=1}^{k} b_j*\gamma = b_1 - \sum_{j=1}^{k} b_j*\gamma$. Since $b_j \in L^1(\mathbb{R}^{2n})$ for j = 1, ..., k by the induction hypothesis and $\gamma \in S(\mathbb{R}^{2n})$, $b_{k+1} \in L^1(\mathbb{R}^{2n})$. Now choose k such that $m - \rho k < -2n$. Then $a_{k+1} \in \Gamma^{m-\rho k}_\rho(\mathbb{R}^{2n}) \subseteq L^1(\mathbb{R}^{2n})$. So $a = a_{k+1} + \sum_{j=1}^{k} b_j \in L^1(\mathbb{R}^{2n})$.

Now suppose $a \in L^1(\mathbb{R}^{2n})$. By induction we construct a sequence of trace class operators B_1, B_2, ... with Weyl symbols b_1, b_2, ... respectively such that $A - \sum_{j=1}^{k} B_j \in G_\rho^{m-\rho k}(\mathbb{R}^n)$ and such that $b_j \in L^1(\mathbb{R}^{2n})$ $\forall j$: Let $B_1 = \int_{\mathbb{R}^{2n}} a(z)C_z dz$. By Proposition 31, $B_1 \in C_1$. By Propositions 40 and42, $b_1 = a*\gamma \in \Gamma_\rho^m(\mathbb{R}^{2n})$ and $a - b_1 \in \Gamma_\rho^{m-\rho}(\mathbb{R}^{2n})$, i.e., $A - B_1 \in G_\rho^{m-\rho}(\mathbb{R}^n)$. Since $a \in L^1(\mathbb{R}^{2n})$ and $\gamma \in S(\mathbb{R}^{2n})$, $b_1 \in L^1(\mathbb{R}^{2n})$. Suppose $B_1, ..., B_k$ and $b_1, ..., b_k$ have been chosen. Let $a_{k+1} = a - \sum_{j=1}^{k} b_j$. Then $a_{k+1} \in L^1(\mathbb{R}^{2n})$ and so $B_{k+1} = \int_{\mathbb{R}^{2n}} a_{k+1}(z)C_z dz \in C_1$ and $b_{k+1} = a_{k+1}*\gamma \in L^1(\mathbb{R}^{2n})$. Furthermore a_{k+1} is the Weyl symbol of $A - \sum_{j=1}^{k} B_j$. So $a_{k+1} \in \Gamma_\rho^{m-\rho k}(\mathbb{R}^{2n})$. Therefore $a - \sum_{j=1}^{k+1} b_j = a - \sum_{j=1}^{k} b_j - b_{k+1} = a_{k+1} - b_{k+1} \in \Gamma_\rho^{m-\rho(k+1)}(\mathbb{R}^{2n})$. So $A - \sum_{j=1}^{k+1} B_j \in G_\rho^{m-\rho(k+1)}(\mathbb{R}^n)$. Now choose k such that $m - \rho k < -2n$. Then $A - \sum_{j=1}^{k} B_j \in C_1$ by Shubin, Proposition 27.3, p.196. So $A = A - \sum_{j=1}^{k} B_j + \sum_{j=1}^{k} B_j \in C_1$.

5. References.

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