AN EXPLORATION OF SOME NON-TONAL PITCH-CLASS SPACES WITH IMPLICATIONS FOR A THEORY OF VOICE LEADING

by

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Abstract

AN EXPLORATION OF SOME NON-TONAL PITCH-CLASS SPACES WITH IMPLICATIONS FOR A THEORY OF VOICE LEADING

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Michael F. Berry

Adviser: Professor Joseph N. Straus

This dissertation develops a theoretical framework suitable for the analysis of neo-tonal music. Neo-tonal music fuses techniques of traditional tonality with elements of atonality: some representative composers include Debussy, Messiaen, and Stravinsky. Chapter one presents a modular-space approach to transformational voice leading and argues for a reconsideration of conventional approaches to pitch-class space. Chapter 2 provides us with the building blocks for pitch-class spaces, using the notion of maximally even sets as a point of departure. Some of the most common scales and sonorities in music are maximally even, or they deviate from maximal evenness ever so slightly. The chapter looks at the construction of maximally even sets and examines why maximally even structures are privileged. Chapter 3 uses maximally even sets as the building blocks for hierarchical pitch-class spaces similar to those discussed in Lerdahl (2001). These pitch-class spaces permit us to discuss the perceived distances between chords as well as to account for non-harmonic tones and chromaticism. Chapter 4 tackles the problem of pieces that feature different pitch-class spaces either presented successively or concurrently. Chapter 5 contains several analytical essays designed to show the theory in practice.

ACKNOWLEDGMENTS

This dissertation grows out of a long-standing interest in the intersection between Schenkerian theory and post-tonal theory. I wish to thank Cynthia Folio for introducing both topics to me and shepherding me through my first ventures into the intersection of the two.

During my doctoral studies, Joseph Straus has been a tremendous mentor and advisor. Shaugn O'Donnell forced me to rethink many of the claims I took for granted in the dissertation. Philip Lambert made many useful comments on the manuscript. I also wish to thank Fred Lerdahl, Jeff Nichols, and Mark Spicer for their insightful comments on the final draft.

I am grateful for the support of my parents and my sister through all of my musical endeavors. Rufus kept my feet warm under the desk on those cold nights. I owe my greatest debt of gratitude to my wife, Laura, who has been tremendously supportive in innumerable ways throughout this project. It is to her that I affectionately dedicate this work with my eternal thanks.

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Chapter 1 : Introduction

This chapter sketches a transformational theory of voice leading that will serve as a starting point for the analysis of neo-tonal music. Neo-tonal music fuses techniques of traditional tonality with elements of atonality: some representative composers include Debussy, Messiaen, and Stravinsky. This theory will be developed in Chapters 2, 3, and 4. Chapter 2 provides us with the building blocks for pitch-class spaces, using the notion of maximally even sets as a point of departure. Some of the most common scales and sonorities in music are maximally even, or they deviate from maximal evenness ever so slightly. The chapter looks at the construction of maximally even sets and examines why maximally even structures are privileged. Chapter 3 uses maximally even sets as the building blocks for hierarchical pitch-class spaces similar to those discussed in Lerdahl (2001). These pitch-class spaces permit us to discuss the perceived distances between chords as well as to account for non-harmonic tones and chromaticism. Chapter 4 tackles the problem of pieces that feature different pitch-class spaces either presented successively or concurrently. Chapter 5 contains several analytical essays designed to show the theory in practice.

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Several recent discussions of voice leading in atonal music have taken Lewin's transformational approach as their point of departure.[1] In this approach, when pitch-class set X is transformed into pitch-class set Y, the transformational pathways along which the

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[1] See, for example, Lewin 1998, 1987, 1982; Klumpenhouwer 1991; O'Donnell 1997; Straus 1997, 2003.

pitch classes travel are understood as a voice leading. These studies rely mainly on transpositional and inversional operators to motivate the voice leadings. When the sets are not related by transposition or inversion, theorists have proposed a variety of ad hoc operations that “fuzzify” or otherwise complicate transposition and inversion.[2] Example 1.1 provides a summary of some of these solutions—the examples are of my own creation, not the work of the authors discussed.

Example 1.1: Some transformational approaches to voice leading

Ex. 1.1.1: Lewin’s “How much like chord 1 is chord 2?”

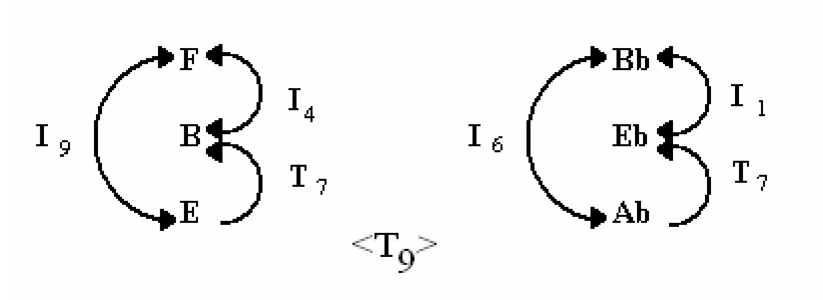
Ex. 1.1.2: Straus’s near transposition

“fairly T_4-ish” “fairly I_2-ish” *T_4

[2] Lewin 1982 says that the sets are “fairly T_n-ish” or “fairly I_n-ish,” and later (Lewin 1998) discusses “pseudo-“ operations; Klumpenhouwer 1991 devises his networks; Straus 1997 proposes “near” operations; and O’Donnell 1997 suggests dual transformations.

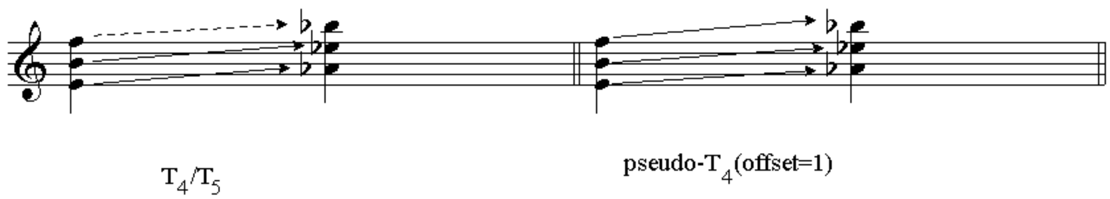
(Example 1.1 continued)

Ex. 1.1.3: Klumpenhouwer networks



Ex. 1.1.4: O'Donnell's dual transformations

Ex. 1.1.5: Lewin's pseudo transposition



Example 1.1.1 illustrates Lewin's question "how much like chord 1 is chord 2?" According to Lewin (1982-3, 336-341), the relationship between the chords can be seen as "fairly T_4-ish" or "fairly I_2-ish." The progression is fairly T_4-ish, as only two of the three elements (E to A♭ and B to E♭) are related by T_4 represented in the example by solid arrows. The remaining elements are related by T_5, represented by the dotted arrow. The progression can also be read as fairly I_2-ish because E and B map onto B♭ and E♭ (respectively) at I_2—shown by the solid arrows—while F maps onto A♭ at I_1—shown by the dotted arrow.

Straus's near transposition (1997) draws voice leadings between sets in which all but one of the progression's constituent pitch classes are related by the same operation. Example 1.1.2 models the two trichords as near-T_4. Here, the solid arrows show the T_4 relation; the dotted arrow represents the aberrant relation

Example 1.1.3 models the two trichords with Klumpenhouwer networks. The pitch-class sets themselves are not related by standard operators, but the networks that describe the interrelationships between the constituent pitch classes are related by $<T_9>$. That is, the transpositional operator remains the same between the pitch-class sets, but the inversional operators are related by $T_{9.}$

O'Donnell's dual transformations (1997) express the voice leading in Example 1.1.4. Two separate transformational operators, T_n and T_{n+x} motivate the voice leadings. In this case, two notes move by T_4 (the solid arrows) and one note moves by T_5 (the dotted arrows).

Example 1.1.5 illustrates Lewin's pseudo transposition (1998), in which deviation from the most maximally uniform voice leading (strict transposition or inversion) is given as an offset number. Here, the trichords are related by pseudo T_4 with an offset of 1.

All of these solutions result in what I see as unnecessary complication of the operators involved. By loosening the restrictions between note connections, anything can be related to anything and the relationships begin to lose their meaning. Furthermore, all of these solutions are rooted in a 12-pitch-class conception of musical space, which may not reflect the musical surface accurately at a given point. My intention, in contrast, is to retain "crisp" transposition and inversion but to employ these operations within and across changing pitch-class spaces. I do not wish to abandon fuzzified operators altogether—there are certainly times and places for them—but what I do wish to emphasize is that the pitch-class space context needs to be taken into account before any discussion of transformational voice leading begins.

Much work has been done to conceptualize and formalize pitch-class spaces outside of the 12-pitch-class space.[3] Andrew Mead (1997-8) describes a way of hearing music in different spaces by examining scalar adjacencies (steps) and non-adjacencies (skips) relative to different pitch spaces. In an attempt to free our ears from the tyranny of absolute interval identification, Mead claims that we perceive steps and skips in various spaces in the same way we do in diatonic and 12-note spaces. Every pitch-class space has its own unique intervallic makeup. An interval class that is a scalar adjacency—a step—in one pitch space may be a leap in another pitch space. The quality of steps and leaps can also vary among pitch spaces. For example, interval class 2 in the whole-tone scale is a "perfect" interval because all steps in the whole-tone scale share the same interval class. In the diatonic scale, interval class two is the larger of two possible steps, and we call it "major."

Matthew Santa (1999a, 1999b) suggests that any musical surface can be viewed as a mosaic of shifting pitch spaces, and he shows connections between the spaces with his MODTRANS operation. Santa argues for a context-sensitive approach to analysis that takes into account the various pitch-class spaces inhabited by a piece of music. The music under consideration in the present work all has a scalar level in between the underlying chromatic collection and the harmonies on the surface. As such, we are not obliged to discuss the music solely in terms of the chromatic collection.

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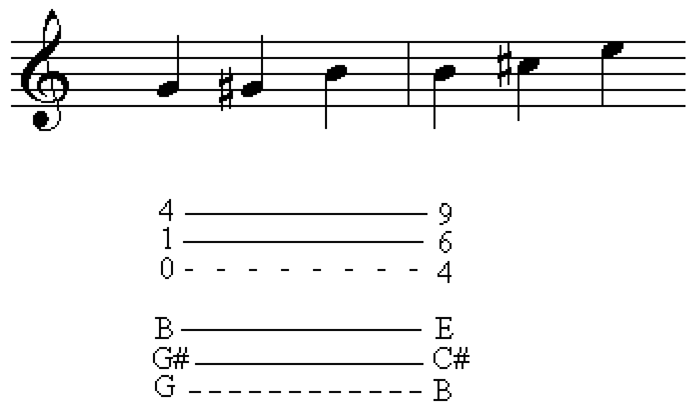
[3] Lewin 1987; Morris 1998, 1995, 1987; Mead 1997-98; Santa 1999a, 1999b; Lerdahl 2001; Neidhöfer 2001, 2005

I propose combining a transformational approach to voice leading in atonal music with recent conceptions of musical space in order to preserve "crisp" operators both within and across well-defined musical spaces, thereby eliminating the need for the fuzzification of traditional operators or the invention of new operators in situations where there is a clearly defined pitch background that is not immediately 12-pitch-class. Almost all Western music has as its ultimate background the 12-note chromatic. When I say "not immediately," I am referring to a situation in which some other scale or larger collection mediates between the foreground harmonies and the 12-note chromatic background. Following a brief introduction to the theory, I will turn to several musical examples that illustrate the theory in action.

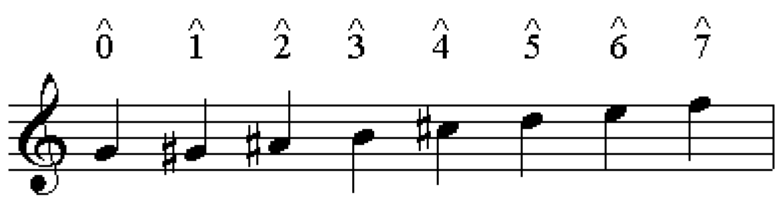
Example 1.2 provides us with a point of entry into the theoretical apparatus. Example 1.2.1 shows two trichords that, in a 12-pitch-class context, cannot be related by transposition or inversion. If we view the trichords as cohabiting a pitch space other than 12-pitch-class space, we can generalize the intervals between the pitches. Both trichords inhabit the same octatonic space, represented by a semitone-first octatonic scale beginning on G, given in Example 1.2.2. The octatonic scale is given above the two trichords, and the pitches are labeled as scale degrees, beginning with G as scale-step zero. (We could, of course, choose any of the transpositions of this scale, and assign zero to the first pitch class. Similarly, a numbering system that labels the first scale degree "1" would also work. At this stage, the assignment of labels is arbitrary.) The octatonic scale features two different-sized scalar adjacencies: a semitone and a whole tone. We can generalize both of these adjacencies into equivalent octatonic *steps*.

Example 1.2: Two trichords

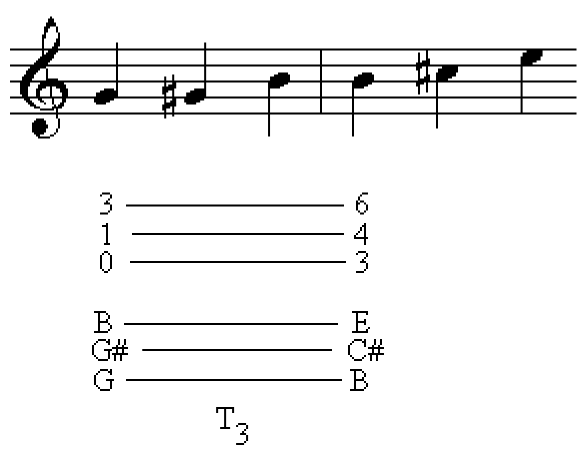
Ex. 1.2.1: Two trichords unrelatable by T or I in 12-pitch-class space



Ex. 1.2.2: The octatonic scale on G



Ex. 1.2.3: Two trichords reconsidered



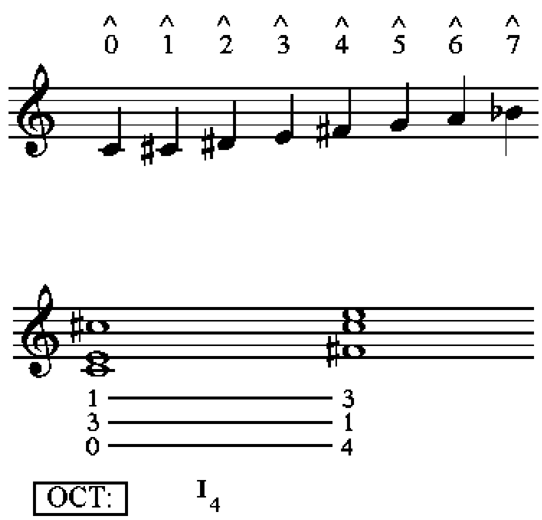
Example 1.2.3 presents these trichords in this new context. Looking again at the first trichord, we see that the distance from G to G# is an octatonic step, and the distance between G# and B is two steps—G# to A# then A# to B. In the second trichord, the distance from B to C# is a step, and the distance from C# to E is two steps. The first trichord becomes $[0,1,3]^8$ and the second, $[3,4,6]^8$, and they are related by T_3 in octatonic space. The superscript number eight next to the set class indicates the cardinality of the space. Subsequent modular transposition or index numbers are given in steps.

Since we can generalize the pitches in modular spaces and we can talk about the intervals between them in terms of steps and skips, we can talk about transformational voice leadings between the sets that appear in these spaces. This generalized model of voice leading permits us to relate step-class sets that belong to different set classes in the twelve pitch-class universe while retaining crisp transformational operators. Step-class sets are subsets of any non-mod 12 pitch-class space in which the pitch classes are generalized into modular step classes or scale degrees.

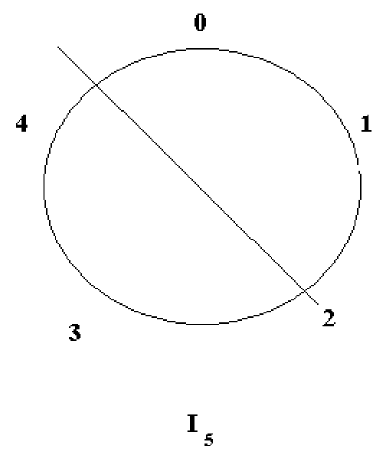
Example 1.2 illustrated transposition in a modular space; Example 1.3 illustrates inversion. The two trichords in 1.3.1 cohabit an octatonic space and are related at I_4 in this octatonic space. Inversion in modular spaces is essentially the same as in 12-pitch-class set theory: step-classes are exchanged around an axis defined by an index number. In the case of modular-space inversion, the index number is the sum of the scale step and its inversional partner. If the index number is greater than the cardinality of the space, it must then be taken modulo the cardinality of the space. Inversion is also possible in pitch-class spaces with an odd cardinality. Example 1.3.2 shows an inversional clockface for pentatonic space, where the axis of inversion bisects a pc and a point between two pcs. The operation shown here is I_4, where 2 maps onto 2, 0 onto 4 and 1 onto 3.

Example 1.3: Inversion in modular spaces

Ex. 1.3.1: Inversion in octatonic space

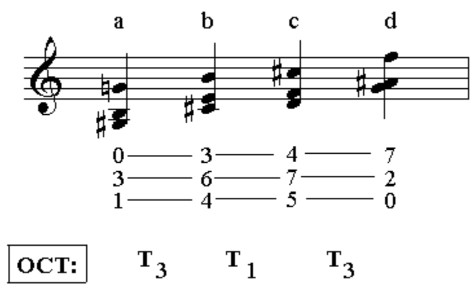


Ex. 1.3.2: A pentatonic (mod 5) clockface

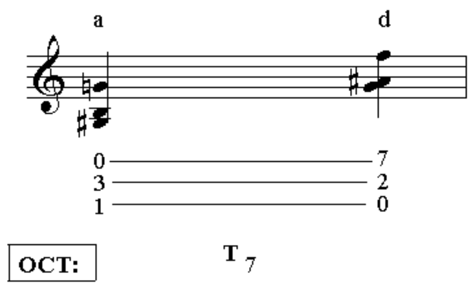


Example 1.4: Composition of operations in modular spaces

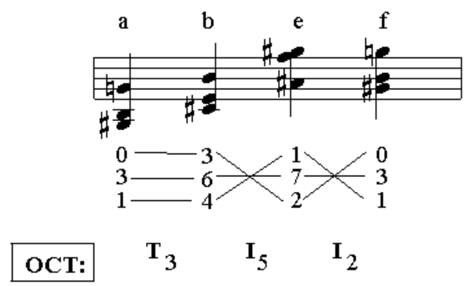
Ex. 1.4.1



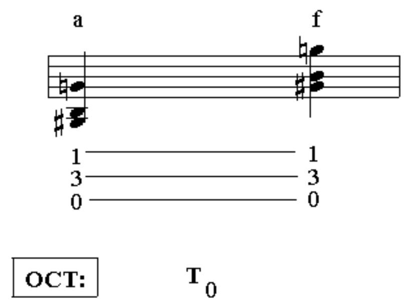
Ex. 1.4.2



Ex. 1.4.3



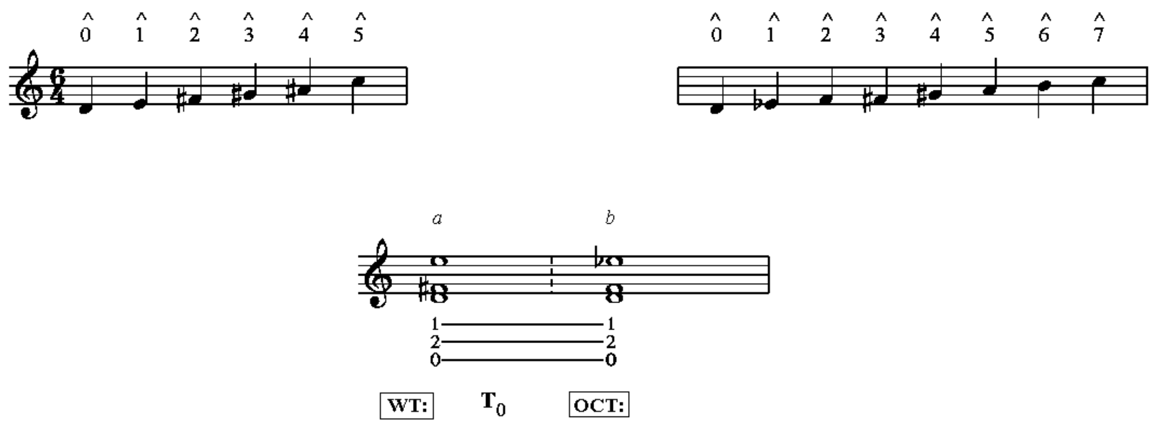
Ex. 1.4.4



Example 1.4 adds two trichords, *c* and *d*, to the two from Example 1.2. All are within the octatonic space defined above, and all four trichords are related by transposition. Example 1.4.1 shows the voice leading between the trichords, and example 1.4.2 shows the total voice leading from *a* to *d*, calculated by summing all three transpositional operators. Example 1.4.3 replaces trichords *c* and *d* with two new trichords, *e* and *f*. Again, *a* and *b* are related by transposition, and *b* and *e* are related by inversion, as are *e* and *f*. Note, though, that the index numbers here are context-dependent and will vary depending on which step class is assigned 0. The operators sum to an overall motion of T_0 in octatonic space from chord *a* to *f*, and this total voice leading

appears in Example 1.4.4. These examples make no prolongational claims: they are intended to show how the operations compose.

Example 1.5: Two different spaces



We have seen how transformations operate within a single pitch-class space. Now, let us explore transformations among sets that belong to different pitch-class spaces. This portion of the theory is indebted to similar work by Matthew Santa. Example 1.5 features two trichords, each of which can be understood in relation to a different space. The trichords are unrelatable by strict transposition or inversion in 12-pitch-class space, and in any one modular space.[4] Let us assume for the moment that the trichords are representative of some longer spans of music in their respective pitch-class spaces. Trichord *a* inhabits a whole-tone space and trichord *b* inhabits an octatonic space. But since both trichords are instances of (012) in their respective spaces, we can trace a voice leading between them, mapping D onto D, E onto E♭, and F# onto F-natural at T_0.

[4] There are any number of ways that these two trichords could be related. We could, of course, create some space that included all six pitch classes present in the example, but for the time being, we will limit ourselves to the most commonly-used collections: the pentatonic, whole-tone, diatonic, octatonic, and enneatonic collections.

In order for transposition and (as we shall see) inversion across spaces to be meaningful, we must clearly define a "tonic" pitch in both cases, which we will label with pitch-class integer 0. For instance, in Example 1.5 the operator is T_0 because the pitch D has been assigned the role of tonic in both cases. If, on the other hand, the D in octatonic space were step-class 0 and the D in whole-tone space were step-class 1, the transpositional operator would be T_1, which seems contrary to the definition of transposition as a measure of distance. Even though the D appears to be retained over the barline, the analysis tells us that there is some pitch-space difference between the two pitches. This is a thorny issue, particularly when inversion is involved, and I will discuss it in detail in Chapter 4.

Essential to the utility of the theory is the need to choose a meaningful pitch-class space background.[5] Formulating hard-and-fast objective rules for determining spaces would be difficult. I propose a more subjective approach that relies on musical intuition. Obviously changing pitch-class spaces on every chord would not yield meaningful results (imagine analyzing a Bach chorale and saying that each chord is the tonic of a different key). Most pitch-class spaces are bound by conventional formal units: the phrase, the exposition, the A section of a rondo, and so on (see Example 1.15 below, where the spaces change at the return of the A section). Obvious changes in musical texture might signal a change of pitch-class space (see Example 1.12 below). I will try to comment on my choices of pitch-class space as Examples arise.

In order for the analytical apparatus to be useful, we must necessarily confine the number of possible pitch-class spaces. It is, of course, possible to create an arbitrary pitch-class space that contains both trichords of Example 1.5. A large number of pitch-

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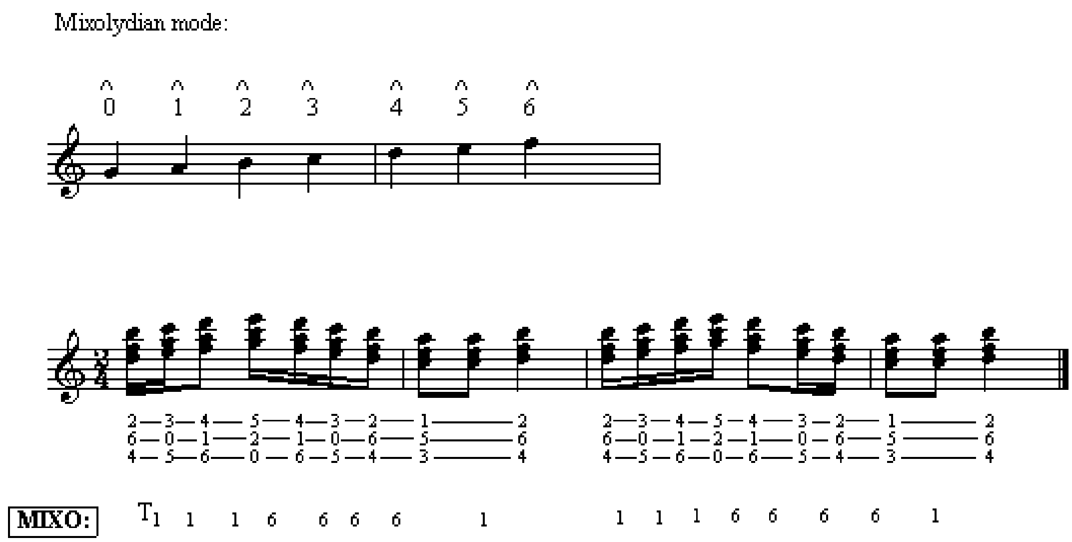
[5] Lerdahl 2001 offers a listener-oriented way of finding the preferred space on pp. 278-80.

class spaces are possible, but the limitless possibilities diminish the usefulness of the system by allowing any chord to be easily related to any other chord by creating a space that contains both. For the present time, I am limiting the pitch-class spaces involved to common, well-known collections: the seven-note diatonic scale and its rotations (the "modes"), Messiaen's modes of limited transposition (which include the whole-tone and octatonic scales), and the pentatonic and hexatonic scales. I have limited myself to these spaces for the time being because of their frequent compositional usage. Chapter 2 will provide the means for expanding the repertoire of available pitch-class spaces.

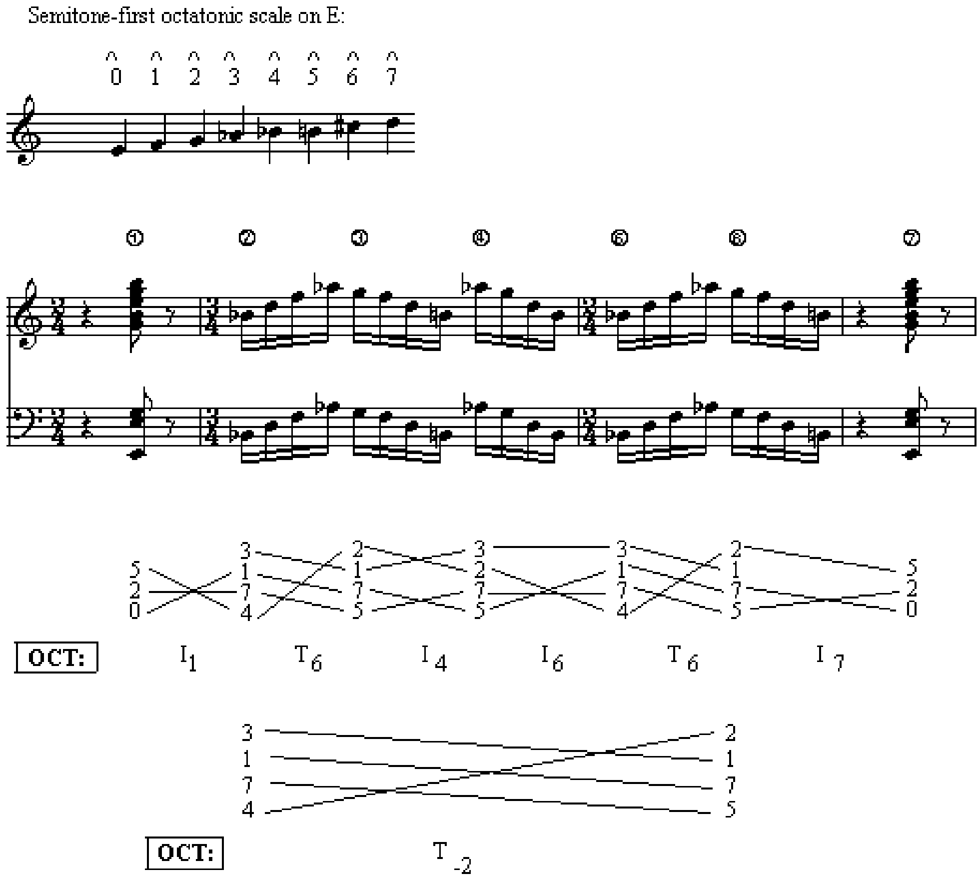
* * *

I turn now to several musical examples that show the theoretical apparatus in action. Example 1.6 is from the Russian Dance in Stravinsky's *Petrouchka*. The Mixolydian scale that accounts for all of the pitches in the excerpt appears above the example, and the scale degrees are numbered from 0 to 6. I have chosen to assign step-class integer 0 to pitch class G because of its salience in the other orchestral parts (not appearing in the diagram). Below the example, I have included the scale-degree integers for each pitch: the corresponding scale degree integers for the first chord, (D,F,B), are 4, 6, and 2, respectively. In the progression from the first chord to the second chord, scale degree 4 in the bottom voice maps onto scale degree 5; in the middle voice, scale degree 6 maps onto scale degree 0; and, in the top voice, scale degree 2 maps onto scale degree 3. Each voice moves up by one step in Mixolydian space—a transposition of one step—despite the fact that the distances traveled by the individual pitches might differ in a twelve-pitch-class context. In the example, I have omitted most of the "Ts" to facilitate reading. The transformation that turns the first chord into the second chord is thus T_1 within a mod-7 space. Our modular space approach illustrates the pandiatonicism nicely and acknowledges that voice and line are one and the same while removing from the discussion any implications of major and minor.

Example 1.6: Stravinsky, "Danse russe" from *Petrouchka*



Example 1.7: Stravinsky, *Symphony of Psalms*: I, mm. 1-4



Example 1.7 is a passage from Stravinsky's *Symphony of Psalms*. The opening measures appear in a pitch-class space defined by the semitone-first octatonic scale beginning on E. I have assigned step-class integer 0 to E in this excerpt due to its perceptual salience in the first chord. This octatonic scale appears above the passage. Again, scale degree integers 0 through 7 in octatonic space are indicated above the pitches. The mod-8 pitch-class integer representation of each chord appears below the excerpt. The first chord maps onto three elements of the second chord at I_1 in an eight-pc space: $0 + 1 = 1$; $2 + 7 = 9$, which, taken mod 8 is equal to 1; and $5 + 4 = 9$, which, mod 8, equals one. Chords 2 and 3 are related by transposition at T_6: each pitch class is transposed down two octatonic steps. Chords 3 and 4 are related by inversion around the

axis defined by octatonic scale degree 2, and chords 4 and 5 are related by inversion around the axis defined by octatonic scale degree 3. Chords 5 and 6 are related by transposition at T_6. Chords 6 and 7 are related by I_7.

A brief digression is warranted here. In discussions of voice leading, we must distinguish between mappings and voices. In the present study, I will define mappings as the result of transformational operators (T and I). Voices are perceptual phenomena, and are projected by unity across domains such as register, timbre, dynamics, and so on.[6] Transformational approaches to voice leading tend to privilege mappings over voices, but in some cases the two are one and the same. This dissertation mainly focuses on mappings, with an occasional nod to the more perceptible voices.

In traditional 12-pitch-class set theory, the minor and major triads are inversionally related: both are members of set class (037). The octatonic-space approach allows us to retain this inversional relationship: see chord 1 to chord 2.[7] A 12-pitch-class approach is unable to relate chords 3 and 4 or chords 4 and 5 under crisp transposition or inversion. Our modular-space approach relates chords 3 and 4 nicely: the G is held invariant, F inverts around the G to become A♭, and D and B-natural exchange places. A similar transformation occurs from chord 4 to chord 5. The A♭ and D are held invariant; G inverts around the Ab and becomes B♭; and the B natural in chord 4 inverts around the D, becoming F.

The overall voice leading appears below the example, showing that the constituent pitch classes of chord 2 descend by two octatonic steps to become the pitch

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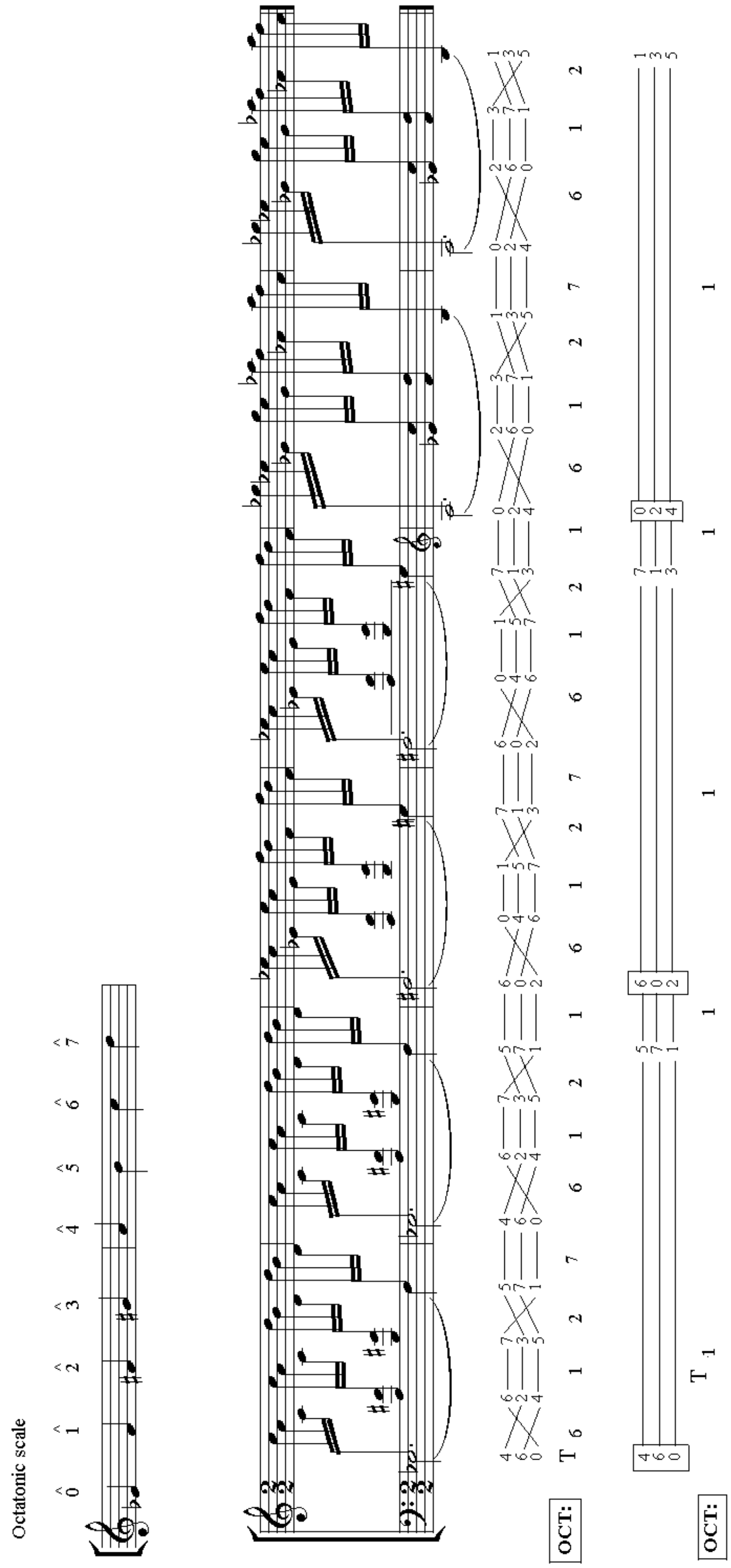
[6] Hasty 1981 discusses domains and segmentation.
[7] The two chords can also be related by T_7 in octatonic space: I have chosen to use the inversional operator to highlight the relationship to the 12-pitch-class universe.

classes of chord 6. The octatonic approach retains the ability to show the equivalence between the "dominant seventh" chords 2 and 6, and the voice leading neatly traces the transformation from "root position" to "first inversion."

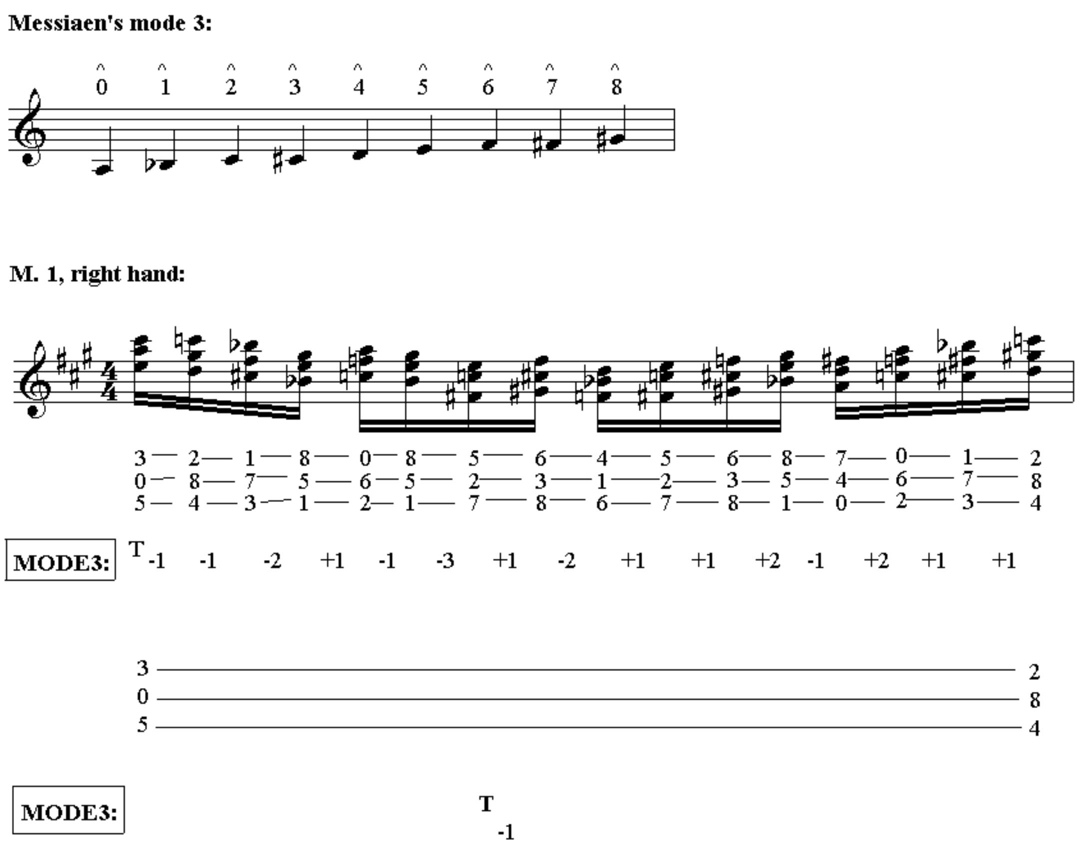
Example 1.8 comes from Debussy's "Jardins sous la pluie," measures 37-42. The music also inhabits an octatonic space, this time, whole-step first beginning on E♭. This octatonic scale appears above the example with the scale degrees labeled. In the example, all of the transformational operators are transposition: I have eliminated most of the "Ts" to facilitate reading. The first chord becomes the second via T_6 in octatonic space; the second becomes the third via T_1, and the third becomes the fourth at T_2. Measures are connected to one another by T_1. Of particular interest here is the relationship between the chord structure and the progression across the measures. The modular set class $(024)^8$ dominates, and the bass notes of each odd-numbered measure outline $(024)^8$. Summing the transpositional operators—6 + 1 + 2 + 1 = 10—and taking them mod 8 yields T_2. Identical transpositional operators obtain if the voice leadings are interpreted against a 12-tone background, but since the octatonic collection dominates, I have chosen to interpret the voice leadings from a mod-8 perspective. The linear projection of the vertical is an example of what Straus (1997) calls "associational" voice leading.

Example 1.8: Debussy, "Jardins sous la pluie," from *Estampes*



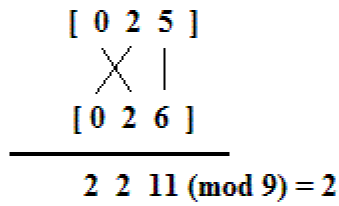
Example 1.9 contains the first measure of Messiaen's piano prelude "Les sons impalpables du rêve..."[8] Messiaen uses his third mode of limited transposition in the right hand—the mode appears directly above the excerpt. I have assigned step-class integer 0 to A, because of the key signature and because of the prominence given to pitch class A throughout the piece. The voice leading for the progression is below the staff. All of the transformational operators in this example are transpositional: I have omitted most of the "Ts" to facilitate reading. In a 12-pitch-class context, the passage includes two different trichords, (037) and (026). For the most part, Messiaen alternates between these two trichords, and, as a result, we cannot trace a simple voice leading through the passage.

Example 1.9: Messiaen, "Les sons impalpables du rêve..." m. 1, right hand



[8] Neidhöfer 2001 and 2005 present a related theory of harmony and voice leading in Messiaen's music.

Viewing the measure in terms of Messiaen's mode 3 results in a progression of step-class $(025)^9$ trichords. Note that, because the mode has only nine members, $[0,2,5]^9$ and $[0,2,6]^9$ are inversionally equivalent at I_2:



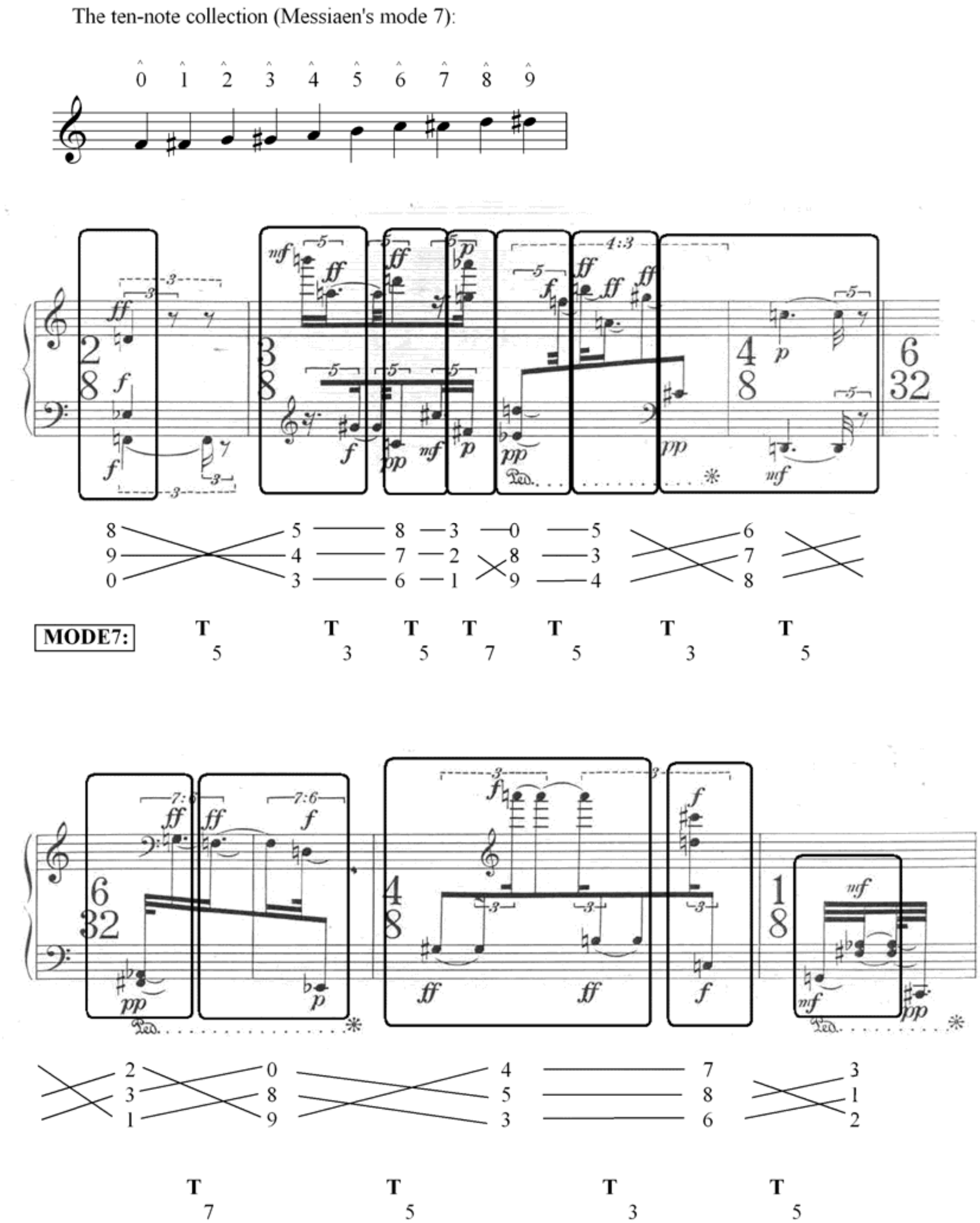
In the context of mode 3, voice and line become one and the same, and the voice leading is motivated by small simple transpositional operators—that is, plus or minus one or two steps.

Example 1.10 comes from the same prelude: it is a passage that serves as the retransition to the A section, and it occurs twice in the work. This passage is based on mode 6, which appears above the passage with its scale steps numbered. The voices notated on the top staff are in an inversional relationship with the voices that are notated on the bottom staff. Messiaen notates the passage in such a way that the wedging is visually apparent, despite crossed hands and registers. As printed on the page, the two outer-most voices are related in a mod-8 pitch-class space by I_3; the middle voices are related by I_4, and the innermost notated voices are also related by I_4.

Example 1.10: Messiaen, "Les sons impalpables du rêve..." retransition

Example 1.11 contains mm. 1-5 from Stockhausen's *Klavierstück II*. The piece essentially comprises all twelve tones, but Stockhausen employs only ten of these until the last measure, when the "missing" notes, E and B♭, appear in the lowest register of the piano. Stockhausen arranges the ten pitches into four different vertical trichords and alters the registral deployment of each in the second statement of the ten notes. In 12-pitch-class space, the first two trichords are members of set class (013), the second two are (012), again, not related by transposition or inversion. In this mod-10 space, which is identical to Messiaen's mode 7, the trichords are all members of set class (012)[10].

Example 1.11: Stockhausen, *Klavierstück* II



The modular space model allows us to trace the voice leadings easily: motion is by transposition at T_3, T_5, or T_7. By viewing the piece as inhabiting a ten-note space, the final two pitches acquire special meaning as residing outside of the controlling collection.

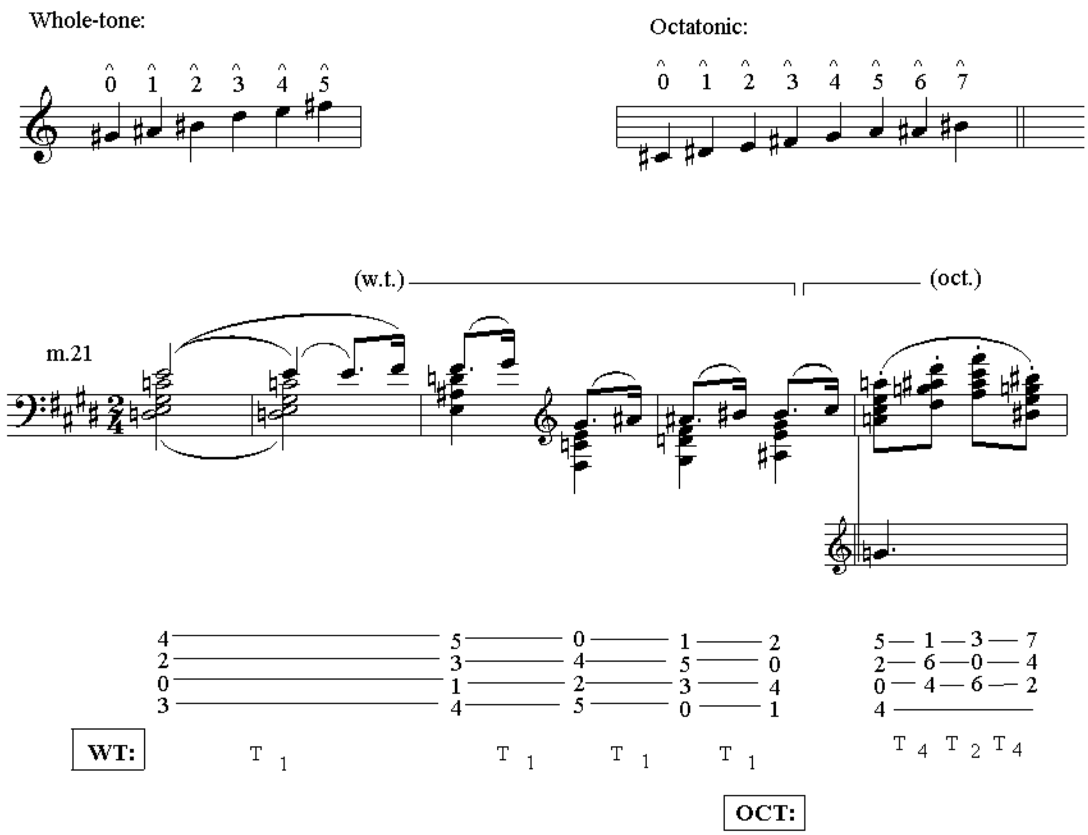
*   *   *

The preceding examples have illustrated how the system describes music that inhabits a particular pitch-class space. But most music to which this system is applicable does not reside in any one space for long periods of time. I turn now to some examples from Debussy and Messiaen that illustrate the potential of the system for describing the relationships that obtain across changing pitch spaces.

Example 1.12 is an excerpt from *Feuilles Mortes*, which is discussed in detail by Richard Bass (1994). According to Bass, the prelude implies a C# minor tonality throughout as a result of its C#-minor key signature, the preponderance of C# "chromatically colored" harmonies, and the concluding C# major triad ("a kind of Picardy third"). This passage begins in a whole-tone space where I have assigned step-class integer 0 to G#. There is no C# present in this collection and the passage appears over a G# ("dominant") pedal point (Bass 1994, 161-4). In the example, the scale appears above the first half of the passage. Debussy then changes the underlying space from whole-tone to octatonic. The octatonic scale used is $OCT_{0,1}$, and I have assigned step-class integer 0 to C# because of the importance of the pitch class throughout the piece. The scale appears above the second half of the example.

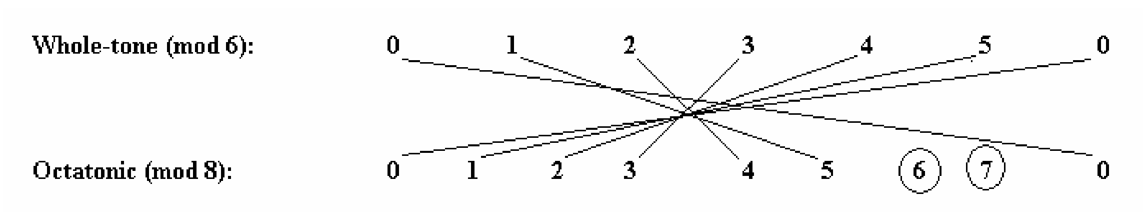
In the whole-tone section, voice and line are one and the same: each sonority is related by simple modular transposition—the tetrachords are also transpositionally related in 12-pitch-class space at T_2. I am treating all of the sixteenth notes in the whole-tone section as anticipations, including the C# just before the octatonic section, which is an anticipation of that collection. The chords in the octatonic section are also related by simple transposition in octatonic space.

Example 1.12: Debussy, *Feuilles mortes*: mm. 21-25



The pitch space changes at the barline between mm. 24 and 25: what was once whole-tone space is now octatonic. Until now, we have only considered transposition and inversion as operations which motivate voice leadings. Employing transposition or inversion across changing pitch spaces—particularly those with different cardinalities—is quite problematic. The final chord in the whole-tone section contains the pitches A#, E, G#, and B# (from bass to soprano); the first chord in the octatonic section consists of the pitches G, C#, E, and A. The two sets cannot be related by transposition, but we can try to relate the sets inversionally. At first glance, we see that it may be possible to relate the sets at I_0 in a mod-6 whole-tone environment: 2 maps onto 4; 0 maps onto 0; 4 maps onto 2; and 1 maps onto 5. This mapping, while mathematically convenient, is problematic due to the differences in cardinalities between the two spaces. Example 1.13 illustrates.

Example 1.13: Inversion from whole-tone to octatonic space

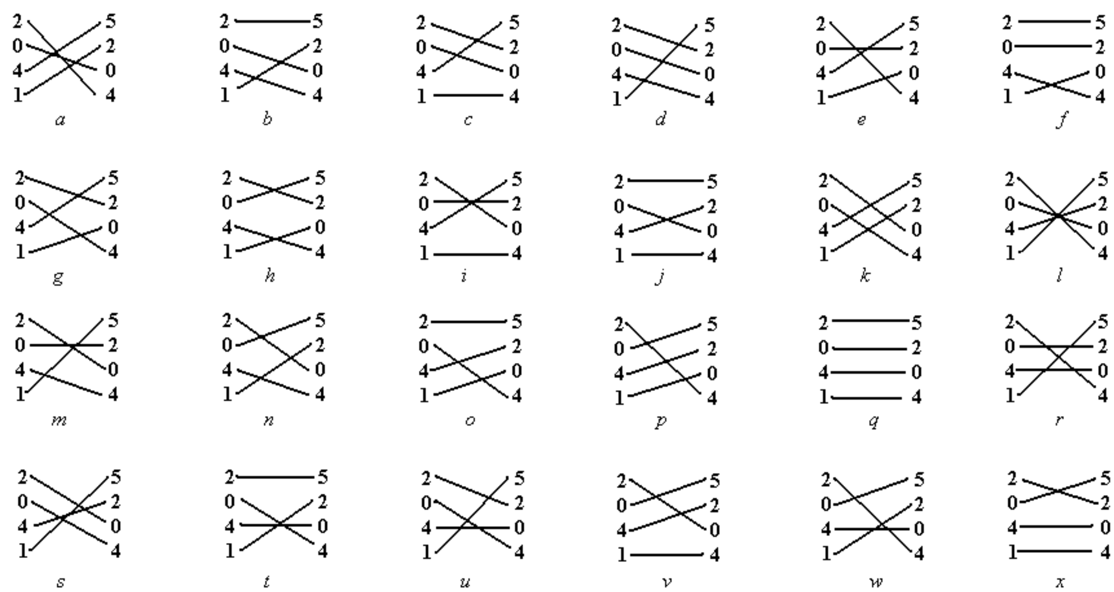


The example shows I_0 from the mod-6 whole-tone space to the mod-8 octatonic space. All of the step classes in the whole-tone scale can find inversional partners in the octatonic scale, but the opposite is not the case: step classes 6 and 7 are left without inversional partners (although step class 6 in the octatonic scale could be paired with 0 in the whole-tone scale, this would result in whole-tone step class 0 having two inversional partners).

In cases where the cardinality of the first space is equal to the cardinality of the second space, we can retain transposition and inversion. Choice of tonic remains a key consideration in the application of these operators.

In cases where we cannot use modular transposition or inversion we must evaluate all of the potential mappings between the two sets.[9] There are 24 possible mappings between the two four-note sets in *Feuilles mortes*, and these mappings appear in Example 1.14. The mappings are labeled *a* through *x* for ease of reference.

Example 1.14: Mappings between (0,1,2,4)[6] and (0,2,4,5)[8]



None of these mappings are strictly transpositional or inversional: the example shows all possible mappings that obtain between the two sets regardless of what motivates the mapping. In other words, the example shows relationships, not operations. Of all of these mappings, only a few make musical sense. Mapping *d* holds the greatest number of step-classes invariant: step-classes 0, 2, and 4 are retained across the changing space, and step-class 1 maps onto step-class 5. This is an example of a functional mapping, where scale-step function is held invariant while the pitches themselves change.

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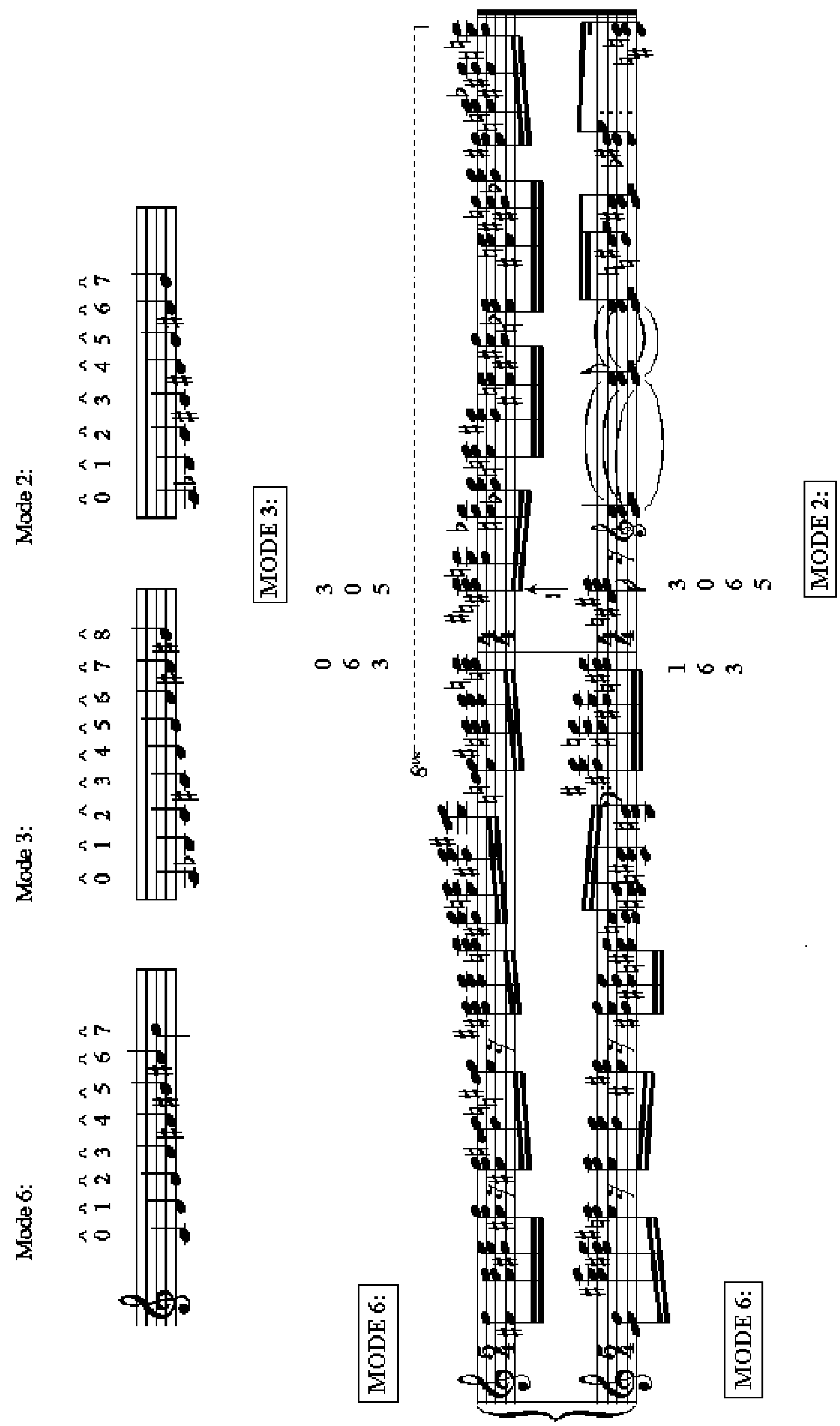
[9] Robert Morris (1998; 178) refers to this as the "total voice leading," consisting of all of the possible moves from any pc in set A to any pc in set B.

Situations like this are quite common in tonal music: C is scale degree 1 in C major, but if we modulate to the dominant, G becomes scale degree 1. The pitch classes are not invariant: the scale degree function is invariant. From a pitch-class invariance point of view, mapping *s* is the most musical. In mapping *s*, B# maps to C#, G# maps to G-natural, E is held invariant, and A# maps to A-natural. Mapping *q* corresponds to the registral lines in the piece: B# maps to A, G# maps to E, E maps to C#, and A# maps to G. Mapping *a* is the most transposition-like (the most "uniform," as per Straus 2003) and it is also the smoothest. That is, the smallest amount of energy is expended by the step-classes in the $(0,1,2,4)^6$ as they move to $(0,2,4,5)^8$. Step-class 0 is held invariant, two step classes move by step-interval class 1 (1→2, 4→5), and one step class moves by step-interval class 2 (2→4). In cases where the pitch-class space changes across the barline—especially when the spaces are of different cardinalities—we must evaluate all of the possibilities and choose the most musical of them. Straus's criteria of uniformity, balance, and smoothness provide us with a convincing means for doing so, and these will be taken up in detail in Chapter 4.

A much more problematic example comes in Messiaen's "Les sons…" after the transition section discussed in Example 1.10 above. As mentioned, the transitional passage is in mode 6, but at the return to the A section that follows, the music becomes bimodal: the right hand is playing in mode 3 and the left hand is playing in mode 2. Example 1.15 presents the passage in question.

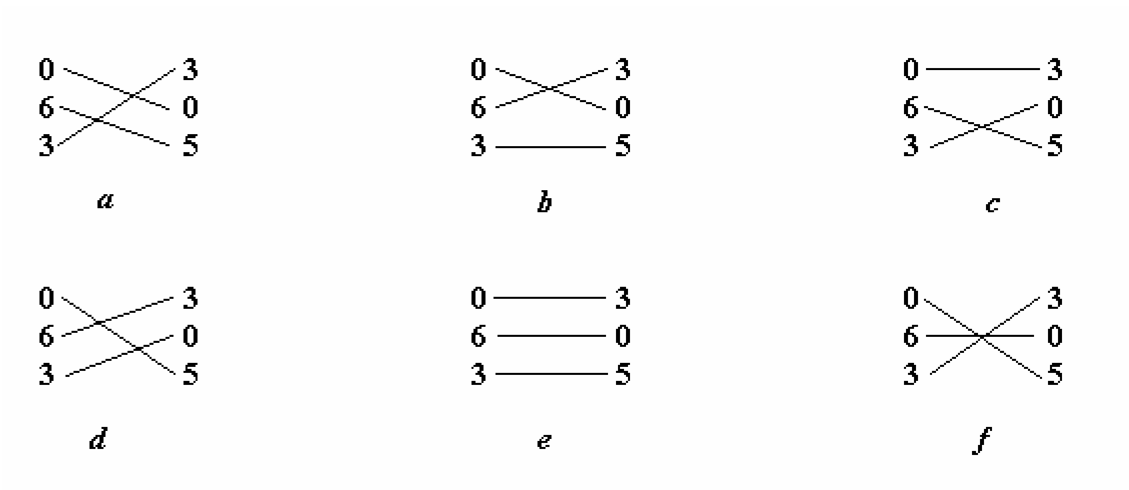
Example 1.15: Transition in "Les sons…," mm. 16-17



Mode 6 is an eight-note collection: (C,D,E,F,F#,G#,A#,B) (labeling C as 0, as in Example 1.10). Mode 2 (the octatonic collection) is also an eight-note collection: here, (A,A#,C,C#,D#,E,F#,G). Mode 3 is a nine-note collection, and I have assigned step-class integer 0 to A, for the reasons discussed in Example 1.9.

There are six possible mappings between any two three-element sets. As with the Debussy, we must explore our alternatives to see which mapping most accurately reflects my interpretation of the music. The six possibilities appear in Example 1.16.

Example 1.16: Mappings between $(0,3,6)^8$ and $(0,3,5)^9$



Mapping *a* is the most transposition-like: step classes 0 and 3 are held invariant, and step class 6 moves by step to step class 5. Mapping *e* maintains the registral lines and also maps the pitch classes in the first set to the nearest pitch class in the second set. In the present analysis, mapping *e* corresponds to the voice leadings in the previous two analyses of the individual passages (Examples 1.9 and 1.10) where voice and registral line are one and the same. Mapping *f* is the most inversion-like: step class 0 inverts onto 5; 6 inverts onto 0; and 3 inverts onto 3.

As we have seen, there are several criteria that we must take into consideration when evaluating the mappings. There are six criteria in particular that I wish to focus on (in no particular order):

- Most transposition-like (Straus's uniformity);
- Most inversion-like (Straus's balance);
- Closest to registral lines;
- Pitch-class invariance;
- Functional mapping;
- Smoothness (measured by offset).

In the most transposition-like, all voices move by roughly the same intervallic distance. The most inversion-like voice leading features most of the voices flipping around the same axis of inversion.[10] Voice leadings may correspond directly to the registral lines, or voice and registral line may diverge. Pitch class invariance attempts to link the pitch class in set *A* with the pitch class closest to it in pitch-class set *B*. In functional mapping, the scale-degree function (whatever that may be in a non-tonal context) is preserved while the pitch classes may vary greatly. Smoothness is a measure of the overall distance traveled by the pitch classes (or step classes) in set *A* as they move to their destinations in set *B*. These criteria will be taken up in greater detail in Chapter 4.

Example 1.17 is a table that compares the voice leadings *a* through *f* in Example 1.16 with the criteria discussed in the previous paragraph. The letters in the left-most column correspond to the mappings given in Example 1.16. Mapping *a* is the most transposition-like; it maps the pitch classes in set *A* onto the closest pitch classes in set *B*;

[10] In the first two cases (transposition-like and inversion-like), all of the attendant baggage that accompanies these two operations across changing pitch spaces (particularly those of different cardinalities) must be taken into consideration.

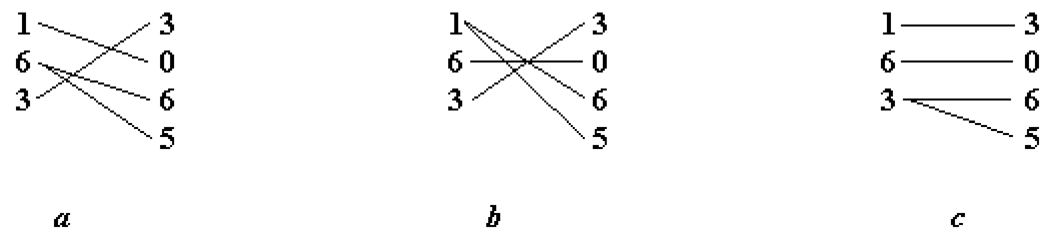
it holds step-class function invariant; and it is the smoothest, with a displacement of one step. Mapping *f* is the most inversion-like, and mapping *e* corresponds most closely to the registral lines.

Example 1.17: Evaluating mappings in Example 1.16

	Most transposition-like (Uniform)	Most inversion-like (Balance)	Registral lines	PC Invariance	Functional mapping	Smoothest
a	X			X	X	X
b						
c						
d						
e			X			
f		X				

In the left hand, there are thirty-six possible mappings between any three-note set and a four-note set. All of these mappings involve one of the pitches in the three-note set splitting into two pitches in the four-note set. Of the thirty-six mappings, I present three that correspond to the criteria given above. In essence, the mapping with the most Xs in the row is the favored interpretation. These mappings appear in Example 1.18, along with a table similar to that in Example 1.17.

Example 1.18: Mapping $(1,3,6)^8$ onto $(0,3,5,6)^8$



	Most transposition-like (Uniform)	Most inversion-like (Balance)	Registral lines	PC Invariance	Functional mapping	Smoothest
a	X				X	X
b		X				
c			X	X		

The passage appearing in Example 1.15 illustrates another problem: the single pitch-class space (mode 6) splits into two pitch-class spaces over the barline: mode 2 in the left hand and mode 3 in the right hand. How are we to apprehend these simultaneous pitch spaces? This question and the question of changing spaces in general will be taken up at length in Chapter 4.

*   *   *

This sketch of a theoretical system has revealed several avenues for further exploration. First, we must examine the nature of voice leading in atonal music. Straus 1997 discusses three types of voice leading in post-tonal music: the prolongational (extensions of Schenker's theories), the associational (finding large-scale projections of locally important sets), and the transformational (dealt with above). Because of the hybrid nature of the repertoire under consideration, our new system will have to engage all of these approaches to some extent.

Second, we need to develop a means of classifying the vast number of pitch-class spaces available to the analyst. In the preceding exposition, I greatly reduced the available choices by including only "commonly used" scales: the diatonic and its modes, the octatonic, hexatonic, whole-tone, pentatonic, and Messiaen's modes of limited transposition. Most of these spaces have the maximal evenness property in common, and, in the case of sets that are not maximally even, we can measure their deviance from maximal evenness. Maximal evenness and deviations from maximal evenness will be the subject of Chapter 2, which will also briefly discuss some perceptual reasons for favoring maximally even collections.

Third, the transformational model is a non-hierarchical system. Some of the music in our neo-tonal repertoire seems to exhibit some degree of tonal hierarchy. Using the notions of maximal evenness found in Chapter 2, we will adapt Fred Lerdahl's model of tonal pitch space to be used with music by these neo-tonal composers. The pitch space model will allow us to show hierarchical structures in the music and permit us to talk about non-harmonic tones and chromaticism in music that is not predominately twelve-tone.

The thorny issue of changing spaces and simultaneous spaces will be taken up in Chapter 4. I will review existing models of changing pitch spaces and synthesize them into a new way of relating pitch spaces of equal and different cardinalities. I will also consider the problems of simultaneous pitch spaces. The conclusion of the chapter generalizes Lerdahl's regional circle-of-fifths rule to apply to shifting pitch spaces. The generalization permits us to measure the perceptual distance between regions as well as chords across different regions.

Chapter 5 is a recapitulation, of sorts. Having developed the theoretical system more fully in the preceding chapters, Chapter 5 consists of several lengthier analytical essays, each devoted to a piece of music. Josef Matthias Hauer's *Nachklangstudie* op. 16 no. IV illustrates how Klumpenhouwer networks can be adapted to uncover voice leadings in an unusual mod-7 space. Messiaen's "Les sons impalpables du rêve…" is a rondo-form piece with very clearly delineated sections. Nearly all of Messiaen's modes of limited transposition are used in the work, as are some other synthetic modes, and they are deployed very clearly by Messiaen both successively and simultaneously. Messiaen's maximally even and near-maximally even collections organize neatly into hierarchical pitch-class spaces. The final work to be considered is Debussy's "Jardins sous la pluie," which features a plethora of different pitch-class spaces deployed horizontally and vertically. The piece illustrates regional distances, and transformational and associational voice leadings across different regions.

Chapter 2 : Perceptual aspects of maximally even sets

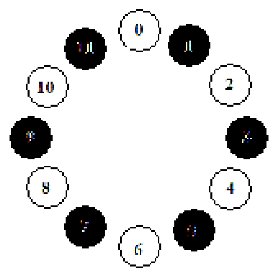
In this chapter, I will explore maximally even (ME) collections[11] in greater depth. I will show that the ME property can alternately facilitate or hinder our ability to navigate pitch spaces. Richmond Browne's concepts of pattern matching and position finding will permit us to evaluate the perceptual characteristics of these collections. I will then examine collections that are parsimoniously related to ME collections with the aim of providing a means of discussing the myriad non-ME sets in terms of an ME "parent." ME sets can be generalized, and second- and third-order ME sets can be derived from the generalized forms. Notions of second- and third-order ME sets are crucial to the next chapter, where I will develop hierarchical pitch-class space models (after Lerdahl) which will bring us closer to a cognitive model of neo-tonal pieces.

Maximally even sets are sets whose elements are distributed such that the maximum possible distance obtains between elements. This distance is measured in terms of some larger, underlying collection (a more formal definition will be given below). Example 2.1 presents the maximally even six-, seven-, eight-, and nine-note collections, represented on pitch-class clockfaces. I am concentrating on these four cardinalities because most of the common referential collections are of cardinalities six, seven, eight, or nine. The open circles indicate pitch classes that are members of the ME collection; the filled-in circles are pitch-classes that are not included in the set. Each open circle is as far

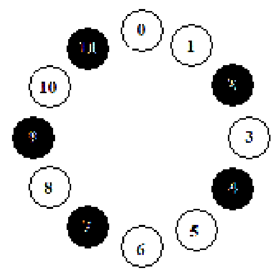
[11] Throughout the chapter I will use "set" or "collection" to refer to an abstract group of pitch classes. I will use "mode" or "scale" to refer to a specific ordering of the collection within an octave.

away as possible from the next nearest open circle. The filled-in pitch classes represent the complements of each of the larger sets, which are similarly ME.

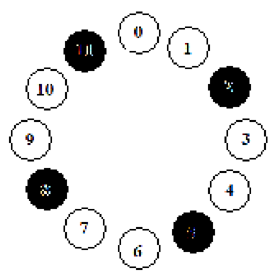
Example 2.1: Some maximally even sets



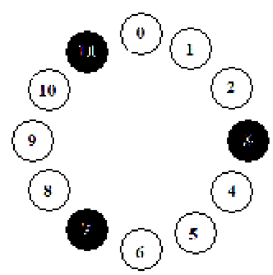
6-note ME collection (6-35)
(whole-tone scale)



7-note ME collection (7-35)
(major/minor scale)



8-note ME collection (8-28)
(octatonic scale)



9-note ME collection (9-12)
(enneatonic scale)

Clough and Douthett provide a more formal way of defining maximal evenness.[12] Let *C* and *D* both be collections of pitch classes such that *D* is a subset of *C*. In the present discussion, *C* will always be the twelve-note chromatic collection, and *D* will be a six-, seven-, eight- or nine-note subset of the chromatic. The ordered pair (*c*,*d*), where *c* is the cardinality of *C* and *d* is the cardinality of *D*, describes a diatonic collection with

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[12] Clough and Douthett 1991, 96-100.

respect to the underlying chromatic collection. Any six-note subset of the 12-pitch-class chromatic collection would be represented as (12,6), seven-note subsets as (12,7), and so on. We define an interval as the distance between two elements of a collection: D_x-D_y or C_x-C_y. The chromatic length (*clen*) of an interval is the distance between any two elements in *C*. The diatonic length (*dlen*) is the distance between any two elements in *D*.

Taking the prime form of the enneatonic collection—(01245689T)—as an example, the clen (D,E) is 2 (two semitones in the twelve-note chromatic universe); the dlen of (D,E) is 1 (one step in this particular nine-note enneatonic collection).[13] We notice, though, that, in the enneatonic collection, one dlen may have several clens: dlen(C#,D) = dlen(D,E) = 1. But clen(C#,D) = 1 and clen(D,E) = 2. Put another way, the dlen is the size of an interval and the clen is the quality. We can say that C# to D and D to E are equivalent enneatonic steps. To better organize this information, Clough and Douthett propose the *spectrum* of a dlen. The spectrum of a dlen (notated <*dlen*>) is the set of all possible clens that correspond to the dlen. For The octatonic scale, <1> = {1,2}; <2> = {3}; <3> = {4,5}; <4> = {6}; <5> = {7,8}; <6> = {9}; <7> = {10,11}.

Clough and Douthett (1991, 95) define a *step* as an interval whose *dlen* = 1. Put another way, a step is the distance between two adjacencies in *D*. We can use this notion of step to generalize any scale. A generalized scale has exactly one step between adjacent members. Consequently, we can assign step-class integers to represent the pitch classes that are members of the generalized scale. Any generalized six-note collection will be labeled $(012345)^6$; any seven-note collection $(0123456)^7$; and so on, regardless of the pitch-class intervals that obtain between elements. In the first whole-tone collection, for

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[13] I should note that the *dlen* of (D,E) is not always 1 in all enneatonic collections. Of the four possible enneatonic collections, only two contain (D,E): (01245689T) and (0234678TE). In the first, *dlen* (D,E) = 1; in the second, *dlen* (D,E) = 2.

example, (02468T) becomes (012345)[6] when generalized, with 0 mapping onto 0, 2 mapping onto 1, 4 mapping onto 2, 6 mapping onto 3, and so on.

Andrew Mead (1997-8) calls for a re-examination of how we name intervals in non-diatonic contexts. Rather than just use the term "major third" to refer to any interval class 4 (for example), Mead suggests we take the interval's scalar context and scale degree difference (*dlen*) into account when naming intervals. In the major scale, interval class 4 is larger of two types of thirds, hence the labeling "major." In the whole-tone scale, interval class 4 is the *only* type of third, and because of this, Mead would argue that we should call it a "perfect" third. (Actually, contextually speaking, all of the intervals in the whole-tone scale are perfect because of the high degree of symmetry in the scale.) Following Mead's lead, then, we can say that in the octatonic collection, we have major and minor seconds; perfect thirds; major and minor fourths; perfect fifths (that are actually tritones!); major and minor sixths; perfect sevenths; and major and minor eighths (I am avoiding the term 'octave' because neither of the *clens* are equal to 12). If all of the spectrums of a given collection are single integers or pairs of consecutive integers—as is the case with the octatonic collection—then the set is ME.

In order to generate ME collections, Clough and Douthett provide the following formula ($[x]$ is the truncation operator, which basically lops off any numbers to the right of the decimal point: it returns the smallest integer not greater than x):

$$D_{c,d} = \{ [0c/d], [1c/d], [2c/d], \ldots [(d-1)c/d] \}$$

Substituting 12 for *c* and 9 for *d* in the equation above will yield the collection (01245689T), the enneatonic collection. We will indicate sets that are maximally even by $M_{c,d}$ (after Clough and Douthett) to distinguish them from all other sets, $D_{c,d}$.

*   *   *

Pattern matching

Any collection of pitch classes can be represented as a string of integers that quantifies the chromatic intervals between successive step-classes in that collection (assuming the elements are placed in ascending order within an octave). The resulting string will always be bounded with hyphens.[14] The major scale is represented by -2212221- —the familiar WWHWWWH pattern of undergraduate theory. Our enneatonic collection can thus be represented as -112112112-: the chromatic distance between step-classes 0 and 1 is -1-; between step-classes 2 and 3 is -2-; and so on. By moving the first element of the string to the last position, we can rotate the collection. Rotations preserve the interval content and thus do not alter the identity of the collection. This procedure is what yields the diatonic modes:

-2212221- =	Major scale
-2122212- =	Dorian
-1222122- =	Phrygian
-2221221- =	Lydian
-2212212- =	Mixolydian
-2122122- =	Aeolian (natural minor)
-1221222- =	Locrian

[14] I enclose these interval strings in hyphens to distinguish them from pitch-class sets (which appear in parentheses) and interval vectors (which appear in angle brackets).

Clough proposes the concept of *interval normal form* (INF) in order to provide a unique identifier to a collection and all of its possible rotations.[15] The INF of a collection is that rotation that has the largest interval in last place. If this does not yield a unique string (as in the case of C major and its modes), then the rotation that also has the smallest interval in first place, second place, and so on until a unique form is identified. In the case of the C major modes, the INF is the Locrian mode, -1221222- (incidentally, equivalent to the normal form of the scale given in most post-tonal theory texts). In the discussion below, I will refer to sets by their INF and I will speak of the rotations of the INF as the *modes* of the collection.

Example 2.2 provides the four ME collections discussed in the previous section. The fourth column presents the INF of the collection and the fifth column includes the possible modes of each collection. Modes are numbered according to the pitch class on which they begin: mode 0 begins on pitch class 0.

Clough and Douthett describe three classes of ME sets. In the following discussion, (*c,d*) indicates the greatest common factor (GCF) of *c* and *d*. In Class A sets, (*c*,*d*) is *d*. In Class B sets, 1 < (*c,d*) < d. In Class C sets, (*c,d*) = 1 (Clough and Douthett 1991, 97). The whole-tone collection is an example of a Class A set; the octatonic and enneatonic are Class B sets; and the diatonic collection is a Class C set. Class C sets will contain *d* unique modes of *c*/(*c,d*) collections. That is, there are twelve possible ME collections where each pitch class is situated within a unique intervallic pattern that places it in an unambiguous position in the scale. If someone played D-E-F-G-A-B-C-D on the piano and we were asked to identify the scale degree (in major) on which the passage began, most of us would answer scale degree 2, because the string comprises the

[15] Clough 1979, 46-47.

unique collection of intervals (WHWWWHW) that we associate with scale degree 2 in the major scale. One could also answer scale degree 4 in minor, because the same pattern of intervals obtains there as well.

Example 2.2: Some common pitch-class spaces and their modes

Scale	(c,d)	INF	Number of unique modes	Modes
Whole-tone (02468T)	(12,6)	-222222-	1	Mode 0: -222222-
Diatonic (013568T)	(12,7)	-1221222-	7	Mode 0: -1221222- Mode 1: -2212221- Mode 3: -2122212- Mode 5: -1222122- Mode 6: -2221221- Mode 8: -2212212- Mode 10: -2122122-
Octatonic (0134679T)	(12,8)	-12121212-	2	Mode 0: -12121212- Mode 1: -21212121-
Enneatonic (01245689T)	(12,9)	-112112112-	3	Mode 0: -112112112- Mode 1: -121121121- Mode 2: -211211211-

Pattern matching becomes more difficult in Class A and B ME sets because there will be a limited number of distinct modes. (Patten matching is also made more difficult when the INF contains more than two step sizes.) These are the modes of limited transposition. An ear-training experiment similar to the one in the previous paragraph might involve someone playing C#-D-E-F-F#-A♭-A- B♭-C on the piano. Asked to identify what *enneatonic* scale degree the string begins with, nine answers are possible. The C# could be scale degrees 2, 5, or 8 in the -112112112- enneatonic scale (mode 0); it could be scale degrees 3, 6, or 9 in the -121121121- enneatonic (mode 1); or it could be scale degrees 1, 4, or 7 in the -211211211- enneatonic scale (mode 2). Because of the limited number of unique modes and the redundancy found in the intervallic representation of the scale, pattern matching tasks are severely compromised in Class A and B ME sets.

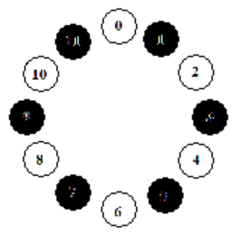
Modal structure is crucial to the notion of pattern matching. If a scale has a number of modes that is less than its cardinality, pattern matching will be hindered because any pattern will have multiple correct locations in the set. The tendency of certain collections to compromise pattern-matching tasks is what results in a lack of "tonal center" or directionality in music that employs these scales.

*   *   *

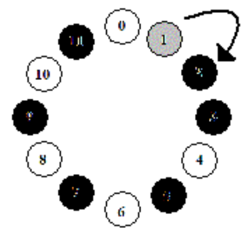
Minimally perturbed ME sets

In Class A and B ME sets, pattern-matching tasks are hindered by the limited number of distinct modes. In many cases, a slight deviation from ME results in a set that is much more listener-friendly. Example 2.3 shows what happens to the six-, eight-, and nine-note ME collections if we displace one of the pitch classes in the collection by semitone. I have omitted the seven-note ME collection from the discussion because it already has a number of distinct modes equal to the cardinality of the set. In the left-hand column are the three ME collections. In the right-hand column are minimally perturbed ME sets: sets that deviate from ME by one semitone. In the diagram, the black circles are again notes that are not present in the large ME collection; the black circles represent the complements. In the right-hand column, the grey circles represent those pitch classes that were not present in the ME set, but are now members of the collection as a result of displacement. In the case of the six-note collection, pitch class 2 in the ME collection is replaced with pitch class 1, creating sc 6-34, Scriabin's "Mystic" chord.

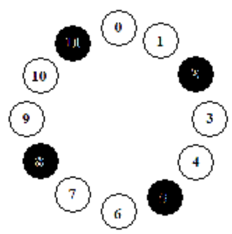
Example 2.3: Minimally perturbed ME sets



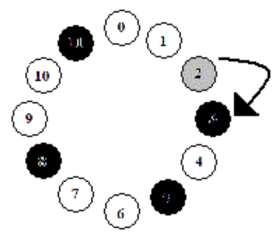
6-note ME collection (6-35)



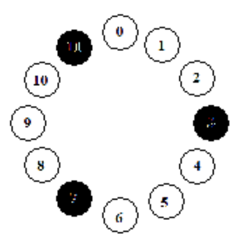
6-note minimally perturbed ME set (6-34)



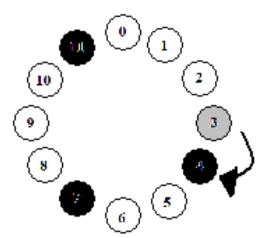
8-note ME collection (8-28)



8-note minimally perturbed ME set (8-27)



9-note ME collection (9-12)



9-note minimally perturbed ME set (9-11)

In the eight-note collection, pitch class 3 is replaced with pitch class 2, which results in the complement of the dominant (or half-diminished) seventh chord. In the nine-note collection, pitch class 4 is replaced with pitch class 3, yielding set class 9-11, the complement of the major/minor triad. Minimally perturbing the larger ME set also minimally perturbs its ME complement.

To examine the impact of this displacement on pattern matching, let us take the enneatonic collection, $M_{12,9}$, as an example. As given in Example 2.2 above, there are only three distinct modes of this collection, and they are given below. Again, I have labeled the mode according to the pitch class on which it begins.

Enneatonic collection: (01245689T) = [C,C#,D,E,F,F#,G#,A,A#]

INF (mode 0):	-112112112-	
mode 1:	-121121121-	
mode 2:	-211211211-	
(mode 4	-112112112-	= INF from 0)

If we minimally perturb one of the elements of the enneatonic collection—that is, exchange one of the elements that borders a gap with its semitone neighbor—we get a collection that facilitates pattern matching. Balzano refers to this property as "uniqueness," defined as a collection in which the ordered set of step classes that follows from any note in a collection is different from the set of step classes of any other pitch in the collection.[16] This new collection ((01235679T); a member of set class 9-11) can be related to the enneatonic by using Douthett and Steinbach's P-relation.[17] Two sets are $P_{m,n}$-related if, of the pitch classes that are not members of X ∩ Y, *m* move by half step and *n* move by whole step. In the example below, I have exchanged pitch class 2 for pitch class 3. The modes are named according to the pitch class on which they begin.

[16] Balzano, 1980. Cited in Lerdahl, 2000: 50.
[17] Douthett and Steinbach, 1998: 243.

$P_{1,0}$-Enneatonic: (01235679T)

INF (mode 0):	-111211212-
mode 1:	-112112121-
mode 2:	-121121211-
mode 3:	-211212111-
mode 5:	-112121112-
mode 6:	-121211121-
mode 7:	-212111211-
mode 9:	-121112112-
mode 10:	-211121121-

Minimally perturbing the enneatonic collection results in a collection that facilitates pattern matching. There are nine distinct modes of this collection, and, as such, each scale step has its own unique string of intervals that places it at a particular point in the collection. To return to our ear-training experiment, if someone were to play the pitches D-E♭-F-F#-G-A- B♭-C-C#, we would in this context be able to say that the example begins on scale degree two of $P_{1,0}$ ($M_{12,9}$).

I should mention that any operation performed on one of these larger ME sets has the same effect on its complement. The complement of the enneatonic collection is the augmented triad, and minimally perturbing one note in the augmented triad yields (037), the major/minor triad. The complement of the octatonic collection is the fully-diminished seventh chord, (0369). Minimally perturbing it results in a dominant- (major/minor) or half-diminished (diminished minor) seventh chord.[18]

Example 2.4 shows the set classes that result from minimally perturbing the six-, eight-, and nine-note ME collections. The modes of the resulting collections are given in the right-most column.

---

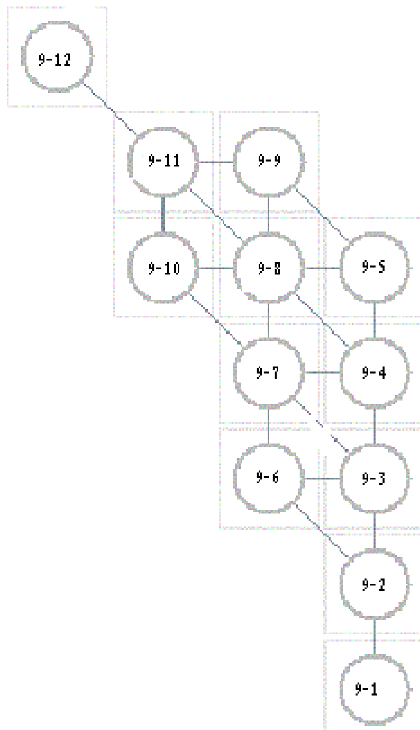
[18] See Douthett and Steinbach's (1998) cube dances and power towers for diagrams of P-related trichords and tetrachords.

Example 2.4: Minimally perturbed six-, eight-, and nine-note ME sets with their modes

Set	Forte number/ Pitch class content	Modes
$P_{1,0}$ ($M_{12,6}$)	6-34: (013579)	INF (mode 0): -122223- mode 1: -222231- mode 3: -222312- mode 5: -223122- mode 7: -231222- mode 9: -312222-
$P_{1,0}$ ($M_{12,8}$)	8-27: (0124578T)	INF (mode 0): -11212122- mode 1: -12121221- mode 2: -21212211- mode 4: -12122112- mode 5: -21221121- mode 7: -12211212- mode 8: -22112121- mode 10: -21121212-
$P_{1,0}$ ($M_{12,9}$)	9-11: (01235679T)	INF (mode 0): -111211212- mode 1: -112112121- mode 2: -121121211- mode 3: -211212111- mode 5: -112121112- mode 6: -121211121- mode 7: -212111211- mode 9: -121112112- mode 10: -211121121-

If we continue to perturb one pitch class by semitone in any of these collections, we can generate a network of parsimoniously related members of a same cardinality. The network in Example 2.5 shows parsimonious relations among all twelve nonachords. The nodes contain the Forte numbers for each nonachord, and lines that link the nodes represent $P_{1,0}$-relations. Set class 9-12, the enneatonic collection, is the only maximally even nonachord, and it is the top-most node of the network. A P-relation line connects 9-12 to 9-11. 9-11 is P-related to three nonachords: 9-8, 9-9, and 9-10. Set class 9-12 deviates from 9-8, 9-9, and 9-10 by two semitones. We can also say that 9-12 is related to those five nonachords by $P_{2,0}$, which tells us that two pitch classes in 9-12 need to move by semitone or one pitch needs to move by two semitones in order to produce 9-8, 9-9, or 9-10. Set class 9-1 is at the bottom of the network because it represents what I will call minimal evenness—that is, it deviates from maximal evenness as much as possible.

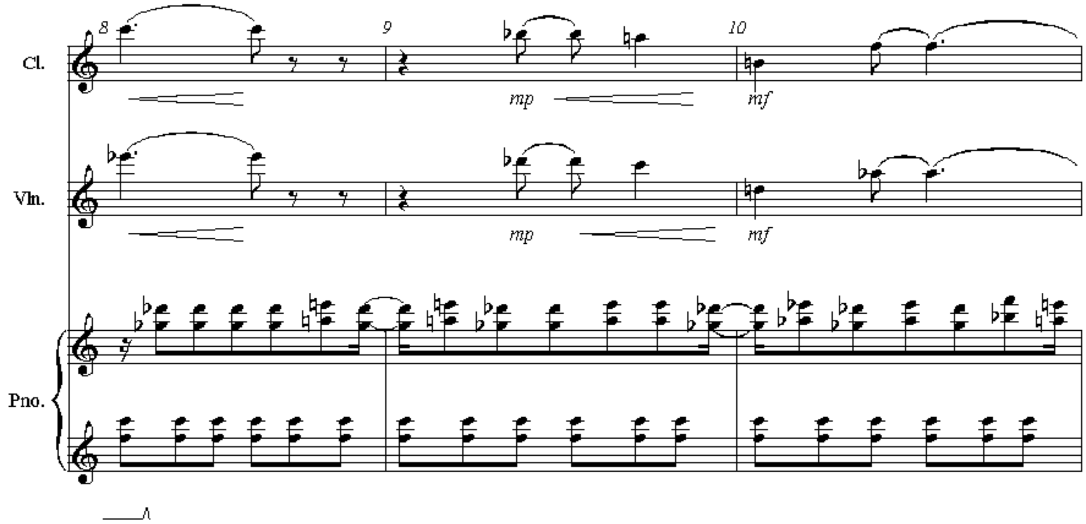
Example 2.5: P-relations among nonachords[19]



[19] This graph and similar graphs that follow were created using Christopher Ariza's *AthenaCL* program, which is available for download at http://www.athenacl.org. Graphs similar to these first appeared in Straus 2003.

Example 2.6: "Pierrot's serenade" from Musgrave's *Pierrot*

(Example 2.6 continued)

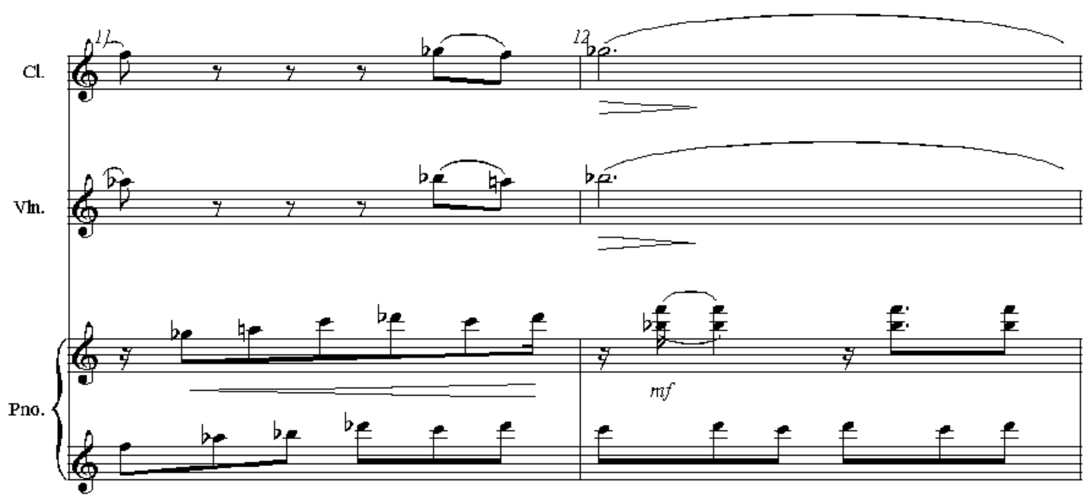


Mode 0 (0,1,2,3,5,6,7,9,T)

Mode 3 = 0 1 3 4 5 6 8 9 T

Mode 0 = 0 1 2 3 5 6 7 9 T

$= P_{1,1}$

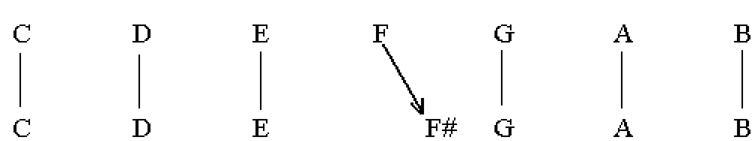


Mode 0 (0,1,2,3,5,6,7,8,10)

"Pierrot's serenade" from Thea Musgrave's *Pierrot* makes extensive use of nine-note collections. The work abounds with members of set class 9-11, and we will see that the P-relation can be used effectively to model the relationships among changing pitch spaces in the work. In the performance notes in the beginning of the score, Musgrave notes that the serenade is in A minor, and we will also explore to what extent we can say that the work is in a key below. The opening of the work appears in Example 2.6: the top staff is clarinet (at concert pitch), the second staff is violin, and the bottom system is piano.

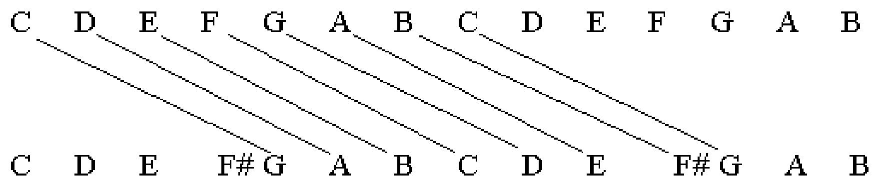
The beginning of the work inhabits Mode 6 of a minimally-perturbed enneatonic collection (the prime form of set class 9-11, rotated to begin on pitch class 6, then transposed up by three semitones), and in m. 8, we see the same material in Mode 0. We can use transpositional operators to relate the sets, but we can also look at them as P-related. In addition to their ability to relate members of different set classes, P-relations can also be used to describe the relationships among transpositions of a set class. In tonal music, the circle of fifths can be interpreted as a series of P-relations: C major becomes G major when F-natural is replaced with F#—a $P_{1,0}$ relation. We can thus set $ME_{(12,7)}P_{1,0} = ME_{(12,7)}T_7$. The relationship between transposing the diatonic collection and P-relations between diatonic collections is illustrated in Example 2.7.

Example 2.7: Transposition versus P-relations between C-major and G-major



As a $P_{1,0}$ relation

(Example 2.7 continued)

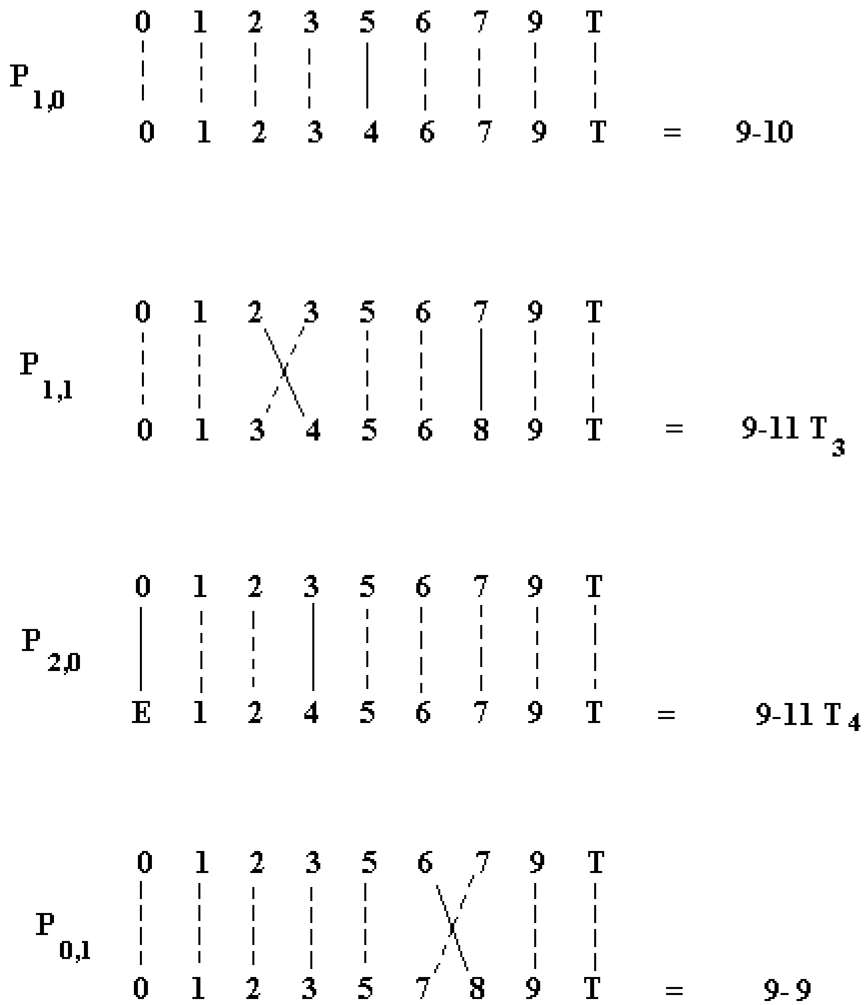


As T_7 relation

In the case of our altered enneatonic, a $P_{1,0}$ relation—the least possible disturbance of the collection—yields a new set class. $P_{0,1}$, in which all pitch classes are retained and one moves by whole step, also yields a new set class, 9-10. $P_{2,0}$ (a displacement of two semitones) and $P_{1,1}$ (a displacement of a semitone and a whole tone, or, three semitones), each produce another member of the same set class. The $P_{1,0}$ relation and its application were discussed above (see Example 2.2). Example 2.8 shows the relationship between transposition and P-relations among members of set-class 9-11, the deviant enneatonic.

Members of set-class 9-11 that are transpositionally related can also be discussed in terms of P-relations. Transpositional operators motivate the transformations; P-relations describe the relationships of the two sets. Sets that are spawned by T_3 are $P_{1,1}$-related. Example 2.9 shows $P_{1,1}$-related members of set class 9-11. From the T_0 form to the T_3 form, all of the pitch classes remain invariant, with the exception of 2, which moves by interval class 2 to 4, and 7, which moves by interval class 1 to 8. The T_3 operator is equivalent to a PR chain in neo-Riemannian theory: the effects can be observed by examining the relationships between the trichord complements of the nine-note sets: a C-major triad transposed up three semitones becomes an E-flat major triad.

Example 2.8: Transposition, P-relations and the deviant enneatonic



Example 2.9: $P_{1,1}$ relations among members of set class 9-11: The T_3 family

T_0	0	1	2	3	5	6	7	9	T
T_3	0	1	3	4	5	6	8	9	T
T_6	0	1	3	4	6	7	8	9	E
T_9	0	2	3	4	6	7	9	T	E
T_0	0	1	2	3	5	6	7	9	T

Other P-relations represent other families: there is, for example, a T_4 family described by the $P_{2,0}$ relation (see Example 2.10). The $P_{2,0}$ relation is equivalent to an LP chain in neo-Riemannian theory. The $P_{1,1}$ relation also describes a non-duplicating T_7 family, which offers a nice analogy to the circle of fifths in tonal music. This family is given in Example 2.11.

Example 2.10: $P_{2,0}$ relations among members of set class 9-11: The T_4 family

T_0	0	1	2	3	5	6	7	9	T
T_4	E	1	2	4	5	6	7	9	T
T_8	E	1	2	3	5	6	8	9	T
T_0	0	1	2	3	5	6	7	9	T

Example 2.11: $P_{1,1}$ relations among members of set class 9-11: The T_7 family

T_0	0	1	2	3	5	6	7	9	T
T_7	0	1	2	4	5	7	8	9	T
T_2	0	2	3	4	5	7	8	9	E
T_9	0	2	3	4	6	7	9	T	E
T_4	1	2	4	5	6	7	9	T	E
T_{11}	1	2	4	5	6	8	9	E	0
T_6	0	1	3	4	6	7	8	9	E
T_1	1	2	3	4	6	7	8	T	E
T_8	E	1	2	3	5	6	8	9	T
T_3	0	1	3	4	5	6	8	9	T
T_{10}	0	1	3	4	5	7	8	T	E
T_5	0	2	3	5	6	7	8	T	E

We can say, then, that the opening of “Pierrot’s serenade” uses the T_3 family of $P_{1,1}$-related members of set class 9-11 (of which mode 6 is representative). In m. 16 there is another instance of set class 9-11, this time in inversion at I_1. The inversion is not P-related to the T_0 form of 9-11 found in mm. 8-16 of the work. In m.18, 9-11 ceases to be the controlling collection. It is transformed by a $P_{1,0}$ relation into 9-10, which manifests itself as (01345679T). Measures 18-20 inhabit 9-10 space, and we return to 9-11 space (again, via $P_{1,0}$ relation) in m. 21.

Musgrave can make the claim that the piece is in A minor because of the deviant ME collection that she employs. Had she used the enneatonic collection, the collection would lack uniqueness and could be interpreted in six possible minor “keys”—A minor, C# minor, F minor, B♭ minor, D minor, and F# minor—because the succession of intervals that proceeds from each “tonic” pitch would be identical. Example 2.12 illustrates the six possible keys.

Example 2.12: Three possible keys using 9-12

A♭ A B♭ C C# D E F F#	= -121121121-	= A minor
C C# D E F F# G# A A#	= -121121121-	= C# minor
E F F# A♭ A B♭ C C# D	= -121121121-	= F minor
A♭ A B♭ C D♭ D E F F#	= -211211211-	= B♭ minor
C C# D E F F# G# A A#	= -211211211-	= D minor
E F F# A♭ A B♭ C C# D	= -211211211-	= F# minor

In each case, the string of intervals that follows the root of the tonic triad is identical: either -121121121- or -211211211-, prohibiting unambiguous pattern matching. Musgrave's choice of pitches results in a collection that allows us to pattern-match unambiguously: (A,B♭,C,C#,D#,E, F,F#,G#) features nine unique strings of intervals such that the tonic pitch A can always be located by the string -121211121-.

Returning now to Example 2.6, we see that Musgrave has the open fifth A4-E5 in the left hand of the piano, and a C6 sustained in the violin part, all of which suggest A minor. The right hand of the piano embellishes the left hand with B♭4-F5, the upper leading tones to the pitches in the right-hand part. The chord can be written with an INF of -12414-. This pattern of intervals can only be found in modes 2 and 6: none of the other modes can spawn a chord with this specific INF. The distinction between modes 2 and 6 is to be found in the way the 4s are divided at the scale level. In mode 2, the first -4- = -112- and the second -4- = -211-. In mode 6, the first -4- = -121- and the second -4- = -121-. The D♭s and E♭s in the violin part unambiguously define mode 6 by splitting the -4- into -121-.

While some ME sets (those where GCF(c,d) = 1) have an intervallic makeup that facilitates pattern matching, ME sets where GCF(c,d) ≠ 1 hinder pattern matching because of their redundant step-interval makeup. Pattern matching can often be enhanced by minimally perturbing a pitch class in the collection, resulting in a collection that obeys many of the preference rules outlined by Lerdahl and others.

*   *   *

Towards minimal evenness

Until now, we have examined sets that are P-related to some ME "parent" set. We can continue to exchange pitch classes, moving down a network such as the one that appears in Example 2.5, until we get to the minimally even collection of a particular cardinality. I will define minimal evenness (MinE) as essentially the opposite of maximal evenness. Minimal evenness occurs when all of the members of a set, *d*, are as close to one another as possible with reference to an underlying collection, *c*. All of the set classes with a -1 suffix on Forte's list (5-1, 6-1, 7-1, 8-1, 9-1) are minimally even with respect to the chromatic collection.

There will come a point on the path from ME to MinE where the collections cease to have the perceptual characteristics discussed above. In most cases, the scale steps cease to be coherent. Coherence is a property discussed first by Balzano and later taken up by Lerdahl.[20] Scale steps cease to be coherent when there is an ambiguity between the distance traveled in the scale (with respect to *c*) and the number of scale steps covered. The diatonic scale is an example of a coherent scale: all of the thirds in the scale are larger than all of the seconds (in terms of semitones). Lerdahl gives the harmonic minor as an example of a non-coherent scale.[21] The augmented second between scale degrees six and seven is the same size as the minor thirds in the scale, and, as a result, creates perceptual confusion when trying to navigate the space. While the harmonic minor scale lacks coherence, it still exhibits uniqueness—that is, it has a number of distinct modes equal to the cardinality of the scale. In fact, because GCF(12,7) = 1, all of the seven-note collections will have the uniqueness property.

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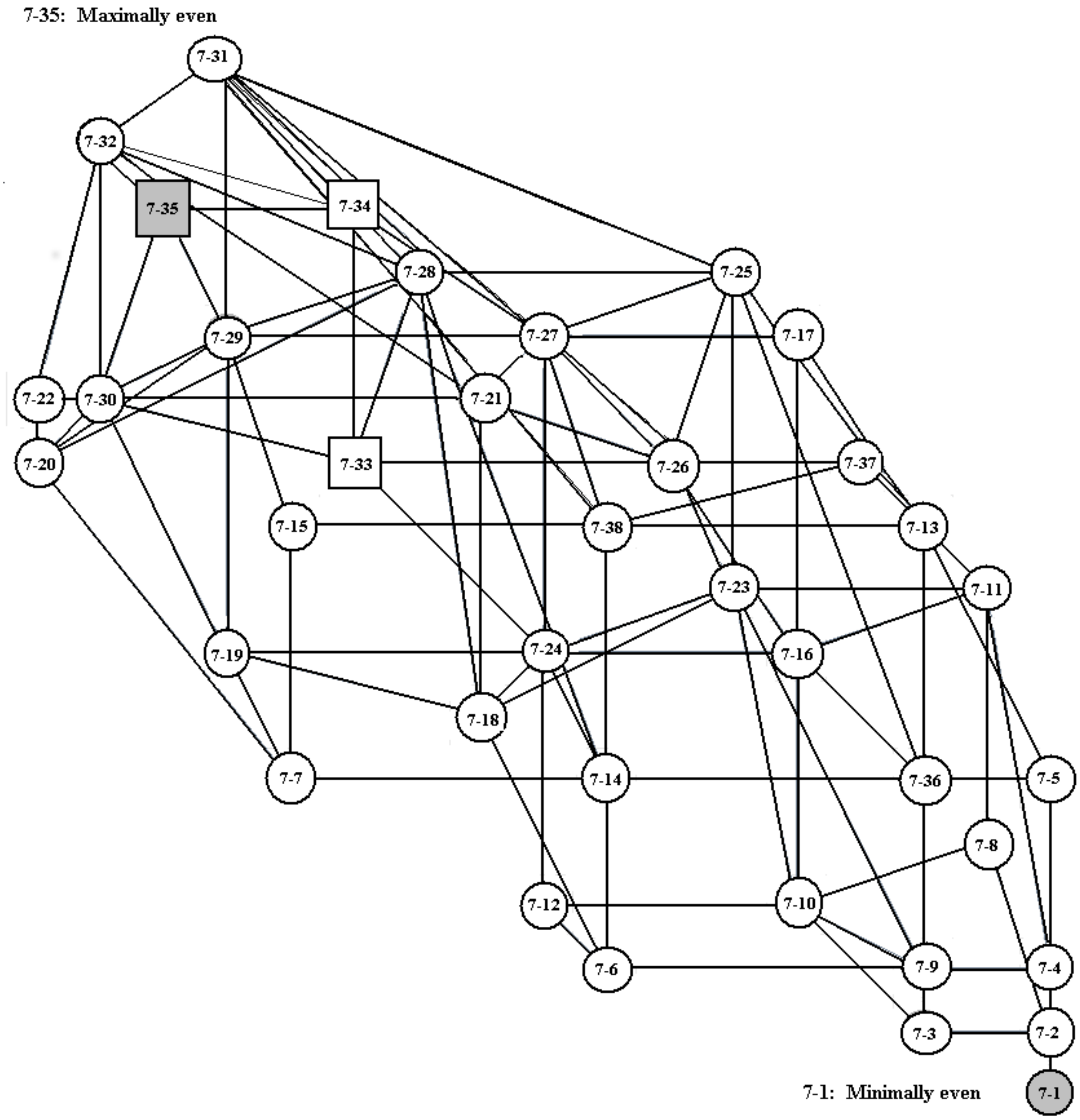
[20] Balzano 1980; Lerdahl 2000.
[21] Lerdahl 2000, 50.

The ME seven-note collection (the major/minor scale) is the only ME set under consideration in this chapter that has the uniqueness property and demonstrates coherence in its scale steps. Perturbations of the collection also have the uniqueness property, but there comes a point where the scale steps cease to be coherent even though the scale remains unique. Example 2.13 contains a map of P-relations among seven-note collections. If the diagram were realized in three dimensions, the ME seven-note collection (7-35) is at the top; the MinE seven-note collection (7-1) is at the bottom—both are shaded in the map. The three seven-note collections that exhibit coherence—7-33, 7-34, and 7-35—appear in square nodes. As we move from maximal evenness towards minimal evenness, the coherence property disappears. 7-32, the harmonic minor scale, is the first collection on the path to lose the coherence property.

Example 2.13: P-related seven-note collections



There are other collections related to the three coherent collections that, following a P-transform, lose the coherence property. Consider the relationship between 7-35 (013568T) and 7-29 (0124679). If we transform pitch class 10 in 7-35 into pitch class 11, or pitch class 8 into 7, the resultant collection is a member of set class 7-29. 7-29 has one scale step of three semitones, three of two semitones, and three of one semitone, making it an incoherent scale. The reverse holds true, of course: 7-29 can be transformed into a coherent scale by displacing one pitch-class by semitone.

* * *

Second-order ME collections and Messiaen's chords

As mentioned above, in the tonal system, the major/minor triad is not maximally even with respect to the chromatic collection, but it is ME with respect to the seven-note diatonic collection that spawns it. Clough and Douthett refer to this relationship as second-order maximal evenness. The relationship between the 12-pitch-class chromatic collection, the seven-note diatonic collection, and the major/minor triad is a unique one: we see first- and second-order maximal evenness, and each generated collection features a number of unique modes equal to the cardinality of *d* or *d*'. The "modes" of the major/minor triad are -223-, -232-, and -322- with respect to *c'*.[22] These properties are one of the reasons that the major/minor tonal system has enjoyed such widespread use—it offers the highest possible degree of perceptual finding aids. The modes of both the three- and seven-note collections facilitate pattern matching, and the intervallic makeup of the seven-tone collection contains one rare interval that aids position-finding. In this section, I will explore second-order ME sets in greater detail in order to see if other chords in other collections have similar properties. Second-order maximal evenness and deviations from it will prove important in Chapter 4 where we will use them to construct hierarchical pitch space models.

[22] See Carey and Clampitt 1996 for a discussion second-order ME sets and related constructions.

To find a second-order ME set, the cardinality *d* is reinterpreted as cardinality *c'* of a new level. Put another way, level *d* is generalized into the step classes of level *c*'. In the case of the major/minor triad, c = 12, $d = c'$ = 7, and d' = 3. The sets emerge as follows:

$M_{12,7}$ = {[0*12/7], [1*12/7], [2*12/7], [3*12/7], [4*12/7], [5*12/7], [6*12/7]}

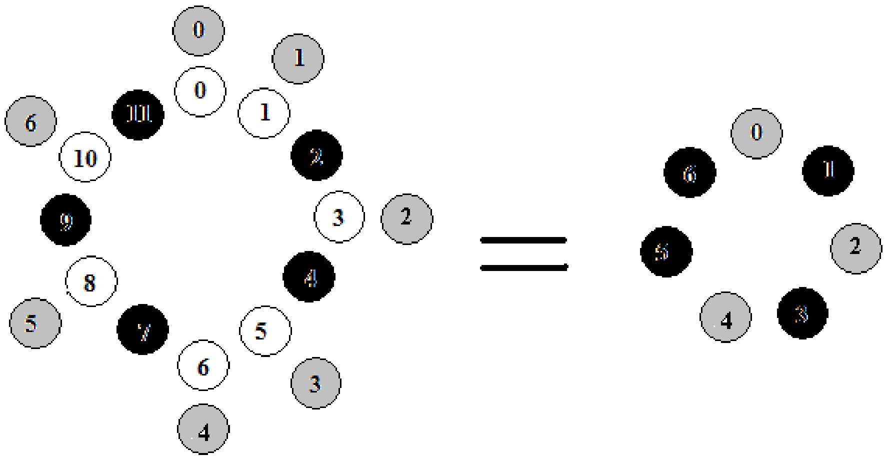
= (013568T)

$M_{7,3}$ = { [0*7/3], [1*7/3], [2*7/3] }

= (0,2,4)

Example 2.14 illustrates second-order maximal evenness using clockfaces. The left-hand side of Example 2.14 shows the relationship of the diatonic scale to the twelve-note chromatic collection. Circles that are blackened indicate chromatic notes that are not part of the seven-note diatonic (in this case, a D♭ major scale). The ring around the twelve-note collection—the grey circles—indicates the generalized diatonic collection. The pitches have been renumbered according to a mod-7 system: in D♭ major, 1 is scale degree 1, 2 is scale degree 2, and so on. The outer ring is presented as a mod-7 clockface on the right-hand side of the example. The second-order ME collection, ((024))—(C,E♭,G♭) in D♭ major—is represented by grey circles, and those pitches of D♭ major that are not members of the set are filled in. Note that the formula to generate an ME set in this case generates a diminished triad. Diatonic transposition will yield major and minor triads in the key of D♭.

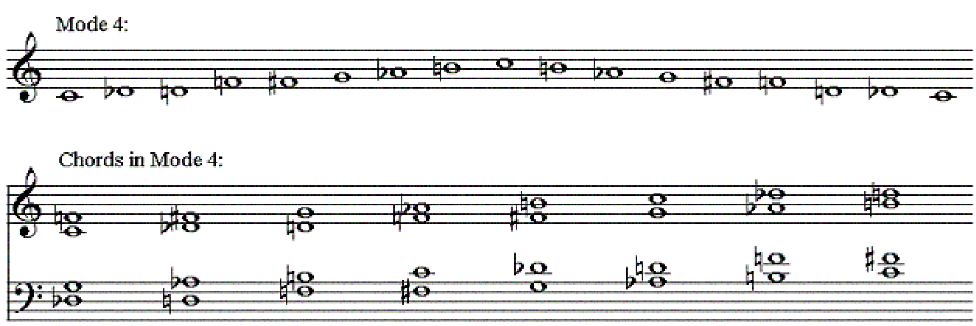
Example 2.14: Second-order ME, the diatonic collection, and the major/minor triad



In his *Traite de rythme, de couleur, et d'ornithologie*, Messiaen provides us with a comprehensive taxonomy of the harmonies that appear in his work.[23] Among these collections are his modes of limited transposition and their representative chords, the chord of the dominant, and an assortment of other verticalities. Example 2.15 presents three of Messiaen's modes and the representative chords for each mode appear beneath the mode.

[23] Messiaen 2002.

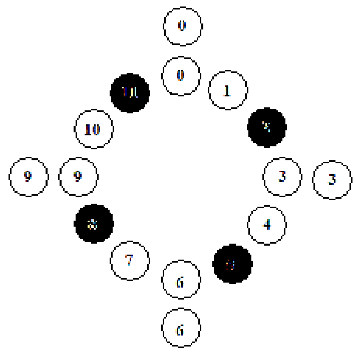
Example 2.15: Messiaen's modes of limited transposition and representative chords (as given by Messiaen)



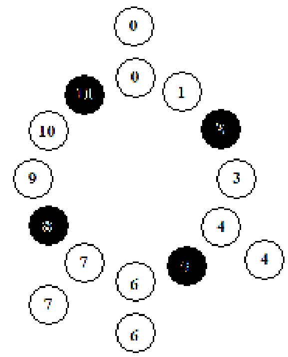
Of the given modes, two are ME: mode 2 (the octatonic collection) and mode 3 (the enneatonic collection). Each of the other collections also hinders pattern-matching tasks because the number of distinct modes is less than the cardinality of the collection. In all of these cases, to facilitate navigation of the pitch space, the second-order set must deviate from maximal evenness.

Example 2.16 permits a comparison of Messiaen's chords in mode 2 and second-order maximal evenness. If we generate a four-note second-order ME collection for mode 2 (the octatonic), we get the set (0369), the fully-diminished seventh chord. This chord appears in Example 2.16.1. Example 2.16.2 shows (0467), the chord Messiaen gives as representative of the mode. The two tetrachords are $P_{1,1}$ related: 0 and 6 remain invariant while 3 moves by semitone to 4 and 9 moves by whole step to 7. Examples 2.16.3 and 2.16.4 illustrate the chords on a generalized mod-8 clockface. In 2.16.3, the second-order ME set is $(0246)^8$. Messiaen's chord is $(0345)^8$. These second-order trichords are $P_{2,0}$ related: 0 and 4 remain invariant while 2 is exchanged with 3 and 5 is exchanged with 6.

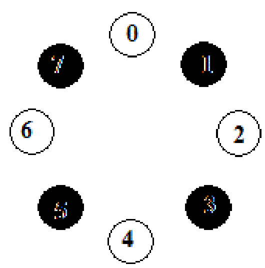
Example 2.16: Second-order maximal evenness and Messiaen's mode 2



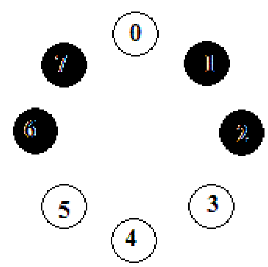
Ex. 2.16.1: Second-order ME collection for 8-28 (C, Eb, F#, A)



Ex. 2.16.2: Messiaen's mode 2 tetrachord (C, E, F#, G)



Ex. 2.16.3: Second-order ME tetrachord on a mod-8 clockface

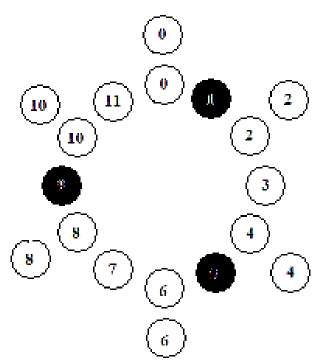


Ex. 2.16.4: Messiaen's tetrachord on a mod-8 clockface

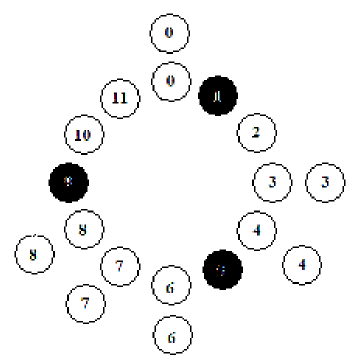
Measurements are taken according to the generalized scale level (the series above the horizontal line). A "semitone" in a generalized pitch space is motion between two adjacent pitches, d and $d\pm1$; a "whole step" in generalized pitch space is motion between two pitches, d and $d\pm2$. The mod-8 representation of Example 2.16.4 also reveals an interesting symmetry in Messiaen's chord that is not immediately apparent from the mod 12 representation: an axis bisects pitch classes 0 and 4, and the pitch classes 3 and 5 are balanced on this axis (4 ± 1).

Example 2.17 is analogous to Example 2.16: it displays the relationship between a six-note ME collection and the chords that Messiaen gives as representative of the mode. Again, 2.17.1 and 2.17.2 represent the chords in 12-pitch-class space and Examples 2.17.3 and 2.17.4 illustrate the ME property on a mod-9 clockface. In mode 3, the ME collection is actually the whole-tone collection. Here, Messiaen's hexachord deviates by $P_{1,1}$: 0, 3, 4, and 6 remain invariant while 1 moves to 2 (an enneatonic half-step) and 7 moves to 5 (an enneatonic whole-step).

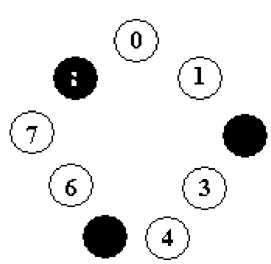
Example 2.17: Second-order maximal evenness and Messiaen's mode 3



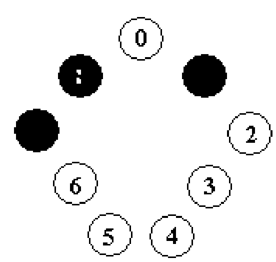
Ex. 2.17.1: Second-order ME hexachord for 9-12 (C, D, E, F#, G#, Bb)



Ex. 2.17.2: Messiaen's mode 3 hexachord (C, Eb, E, F#, G, Ab)



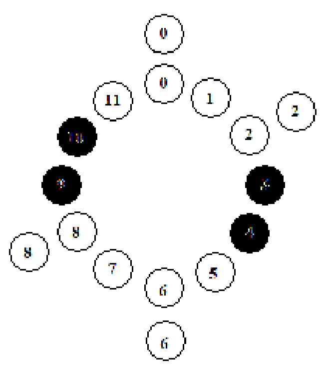
Ex. 2.17.3: Second-order ME hexachord on mod-9 clockface



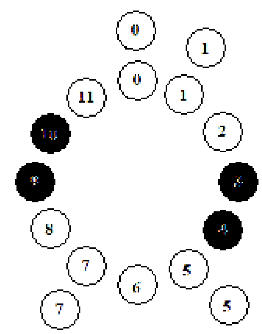
Ex. 2.17.4: Messiaen's hexachord on a mod-9 clockface

Modes 2 and 3 are both ME, while mode 4 is not. Mode 4 is an eight-note set (a member of set class 8-9) that is the complement of (0167). It can be related to the octatonic collection by $P_{0,2}$: it deviates from ME by four semitones (two whole steps). Example 2.18 illustrates the relationship between an ME tetrachord in mode 4 and the chord Messiaen gives. Whereas mode 4 deviates quite a bit from ME, Messiaen's tetrachord deviates only by $P_{1,0}$: pitch class 7 in the generalized mod-8 system exchanges with pitch class 0.

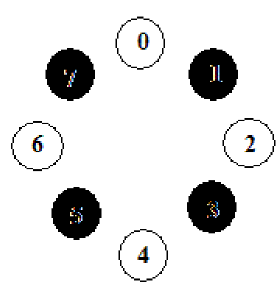
Example 2.18: Second-order ME and Messiaen's mode 4



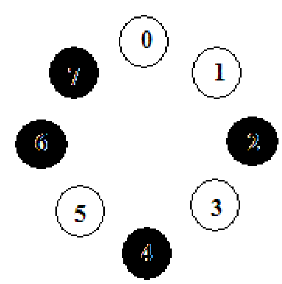
Ex. 2.18.1: ME tetrachord in mode 4 (C, D, F#, G#)



Ex. 2.18.2: Messiaen's mode 4 tetrachord (C, Db, F, G)



Ex. 2.18.3: ME tetrachord on a mod-8 clockface



Ex. 2.18.4: Messiaen's tetrachord on a mod-8 clockface

The benefit of all this is much the same as above: in cases where the collection is maximally even, a second-order collection that deviates from ME is desired so as to facilitate pattern matching. In cases where the collection is not ME, the second-order collection can be ME without compromising pattern matching. In the next chapter, we will see how first- and second-order ME sets relate to the hierarchical organization of neo-tonal music using a model of pitch-class space developed by Fred Lerdahl.

Chapter 3 : Non-tonal pitch-class spaces and hierarchical structure

Generalizing pitch spaces

Lerdahl's theory of pitch-class spaces (Lerdahl 2001) gives us a highly-developed method of examining the perceptual organization of tonal pitch-class spaces, generally assuming a diatonic reference. Lerdahl's system provides a concise method of illustrating the chords that arise from these spaces and the perceptual distances between them. The theory is easily generalized to non-diatonic pitch-class spaces.

The foundation of the theory is provided by what Lerdahl calls the "basic space" (Lerdahl 2001, 47-70).[24] Example 3.1 presents a basic space for the tonic triad in the key of C major.

Example 3.1: Basic space of the tonic triad in C major

level a:	C												(C)
level b:	C							G					(C)
level c:	C				E			G					(C)
level d:	C		D		E	F		G		A		B	(C)
level e:	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	(C)

In the example, level e contains the twelve-note chromatic pitch-class space that forms the basis for most Western music. Level d is the scale level (also called the regional level), and it emerges from the background of level e. In Example 3.1, level d contains the C-major scale. Level d can contain any major or minor scale, any of the church modes, and hexatonic or octatonic scales (Lerdahl 2001, ch. 6). Level c is the chord level,

[24] The following two paragraphs are adapted from these pages.

and in the example above contains the pitch classes of the C-major triad. Level b is the fifth level, and it contains the fifth that frames the triad: the tonic and dominant pitch classes (the root and the fifth). The fifth deserves its own level because of its importance in tonal music: it is the generator of the major scale, root motion by fifth is quite common, and the fifth frames triads. Level a is the octave level.

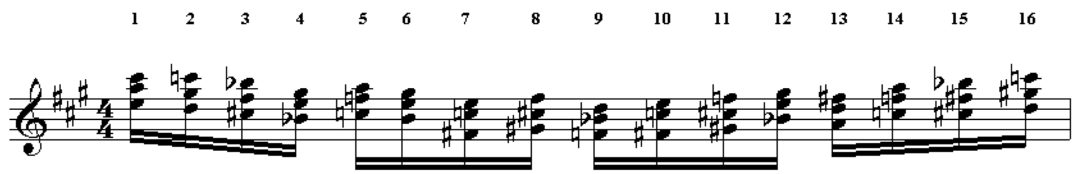
The space neatly illustrates many of our intuitions about tonal music. First, pitches that are most structurally important appear most often in the diagram. In the case of the C-major triad, pitch-class C naturally occurs more than any of the other pitch classes. Conversely, A# (or B♭) which is not in the key of C major appears only once in the space, and it appears below (subservient to) the scale level. Chord distances can be calculated algebraically by determining the number of new pitch classes that appear in each level when a shift to a new basic space occurs. A change of chord will introduce new pitch classes at levels a-c; a change of key will introduce new pitch classes at level d. In what follows, I intend to show how this model can describe our experiences of neo-tonal music just as it does for tonal music.

Example 3.2 is a reproduction of Example 1.9, and it presents a passage from Messiaen's prelude "Les sons impalpables du rêve..." Some of the trichords in the passage are tertian triads and some are not. But all of the trichords are subsets of Messiaen's mode 3 and are members of the same step-class set. Mode 3 is also known as the enneatonic scale (the complement to the augmented triad). Above the example, I have extracted the scale and labeled the pitch classes according to their scale degree relative to a hypothetical tonic, A.

Example 3.2: "Les sons impalpables du rêve..."

Messiaen's mode 3:

M. 1, right hand:



The key signature suggests A major, and the first trichord confirms this, but the preponderance of non-diatonic trichords that follow it suggest viewing this familiar object in a different light. Because of this discrepancy—the commingling of diatonic triads and non-diatonic trichords—we cannot view the passage tonally. At the same time, because only nine pitch classes are present, we are not compelled to view the passage in terms of the 12-note chromatic collection either. This passage and others like it reside within a scale structure that is capable of generating hierarchical relationships not unlike those found in tonal music, and I intend to show how these hierarchical relationships can be modeled and how the perceptual distances between the chords can be calculated.

Example 3.3: Pitch-class space representation of the first chord

Level a:	A											
Level b:	A				C#			E				
Level c:	A	B♭		C	C#	D		E	F	F#		G#
Level d:	A	B♭	B	C	C#	D	D#	E	F	F#	G	G#

Level a:	0											
Level b:	0				4			7				
Level c:	0	1		3	4	5		7	8	9		
Level d:	0	1	2	3	4	5	6	7	8	9	10	11

Example 3.3 contains a pitch space for the first chord of the Messiaen example above, represented as both pitch classes and pitch-class integers. Level d is the basic level: the 12-pitch-class chromatic universe in this case.[25] The only constraints on level d are that the distance between the pitches be equal and that there be a modulus that replicates pitches at the octave. Level c is the scale level. The pitch classes at level d in this case correspond to the pitch classes of Messiaen's mode 3. Level b is the chord level, and it contains the pitch classes that correspond to the first chord in the right hand. Level a is the octave level. In Example 3.3, I have placed the pitch class A in level a to reflect the "A-major-ness" of this first trichord.

[25] In Example 3.3, I have re-lettered the levels because I have omitted level b in the diagram: level b is the fifth level and appears in Lerdahl's diagrams for reasons mentioned above. In most neo-tonal music, there is no particular interval that corresponds to the fifth. Level d in my examples corresponds to level e in Lerdhal's; level c (the scale level) corresponds to level d in Lerdahl's, and so on.

Example 3.4: Generalized model of example 3.3

Level a:	0												SC space
Level b:	0				3			5					
Level c’:	0	1		2	3	4		5	6	7		8	
Level c:	0	1		3	4	5		7	8	9		11	PC space
Level d:	0	1	2	3	4	5	6	7	8	9	10	11	

Example 3.4 presents a generalized version of Example 3.3 that is more conducive to our modular-space voice-leading approach. This model is a departure from Lerdahl’s in its generalization of levels b and a. I have inserted a level c’ between levels c and b: levels c and c’ are separated by a horizontal line. Above the line, the numbers represent step classes; below the line, numbers represent pitch classes. Level b now features the generalized set [0,3,5], an instance of step-class set $(025)^9$ (a T_n type in 9-space). All of the trichords in the Messiaen example are members of $(025)^9$.

Example 3.5: Pitch-class spaces for chords 1-10

Chord 1 (Θ):

level a	A								
level b	A					E			
level c	A			C#		E			
level d’	A	B♭	C	C#	D	E	F	F#	G#

Chord 2 (VIII):

level a									G#
level b					D				G#
level c			C		D				G#
level d’	A	B♭	C	C#	D	E	F	F#	G#

Chord 3 (VII):

level a								F#	
level b				C#				F#	
level c		B♭		C#				F#	
level d’	A	B♭	C	C#	D	E	F	F#	G#

Chords 4 & 6 (V):

level a						E			
level b		B♭				E			
level c		B♭				E			G#
level d’	A	B♭	C	C#	D	E	F	F#	G#

(Ex. 3.5 cont'd)

Chord 5 (VI):

level a							F		
level b			C				F		
level c	A		C				F		
level d'	A	B♭	C	C#	D	E	F	F#	G#

Chord 7 (II):

level a			C						
level b			C					F#	
level c			C			E		F#	
level d'	A	B♭	C	C#	D	E	F	F#	G#

Chord 8 (III):

level a				C#					
level b				C#					G#
level c				C#			F		G#
level d'	A	B♭	C	C#	D	E	F	F#	G#

Chord 9 (I):

level a		B♭							
level b		B♭					F		
level c		B♭			D		F		
level d'	A	B♭	C	C#	D	E	F	F#	G#

Chord 13 (IV):

level a					D				
level b	A				D				
level c	A				D			F#	
level d'	A	B♭	C	C#	D	E	F	F#	G#

Example 3.5 presents pitch-class spaces for all nine of the enneatonic trichords used by Messiaen. All nine trichords appear in the first measure of the piece, with some repetitions. In the diagrams, I have omitted levels e and d since they are held constant throughout. Level d', which appears at the bottom of each model, provides a sufficient point of reference. The chords are labeled 1 through 13, and each is accompanied by a Roman numeral in parentheses.[26] A previous analysis of the piece (see chapter 1) illustrated that voice and line are one and the same in this excerpt. As such, I have elected to privilege the middle note of each trichord, viewing the passage as a series of parallel trichords all in second inversion.[27] Roman numerals are applied just as they would be in

[26] With apologies to both ancient civilizations, I have adopted the capital Greek letter theta to serve as Roman numeral zero.

[27] I am using the term "second inversion" casually here. The first trichord, an A-major triad, appears in second inversion. Since the voices and mappings are one and the same in this passage (i.e., the passage

tonal music: the triad built on enneatonic scale degree 0 is Θ, the triad built on scale degree 1 is I, and so on. The Roman numerals are not intended to designate functionality or quality—they are simply for ease of reference in the following discussion.

According to Lerdahl, any pitch space is constrained by a number of well-formedness conditions and preference rules, and these rules are listed in Example 3.6. A space must consist of a closed group of pitch classes with a modulus that replicates the pitches at the octave. Basic spaces are tonal-hierarchical systems that embody organizational principles that extend beyond individual works. This is in contrast to an event hierarchy, which is derivative from an individual work. Schenkerian analyses and prolongational reductions from *Generative Theory of Tonal Music* belong to this second category. These tonal-hierarchical properties are acquired and accumulated through listening experiences.

---

features planing in mode 3), I am taking the middle note of each triad as the "root" for purposes of assigning Roman numerals.

Example 3.6: Rules governing basic spaces

Well-formedness conditions: A basic space must:

1. Consist of a closed group of PCs, with a single modulus operative at all levels.
2. Be hierarchically organized such that:
    a. At the lowest level, intervals between adjacent PCs are equal.
    b. Every PC above the lowest level is also a PC at all lower levels.
    c. A PC that is relatively stable at level L is also a PC at L+1.
    d. L+1 has fewer PCs than L.

Preference conditions: A basic space preferably:

1. Correlates height of level in the space with the degree of sensory consonance of adjacent intervals within a level.
2. Has almost half the PCs at L+1 as it does at L.
3. Expresses maximal evenness in its distribution of PCs at a given level (alternatively, has coherent steps at a given level).
4. Expresses a unique distribution of directed intervals.
5. Has two step sizes at the scale level.
6. Has two intervallic generators at the scale level, such that the starting point is reached again without intervening PC duplication.
7. Has steps no larger than ic2 between adjacent PCs at the scale level.

(Lerdahl 2001, 272-273)

The spaces constructed in Example 3.5 are well-formed according to Lerdahl's rules, but they violate many of the preference rules. Level c violates the almost-half rule: three is less than half of nine. The step classes of levels c and b are not maximally evenly distributed, nor are the steps coherent. There are two step sizes at the scale level (level d): interval class 1 and interval class 2. The scale does not have two intervallic generators at level d; however, generic interval class (gic) 4 and 5 will generate the generic nine-note collection found in level d' without duplicating step classes:

gic 5:	0	5	1	6	2	7	3	8	4
gic 4:	0	4	8	3	7	2	6	1	5

There are no steps greater than interval class 2 at the scale level. We could place generic interval class 4 or 5 in level b of our basic space, but doing so at this point would not drastically affect the outcome of the calculations.

A few points of interest arise concerning the pitch space as it appears above. The maximally even three-note collection in level c with respect to a generalized level d would be $(036)^9$ (which is inversionally symmetrical in 9-space), and $(025)^9$ deviates from this by one step: the two trichords are $P_{1,0}$-related. Here we see a case of the scale level being ME and the trichords built above it deviating minimally from ME. As discussed in the previous chapter, such a construction facilitates pattern-matching and position-finding. $(04)^9$ would divide level b maximally evenly with reference to level d'. A level b that is divided $(06)^9$ deviates from maximally even by two steps ($P_{2,0}$) with respect to level c. Sets in which the cardinality of the emergent set differs by only one from the background set are considered trivially ME: this is the case in levels c-b and levels b-a. Since level b contains two pcs and level c contains only three pcs, the two pcs of level b will naturally be ME. Any level that contains only one pc will also be trivially ME. Constructing a pitch space out of a combination of any of these maximally even divisions would result in a space that violates well-formedness rules 2b and c above. The deviation of level b is not a big issue, however, as Lerdahl's tonal pitch spaces feature a level b that is not maximally even, namely, (07). It deviates from maximal evenness by one semitone.

*   *   *

Lerdahl's theory also provides us with a means of measuring the relative perceptual distance between the chords. Lerdahl's chord distance rule is as follows:

> CHORD DISTANCE RULE (full version) $\delta(x \rightarrow y) = i + j + k$, where $\delta(x \rightarrow y)$ = the distance between chord x and chord y; i = the number of applications of the regional circle-of-fifths rule needed to shift the diatonic collection that supports x into the diatonic collection that supports y; j = the number of applications of the chordal circle-of-fifths rule needed to shift x into y; and k = the number of distinctive pcs in the basic space of y compared to the basic space of x (Lerdahl 2001, 60).

The lower the value of $\delta(x \rightarrow y)$, the "closer" the chords appear to be; the higher the value of $\delta(x \rightarrow y)$, the "further apart" the two chords seem. A C major tonic chord (I) is going to sound much closer to an A minor chord (vi) (because of the two retained pitches) than to, say, a D major chord (V/V), which has the same quality as the C major triad but no common pitches. Here, chord distance is different from voice-leading or transpositional distance. Transpositional distance is a discrete measurement: C is two semitones away from D. The numerical values yielded by the calculations should be read as magnitudes: if the distance from chord 1 to chord 2 is 4, the chords are pretty similar (close); if the distance is 13, the chords are quite different (distant). The chord distance rule as stated above works well for tonal music. Lerdahl presents amended versions of the rule for octatonic and hexatonic tonal music, both of which stem from the cyclic nature of these two scales (Lerdahl 2001, ch. 6). The octatonic scale, for instance, comprises two chains of triads whose roots lay a minor third apart: C-E♭-F#-A and Cm-E♭m-F#m-Am. A C major triad can become an E♭ major triad by moving two octatonic steps to the right; a C major triad can become a C minor triad by exchanging the third, E, with its immediate level d neighbor, E♭. Since atonal music is typically considered non-hierarchical, Lerdahl proposes a flat-space distance rule in which the only operative space is level e, the chromatic collection (Lerdahl 2001, ch. 8).

It seems to me that the distinction between chromatic and atonal need not be so black and white. Certain music, such as that by Messiaen, Debussy, and a number of other composers of the early twentieth century seems to have vestiges of tonality buried underneath the apparently non-tonal surface. Messiaen's prelude is a case in point. That Messiaen included a key signature and begins the piece on what would be the tonic triad in that key support such an assertion. Messiaen even claims that the piece is in A major (Messiaen 1944, 55). (To take another example, the Musgrave piece analyzed in Chapter 2 is, according to the composer, in A minor.) The chords in the left hand in the opening measures are vaguely rooted in A major, and the piece ends with a very strong A octave in the lowest register of the piano. In the section that follows, I will show how Lerdahl's pitch spaces can be made to accommodate such transitional music.

* * *

Example 3.7 presents an enneatonic chordal pitch space as found in "Les sons…". The Roman numerals represent the chords as listed above in Example 3.5. The horizontal rows present the chords arranged according to the gic 5 cycle; the columns present the chords in ascending stepwise order. The legend to the right of the example spells out the chords that correspond to the Roman numerals.

Example 3.7: Mode 3 chordal pitch space

VI		II		VII		III
	Θ		V		I	
VII		III		VIII		IV
	I		VI		II	
VIII		IV		Θ		V
	II		VII		III	
Θ		V		I		VI
	III		VIII		IV	
I		VI		II		VII
	IV		Θ		V	
II		VII		III		VIII
	V		I		VI	
III		VIII		IV		Θ
	VI		II		VII	
IV		Θ		V		I
	VII		III		VIII	
V		I		VI		II
	VIII		IV		Θ	

Θ = (A,C#,E)
I = (Bb,D,F)
II = (C,E,F#)
III = (C#,F,G#)
IV = (D,F#A)
V = (E, G#, Bb)
VI = (F,A,C)
VII = (F#,Bb,C#)
VIII = (G#,C,D)

We can trace the chord progression of the first measure of the Messiaen prelude through the pitch space in Example 3.7. We can also correlate the voice leading with the distances between the chords. In order to examine the distances between the chords in Messiaen's prelude, we must amend the chord distance rule. The regional circle-of-fifths rule will not concern us here: I will take up this rule in the next chapter. For the purposes of our calculations, the value of *i* will equal 0. The chordal circle-of-fifths rule must be changed to correspond to our intervallic generator, generic interval class 5. Beginning with the generalized (0,3,5), the "circle-of-fifths" rule generates the following chords in order, as given in Example 3.8. In the example, the step-class representation of the trichord is given in the left-most column. The second column includes the pitch classes found in each trichord. The third column presents the Roman numeral of the chord. The right-most column contains the chord number, which corresponds to Example 3.5 above.

Example 3.8: Trichords generated by generic interval class 5

Step class	Pitch class	Roman #	Chord #
[0,3,5]	[A,C#,E]	Θ	1
[5,8,1]	[E,G#,B♭]	V	4,6
[1,4,6]	[Bb,D,F#]	I	9
[6,0,2]	[F,A,C]	VI	5
[2,5,7]	[C,E,F#]	II	7
[7,1,3]	[F#,B♭,C#]	VII	3
[3,6,8]	[C#,F,G#]	III	8
[8,2,4]	[G#,C,D]	VIII	2
[4,7,0]	[D,F#,A]	IV	13

The second chord results from (0+5, 3+5, 5+5) all taken mod 9. The same process generates the remaining chords. Beginning with [0,3,5], then, one application of the "circle-of-fifths" rule produces [5,8,1]. The perceived distance between [0,3,5] and [5,8,1] then is 0 + 1 + 6 = 7.

Example 3.9 calculates the chord distances for each pair of adjacent chords in the Messiaen prelude using our modified chord distance rule.

Example 3.9: Chord distances in Messiaen's "Les sons…"

level a									G#
level b					D				G#
level c			C		D				G#
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(1→2) = 0 + 7 + 6 = 13

level a								F#	
level b				C#				F#	
level c		B♭		C#				F#	
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(2→3) = 0 + 7 + 6 = 13

level a						E			
level b		B♭				E			
level c		B♭				E			G#
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(3→4) = 0 + 5 + 5 = 10

level a							F		
level b			C				F		
level c	A		C				F		
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(4→5) = 0 + 2 + 6 = 8

level a						E			
level b		B♭				E			
level c		B♭				E			G#
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(5→6) = 0 + 7 + 6 = 13

level a			C						
level b			C					F#	
level c			C			E		F#	
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(6→7) = 0 + 2 + 6 = 8

level a				C#					
level b				C#					G#
level c				C#			F		G#
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(7→8) = 0 + 2 + 6 = 8

level a		B♭							
level b		B♭					F		
level c		B♭			D		F		
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(8→9) = 0 + 5 + 5 = 10

level a			C						
level b			C					F#	
level c			C			E		F#	
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(9→10) = 0 + 2 + 6 = 8

level a				C#					
level b				C#					G#
level c				C#			F		G#
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(10→11) = 0 + 2 + 6 = 8

level a						E			
level b		B♭				E			
level c		B♭				E			G#
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(11→12) = 0 + 4 + 5 = 9

level a					D				
level b	A				D				
level c	A				D			F#	
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(12→13) = 0 + 7 + 6 = 13

level a							F		
level b			C				F		
level c	A		C				F		
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(13→14) = 0 + 4 + 5 = 9

level a								F#	
level b				C#				F#	
level c		B♭		C#				F#	
level d'	A	B♭	C	C#	D	E	F	F#	G#

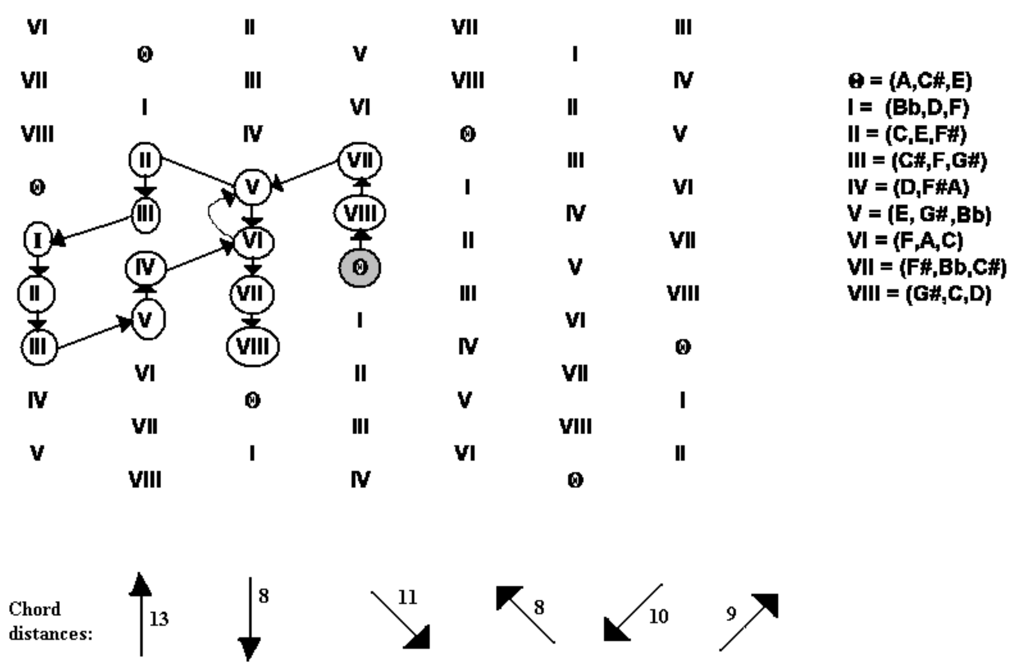
δ(14→15) = 0 + 2 + 6 = 8

level a								F#	
level b					D			F#	
level c			C		D			F#	
level d'	A	B♭	C	C#	D	E	F	F#	G#

δ(15→16) = 0 + 2 + 6 = 8

All of the possible perceptual distances between any two chords in Messiaen's mode 3 can be traced through the map in Example 3.7. Example 3.10 reproduces Example 3.7 and traces the chord progression from the Messiaen example through the space. I have added a second legend to this example that identifies the chord distance values as they occur in the space. Reading from top to bottom, the distance between the chords is 8; from bottom to top, 13. From right to left, the distance is 5; from left to right, the distance is 12. The diagonals read as follows: NE→SW is 10; NW→SE is 11; SE→NW is 8; and SW→NE is 9. The presentation of the space is necessarily skewed: it would be impossible to draw the space so that the difference between reading from left to right and right to left were apparent.

Example 3.10: A progression through enneatonic pitch space



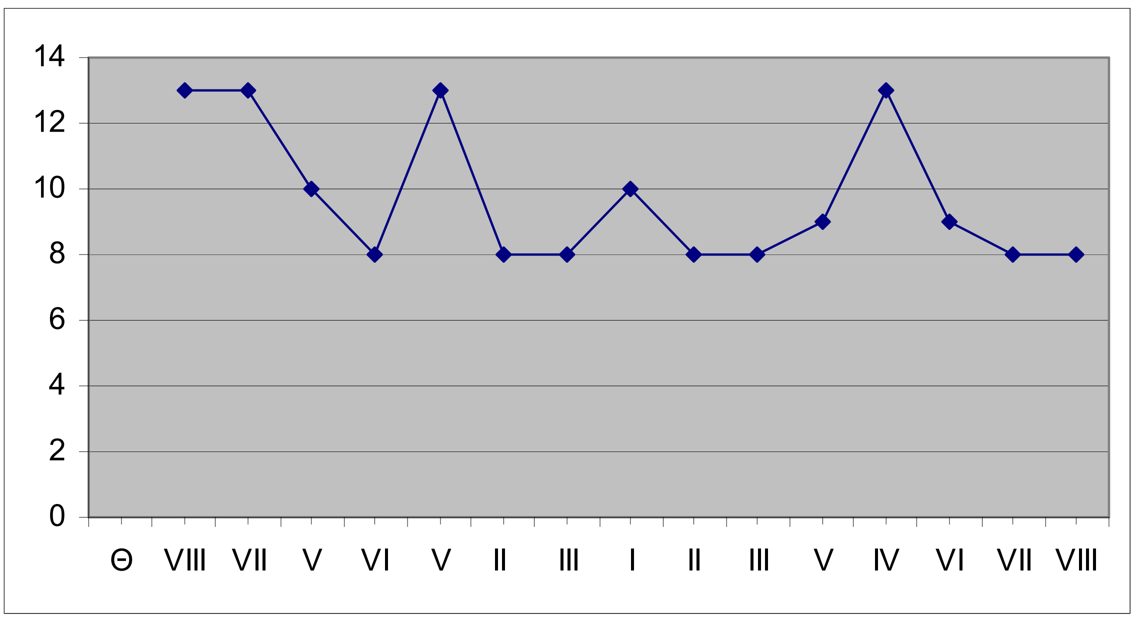
In the diagram, I have traced the chord progression through measure 1 of the Messiaen excerpt so that the voice leadings and perceptual distances correspond. Example 3.11 reproduces Example 3.2 with the addition of Roman numerals.

Example 3.11: Harmonic progression in m. 1 of "Les sons…"

In the enneatonic pitch space, the V chord that appears on the last sixteenth note of beat one is embellished by a neighboring VI chord. I say that it is embellished because it is the only chord in the progression that is returned to in Example 3.10. This embellishment strengthens the V chord's presence in the progression, lending it structural weight. The I chord in the middle of the measure is also particularly salient: it is the lowest chord in the progression and occurs in a relatively metrically strong place in the measure. If we extract the first chord of the progression (Θ), the V chord and the I chord, we can talk about an underlying structural progression of Θ-V-I-Θ. I should mention that the Θ chord that appears on the downbeat of m. 2 is not the same Θ chord that began the piece. This Θ chord appears directly below the VIII chord that concludes the progression in Example 3.10. That measure two does not begin on the same "tonic" provides a nice way of hearing the passage: the Θ chord sounds different now that other chords have contextualized it.

Example 3.12 plots the chord distance values in the Messiaen as calculated in Example 3.9. The *x*-axis contains the chords in the order in which they appear in the first measure of the Messiaen. The *y*-axis contains the perceptual distances between the chords. For the most part, the chords move either up or down by step, and this is evidenced by the preponderance of 8s (up a step) and 13s (down a step). A distance value

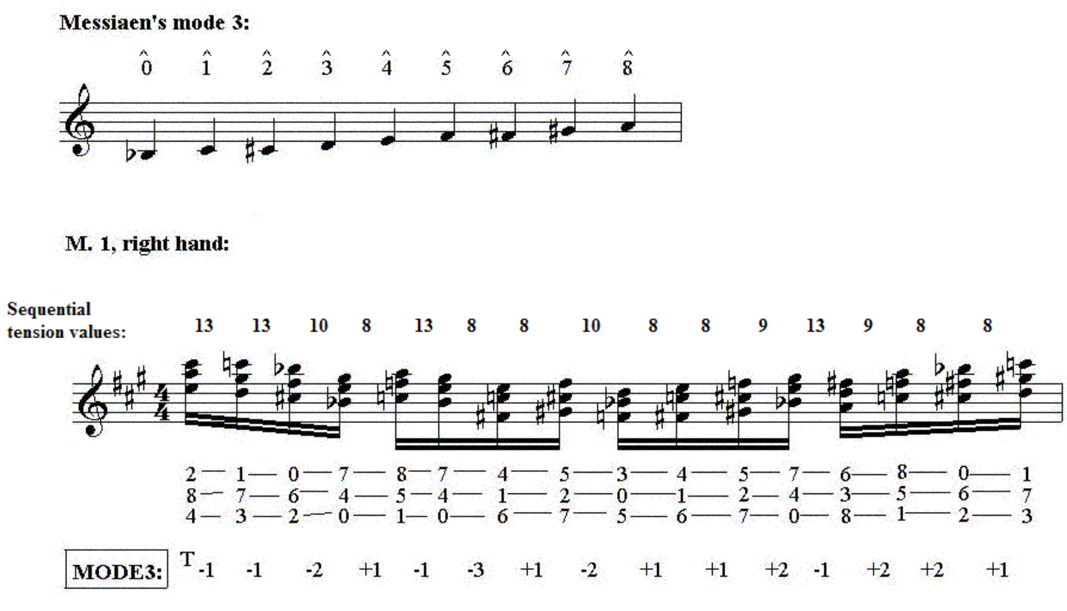
Example 3.12: Chord distance values in Messiaen's "Les sons…"



of 8 in the context of mode 3 is relatively low: 6 is the lowest possible value for two non-identical chords. On the other hand, 13 represents the maximum possible distance between two chords. For purposes of comparison, in any major key, the dominant and subdominant are closest to the tonic (that is, the least amount of sequential tension exists between the tonic and IV or V), each having a chord distance value of 5. II and vii exhibit the highest degree of sequential tension relative to the tonic, each with a value of 8. It is worth noting that, contrary to our intuition, the dominant is actually as close as it can be to the tonic in terms of perceptual distance; the chords that lay only a step away—ii and vii—create the most sequential tension relative to the tonic.

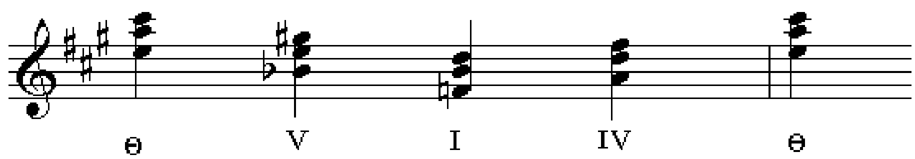
Having determined chord distance values for the Messiaen, I turn now to the interaction between this and the voice leading of the passage. Example 3.13 reproduces Example 1.9, which diagrams the voice leading of the passage through enneatonic space. The voice leading values in enneatonic space appear below the staff, and the chord distance values appear above the staff. From this example, we can see that voice-leading distance and perceptual distance are not necessarily equivalent. Perceptual distance is governed by the number of common tones shared by two chords regardless of the transformations needed to produce those pitches. It is significant to note that none of the chords in the Messiaen example share more than one common tone with any other chord. Direction is important in determining perceptual distance. From a voice-leading point of view, we can say that +8 is the same as -1 in a mod 9 system. But in terms of chord distance, ascending by step does not necessarily yield the same value as descending by step. According to Lerdahl, nonprototypical elements are judged to be closer to their prototypes, but not the other way around (2001, 78-9). In a tonal context, the leading tone will sound closer to the tonic than the tonic will to the leading tone, largely due to harmonic context. Stepwise voice leading will usually result in the greatest perceptual distance between chords, since chords that are one scale degree apart are not likely to have many pitch classes in common. Chords that are closely related from a voice-leading perspective are not necessarily perceptually proximate.

Example 3.13: Voice leading in m. 1 of Messiaen's "Les sons…"



Lerdahl also uses the chord distance rule as a measure of sequential tension: the tension between chord *x* and chord *y* is the value of $\delta(x\rightarrow y)$. That is, sequential tension is equal to the perceptual distance between chords. But sequential tension does not represent hierarchical tension. Hierarchical tension is the perceived distance between a chord and the chord that governs a passage. In order to determine hierarchical tension, we must take steps towards a time-span reduction of the passage in question. In the absence of the stability conditions of tonal music (it is impossible to determine conclusively a "tonic" chord in the Messiaen because of the lack of position-finding rare intervals in the enneatonic scale), we must for the moment rely on salience criteria to determine the head of the time span. In the case of the Messiaen, the chord that governs the span is the first chord of the measure. As per the preference rules set forth in Lerdahl (1989, 73-74) the first chord of the measure is privileged because it occurs in a strong metrical position, it is registrally prominent, and it appears next to a large grouping boundary (namely, the beginning of the piece). The V and I chords are also important for the reasons detailed above. I have also included the IV chord that occurs on beat four: it is in a relatively metrically strong position, and it is approached by leap (chord 11→chord 12 = T_2; chord 12→chord 13 = T_8; see Example 3.13 for voice leading). We can use the Θ-V-I-IV-Θ progression to create a time-span reduction of the piece, which appears in Example 3.14.

Example 3.14: Time-span reduction of measure one of "Les sons…"



This time-span reduction now serves as the starting point for a prolongational analysis.

Lerdahl gives three types of chord-to-chord motion: strong prolongation, weak prolongation, and progression (Lerdahl 2001, 14-15).[28] In strong prolongation, all of the notes in chord one appear in chord two. In weak prolongations, some of the pitches of chord one appear in chord two. In a progression, all of the pcs in chord two differ from those in chord one. Example 3.15 is a prolongational reduction of measure one of the Messiaen.

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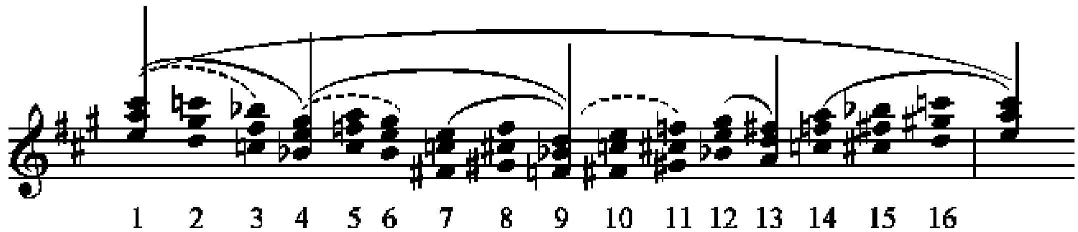
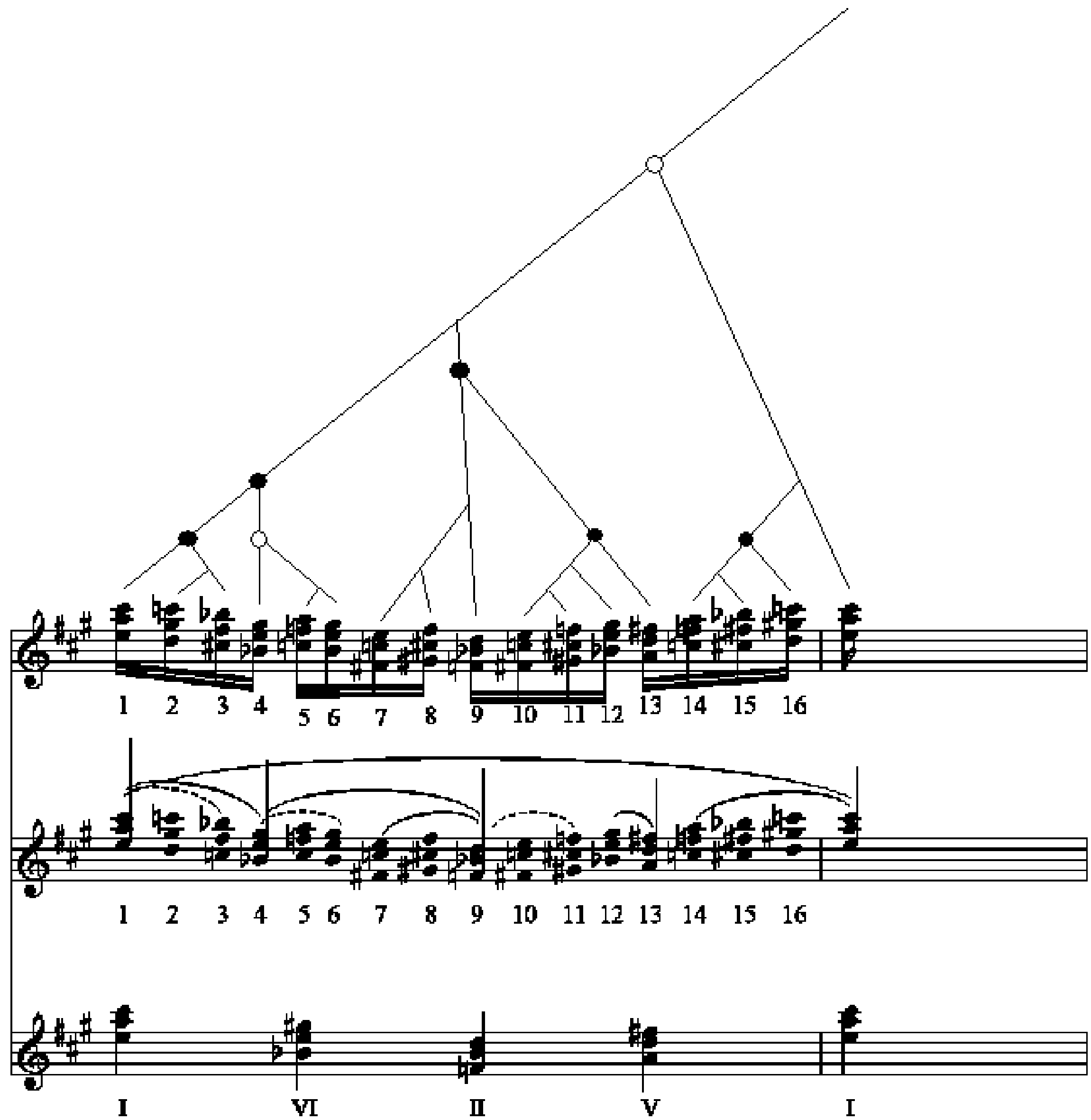
[28] see also Lerdahl 1989.

Example 3.15: Prolongational analysis of "Les sons…," m.1



In Example 3.15, all of the structural chords from the time-span reduction are stemmed: they have the greatest structural weight for the reasons detailed above. All of the structural chords share one pc in common, resulting in a large-scale weak prolongation. Chord 1 to chord 3 is a weak prolongation because the C# is common to both chords. A dotted slur shows the embellishment of chords 4 and 6 by chord 5. Chords 7 and 8 embellish chord 9, highlighting it by leaping down to it (chord 8→chord 9 = T_7) after a step up (chord 7→chord 8 = T_1). Chord 13 is similarly embellished.

Example 3.16: Prolongational analysis of "Les sons…," m. 1



We can combine the time-span and prolongational reductions into a tree diagram. The tree diagram will provide us with the means for calculating hierarchical tension, and it appears in Example 3.16. In the tree, right-branches indicate increasing tension; left branches indicate increasing relaxation. Chord 1 is the head of the span. The black dot at the junction of this main branch and the right branch that leads to V shows that this is a weak prolongation: the pitch E is retained from Θ to V and the other two pitches deviate minimally. The open circle indicates a strong prolongation: the two chords are identical. Junctions without circles indicate progressions: no pitch classes are retained.

We are now in a position to calculate hierarchical tension values. Lerdahl's hierarchical tension rule is as follows:

> HIERARCHICAL TENSION RULE (long version): $T_{loc}(\mathbf{y}) = T_{diss}(\mathbf{y}) + \delta(x_{dom} \rightarrow \mathbf{y})$; and $T_{glob}(\mathbf{y}) = T_{loc}(\mathbf{y}) + T_{inh}(x_{dom})$. (Lerdahl 2001, 151)

In the formulas, **y** is the target chord and x_{dom} is the chord that directly dominates **y** in the prolongational tree. The first formula calculates the local tension associated with **y**: $T_{diss}(\mathbf{y})$ is the surface tension associated with **y** and is dependent upon a number of features—scale degree in the top voice, inversion of the chord, and non-harmonic tones. $\delta(x_{dom} \rightarrow \mathbf{y})$ is the perceived distance between **y** and the chord that directly dominates it in the prolongational tree. In the second formula, $T_{glob}(\mathbf{y})$ is the global tension associated with **y**, and it is calculated by adding the value of the first formula to $T_{inh}(x_{dom})$, the tension value that **y** inherits from the chord that directly governs **y** in the tree.

Example 3.17 shows the hierarchical tension values associated with the Θ-V-I-IV-Θ progression. I have set the value of $T_{diss}(\mathbf{y})$ to 0 in all cases because all of the chords are in the same inversion (whatever that may be) so any influence that scale degree in the soprano or bass may have had is nullified. Non-harmonic tones are also not an issue here. The value of *i* is always going to be zero, as there is no regional movement in the passage.

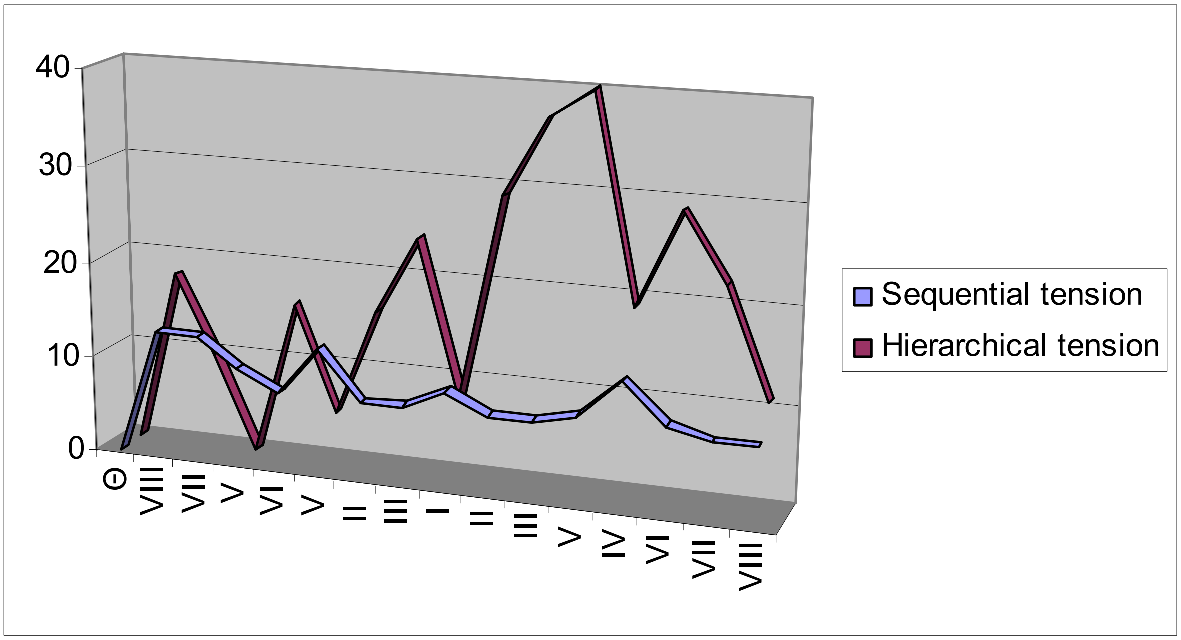
Example 3.17: Hierarchical tension values in m. 1 of "Les sons…"

	pitch-space distance *i*	*j*	*k*	*local total*	*inherited value*	*global total*
T_{glob}("I"→1)	0	0	0	0	0	0
T_{glob}(1→3)	0	5	5	10	0	10
T_{glob}(1→6)	0	1	4	5	0	5
T_{glob}(1→9)	0	2	6	8	0	8
T_{glob}(3→2)	0	2	6	8	10	18
T_{glob}(4→5)	0	2	6	8	5	13
T_{glob}(6→4)	0	0	0	0	0	0
T_{glob}(7→8)	0	2	6	8	16	24
T_{glob}(9→7)	0	2	6	8	8	16
T_{glob}(9→13)	0	6	5	11	8	19
T_{glob}(10→11)	0	2	6	8	29	37
T_{glob}(10→12)	0	6	5	11	29	40
T_{glob}(10→13)	0	5	5	10	19	29
T_{glob}(14→15)	0	2	6	8	21	29
T_{glob}(14→16)	0	4	6	10	11	22
T_{glob}(14→1)	0	6	5	11	0	11

The global total figure gives us a relative measure of how far away from the "tonic" we are. This figure peaks around chords 10 and 11, about two-thirds of the way through the measure. The values diminish as we get closer to the beginning of the next measure, and another statement of the "tonic" trichord.

Example 3.18 provides us with a means of comparing sequential and hierarchical tension measurements. Sequential tension peaks when descending stepwise motion prevails. In the Messiaen, it spikes sharply upward in the beginning of the measure and levels off over the remainder of the passage. Hierarchical tension, on the other hand, peaks as we get further away from the tonic both temporally and in the domain of pitch-class space (see Example 3.12 above).

Example 3.18: Sequential vs. hierarchical tension in "Les sons…"



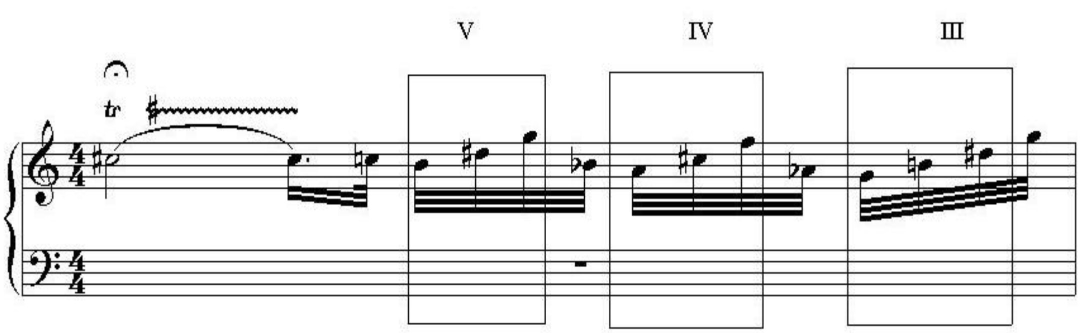
Our preliminary investigation into the Messiaen shows that it is possible to model tension, harmonic progression, and prolongation for neo-tonal music. The process requires quite a bit of theoretical heavy lifting, but the results demonstrate that vestiges of hierarchical tonal structures are present in this music. In the absence of the stability conditions of tonal music—a scale that facilitates pattern-matching and position finding, a generating interval, and so on—we can interpret the music hierarchically based on contextual features inherent to individual pieces of music as opposed to an overarching system like traditional tonality. It is, in fact, our ability to synthesize tonal-hierarchical and event-hierarchical systems that enables us to hear the Messiaen and other pieces like it as hierarchical. By superimposing an unfamiliar event hierarchy (an individual piece) onto our accumulated knowledge of the tonal system and its hierarchical properties we are able to adapt our hearing and understand a new hierarchical system in terms of that with which we are familiar.

* * *

Space invaders

In the remainder of the chapter, we will look at a passage from Debussy's piano piece *L'isle joyeuse*. The piece is interesting because Debussy sprinkles what appear to be non-harmonic tones into the overwhelmingly whole-tone context of the piece. We can use Lerdahl's melodic attraction formula to model the progression. Example 3.19 contains the first measure of the piece.

Example 3.19: Debussy, *L'isle joyeuse*, m. 1



I have chosen here to assign step-class integer 0 to C#, due to its perceptual salience: it is a long note in a strong metrical position, and it is the first note we hear. The remaining scale degrees are assigned as follows:

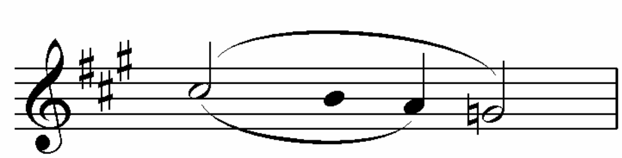
C#	D#	F	G	A	B
^	^	^	^	^	^
0	1	2	3	4	5

In the example, I have placed three trichords in boxes, and labeled them with Roman numerals corresponding to the scale degrees on which the chords are built. Chords V and III have the same pitch-class content, but the pitches are presented in a different registral order, and the G occurs in a different octave. Disregarding the chromatic passing tones (C, B♭, and A♭), the voice leading is a simple stepwise descent through the whole-tone scale. V becomes IV at T_5 and IV becomes III at T_5, manifest as T_{-1} in pitch space. The chords are as proximate as possible from a mod-6 voice-leading perspective.

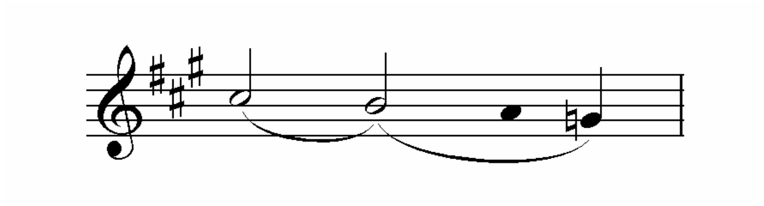
There are two ways that we can interpret this passage. Example 3.20 illustrates these two interpretations. In both cases, the reductions consist of the most perceptually salient pitches in the measure.

Example 3.20: Two reductions of *L'isle joyeuse*, m. 1

Ex. 3.20.1:



Ex. 3.20.2:



In Example 3.20.1, the C# is prolonged through a passing tone, B, as it moves down to A. The G at the end of the measure represents a change of harmony at the middleground level. In this example, we see what we might call in this context a prolongation of the "tonic" triad, C#-F-A, followed by a "dominant" harmony based on G. In Example 3.20.2, the "tonic" triad is established by the C# on the downbeat, and the "dominant" triad is prolonged as the B passes through A on its way to G. When I say tonic and dominant, I am not using these terms in their traditional sense, rather, I am using them to emphasize the opposition between two harmonic entities. Because of the nature of the whole-tone scale, we can speak of the tonic triad as containing three of the six pitch classes, and the dominant triad as containing the remaining three pitch classes. In this particular case, each trichord is a maximally even three-note subset of the whole-tone scale.

We have three whole-tone spaces with which we need to contend: a I/WT_1 space (the "tonic" chord), as well as III/WT_1 and V/WT_1 (the "dominant" chords). Example 3.21 diagrams these spaces.

Example 3.21: Whole-tone "tonic" and "dominant" pitch-class spaces

													anchoring strength
level a:	C#												4
level b:	C#				F				A				3
level c:	C#		D#		F		G		A		B		2
level d:	C#	D	D#	E	F	F#	G	G#	A	B♭	B	C	1

I/WT_1

													anchoring strength
level a:							G						4
level b:			D#				G				B		3
level c:	C#		D#		F		G		A		B		2
level d:	C#	D	D#	E	F	F#	G	G#	A	B♭	B	C	1

III/WT_1

													anchoring strength
level a:											B		4
level b:			D#				G				B		3
level c:	C#		D#		F		G		A		B		2
level d:	C#	D	D#	E	F	F#	G	G#	A	B♭	B	C	1

V/WT_1

I have omitted level b in the diagram and re-lettered the levels accordingly. Notice that this pitch-class space features first- and second-order maximal evenness: The whole-tone scale is ME with respect to the twelve-note chromatic, and the augmented triad is ME with respect to the whole-tone scale. As discussed in the previous chapter, such a structure hinders the perceptual tasks of pattern matching and position-finding.

In any pitch-class space, each pitch class has a certain anchoring strength that is relative to its prominence in the space. In tonal music, the tonic naturally has the strongest anchoring power, followed by elements of the tonic triad. Other scale degrees we perceive as more transient, and chromatic non-harmonic tones are the least stable.

Lerdahl assigns each level of a pitch-class space an anchoring strength: level d has an anchoring strength of 1, level c has an anchoring strength of 2, and so on. The values for our whole-tone space appear on the right-hand side of Example 3.21. The anchoring strength of a pitch class is equal to the highest level in which it appears.

We can use anchoring strength to calculate values for melodic attraction. Lerdahl's melodic attraction rule is as follows:

> MELODIC ATTRACTION: $\alpha(p_1 \rightarrow p_2) = {}^{s2}/_{s1} \times {}^{1}/n^2$, where p_1 and p_2 are pitches, with $p_1 \neq p_2$; $\alpha(p_1 \rightarrow p_2)$ = the attraction of p_1 to p_2; s_1 = the anchoring strength of p_1 and s_2 = the anchoring strength of p_2 in the current configuration of the basic space; and n = the number of semitone intervals between p_1 and p_2 (Lerdahl 2001, 163).

The higher the melodic attraction value, the stronger the perceived "pull" between the two pitches. Example 3.22 calculates two sets of melodic attraction values. Example 3.22.1 calculates the values per the analysis in Example 3.20.1 and Example 3.22.2 corresponds to the analysis in Example 3.20.2.

Example 3.22: Melodic tension values in m. 1 of *L'isle joyeuse*

Ex. 3.22.1:

α(C# → C) = ¼ x $1/1^2$	= 0.25
α(C → B) = ½ x $1/1^2$	= 0.5
α(B →D#) = 2/2 x $1/4^2$	= 0.0625
α(D# → G) = 2/2 x $1/4^2$	= 0.0625
α(G → B♭) = ½ x $1/3^2$	= 0.056
α(B♭ → A) = 3/1 x $1/1^2$	= 3
α(A → C#) = 4/3 x $1/4^2$	= 0.083
α(C# → F) = ¾ x $1/4^2$	= 0.0469
α(F → A♭) = 1/3 x $1/3^2$	= 0.037
α(A♭ → G) = 4/1 x $1/1^2$	= 4
α(G → B) = 3/4 x $1/4^2$	= 0.0469
α(B →D#) = 3/3 x $1/4^2$	= 0.0625
α(D# → G) = 4/3 x $1/4^2$	= 0.083

(Ex. 3.22 cont'd.)

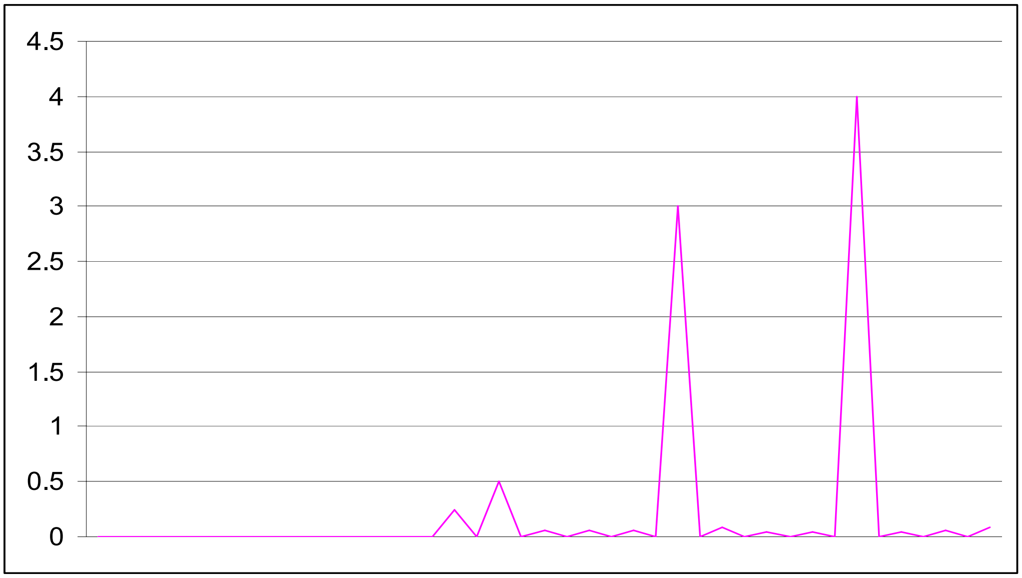
Ex. 3.22.2:

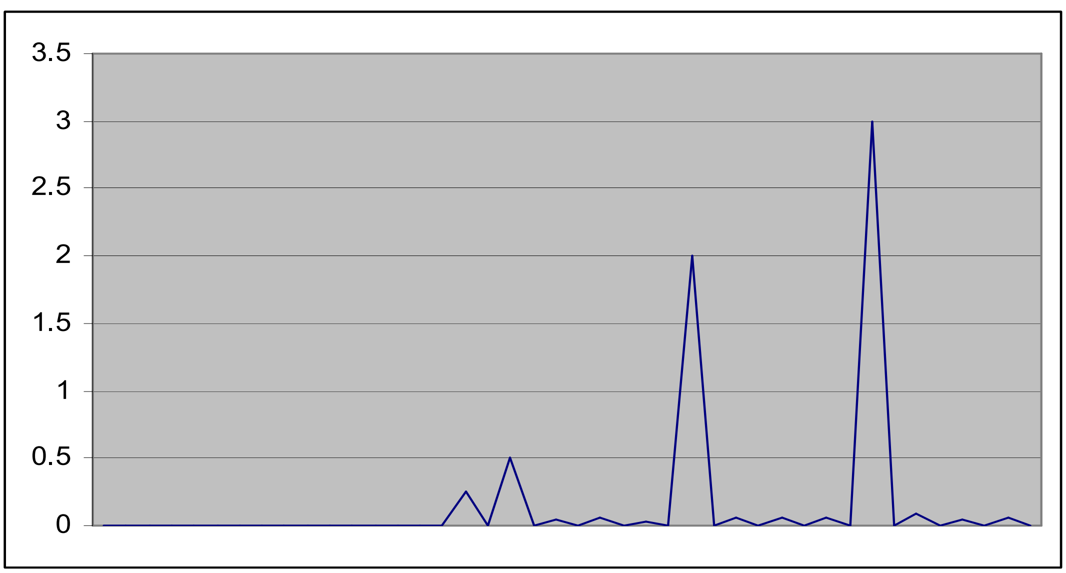
α(C# → C) = ¼ x $1/1^2$	= 0.25
α(C → B) = ½ x $1/1^2$	= 0.5
α(B →D#) = 3/4 x $1/4^2$	= 0.0469
α(D# → G) = 3/3 x $1/4^2$	= 0.0625
α(G → B♭) = 1/3 x $1/3^2$	= 0.037
α(B♭ → A) = 2/1 x $1/1^2$	= 2
α(A → C#) = 2/2 x $1/4^2$	= 0.0625
α(C# → F) = 2/2 x $1/4^2$	= 0.0625
α(F → A♭) = 1/2 x $1/3^2$	= 0.056
α(A♭ → G) = 3/1 x $1/1^2$	= 3
α(G → B) = 4/3 x $1/4^2$	= 0.083
α(B →D#) = 3/4 x $1/4^2$	= 0.0469
α(D# → G) = 3/3 x $1/4^2$	= 0.0625

The strongest attractions occur between the non-harmonic tones and the notes that occur on strong beats of the measure, with the strongest attractions occurring between the B♭ and the A and the A♭ and G. In the first analysis, the largest spike occurs at the change of harmony. The next largest spike occurs within the tonic prolongation moving from the passing "dominant" to the A, a member of the "tonic" triad. In the second analysis, the spikes occur at the same places, but are attenuated. The first spike, the smaller of the two, occurs at the change of harmony. The second occurs in the context of the "dominant" prolongation as the line passes through the A on its way to G.

Example 3.23 plots both sets of values from Example 3.22 and superimposes the graphs over the first measure. The top graph corresponds to Example 3.22.1; the lower graph corresponds to Example 3.22.2

Example 3.23: Melodic attraction in m. 1 of *L'isle joyeuse*





In cases where first- and second-order maximal evenness are present, position-finding tasks can be aided when non-harmonic tones are present. As we saw in Chapter 2, one of the two levels (c or d) in a pitch-class space should deviate from maximal evenness in order to facilitate pattern-matching tasks. In the previous section, I have shown that pattern matching can be facilitated by the presence of non-harmonic tones in cases where both levels are ME. Non-harmonic tones result in spikes in melodic attraction values, and these spikes correspond to motion to a structural pitch class.

* * *

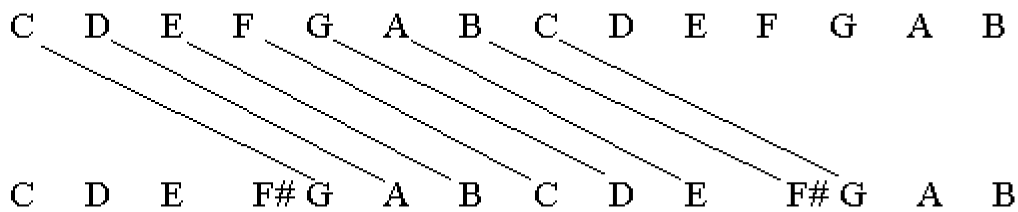
I have shown in this chapter how first- and second-order ME sets can combine to form hierarchical pitch-class spaces. These pitch-class spaces permit us to model perceptual distances between chords and melodic pitches. In the next chapter, we will examine the relationships between these pitch-class spaces and what happens when they occur successively or simultaneously in a piece of music.

Chapter 4 : Changing spaces

Two possible situations exist when one pitch-class space becomes another. The first pitch-class space could be of the same cardinality as the second, or the first pitch-class space could be of a different cardinality than the second pitch-class space. In the first case, each and every generalized step class of pitch-class space *A* can be mapped directly onto a unique generalized step class of the second space *B* in a one-to-one relationship. There are two possible kinds of mappings that obtain in this situation. The first is a transpositional mapping where *b*, an element of *B*, is equal to $a + n$. This type of mapping occurs most often in tonal music, where scale degree one—the tonic—of *A*, is mapped onto scale degree one in the new key, *B*. All of the elements of *A* are transposed by some constant, *n*. The equivalence here is one of function, but not pitch-class. In neo-tonal music, functional mapping will not concern us in most cases, since (as we have seen) it is relatively difficult to ascribe functions to elements of non-diatonic collections. The alternative is direct pitch-class mapping (not unlike the P-relation, discussed in Chapter 2). In direct pitch-class mapping, pitch classes that are common to the two spaces are held invariant and the relationship between elements that are elements of *A* but not of *B* can be described in several ways, which I will discuss momentarily.

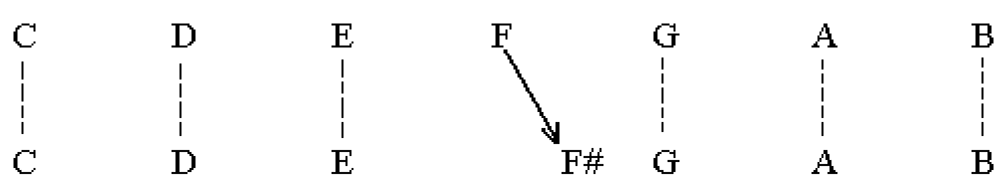
Let us look at an example. If we want to map the C-major scale onto the G-major scale, we can do this in two ways. We can transpose each of the elements in the C-major scale up by seven semitones (or down by five). This is an example of functional mapping that I mentioned above. In this approach, scale degree 1 in C major becomes scale degree 1 in G major (the pitch C becomes the pitch G), D becomes A, and so on. Each pitch in G major is seven semitones away from a pitch in C major. This approach is illustrated in Example 4.1, which is a reproduction of Example 2.7.

Example 4.1: Functional mapping between C major and G major



Another way we can relate these two collections is through the P relation. C major is $P_{1,0}$ related to G major because all of the pitches in C major appear in G major, with the exception of F, which moves by semitone to F#. Example 4.2 illustrates this approach. Dotted lines indicate pitch classes that remain invariant across the collection. F moves to F#, a semitone away.

Example 4.2: P-relating C major and G major



The question of when exactly does one pitch-class space become another is an important one, and I will return to it below.

Straus (2003) provides several methods of describing the relationship between sets that are not necessarily members of the same set class (or even of the same cardinality). Of the three measures he provides, we are going to be concerned chiefly with smoothness. Straus defines smoothness as the sum of the intervals traversed by each pc in collection *X* as it moves to its destination in *Y* (Straus 2003, 320). The smoothness of a voice leading can be presented in vector form.[29] Each vector has seven places, corresponding to the interval classes 0 through 6. The smoothness vector for Example 4.2 would be <6,1,0,0,0,0,0>, which indicates that, of the seven pitch classes present in the C major collection, six did not move at all (moved by zero semitones) to their position in the G major collection, and one pc moved by one semitone to arrive at its new position (F$\rightarrow$F#). The smoothest relationships are going to be those whose vector representation is most left-packed: the relationship in Example 4.2 is quite smooth, since the only positive entries in the vector appear in the 0 and 1 places.

Related to the notion of smoothness is the notion of displacement. According to Straus, displacement is the total number of semitones traversed by all voices that participate in the voice leading. Using the smoothness vector <*a,b,c,d,e,f,g*> , we can calculate displacement with the formula:

$$\text{total displacement} = (0*a) + (1*b) + (2*c) + (3*d) + (4*e) + (5*f) + (6*g)$$

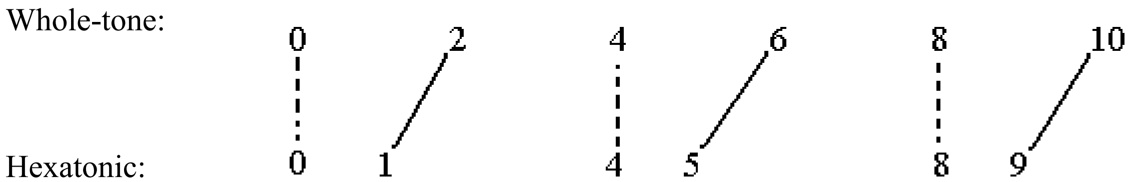
[29] Ariza 2002 calculates smoothness, uniformity, and balance vectors and can rank relationships between set classes accordingly.

In the case of Example 4.2, the total displacement is 1, because (0*6) + (1*1) = 1.

Even though the relationships between pitch-class spaces are not necessarily voice leadings, I am drawing on the transformational voice-leading literature to develop ways of relating pitch-class spaces. As I see it, studies of post-tonal voice leading are (and should be) chiefly concerned with the relationships between elements that inhabit level c (the chordal level) of Lerdahl's basic space. These relationships occur much closer to the musical surface and at a much more rapid rate than pitch-class space changes. On the other hand, relationships between pitch-class spaces occur at a deeper, more middleground level of structure: the scalar level of Lerdahl's basic space. So while pitch-class space changes are not identical to voice leadings, I am proceeding in this chapter by way of analogy *to* voice leading in my efforts to relate pitch-class spaces.

In the preceding examples, the original set and the destination set were both members of the same set class. We can employ a similar procedure if the original and destination sets are of the same cardinality but members of different set classes. If we wanted to map a whole-tone scale onto a hexatonic scale, the pitch classes 0, 4, and 8 would remain invariant between the two; pitch classes 2, 6, and 10 in the whole-tone scale would all move by semitone to pitch classes 1, 5, and 9 in the hexatonic scale, a P_3 relation. Example 4.3 illustrates the transformation. Dotted lines again indicate invariance, and solid lines indicate displacement by semitone.

Example 4.3: P-relating the whole-tone and hexatonic scales



Smoothness: <3,3,0,0,0,0,0>

Displacement = 3

Here, the smoothness vector indicates that, of the six pitch classes involved, three are held invariant and three move by semitone. In cases where both pitch-class spaces are of the same cardinality, we will hold common pitch classes invariant and move other pitch classes the shortest possible distance to the closest pc in the new space.

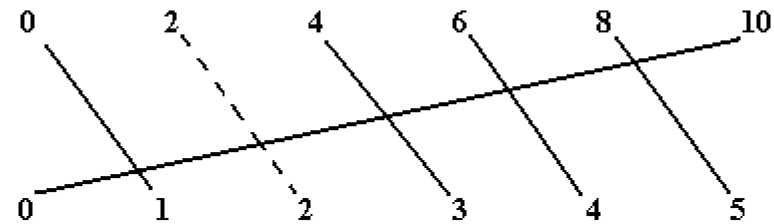
Example 4.4 shows four possible relationships between the whole-tone scale (the six-note maximally even collection) and set class 6-1 (the six-note minimally even collection). All of the relationships have a displacement value of 9. Example 4.4.1 has only one pitch class held invariant (pitch class 2); Examples 4.4.2 and 4.4.3 each have two pitch classes held invariant; and Example 4.4.4 has three pitch classes held invariant. Despite the fact that only one pitch class is held invariant, Example 4.4.1 presents the smoothest relationship between the two spaces. The smoothness vector in Example 4.4.1 is the most left-packed of the bunch, indicating that the pitch classes in this relationship do the least amount of work: more pitch classes move by smaller distances in this Example as opposed to Example 4.4.4, where a few pitch classes are called upon to move great distances. These examples illustrate that displacement alone is not always sufficient to measure distances between two pitch-class spaces.

Example 4.4: Relating 6-35 and 6-1

Ex. 4.4.1:

Whole-tone:

6-1:



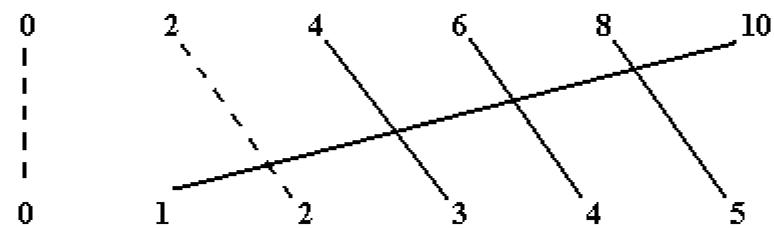
Smoothness: <1,2,2,1,0,0,0>

Displacement = 9

Ex. 4.4.2:

Whole-tone:

6-1:



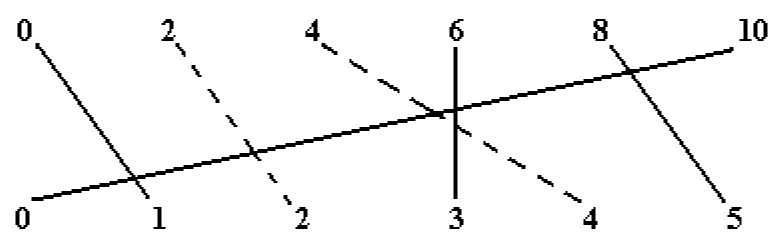
Smoothness: <2,1,1,2,0,0,0>

Displacement = 9

Ex. 4.4.3:

Whole-tone:

6-1:

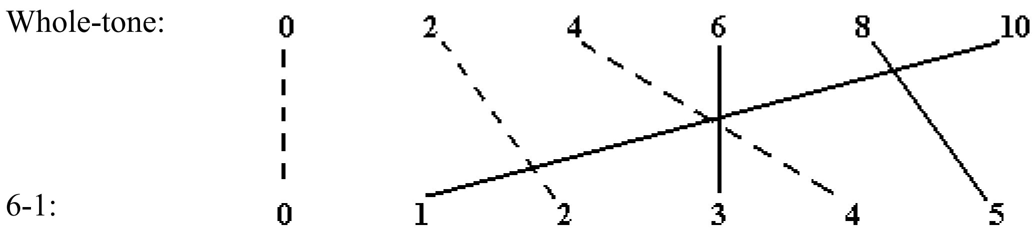


Smoothness: <2,1,1,2,0,0,0>

Displacement = 9

Ex. 4.4 cont'd.

Ex. 4.4.4:



Smoothness: <3,0,0,3,0,0,0>

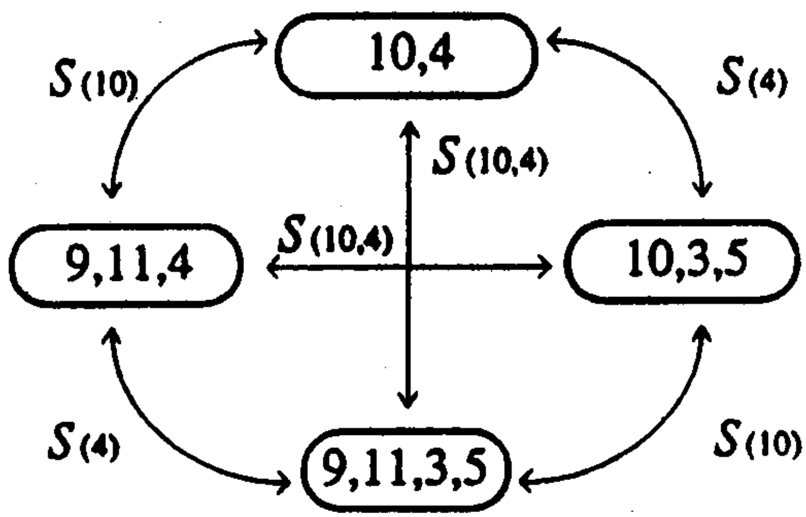
Displacement = 9

When the original set and the destination set are of different cardinalities, we must find a way of modeling the one-to-many or many-to-one mappings that will invariably arise. To do this, I am adopting Callender's (1998) split transformation. Callender defines the split transformation as follows:

> Let X and Y be pcsets such that $|X \setminus Y| = 1$ and $|Y \setminus X| = 2$. X and Y are $S_{(x)}$-related (written $X\,S_{(x)}\,Y$ or $Y\,S_{(x)}\,X$) if for $x \in X \setminus Y$ and every $y \in Y \setminus X$, $x - y \equiv \pm 1 \pmod{12}$. (If x is not specified, then we write $X\,S\,Y$ or $Y\,S\,X$) (Callender 1998, 224).

There must be one element that belongs to X that does not belong to Y, and there must be two elements of Y that do not belong to X. The unique element of X, x, splits to become the two elements of Y that are not present in X. Callender's Example 6, given here as Example 4.5, illustrates the operation. Moving clockwise around the circle, (10,4) $S_{(4)}$ = (10,3,5) (4 splits into 3 and 5); (10,3,5) $S_{(10)}$ = (9,11,3,5) (10 splits into 9 and 11); (9,11,3,5) $S_{(4)}$ = (9,11,4) (3 and 5 fuse into 4); and (9,11,4) $S_{(10)}$ = (10,4) (9 and 11 fuse into 10). Notice the operation is associative, as indicated by the arrows. The vertical line shows us that multiple simultaneous *S* transforms can occur: (10,4) $S_{(10,4)}$ yields (9,11,3,5) (also associative).

Example 4.5: The split transformation (Callender's Example 6, p. 225)



Callender's split transformations handily relate spaces of different cardinalities. Applying $S_{(2,8)}$ to the whole-tone scale, (02468T) yields the octatonic scale, (0134679T): 2 splits into 1 and 3; 8 splits into 7 and 9. As Callender does, I am restricting the possible split operations to those that produce the input pitch class ± 1; that is, 6 splits into 5 and 7; it cannot split into 5 and 6 or 6 and 7. Example 4.6 illustrates the process in mod-12 space. The smoothness vector and displacement values are given below the example.

Example 4.6: Relating two pitch-class spaces of different cardinalities

Whole-tone:	0	2		4	6	8		T
$S_{(2,8)}$	|	/	\	|	|	/	\	|
Octatonic:	0	1	3	4	6	7	9	T

smoothness: <4,4,0,0,0,0,0>

displacement = 4

In some cases we will need to employ a combination of the P-relation and split transformations. Consider the relationship between a pentatonic scale, (02479) and a whole tone scale, (02468T). The whole-tone scale has a cardinality that is one greater than the pentatonic scale. Three pitch classes are held invariant between the two scales, two pitch classes differ by one semitone, and one pitch class is added. Example 4.7 shows how the two spaces can be related using P- and S- transformations.[30] The smoothness vector and total displacement are given below the example. It is interesting to note that it takes exactly the same amount of work (read: displacement) to turn the pentatonic scale into the whole-tone scale as it does to turn the whole-tone scale into the hexatonic scale (see Example 4.3 above). The two operations have the same smoothness vector as well.

Example 4.7: Mapping the pentatonic onto the whole-tone

Pentatonic:	0	2	4	7		9
$S_{(7)}$, $P_{1,0}$	|	|	|	/	\	|
Whole-tone:	0	2	4	6	8	10

smoothness: <3,3,0,0,0,0,0>

displacement = 3

[30] In the example, pc 9 in the pentatonic collection could just as easily split onto pcs 8 and 10 in the whole-tone collection and pc 7 would be P-related to pc 6.

* * *

Adapting Lerdahl's regional circle-of-fifths rule permits us to measure perceived distances between different pitch-class spaces—either successive or simultaneous—and to measure the tension between chords in different pitch-class spaces (for example, a trichord in whole-tone space as compared to a different trichord in octatonic space).

In the previous chapter, we encountered Lerdahl's chord distance rule, which I reproduce here for convenience:

> CHORD DISTANCE RULE (full version) $\delta(x \rightarrow y) = i + j + k$, where $\delta(x \rightarrow y)$ = the distance between chord x and chord y; i = the number of applications of the regional circle-of-fifths rule needed to shift the diatonic collection that supports x into the diatonic collection that supports y; j = the number of applications of the chordal circle-of-fifths rule needed to shift x into y; and k = the number of distinctive pcs in the basic space of y compared to the basic space of x (Lerdahl 2001, 60).

Until now, the value of *i* has always been zero: transformations have taken place within a single pitch-class space. Now that we have opened the door to changing pitch-class spaces, we must make room for this in our rule.

Lerdahl defines the regional circle-of-fifths rule as follows:

> REGIONAL CIRCLE-OF-FIFTHS RULE: Move the pcs at level d of the basic space seven steps to the right (mod 12) on level e or seven steps to the left (Lerdahl 2001, 59).

The value of *i* then will be equal to the number of new pitch classes at level d after the rule is applied. Lerdahl gives the example of mapping a C major scale into a G major scale. This operation requires one application of the regional circle-of-fifths rule, and generates one new pitch class, F#, so the value of *i* in this case is 1.

There is some overlap here between the circle-of-fifths rule and Straus's displacement. As shown above in Examples 4.1 and 4.2, there are two ways the relationship between C major and G major can be modeled (actually, there are 5,040 ways the two sets can be related). T_7 is the most uniform (most transposition-like), but it is hardly the smoothest: the displacement value is 49 (7 pitch classes move by 7 semitones; it is actually the least smooth voice leading between the two sets). One application of Lerdahl's regional circle-of-fifths rule thus creates a displacement value of 49, but, by definition, the value of *i* is only 1. The smoothest voice leading, where all of the pitch classes are retained and F is replaced with F# (modeled in Example 4.2), has a displacement value of 1: it is the second most uniform voice leading between the two sets.[31] If we take the displacement value mod 12, we get a value that is equal to the number of applications of the regional circle-of-fifths rule: 49 mod 12 = 1.[32]

It appears to be mathematical coincidence that the number of applications of Lerdahl's rule is equal to the number of new pitch classes introduced at level d, which is also equal to the displacement of the smoothest voice leading between the two collections. All of these measure different things: the first is simply the number of times something happens; the second is how many things are in *Y* that are not in *X*; the third is a

[31] The other terms of Lerdahl's (*j* and *k*) account for the changes in hierarchy as the spaces change.

[32] See Straus 2003, fn. 39 for a comparison of Straus's and Cohn's approach to this problem. Straus does not use modular arithmetic to calculate total displacement; Cohn does. My approach embraces both methods (since both are equivalent measures) depending on the situation.

measure of the distance traveled multiplied by the number of things traveling that distance.

Since we are dealing with pitch-class spaces that are members of different set classes and even different cardinalities, all of which have different (or no) intervallic generators such as the circle of fifths, a rule that instructs us to "move the pitches at level d seven steps to the right or seven steps to the left" is of little use to us. But since the values generated by that rule correspond to the displacement value in Straus's smoothness measure, we can substitute "number of applications of the regional circle-of-fifths rule" with "displacement value of the smoothest voice leading" in a revised regional circle-of-fifths rule (I have renamed it the regional distance rule to avoid confusion and because the idea of the circle of fifths does not apply in most non-tonal cases):

> REGIONAL DISTANCE RULE: Given two collections *X* and *Y* at level d of the basic space, the regional distance value, *i*, is equal to the displacement value of the smoothest voice leading between the two collections.

By using the displacement value instead of the number of new pitch classes introduced, the value from *X* to *Y* will be identical to the value from *Y* to *X*. Had we used the number of new pitch classes introduced, going from the whole-tone scale to the octatonic scale would yield a different value than going from octatonic to whole-tone. A case certainly could be made for different perceptual distances between whole-tone to octatonic and octatonic to whole-tone. This would be more the job of similarity relations, since the difference would largely stem from the differing intervallic structures of the two collections.

An interesting problem arises when calculating the distance from whole-tone to enneatonic pitch-class spaces. In cases where the WT collection is a literal subset of the enneatonic, three new pitch classes will be introduced when WT→ENN, but the spaces will be virtually indistinguishable when moving ENN→WT because no new pitch classes are introduced: the distance ENN→WT = 0. Example 4.8 illustrates. Moving from $ENN_{0,1,2}$ to WT_0 (that is, from the enneatonic collection that begins with 012 to the whole-tone collection that contains pc 0), the distance is 0. The distance from $ENN_{0,1,2}$ to WT_1 is 3: the three new pitches that are introduced form the complement of $ENN_{0,1,2}$.

Example 4.8: $ENN_{0,1,2}$ mapping into WT_0

whole-tone

level d:	0		2		4		6		8		10	
level e:	0	1	2	3	4	5	6	7	8	9	10	11

enneatonic

level d:	0	1	2		4	5	6		8	9	10	
level e:	0	1	2	3	4	5	6	7	8	9	10	11

enneatonic

level d:	0	1	2		4	5	6		8	9	10	
level e:	0	1	2	3	4	5	6	7	8	9	10	11

WT_0 (no new pitch classes)

level d:	0		2		4		6		8		10	
level e:	0	1	2	3	4	5	6	7	8	9	10	11

enneatonic

level d:	0	1	2		4	5	6		8	9	10	
level e:	0	1	2	3	4	5	6	7	8	9	10	11

WT_1 (three new pitch classes)

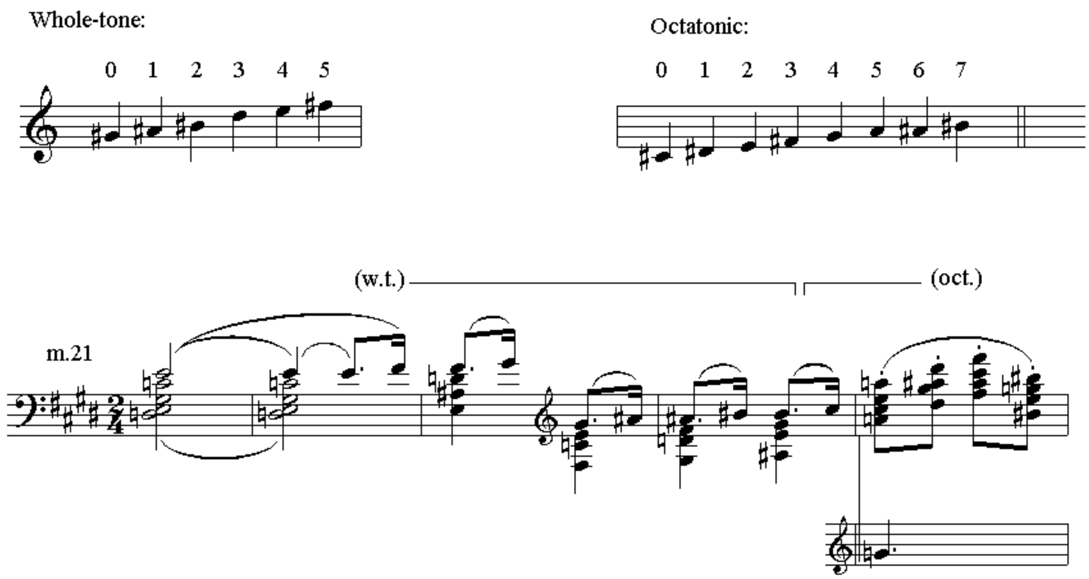
level d:		1		3		5		7		9		11
level e:	0	1	2	3	4	5	6	7	8	9	10	11

In general, if the destination pitch-class space is a subset of the original pitch-class space, no perceptible shift in space will occur. The question then arises as to whether a shift in the underlying collection has taken place at all. This is an analytical question, best examined on a case-by-case basis. The following section of the paper contains a number of analyses chosen to illustrate the measurements detailed above.

* * *

Example 4.9 reproduces Example 1.12, the excerpt from *Feuilles mortes* discussed in Chapter 1. There are several factors that we must take into account when determining if a pitch-class space changes. In the Debussy, the change in pitch-class space is highlighted by changes in other musical features. The dotted-eighth–sixteenth rhythm changes to straight eighth notes after the barline at m. 25; the bass ostinato in mm. 19-24 (not present in Example 4.9) ceases; and the crescendos that appear in mm. 19-24 are thwarted on the downbeat of m. 25, where the dynamic becomes *ppp*. More importantly, the (0248) sonority that pervades mm. 19-24 is replaced by (037) at the downbeat of m. 25. (0248) is not an octatonic subset, nor is (037) a whole-tone subset, so the contrast is well marked between the two sections.

Example 4.9: Debussy, *Feuilles mortes*, mm. 21-5



Example 4.10, a reproduction of Example 4.6, illustrates the transition from WT_0 to $OCT_{0,1}$. The total displacement when moving from WT_0 to $OCT_{0,1}$ is 4. Pitch class 2 splits, traveling one semitone in either direction to pitch classes 1 and 3, and pitch class 8 behaves in the same way, splitting into pitch classes 7 and 9.

Example 4.10: Pitch-class space transition in mm. 21-5

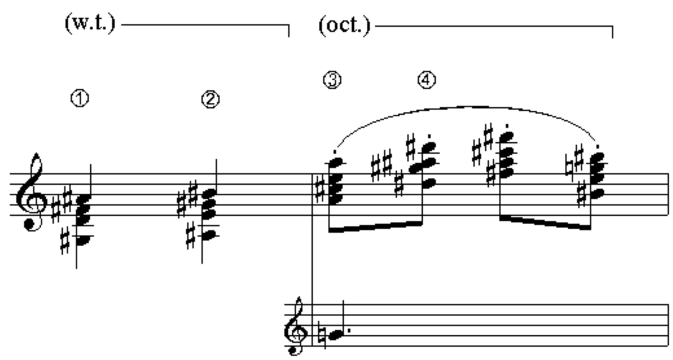
Whole-tone:	0	2		4	6	8		T
$\mathbf{S_{(2,8)}}$	|	/	\	|	|	/	\	|
Octatonic:	0	1	3	4	6	7	9	T

smoothness: <4,4,0,0,0,0,0>

displacement = 4

Example 4.11: Chord distances in *Feuilles mortes*, mm. 24-5

Ex. 4.11.1: mm. 24-5



Ex. 4.11.2: Pitch-class spaces for chords 1-4

	level a:									8			
	level c:			2				6		8		10	
chord 1:	level d:	0		2		4		6		8		10	
	level e:	0	1	2	3	4	5	6	7	8	9	10	11
	level a:											10	
	level c:	0				4				8		10	
chord 2:	level d:	0		2		4		6		8		10	
	level e:	0	1	2	3	4	5	6	7	8	9	10	11
	level a:								7				
	level c:		1			4			7		9		
chord 3:	level d:	0	***1***		***3***	4		6	***7***		***9***	10	
	level e:	0	1	2	3	4	5	6	7	8	9	10	11
	level a:								7				
	level c:				3				7			10	
chord 4:	level d:	0	1		3	4		6	7		9	10	
	level e:	0	1	2	3	4	5	6	7	8	9	10	11

In this case, the value of the displacement (4) is equal to the regional distance between the spaces. We can include this measure in our calculation of chord distances over the change in space. In Example 4.11.1, only mm. 24-5 of the Debussy are included. The sixteenth-note anticipations have been removed for the moment. Each chord has a number over it, and the number refers to the pitch-class space constructed in Example 4.11.2. Those pitch classes that are new to a space (in levels a and c) are underlined. The pitch-class space changes between chords two and three from whole-tone to octatonic. In the pitch-class space for chord 3, all of the new pitch classes introduced at level d—those which are the products of split transformations—are indicated by boldface and italic type.

We are now in a position to calculate chord distances across regions. Example 4.12 provides the values for *i*, *j*, and *k* for the progression. *i* is the number of pitch classes new to levels a and c; *j* will either be equal to 1 or 0 for reasons to be discussed momentarily; and *k* is the displacement from pitch-class space *X* to pitch-class space *Y*.

Example 4.12: Chord distances for chords 1-4

$$\delta(\text{x}\rightarrow\text{y}) = i + j + k$$
$$\delta(1\rightarrow2) = 0 + 0 + 3 = 4$$
$$\delta(2\rightarrow3) = 4 + 1 + 4 = 9$$
$$\delta(3\rightarrow4) = 0 + 0 + 2 = 3$$

The value of *j* is problematic in the examples. Because not all common collections can be derived from a single generating interval, we must find a comparable means of measuring intra-regional distance. For octatonic and hexatonic collections, Lerdahl sets *j* equal to the number of applications of a chord-quality preserving operation (“move the pcs at levels a-c two steps to the right […] or two steps to the left”) to a triad (Lerdahl

252-60). These values are acceptable when we are dealing with triadic/octatonic or triadic/hexatonic space and chord quality is an issue, but most of the music with which the present study is concerned is not triadic. Conceivably, we could develop a system of chord qualities for non-triadic hexatonic, octatonic, whole-tone, and enneatonic music and develop operations that preserve these qualities, but such a study is beyond the scope of the present investigation.[33]

As was discussed in Chapter 2, there are three classes of ME sets. Class A sets are those where (c,d) is d; class B sets are those where $1 < (c,d) < d$; and Class C sets are those where $(c,d) = 1$ (Clough and Douthett 1991, 97). Class A and Class C sets feature a single intervallic generator g (mod d). To determine the value of j within Class A and Class C universes, we will use the number of times pc set X has to be transposed by g in order to map into pc set Y. In Class B scales, j will equal the mod-d transpositional distance between the two pc sets.

Lerdahl offers corollaries to his octatonic and hexatonic inter-regional chord distances, and we need to account for these here as well. Because of the limited transposition of the octatonic and hexatonic scales (and, by extension, the whole-tone and enneatonic collections), it is possible that motion to another one of the collections will eliminate the "tonic" pitch. As a result, Lerdahl suggests setting j equal to 1 when calculating distances across regions (Lerdahl 258-60). I will retain this convention here, in a slightly expanded version: when calculating distances between chords across regions, (a) set $j = 1$ if pitch-class set X is a subset of region A but not a subset of region B, and (b) set $j = 0$ if pitch-class set X is a subset of both Regions A and B. Taking the whole-

[33] Quinn (2004) constructs such a system.

tone scale as an example, if we move from WT_0 to WT_1, we must set *j* equal to 1 because WT_1 excludes all of the pitch classes found in WT_0.

The introduction of a new pitch class does not always indicate a change in pitch-class space. A foreign pitch may simply be some sort of "non-harmonic" tone (as in Debussy's *L'isle joyeuse* in the previous chapter). In *Feuilles mortes*, we can now read the C# at the end of m. 24 as an anticipation. The Debussy analysis serves as an example of some factors that need to be taken into account when determining when the underlying pitch-class space changes. Each musical work will present its own unique set of circumstances. In some cases, the change will be very clear (as I believe it is in *Feuilles mortes*); in other cases, the change may be obscured.

Example 4.13 details the pitch-class spaces found in Debussy's "Jardins sous la pluie." The left-most column is the measure in which the collection assumes control; the second column is the common name for the controlling collection, and the third column includes all of the pitch classes in the collection. I have listed the pitch classes out of order in some cases to highlight the invariance (or lack thereof) across collections. The distance measures in the fourth column were calculated using the regional distance rule. In the case of scales with a given "tonic" (e.g. E Aeolian, F# Ionian—in contrast to the whole-tone and octatonic collections) the tonic note was counted three times: it appears in levels a, b, and c of the basic space as well as in level d.

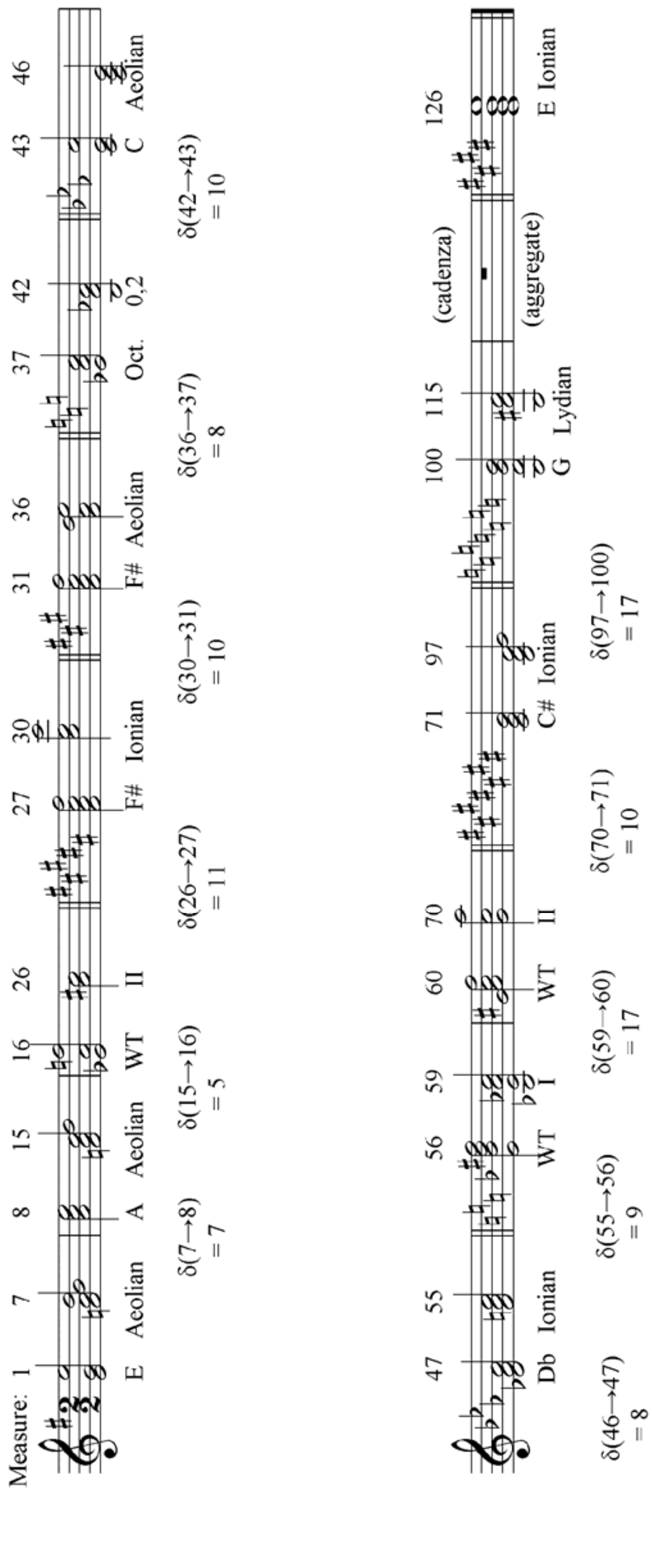
Example 4.13: Pitch-class spaces in "Jardins sous la pluie"

Measure	Collection	Pitch classes (level d)	Distance
1	E Aeolian	E F# G A B C D	-
6	F Lydian	E F G A B C D	4
8	A Aeolian	E F G A B C D	3
16	Whole-tone I & II	E F# G# A# C D/F G A B C# D#	3/6
27	F# Ionian	E# F# G# A# B C# D#	From WT I: 7 From WT II: 6
31	F# Aeolian	E F# G# A B C# D	3
37	Octatonic II	E♭ F F# G# A B C D	3
43	C Aeolian	E♭ F G A♭ B♭ C D	6
47	D♭ Ionian	E♭ F G♭ A♭ Bb C Db	5
56	Whole-tone I	E F# G# A# C D	3
60	Whole-tone II	E# G A B C# D#	6
71	C# Ionian	E# F# G# A# B# C# D#	7
100	G Lydian	E F# G A B C# D	8
126	E Ionian	E F# G# A B C# D#	5

Example 4.14 contains a chordal reduction of the piece. The reduction is by no means prolongational or Schenkerian; rather, I have essentially taken the first and last chords from each of the sections detailed above in order to show chord connections across the changing spaces. For the most part, voice leadings within the sections can be modeled using the apparatus outlined in Chapter 1. The number of the measure in which each chord appears is given above the chord. In the diagram, each measure corresponds to one of the pitch-class spaces given above. The name of the pitch-class space appears below the example. Below each barline, the distance values for the chords, $\delta(a \rightarrow b)$, appears. The distance is calculated using the full version of the chord distance rule, taking into account regional distances (displacement from X to Y) as well as new pitches at levels a-d. The value of j was calculated as per the process stated above. In the tonal/modal spaces, the root of each chord counts three times beyond its appearance in level d; the fifth counts two times beyond its appearance in level d. In the octatonic and

whole-tone spaces, I have excluded levels a and b from the calculations due to the lack of stability and tonal centricity usually associated with these spaces.

Example 4.14: Chordal reduction of Debussy's "Jardins sous la pluie"



Two of the calculations require further explanation. In calculating the distance from m. 7 to m. 8, and in calculating the distance from m. 30 to m. 31, non-harmonic tones had to be taken into consideration. Example 4.15 shows the pitch-class spaces used to calculate the distances in both cases.

Example 4.15: Calculating the distance from m. 7 to m. 8 and from m. 30 to m. 31

Ex. 4.15.1: m. 7 to m. 8

											11
					(5)						11
		2			(5)				9		11
0		2		4	(5)	6	7		9		11
0	1	2	3	4	5	6	7	8	9	10	11

E Aeolian

									9		
				4					9		
0				4					9		
0		2		4	5		7		9		11
0	1	2	3	4	5	6	7	8	9	10	11

A Aeolian

Ex. 4.15.2: m. 30 to m. 31

	1										
	1							(8)			
	1			(4)				(8)			
	1		3	(4)	5	6		8		10	11
0	1	2	3	4	5	6	7	8	9	10	11

F# Ionian

						6					
	1					6					
	1					6			9		
	1	2		4		6		8	9		11
0	1	2	3	4	5	6	7	8	9	10	11

F# Aeolian

In Example 4.15.1, pitch class 5 appears in parentheses because it is not part of the controlling E-Aeolian space—it is a non-harmonic tone. Therefore, it does not participate in the displacement calculation: it cannot be held invariant as the space shifts to A-Aeolian space. It is a new pitch class at level d and I have counted it as such even despite its high perceptual salience in E-Aeolian space. The same thing occurs from m. 30 to m. 31 (Ex. 4.15.2): pitch-class 4 is a non-harmonic tone in F#-Ionian space, again despite its relatively high degree of salience. It is counted as a pitch class new to F#-Aeolian space.

Pitch-class 8 appears in parentheses in F#-Ionian space because it is not written in the score, but is perceived as part of the chord (the fifth) none the less.

“Jardins sous la pluie” is the subject of one of the analytical essays in the next chapter. I will deal with the issues of non-harmonic tones and changing spaces in the piece in much greater depth there.

* * *

The final situation we must address concerns simultaneous spaces. Here, we must shift our orientation from a pitch-*class* space perspective to a pitch-space perspective: if two spaces occur simultaneously, then one must clearly be higher or lower than the other.

Example 4.16 comes from Messiaen’s “Les sons impalpables du rêve…” I have diagrammed the form of the work (a seven-part rondo) and included the modes employed by Messiaen in each section. The larger formal boundaries are given in the left-hand column. The measure numbers that correspond to these larger sections are given in the middle column. The pitch material used in each section/subsection is given in the right-hand column. Mode 2 is the octatonic scale: its first transposition is $OCT_{0,1}$. Mode 3 is the enneatonic scale: its first transposition is $ENN_{0,1,2}$. Mode 6 in its first transposition contains the pitch classes [C,D,E,F,F#,G#,A#,B].

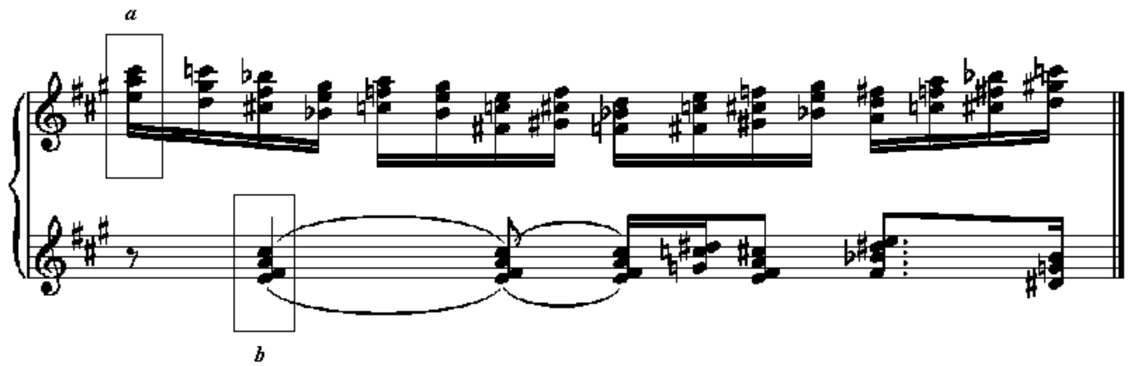
Example 4.16: Formal organization of Messiaen's "Les sons…"

Formal unit	mm.	Pitch material
A	1	RH: mode 3, 3rd trans. LH: mode 2, 1st trans.
B	7	mode 2, 1st trans.
	8	mode 2, 2nd trans.
	10	mode 3, 3rd trans.
(trans)	16	mode 6, 1st trans.
A	17	RH: mode 3, 3rd trans. LH: mode 2, 1st tras.
C	22	mode 6, 5th trans.
B2	34	mode 3, 3rd trans.
A	43	RH: mode 3, 3rd trans. LH: mode 2, 1st tras.
B	49	mode 2, 1st trans.
	50	mode 2, 2nd trans.
		mode 3, 3rd trans.
(trans)	58	mode 6, 1st trans.
A	59	RH: mode 3, 3rd trans. LH: mode 2, 1st tras.
Coda	65	mode 2, 1st trans.

In this piece, we have simultaneous pitch-class spaces occurring in the A sections: mode 3/3rd transposition in the right hand and mode 2/1st transposition in the left hand. In the A section, the two hands are separated by roughly an octave, so the distinction between the musical lines is evident. In both hands, the chords are in close position, fusing the pitches that participate in the individual polyphonic lines in each hand into homorhythmic gestalts. We can calculate regional distances between these simultaneous pitch-class spaces in almost the same way that we calculate the distance between successive pitch-class spaces. The distance value is only the regional distance value (the displacement value of the smoothest voice leading), which in this case equals 4. To determine the distance between any two chords in these regions, we must also take into account levels a through c of each chord's basic space (as we did with "Jardins sous la pluie," above).

We must amend the other terms in the chord distance rule in order for the rule to be applicable to simultaneous pitch spaces. The value of j is set to 0 because the two spaces share many of the same pitch classes: the trichord in the right hand is actually a subset of the trichord in the left hand. The value of k will be calculated in a similar manner as the value of i. Since the pitch spaces can be read top-to-bottom as well as bottom-to-top, we must reconcile both new-pitch counts. To do this, we will again take the absolute value of the difference between the number of new pitch classes introduced from X to Y and from Y to X. As an example, if we wanted to calculate the perceptual distance between the first chord in the right hand—[A,C#,E]—and the first chord in the left hand—[E,F#,A,C#]—in "Les sons…" we would calculate the distance as in Example 4.17. In Example 4.17, Messiaen indicates that the right-hand part be played an octave higher than it is written.

Example 4.17: Perceived distance between chords in m. 2 of "Les sons…"



a = [A,C#,E]; *b* = [E,F#,A,C#]

Chord distance rule = $i + j + k$

$i = \delta(X/Y) = 4$
$j = 0$

	1			4					9		
0	1	2		4	5	6		8	9	10	
0	1	2	3	4	5	6	7	8	9	10	11

a: mode 3/3rd transposition

	1			4		$\underline{6}$			9		
0	1		$\underline{3}$	4		6	$\underline{7}$		9	10	
0	1	2	3	4	5	6	7	8	9	10	11

b: mode 2/1st transposition

New pitch classes ($X \rightarrow Y$) = 3

	1			4		6			9		
0	1		3	4		6	7		9	10	
0	1	2	3	4	5	6	7	8	9	10	11

b: mode 2/1st transposition

	1			4					9		
0	1	$\underline{2}$		4	$\underline{5}$	6		$\underline{8}$	9	10	
0	1	2	3	4	5	6	7	8	9	10	11

a: mode 3/3rd transposition

New pitch classes ($Y \rightarrow X$) = 3

$k = | 3 - 3 | = 0$

$\delta(a/X \rightarrow b/Y) = i + j + k = 4$

The calculations in the Example are intuitively satisfying: [A,C#,E] should not sound terribly different from [E,F#,A,C#], regardless of the differences in the underlying collections. These calculations do not take into account the registral distance between the chords. To account for register, we must add a fourth term to the chord distance rule. Because we are discussing pitch space here and not pitch-class space, I will substitute MIDI note numbers in place of pitch-class integers.[34] MIDI note numbers offer a standard way of discussing pitch space as opposed to pitch-class space. The MIDI standard assigns the number 60 to middle C (C4). C#4/D♭4 would be 61; B3 would be 59, and so on. Example 4.18 lays out the chords in pitch space.

Example 4.18: Pitch-space representation of chords a and b

```
                    |-------------------b--------------------|
                    64       66            69                 73
60   61        63   64       66   67       69   70       72   73
60   61   62   63   64   65  66   67   68  69   70   71  72   73
(C4)                                                     (C5)

|--------------------a-------------------|
88                       93                  97
88   89   90        92   93   94        96   97   98
88   89   90   91   92   93   94   95   96   97   98   99
                                       (C7)
```

In Example 4.18, middle C (C4) appears at the left-hand side. With respect to chords *a* and *b*, the pitch space changes from octatonic (MIDI notes 60-73) to enneatonic (MIDI notes 76-87) somewhere in the range of notes 74-87. We can assume that the octatonic space continues on infinitely to the left and that the enneatonic space continues

[34] Other systems are of course possible: the calculations would end up being the same if we used a system that (for example) assigned 0 to middle C. Pitches above middle C would be represented by positive integers and pitches below middle C would be represented by negative integers: B3 would then be -1; C#4 would be 1.

on infinitely to the right. Over the course of the measure, the left hand contains notes as high as E5 (MIDI note 76); the right hand contains notes as low as F5 (MIDI note 77), so we can locate the junction of the two spaces around MIDI notes 76/77, or between E5 and F5. In order to find the registral distance between chords *a* and *b*, we will use the following formula:

> REGISTRAL DISTANCE RULE (first version): $\delta_{reg.}(a/X \rightarrow b/Y)$ (in semitones) = [(highest pitch in *a* + lowest pitch in *a*)/2 – (highest pitch in *b* + lowest pitch in *b*)/2]. The registral distance between a chord, *a*, in pitch space *X* and a chord, *b*, in pitch space *Y* is equal to the average distance between the highest and lowest pitches of the two chords, given in semitones.

The distance between the two chords in the Messiaen excerpt would be calculated as follows: [(97 + 88)/2 - (73 + 64)/2] = 24 semitones. The chords are basically two octaves apart. In order to keep the distance values in the chord distance rule within reason (it seems strange to say that, since the chords are two octaves apart, the distance value jumps from 4—see Example 4.17—to 28), we convert the number of semitones into octaves and use that value as a fourth term, *l*, in our chord distance rule. We will amend the rule above by changing the denominator in the formula to 12 to give us the number of octaves between the chords:

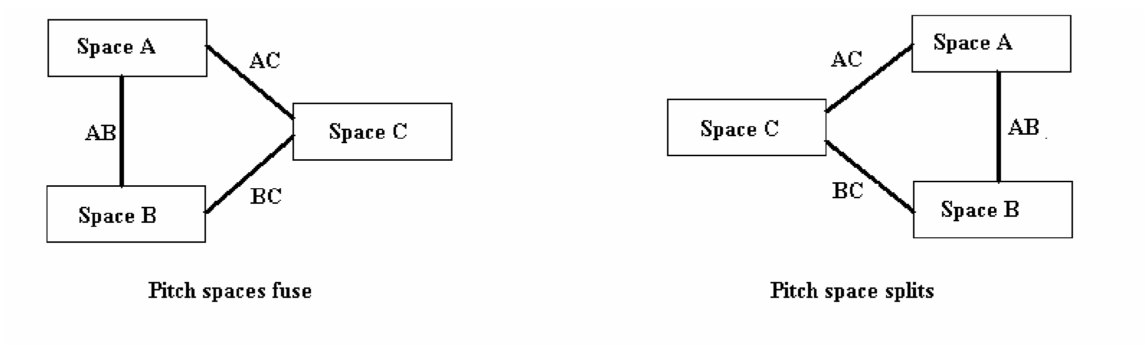
> REGISTRAL DISTANCE RULE (final version): $\delta_{reg.}(a/X \rightarrow b/Y)$ (in octaves) = [(highest pitch in *a* + lowest pitch in *a*)/2 – (highest pitch in *b* + lowest pitch in *b*)/2]/12. The registral distance between a chord, *a*, in pitch space *X* and a chord, *b*, in pitch space *Y* is equal to the average distance between the highest and lowest pitches of the two chords, given in octaves.

The total distance between chords *a* and *b* in the Messiaen example above would be $i + j + k + l = 4 + 0 + 0 + 2 = 6$.

The final problem we need to address involves several pitch-class spaces fusing into a single pitch-class space or a single pitch-class space splitting into several pitch-class spaces. Example 4.19 diagrams the two possible situations. In the example, the distances from space to space are indicated by pairs of letters that refer to the spaces they connect.

Example 4.19: Splitting and fusing pitch-class spaces

Ex. 4.19.1 **Ex. 4.19.2**



From the A section to the B section of "Les sons…" two registrally distinct pitch spaces fuse into one pitch-class space. With reference to Example 4.19.1, space A is mode 3/3rd transposition; space B is mode 2/1st transposition; and space C is mode 2/1st transposition. The distance from space A to space B (AB) is going to equal the displacement from A to B plus the number of octaves between the spaces (the sum of terms *i* and *l* of the regional distance rule). Here, distance AB = 6. Distance BC equals 0 because the space does not change: every element of the pitch-class space B can map directly onto a member of pitch-class space C.

* * *

In this chapter I have presented a variety of situations involving changing pitch- and pitch-class spaces and I have provided methods for describing voice leading across these spaces as well as formulas for calculating distances between the spaces. At this point, I have finished outlining the theoretical system. In Chapter 5, we will see some analytical applications of the theory as well as suggestions for expanding the present research.

Chapter 5 : Extending the theory

The final chapter contains several analyses that illustrate the analytical system presented in the preceding four chapters and offers some closing remarks and possibilities for future study. The first analysis is of Messiaen's "Les sons impalpables de rêve...," a piece that I have touched on at several points in the dissertation. The analysis in this chapter synthesizes the findings of the previous chapters to present an overall picture of the piece. The second analysis is of Debussy's "Jardins sous la pluie," again, a piece that I have touched on several times in the preceding chapters. I have included it here to show how transformational and associational voice leading can interact within and across changing regions.

* * *

I have already discussed excerpts from Messiaen's "Les sons impalpables du rêve…" in previous chapters. In this chapter, I supplement the observations made earlier in the dissertation with the aim of providing an analysis of the entire piece. I begin with a discussion of the form of the prelude and the modal organization. Having discussed the A section of the prelude in several places in the previous chapters, I will focus on sections B and C in this chapter.

The prelude is in rondo form. Example 5.1 contains a chart of the form along with the modes present in each section.

Example 5.1: Form of "Les sons impalpables du rêve…"

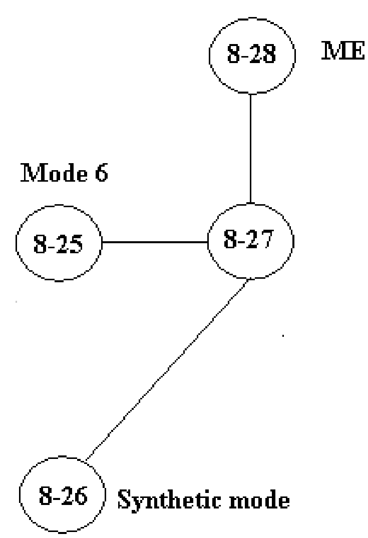
Section	Measures	Mode	Musical characteristics
A	1-6	mode 3 (r.h.) mode 2 (l.h.)	r.h. ostinato in F5-C#7 register; chordal melody in l.h.
B	7-16	mode 2 (mm. 7-8) mode 3 (mm. 10-15) mode 6 (m. 16)	mm. 7-8: r.h. lyrical melody; l.h. eighth-note accomp. mm. 10-15: staccato melody in bass; 16th-notes in r.h. m. 16: expanding wedge
A	17-21	mode 3 (r.h.) mode 2 (l.h.)	r.h. ostinato in F5-C#7 register; chordal melody in l.h.
C	22-33	mode 6	canon in inversion in outer voices; 16th-note accomp. between voices
B'	34-43	mode 3	staccato melody in r.h.; 16th-notes in bass
A	44-48	mode 3 (r.h.) mode 2 (l.h.)	r.h. ostinato in F5-C#7 register; chordal melody in l.h.
B	49-58	mode 2 (mm. 49-50) mode 3 (mm. 51-57) mode 6 (m. 58)	mm. 7-8: r.h. lyrical melody; l.h. eighth-note accomp. mm. 10-15: staccato melody in bass; 16th-notes in r.h. m. 16: expanding wedge
A	59-64	mode 3 (r.h.) mode 2 (l.h.)	r.h. ostinato in F5-C#7 register; chordal melody in l.h.
Coda	65-68		quasi-cadenza

The prelude is one of Messiaen's first experiments with polymodality. The ostinato (which Messiaen calls a "pedal group")[35] in the right hand of the A section is in mode 3 and the left hand is in mode 2. For the most part, the texture of the prelude is melody-plus-accompaniment, but the polyphonic C section features an accompanied canon in inversion in mode 6.

[35] Messiaen 1944, 55: "Instead of one sustained note foreign to the chords which surround it, we shall have a repeated music (repetition and sustaining are equivalent), foreign to another music situated above or below it; each of these musics will have its own rhythm, melody, harmonies… In this example [from the beginning of "Les sons…"], the music of the upper staff repeats itself from measure to measure, independent of the music of the lower staff; it is a pedal group."

In the B section, which begins in m. 7, the roles of the right and left hand are reversed: the right hand has a melody in mode 2 for three measures and the left hand has an eighth-note blocked-chord accompaniment. In m. 10, the hands switch again: the right hand has a sixteenth-note ostinato figure and the left hand has a syncopated melody. Messiaen uses a synthetic mode here, comprising the pitch classes (C#,D,E,F#,G,G#,B♭,C). The mode is a member of set class 8-26, and it deviates from the 8-note ME collection by 2 semitones. Example 5.2 contains a section of an octachordal set-class space. Set class 8-28 (the octatonic collection) appears at the top of the network. Messiaen's mode 6, set class 8-25, appears in the left-most node. Set class 8-26, the synthetic mode that appears in the B section of "Les sons…" appears in the southwest node. The synthetic mode deviates from both 8-28 and 8-26 by two semitones. As discussed in Chapter 2, this synthetic mode is a collection that can facilitate pattern-matching because of its deviance from ME.

Example 5.2: Octachordal set-class space



In section C of the prelude, Messiaen dispenses with the A-major key signature in favor of one with no sharps and flats (A minor, it would seem). Messiaen constructs a canon in inversion using the fifth transposition of his mode 6. The canon is at the interval of an eighth note. The canon appears in the outer voices of the texture, and a sixteenth-note accompaniment figure slithers around between the outer parts. Example 5.3 presents the canon without the accompaniment. The canon has been metrically normalized—that is, the two voices are aligned. The step-class integers for mode 6 appear above the top voice and below the lower voice. In the example, step-class integer 0 is assigned to the pitch-class A, reflecting the "tonic" nature of that pitch. The other pitches are assigned as follows: 0 = A, 1 = B♭, 2 = C, 3 = D, 4 = D#, 5 = E, 6 = F#, and 7 = G#. Between the two staves, I have included the mod-8 index number that describes the inversional relationship between the upper and lower voice. Again, this index number is contextual, not fixed: if we had assigned step-class integer 0 to some other pitch class, the index of inversion would be different.

Of particular interest in the example is the changing index number. An index number of 5 places the axis of inversion between pitch-classes C and D; an index number of 4 places the axis of inversion through pitch-class C; and an index number of 6 places the axis through pitch-class D. If the voices were exactly inversional, the index number would be the same throughout. Instead, the axis oscillates between C and D, occasionally stopping between the two pitch classes. From a mod-12 perspective, the index numbers are not consistent either.

Example 5.3: Canon from "Les sons…"

Example 5.4 is similar to Example 5.3, but replaces the step-class integers with mod-8 ordered step-class intervals. Only the first four measures of the excerpt appear in this example.

Example 5.4: Canon from "Les sons…" with directed step-class intervals

From this example we can see that, despite the apparent similarities in motion between the two voices—the lower voice moves up a third and up a sixth, and the upper voice moves down a third and then down a sixth, and so on—from a mod-8 perspective, the voice leadings between the two lines do not agree.

Example 5.5 presents a recomposed version of the canon. The top line is as Messiaen wrote it. I have replaced some of the pitches in the bottom line with other mode 6 pitches to preserve a mod-8 index number of 5 throughout. I have chosen 5 as the index number because it preserves the A-E dyad on the downbeat: this dyad is a contributing factor to the "A-minor-ness" of the canon and the overall tonality of the prelude. Again, mod-8 step-class integers appear below the bottom voice, and I have circled the integers that represent step-classes that differ from those in Example 5.3.

Example 5.5: Recomposed canon from "Les sons…"

More than half of the pitches had to be altered to present a true canon in inversion in a mod-8 pitch-class space. In light of this, it appears that Messiaen liberally altered pitches in the bottom voice to make what would have been an exact inversion with a fixed axis of inversion into a canon that more clearly suggested A minor. In most cases, where the index number would have warranted a D (step-class 3) in the bottom voice—see, for example, the first measure of the canon—Messiaen replaced the D with a C. The total pitch-class content of the first measure as Messiaen has written it is an A-minor triad. This process is common in tonal music; it manifests itself in fugal writing as a tonal answer.

This analysis of the canon in "Les sons…" has shown us several things. First, we have seen how a modular-space approach can reveal aspects of contrapuntal structure. Most of the examples in the previous chapters have looked at motion among trichords or tetrachords, arranged neatly as vertical stacks of three or four notes. In this example, we see the utility of the system for analyzing lines instead of chords. Second, we have seen how Messiaen's synthetic mode can be related to his other modes of limited transposition. The material in Chapter 2 has provided us with a way to classify collections in terms of a maximally even "parent" collection, in this case, the octatonic. Third, the analysis tells us about Messiaen's compositional technique and modal/tonal thinking. In the analysis of this prelude in Chapter 3, we learned that Messiaen conceived of the work in A major. In that analysis, we saw that Messiaen constructed his chords by taking every other note of his mode 3 and arranging them in a stack. Some of the resulting chords are major, some are minor, and some are non-tonal trichords: implications of major and minor were not nearly so important in the A section. In the C section, Messiaen carefully altered the strict canon that appears in Example 5.5 to produce the more tonally suggestive counterpoint that appears in Example 5.3.

*　　*　　*

The final analytical essay in this chapter examines Debussy's "Jardins sous la pluie," from *Estampes*. This piece has also been touched on earlier in the dissertation (see Chapters 1 and 4). In this analysis, I will discuss the different pitch-class spaces that are present in the work. I will then talk about non-harmonic tones that appear in these spaces, and the influence that these NHTs have on the voice leading throughout the prelude.

The form of the piece is given in the table in Example 5.6. This table differs slightly from the table that appears in Example 4.13. Example 4.13 is intended to highlight the different collections that Debussy used; Example 5.6 is more concerned with formal structure and thematic material than it is with collection.

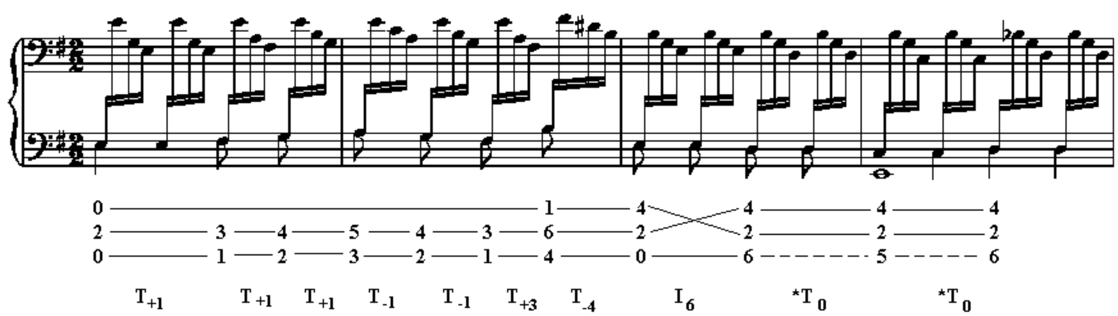
Example 5.6: Form of "Jardins sous la pluie"

Formal unit	Measures	Collections employed
A	1-15	E minor (mm. 1-7) A minor (mm. 8-15)
transition	16-26	WT_0, WT_1
A^1	27-36	F# major (mm. 27-30) F# minor (mm. 31-36)
trans.	37-42	$OCT_{0,2}$
A^2	43-55	C minor (mm. 43-46) Db major (mm. 47-52) chromatic (mm. 52-56)
trans.	56-70	WT_0, WT_1 (mm. 56-63) chromatic (mm. 64-70)
A+B	71-99	C# major/minor
A^3	100-115	G Lydian
trans.	116-125	chromatic
A+B	126-157	E major

In both the A and B sections, Debussy uses traditional tonal material—major and minor collections in a variety of keys—with the exception of the A^3 section, which is modal. Debussy uses symmetrical scales (the whole-tone and octatonic) in the transitional sections. The choice of symmetrical or asymmetrical collection is significant. As we have seen in Chapter 2, asymmetrical collections facilitate pattern-matching and position-finding tasks; symmetrical collections hinder these processes. It makes sense, then, that transitional passages between major formal units (which are tonally stable) be tonally unstable.

In analyzing the tonal sections, we will use a mod-7 approach exclusively. Pitch classes can be inflected in this environment, but the step-class integer representation will remain the same. Example 5.7 contains mm. 1-4 of “Jardins” to illustrate the approach.

Example 5.7: “Jardins sous la pluie,” mm. 1-4



Of interest in this passage is the D# in m. 2 and the B♭ in m. 4. Neither of these pitch classes are present in the E (natural) minor collection. If we assign step-class integer 0 to pitch-class E (to reflect its tonic nature), then D-natural would be assigned step-class integer 6, and B-natural would be assigned step-class interval 4. We can thus describe the D# as an altered step-class 6, and the B♭ as an altered step-class 4.[36] In this case, even though the pitches are altered, their function remains unchanged.[37] The voice leading appears below the excerpt in Example 5.7.

[36] To my ears, the B♭ sounds like a lower neighbor to the B-natural in the first half of the measure. Conceivably, we could respell the B♭ as an A#, which would make it an altered step-class 3.
[37] See Harrison 1994 and Santa 1999a for more on scale degree function and altered pitches.

In the first two measures, I am treating the high E as a pedal tone and therefore not factoring it in to the voice leadings. The E is replaced by an F# at the end of the second measure, calling our attention to it and thus bringing it into the voice leadings. The bottom two voices move in parallel thirds (step-interval class 2 in mod-7 space). In m. 3, the chords are related by inversion: the G and B exchange places and E is replaced by D. From m. 3 to m. 4, the G and B are held invariant while the D moves down by one mod-6 step to C. The C then moves back up to the D in the second half of the measure. I have used Straus's near transposition to model the motions in the last two measures.[38] In near transposition, all of the voices but one travel the same intervallic distance. Here, two voices are retained (T_0) and one moves by a single mod-7 step (T_1).

The first transitional passage (mm. 16-26) features the two whole-tone collections. For the most part, the collections alternate every two beats. The chords that Debussy chooses are related by transposition, but the transposition number will vary depending upon which pitch-class in each collection is assigned to step-class integer 0. The symmetrical nature of the whole-tone collection coupled with Debussy's use of it in sequential passages makes assigning 0 difficult. For the time being, I am going to assign step-class integer 0 to pitch-class C in WT_0 and C#/D♭ in WT_1.

Example 5.8 illustrates one possible reading of the voice leadings in mm. 16-19. I have only considered the voice leadings in the right hand for the moment: we could easily figure in the left hand and adjust the calculations accordingly. In each case, I have chosen the smoothest voice leading between the two chords, and the total mod-6 displacement appears below the mapping.

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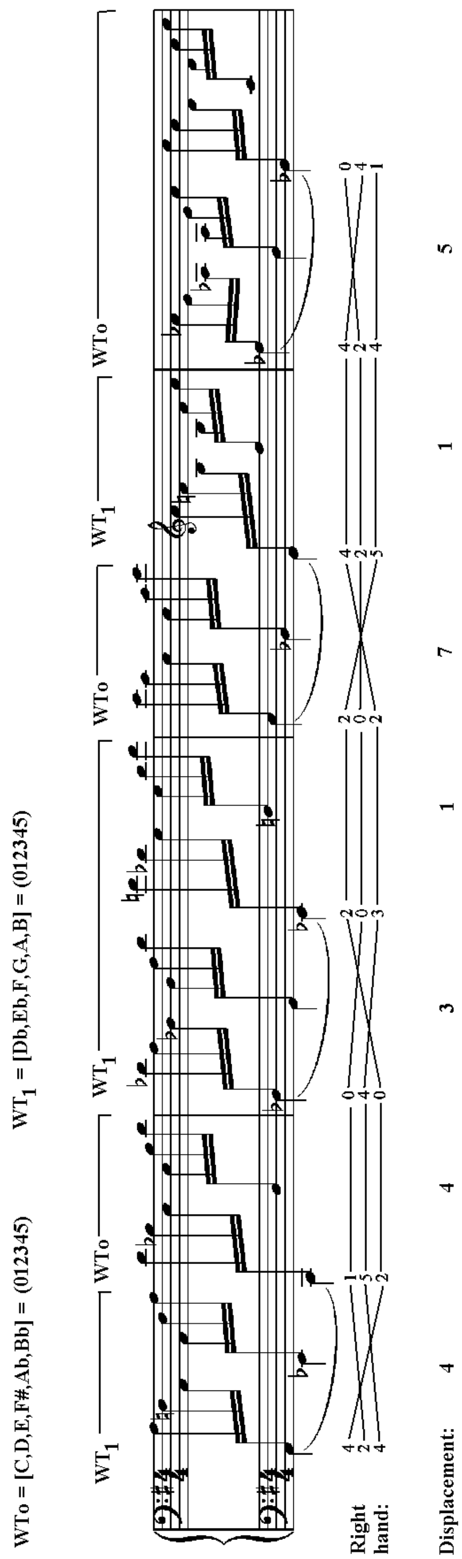
[38] Straus 1997.

Example 5.8: Voice leadings in mm. 16-19

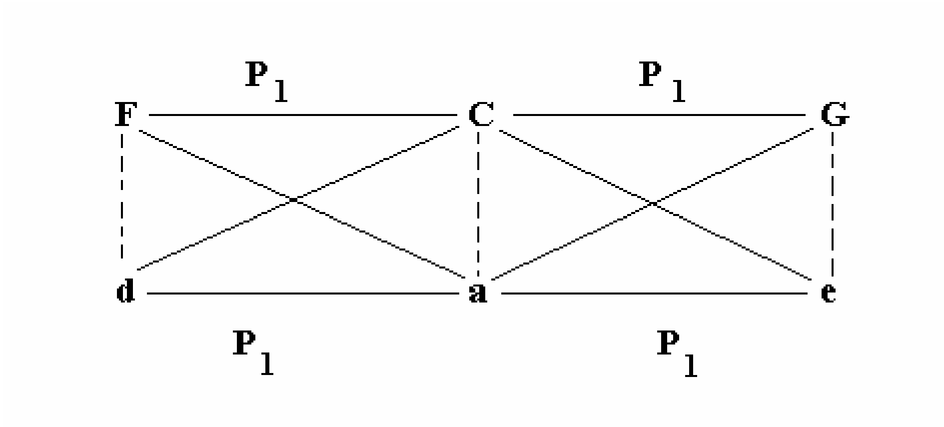


The mappings move between WT_0 and WT_1 in some cases, but the mod-6 displacement does not need to be adjusted across the space. In 12-pitch-class space, a regional distance value of 6 would have to be added to the mod-12 displacement values when calculating chord distances.

In m. 24, the second whole-tone collection assumes power and is whittled down to an oscillating A-C# dyad. This dyad serves as common-tone transitional material into the next A section, which is initially in F# major, but quickly becomes F# minor. We could say that the A-natural in the dyad is an altered step-class 2, or, from a 12-pc point of view, that it is a chromatic lower neighbor to the A# of F# major.

Having surveyed some of the voice leading strategies employed by Debussy, I turn now to his choice of collections in the A and B sections of the piece. In each of the first three A sections, Debussy employs pairs of related collections. In tonal theory, we talk about modulation to closely related or distantly related keys. Closely related keys are those that differ by one sharp or flat, and their relative major/minor keys (put another way, the keys that share the same diatonic collection). We can show the relationships between closely related keys with the P-relation. Example 5.9 shows the closely related keys to C major as well as the P-relations that relate the collections. In the diagram, solid lines represent P_1-relations and dotted lines represent P_0-relations (identical diatonic collections, in this case). F major is P_1-related to C major, which is P_1-related to G major. By extension, we can say that F major and G major are P_2 related: they differ by two semitones. Notice, too, that D minor and C major are P_1-related, as are F major and A minor, C major and E minor, and A minor and G major.

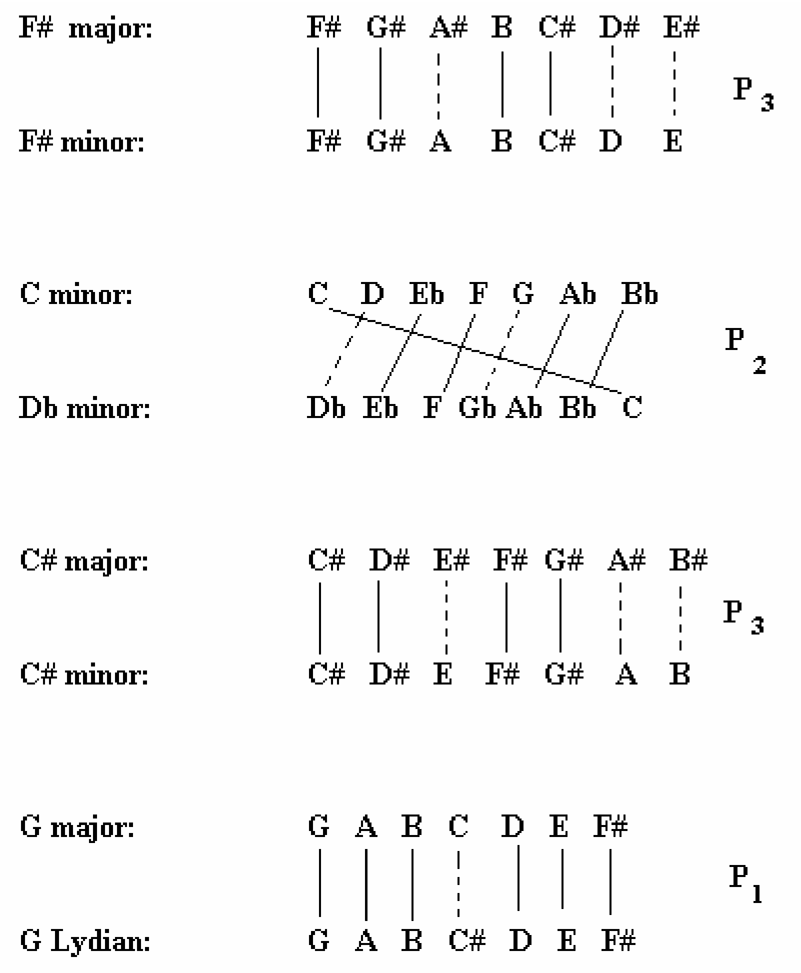
Example 5.9: Closely related keys and P-relations in C major



In the first A section, Debussy begins in E minor and moves to A minor, a closely related key because it differs by only one accidental. The two keys are P_1-related.

Example 5.10 models the relationships between the other key pairs in the formal units of "Jardins sous la pluie" using P-relations. In section A^1 (mm. 27-36), Debussy begins in F# major and switches to its parallel, F# minor. The two keys are P_3-related, as shown in the Example. The relationship between the keys in section A^2 is less obvious. On the surface, C minor and D♭ major are distantly related keys, but the two keys are actually P_2-related: they are no more distant than F major and G major. From C minor to D♭ major, five pitch classes are retained and two move by semitone to their new home in D♭ major: D is replaced with D♭ and G is replaced with G♭. The result is two P_2-related collections.

Example 5.10: P-relations between collection pairs in "Jardins…"



In the B section, Debussy modally mixes C# major and minor. Here, too, we can describe the collections as P-related: E# and E differ by semitone, A# and A differ by semitone, and B# and B differ by semitone. Thus parallel keys, which are not closely related in the traditional sense, are related by P_3.

In the A^3 section, Debussy uses only one collection, but it is neither major nor minor. G Lydian differs from G major by only one semitone, and as soon as the C# is introduced in m. 103, we notice the tension created by this intruder. G Lydian is P_1-related to G major, and the relationship is shown in the Example. G Lydian is actually the same collection of pitch classes as D major, which is obviously closely related to G major.

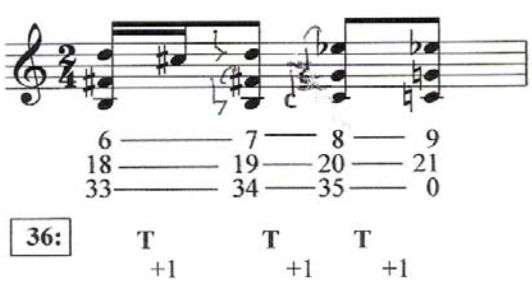
In the transitional sections, the distance between the pairs of collections that Debussy chooses is much greater than the distances between the key pairs in the tonal sections. For instance, as we have seen, the two whole-tone collections differ from one another by six pitch classes (P_6). Octatonic collections differ from one another by four pitch classes (P_4).

* * *

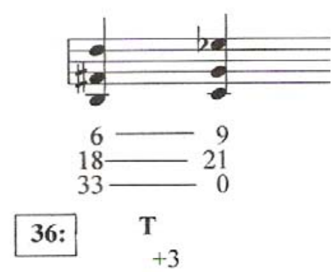
I close by offering some suggestions for future study. One logical extension of the system presented in the preceding chapters would be to examine microtonal music and/or music that uses alternate tuning systems. The precise frequency ratios between pitches in, say, a 1/4-comma mean-tone piece could facilitate position-finding in the same way that the deviant ME modes did in Chapter 2. Example 5.11 contains two passages from Alois Hába's violin suite, op. 85a (1955). The piece is *im Sechsteltonsystem*: Hába divides the octave into thirty-six equal steps, to each of which I have assigned a scale-step number. The scale appears at the top of the page. The bracketed pitches are enharmonically equivalent and thus are assigned the same scale-step number. Example 5.11.1 presents a succession of three trichords from the first movement of the work—they are related at T_1 in thirty-six space and the overall voice leading, given in Example 5.11.2, is T_3. Example 5.11.3 is a sequence that appears in the middle of the fourth and final movement. The overall voice leading is given in Example 5.11.4. These excerpts from the Hába suite show that the theory can be extended to model a work based on scales whose steps are smaller than a semitone.

Example 5.11: Alois Hába, violin suite, op. 85

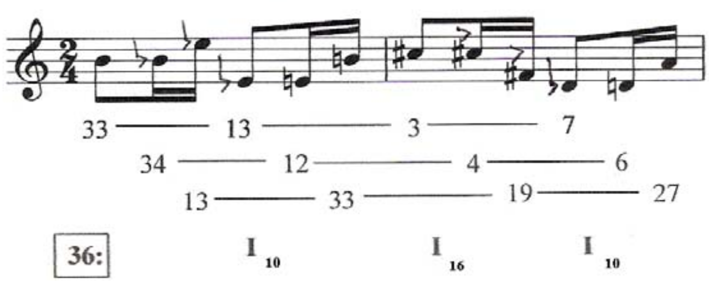
5.11.1. First movement, m. 27



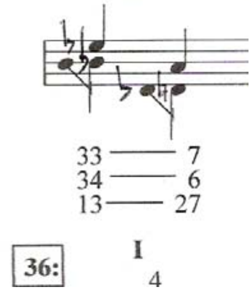
5.11.2. Total voice leading



5.11.3. Last movment, mm. 34-5



5.11.4. Total voice leading



In Chapter 1, I outlined a theory of transformational voice leading for music that inhabits non-mod-12 pitch-class spaces. Because transposition and inversion are well defined within these non-mod-12 spaces, we can use these operators to create Klumpenhouwer networks in non-mod-12 spaces. Mod-7 K-nets offer a convincing account of the voice leading in Josef Matthias Hauer's *Nachklangstudie* op. 16 no. IV. I begin with a discussion of the pitch-class space that Hauer creates.

Hauer's *Nachklangstudien* predate most of his experiments with the 12-tone method. Of the five pieces that comprise op. 16, one uses only six pitch classes; three use 10 or 11 pitch classes, and one—number IV, the focus of this section—uses only seven pitch classes. (The techniques found in the present study would doubtless prove useful in analyzing the other four pieces.) A score of the entire work appears in Example 5.12.[39] Hauer constructs the piece from the following seven pitch classes: A♭, B♭, B, D♭, E♭, E, and G. Taken together, the pitch classes are an instance of set class (0134689), 7-32 on Forte's list of set classes.

Hauer's collection can be ordered as a harmonic minor scale starting on A♭. There are moments in the piece where he shows an awareness of this: the piece begins on G# and ends with what appears to be a V-I cadence in A♭ minor. For the most part however Hauer picks and chooses from the collection. The lack of internal cadences and absence of any tonal triadic structures argue for an analysis that is more post-tonal than tonal. For this, we can adapt set theory for use in a musical system that only employs seven tones. [40]

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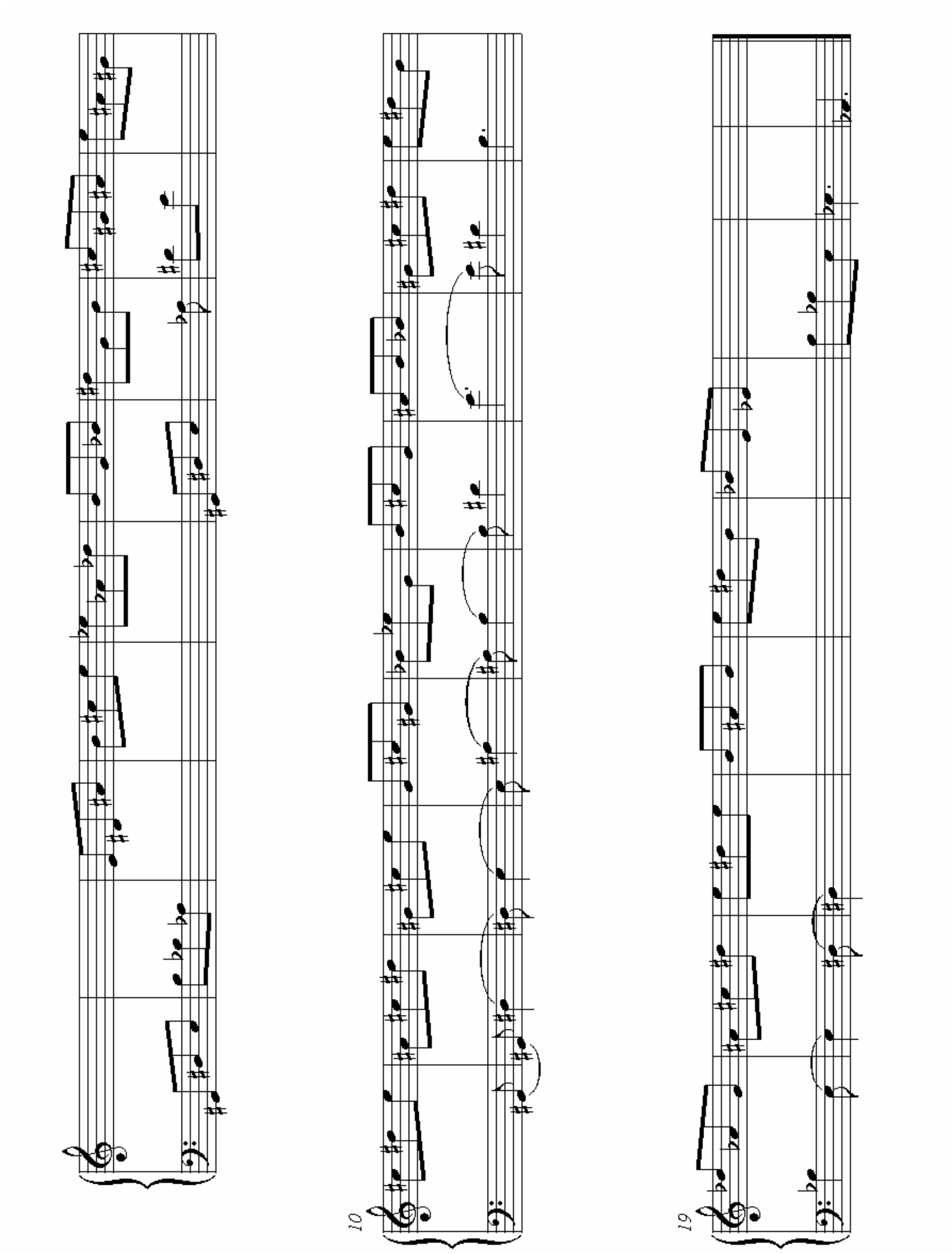
[39] The example reflects Hauer's notation: there are no rests. The left hand plays a duplet against the right hand's triple in m. 8.
[40] See Santa 1999a, 1999b and Neidhöfer 2005 for a variety of non-mod-12 analyses.

Example 5.12: Hauer, *Nachklangstudie* op. 16 no. IV



We can generalize Hauer's collection by assigning step-class integers to each element of the collection. Example 5.13 shows the collection with its step-class integers. I have arbitrarily assigned step class-integer 0 to A♭: the following analysis would yield the same results if 0 was assigned to any other pitch in the collection.

Example 5.13: Hauer’s scale generalized

pitch:	A♭	B♭	B	D♭	E♭	E	G
	^	^	^	^	^	^	^
scale degree mod 7:	0	1	2	3	4	5	6
size in semitones:	2	1	2	2	1	3	1

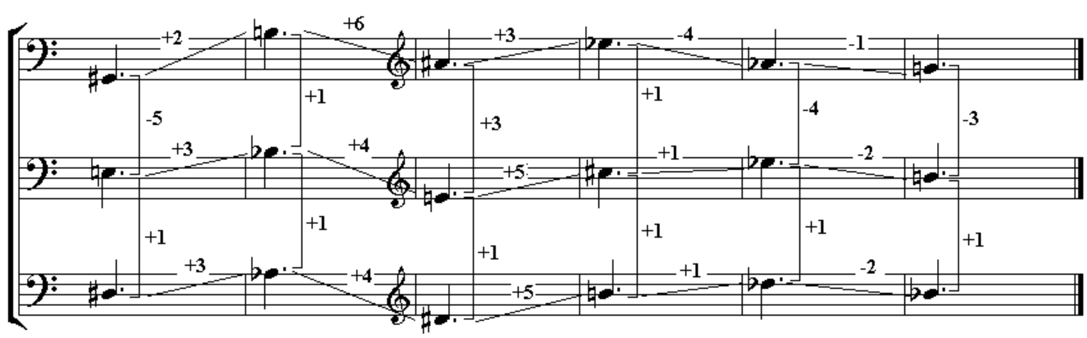
Klumpenhouwer networks provide us with a way of elucidating the voice leading in the piece. They are particularly useful when pitch-class sets cannot be related by transposition or inversion. K-nets do not relate trichords, per se: they relate *interpretations* of trichords by modeling the dynamic interrelationships between the constituent step classes. All trichordal K-nets consist of a T-arrow and two I-arrows. The T-arrow describes the transpositional relationship between the two step classes it relates; the I-arrows describe the inversional relationships between the two step classes they relate. Example 5.14 is a K-net analysis of mm. 1-6 of the piece. Notice that in these first six measures, three of the four possible mod-7 trichords are represented. Measure 1 is an instance of (013); m. 2, (012); m. 3, (014); m. 4, (013); m. 5, (014); and m. 6 (013).

Example 5.14: Mod-7 K-net analysis of mm. 1-6

The networks of all six trichords share a common T_1 arrow. From the first trichord to the second, the index numbers of the inversional operators differ by 5; from the second to the third, the index numbers increase by 3; from the third to the fourth the index numbers increase by 2; and so on. Although they are not members of the same set class (in mod-7 or mod-12 space), the networks that describe the trichords are all positively isographic. That is to say, they share a common T-arrow, and the values of the I-arrows differ by the same number. Two of the networks—those in mm. 4 and 6—are strongly isographic. They have identical T- and I-values.

By interpreting the trichords as networks, we force the step classes into a registral order that may or may not accurately represent their deployment in the piece. This registral ordering produces independent lines of pitch classes that we can treat as voices. Example 5.15 presents the voice leadings that emerge from the registral ordering of the K-nets.

Example 5.15: Voice leadings derived from K-net analysis of mm. 1-6



In Example 5.15, I have placed the voices on three separate staves. The measures of the example correspond to mm. 1-6 of the piece—the same as Example 5.14. In

Example 5.15, the trichords are laid out vertically instead of horizontally, as they appear in the piece, and their placement corresponds to the location of their network node in Example 5.14.[41] The horizontal lines between the notes show the number of mod-7 steps traversed by the note (here I am using simple intervals, not compound). The vertical brackets between staves show the mod-7 distance between the pitch classes. The plus and minus signs indicate direction. Negative numbers on the vertical brackets indicate voice crossings. The distance between the pitches in the top and bottom staves can be calculated by adding the two bracket values. The example makes apparent the shared T_1-arrow between the bottom two voices: the dyads in the bottom and middle staves both move by the same horizontal distances.

All of the trichords in mm. 1-6 share a mod-7 interval class 1, as indicated by the T-arrows. The persistence of step-class interval 1 in all of these trichords provides a constant frame of reference against which we can judge the other pitch classes. Because all of these trichords share the same T-arrow, we can say that these trichords all belong to the 1-family of K-nets. The number indicates the value of the T-arrow. In mod-7 space, there are three possible trichordal K-families: the 1-family, the 2-family, and the 3-family. Other would-be families are inversions of the 1- 2- and 3-families, and will not be discussed here. We can divide the families further into K-classes, which are defined according to the values of their I-arrows.

Klumpenhouwer networks are a remarkably flexible analytical tool. I have shown here how they can be adapted for use in non-mod 12 environments. K-nets are valuable for uncovering voice leading in pieces where a standard transformational approach—one

[41] Conceivably, the Klumpenhouwer networks can be seen as order permutations of the trichords—a more horizontal phenomenon—instead of registral orderings.

in which the voices are activated by transposition or inversion—cannot adequately describe the voice leadings. Further investigation into Hauer's piece using K-nets might examine the piece for recursive networks—that is, networks of networks. Not only can pitches be related by transposition and inversional networks, but the networks themselves can be modeled in the same way. The flexibility and utility of K-nets in non-mod 12 environments provides an important new addition to our analytical toolkit.

The theory presented here could also be applied with little modification to the domain of rhythm.[42] Clough and Douthett 1991 provide a formula for generating ME sets of cardinality *d* with respect to some underlying collection *c*, whose cardinality is greater than that of *d*. Until now, the ME property has only been explored in the realm of pitch-class space. But the formula and ideas can be equally applied to the domains of rhythm and meter. I should emphasize that I am not equating rhythm and meter on one hand and pitch on the other, rather I am suggesting that we can use the same terminology to discuss both musical features. We often talk about intervals in time and intervals in pitch-class space: we are simply using two different units of distance measurement.

Example 5.16 contains a diagram with several levels of dots. Level c is at the bottom of the diagram, and it contains eight dots. Level b is above level c, and it contains four dots. Level a is above level b and it contains two dots.

Example 5.16: Three levels of equally spaced dots

level a:	•				•			
level b:	•		•		•		•	
level c:	•	•	•	•	•	•	•	•

[42] See London 2002 for an important discussion of dissimilarities between analysis of pitch and rhythm.

The dots at level c are equidistant, and we will call the distance between any two adjacent dots at level c some arbitrary distance, *x*. The distance from the first dot in level c to the third dot in level c is then 2*x*, and thus the distance between any two adjacent dots in level b is 2*x*. The distance between any two adjacent dots in level a is then 4*x*.

The dots at level b are maximally even with respect to the underlying collection c. That is, each dot in level b is as far away from any other dot in level b as it possibly can be, with the distances between the dots measured in terms of level c.

The diagram above can be translated into rhythmic values. The result looks similar to tables found in music fundamentals textbooks in introductory rhythm chapters. The translated table appears in Example 5.17.

Example 5.17: Translated table

level a:	𝅗𝅥				𝅗𝅥			
level b:	♩		♩		♩		♩	
level c:	♪	♪	♪	♪	♪	♪	♪	♪

The table now resembles a measure of 4/4 time. We can describe quarter notes as being maximally evenly distributed with respect to an underlying eighth-note pulse, and we can say that the half notes in level a are maximally evenly distributed with respect to the quarter notes in level b. More specifically, the *attack points* are maximally evenly distributed from level to level. That is to say that any variation in the duration of the quarter note (i.e., eighth note-eighth rest; dotted-eighth note–sixteenth rest; etc.) will not affect the maximal evenness property exhibited by the attack points. Because level b is ME with respect to level c, and level a is ME with respect to level b, we say that level a exhibits second-order maximal evenness with respect to level c.

Example 5.18 presents another way of expressing the same table. This time, each note is replaced with an integer. The first note of each level is labeled 0. At level c, attack points are labeled 0 through 7; at levels b and a, attack points are labeled according to their position relative to level c.

Example 5.18: The table reinterpreted again

level a:	0				4			
level b:	0		2		4		6	
level c:	0	1	2	3	4	5	6	7

We are now in a position to talk about rhythms as sets. $(0246)^8$ represents the attacks of the quarter notes at level b with respect to the underlying eighth notes at level c. Similarly, $(04)^8$ represents half notes with respect to the underlying eighth note collection. As we did with scales in Chapter 2, level b could be generalized into $(0123)^4$, and then the half notes could be written as the dyad $(02)^4$.

The rhythm at level b can be rotated and the ME property will not be affected. The rotation is shown in Example 5.19, which, for the moment, omits level a. Put another way, we can say that the rhythm $(0246)^8$ (level b in Example 5.18, above) is transposed by one eighth note, producing $(1357)^8$.

Example 5.19: Rotating level b

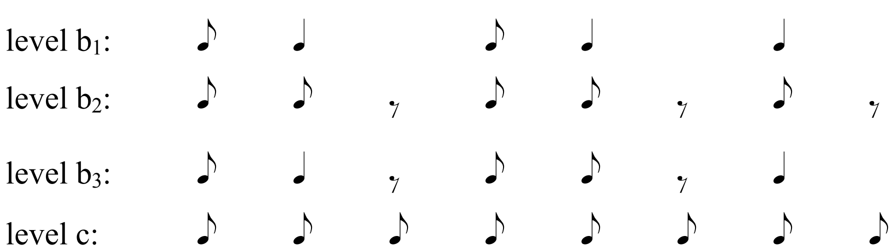
level b:
level c:

If we think in modular terms and tie the last eighth note around to the first eighth note, the rotated four-note collection is still ME with respect to the underlying eighth-note pulse. If we take the first attack of level c as initiating beat one of a 4/4 measure, then we can say that level b in Example 5.19 is syncopated.

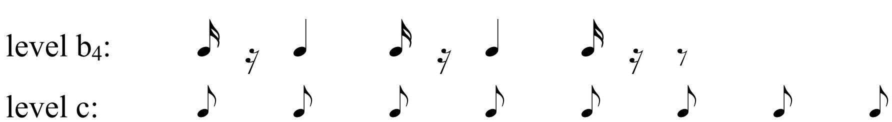
If we rotate the collection again, shifting level b to the left by one eighth note, the result will look similar to the level b given in Example 5.17 above—four quarter notes on the beats of a 4/4 measure. Because rotation yields only two distinct patterns (out of a possible four), we can say that level b is a mode of limited transposition, just as we saw with scales in Chapter 2. In set-theoretical terms, we can take the set $(0246)^8$ and transpose it by one eighth note, we get $(1357)^8$. If we transpose it again by T_1 (mod 8), we get $(2460)^8$.

Because we are measuring from attack points, there are several ways a maximally even rhythm can be expressed. The ME five-attack point set in a mod-8 space ($M_{8,5}$) is (01346). Example 5.20 shows several possible ways of expressing $M_{8,5}$.

Example 5.20: Expressing $M_{8,5}$ (01346) rhythmically



Example 5.21: $M_{8,5}$ expressed differently



Example 5.21 shows a slightly different way of expressing $M_{8,5}$. In Example 5.21, I have replaced some of the eighth notes with sixteenth notes followed sixteenth rests. The attack points are still ME with respect to the underlying eighth notes, but they suggest a level of subdivision below level c. We can insert a level d below the level c in Example 5.21, and this level d will consist of 16 sixteenth notes. We can then say that level c is $M_{16,8}$ and that level b_4 is still $M_{8,5}$. Level b_4 exhibits second-order maximal evenness with respect to level d.

An example from Messiaen's *Quator pour la fin du temps* illustrates how this system might be employed. Example 5.22 includes the first 10 measures of "Abîme des oiseaux." Below the example, I have created "rhythm spaces" for each of the measures. In each rhythm space, the bottom level comprises sixteenth notes, and the attack points are placed with respect to this subdivision. Notice that mm. 1 and 3 are identical, as are mm. 2 and 4. Measures 5 and 9 are trivially ME: a singleton in any level is ME with itself. The attack points in mm. 1 and 3 are not maximally evenly distributed. $M_{15,4}$ is $(037T)^{15}$, while Messiaen's rhythm is a member of set class $(0135)^{15}$ in mod-15 space. We can calculate deviation from the ME collection as we did in previous chapters: Messaien's rhythm deviates from ME by eight sixteenth notes—coincidentally the length of a half note.

Example 5.22: Messiaen, "Abîme des oiseaux," mm. 1-10

	0										10		12		14
mm. 1, 3	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	0		2		4	5		7		9					
mm. 2, 4	0	1	2	3	4	5	6	7	8	9					
m. 5	trivially ME														
	0		2												
m. 6	0	1	2	3	4	5	6	7							
	(0)		2		4		6		8		10		12	13	
m. 7	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	0		2												
m. 8	0	1	2	3	4	5									
m. 9	trivially ME														
	0						6						(12)		
m. 10	0	1	2	3	4	5	6	7	8	9	10	11	12	13	

The rhythms in mm. 2 and 4 are ME. In both cases, the attack points are members of set class $(013568)^{10}$ in a mod-10 rhythmic space. Messiaen's rhythm is $(013568)^{10}$ at T_4. This particular rhythm, $M_{10,6}$ is a mode of limited transposition because the GCF(10,6) $\neq$ 1. In m. 7, I enclosed 0 in parentheses to indicate that it is tied to the previous note: it is not a new attack point. If we include the 0 in *d*, the cardinality is 8, and we see that $M_{15,8}$ is $(013579ER)^{15}$ (the R stands for 13). Messiaen's rhythm is a transposition of this ME set class at T_{12}. Conceivably, we could eliminate the 0, counting it as part of the overall duration of the previous gesture, and calculate deviance from $M_{15,7}$. I choose not to do this here simply for the sake of convenience: this type of discrepancy would be an important avenue of study in future work on ME rhythm.

Measures 6 and 8 are also similar in that both contain the rhythmic dyad (02). In both cases, despite the differing cardinality of *c*, the ME dyad would be (03). In both measures, the rhythm Messiaen writes deviates from ME by one sixteenth note. Again, the calculations in m. 6 could be altered if we choose to include the value of the tied note in the overall length of the "measure."

The rhythm in m. 10 is not ME. It deviates from $M_{14,3}$ by five sixteenth notes. I have enclosed the 12 in parentheses here because an eighth rest falls on the twelfth sixteenth note of the measure. Again, we could argue for an analysis that compares Messiaen's rhythm to $M_{14,2}$.

As to what all of this might tell us about the music, I offer the following explanation. As shown in Chapter 2, composers tend to favor collections that are ME for any number of reasons. The problem with ME collections is that, more often than not, they tend to hinder pattern-matching tasks. The same is true in the domain of rhythm. In

Example 5.17, we saw evenly-spaced eighth notes with ME-spaced quarter notes atop them with ME-spaced half notes on top of them. The pattern is endlessly repeatable with any half note able to serve as the downbeat of some 4/4 measure. By breaking from the ME pattern we create a sense of rhythm within the meter. We can measure the goings-on in level b according to the unchanging eighth-note reference at level c—it is, in fact, these deviations in level b that suggest a level c at all.

Since perceptually we tend to favor ME structures, we might hear the first measure of the Messiaen as being rhythmically "dissonant" because of its great deviation from ME.[43] The long half note on the downbeat further hinders our attempts to find a metrical framework right away. Rather, we must wait until the very end of the measure, as m. 1 becomes m. 2, until we have a viable metrical framework. That m. 2 is ME provides a bit of relief, or rhythmic "consonance," especially because the eighth-eighth-sixteenth figure is repeated.

* * *

In the preceding work, I have demonstrated how reconsidering pitch-class space in post-tonal music can yield new insights into voice leading and hierarchical structure—insights that may have been obscured by a traditional twelve-note approach. It is my hope that the analytical system detailed here will encourage others to view this repertoire in a new light.

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[43] I am not using rhythmic consonance and dissonance in the same way that Krebs 1999 does.

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