STRING THEORY, COSMOLOGY AND BRANY GEOMETRY

by

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A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

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Abstract

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by

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Motivated by cosmological applications in this thesis we describe several string theory based models of the early Universe. The major property of these models is that they lead to inflationary-like expansion for early times. The interaction properties of fundamental strings, leading to the velocity dependent potentials are used to describe this accelerating expansion rate. Other types of extended objects such as fivebranes dual to fundamental strings are shown to lead to the similar cosmological implications. Our findings are consistent with recent astronomical observations of an accelerated expansion of the Universe and predict an asymptotically constant late time expansion rate.

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CONTENTS

1. INTRODUCTION

String theory and cosmology are among the most ambitious intellectual projects ever undertaken. String theory seeks to describe all the fundamental elements of nature and their interaction in the framework of a single theory. Cosmology, on the other hand, seeks to describe the origin and evolution of the universe, providing techniques for determining its structure and composition at present, in the past and in future.

The theories which underlie our current understanding of nature are quantum field theory and general relativity. As was shown in the late 1940s, when quantum electrodynamics was constructed, quantum field theory is the correct framework for quantizing electromagnetism [1], [2], [3]. In the late 1960s electromagnetism was unified with weak interactions when the theory of electroweak interactions was constructed [4], [5]. By the early 1970s it was understood that the strong interactions are also described by quantum field theory and the proof of renormalizability of gauge theories was obtained [6]. Electroweak theory, which is based on the spontaneous breaking of gauge symmetry group $SU(2) \otimes U(1)$ down to $U(1)$, and quantum chromodynamics based on group $SU(3)$, became generally accepted theories of electroweak and strong (nuclear) interactions respectively. The modern theory of elementary particles, which is based on $SU(3) \otimes SU(2) \otimes U(1)$ gauge symmetry spontaneously broken down to $SU(3) \otimes U(1)$

became the Standard Model of particle interactions. This theory was repeatedly confirmed and refined experimentally. In fact, all the accelerator experiments conducted up to the time of writing of this dissertation are in agreement with the Standard Model of elementary particles.

Despite its impressive success, the Standard Model is surely not complete, and cannot be considered as the final theory of elementary particles but rather as some approximation to it. First of all, there are at least 18 free parameters in the theory such as masses of some basic particles or several coupling constants of the theory. The values of these parameters are inserted into the theory based on experimental evidence and neither their values nor even the choice of the gauge groups are determined by the underlying theory itself. Gravity is automatically absent in the Standard Model, since Einstein's general relativity is a purely classical theory which is not renormalizable[1]. The Standard Model is in a certain sense unnatural: some of the parameters of the theory are much smaller than the others without any theoretical explanation why. It is these problems, rather then any experimental evidence that drive us to look for more fundamental theories beyond the Standard Model, incorporating it in certain limits.

String theory [7], [8], on the other hand, is the most serious candidate for quantum theory of gravity, and it is believed that it contains the Standard Model of elementary particles as one of its limits. In string theory all particles (i.e. zero-dimensional objects) of quantum field theory are replaced by the modes of oscillations of one-dimensional objects, or strings. Basically string theory is a theory of interacting vibrating strings:

[1] *It should be noted that in addition to nonrenormalizability, classical theory of gravity breaks down at the singularities of general relativity.*

open (with some finite length) or closed (vibrating loops), propagating in space-time. But because space-time itself is dynamical, its properties in turn are determined by propagating strings themselves [2]. String theory incorporates all the ideas of particle physics and general relativity in its mathematical framework, and does it with a structure more elegant and unified than in quantum field theory. Although lacking a full nonperturbative formulation, string theory has been highly developed and understood during the last 30 years, and needs a way to be confronted with experiments. Out of three generally accepted possibilities to experimentally test string theory:

1) low energy phenomenology experiments on the future colliders such as Large Hadron Collider (LHC) at CERN,

2) table-top experiments testing deviations from inverse square law of gravity at submillimeter distances, which follow from various large extra dimensions [9] and brane-world scenarios [47],

3) cosmological implications;

cosmology may turn out to be the most serious candidate. Since low energy phenomenology can shed the light on string theory only if supersymmetry is discovered at low energies and its properties allow to reconstruct its high energy behavior, or if string scale is low enough to be probed directly at LHC (which is extremely unlikely). As to the table-top experiments seeking deviations from inverse square law of gravity, they have been conducted for several years by now, and still resulted in no deviations from this law.

[2] The concept of a dynamical space-time (when its properties are determined by matter propagating through it) is not a new or unusual one and originates from Einstein's theory of gravity

In cosmology there are also many open questions. However, over the past several years cosmology has been enjoying one of the most productive periods yet. Over the past five years several cosmological parameters were pinned down with an accuracy which greatly exceeds that for the previous observations (some of these recent measurements are discussed in section 2.3). Modern cosmology is based on the highly successful standard hot Bing Bang cosmology (see for example [10], [25], [26], [57] and references therein) and may extend our understanding of the Universe to times as early as $10^{-32} sec$ when the largest structures in the Universe were still subatomic quantum fluctuations. These new observations (see e.g. [59], [60] and references therein) raised the status of cosmology to the science of precise experimental tests.

One of the most important questions addressed by cosmology is the evolution of the Universe. According to the standard cosmological model, our Universe started with the Big Bang, which is an initial singularity of the theory, when the entire Universe was localized in an extremely small region. Modern science is not in a position to address the state of the Universe at the moment of the Big Bang. There is even no notion of the 'moment' since our usual concepts of space and time, based on the classical theory of gravity cannot be applied to the description of Big Bang physics, where quantum effects were predominant[3] . According to the standard cosmology, the Big Bang was followed by *inflation* [12], [13], [14], which is an exponential expansion of the Universe. Inflation was followed by a much slower expansion which is satisfactorily described by the Friedmann-Robertson-Walker (FRW) cosmological model. This model describes

[3] It should be noted that some modern theories seek to address the state of the Universe before the Big Bang [11], others do not even require it as an initial state of the Universe

the evolution of the universe after the inflationary expansion ended and up to the present time.

Inflation offers solutions to the horizon, flatness and monopole problems. Moreover it could also explain the possible CMB anisotropies and therefore cosmological structure formation. If this period of exponential expansion was long enough it would flatten the Universe, solving the flatness problem. In addition this fast expansion would dilute the density of many massive objects (such as magnetic monopoles), making impossible the over-closure of the Universe by these objects and also explaining why we cannot observe them. At present there is no satisfactory explanation of inflation and of why and how inflationary period has ended.

There are several others fundamental questions in cosmology (some of them are discussed in more detail in Chapter 2): the initial singularity, the cosmological constant problem, realization of de Sitter or quintessential backgrounds, physical interpretation of dark matter and dark energy, topology and evolution of the Universe, and even such a basic question as *why do we live in three special dimensions?*. All of these questions still remain unanswered, and some of them are probably closely connected and intertwined.

To approach those questions we probably need to use insights of some fundamental theory such as string theory. This was realized as early as the 1980s when the first papers on string cosmology started to appear. Recently the subject became even more popular due to the fast progress in both observational cosmology and string theory.

For example in the Brandenberger-Vafa scenario [15] $T-$duality and winding modes have interesting implication for early universe cosmology, including the possible determination of critical dimension of spacetime. It is also possible that inflationary-like behavior can be obtained in a system consisting of a pair of $D-$branes, approaching each other [16]. It is likely that the solution to the problem of the *graceful exit* from inflation can be obtained based on theories of this type, since it is known from string theory that after a pair of $D-$branes gets to a critical distance, an open string mode becomes tachyonic, thus providing an instability which is needed to end inflation [17].

Tachyon condensation [18], [19], [20] can also have important implications for cosmology [21]. Moreover some recent string theory based models such as an *ekpyrotic* universe [22], [23], *cyclic* universe [24], *large extra dimensions* [9] and *brane-world* scenarios [47] can provide answers to some of the cosmological questions mentioned above.

To conclude, this is a very interesting time to think about cosmology from a string theory perspective. These two subjects: cosmology and string theory, complement each other in a number of ways. Cosmology needs an underlying theory to approach its basic questions (some of which are mentioned above and in Chapter 2). String theory, on the other hand has been highly developed and understood over the last two decades and needs to find ways to be confronted with experiments. Not only the study of cosmological implications of string theory can shed some light onto a better understanding of theory itself, but also cosmology may turn out to be the only way to experimentally probe string theory. Moreover cosmology is closely connected to particle physics and therefore cosmological insights can be very useful to string theory, struggling to find ways to relate itself to particle phenomenology.

The dissertation is organized as follows:

In Chapter 2, basic cosmological models are discussed together with open questions of modern cosmology and results of recent astronomical observations. As a result of these observations several cosmological parameters were determined with greatly increased precision.

Chapter 3 is based on the papers [32], [33]. In these papers some simple string theory based models of the early Universe were constructed, with the major property that they lead to inflationary-like expansion for early times. The interaction properties of fundamental strings, leading to the velocity dependent potentials, are used to describe this accelerating expansion rate. Some other types of extended objects such as fivebranes dual to fundamental strings are shown to lead to similar cosmological implications. These findings are consistent with recent astronomical observations [43] and predict an asymptotically constant late time expansion rate. In addition references to some reviews on stringy/brany cosmology are given in the beginning of this chapter, and possible interplay between cosmology and stringy/brany geometry is discussed at the end.

Chapter 4 is an extended version of [27]. In this chapter superfield equations of motion for $D = 4$ Dirichlet super-3-brane are obtained from the generalized action principle. The geometric equations containing fermionic superembedding conditions and constraints on the generalized field strength of Abelian gauge fields are separated from the proper dynamical equations and are found not to contain these equations among their consequences. This points to the existence of a superfield action for this type of brane, which is constructed at the end of the chapter. In addition we obtain a set of superfield equations and show that they involve an $SL(2,C)$ group valued superfield. The Cayley image of this superfield coincides (on the mass shell) with the field strength tensor of the world volume gauge field characteristic for the Dirichlet branes. The superfield description of the super-$D3$-brane obtained in this Chapter, which is a nonlinear (Born-Infeld) generalization of supersymmetric Yang-Mills theory is equivalent to the partial spontaneous breaking of $D=4$, $N=2$ supersymmetry down to $D=4$, $N=1$.

2. STANDARD COSMOLOGY IN MODERN INTERPRETATION

2.1 Friedmann-Robertson-Walker Cosmology

From all large scale structure astronomical observations it follows that the Universe appears to be homogeneous (the same at every point) and isotropic (the same in every direction) along the preferred set of spatial hypersurfaces. Of course homogeneity and isotropy are only approximate, but they become increasingly good approximations on larger length scales, allowing us to decribe spacetime on cosmological scales by the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right) \tag{2.1}$$

where the scale factor $a(t)$ describes the relative size of spacelike hypersurface at different times t, and the curvature parameter k is $+1$ for positively curved spacelike hypersurfaces, 0 for flat hypersurfaces, and -1 for negatively curved hypersurfaces. These possibilities are more formally known as "closed", "flat" and "open" universes respectively, in reference to the spatial topology, but there are problems with such designations. First, the flat and negatively-curved spaces may in fact be compact manifolds obtained by global identification of their noncompact relatives [54], [55],

[56]. Second, there is confusion in the use of "open/closed" in referring to spatial topology and the evolution of the universe. If such universes are dominated by matter or radiation, the negatively curved ones will expand forever and the positively curved ones will recollapse, but more general sources of energy/momentum will not respect this relationship.

General relativistic energy-momentum tensor for a *perfect fluid* is of the form

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p g_{\mu\nu} \tag{2.2}$$

where $u_\mu = \frac{dx_\mu}{d\tau}$ are 4−velocities, $d\tau = \frac{1}{c}\sqrt{\eta_{\mu\nu}dx^\mu dx^\nu}$ is a *proper* time, c is a speed of light (which, unless specifically indicated, is taken as $c = 1$ throughout the major portion of this dissertation), p is a pressure of the fluid and ρ is its total energy density (for example $\rho = \rho_{mat} + \rho_{rad}$ for the system consisting of matter and radiation). The flat Minkowski metric is taken as

$$\eta_{\mu\nu} = diag(1, -1, -1, -1) \tag{2.3}$$

$T_{\mu\nu}$ satisfies the vanishing covariant divergence condition

$$D_\mu T^{\mu\nu} = 0 \tag{2.4}$$

where

$$D_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma}T^{\sigma\nu} + \Gamma^\nu_{\rho\sigma}T^{\rho\sigma} \tag{2.5}$$

The full Einstein equations (including a cosmological constant term) are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.6}$$

If we want to solve the Einstein equations for $g_{\mu\nu}$ given by FRW metric (2.1) with $T_{\mu\nu}$ for perfect fluid (2.2) we just need to use

$$g_{\mu\nu} = diag\left(1, -\frac{a^2}{1-kr^2}, -a^2r^2, -a^2r^2\sin^2\theta\right) \tag{2.7}$$

which follows from (2.1) and compute metric connections (Christoffel symbols)

$$\Gamma^{\sigma}_{\mu\nu} = \frac{g^{\rho\sigma}}{2}\left(\frac{\partial g_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial g_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\rho}}\right), \tag{2.8}$$

curvature tensor

$$R^{\mu}{}_{\sigma\beta\alpha} = \frac{\partial\Gamma^{\mu}_{\sigma\alpha}}{\partial x^{\beta}} - \frac{\partial\Gamma^{\mu}_{\sigma\beta}}{\partial x^{\alpha}} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\sigma\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\sigma\beta} \tag{2.9}$$

and the corresponding Ricci tensor

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} \tag{2.10}$$

The 00 component of the Einstein equations (2.6) (which is sometimes called *first Friedmann* equation) is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{tot} \tag{2.11}$$

where ρ_{tot} is the total energy density of matter, radiation and vacuum

$$\rho_{tot} = \rho_{mat} + \rho_{rad} + \rho_{vac} \tag{2.12}$$

and the vacuum energy density is defined by

$$\rho_{vac} \equiv \frac{\Lambda}{8\pi G} \tag{2.13}$$

The 11, 22 and 33 components of Eq. (2.6) give the same equation (sometimes called *second Friedmann* equation)

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi Gp \tag{2.14}$$

Equation (2.14) is linearly related to (2.11) and the relation that comes from the vanishing of the covariant divergence of the energy-momentum tensor (2.4) which leads to

$$\dot{p}a^3 = \frac{d}{dt}\left[a^3(\rho + p)\right] \tag{2.15}$$

Therefore only two out of three equations (2.11), (2.14) and (2.15) are linearly independent. As two independent equations Eq. (2.11) is usually chosen together with (2.15).

Eq. (2.15) can be written as

$$\frac{d}{dt}\left(\rho a^3\right) = -p\frac{d}{dt}a^3 \tag{2.16}$$

which has a simple physical interpretation: the rate of change of total energy in a volume element of size $V = a^3$ is equal to $-pdV$.

Usually density ρ and pressure p can be related by the *equation of state*

$$p = \alpha\rho, \quad \alpha = const \tag{2.17}$$

For radiation $p = \frac{\rho}{3}$, for non-relativistic matter ρ is dominated by the rest mass energy mc^2 which is much larger than the pressure (proportional to the velocity $v \ll c$). Thus, to a good approximation nonrelativistic matter is pressureless and $\alpha = 0$. This approximation is sometimes referred to as a *dust Universe.*

For a vacuum (as explained in Appendix A) $\alpha = -1$ and

$$p = -\rho \tag{2.18}$$

Substituting Eq. (2.17) to (2.16) and solving for the density ρ as a function of scale factor a we obtain

$$\rho = const \cdot a^{-3(1+\alpha)}. \tag{2.19}$$

Therefore for a radiation dominated Universe (i.e. the first few hundred thousand years after the Big Bang)

$$\rho_{rad} \propto \frac{1}{a^4} \tag{2.20}$$

and for a matter dominated Universe

$$\rho_{mat} \propto \frac{1}{a^3}. \tag{2.21}$$

The physical interpretation of Eqs (2.21) and (2.20) is straightforward: since baryonic matter is not spontaneously created or destroyed, its density at any time t is diluted proportionally to the volume factor $a^3(t)$ as the scale factor $a(t)$ increases. For photons, there is an additional factor of $a(t)$ since the energy of each photon gets redshifted due to the expansion.

For a vacuum

$$\rho \propto const \tag{2.22}$$

thus the cosmological constant gives a contribution to the energy density which is independent of the scale factor $a(t)$

2.2 Time Evolution of the Scale Factor $a(t)$

Subtracting Eq. (2.11) from Eq. (2.14) we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.23}$$

from which it is easy to see that if matter alone drove the Universe's expansion the expansion rate would be decelerating (because for matter $p \simeq 0$ and $\rho > 0$), that contradicts recent astronomical observations (which are discussed in the next chapter). According to these observations the Universe expands with an accelerating rate. However, if vacuum energy plays an important role then the expansion may accelerate (because $\rho + 3p < 0$ leads to $\ddot{a} > 0$).

Taking equation of state (2.17) into account, Eq. (2.23) can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+3\alpha)\rho \tag{2.24}$$

Eq. (2.24) can be used to determine the time evolution of the scale factor for both matter domination ($\alpha = 0$) and radiation domination ($\alpha = \frac{1}{3}$) for early times t. Indeed by making an ansatz

$$a \propto t^{\beta}, \quad \beta = const, \tag{2.25}$$

substituting it into (2.24) and using $\rho(a)$ given by (2.19) one obtains

$$\beta(\beta - 1) \propto t^{-3(1+\alpha)\beta+2} \tag{2.26}$$

but *l.h.s.* of (2.26) is time independent, thus $\beta = \frac{2}{3(1+\alpha)}$ and from (2.25) one obtains time evolution of the scale factor

$$a(t) \propto t^{\frac{2}{3(1+\alpha)}}. \tag{2.27}$$

That is for matter domination

$$a(t) \propto t^{2/3} \tag{2.28}$$

and for radiation domination

$$a(t) \propto \sqrt{t}. \tag{2.29}$$

The time evolution of the scale factor for a vacuum dominated Universe can also be easily obtained. Indeed the Friedmann equation (2.11) for a vacuum takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} \tag{2.30}$$

where we took (2.13) and (2.12) into account as well as the fact that for a vacuum $\rho_{mat} = \rho_{rad} = 0$.

For $k = 0$ (2.30) gives

$$a(t) = e^{Ht} \tag{2.31}$$

where the time dependent Hubble's parameter is given by

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} = \sqrt{\frac{\Lambda}{3}} \tag{2.32}$$

Eq. (2.31) shows that scale factor $a(t)$ increases exponentially with time. This behavior is called *inflation* and is caused by the negative pressure associated with the cosmological constant term.

It is easy to see that the curvature term $\frac{k}{a^2}$ in (2.30) can be neglected even for $k = 1$ and $k = -1$, because the exponential growth of the scale factor makes the Universe look flat.

Therefore we obtained the time evolution of the scale factor $a(t)$ for a matter, radiation and vacuum dominated Universe (Eqs. (2.28), (2.29) and (2.31) respectively).

From (2.11) and (2.23) it is easy to obtain the time evolution of the scale factor for an arbitrary (not necessary early) time t [57].

For radiation domination

$$a(t) = \sqrt{c'}\sqrt{1 - \left(1 - \frac{t}{\sqrt{c'}}\right)^2}, \qquad for \quad k = 1,$$

$$a(t) = (4c')^{1/4}\sqrt{t}, \qquad for \quad k = 0,$$

$$a(t) = \sqrt{c'}\sqrt{\left(1 + \frac{t}{\sqrt{c'}}\right)^2 - 1}, \qquad for \quad k = -1$$

where $c' = \frac{8\pi\rho a^4}{3}$.

For matter domination one can only solve (2.11) and (2.23) in parametric form when $k = \pm 1$ and explicitly when $k = 0$

$$a(\eta) = \frac{1}{2}C(1 - \cos\eta), \quad t(\eta) = \frac{1}{2}C(\eta - \sin\eta), \quad for \quad k = 1$$

$$a(\eta) = \frac{1}{2}C(\cosh\eta - 1), \quad t(\eta) = \frac{1}{2}C(\sinh\eta - \eta), \quad for \quad k = -1$$

$$a(t) = \left(\frac{9C}{4}\right)^{1/3} t^{2/3}, \quad for \quad k = 0$$

where $C = \frac{8\pi\rho a^3}{3}$.

The more general solution for an arbitrary mixture of matter, radiation and vacuum energy cannot be given in closed form, but certain interesting cases may be studied. In particular it may be interesting to study late-time behavior of the scale factor $a(t)$. One immediate simplification is that we know that radiation plays no major role today and due to continuing redshift of the cosmic microwave background (CMB) radiation will be even less important in future.

Thus $\rho_{rad} = 0$ and (2.12) becomes

$$\rho_{tot} = \rho_{mat} + \rho_{vac} \tag{2.33}$$

First let us consider $\Lambda = 0$ (i.e. $\rho_{vac} = 0$) case. From (2.21) it then follows that

$$\rho_{mat}(t) = \rho_0 \frac{a_0^3}{a(t)^3} \tag{2.34}$$

where $a_0 \equiv a(t_0)$, $\rho_0 \equiv \rho(t_0)$ are the present values of the scale factor and matter density respectively. Substituting (2.34) into (2.11) we obtain

$$\dot{a}^2 = \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a(t)} - k \tag{2.35}$$

If we suppose (unrealistically) that $\rho_0 = 0$ we get that $k = -1$. This is the so called *Milne* model. In this case the solutions of (2.35) are

$$a_{Milne}(t) = \pm t \tag{2.36}$$

The '+' sign in (2.36) corresponds to a linearly expanding Universe; the '-' can be interpreted as a contraction, since then $\dot{a}(t) < 0$.

For $\rho_0 > 0$ and early time $a(t) \propto t^{2/3}$ (see (2.28)). The solutions for an arbitrary t will depend on the curvature term k, for example for $k = 0$ Eq. (2.35) can be easily solved to give (2.28) once again

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3} \tag{2.37}$$

this is the so called *Einstein-de Sitter* model.

If $\rho_0 > 0$ and $k = -1$, $\dot{a}(t)^2$ is always positive, which means an ever-increasing $a(t)$. That means that the matter term in (2.35) will eventually become negligible compared to the curvature term. Therefore for late times this model will expand similar to the Milne model with $a(t) \propto t$.

If $k = +1$ and $\rho_0 > 0$ we can see that $\ddot{a} \leq 0$ for any value of $a(t)$. That means that the Universe will start to contract when a becomes equal to some *critical* value $a = a_{crit}$. Therefore this model leads to an *oscillating* Universe, where the Big Bang is followed first by an expansion and then by contraction to a 'Big Crunch'.

If $\Lambda \neq 0$ we should replace (2.35) by

$$\dot{a}^2(t) = \frac{8\pi G \rho_0 a_0^3}{3} \frac{1}{a(t)} - k + \frac{\Lambda a^2(t)}{3} \tag{2.38}$$

An important property of the cosmological constant which can be seen from (2.38) is that it is completely negligible for small $a(t)$ (early time), but eventually it dominates matter and curvature for large $a(t)$ (late times).

This leads to the major puzzle: *why is it that we live in such a special epoch when* $\rho_{vac} = \frac{\Lambda}{8\pi G}$ *is of the same order as the present value of matter density* $\rho_0 = \rho_{mat}(t_0)$*?* This and other results of modern cosmology are discussed in more detail in the next section.

If $\Lambda < 0$, it follows from Eq. (2.38) that $a(t)$ cannot become arbitrary large. It is also easy to show that in this case we have an oscillating Universe once again. This conclusion does not depend on k, although the value of a_{crit} does.

For $\Lambda \geq 0$ for both $k = 0$ and $k = -1$ we see from (2.38) that the Universe enters a period of an exponential expansion similar to (2.31). For $\Lambda \geq 0$ and $k = +1$ it is possible to fine-tune Λ in such a way that $\dot{a} = 0$ and $\ddot{a} = 0$ simultaneously. This corresponds to a static solution. This simple observation in fact motivated Einstein to introduce the cosmological constant term in the first place. This last model of course is not realistic since it does not agree with observations (in particular predicts no redshifts).

2.3 New Cosmology

This section is intended as a brief overview of recent advances in cosmology, here we also discuss refined values of some cosmological parameters.

Modern cosmology incorporates the highly successful standard hot big-bang cosmology, and may extend our understanding of the Universe to times as early as $10^{-32} sec$ after the Big Bang, when the largest structures in the Universe were still subatomic quantum fluctuations.

Modern cosmology is characterized by:

a) Flat, critical density accelerating Universe;

b) Early period of rapid expansion (inflation);

c) Density inhomogeneities produced from quantum fluctuations during inflation;

d) Composition: 2/3 dark energy; 1/3 dark matter; 1/200 bright stars;

e) Matter content: $(29\pm4)\%$ cold dark matter; $(4\pm1)\%$ baryons; $\geq 0.3\%$ neutrinos;

f) Present temperature $T_0 = 2.725 \pm 0.001K$;

g) Present age $t_0 = (14 \pm 1) \cdot 10^9 yrs$;

h) Present value of Hubble's parameter $H_0 = (72 \pm 7)\frac{km/s}{Mpc}$.

The first Friedmann equation (2.11) can be rewritten in the form

$$\frac{k}{H^2a^2} + 1 = \frac{\rho_{tot}}{\rho_{crit}}, \quad \rho_{crit} \equiv \frac{3H^2}{8\pi G} \tag{2.39}$$

or

$$1 = \Omega_m + \Omega_k + \Omega_\Lambda \tag{2.40}$$

$$\Omega_m \equiv \frac{8\pi G\rho_{mat}}{3H^2}, \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H^2}, \quad \Omega_k \equiv -\frac{k}{H^2a^2}.$$

For any other energy component such as *quintessence*, a term Ω_Q should be added to the *r.h.s.* of (2.40). The position of the first acoustic peak in the multiple power spectrum of the anisotropy of the CMB provides techniques for determining the global curvature of the Universe (see [58]). The result is that the curvature radius of the Universe is at least 50 times greater than the Hubble radius, therefore the Universe appears to be spatially flat and $\Omega_k \simeq 0$.

Therefore Eq. (2.40) can be written as $1 = \Omega_m + \Omega_\Lambda$[1]. There are several observational techniques for determining Ω_m/Ω_Λ (see [59], [60] and references therein). The interesting fact about these observations is that they seem to be inconsistent with each other, except when

$$\Omega_m \simeq 1/3, \quad \Omega_\Lambda \simeq 2/3 \tag{2.41}$$

and that is the major reason why (2.41) are generally accepted values at present.

There are also several techniques to determine the constituents of matter density Ω_m itself [60]. CMB and cluster of galaxies method allow a determination of the ratio of the total matter density to that in baryons alone, with the result that $\Omega_B = (4 \pm 1)\%$ (baryonic), $\Omega_{DM} = (29 \pm 4)\%$ (dark matter) and $\Omega_m \simeq \Omega_B + \Omega_{DM}$. Dark matter presumably consists of a new form of matter, with the axion and neutralino as the leading candidates, but there is no satisfactory quantitative explanation as to the nature of dark matter at present.

Another interesting observation is that most of the baryons are optically dark. That

[1] Strictly speaking Ω_Λ in (2.40) does not have to be associated with vacuum energy. Usually it is denoted by Ω_X and is called *energy density of dark energy*. Therefore Ω_Λ can be replaced by Ω_X in some of the formulas of this section. We will see however that vacuum energy is probably the best candidate for the role of a dark energy, and this fact is reflected in our notations.

is because $\Omega_B \simeq 0.04$, but for bright stars $\Omega_{stars} = 0.005$. In clusters, the dark baryons have been identified: they exist as hot, x-ray emitting gas. Elsewhere, the dark baryons have not been identified. Some qualitative assumptions are that dark baryons are likely to exist as hot gas associated with galaxies, but this gas has not yet been detected.

From (2.41) we see that dark matter accounts only for 1/3 of the critical density of the Universe. Another 2/3 of Universe's density goes under the name of *dark energy* Ω_X (see the footnote on the previous page).

Dark energy may be thought of as a causative agent for the current epoch of accelerated expansion of the Universe [43]. As was already mentioned, according to the standard FRW cosmology Universe accelerates if the *r.h.s.* of Eq. (2.23) is positive, that means $\rho + 3p < 0$. Therefore any causative agent of accelerated expansion must have negative pressure with a magnitude comparable to its energy density. In addition it should not show its presence in galaxies and clusters of galaxies (otherwise it would have probably been detected). Thus dark energy must be relatively smoothly distributed. Therefore dark energy should have the following defining properties:

1) it emits/absorbs no light,

2) it has large negative pressure of order $p_X \sim -\rho_X$,

3) it does not cluster significantly with matter on scales at least as large as cluster of galaxies.

Since its pressure is comparable in magnitude to its energy density, it is more "energy-like" than "matter-like" (because as we mentioned in the previous section, for matter $p \ll \rho$).

Due to property 2) of dark energy, vacuum energy with the defining property $p_\Lambda = -\rho_\Lambda$ (see Appendix A) is the most serious candidate for the role of dark energy at present. There is a serious problem behind this reasoning though, which is sometimes called *the cosmological constant problem.*

If we assume that vacuum energy is indeed a dark energy then its present density ρ_Λ can be expressed through the present value of Hubble's parameter H_0, then $\Omega_X \equiv \Omega_\Lambda = \frac{\Lambda}{3H_0^2}$. Taking into account that $\Omega_X \simeq 2/3 \sim 1$ we see that $\Lambda \sim 3H_0^2$. Then the present density of vacuum energy $\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \sim \frac{H_0^2}{G}$. Substituting the present value of Hubble's parameter H_0 and Newton's constant G taken in natural units ($\hbar = c = 1$) this becomes

$$\rho_\Lambda \sim 10^{-46} GeV^4 \tag{2.42}$$

Physically, this should correspond to m^4, where m is some fundamental mass scale in nature. The only fundamental mass scale we know is the mass scale of gravity or *Planck mass* $m_{Pl} = 1.2 \cdot 10^{19} GeV$. Therefore, for dimensional reasons one could expect a vacuum energy density of order $m_{Pl}^4 \sim 10^{76} GeV^4$ to emerge from an eventual quantum theory of gravity. This leads to an unprecedented inconsistency since then the ratio of observed vacuum energy to expected one is of the order of

$$\frac{(\rho_{vac})_{obs}}{(\rho_{vac})_{expected}} \sim \frac{10^{-46} GeV^4}{10^{76} GeV^4} \sim 10^{-122}$$

This is one of the most serious problems of modern cosmology.

Another important problem which we already mentioned in the previous chapter is that Ω_X and Ω_m evolve at different rates. The period of time when Ω_m is of the same order as Ω_X is an extremely brief one in the cosmological time scale. Therefore the fundamental question is: *why do we just happen to live at the time when dark matter and dark energy have comparable densities?*. This question still remains unanswered.

3. STRINGY AND BRANY COSMOLOGY

3.1 Fundamental Strings

This chapter is based on papers [32], [33], where some simple string theory models of the early Universe were proposed. These models are based on interaction properties of fundamental strings and some other types of extended objects such as fivebranes dual to fundamental strings. A detailed discussion of the current status of string/brane cosmology is given in the reviews [28], [29], [30], [44], [31], [11].

In a recent paper [49], it was shown that the velocity-dependent forces between parallel fundamental strings in D spacetime dimensions, with certain initial conditions, lead to an expanding universe in $D-1$ dimensions. These findings were consistent with recent observations [43] of an accelerating universe, and predict an asymptotically constant late time expansion rate.

We start with the action $S = I_D + S_2$, where

$$I_D = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{1}{12}H_3^2\right) \tag{3.1}$$

is the D-dimensional string low-energy effective spacetime action and

$$S_2 = -\frac{\mu}{2}\int d^2\zeta\left(\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^M\partial_\nu X^N g_{MN} + \epsilon^{\mu\nu}\partial_\mu X^M\partial_\nu X^N B_{MN}\right) \tag{3.2}$$

is the two-dimensional worldsheet sigma-model source action. g_{MN}, B_{MN} and ϕ are the spacetime sigma-model metric, antisymmetric tensor and dilaton, respectively, while $\gamma_{\mu\nu}$ is the worldsheet metric. $H_3 = dB_2$ and μ is the string tension. The fundamental string solution to the combined action, representing stationary macroscopic strings parallel to the x^1 direction, is given by [41]

$$ds^2 = h^{-1}\left(-dt^2 + (dx^1)^2\right) + \delta_{ij}dx^i dx^j, \tag{3.3}$$
$$e^{-2\phi} = h = 1 + \frac{k_n}{r^n}, \qquad B_{01} = -h^{-1}$$

where $n = D-4$, $r^2 = x^i x_i$ and the indices i and j run through the $D-2$-dimensional space transverse to the string. The constant $k_n = 2\kappa^2 T_1/n\Omega_{n+1}$, where $T_1 = \mu$ is the tension of the string, equal to its mass/length, and Ω_{n+1} is the volume of S^{n+1}, the $n+1$-dimensional unit sphere.

This solution can be extended to a multi-static string solution owing to the existence of a zero-force condition. This condition in turn arises from the cancellation between the attractive gravitational and dilatational forces of exchange with the repulsive antisymmetric field exchange, and is based on the existence of supersymmetry and the saturation of a BPS bound [45].

It was subsequently shown that, in addition to the zero static force, the leading order ($O(v^2)$) velocity-dependent forces cancel for moving strings as well [42] (see also [40]). This result too is associated with the existence of higher supersymmetry [46]. Following [46], it is straightforward to verify that the four-point amplitude corresponding to the scattering of two such fundamental string states approaches zero in the small velocity limit. This is identical to the result found for the $a = \sqrt{3}$ black holes, which also preserve half of the total spacetime symmetries, the maximum for such black hole, string or p-brane solutions.

The Lagrangian for a test fundamental string moving in the background of a parallel source string is then given by [49],[42]

$$\mathcal{L} = -mh^{-1}\left(\sqrt{1 - h\dot{x}^2} - 1\right), \tag{3.4}$$

where m is the mass of the string, $\dot{x}^2 = \dot{x}^i\dot{x}_i$ and the "$\cdot$" represents a time derivative. It was shown in [49] that the velocity-dependent force following from this Lagrangian is repulsive whenever the strings are moving away from each other, and this leads to a further separation of the strings. Since this type of interaction occurs for any two strings, if we start with any number of close, parallel strings initially moving apart in the transverse space, they will continue to do so indefinitely and will fuel an expanding universe in the $D-2$-dimensional transverse space and therefore in the $D-1$-dimensional spacetime orthogonal to the strings. For example, five-dimensional fundamental strings lead to an expanding universe in $D = 4$ spacetime dimensions.

The toy model presented in [49] consisted of a large number of fundamental strings initially very close to each other. Each pair of such strings interacts as above, so that an initial outward propagation of the strings tends to further push them apart in the transverse space. In a mean-field approximation, the effective force on each string was approximated by that of a single, very large source fundamental string whose Noether charge k is equal to the total charge of all of the strings in the D-dimensional space. The distance r between the test string and the source string in this model then represents the approximate average position of the strings, and hence the size of the universe.

The time dependence of r at both early and late times was determined [49] for this expanding model.

In this chapter, we expand on the results of [49] to obtain exact solutions for the radial position as a function of time for the mean-field approximation for $D = 5$ and $D = 6$ for the case of zero angular momentum. We also consider a spherically symmetric toy model and obtain similar results in the very early and late time limits as well, to give further support to the mean-field approximation results. In addition we investigate whether the feature of an accelerating universe holds for different types of p-branes. We first verify that the string result holds for parallel fivebranes moving apart and then consider strings moving in a fivebrane background. We find an explicit solution in the mean-field approximation for an expanding, accelerating universe consisting of strings moving in the fivebrane background.

3.2 Generalized p-Branes

Before outlining these solutions, it is interesting to note that the Lagrangian (3.4) also arises whenever we consider the motion of a maximally supersymmetric p-brane in the background of a parallel, identical p-brane. For example, (3.4) is the same Lagrangian one obtains for a test fivebrane moving in the background of a parallel source fivebrane or for a D0-brane moving in the background of a source D0-brane (see, e.g., [50]). This can be seen immediately either from supersymmetry, or through dimensional reduction [51]. For the case of the fivebrane, for example, one replaces the two form B_{01} with a six-form A_{012345} and proceeds in the same manner to obtain (3.4), where now m

represents the mass of the test fivebrane moving in the background of a parallel source fivebrane.

The relevant dimension is the number of transverse dimensions, given by $n = D - p - 3$, since the harmonic function $h = 1 + k_n/r^n$ depends only on n. Here

$$k_n = \frac{2\kappa_D^2 T_p}{n\Omega_{n+1}}, \tag{3.5}$$

where T_p, the tension of the p-brane, is equal to its mass/p-volume [40]. Compactifying $q \leq p$ dimensions, we can relate the D-dimensional Newton's constant G_D to the $D-q$-dimensional Newton's constant G_{D-q} via [40]

$$G_D = \kappa_D^2 = \kappa_{D-q}^2 V_q = G_{D-q} V_q, \tag{3.6}$$

where V_q is the compactified q-volume. Since $T_p = m/V_p$, it follows that

$$k_n = \frac{2\kappa_{D-q}^2 T_{p-q}}{n\Omega_{n+1}}, \tag{3.7}$$

which is just (3.5) for a $p-q$-brane. In particular, for $q = p - 1$, we recover the string formula. For $q = p$, we obtain the formula for D0-branes. As long as the longitudinal directions of the branes are held parallel, the dynamics are independent of p, the dimension of the branes, and depend only on the number of transverse directions.

In what follows, we will consider parallel strings for simplicity, keeping in mind that we could equally well consider, say, D0-branes in one less dimension.

3.3 Mean-Field Approximation

For the case in which we replace the total repulsive force on a single string by an effective, large string at the origin, it was shown in [49] that[1]

$$\dot{r}^2 = \frac{\rho(h\rho+2)}{(h\rho+1)^2}, \tag{3.8}$$

where E is the constant total energy of the string, and where $\rho = E/m$ is the ratio of the energy to the rest energy of the test string. We have set the angular momentum $l = 0$. Following, Chebyshev's Theorem[2], only the two cases of $D = 5$ $(n = 1)$ and $D = 6$ $(n = 2)$ can be integrated exactly. A straightforward integration for $n = 1$ yields

$$\left(\frac{\rho+3}{\rho+1}\right)\ln\left(\sqrt{\frac{r}{a}}+\sqrt{\frac{r}{a}+1}\right)+\sqrt{\frac{r}{a}}\sqrt{\frac{r}{a}+1} = \sqrt{\frac{(\rho+2)^3}{\rho\,(\rho+1)^2}}\left(\frac{t-t_0}{k_1}\right), \tag{3.9}$$

where $a = \frac{\rho k_1}{\rho+2}$. For small r (or early time t),

$$r \simeq \left(\frac{\rho+2}{\rho+1}\right)^2 t^2/k_1 \tag{3.10}$$

while for large r (or late time), $r \simeq \sqrt{\rho(\rho+2)/(\rho+1)^2}t$, both in agreement with the findings of [49][3]. In this three-dimensional transverse space, $r(t)$ represents the mean

[1] This equation can be easily obtained from the Hamiltonian (3.13) for the system of test string moving in the source string background, when solved with respect to $\dot{x}^2$, since this Hamiltonian is nothing but the conserved energy E of this two-string system.

[2] In order to integrate (3.8) we need to evaluate integrals of the form $\int x^m(a+bx^n)^p dx$ (so called binomial differentials), where m, n, p are any rational numbers and a, b-any constants. Chebyshev proved that integrals of this form can be expressed through algebraic, logarithmic and inverse circular functions in only three cases:
1) p is an integer (positive, negative or zero)
2) $\frac{m+1}{n}$ is an integer
3) $\frac{m+1}{n} + p$ is an integer.

[3] The constant energy E does *not* include the constant rest energy m. It is straightforward to show that, in the late time limit, $\gamma = \left(1-\beta^2\right)^{-1/2} = \rho + 1$, so that the interaction energy is ultimately transformed into kinetic energy ($= (\gamma - 1)\,m$).

size of the universe in this toy model. Restoring factors of the speed of light c in (3.5), it follows that $k_1 = \frac{2G_5M}{Lc^2\Omega_2} = \frac{G_4M}{2\pi c^2}$, where L is the length of each string and M is the mass of the source string, representing the effective total mass of the universe in the mean-field approximation.

In [49], it was claimed that $r << k_1$ and $r >> k_1$ corresponded to early and late times (relative to the current epoch), respectively, in the expansion of the universe as implied in (3.9). Let us verify this assumption using estimates of cosmological parameters obtained in [52]. The current matter density ρ is very close to the critical density $\rho_0 = 3H_0^2/8\pi G_4$, and the age of the universe $t_0 = H_0^{-1}$, where H_0 is the Hubble's constant at present time. Ignoring numerical factors of $O(1)$, we take the current size of the universe $r_0 \sim ct_0$ and its mass as $M \sim \rho r_0^3$. It is then straightforward to show that k_1 is at most an order of magnitude less than r_0. It follows that $r << k_1$ corresponds to much earlier times than present $t << k_1/c$ and $r >> k_1$ corresponds to much later times $t >> k_1/c$. The model for $n = 1$ has the defect that the ratio of relative velocities to relative positions is not immediately a spatial constant, unless the spatial dimensions of the universe are restricted to two (see next section).

For the more interesting case of $n = 2$, a straightforward integration yields

$$\sqrt{r^2+a} + \sqrt{a}\left(\frac{\rho+2}{\rho+1}\right)\ln\left(\frac{r+\sqrt{r^2+a}-\sqrt{a}}{r+\sqrt{r^2+a}+\sqrt{a}}\right) = \sqrt{\frac{\rho(\rho+2)}{(\rho+1)^2}}(t-t_0), \tag{3.11}$$

where again $a = \frac{\rho k_2}{\rho+2}$ and t_0 is a constant. For small r,

$$r \simeq r_0 \exp\frac{t}{\sqrt{k_2}}, \tag{3.12}$$

while for large r we again find $r \simeq \sqrt{\rho(\rho+2)/(\rho+1)^2}t$, both again in agreement with the findings of [49]. In this four-dimensional transverse space, the three-dimensional universe may be regarded as an expanding spherical shell with radius $r(t)$. A subtle point here arises as to the connection between G_5 and G_4. In going from a five-dimensional universe to a four-dimensional one whose constant time slices consist of the expanding three-sphere, it follows that $G_5 \sim G_4 r(t)$. So G_5 and G_4 cannot both be constant. For constant G_5, $k_2 = G_5 M/2\pi^2 c^2$ (from (3.5)) is also constant, but the four-dimensional Newton's law changes with time as the universe expands. The alternative picture is to demand a constant G_4, but then allow for changing k_2, so that the mean-field model in this case should be thought of as an approximation to a cosmological p-brane solution with k_2 a function of time. In either case, within the limits of these toy models, a straightforward calculation shows that $k_2 \sim \bar{r}^2$, where $\bar{r}$ is the current size of the universe. It again follows that the early ($r << \sqrt{k_2}$) and late ($r >> \sqrt{k_2}$) time limits are valid, as in the $n = 1$ case. Especially interesting features of the $n = 2$ case, other than allowing for a spatially constant Hubble's constant, are the inflationary expansion at early times and the asymptotically constant expansion rate for late times. This latter feature is generic to these string/brane models, since the velocity-dependent forces vanish at asymptotically large distances.

3.4 Spherical Shell Model

Now consider the following model of a string-seeded universe for both $n = 1$ and $n = 2$. N parallel, identical strings, with $N >> 1$, are all located at the same distance R from the center of the transverse $D - 2$ dimensional space and move with the same, purely radial, velocity $\vec{v} = \dot{R}\hat{R}$ outward from the center. We would again like to determine

$R(t)$ and to compare our findings with the mean-field approximation. Before doing so, we note that such a model is consistent with the cosmological observation of a Hubble's constant. In the $n = 1$ ($n = 2$) case, the spatial universe consists of an expanding 2-sphere (3-sphere). It is straightforward to show that the relative position is given by $r_{21} = |\vec{r}_{21}| = |\vec{r}_2 - \vec{r}_1| = 2R\sin\theta/2$, where θ is the angle between the position vectors $\vec{r}_1$ and $\vec{r}_2$. Similarly, the relative speed between two strings $v_{21} = |\vec{v}_{21}| = |\vec{v}_2 - \vec{v}_1| = 2v\sin\theta/2$, where $\vec{v}_1$ and $\vec{v}_2$ are the velocities of the two strings. It follows that $v_{21}/r_{21} = v/R$ is a constant over each sphere (or for a given time slice), representing the Hubble's constant for this model.

The Hamiltonian for the system of test string moving in the source string background can be easily obtained from the Lagrangian (3.4) of this system[4]

$$H = \frac{m}{h}\left(\frac{1}{\sqrt{1 - h\dot{x}^2}} - 1\right), \tag{3.13}$$

For $D = 5$ ($D = 6$) this model is equivalent to a system of particles on the surface of a 2-sphere (3-sphere) with two-particle energy of interaction. This interaction energy is just the difference between the conserved total energy of the 2-particle system given by (3.13) and the kinetic energy of the test string, since the source string is assumed to be stationary in the mean-field approximation. Note also that in the mean-field approximation (see also [49]), the energy (3.13) was taken to be a constant, which led to a solution for the motion of a single test string in the background of a much

[4] It is convenient to use this velocity dependent form of the Hamiltonian. In the same way, one can easily obtain the conventional form $H = H(r,p)$ by using the expression for the momenta $p_i = \frac{m\dot{x}_i}{\sqrt{1-h\dot{x}^2}}$. Then $H = \frac{m}{h}\left(\sqrt{1 + h\frac{p^2}{m^2}} - 1\right)$.

larger source string, which approximated the aggregate effect of the velocity-dependent forces of all the other strings. In the shell model, the interaction energies obtained from (3.13) are not individually constant but must be added into a total energy for the system, which is then set equal to a constant total energy. The center of mass of the system remains at the center of the sphere. The interaction energy of one string in the background of another is then given by

$$E_{int12} = \frac{m}{h_{12}}\left(\frac{1}{\sqrt{1-h_{12}\dot{r}_{12}^2}}-1\right) - \frac{m\dot{r}_{12}^2}{2}, \tag{3.14}$$

where $h_{12} = 1 + \frac{k_n}{r_{12}^n}$ and r_{12} is the relative position of the strings[5].

For $D = 5$ we assume for simplicity that the 1^{st} string is located at the north pole of the 2-sphere with radius R. Then, by symmetry, the energy of interaction between the 1^{st} string and dN strings which are located inside the belt with azimuthal angles between θ and $\theta + d\theta$ is given by

$$dE_{int12} = dN\left[\frac{m}{h_{12}}\left(\frac{1}{\sqrt{1-h_{12}\dot{r}_{12}^2}}-1\right) - \frac{m\dot{r}_{12}^2}{2}\right], \tag{3.15}$$

where $dN = \frac{N}{2}\sin\frac{\theta}{2}d\theta$, $r_{12} = 2R\sin\frac{\theta}{2}$, $\dot{r}_{12} = 2\dot{R}\sin\frac{\theta}{2}$ and $h_{12} = 1 + \frac{k_1}{r_{12}}$

In order to obtain the energy of interaction between the 1^{st} string and all other strings we need to integrate (3.15) over θ from $\theta = 0$ to $\theta = \pi$. This can be done explicitly, but leads to a rather complicated, and not especially illuminating, expression. In order to make a connection with the early-time mean-field approximation, we first

[5] Strictly speaking, we should use the more cumbersome relativistic form of the kinetic energy. However, since most of the subsequent analysis involves the early time expansion, the non-relativistic approximation used for this model is valid. Furthermore, as we shall see later, the result for late times is not affected by making this simplification either.

make the assumption that $\frac{k_1}{r_{12}} >> 1$. This means that the distance between any two strings $r_{12} << k_1$ and our model can describe the system during the time when this condition holds, i.e. early times.

Thus we replace $h_{12} \simeq \frac{k_1}{r_{12}}$ in (3.15) . The integration over θ is easy to perform, and we obtain for the energy of interaction between the 1^{st} string and all the other strings

$$E_{1int} = -\frac{4mNR}{k_1}\left\{\frac{2}{15a^3}\left[\sqrt{1-a}\,(3a^2+4a+8)-8\right]+\frac{a}{8}+\frac{1}{3}\right\} \tag{3.16}$$

where

$$a = \frac{2\dot{R}^2 k_1}{R}. \tag{3.17}$$

Note that a is restricted to be in the domain $0 \leq a \leq 1$ for this model to be valid.

By symmetry, the total energy of interaction of N strings is

$$E_{int} = NE_{1int} \tag{3.18}$$

and the total conserved energy of the system is

$$NE_1 = E = E_{int} + N\frac{m\dot{R}^2}{2}, \tag{3.19}$$

where E_1 is the total energy of a single string.

First assume $a << 1$ and expand the total interaction energy of the system (3.19) in powers of a

$$E_{int} = \frac{3mR}{10k_1}N^2\left[a^2 + O(a^3)\right]. \tag{3.20}$$

Thus up to 2^{nd} order in a, Eq. (3.19) takes the form

$$\rho = \frac{3NR}{10k_1}a^2 + \frac{R}{4k_1}a \tag{3.21}$$

where $\rho = E_1/m = E/mN$ is again the ratio of the total energy (not including the rest energy) of each string to its rest energy.

Solving the quadratic equation in (3.21) for a, and using $R << k_1$, it easily follows that the linear term in a in (3.21) may be dropped. It then follows that up to an $O(1)$ numerical factor, $a \simeq \sqrt{\frac{k_1\rho}{NR}} << 1$. Since ρ is at least of $O(1)$, it follows from $a << 1$ that $R >> \frac{k_1}{N}$. Dropping the 2^{nd} term in the *r.h.s.* of (3.21), solving (3.21) with respect to $\dot{R}$ and integrating we obtain

$$R \simeq \left(\frac{\rho}{kN}\right)^{1/3} t^{4/3}. \tag{3.22}$$

Another domain of interest is when $a \simeq 1$. This corresponds to an even earlier time, since from (3.22) it follows that $a \sim t^{-2/3}$, so that an earlier time corresponds to larger a. For $a \longrightarrow 1$, the solution $R = R(t)$ is easily obtained from the definition of a (3.17)

$$R \simeq \frac{t^2}{k_1} \tag{3.23}$$

and is valid for $R \sim k_1/N$, which can be seen from (3.19) with E_{int} given by (3.18) and E_{1int} by (3.16) with $a \simeq 1$. Note that this quadratic expansion in time is consistent with the early-time approximation of the mean-field model.

For large R ($R >> k_1$), we can assume that $h = 1 + k_1/R \simeq 1$, so that $\partial H/\partial R = 0$.

From (3.15), (3.19) it then follows that the constant energy E_1 depends only on $\dot{R}$ and N [6] so that $\dot{R}$ depends only on N and ρ. So the radial velocity $\dot{R}$ is constant. Thus $R \propto t$ for large R, again in agreement with the mean-field approximation.

As mentioned above, for $D = 6$, the transverse motion of parallel strings is equivalent to the motion of particles with two particle energy of interaction given by (3.14) with

$$h_{12} = 1 + \frac{k}{r_{12}^2} \tag{3.24}$$

where $r_{12} = 2R \sin\frac{\chi}{2}$, $\dot{r}_{12} = 2\dot{R}\sin\frac{\chi}{2}$ and χ is the azimuthal angle, where we assume that the 1^{st} string is located at the north pole of the 3-sphere.

If we assume (as before) that the $N >> 1$ strings are distributed homogeneously on the surface of the 3-sphere, then the number of strings located inside the belt with azimuthal angles between χ and $\chi + d\chi$ is

$$dN = \frac{2N}{\pi} \sin^2 \chi d\chi \tag{3.25}$$

and the energy of interaction between the 1^{st} string (at the north pole) and dN strings inside the belt is given by the same expression (3.15) with h_{12} given by (3.24) and dN by (3.25). Assuming (as for $D = 5$) that $\frac{k_2}{r_{12}^2} >> 1$ (i.e. $R << \sqrt{k_2}$) and replacing $h_{12} \simeq \frac{k_2}{r_{12}^2}$ we obtain

$$dE_{int} = \frac{2Nm}{\pi} \sin^2 \chi d\chi \left[\frac{4R^2 \sin^2 \frac{\chi}{2}}{k_2} \left(\frac{1}{\sqrt{1 - \frac{k_2 \dot{R}^2}{R^2}}} - 1 \right) - 2\dot{R}^2 \sin^2 \frac{\chi}{2} \right] \tag{3.26}$$

[6] This last statement is obviously also valid for the relativistic expression for the kinetic energy.

integrating this expression over χ from 0 to π we obtain the energy of interaction between the 1^{st} string and all other strings

$$E_{int} = \frac{2mNR^2}{k_2}\left(\frac{1}{\sqrt{1-\frac{k_2\dot{R}^2}{R^2}}} - 1\right) - mN\dot{R}^2 \tag{3.27}$$

and conservation of energy condition can be written in the same form (3.19) as before with E_{int} given by (3.27) or

$$\frac{1}{\sqrt{1-b}} - 1 - \frac{b}{2} = \frac{\rho k_2}{2NR^2} \tag{3.28}$$

where

$$b = \frac{k_2\dot{R}^2}{R^2}, \tag{3.29}$$

where $0 < b < 1$. For $R << \sqrt{k_2/N}$, it follows from (3.28) that $1 - b << 1$. Alternatively, expanding (3.28) in powers of $1-b$ and keeping only the first nonvanishing term we easily obtain that condition $b \simeq 1$ (or $1 - b << 1$) leads to $R << \sqrt{k_2/N}$. From the definition of b (3.29) we see that $R \simeq R_0 \exp \frac{t}{\sqrt{k_2}}$ in this case, again corresponding to the exponentially inflationary expansion in the $D = 6$ mean-field model.

On the other hand, the condition $\frac{\rho k_2}{2NR^2} << 1$ or $R >> \sqrt{\frac{k_2}{N}}$ (again ρ is at least of $O(1)$) leads to $b << 1$, which can also be easily seen from (3.28). As before, expanding (3.28) in powers of b, keeping the lowest nonvanishing term and then integrating (in order to get $R = R(t)$) we obtain

$$R \simeq \sqrt{\frac{\rho}{12Nk_2}}\, t^2 \tag{3.30}$$

For large R (i.e. $R >> \sqrt{k_2}$) we can drop the 1 in Eq (3.24). In this case we obviously obtain the same result as for $D = 5$ (and generally any D): expansion with constant radial speed, once again in agreement with the mean-field limit. We emphasize again that the nonrelativistic approximation for the kinetic energy in both cases does not affect the results for either early or late times.

An interesting possibility in this case is that the moving strings in $D = 6$ lead to an expanding five-dimensional universe, in which an effective four-dimensional brane universe resides, following [47]. The asymptotic late time expansion rate is also intriguing, and may represent a testable prediction for this type of model.

One possible advantage to the type of asymptotically flat universe shown in these models is that, in contrast to a de Sitter universe, S-matrices would be well-defined [53]. At the same time, these models allow for an accelerating universe without assuming the existence of a cosmological constant. One can regard the changing acceleration as corresponding to an effective cosmological "constant" which varies with time. For example, the $D = 6$ ($n = 2$) model which has exponential growth in the early universe, has a constant effective cosmological constant, $\Lambda \sim k_2^{-1}$ ([44]) which, however, is due entirely to the velocity-dependent forces between the strings/branes. At very late times, the effective cosmological constant is zero. It is then a straightforward but tedious exercise to determine the exact time dependence of the effective cosmological constant in these models.

Further investigations of this type of model are clearly merited. More complicated and far more realistic models, possibly involving different species of branes could be considered. Furthermore, it would be interesting to go beyond the analytic, classical results obtained above, using a possible combination of numerical computations, quantum string effects and nonequilibrium thermodynamics. Nevertheless, it is likely that the leading order behavior of the type of accelerating string/brane universe considered above is well-described in the classical approximation. Needless to say, these results await further verification and a better understanding of the underlying many-body interactions.

3.5 Mean-Field Approximation for String/Fivebrane and Fivebrane/Fivebrane Systems.

In [32] it was shown that some simple models of a string-seeded universe in $D = 5, 6$ have the same early time behavior (3.10),(3.12) as corresponding source/test string systems. This suggests that the mean-field approximation of [49] provides a valid description of the early time expansion rate for these systems. Late time expansion rates for these models can also be obtained from conservation of energy and may represent a testable prediction (see [32] for further discussions).

The fundamental string in $D = 10$ is a solution of 3−form version of $D = 10, N = 1$ supergravity. The dual 7−form version of this theory [36] corresponds to supergravity coupled to the fivebrane σ−model [34], [40]. The action for the supergravity fields $(g_{MN}, A_{MNPQRS}, \phi)$ is now

$$I_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\cdot 7!}e^{\phi}K_7^2\right) \tag{3.31}$$

where $K_7 = dA_6$. I_{10} is the same action as I_D in (3.1) for $D = 10$ provided H_3 and K_7 are related via the duality transformation

$$K_7 = {}^*H_3 e^{-\phi}. \tag{3.32}$$

The fivebrane sigma-model action is given by [34], [35]

$$S_6 = -T_6 \int d^6\zeta \left(\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^M \partial_\nu X^N g_{MN} e^{-\phi/6} - 2\sqrt{-\gamma} + \right. \tag{3.33}$$
$$\left. \frac{1}{6!}\epsilon^{\mu_1\mu_2\ldots\mu_6}\partial_{\mu_1}X^{M_1}\partial_{\mu_2}X^{M_2}\partial_{\mu_3}X^{M_3}\partial_{\mu_4}X^{M_4}\partial_{\mu_5}X^{M_5}\partial_{\mu_6}X^{M_6}A_{M_1M_2\ldots M_6}\right)$$

where T_6 is the fivebrane tension. The fivebrane sigma model metric is related to the canonical metric via $g^f_{MN} = e^{-\phi/6} g_{MN}$.

The fundamental fivebrane solution to the equations of motion of the combined action $S_{fivebrane} = I_{10} + S_6$ is given by

$$ds^2 = e^{-\phi/2}\left(-dt^2 + (dx^1)^2 + \ldots + (dx^5)^2\right) + e^{3\phi/2}\delta_{mn}dx^m dx^n, \tag{3.34}$$

$$e^{2\phi} = 1 + \frac{\tilde{k}_2}{r^2}, \qquad A_{012345} = -e^{-2\phi}$$

where $\tilde{k}_2 = \frac{\kappa_{10}^2 T_6}{\Omega_3}$, $m, n = 6, 7, 8, 9$ and r is the radial coordinate in the four-dimensional space transverse to the six-dimensional worldvolume of the fivebrane.

The Lagrangian for a test fivebrane moving in the background of a parallel source-fivebrane [39] can be easily obtained from (3.33), (3.34) and in term of the proper time of the test fivebrane takes the form

$$\mathcal{L}_6 = -me^{-2\phi}\left(\sqrt{1 - e^{2\phi}\dot{r}^2} - 1\right), \tag{3.35}$$

where m is the mass of the test fivebrane. Thus the Lagrangian (3.35) for this two-fivebrane system in $D = 10$ has exactly the same form as the Lagrangian for the test string moving in the source string background in $D = 6$. This fact is a consequence of string/fivebrane duality in $D = 10$ or even string/string duality in $D = 6$, once the fivebrane is reduced to a dual string in $D = 6$ [40]. Note that we could have obtained the identical result as above starting directly with the dual string in $D = 6$. So the early time dependence $r = r(t)$ (where r is the radial coordinate for the four-dimensional space transverse to the six-dimensional worldvolume) has the form $r \simeq r_0 \exp \frac{t}{\sqrt{\tilde{k}_2}}$ where r_0 is the initial distance between the fivebranes in this four-dimensional space. In the mean-field approximation, the position of the test-fivebrane represents the average fivebrane-fivebrane distance and hence the scale size of the universe (see [49], [32]).

We now wish to investigate whether the above scenario holds for velocity-dependent forces between different branes. The most natural case to consider is that of a brane propagating in the background of a dual brane. In particular, we consider a test string moving in the background of a fivebrane. Suppose the fivebrane is oriented along $x^\alpha = \zeta^\alpha$ ($\alpha = 1, 2, \ldots, 5$). We assume that the test string lies either parallel or antiparallel to one of the fivebrane directions, say x^1. Viewed as a background for string propagation, the fivebrane is a nonsingular solution of the spacetime action I_{10} alone, without the need for a source term (since no singularity is present in the string frame). The metric and dilaton are then given by (3.34), with the three-form given by the duality transformation (3.32). Since, from (3.34), the only nonvanishing components of K are of the form $K_{012345m}$ where the directions $m = 6, 7, 8, 9$ are transverse to the fivebrane, by dualizing we obtain that the only nonzero components of $H_3 = dB_2$ are $H_{pqs}(r)$ where again $p, q, s = 6, 7, 8, 9$. Thus the only nonvanishing components of B_{MN} occur when $M, N = 6, 7, 8, 9$. It then follows that the Wess-Zumino-Witten term in the action (3.2) for the test string in the fivebrane background $\epsilon^{\mu\nu}\partial_\mu X^M \partial_\nu X^N B_{MN}$ vanishes. Replacing the fields of the fivebrane from (3.34) into (3.2) we obtain the Lagrangian [39]

$$\mathcal{L} = -m\sqrt{1 - e^{2\phi}\dot{r}^2} \tag{3.36}$$

where m is the mass of the test string and $e^{2\phi}$ is given by (3.34). The Hamiltonian of this test string/source fivebrane system also does not depend on time and thus represents the conserved energy

$$E \equiv H = \frac{m}{\sqrt{1 - e^{2\phi}\dot{r}^2}}. \tag{3.37}$$

Integrating (3.37) over time we obtain

$$\sqrt{r^2 + \tilde{k}_2} - \sqrt{\tilde{k}_2}\ln\frac{\sqrt{r^2 + \tilde{k}_2} + \sqrt{\tilde{k}_2}}{r} = \sqrt{\alpha}t + const, \tag{3.38}$$

where $\alpha = 1 - \frac{m^2}{E^2}$.

For early times ($r << \sqrt{\tilde{k}_2}$) we obtain from (3.38)

$$r \simeq r_0 \exp\left(\sqrt{\frac{\alpha}{\tilde{k}_2}}t\right) \tag{3.39}$$

i.e. the same type of early time dependence as for parallel strings in $D = 6$. Once again, for late times we obtain $r \propto t$ as a general feature of this type of model [49], [32]. We would expect this feature to persist for any dual pair of p-brane, in particular for fivebranes propagating in the background of a string.

Since similar results seem to hold for branes propagating in either identical or dual brane backgrounds, this strongly suggests that the type of velocity-dependent forces present in the above cases are generic to branes and would lead to an accelerating expanding universe for various different types of branes simultaneously propagating in the same background. This is also supported by the compositeness feature of brane solutions, which allows for the construction of arbitrary brane solution from fundamental brane building blocks (of which the fundamental string and fivebrane are examples) [37], [38]. Another interesting question is whether these scenarios hold in the context of general relativity, independent of string theory. A possible extension of these results is to consider different orientations of the various branes, in which case the zero-force condition no longer holds in the static limit. Finally, it is worthwhile to go beyond the mean-field approximation and investigate the many-body problem directly (see [32] for some simplified models), also taking into account quantum interactions.

Thus as was explained in the Introduction and in this Chapter, string theory based cosmological models may turn out to be the most suitable framework in an attempt to quantitatively explain many of those fundamental cosmological puzzles partially mentioned in Chapter 2. Not only is the string theory based approach to cosmology an attractive one aesthetically, as an example of a unified description of large and small scale physics in the framework of a single theory, but it also provides powerful mathematical techniques many of which are highly developed and understood. These are the major reasons for the recent upsurge of interest in stringy/brany cosmology and that is why further investigations on brany geometry and the possible interplay between string theory and cosmology are definitely worth further investigation.

In addition to that, some types of extended objects (such as p-branes, D-branes, etc.) have been proven to be of paramount importance to string theory itself. In light of possible applications of string theory to cosmology it is also important to better understand the geometry and dynamics of these objects, revealing their mathematical characteristics. In Chapter 4 we present a detailed description of one particular type of these extended objects, space-filling Dirichlet super-3-brane, based on the so called *superembedding* approach.

4. SUPEREMBEDDING APPROACH

The superembedding approach is the geometrical approach for the description of the dynamics of superparticles, superstrings, super–p–branes, Dirichlet branes and the M5-brane, which is based on a generalization of surface theory for the description of the embedding of supersurfaces into target superspaces. The application of surface theory in string theory is rather natural, since a string is a one–dimensional relativistic object which sweeps a two–dimensional surface (worldsheet) when it propagates in a (target) space–time. The dynamics of the string completely determines the geometrical properties of the worldsheet, describing its embedding into the target space, and vice versa, specifying geometrical properties of the embedding of a surface into a target space, one can, in principle, get the full information about the dynamics of a string whose worldsheet is associated with this surface.

Being manifestly supersymmetric in both, the superworldvolume of the brane and the target superspace, superembedding approach unifies the Neveu–Schwarz–Ramond and the Green–Schwarz approach for the description of string/brane actions and provides the fermionic κ–symmetry of the Green–Schwarz–type superbrane actions with a clear geometrical meaning of standard worldvolume local supersymmetry.

The dynamics of superbranes is encoded in a generic superembedding condition.

Depending on the superbrane and the target–space dimension, the superembedding condition produces either only off–shell constraints (as in the case of $N = 1$ superparticles and $N = 1$ superstrings), or also results in the full set of the superbrane equations of motion (as, for example, in the case of the M–theory branes). In the first case worldvolume superspace actions for the superbranes can be constructed, while in the second only components of generalized superfield actions are known.

In this Chapter we describe the properties of the $D = 4$ Dirichlet super-3-brane using the superembedding approach.

The four–dimensional field theoretical model which possessed space–time supersymmetry was proposed in [61]. The model was constructed in such a way that (as we assume to be realized in nature) supersymmetry is broken spontaneously with a "neutrino" playing the role of the associated fermionic Goldstone particle. This was the example of a mechanism of spontaneous breaking of global supersymmetry which was generalized in [62] to the super–Higgs effect in the supergravity model with spontaneously broken local supersymmetry [63].

Later on it has been realized [64] that models of this type describe supersymmetric effective field theories exhibiting partial supersymmetry breaking on the worldvolumes of branes. This subject has recently faced a significant revival of interest due to the extensive study of various aspects of brane physics (see e.g. [65, 66, 67, 68, 69, 70]).

The aim of this part of the dissertation is to describe the peculiarities of the superembedding description [71, 72, 73, 74, 75, 76] of a space filling D(irichlet)3–brane propagating in an $N = 2$, $D = 4$ superspace and to establish the relationship of this covariant geometrical formulation with a Goldstone superfield formulation of $N = 1$, $D = 4$ supersymmetric Dirac–Born–Infeld theory [65, 69, 70] based on methods of nonlinear realizations of supersymmetry.

We will show that the superembedding conditions and worldvolume gauge field constraints do not put D3–brane dynamics in $D = 4$ on the mass shell, in contrast, for example, to the case of a D3–brane [74, 77] and a D9–brane [78] in type IIB D=10 supergravity. A geometrical consequence of these conditions is the Grassmann analyticity of both the $N = 1$, $D = 4$ superworldvolume and $N = 2$, $D = 4$ target superspace. This is an extension to the D3–brane of the relationship between the superembedding condition and Grassmann analyticity properties of supermanifolds observed previously for $N = 1$, $D = 4$ superparticle and an $N = 1$, $D = 4$ superstring [71, 79, 80, 81].

The fact that the superembedding conditions are off–shell constraints will allow us to construct a worldvolume superfield action for the space filling D3–brane which we briefly discuss at the end of this chapter.

4.1 Notations, Constraints and Superembedding Conditions

We use the formalism of two-dimensional Weyl spinors both in $N = 1$, $D = 4$ superworldvolume and in $N = 2$, $D = 4$ target superspace. Since the D3–brane in $D = 4$ is a space filling brane we can always gauge fix local Lorentz rotations in the worldvolume in such a way that they coincide with Lorentz transformations in the target superspace. So there is no need to distinguish between the vector and spinor indices corresponding to the tangent spaces of superworldvolume and target superspace. Small letters of the Greek and Latin alphabet stand, respectively for spinor and vector indices, e.g. $\alpha, \dot{\alpha} = 1, 2$; $a, b = 0, 1, 2, 3$. Capital Latin letters denote both the spinor and vector indices. The curved target superspace indices denoted by the letters from the second half of the alphabets and underlined to indicate that the worldvolume and target space superdiffeomorphism groups are a priori independent.

We denote the coordinate of the flat $D = 4,\ N = 2$ superspace as

$$Z^{\underline{M}} = (X^{\underline{m}}, \Theta^{I\underline{\mu}}, \bar{\Theta}^{I\underline{\dot{\mu}}}) \tag{4.1}$$

$$\underline{m} = 0, 1, 2, 3, \quad \underline{\mu}, \underline{\dot{\mu}} = 1, 2, \quad I = 1, 2$$

The supervielbein form of this superspace

$$E^A(Z) = (E^a, E^{I\alpha}, \bar{E}^{I\dot{\alpha}}) \tag{4.2}$$

$$a = 0, 1, 2, 3, \quad \alpha, \dot{\alpha} = 1, 2, \quad I = 1, 2$$

can be identified with the coordinate supervielbein $(\Pi^m, d\Theta^{I\mu}, d\bar{\Theta}^{I\dot{\mu}})$

$$E^a \equiv \Pi^m \delta_m^a \tag{4.3}$$

$$\Pi^m = dX^m - id\Theta^{I\alpha}\sigma^m_{\alpha\dot{\alpha}}\bar{\Theta}^{I\dot{\alpha}} + i\Theta^{I\alpha}\sigma^m_{\alpha\dot{\alpha}}d\bar{\Theta}^{I\dot{\alpha}} \tag{4.4}$$

and

$$E^{I\alpha} = d\Theta^{I\mu}\delta_\mu^\alpha, \quad \bar{E}^{I\dot{\alpha}} = d\bar{\Theta}^{I\dot{\mu}}\delta_{\dot{\mu}}^{\dot{\alpha}} \tag{4.5}$$

Thus the expression for the torsion forms are

$$d\Pi^m = -2id\Theta^{I\alpha}\sigma^m_{\alpha\dot{\alpha}}d\bar{\Theta}^{I\dot{\alpha}} \Rightarrow T^a = -2i(E^{1\alpha}\wedge\bar{E}^{1\dot{\alpha}} + E^{2\alpha}\wedge\bar{E}^{2\dot{\alpha}})\sigma^a_{\alpha\dot{\alpha}} \tag{4.6}$$

$$T^{I\alpha} = dE^{I\alpha} = 0, \quad \bar{T}^{I\dot{\alpha}} = d\bar{E}^{I\dot{\alpha}} = 0$$

The basic volume form written in terms of the vielbeins is denoted by

$$E^{\wedge 4} \equiv \frac{1}{4!}\epsilon_{a_1 \dots a_4} E^{a_1} \wedge \cdots \wedge E^{a_4} \equiv \frac{1}{4!}\epsilon_{a_1 \dots a_4} \Pi^{a_1} \wedge \cdots \wedge \Pi^{a_4} \tag{4.7}$$

In further calculation it is convenient to use the notations

$$E_a^{\wedge 3} \equiv \frac{1}{3!}\epsilon_{a a_1 a_2 a_3} E^{a_1} \wedge E^{a_2} \wedge E^{a_3}, \quad E_{ab}^{\wedge 2} \equiv \frac{1}{4}\epsilon_{a b a_1 a_2} E^{a_1} \wedge E^{a_2} \tag{4.8}$$

and their 'multiplication table'

$$E_a^{\wedge 3} \wedge E^b = -\delta_a^b E^{\wedge 4}, \quad E_{ab}^{\wedge 2} \wedge E^c = -\delta_{[a}^c E_{b]}^{\wedge 3} \tag{4.9}$$

The world volume superspace of the super-$D3$-brane is defined by

$$z^M = \{\xi^m, \eta^\mu, \bar{\eta}^{\dot{\mu}}\}, \quad m = 0,1,2,3, \quad \mu, \dot{\mu} = 1,2 \tag{4.10}$$

and the intrinsic supervielbein forms of the world volume are

$$e^A = (e^a, e^\alpha, \bar{e}^{\dot{\alpha}}) \equiv dz^M e_M^A, \qquad a = 0,1,2,3, \quad \alpha, \dot{\alpha} = 1,2 \tag{4.11}$$

The pull-backs of target space supervielbein forms onto the world volume superspace are denoted by the same symbols $E^a, E^{I\alpha}, \bar{E}^{I\dot{\alpha}}$ and for instance fermionic forms can be decomposed as follows:

$$E^{I\alpha} = e^\beta E_\beta^{I\alpha} + \bar{e}^{\dot{\beta}} C_{\dot{\beta}}^{I\alpha} + e^b E_b^{I\alpha}, \qquad \bar{E}^{I\dot{\alpha}} = \bar{e}^{\dot{\beta}} \bar{E}_{\dot{\beta}}^{I\dot{\alpha}} + e^\beta \bar{C}_\beta^{I\dot{\alpha}} + e^b \bar{E}_b^{I\dot{\alpha}} \tag{4.12}$$

In the case of the space filling superbranes the basic superembedding condition reads [78] that the superembedding of the brane superworldvolume into a target superspace is carried out in such a way that (using local Lorentz transformations on the worldvolume and in target superspace) it is always possible to choose the vector component e^a of a worldvolume supervielbein to coincide with the pullback of the vector component E^a of a target space supervielbein. Namely,

$$e^a = E^a(Z(z)). \tag{4.13}$$

Note that by imposing (4.13) we have identified the group of local Lorentz rotations in the tangent space of the superworldvolume with that of the target space Lorentz group, while the worldvolume superdiffeomorphisms still remain an independent group of transformations, and can be used (as we will do at the final stage of our analysis), to impose a physical gauge

$$\xi^m = X^m, \quad \eta^\mu = \Theta^{1\mu}, \quad \bar{\eta}^{\dot{\mu}} = \bar{\Theta}^{1\dot{\mu}}. \tag{4.14}$$

In this gauge the theory remains manifestly invariant under $N = 1$, $D = 4$ supersymmetry associated with the supertranslations along η^μ and $\bar{\eta}^{\dot{\mu}}$, while the second target space supersymmetry associated with the Θ^2 translations is realized nonlinearly in the transformation law of $\Theta^{2\mu}(z)$, which implies its spontaneous breaking,

$$\delta\Theta^2 = \epsilon^2 + i(\epsilon^2\sigma^a\bar{\Theta}^2 + \Theta^2\sigma^a\bar{\epsilon}^2)\partial_a\Theta^2. \tag{4.15}$$

Thus $\Theta^{2\mu}(z)$ is the Volkov–Akulov Goldstone fermion associated with the half of $N = 2$, $D = 4$ supersymmetry spontaneously broken by the D3–brane.

As a consequence of (4.13) the pullback of E^a along the Grassmann directions (4.11) of the superworldvolume is zero. If the target superspace is flat (which is the case we are interested in)

$$E^a = dX^a - id\Theta^{I\alpha}\sigma^a_{\alpha\dot{\alpha}}\bar{\Theta}^{I\dot{\alpha}} + i\Theta^{I\alpha}\sigma^a_{\alpha\dot{\alpha}}d\bar{\Theta}^{I\dot{\alpha}}, \tag{4.16}$$

and eq. (4.13) implies

$$E_{\alpha}{}^{a} = \mathcal{D}_{\alpha}X^{a} - i\mathcal{D}_{\alpha}\Theta^{I}\sigma^{a}\bar{\Theta}^{I} - i\Theta^{I}\sigma^{a}\mathcal{D}_{\alpha}\bar{\Theta}^{I} = 0,$$

$$E_{\dot{\alpha}}{}^{a} = \bar{\mathcal{D}}_{\dot{\alpha}}X^{a} - i\mathcal{D}_{\alpha}\Theta^{I}\sigma^{m}\bar{\Theta}^{I} - i\Theta^{I}\sigma^{a}\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\Theta}^{I} = 0, \tag{4.17}$$

where $\mathcal{D}_{\alpha}$ and $\bar{\mathcal{D}}_{\dot{\alpha}}$ are worldvolume covariant derivatives.

4.2 Generalized Action for $D = 4$ Super-D3-Brane

The generalized action for the super-D3-brane in $D = 4$, $N = 2$ superspace has the form

$$S = \int_{\mathcal{M}^4} \mathcal{L} = \int_{\mathcal{M}^4} (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{WZ}) \tag{4.18}$$

where

$$\mathcal{L}_0 = E^{\wedge 4}\sqrt{-det(\eta + F)_{ab}} \tag{4.19}$$

$$\mathcal{L}_1 = Q_2 \wedge (\mathcal{F} - F) \equiv Q_2 \wedge (dA - B_2 - \frac{1}{2}E^{b} \wedge E^{a}F_{ab}) \tag{4.20}$$

and $\mathcal{M}^4$ is an arbitrary bosonic surface in $D = 4$, $N = 1$ world volume superspace (4.10) $\mathcal{M}^4 : \{\xi^m, \eta^\mu = \eta^\mu(\xi) \quad \bar{\eta}^{\dot{\mu}} = \bar{\eta}^{\dot{\mu}}(\xi)\}$ which is specified by $\eta^\mu = \eta^\mu(\xi), \bar{\eta}^{\dot{\mu}} = \bar{\eta}^{\dot{\mu}}(\xi)$

In (4.19), (4.20) $F_{ab} = -F_{ba}$ is an auxiliary antisymmetric tensor field, $A = d\xi^m A_m(\xi)$ is the gauge field of $D3$-brane, Q_2 is 2-form Lagrange multiplier, B_2 is defined as

$$B_2 = 2i\Pi^a \wedge (d\Theta^1\sigma_a\bar{\Theta}^1 - d\Theta^2\sigma_a\bar{\Theta}^2) + 4(d\Theta^1\sigma^a\bar{\Theta}^1) \wedge (d\Theta^2\sigma_a\bar{\Theta}^2) \tag{4.21}$$

is the NS-NS (or Kalb-Ramond) gauge field, which play the role of the Wess-Zumino term in $D = 4$ superstring action. Its field strength has the form

$$H_3 \equiv dB_2 = 2iE^a \wedge (E^{1\alpha} \wedge \bar{E}^{1\dot{\alpha}} - E^{2\alpha} \wedge \bar{E}^{2\dot{\alpha}})\sigma_{a\alpha\dot{\alpha}} \tag{4.22}$$

and

$$\mathcal{L}_{WZ} = C_4 + C_2 \wedge \mathcal{F} + \frac{1}{2} C_0 \wedge \mathcal{F} \wedge \mathcal{F} \tag{4.23}$$

is the Wess-Zumino term, where $\mathcal{F} = dA - B_2$ is the generalized field strength of the Abelian gauge field A_m. The role of Lagrange multiplier is to identify this field strength with the 2-form $\frac{1}{2}\Pi^m \wedge \Pi^n F_{nm}$ constructed from the auxiliary tensor F_{nm}

$$\frac{\delta S}{\delta Q_2} = 0 \Rightarrow \quad F \equiv \frac{1}{2} E^b \wedge E^a F_{ab} = \mathcal{F} \equiv dA - B_2 \tag{4.24}$$

C_4, C_2, C_0 are the RR gauge fields whose flat superspace form is not essential (e.g. $C_0 = 0$). For our purposes we really need only the external derivative of the Wess-Zumino form (4.23) (see the next section for more details)

$$d\mathcal{L}_{WZ} = 2i(E^{1\alpha} \wedge \bar{E}^{2\dot{\alpha}} - E^{2\alpha} \wedge \bar{E}^{1\dot{\alpha}}) \wedge \hat{\sigma}^{(3)}_{\alpha\dot{\alpha}} + 2i(E^{1\alpha} \wedge \bar{E}^{2\dot{\alpha}} + E^{2\alpha} \wedge \bar{E}^{1\dot{\alpha}}) \wedge \hat{\sigma}^{(1)}_{\alpha\dot{\alpha}} \wedge \mathcal{F} \tag{4.25}$$

where

$$\hat{\sigma}^{(1)}_{\alpha\dot{\alpha}} = E^a \sigma_{a\alpha\dot{\alpha}} \equiv \Pi^m \sigma_{m\alpha\dot{\alpha}}, \quad \hat{\sigma}^{(3)}_{\alpha\dot{\alpha}} = \frac{1}{3!} E^{a_1} \wedge E^{a_2} \wedge E^{a_3} (\sigma_{a_3 a_2 a_1})_{\alpha\dot{\alpha}} \tag{4.26}$$

4.3 External Derivative of the Lagrangian Form and κ-Symmetry

The external differential of the kinetic term (4.19) can be presented in the form:

$$d\mathcal{L}_0 = 2i\sqrt{-det(\eta + F)_{ab}} E_a^{\wedge 3} \wedge (E^{1\alpha} \wedge \bar{E}^{1\dot{\alpha}} + E^{2\alpha} \wedge \bar{E}^{2\dot{\alpha}}) \sigma^a_{\alpha\dot{\alpha}} + \tag{4.27}$$

$$\sqrt{-det(\eta + F)_{ab}}\; E^{\wedge 2}_{cd} (\eta + F)^{-1\; cd} \wedge \frac{1}{2} E^b \wedge E^a \wedge dF_{ab}$$

(we have used general expression for torsion (4.6) and Eqs. (4.9)) The external differential of the Lagrange term (4.20) is

$$d\mathcal{L}_1 = dQ_2 \wedge (\mathcal{F} - F) + Q_2 \wedge d(\mathcal{F} - F) = \tag{4.28}$$

$$dQ_2 \wedge (\mathcal{F} - F) - Q_2 \wedge (H_3 + E^b \wedge T^a F_{ab} + \frac{1}{2} E^b \wedge E^a \wedge dF_{ab})$$

Using general expression for torsion (4.6), H_3 (4.22), identities (4.9) and collecting Eqs. (4.27),(4.28) we finally obtain (in all subsequent calculations we for simplicity denote $\sqrt{-det(\eta + F)_{ab}}$ as $\sqrt{\ldots}$).

$$d\mathcal{L}_0 + d\mathcal{L}_1 = dQ_2 \wedge (\mathcal{F} - F) + (Q_2 - \sqrt{\ldots}\, E^{\wedge 2}_{cd} (\eta + F)^{-1\ cd}) \wedge d(\mathcal{F} - F) + \tag{4.29}$$

$$2i\sqrt{\ldots}\, E^{\wedge 3}_a \wedge [E^{1\alpha} \wedge \bar{E}^{1\dot{\alpha}} \sigma_{b\alpha\dot{\alpha}} (\eta + F)^{-1\ ba} + E^{2\alpha} \wedge \bar{E}^{2\dot{\alpha}} \sigma_{b\alpha\dot{\alpha}} (\eta - F)^{-1\ ba}]$$

To simplify subsequent calculations we introduce auxiliary variables $h_\alpha^{\ \beta}$ which (by definition) satisfy

$$(h\sigma^a \bar{h}^T)_{\alpha\dot{\alpha}} = \sigma^b_{\alpha\dot{\alpha}} k_b^{\ a} \tag{4.30}$$

where

$$k_b^{\ a} = \delta_b^{\ a} - 2((\eta + F)^{-1} F)_b^{\ a} \equiv ((\eta + F)^{-1}(\eta - F))_b^{\ a} \equiv ((\eta - F)(\eta + F)^{-1})_b^{\ a} \tag{4.31}$$

$$(k^T)_b^{\ a} \equiv k^a_{\ b} = (k^{-1})_b^{\ a} \Leftrightarrow k_b^{\ a} \in SO(1,3)$$

Here (4.31) is the Cayley construction for the pseudoorthogonal ('η-orthogonal' or Lorentz group valued) matrix $k_b^{\ a}$ for the antisymmetric Cayley image F_{ab}. Using (4.30), (4.31) one can present (4.29) in the form

$$d\mathcal{L}_0 + d\mathcal{L}_1 = dQ_2 \wedge (\mathcal{F} - F) + (Q_2 - \sqrt{\ldots}\, E^{\wedge 2}_{cd} (\eta + F)^{-1\ cd}) \wedge d(\mathcal{F} - F) + \tag{4.32}$$

$$2i\sqrt{\ldots} E^{\wedge 3}_a (\eta + F)^{-1\ ab} (E^{2\alpha} - E^{1\beta} h_\beta^{\ \alpha}) \wedge (\bar{E}^{2\dot{\alpha}} - \bar{E}^{1\dot{\beta}} \bar{h}_{\dot{\beta}}^{\ \dot{\alpha}}) \sigma_{b\alpha\dot{\alpha}} +$$

$$2i\sqrt{\ldots} E^{\wedge 3}_a (\eta + F)^{-1\ ab} [E^{2\alpha} \wedge \bar{E}^{1\dot{\alpha}} (\sigma_b \bar{h}^T)_{\alpha\dot{\alpha}} + E^{1\alpha} \wedge \bar{E}^{2\dot{\alpha}} (h\sigma_b)_{\alpha\dot{\alpha}}]$$

As a requirement of $\kappa-$symmetry we should choose $d\mathcal{L}_{WZ}$ in such a form to cancel the lowest line in (4.32) The most general expression for $d\mathcal{L}_{WZ}$ is

$$d\mathcal{L}_{WZ} = 2i(E^{1\alpha}\wedge\bar{E}^{2\dot{\alpha}}\mp E^{2\alpha}\wedge\bar{E}^{1\dot{\alpha}})\wedge\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}}+2ia(E^{1\alpha}\wedge\bar{E}^{2\dot{\alpha}}+E^{2\alpha}\wedge\bar{E}^{1\dot{\alpha}})\wedge\hat{\sigma}^{(1)}_{\alpha\dot{\alpha}}\wedge\mathcal{F} \quad (4.33)$$

where $\hat{\sigma}^{(1)}_{\alpha\dot{\alpha}}$, $\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}}$ are defined by (4.26)

The requirement for (4.25) to be the closed superform ($dd\mathcal{L}_{WZ} = 0$) fixes the sign in the first term and the (mutual) coefficient in the second uniquely [1]. Therefore we obtain Eq. (4.25) which can be written as

$$\pm d\mathcal{L}_{WZ} = 2i(E^{1\alpha}\wedge\bar{E}^{2\dot{\alpha}}-E^{2\alpha}\wedge\bar{E}^{1\dot{\alpha}})\wedge\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}}+2i(E^{1\alpha}\wedge\bar{E}^{2\dot{\alpha}}+E^{2\alpha}\wedge\bar{E}^{1\dot{\alpha}})\wedge\hat{\sigma}^{(1)}_{\alpha\dot{\alpha}}\wedge F+ \quad (4.34)$$

$$2i(E^{1\alpha}\wedge\bar{E}^{2\dot{\alpha}}+E^{2\alpha}\wedge\bar{E}^{1\dot{\alpha}})\wedge\hat{\sigma}^{(1)}_{\alpha\dot{\alpha}}\wedge(\mathcal{F}-F)$$

However there is still an ambiguity as to the overall sign in (4.34).

Taking into account Eqs. (4.32), (4.34) and requirement of $\kappa-$symmetry we obtain for the external differential of $\mathcal{L}$

$$d\mathcal{L} = d(\mathcal{L}_0 + d\mathcal{L}_1 + \mathcal{L}_{WZ}) = \quad (4.35)$$

$$(dQ_2\pm 2i(E^{1\alpha}\wedge\bar{E}^{2\dot{\alpha}}+E^{2\alpha}\wedge\bar{E}^{1\dot{\alpha}}))\wedge\hat{\sigma}^{(1)}_{\alpha\dot{\alpha}}\wedge(\mathcal{F}-F)+(Q_2-\sqrt{\ldots}E^{\wedge 2}_{cd}(\eta+F)^{-1\ cd})\wedge d(\mathcal{F}-F)+$$

$$2i\sqrt{\ldots}E^{\wedge 3}_a(\eta+F)^{-1\ ab}(E^{2\alpha}-E^{1\beta}h_\beta{}^\alpha)\wedge(\bar{E}^{2\dot{\alpha}}-\bar{E}^{1\dot{\beta}}\bar{h}_{\dot{\beta}}{}^{\dot{\alpha}})\sigma_{b\alpha\dot{\alpha}}.$$

To derive (4.35) we have used the fact that as a requirement of $\kappa-$symmetry the lowest line in (4.32) should be cancelled by the first one of the Wess-Zumino term (4.34).

[1] Indeed, taking into account that $d\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}}$ may be presented in the form

$$d\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}} = -i(\hat{\sigma}^{(1)\gamma}_{\dot{\alpha}}\wedge\hat{\sigma}^{(1)}_{\gamma\dot{\beta}}\epsilon_{\alpha\beta}+\hat{\sigma}^{(1)}_{\alpha\dot{\gamma}}\wedge\hat{\sigma}^{(1)\dot{\gamma}}{}_{\beta}\epsilon_{\dot{\alpha}\dot{\beta}})(E^{1\beta}\wedge\bar{E}^{1\dot{\beta}}+E^{2\beta}\wedge\bar{E}^{1\dot{\beta}})$$

we obtain that one should fix upper sign and the value of $a = 1$ in the Eq. (4.33).

As a consequence of this requirement we obtain the conditions

$$E_a^{\wedge 3}\sqrt{\ldots}(\sigma_b\bar{h}^T)_{\alpha\dot{\alpha}}(\eta - F)^{-1ba} = \pm(\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}} - \hat{\sigma}^{(1)}_{\alpha\dot{\alpha}} \wedge F) \tag{4.36}$$

$$E_a^{\wedge 3}\sqrt{\ldots}(h\sigma_b)_{\alpha\dot{\alpha}}(\eta - F)^{-1ba} = \mp(\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}} + \hat{\sigma}^{(1)}_{\alpha\dot{\alpha}} \wedge F) \tag{4.37}$$

which can be used to explicitly express $h_\alpha{}^\beta$ in terms of F_{ab}.

4.4 Spin-Tensor $h_\alpha{}^\beta$ as a Function of the Auxiliary Tensor Field F_{ab}.

Taking into account that $\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}}$ and $\hat{\sigma}^{(1)}_{\alpha\dot{\alpha}} \wedge F$ can be presented in the form

$$\hat{\sigma}^{(3)}_{\alpha\dot{\alpha}} = iE_a^{\wedge 3}\sigma^a_{\alpha\dot{\alpha}}, \quad \hat{\sigma}^{(1)}_{\alpha\dot{\alpha}} \wedge F = -iE_a^{\wedge 3}F^{*ab}\sigma_{b\alpha\dot{\alpha}} \tag{4.38}$$

where Hodge operation is defined as

$$F^{*ab} = \frac{i}{2}\epsilon^{abcd}F_{cd} \tag{4.39}$$

one can obtain from Eqs. (4.36),(4.37)

$$\pm i\sqrt{\ldots}(h\sigma^a)_{\alpha\dot{\alpha}} = \sigma^a_{\alpha\dot{\alpha}} + \sigma_{b\alpha\dot{\alpha}}(F^{*ba} - F^{ba}) - \sigma^b_{\alpha\dot{\alpha}}F^*_{bc}F^{ca} \tag{4.40}$$

Converting F_{ab} from vector to spinor indices and taking into account its antisymmetry properties, one can decompose it on the irreducible parts

$$F_{ab} \Leftrightarrow F_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}} \equiv \sigma^a_{\alpha\dot{\alpha}}F_{ab}\tilde{\sigma}^{b\beta\dot{\beta}} = 2\delta^\beta_\alpha\bar{f}_{\dot{\alpha}}{}^{\dot{\beta}} + 2\delta^{\dot{\beta}}_{\dot{\alpha}}f_\alpha{}^\beta \tag{4.41}$$

$$F^*_{ab} \Leftrightarrow F^*_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}} \equiv \sigma^a_{\alpha\dot{\alpha}}F^*_{ab}\tilde{\sigma}^{b\dot{\beta}\beta} = 2\delta^\beta_\alpha\bar{f}_{\dot{\alpha}}{}^{\dot{\beta}} - 2\delta^{\dot{\beta}}_{\dot{\alpha}}f_\alpha{}^\beta \tag{4.42}$$

where

$$f_\alpha{}^\beta = \frac{i}{4}F_{ab}(\sigma^{ab})_\alpha{}^\beta, \quad \bar{f}_{\dot{\alpha}}{}^{\dot{\beta}} = -\frac{i}{4}F_{ab}(\tilde{\sigma}^{ab})_{\dot{\alpha}}{}^{\dot{\beta}} \tag{4.43}$$

and $\sigma^{ab} = -\sigma^{ab}$ is defined as follows

$$\sigma^a\tilde{\sigma}^b = \eta^{ab} + i\sigma^{ab}, \quad \tilde{\sigma}^a\sigma^b = \eta^{ab} + i\tilde{\sigma}^{ab} \tag{4.44}$$

Using decomposition on the irreducible parts (4.41),(4.42) and trivial identities

$$f_\alpha{}^\gamma f_\gamma{}^\beta = -\frac{1}{2}\delta_\alpha^\beta f^2, \quad \bar{f}_{\dot\alpha}{}^{\dot\gamma}\bar{f}_{\dot\gamma}{}^{\dot\beta} = -\frac{1}{2}\delta_{\dot\alpha}^{\dot\beta}\bar{f}^2 \tag{4.45}$$

where

$$f^2 = f^{\alpha\beta}f_{\alpha\beta}, \quad \bar{f}^2 = \bar{f}^{\dot\alpha\dot\beta}\bar{f}_{\dot\alpha\dot\beta} \tag{4.46}$$

we obtain

$$F^{*}_{\alpha\dot\alpha}{}^{\gamma\dot\gamma}F_{\gamma\dot\gamma}{}^{\beta\dot\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}(f^2 - \bar{f}^2) \Rightarrow F^*_{bc}F^{ca} = \frac{1}{2}(f^2 - \bar{f}^2)\delta_b^a \tag{4.47}$$

therefore

$$F^{*}_{\alpha\dot\alpha}{}^{\gamma\dot\gamma}F_{\gamma\dot\gamma}{}^{\alpha\dot\alpha} = 8(f^2 - \bar{f}^2) \Rightarrow F^*_{ab}F^{ab} = 2(f^2 - \bar{f}^2) \tag{4.48}$$

Taking into account Eqs. (4.41), (4.42) and (4.47) we obtain general expression for $h_\alpha{}^\beta$ in terms of F_{ab}:

$$h_\alpha{}^\beta = \frac{\mp i}{\sqrt{\cdots}}[\delta_\alpha^\beta(1 - \frac{f^2}{2} + \frac{\bar{f}^2}{2}) - 2f_\alpha{}^\beta] \tag{4.49}$$

$$\bar{h}_{\dot\alpha}{}^{\dot\beta} = \frac{\pm i}{\sqrt{\cdots}}[\delta_{\dot\alpha}^{\dot\beta}(1 - \frac{\bar{f}^2}{2} + \frac{f^2}{2}) - 2\bar{f}_{\dot\alpha}{}^{\dot\beta}] \tag{4.50}$$

Now we would like to show that the expressions (4.49) and (4.50) satisfy the condition (4.30).

For this purpose one needs the general decomposition

$$-det(\eta + F)_{ab} = 1 + \frac{1}{2}F_{ab}F^{ab} - (\frac{1}{8}\epsilon^{abcd}F_{ab}F_{cd})^2. \tag{4.51}$$

Then taking into account Eq. (4.48) and the fact that

$$\frac{1}{2}F_{ab}F^{ab} = -\frac{1}{8}F_{\alpha\dot\alpha}{}^{\beta\dot\beta}F_{\beta\dot\beta}{}^{\alpha\dot\alpha} = f^2 + \bar{f}^2 \tag{4.52}$$

Eq. (4.51) can be expressed in the following form

$$-det(\eta + F) = 1 + f^2 + \bar{f}^2 + \frac{1}{4}(f^2 - \bar{f}^2)^2 \tag{4.53}$$

Therefore using Eqs. (4.49) and (4.50) one can prove that

$$h_\alpha{}^\gamma \bar{h}_{\dot{\alpha}}{}^{\dot{\gamma}} (\eta + F)_{\gamma\dot{\gamma}}^{\beta\dot{\beta}} = (\eta - F)_{\alpha\dot{\alpha}}^{\beta\dot{\beta}} \tag{4.54}$$

which coincides with (4.30) after transition from spinor to vector notations.

4.5 Fermionic Superembedding Equations and Equations of Motion

The worldvolume supervielbeins of the $D3-$brane worldvolume superspace induced by the embedding can be chosen as follows

$$e^a = E^a \equiv \Pi^m \delta_m^a \tag{4.55}$$

$$e^\alpha = E^{1\alpha} \equiv d\Theta^{1\mu}\delta_\mu^\alpha, \quad \bar{e}^{\dot{\alpha}} = \bar{E}^{1\dot{\alpha}} \equiv d\bar{\Theta}^{1\dot{\mu}}\delta_{\dot{\mu}}^{\dot{\alpha}} \tag{4.56}$$

This reflects the possibility to identify the Grassman coordinate of the world volume superspace η^μ (or $\bar{\eta}^{\dot{\mu}}$) with the coordinate function $\Theta^{1\mu}$ (or $\bar{\Theta}^{1\dot{\mu}}$) and corresponds to the possibility to fix a gauge for the κ-symmetry as

$$\Theta^{1\mu}|_{\eta^\mu=0} = 0, \quad \bar{\Theta}^{1\dot{\mu}}|_{\bar{\eta}^{\dot{\mu}}=0} = 0 \tag{4.57}$$

Thus the generic expression for the pull-backs of the rest of fermionic 1-forms (4.12) may be presented in the form:

$$E^{2\alpha} = E^{1\beta}E_\beta{}^{2\alpha} + \bar{E}^{1\dot{\alpha}}C_{\dot{\alpha}}{}^{2\alpha} + E^a E_a^{2\alpha}, \quad \bar{E}^{2\dot{\alpha}} = \bar{E}^{1\dot{\beta}}\bar{E}_{\dot{\beta}}{}^{2\dot{\alpha}} + E^{1\alpha}\bar{C}_\alpha{}^{2\dot{\alpha}} + E^a\bar{E}_a{}^{2\dot{\alpha}}, \tag{4.58}$$

where

$$E_{\beta}{}^{2\alpha} = \mathcal{D}_{\beta}\Theta^{2\alpha}, \quad C_{\dot\alpha}{}^{2\alpha} = \bar{\mathcal{D}}_{\dot\alpha}\Theta^{2\alpha}, \quad E_{a}^{2\alpha} = \mathcal{D}_{a}\Theta^{2\alpha} \tag{4.59}$$

and the analogous expressions are valid for the complex conjugate superfields.

The possibility of identifying the worldvolume supervielbein with the "pulled back" components (4.13) and (4.56) of the target space supervielbein implies that in flat target superspace the induced superworldvolume geometry is also flat, and that the worldvolume spin connection is zero. This is natural, since the brane worldvolume completely fills in (or coincides with) the bosonic core of the flat target superspace. We should note that though the superworldvolume is flat the supervielbein e^A defined by (4.13) and (4.56) differs from the standard flat superspace basis

$$e_0^A = (d\xi^a - id\eta\sigma^a\bar\eta + i\eta\sigma^a d\bar\eta, \ d\eta^{\alpha}, \ d\bar\eta^{\dot\alpha}). \tag{4.60}$$

This, in particular, implies that the superworldvolume covariant derivatives $\mathcal{D}_A$ in (4.17) and (4.59) associated with the basis (4.13) and (4.56) differ from conventional flat covariant derivatives and form a more complicated superalgebra.

The integrability of (4.5) and (4.58) requires some differential relations between the components $E_{\beta}{}^{2\alpha}$, $C_{\dot\alpha}{}^{2\alpha}$ and $E_{a}^{2\alpha}$ of the superforms (4.5), however the integrability of the superembedding condition (4.13), which implies that the worldvolume torsion is the pullback of the target space torsion

$$de^a = dE^a \equiv T^a = -2iE^{\alpha 1} \wedge \bar{E}^{\dot\alpha 1}\sigma^a_{\alpha\dot\alpha} - 2iE^{\alpha 2} \wedge \bar{E}^{\dot\alpha 2}\sigma^a_{\alpha\dot\alpha}, \tag{4.61}$$

does not put any further restrictions on $E_{\beta}{}^{2\alpha}$, $C_{\dot\alpha}{}^{2\alpha}$, and $E_{a}^{2\alpha}$, and these superfields are still too general to be associated with the physical modes of the D3–brane, which form a gauge vector supermultiplet.

The situation when the basic superembedding condition is not sufficient to determine the dynamics of the brane even off the mass shell is a generic one for the space–filling [78] and codimension one [82] branes. In such cases, for the superembedding to describe the superbrane dynamics, an additional constraint should be imposed [2]. In our case this is a constraint on an 'extended' field–strength two–form

$$F_2 = dA - B_2 \tag{4.62}$$

of a worldvolume gauge field $dz^M A_M(z)$ living on the D3-brane, and the two–form B_2 is the pullback of an "NS–NS" gauge superfield of an $N = 2$, $D = 4$ supergravity which the D3 brane couples to.

Gauge field constraints for the $D = 4$ super–$D3$–brane intrinsic gauge field produced by the generalized action have the form (4.24). Their integrability conditions can be written as

$$J_3 \equiv \frac{1}{3!} e^A \wedge e^B \wedge e^C J_{CBA} \equiv d(\mathcal{F} - F) = 0 \tag{4.63}$$

or

$$-H_3 - E^a \wedge T^b F_{ba} - \frac{1}{2} E^a \wedge E^b \wedge dF_{ba} = 0 \tag{4.64}$$

Substituting Eqs. (4.6),(4.22) into (4.64) and taking (4.58) into account, one can find from $dim2$ term ($\propto E^a \wedge E^{1\alpha} \wedge E^{1\beta}$) of the integrability condition (4.64)

$$E^{2\gamma}_{(\alpha|} \sigma^b_{\gamma\dot{\gamma}} \bar{C}^{2\dot{\gamma}}_{|\beta)} (\eta - F)_{ba} = 0 \tag{4.65}$$

[2] Note that these additional constraints are reproduced by the generalized action [73, 77] on the same footing as the basic superembedding conditions and the dynamical equations of motion.

For nondegenerate matrix $(\eta - F)$ (which is the general assumption of the Born-Infeld-like models) this equation has only two solutions:

$$\bar{C}_{\beta}^{2\dot{\gamma}} = 0 \tag{4.66}$$

or

$$E_{\alpha}^{2\beta} = 0 \tag{4.67}$$

Thus there are two possibilities for the choice of the fermionic superembedding conditions (in the following we denote $E_a^{2\alpha} \equiv \phi_a^{\alpha}$)

$$E^{2\alpha} = E^{1\beta} E_{\beta}^{2\alpha} + E^a \phi_a^{\alpha}, \quad \bar{E}^{2\dot{\alpha}} = \bar{E}^{1\dot{\beta}} \bar{E}_{\dot{\beta}}^{2\dot{\alpha}} + E^a \bar{\phi}_a^{\dot{\alpha}} \tag{4.68}$$

or

$$E^{2\alpha} = \bar{E}^{1\dot{\alpha}} C_{\dot{\alpha}}^{2\alpha} + E^a \phi_a^{\alpha}, \quad \bar{E}^{2\dot{\alpha}} = E^{\alpha} \bar{C}_{\alpha}^{2\dot{\alpha}} + E^a \bar{\phi}_a^{\dot{\alpha}} \tag{4.69}$$

We choose superembedding condition (4.68) (i.e. solution (4.66)) because the choice (4.67) leads to the trivial solution of the superembedding condition, which does not describe any physical dynamical system (see Appendix D for further details).

Another $dim2$ term ($\propto E^a \wedge E^{1\alpha} \wedge \bar{E}^{1\dot{\alpha}}$) of the integrability condition (4.64) yields the equation

$$E_{\alpha}^{2\beta} \sigma^a_{\beta\dot{\beta}} \bar{E}_{\dot{\alpha}}^{2\dot{\beta}} = \sigma^b_{\alpha\dot{\alpha}} ((\eta - F)(\eta + F)^{-1})_b{}^a \tag{4.70}$$

which together with the higher dimension integrability conditions

$$J_{\alpha cb} = D_{\alpha} F_{cb} - 4i E_{\alpha}^{2\beta} \sigma^a_{\beta\dot{\beta}} \bar{\phi}^{\dot{\beta}}_{[c|} (\eta - F)_{|b]a} = 0 \tag{4.71}$$

$$J_{dcb} = -D_{[d} F_{cb]} + 4i \phi^{\alpha}_{[d} \sigma^a_{\alpha\dot{\alpha}} \bar{\phi}^{\dot{\alpha}}_{c} (\eta - F)_{|b]a} = 0 \tag{4.72}$$

leads to the following identification (see Appendix C for more details)

$$E_{\alpha}^{\ 2\beta} = h_{\alpha}^{\ \beta} \tag{4.73}$$

Thus superembedding equations take the form

$$E^{2\alpha} = E^{1\beta} h_{\beta}^{\ \alpha} + E^{a} \phi_{a}^{\alpha}, \quad \bar{E}^{2\dot{\alpha}} = \bar{E}^{1\dot{\beta}} \bar{h}_{\dot{\beta}}^{\ \dot{\alpha}} + E^{a} \bar{\phi}_{a}^{\dot{\alpha}} \tag{4.74}$$

We would like to emphasize that for the derivation of (4.74) from the integrability conditions (4.64) we have used only (4.55), (4.56) and the definition of the auxiliary variables $h_{\alpha}^{\ \beta}$.

To obtain bosonic superfield equations of motion we should consider contractions of (4.35) with a variation symbol obeying:

$$i_{\delta} E^{I\alpha} = 0 \quad and \quad i_{\delta} \bar{E}^{I\dot{\alpha}} = 0. \tag{4.75}$$

Taking the only nonvanishing contraction in (4.35) to be $i_{\delta} dQ_2 \equiv \delta Q_2$, we obtain

$$\frac{\delta S}{\delta Q_2} \equiv \frac{i_{\delta} d\mathcal{L}}{i_{\delta} dQ_2} = 0 \Rightarrow F \equiv \frac{1}{2} E^{b} \wedge E^{a} F_{ab} = \mathcal{F} \equiv dA - B_2 \tag{4.76}$$

(which is just (4.24))

If the only nonvanishing variation is $i_{\delta} dF_{ab} = \delta F_{ab}$ one can obtain from (4.35)

$$Q_2 = \sqrt{\ldots} E_{ab}^{\wedge 2} (\eta + F)^{-1\ ab} \tag{4.77}$$

In the similar manner taking the only nonvanishing variation to be $i_{\delta} dA = \delta A$, one obtains from (4.35) the equation corresponding to the variation of the world volume gauge superfield

$$\frac{\delta S}{\delta A} \equiv \frac{i_{\delta} d\mathcal{L}}{i_{\delta} A} = 0 \Rightarrow dQ_2 = \mp 2i (E^{1\alpha} \wedge \bar{E}^{2\dot{\alpha}} + E^{2\alpha} \wedge \bar{E}^{1\dot{\alpha}}) \wedge \hat{\sigma}_{a\dot{\alpha}}^{(1)} \tag{4.78}$$

One can easily prove that the equations of motion following from (4.35) for the only non vanishing contraction $\delta X^m \equiv i_\delta E^a \delta_a^m$ are the sequence of the fermionic equations of motion, which can be obtained from the second line of (4.35), if we set $i_\delta E^a = 0, i_\delta E^{I\alpha} = \delta\Theta^{I\alpha} = 0, i_\delta \bar{E}^{I\dot{\alpha}} = \delta\bar{\Theta}^{I\dot{\alpha}} \neq 0, (I = 1,2)$ (we should also repeat this procedure for $E^{I\alpha}$, i.e. $i_\delta E^a = 0, i_\delta \bar{E}^{I\dot{\alpha}} = \delta\bar{\Theta}^{I\dot{\alpha}} = 0, i_\delta E^{I\alpha} = \delta\Theta^{I\alpha} \neq 0, (I = 1,2)$)

Due to nondegeneracy of the matrix $\bar{h}_{\dot{\alpha}}^{\dot{\beta}}$ we obtain

$$\frac{\delta S}{\delta\bar{\Theta}^{I\dot{\mu}}} \equiv \frac{i_\delta d\mathcal{L}}{i_\delta \bar{E}^{I\dot{\mu}}} = 0 \Rightarrow E_a^{\wedge 3} \wedge (E^2 - E^1 h)^\alpha \sigma_{b\alpha\dot{\alpha}} (\eta - F)^{-1\ ba} = 0 \tag{4.79}$$

and its complex conjugate

$$\frac{\delta S}{\delta\Theta^{I\mu}} \equiv \frac{i_\delta d\mathcal{L}}{i_\delta E^{I\mu}} = 0 \Rightarrow E_a^{\wedge 3} \wedge (\bar{E}^2 - \bar{E}^1 \bar{h})^\alpha \sigma_{b\alpha\dot{\alpha}} (\eta - F)^{-1\ ba} = 0 \tag{4.80}$$

Decomposing $E^{\alpha 2}$ in (4.79) into basic forms (see also (4.68)) we obtain

$$E_a^{\wedge 3} \wedge E^{\beta 1} (E_\beta^{2\alpha} - h_\beta{}^\alpha) \sigma_{b\alpha\dot{\alpha}} (\eta - F)^{-1\ ba} = 0 \Rightarrow \quad E_\beta^{2\alpha} = h_\beta{}^\alpha \tag{4.81}$$

$$E_a^{\wedge 3} \wedge E^c E_c^{2\alpha} (\eta - F)^{-1\ ba} \sigma_{b\alpha\dot{\alpha}} = 0. \tag{4.82}$$

Therefore identification (4.73) is consistent with the Eq. (4.79). And from (4.82) one obtain the fermionic dynamical equations of motion

$$\sigma_{a\alpha\dot{\alpha}} (\eta - F)^{-1\ ab} \phi_b^\alpha = 0, \quad \sigma_{a\alpha\dot{\alpha}} (\eta - F)^{-1\ ab} \bar{\phi}_b^{\dot{\alpha}} = 0 \tag{4.83}$$

The $dim2$ integrability condition ($\propto E^a \wedge E^{1\alpha} \wedge \bar{E}^{1\dot{\alpha}}$) leads to the Eq. (4.70), or taking (4.73) into account

$$(h\sigma^a \bar{h}^T)_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^b ((\eta - F)(\eta + F)^{-1})_b{}^a \tag{4.84}$$

which means that

$$h_\alpha^\beta \quad \in \quad SL(2, C) \times U(1) \tag{4.85}$$

i.e. it is the complex nondegenerate matrix with $|det(h)| = 1 \Rightarrow det(h) = e^{2i\alpha(z)}$.

Hence

$$h^{-1}{}_{\alpha}{}^{\gamma} dh_{\gamma}{}^{\beta} = \frac{1}{2}(k^{-1}dk)^{ab}\sigma_{ab\alpha}{}^{\beta} + id\alpha(z)\delta_{\alpha}{}^{\beta} \tag{4.86}$$

with some superfield $\alpha(z)$ which can be identified with parameters of the phase transformations of the h matrix, that leave the relation (4.84) invariant.

Now we would like to show that the relation

$$\alpha(z) = const$$

can be considered as superfield equation of motion.

Contracting indices in (4.86) we obtain

$$h^{-1}{}_{\alpha}{}^{\gamma} dh_{\gamma}{}^{\alpha} = 2id\alpha(z) \tag{4.87}$$

Then one has to consider the integrability conditions for superembedding equations (4.74)

$$I_2^{\alpha} = d(E^{2\alpha} - E^{1\beta}h_{\beta}{}^{\alpha} - E^a\phi_a^{\alpha}) = 0, \qquad \bar{I}_2^{\dot\alpha} = d(\bar{E}^{2\dot\alpha} - \bar{E}^{1\dot\beta}\bar{h}_{\dot\beta}{}^{\dot\alpha} - E^a\bar{\phi}_a^{\dot\alpha}) = 0 \tag{4.88}$$

The only independent components of (4.88) are $dim1/2$ components (see Appendix B)

$$I_{\beta\gamma}^{\alpha} = 0, \qquad I_{\beta\dot\gamma}^{\alpha} = 0$$

the first of which leads to the equations

$$D_{(\beta}h_{\gamma)}^{\alpha} = 0 \tag{4.89}$$

and the second yields in

$$\bar{D}_{\dot\gamma}h_{\beta}{}^{\alpha} = 4i\sigma_{\beta\dot\gamma}^{b}(\eta + F)_{b}^{-1}{}^{a}\phi_a^{\alpha} \tag{4.90}$$

(to derive the final form of (4.90) the Eqs. (4.84) have been used). Then, using (4.84) and (4.87), one obtains

$$\sigma_{b\beta\dot{\beta}}(\eta - F)^{-1\ ba}\phi_a^\beta = \frac{1}{2}(\bar{D}_{\dot{\gamma}}\alpha(z))\bar{h}^{-1\dot{\gamma}}{}_{\dot{\beta}} \tag{4.91}$$

Where the left hand side is the same as in (4.83), but in general it is non-zero, since $\alpha(z)$ is not a constant in general. Thus in the case of the space-filling $D3$-brane the superembedding conditions and the field strength constraint do not produce dynamical equations of motion and, therefore, leave the theory off the mass shell. The equations of motion arise only if in addition we put $\alpha(z) = const.$ Then on the mass shell a spin tensor h becomes an $SL(2, C)$ valued matrix.

$$det(h_\alpha{}^\beta) = 1 \tag{4.92}$$

4.6 Grassmann Analyticity

As we have already mentioned, the superembedding conditions (4.13), (4.16), (4.17), (4.56), (4.58), (4.59), (4.5) and the gauge field constraint (4.24) and its integrability conditions result in double analyticity, i.e. Grassmann analyticity both in the worldvolume and in target superspace, the phenomenon which was used in [79] as a characteristic principle for some types of superembeddings, describing for instance certain superparticles and superstrings [71, 79, 80, 81]. Indeed, since the integrability of the constraints leads to the solution (4.66), from (4.56), (4.58), (4.59), (4.5) it follows that $\Theta^{I\alpha}$ are chiral worldvolume superfields, i.e.

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Theta^{I\alpha} = 0, \qquad \mathcal{D}_\alpha\bar{\Theta}^{I\dot{\alpha}} = 0. \tag{4.93}$$

Then eqs. (4.17) take the form

$$\mathcal{D}_\alpha(X^a - i\Theta^I\sigma^a\bar{\Theta}^I) = 0, \quad \bar{\mathcal{D}}_{\dot\alpha}(X^a + i\Theta^I\sigma^a\bar{\Theta}^I) = 0, \tag{4.94}$$

or

$$X^a = \frac{1}{2}(X^a_R + X^a_L), \quad X^a_L - X^a_R - 2i\Theta^I\sigma^a\bar{\Theta}^I = 0, \tag{4.95}$$

where $X^a_R = X^a - i\Theta^I\sigma^a\bar{\Theta}^I = \overline{(X^a_L)}$ are complex conjugate chiral worldvolume superfields

$$\bar{\mathcal{D}}_{\dot\alpha}X^a_L = 0, \qquad \mathcal{D}_\alpha X^a_R = 0. \tag{4.96}$$

The equations (4.95) are nothing but the definition of complex coordinates $Z^M_L = (X^a_L = X^a + i\Theta^I\sigma^a\bar{\Theta}^I, \Theta^{I\alpha})$ of a chiral subspace of the $N = 2$, $D = 4$ superspace, which in their turn are chiral superfields in the $N = 1$, $D = 4$ superworldvolume.

We thus have obtained that the conditions imposed on the embedding of the D3–brane into the target superspace imply that the superembedding is performed in such a way that the chiral subsuperspace of the superworldvolume gets mapped into the chiral subsuperspace of the target superspace.

4.7 Geometric Equations in Linear Approximation

Bosonic vielbein (4.4) can be written in the form

$$\Pi^m = dX^m_L - 2id\Theta^{I\alpha}\sigma^m_{\alpha\dot\alpha}\bar{\Theta}^{I\dot\alpha}, \quad I = 1, 2 \tag{4.97}$$

where

$$X^m_L = X^m + i\Theta^{I\alpha}\sigma^m_{\alpha\dot\alpha}\bar{\Theta}^{I\dot\alpha}, \quad I = 1, 2 \tag{4.98}$$

In the static gauge

$$\Theta^{1\alpha} = \eta^{\alpha} \tag{4.99}$$

we should take $\Theta^{2\alpha}$ in the linearized approximation as follows

$$\Theta^{2\alpha}(z) = \eta^{\alpha} + W^{\alpha}(z) \tag{4.100}$$

where W^{α}(z) is a Grassman Goldstone fermion superfield for partially broken $N = 2$ supersymmetry.

This choice can be understood with the help of the following reasoning: when there is no gauge field on the $D3$-brane worldvolume $F_{ab}(z) = 0$, the integrability condition (4.64) of the gauge field constraint (4.24) reduces to

$$2iE^{a} \wedge (E^{1\alpha} \wedge \bar{E}^{1\dot{\alpha}} - E^{2\alpha} \wedge \bar{E}^{2\dot{\alpha}})\sigma_{a\alpha\dot{\alpha}} = 0, \tag{4.101}$$

which is satisfied if we choose $E^{1\alpha} = E^{2\alpha}$ along superworldvolume. Hence, in the static gauge (4.99) this configuration of the $D3$-brane can be associated with the map $\eta^{\alpha} = \Theta^{1\alpha} = \Theta^{2\alpha}$ and 'fluctuations' around this solution are described by the superfield W^{α}.

Substituting (4.99) and (4.100) in the bosonic vielbein form (4.97) we obtain (in the linearized approximation)

$$\Pi^{m} = w^{m} - 2i(d\eta\sigma^{m}\bar{W} + W\sigma^{m}d\bar{\eta}) \tag{4.102}$$

where

$$w^{m} = d\xi^{m} - 4id\eta\sigma^{m}\bar{\eta} \tag{4.103}$$

is a bosonic vielbein of the flat world volume superspace and

$$\xi^{m} = X_{L}^{m} - 2iW\sigma^{m}\bar{\eta} \tag{4.104}$$

is the bosonic gauge fixing condition.

Below we will see that only the 0^{th} order term in (4.102) is essential for linear approximation theory.

The lowest dimensional components of the induced world volume torsion in the linearized approximation can be obtained from (4.6). Indeed taking into account Eqs. (4.99),(4.100) one obtains for the pull-back of the target space fermionic forms (in the linear approximation)

$$E^{2\alpha} = e^{\beta}(\delta_{\beta}^{\alpha} + D_{\beta}W^{\alpha}) + \bar{e}^{\dot{\beta}}\bar{D}_{\dot{\beta}}W^{\alpha} + w^{b}\partial_{b}W^{\alpha} \quad and \quad c.c. \tag{4.105}$$

Therefore we obtain from (4.6) the following expressions for the torsion components

$$t_{b\alpha}^{a} = -4i\partial_{b}\bar{W}^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^{a} \tag{4.106}$$

$$t_{\alpha\beta}^{a} = -4iD_{(\beta|}\bar{W}^{\dot{\alpha}}\sigma_{|\alpha)\dot{\alpha}}^{a} \tag{4.107}$$

$$t_{\alpha\dot{\alpha}}^{a} = -4i(2\sigma_{\alpha\dot{\alpha}}^{a} + D_{\alpha}W^{\beta}\sigma_{\beta\dot{\alpha}}^{a} + \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^{a}) \tag{4.108}$$

Comparing the linear superembedding Eq. (4.105) taken in the static gauge (4.99), (4.100) with its nonlinear counterpart (4.58) we see that in the linear approximation

$$h_{\beta}{}^{\alpha}|_{lin} = \delta_{\beta}^{\alpha} + D_{\beta}W^{\alpha} \quad C_{\dot{\beta}}^{2\alpha}|_{lin} = \bar{D}_{\dot{\beta}}W^{\alpha} \quad \phi_{b}{}^{\alpha}|_{lin} = \partial_{b}W^{\alpha}. \tag{4.109}$$

In addition taking into account that

$$dw^{a} \wedge d\eta^{\alpha}\sigma_{a\alpha\dot{\alpha}}\bar{W}^{\dot{\alpha}} = 0$$

one can present H_3 defined by (4.22) in the linearized approximation as follows

$$H_3 \equiv -dB_2 = -d(w^{a} \wedge d\eta^{\alpha}F_{a\alpha} + w^{a} \wedge d\bar{\eta}^{\dot{\alpha}}F_{a\dot{\alpha}}) \tag{4.110}$$

where we define

$$F_{a\alpha} = -2i\sigma_{a\alpha\dot{\alpha}}\bar{W}^{\dot{\alpha}}, \quad F_{a\dot{\alpha}} = -2i\sigma_{a\alpha\dot{\alpha}}W^{\alpha} \tag{4.111}$$

i.e. in the linearized approximation

$$dA \equiv \mathcal{F} + B_2 = F + B_2 = \frac{1}{2}w^a \wedge w^b F_{ba} + w^a \wedge d\eta^\alpha F_{a\alpha} + w^a \wedge d\bar{\eta}^{\dot{\alpha}} F_{a\dot{\alpha}}. \tag{4.112}$$

Thus we obtained that superfield F_{AB} satisfies the usual SYM constraints ($F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = F_{\alpha\dot{\beta}} = 0$).

Considering linearized integrability condition

$$d(\mathcal{F} - F) = 0 \tag{4.113}$$

one obtains from $dim2$ ($\propto w^a \wedge d\eta^\alpha \wedge d\eta^\beta$) component

$$D_\beta \bar{W}^{\dot{\alpha}} = 0, \tag{4.114}$$

which is just the linearized version of the Eq. (4.65).

Another $dim2$ ($\propto w^a \wedge d\bar{\eta}^{\dot{\alpha}} \wedge d\eta^\beta$) component of Eq. (4.113) can be presented in the form (which is just the linearized version of Eq. (4.70))

$$2\sigma^b_{\alpha\dot{\alpha}}F_{ba} + \sigma_{a\beta\dot{\alpha}}D_\alpha W^\beta + \sigma_{a\alpha\dot{\beta}}\bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\beta}} = 0 \tag{4.115}$$

and its scalar part (which is the only independent condition (see Appendix B)) is

$$D_\alpha W^\alpha + \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} = 0. \tag{4.116}$$

Linearized fermionic dynamical equation of motion (4.83) can be presented in the form

$$\sigma^a_{\alpha\dot{\beta}}\partial_a \bar{W}^{\dot{\beta}} = 0 \quad and \quad c.c. \tag{4.117}$$

To get the equations of motion for the gauge fields it is sufficient to take a divergence of the Eq. (4.115) and then use (4.117) to get the standard Yang-Mills equation

$$\partial^a F_{ab} = 0. \tag{4.118}$$

Taking into account (4.114) and flat spinor derivatives algebra $\{D_\alpha, \bar{D}_{\dot\beta}\} = 2i\sigma^a_{\alpha\dot\beta}\partial_a$ we obtain

$$\bar{D}_{\dot\beta} D_\alpha W^\alpha = 0 \quad and \quad c.c. \tag{4.119}$$

As a consequence of Eqs. (4.116), (4.119) we also have

$$D_\beta D_\alpha W^\alpha = 0. \tag{4.120}$$

Eqs. (4.119), (4.120) restrict $D_\alpha W^\alpha$ as follows

$$D_\alpha W^\alpha = ia \tag{4.121}$$

where a is an arbitrary real constant.

We have thus demonstrated that the choice of the basic superembedding condition (4.13) and the gauge field constraint (4.24) are consistent with the linearized limit of the D3–brane model which is just $N = 1$, $D = 4$ supersymmetric Maxwell theory.

4.8 The D3–brane action

We now present a worldvolume superfield action which we assume to produce (upon integrating over Grassmann coordinates and solving for the auxiliary fields) the standard action [83] for the D3–brane coupled to an $N = 2$ supergravity.

The D3–brane couples to supergravity fields via the worldvolume pullback of the Wess–Zumino form [83]

$$\hat{C} = C_4 + F_2 \wedge C_2 + \frac{1}{2} F_2 \wedge F_2 C_0, \tag{4.122}$$

where F_2 is defined in (4.62) and C_p (p=0,2,4) are 'Ramond-Ramond' p–form fields.

Since, as we have shown in Section 4.6, the superembedding conditions imply chirality of the worldvolume superfields, we assume the action to obtained by integration over $N = 1$, $D = 4$ chiral superspace $Z_L = (\xi^m_L, \eta^\alpha)$ of an appropriate pullback component of $\hat{C}$. Such a structure is prompted by the form of the worldvolume superfield actions for a heterotic string [84] and a supermembrane [85]. Due to dimensional reasons the Lagrangian is constructed with the use of $\hat{C}_{\dot\alpha\dot\beta ab}$, and the action (accompanied by the superembedding condition (4.24)) has the following form

$$S = \int d^2\xi_L d^2\eta \mathcal{E}_L \sigma^{ab\dot\alpha\dot\beta} \hat{C}_{\dot\alpha\dot\beta ab} \quad + \quad h.c. \tag{4.123}$$

where $\mathcal{E}_L = sdet(e_M^A)\ det(\eta_{ab} - F_{ab})\ det^{-1} h_\beta{}^\alpha$ is the chiral measure ($\mathcal{D}_{\dot\alpha}\mathcal{E}_L = 0$), and σ^{ab} is the antisymmetrized product of the Pauli matrices.

Upon integration over η (4.123) should produce both the Dirac–Born–Infeld and the Wess–Zumino term of the standard D3–brane action. In the static gauge (4.14) and in the linearized limit (4.100), (4.114) and (4.116) the action (4.123) reduces to the superfield Maxwell action.

To conclude, using the superembedding approach we have shown that the off–shell dynamics of the D3–brane in $N = 2$, $D = 4$ target superspace is described by the worldvolume superfield (superembedding) conditions (4.13) and (4.24). In the static gauge they reduce to nonlinear off–shell constraints on the spinor (Goldstone) superfield strength of the Dirac–Born–Infeld supermultiplet which generalize the Maxwell superfield linear constraints. This establishes the link of the superembedding formulation of the D3–brane with the nonlinear realization method used by Bagger and Galperin [65] to construct the $N = 1$, $D = 4$ superfield formulation of the Dirac–Born–Infeld theory as a Volkov–Akulov–type model exhibiting partial breaking of $N = 2$ supersymmetry down to $N = 1$.

APPENDIX

A. EQUATION OF STATE FOR A VACUUM

Eq. (2.18) can be easily understood from the following consideration. The perfect fluid has the property that viscous forces (which correspond to non-vanishing T^{ij} with $i \neq q$) are absent, and in the rest frame of a fluid element (*comoving frame*) the energy-momentum tensor takes the form

$$T_{ij} = p\delta_{ij}, \quad T_{i0} = 0, \quad T_{00} = \rho \tag{A.1}$$

i.e.

$$T_{\mu\nu} = diag(\rho, p, p, p) \tag{A.2}$$

Einstein equations (2.6) for a vacuum ($T_{\mu\nu} = 0$) become

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu} \tag{A.3}$$

and can be brought to the conventional form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T^{\Lambda}_{\mu\nu} \tag{A.4}$$

by defining

$$T^{\Lambda}_{\mu\nu} \equiv \frac{\Lambda}{8\pi G}g_{\mu\nu} \equiv \rho_{\Lambda} g_{\mu\nu}. \tag{A.5}$$

It is always possible to find a local *free fall* frame where $g_{\mu\nu} = \eta_{\mu\nu} \equiv diag(1, -1, -1, -1)$. Therefore in this local frame effective 'energy-momentum' tensor for a vacuum takes the form

$$T^{\Lambda}_{\mu\nu} = diag(\rho_\Lambda, -\rho_\Lambda, -\rho_\Lambda, -\rho_\Lambda) \tag{A.6}$$

and comparing Eq. (A.6) to the energy-momentum tensor of a perfect fluid in the comoving frame (A.2) we see that Eq. (2.18) is indeed satisfied.

B. INTERDEPENDENCE OF THE HIGHER DIMENSIONAL INTEGRABILITY CONDITIONS

In this section we investigate the higher dimensional integrability conditions

$$J_4 \equiv dJ_3 \equiv \frac{1}{4!} e^A \wedge e^B \wedge e^C \wedge e^D J_{DCBA} = 0 \tag{B.1}$$

for the integrability conditions J_3

$$J_3 \equiv d(\mathcal{F} - F) \equiv \frac{1}{3!} e^A \wedge e^B \wedge e^C J_{CBA} = 0 \tag{B.2}$$

(so called 'integrability condition for the integrability condition' technique)

The component form of (B.1) is

$$J_4 \equiv \frac{1}{4} J_{ABCD} = D_{[A} J_{BCD\}} + \frac{3}{2} t_{[AB|}^{E} J_{E|CD\}} = 0 \tag{B.3}$$

Taking into account the expression for the induced world volume torsion which can be obtained from (4.6) using superembedding equations (4.74)

$$t^a_{\beta\dot\gamma} = -8i\sigma^b_{\beta\dot\gamma}(\eta + F)^{-1}_b{}^a, \quad t^a_{\beta c} = -4i(h\sigma^a\phi_c)_\beta \tag{B.4}$$

$$t^a_{bc} = -4i(\phi_{[b|}\sigma^a\bar\phi_{|c]}), \quad t^a_{\alpha\beta} = 0$$

$$t^\alpha \equiv de^\alpha = 0$$

and assuming that all integrability conditions (B.2) of dimension 5/2 are satisfied

$$J_{\alpha cd} = 0 \tag{B.5}$$

we obtain from the component (B.3) of dimension 3 ($J_{\alpha\dot{\beta}cd}$)

$$t^{e}_{\alpha\dot{\beta}} J_{ecd} = 0 \Rightarrow J_{ecd} = 0 \tag{B.6}$$

In the same manner using *dim* 5/2 component of (B.3) $J_{\alpha\beta\dot{\gamma}a}$ and assuming that all integrability conditions (B.2) of *dim* 2 are satisfied

$$J_{\alpha\beta a} = J_{\dot{\alpha}\beta a} = 0 \tag{B.7}$$

we obtain

$$t^{e}_{\alpha\beta} J_{e\dot{\gamma}a} = 0 \Rightarrow J_{e\dot{\gamma}a} = 0 \tag{B.8}$$

If we assume that *dim* 3/2 components of the (B.2) are satisfied

$$J_{\alpha\beta\gamma} = J_{\dot{\alpha}\beta\gamma} = 0 \tag{B.9}$$

we can obtain from *dim* 2 component of (B.3) $J_{\alpha\beta\dot{\gamma}\dot{\delta}}$

$$t^{e}_{\alpha\beta} J_{e\dot{\gamma}\dot{\delta}} = 0 \Rightarrow J_{e\dot{\gamma}\dot{\delta}} = 0 \tag{B.10}$$

Taking (B.9) into account we obtain from *dim* 2 component of (B.3) $J_{\alpha\beta\gamma\dot{\delta}}$

$$t^{a}_{(\alpha\beta|} J_{a|\gamma)\dot{\delta}} = 0 \tag{B.11}$$

I.e. $J_{a\gamma\dot{\delta}}$ can in principle be independent of the lower dimensional components (B.9) (in fact it can be shown that only its scalar part $\tilde{\sigma}^{a\dot{\delta}\gamma} J_{a\gamma\dot{\delta}}$ is independent).

Thus we have shown that all the contents of the *dim* 3, *dim* 5/2 and *dim* 2 ($J_{a\alpha\beta}$ and their complex conjugate) integrability conditions (B.2) can be obtained from the corresponding action by covariant Grassman derivatives on their components of *dim* 3/2 (B.9). Therefore *dim* 3/2 integrability conditions as well as the scalar part of the $dim2$ component $J_{a\alpha\dot\beta}$ are independent. Taking into account that in the case under consideration (B.9) are satisfied identically we obtain that the only independent component of (B.2) is $J_{a\gamma\dot\delta}$.

Using the same technique for the superembedding conditions (4.74) we obtain their integrability conditions (4.88) which can be written in the component form as follows

$$I_2^{\alpha} \equiv \frac{1}{2} e^A \wedge e^B I^{\alpha}_{BA} = 0. \tag{B.12}$$

The integrability conditions for (B.12) are

$$I_3^{\alpha} = \frac{1}{3} e^A \wedge e^B \wedge e^C I^{\alpha}_{CBA} = 0 \tag{B.13}$$

and can be presented in the component form

$$\frac{1}{3} I^{\alpha}_{ABC} \equiv D_{\{A} I^{\alpha}_{BC]} + t_{\{AB|}{}^{D} I^{\alpha}_{D|C]} = 0 \tag{B.14}$$

Now assuming that all the integrability conditions (B.12) of $dim1$ are satisfied ($I^{\alpha}_{\beta a} = 0$) one obtains from the $dim3/2$ ($I^{\alpha}_{\beta\dot\gamma a}$) component of (B.14)

$$t^{a}_{\beta\dot\gamma} I^{\alpha}_{da} = 0 \Rightarrow \quad I^{\alpha}_{da} = 0 \tag{B.15}$$

Thus integrability conditions of $dim3/2$ follow from the integrability conditions of lower dimension.

In the same manner assuming that $I^{\alpha}_{\beta\dot\gamma} = 0, \quad I^{\alpha}_{\beta\gamma} = 0$ and their integrability conditions (B.14) are satisfied we obtain

$$t^{a}_{\beta\gamma} I^{\alpha}_{a\delta} = 0 \Rightarrow I^{\alpha}_{a\delta} = 0 \tag{B.16}$$

In other words components of $dim3/2$ (I^{α}_{ab}) and $dim1$ ($I^{\alpha}_{\beta a}, \quad I^{\alpha}_{\dot\beta a}$) of the integrability condition (B.12) are dependent and the only independent set of components of this integrability condition are

$$I^{\alpha}_{\beta\gamma} = 0, \quad I^{\alpha}_{\beta\dot{\alpha}} = 0 \tag{B.17}$$

C. FERMONIC SUPEREMBEDDING CONDITIONS FROM GAUGE FIELD CONSTRAINTS

If one extracts the expressions

$$\mathcal{E}^{\alpha} \equiv E^{2\alpha} - E^{1\beta} h_{\beta}{}^{\alpha} - E^{a}\phi_{a}^{\alpha} = E^{1\beta}(E_{\beta}^{2\alpha} - h_{\beta}{}^{\alpha}) \quad and \quad c.c. \tag{C.1}$$

from all the terms in $l.h.s.$ of (4.64), one obtains

$$J_3 \equiv d(\mathcal{F} - F) = -2iE^{b} \wedge E^{1\alpha} \wedge \bar{E}^{1\dot{\alpha}}[\sigma^{a}_{\alpha\dot{\alpha}}(\eta - F)_{ab} - (h\sigma\bar{h}^{T})_{\alpha\dot{\alpha}}(\eta + F)_{ab}] + \tag{C.2}$$

$$\frac{1}{2}E^{b} \wedge E^{c} \wedge E^{1\alpha} J_{\alpha cb} + \frac{1}{2}E^{b} \wedge E^{c} \wedge \bar{E}^{1\dot{\alpha}} J_{\dot{\alpha}cb} - \frac{1}{2}E^{b} \wedge E^{c} \wedge E^{d} J_{dcb} +$$

$$2iE^{b} \wedge [E^{1\beta} \wedge \bar{E}^{1\dot{\beta}}((E_{\beta}^{2\alpha} - h_{\beta}{}^{\alpha})\bar{E}_{\dot{\beta}}^{2\dot{\alpha}} + (\bar{E}_{\dot{\beta}}{}^{2\dot{\alpha}} - \bar{h}_{\dot{\beta}}{}^{\dot{\alpha}})h_{\beta}{}^{\alpha}) +$$

$$E^{1\beta} \wedge E^{c}(E_{\beta}^{2\alpha} - h_{\beta}{}^{\alpha})\bar{\phi}_{c}^{\dot{\alpha}} + \bar{E}^{1\dot{\beta}} \wedge E^{c}(\bar{E}_{\dot{\beta}}^{2\dot{\alpha}} - \bar{h}_{\dot{\beta}}{}^{\dot{\alpha}})\phi_{c}^{\alpha}]\sigma^{a}_{\alpha\dot{\alpha}}(\eta + F)_{ab} = 0$$

where

$$J_{\alpha cb} = D_{\alpha}F_{cb} - 4i(h\sigma^{a}\bar{\phi}_{[c|})_{\alpha}(\eta - F)_{|b]a} \tag{C.3}$$

$$J_{dcb} = -D_{[d}F_{cb]} + 4i(\phi_{[d}\sigma^{a}\bar{\phi}_{c})(\eta - F)_{b]a} \tag{C.4}$$

If we assume that

$$(h\sigma^{a}\bar{h}^{T})_{\alpha\dot{\beta}} = \sigma^{b}_{\alpha\dot{\beta}}((\eta - F)(\eta + F)^{-1})_{b}{}^{a} \tag{C.5}$$

is satisfied than the first line in (C.2) vanishes identically, as well as the second line which vanishes due to the fact that (C.3) and (C.4) are equal to zero in this case (see

(4.71), (4.72)). Therefore in this case (C.2) takes the form

$$J_3 = 2iE^b \wedge E^{1\beta} \wedge \bar{E}^{1\dot{\beta}}[(E^{2\alpha}_{\beta} - h_{\beta}{}^{\alpha})\bar{E}^{2\dot{\alpha}}_{\dot{\beta}} + (\bar{E}_{\dot{\beta}}{}^{2\dot{\alpha}} - \bar{h}_{\dot{\beta}}{}^{\dot{\alpha}})h_{\beta}{}^{\alpha}]\sigma^a_{\alpha\dot{\alpha}}(\eta + F)_{ab} + \tag{C.6}$$

$$2iE^b[E^c \wedge E^{1\beta}(E^{2\alpha}_{\beta} - h_{\beta}{}^{\alpha})\bar{\phi}^{\dot{\alpha}}_c + c.c.)]\sigma^a_{\alpha\dot{\alpha}}(\eta + F)_{ab} = 0.$$

Decomposing (C.6) onto the basic forms of the worldvolume we obtain that it contains three equations:

$$E^{2\alpha}_{\beta}\sigma^a_{\alpha\dot{\alpha}}\bar{E}^{2\dot{\alpha}}_{\dot{\beta}} = h_{\beta}{}^{\alpha}\sigma^a_{\alpha\dot{\alpha}}\bar{h}_{\dot{\beta}}{}^{\dot{\alpha}} \tag{C.7}$$

$$(E^{2\alpha}_{\beta} - h_{\beta}{}^{\alpha})\sigma^a_{\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}}_{[c}(\eta + F)_{b]a} = 0 \quad and \quad c.c. \tag{C.8}$$

The general solution of Eq. (C.7) is

$$E^{2\beta}_{\alpha} = \pm e^{i\lambda}h_{\alpha}{}^{\beta} \tag{C.9}$$

where λ is an arbitrary real superfield.

Taking into account that Eqs. (C.7), (C.8) should be satisfied simultaneously one should further restricts (C.9) to the form:

$$E^{2\beta}_{\alpha} = h_{\alpha}{}^{\beta} \quad and \quad c.c. \tag{C.10}$$

Thus the condition (C.10) is the general sequence of the Eq. (C.5) and its integrability conditions (C.2)

D. ALTERNATIVE CHOICE OF THE SUPERVIELBEIN PULLBACKS AND TRIVIAL DYNAMICS

In this section we prove that the solution (4.67) of (4.65) leads to the trivial solution of the superembedding condition.

Under assumption

$$E_{\alpha}^{2\beta} \equiv h_{\alpha}^{2\beta} = 0 \tag{D.1}$$

the generic expression for the pullbacks of fermionic one-form $E^{2\alpha}$ and $\bar{E}^{2\dot{\alpha}}$ takes the form (4.69)

$$E^{2\alpha} = \bar{E}^{1\dot{\alpha}} C_{\dot{\alpha}}^{2\alpha} + E^{a} \phi_{a}{}^{\alpha}, \quad \bar{E}^{2\dot{\alpha}} = E^{1\alpha} C_{\alpha}^{2\dot{\alpha}} + E^{\alpha} \bar{\phi}_{a}^{\dot{\alpha}} \tag{D.2}$$

(for simplicity we denote $C_{\dot{\alpha}}^{2\alpha}$ as $C_{\dot{\alpha}}^{\alpha}$ below).

In this case the $a\alpha\dot{\alpha}$ component of (4.64) (which is the integrability condition of (4.24)) takes the form (cf. with (4.70))

$$C_{\dot{\alpha}}{}^{\beta} \sigma_{\beta\dot{\beta}}^{c} \bar{C}_{\alpha}{}^{\dot{\beta}} = \sigma_{\alpha\dot{\alpha}}^{b} k_{b}{}^{c} \tag{D.3}$$

which can be written in vector notations (C^{m} *and* $\bar{C}^{n}$) introduced by

$$C_{\dot{\alpha}}{}^{\beta} = C^{m} \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\sigma}_{m}^{\dot{\beta}\beta}, \quad C_{\alpha}{}^{\dot{\beta}} = \bar{C}^{n} \epsilon^{\dot{\beta}\dot{\alpha}} \sigma_{n\alpha\dot{\alpha}} \tag{D.4}$$

as

$$k_{ca} = C_a\bar{C}_c + C_c\bar{C}_a - \eta_{ac}C_n\bar{C}^n + i\epsilon_{camn}C^m\bar{C}^n \tag{D.5}$$

According to its definition (4.31) k has the property

$$k_{ab} = k^{-1}_{ab} \Rightarrow k^{-1ab}k_{bc} = \delta^b_a. \tag{D.6}$$

Substituting k from (D.5) into (D.6) we obtain

$$\delta^b_a C^2\bar{C}^2 = \delta^b_a \Rightarrow C^2\bar{C}^2 = 1. \tag{D.7}$$

Next, from (D.5) and (D.7) one can easily obtain

$$C^2\bar{C}^b = k^{ab}C_a \tag{D.8}$$

and

$$k^{ab}C_a\bar{C}_b = 1. \tag{D.9}$$

Complex vector C^a can be decomposed into the real parts as

$$C^b = A^b + iB^b \tag{D.10}$$

where A^b, B^b are two real vectors. Substituting (D.10) into (D.8) we obtain two equations

$$C^2A^b = k^{ab}A_a \tag{D.11}$$

and

$$C^2B_b = -k_{ab}B^a. \tag{D.12}$$

Contracting (D.11) with B_b we obtain

$$C^2A^bB_b = k^{ab}A_aB_b \tag{D.13}$$

or substituting (D.12) into (D.13)

$$k^{ab}(A_a B_b + B_a A_b) = 0. \tag{D.14}$$

From the other hand, substituting (D.10) into (D.9) we obtain two equations

$$k^{ab}(A_a A_b + B_a B_b) = 1 \tag{D.15}$$

and

$$k^{ab}(B_a A_b - A_a B_b) = 0. \tag{D.16}$$

From (D.15) and (D.16) one can obtain that

$$k^{ab} B_a A_b = 0. \tag{D.17}$$

Substituting (D.12) into (D.17) we obtain

$$k^{ab} B_a A_b = (k_{ab} B^a) A^b = -C^2 B_b A^b = 0 \tag{D.18}$$

thus if we assume that $C^2 \neq 0$ (i.e. $A, B \neq 0$ simultaneously) we obtain that

$$B_b A^b = 0 \tag{D.19}$$

is the general sequence of (D.18). In this case

$$C^2 = \bar{C}^2 = C\bar{C} = A^2 + B^2 \tag{D.20}$$

and

$$C^a = A^a, \quad B^a = 0 \tag{D.21}$$

is the general solution of (D.20).

Substituting (D.21) into (D.5) one obtains

$$-A^2\eta^{ab} + 2A^aA^b = k^{ab} \tag{D.22}$$

or (contracting (D.22) with A_a)

$$A^2A^b = k^{ab}A_a, \tag{D.23}$$

but it follows from (D.7) that

$$A^2 = 1 \quad or \quad A^2 = -1. \tag{D.24}$$

We will assume that $A^2 = 1$ in the following (consideration below can be easily repeated for the $A^2 = -1$ case and leads to the similar results). From (D.24) and (D.23) it follows that

$$A^b = k^{ab}A_a. \tag{D.25}$$

Decomposing k^{ab} as

$$k^{ab} = \eta^{ab} + M^{ab} \tag{D.26}$$

(where M^{ab} is some unknown matrix) we obtain from

$$k^{ab}k_{bc} = \delta^a_c \tag{D.27}$$

(which follows from (D.25)) that

$$M^{ab}(2\eta_{bc} + M_{bc}) = 0. \tag{D.28}$$

This equation has only two solutions: either

$$M^{ab} = 0 \Rightarrow \quad k^{ab} = \eta^{ab} \tag{D.29}$$

or

$$M^{ab} = -2\eta^{ab} \Rightarrow \quad k^{ab} = -\eta^{ab} \tag{D.30}$$

The choice (D.30) contradicts to (D.25). Taking into account that (as it follows from (4.31))

$$k^{ab} = \eta^{ab} - 2((\eta + F)^{-1}F)^{ab} \tag{D.31}$$

we obtain from (D.29) that

$$[(\eta + F)^{-1}F]^{ab} = 0 \tag{D.32}$$

which leads to the trivial solution

$$F^{ab} = 0 \tag{D.33}$$

because $det(\eta + F) \neq 0$ is the general assumption of Born-Infeld-like models.

Therefore we have proved that the assumption (D.1) leads to trivial solution (D.33) and should be omitted from further consideration. Therefore only the solution (4.66) (and its sequences (D.2)) describe nontrivial brane dynamics.

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