

# Holomorphic Motions and Extremal Annuli

by

Zhe Wang

A dissertation submitted to the Graduate Faculty in Mathematics  
in partial fulfillment of the requirements for the degree of Doctor  
of Philosophy, The City University of New York.

2011

©2011

Zhe Wang

All Rights Reserved

---

This manuscript has been read and accepted for the  
Graduate Faculty in Mathematics in satisfaction of the  
dissertation requirement for the degree of Doctor of Philosophy.

*Yunping Jiang*  
\_\_\_\_\_

\_\_\_\_\_  
*Date*

\_\_\_\_\_  
*Chair of Examining Committee*

*Josef Dodziuk*  
\_\_\_\_\_

\_\_\_\_\_  
*Date*

\_\_\_\_\_  
*Executive Officer*

*Yunping Jiang*  
\_\_\_\_\_

*Frederick P. Gardiner*  
\_\_\_\_\_

*Linda Keen*  
\_\_\_\_\_

*Supervisory Committee*

## Abstract

## Holomorphic Motions and Extremal Annuli

by

Wang, Zhe

Advisor: Professor Frederick P. Gardiner and Professor Yunping Jiang

## Abstract:

Holomorphic motions, soon after they were introduced, became an important subject in complex analysis. It is now an important tool in the study of complex dynamical systems and in the study of Teichmüller theory. This thesis serves on two purposes: an expository of the past developments and a discovery of new theories.

First, I give an expository account of Slodkowski's theorem based on the proof given by Chirka. Then I present a result about infinitesimal holomorphic motions. I prove the  $|\varepsilon \log \varepsilon|$  modulus of continuity for any infinitesimal holomorphic motion. This proof is a very well application of Schwarz's lemma and the estimate of Agard's formula for the hyperbolic metric on the thrice punctured sphere. One application of this result is that, after the integration of an infinitesimal holomorphic motion, it leads to the Hölder continuity property of a quasiconformal homeomorphism. This will be presented in Chapter 3.

Second, I compare the proofs given by both Slodkowski and Chirka. Then I construct a different extension of a holomorphic motion in the frame work of Slodkowski's proof by using the method in Chirka's proof. This gives some opportunity for me to discuss the uniqueness in the extension problem for a holomorphic motion. This will be presented in Chapter 4.

Third, I discuss the universal holomorphic motion for a closed subset of the Riemann sphere and the lifting property in the Teichmüller theory. One application of this discussion is the proof of the coincidence of Teichmüller's metric and

Kobayashi's metric, a result due to Royden and Gardiner, given by Earle, Kra, and Krushkal by using Slodkowski's theorem. This will be presented in Chapters 5 and 6.

Fourth, I study the complex structure of the universal asymptotically conformal Teichmuller space. I give a direct and new proof of the coincidence of Teichmuller's metric and Kobayashi's metric on the universal asymptotically conformal Teichmuller space, a result previously proved by Earle, Gardiner, and Lakic. The main technique that I have used in this proof is Strebel's frame mapping theorem. This will be presented in Chapter 7.

Finally, in Chapter 8, I study extremal annuli on a Riemann sphere with four points removed. By using the measurable foliation theory, the Weierstrass P-function, and the variation formula for the modulus of an annulus, I prove that the Mori annulus maximize the modulus for the two army problem in the chordal distance on the Riemann sphere. Gardiner and Masur's minimum axis is also discussed in this chapter.

Most of the results in this thesis have been published in several research papers jointly with Fred Gardiner, Jun Hu, Yunping Jiang, and Sudeb Mitra.

## Acknowledgments

I am grateful to my advisers Prof. Frederick P. Gardiner and Prof. Yunping Jiang for sharing their insight and experience in mathematics and life with me, especially the generous advice they gave me throughout working on all papers in this dissertation.

I also indebted to Jun Hu and Sudeb Mitra for introducing me the earthquake theorem and universal holomorphic motion. It is a wonderful experience working with them.

Linda Keen is the leader of our big complex analysis group at Graduate Center of CUNY, she always helps and encourages young people including me to share ideas and give lectures in the complex analysis and dynamical systems student seminar and the complex analysis and dynamics seminar. I had many instructive conversations with her during or after my talks in these seminars.

Our seminars provided a great platform for people in the great New York area, in particular for students, to learn the modern theory in complex analysis and dynamical systems. During the last few years, I was benefited by these two seminars and had a great opportunity to participate in many interesting discussions with Ara Basmajian, Mike Beck, Reza Chamanara, Zeno Huang, Nikola Lakic, Dragomir Saric, Saeed Zakeri, Tao Chen, Oleg Muzician, Viveka Erlandsson, Gerardo Jimenez, Robert Suzzi Valli, Ozgur Evren, Hengyu Zhou and all others in the seminars.

Frederick, Reza and Jun also help me a lot for my teaching at Brooklyn college. Thank you for sharing your experience, homework problems, lecture notes and exams with me.

Finally, I would like to express my deepest gratitude to my family, specially to my dad, my mom and my wife. To my parents, I would like to thank them for teaching me to think about all sides of every issue and to my wife, Yunchun Hu, I would like to thank her for supportive and understanding.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Metrics of constant curvature . . . . .	1
1.2	Definition of quasiconformal map . . . . .	3
1.3	Modulus of a rectangle and modulus of an annulus . . . . .	6
1.4	Quadratic differential . . . . .	10
1.5	Beltrami equation . . . . .	15
1.6	Universal Teichmüller space $T(\Delta)$ or $T(H)$ . . . . .	17
1.7	Beurling-Ahlfors Extension . . . . .	20
1.8	Complex structure of Teichmüller space . . . . .	22
1.8.1	Schwarzian derivative and Bers embedding . . . . .	22
1.8.2	Complex structure of $T_0(\Delta)$ and $AT(\Delta)$ . . . . .	26
1.9	Measured foliation . . . . .	27
<b>2</b>	<b>Holomorphic Motion Theorem</b>	<b>33</b>
2.1	The $\mathcal{P}$ -Operator and the Modulus of Continuity . . . . .	35
2.2	Extensions of holomorphic motions for $0 < r < 1$ . . . . .	39
2.3	Controlling quasiconformal dilatation . . . . .	49
2.4	Extension of holomorphic motions for $r = 1$ . . . . .	53
<b>3</b>	<b>The <math> \varepsilon \log \varepsilon </math> continuity of a holomorphic motion</b>	<b>57</b>
3.1	Agard's formula for $\rho_{0,1}$ and its lower bound . . . . .	57
3.2	The $ \varepsilon \log \varepsilon $ continuity of a holomorphic motion . . . . .	61
<b>4</b>	<b>Different extensions of holomorphic motion</b>	<b>67</b>

---

4.1	From Green's Theorem to Chirka's proof . . . . .	68
4.2	Different ways to construct $\varphi(z, w)$ . . . . .	74
4.3	Examples of holomorphic motions . . . . .	76
<b>5</b>	<b>Lifting map of Teichmuller space</b>	<b>79</b>
5.1	Teichmüller space of a closed set . . . . .	79
5.2	The lifting theorem and Universal holomorphic motions . . . . .	81
5.2.1	Universal holomorphic motions . . . . .	82
5.2.2	A proposition . . . . .	84
5.3	Proof of the lifting theorem . . . . .	86
5.4	Some concluding remarks . . . . .	90
<b>6</b>	<b>Kobayashi's and Teichmuller metrics on <math>T(\mathcal{R})</math></b>	<b>93</b>
6.1	Kobayashi's and Teichmuller metrics . . . . .	93
6.2	The Lifting Problem . . . . .	97
<b>7</b>	<b>Kobayashi's and Teichmüller's Metrics on <math>\mathcal{T}_0</math></b>	<b>103</b>
7.1	Kobayashi's metric $\geq$ Teichmüller's metric on $\mathcal{T}_0$ . . . . .	103
7.2	Extremal maps of points in $\mathcal{T}_0$ . . . . .	105
7.3	The proof of Theorem 38, $d_K = d_T$ on $\mathcal{T}_0$ . . . . .	107
7.4	Approaching by frame mapping . . . . .	109
<b>8</b>	<b>Extremal Annuli on the Sphere</b>	<b>113</b>
8.1	Extremal Length and Modulus of Annulus . . . . .	113
8.2	The intersection inequality . . . . .	118
8.3	The minimal axis theorem . . . . .	119
8.4	The Teichmüller annulus . . . . .	123
8.5	The Mori annulus . . . . .	129
8.6	Comparison of the Mori and Teichmüller annuli . . . . .	134
8.7	Pairs of extremal of annuli on a four times punctured sphere . . . . .	137
8.8	Mori type extremal problems . . . . .	138

<b>Contents</b>	<b>ix</b>
<hr/>	
<b>A Papers</b>	<b>141</b>
A.1 Papers . . . . .	141
A.2 Preprints . . . . .	142
<b>Bibliography</b>	<b>143</b>

# List of Figures

1.1	A Quasiconformal mapping . . . . .	4
1.2	Schwarz reflection of conformal maps . . . . .	6
1.3	Topological triangles . . . . .	7
1.4	A topological annulus . . . . .	9
1.5	Horizontal and Vertical trajectories . . . . .	11
1.6	Trojecotries for four punctured Riemann sphere . . . . .	12
2.1	Holomorphic motion . . . . .	40
2.2	Smooth function for Chirka's operator . . . . .	42
8.1	Mori's annulus . . . . .	117
8.2	Teichmuller minimal axis . . . . .	124
8.3	Weierstrass P function . . . . .	125
8.4	Mori's minimal axis . . . . .	133
8.5	Extremal domains . . . . .	135
8.6	Six punctured Riemann sphere . . . . .	139

# Introduction

## Contents

1.1	Metrics of constant curvature . . . . .	<b>1</b>
1.2	Definition of quasiconformal map . . . . .	<b>3</b>
1.3	Modulus of a rectangle and modulus of an annulus . . . . .	<b>6</b>
1.4	Quadratic differential . . . . .	<b>10</b>
1.5	Beltrami equation . . . . .	<b>15</b>
1.6	Universal Teichmüller space $T(\Delta)$ or $T(H)$ . . . . .	<b>17</b>
1.7	Beurling-Ahlfors Extension . . . . .	<b>20</b>
1.8	Complex structure of Teichmüller space . . . . .	<b>22</b>
	1.8.1 Schwarzian derivative and Bers embedding . . . . .	22
	1.8.2 Complex structure of $T_0(\Delta)$ and $AT(\Delta)$ . . . . .	26
1.9	Measured foliation . . . . .	<b>27</b>

Suppose  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is the extended complex plane. For any real number  $r > 0$ , we let  $\Delta_r$  be the disk centered at the origin in  $\mathbb{C}$  with radius  $r$  and  $\Delta$  be the disk of unit radius.

## 1.1 Metrics of constant curvature

For a metric  $\rho(z)|dz|$  of class  $C^2$ , the quantity

$$\kappa(\rho) = -\rho^2 \Delta \log \rho,$$

where  $\Delta$  is the Laplacian ( $\Delta\mu = \partial^2\mu/\partial x^2 + \partial^2\mu/\partial y^2$ ), is known as the Gaussian curvature, which is conformal invariant. To calculate the curvature, it is convenient to use the complex form of the Laplacian:

$$\mu_{xx} + \mu_{yy} = \Delta\mu = 4 \frac{\partial^2\mu}{\partial z\partial\bar{z}}.$$

A metric of the constant curvature 4 for the sphere is the spherical metric

$$ds = \frac{|dz|}{1 + |z|^2}.$$

In the case of the complex plane  $\mathbb{C}$ , the Euclidean metric  $|dz|$  is complete and has constant curvature 0. For the punctured plane  $\mathbb{C} \setminus \{0\}$ , the metric  $|z|^{-1}|dz|$  is complete and has zero curvature.

We turn now to the more interesting case, that of the hyperbolic plane  $H = \{z : \text{Im}z > 0\}$ . Here, the infinitesimal metric is

$$\rho(z)|dz| = \frac{|dz|}{|z - \bar{z}|}.$$

It is called, interchangeably, either the non-Euclidean metric or the hyperbolic metric or the Poincare metric.

The poincare metric for the unit disk  $\Delta$  is

$$\rho(z)|dz| = \frac{|dz|}{1 - |z|^2}.$$

These two poincare metrics has constant curvature -4. Global poincare metric on the unit disk is

$$d(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.$$

In general, if  $\Omega$  is an arbitrary simply-connected domain and  $f$  is a biholomorphic map from  $\Omega$  onto  $H$  (or  $\Delta$ ), we define the poincare metric of  $\Omega$ ,

$$\rho_{\Omega}(z) = \rho(f(z))|f'(z)|.$$

Moreover, the infinitesimal form  $\rho_R(z)|dz|$  of the Poincare metric on a Riemann surface  $R$  is given by the formula

$$\rho_R(z_0) = \inf \frac{1}{|g'(0)|}$$

where the infimum is taken over all holomorphic functions  $g$  in the unit disk mapping into the surface  $R$  with  $g(0) = z_0$ .

The largest subdomain of the Riemann sphere carrying a hyperbolic metric is the sphere with three points removed. The infinitesimal form  $\rho_{0,1}(z)|dz|$  of the Poincare metric on the three times punctured sphere  $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$  is given by the Agard's formula [1]:

$$\rho_{0,1}(z_0)^{-1} = \frac{1}{\pi} \int \int_{\overline{\mathbb{C}}} \frac{|z_0(z_0 - 1)|}{|\zeta(\zeta - 1)(\zeta - z_0)|} d\xi d\eta,$$

where  $\zeta = \xi + i\eta$ .

For more properties about Poincare metrics, please read the book of Linda Keen and Nikola Lakic [47]. In Chapter 3, I will use the Poincare metric  $\rho_{0,1}(z)|dz|$  to show the  $|\varepsilon \log \varepsilon|$  continuity of a holomorphic motion. And the above Agard's formula also can be proved by the extension theorem of holomorphic motions, I will give the details of the proof in Section 3.1.

## 1.2 Definition of quasiconformal map

Let  $w = f(z)$  ( $w = u + iv$  and  $z = x + iy$ ) be a  $C^1$  homeomorphism from one region to another. At a point  $z_0$ , it induces a linear mapping of differentials

$$du = u_x dx + u_y dy$$

$$dv = v_x dx + v_y dy$$

which we can also write in complex form

$$dw = f_z dz + f_{\bar{z}} d\bar{z}$$

with  $f_z = \frac{1}{2}(f_x - if_y)$  and  $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$ .

The Jacobian

$$J(z) = |f_z|^2 - |f_{\bar{z}}|^2$$

is positive for an orientation preserving mapping. In this dissertation, We only consider orientation preserving maps, i.e.,  $|f_{\bar{z}}| < |f_z|$ . I introduce now the *complex*

dilatation

$$\mu_f(z) = \frac{f_{\bar{z}}}{f_z},$$

and the dilatation

$$D_f(z) = \frac{1 + d_f}{1 - d_f} = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1.$$

Then locally we have  $(|f_z| - |f_{\bar{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|)|dz|$ . This inequality implies that  $f$  maps an infinitesimal circle around  $z$  to an infinitesimal ellipse around  $w = f(z)$ . And  $D_f$  is the ratio of the major axis to the minor axis.

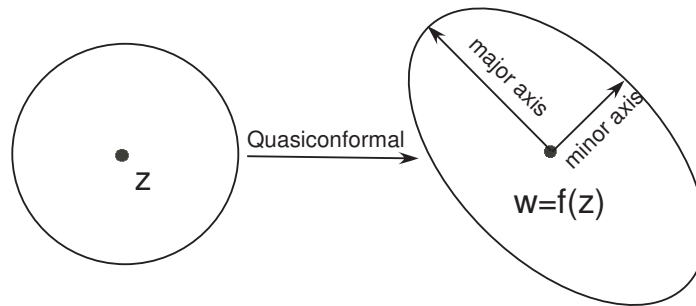


Figure 1.1: A Quasiconformal mapping

**Definition 1** ( $C^1$  Quasiconformal Map). A  $C^1$  homeomorphism  $f(z)$  is said to be quasiconformal if  $D_f$  is bounded, and it is  $K$ -quasiconformal if  $D_f \leq K$ .

Sometimes it is more convenient to consider  $d_f$ , and the condition  $1 \leq D_f \leq K$  in this definition is equivalent to the condition  $0 \leq d_f = |\mu_f| \leq k = \frac{K-1}{K+1} < 1$ . The simplest examples of  $C^1$  quasiconformal map are complex affine maps.

**Example 1.**

$$f(z) = z + \frac{1}{2}\bar{z}$$

is a 3-quasiconformal map on the complex plane.

Actually, the condition  $C^1$ -map is not necessary for quasiconformal mappings. We have following three equivalent and more general definitions for quasiconformality.

**Definition 2** (Analytic Definition). *The homeomorphism  $f$  from a plane domain  $\Omega$  to a plane domain  $f(\Omega)$  is quasiconformal if there exists  $0 \leq k < 1$  such that*

1.  $f$  has locally integrable, distributional derivatives  $f_z$  and  $f_{\bar{z}}$  on  $\Omega$ .
2.  $|f_{\bar{z}}| \leq k|f_z|$  almost everywhere.

**Definition 3** (Geometric Definition). *An orientation preserving homeomorphism  $f$  from a plane domain  $\Omega$  to a plane domain  $f(\Omega)$  is  $K$ -quasiconformal, if for every topological rectangle (quadrilateral)  $Q$  contained in  $\Omega$ ,*

$$K^{-1} \text{mod}(Q) \leq \text{mod}(f(Q)) \leq K \text{mod}(Q),$$

where  $\text{mod}(Q)$  is the modulus for  $Q$ .

(please read the Section 1.3 for more details about definitions and properties of  $\text{mod}(Q)$ .)

For the last equivalent definition of a quasiconformal mapping, we use the ratio distortion (or cross ratio distortion).

**Definition 4.** *Suppose  $H : \bar{\mathbb{C}} \mapsto \bar{\mathbb{C}}$  is an orientation-preserving homeomorphism such that  $H(\infty) = \infty$ . Then one of the definitions of quasiconformality [51] of  $H$  is that*

$$\limsup_{r \rightarrow 0} \sup_{a \in \mathbb{C}} \frac{\sup_{|z-a|=r} |H(z) - H(a)|}{\inf_{|z-a|=r} |H(z) - H(a)|} < \infty.$$

In [62], Sullivan and Thurston used this definition to show that a holomorphic motion is quasiconformal for any fixed value of the time parameter  $t$  for  $|t| < 1$ . It is easy to see that Definition 1 is a special case of Definition 2. The fact that Definition 2 is equivalent to Definition 3 was proved by Ahlfors in [3]. In the book of Lehto [52], there are more discussions about Definition 4 of a quasiconformal map.

Now let us look at some basic properties of a quasiconformal map:

1.  $f$  and  $f^{-1}$  are simultaneously  $K$ -quasiconformal.
2. The composition of a  $K_1$ -quasiconformal mapping and a  $K_2$ -quasiconformal mapping is  $K_1 K_2$ -quasiconformal.

3.  $\mu = 0$  a.e. if and only if  $f$  is conformal.
4. Let  $\mu, \sigma$ , and  $\tau$  be the complex dilatations of quasiconformal maps of  $f^\mu, f^\sigma$  and  $f^\tau$  with  $f^\tau = f^\sigma \circ (f^\mu)^{-1}$ . Then

$$\tau = \left( \frac{\sigma - \mu}{1 - \bar{\mu}\sigma} \frac{1}{\theta} \right) \circ (f^\mu)^{-1}$$

where  $\theta = \frac{\bar{p}}{p}$  and  $p = \frac{\partial}{\partial z} f^\mu(z)$ .

**Proposition 1.**  $\{f_n\}$  is a family of normalized  $K$ -quasiconformal maps fixing  $0, 1, \infty$ , then  $\{f_n\}$  is a normal family, i.e., there is a converging subsequence  $\{f_{n_i}\} \rightarrow f$ , where the limit  $f$  is a  $K$ -quasiconformal map or a constant map.

For the proof of Proposition 1 and previous properties, please read [51] and [52]. It is also helpful to use the Geometric Definition to understand these properties.

### 1.3 Modulus of a rectangle and modulus of an annulus

**Definition 5.** For a rectangle  $S$  with length  $a$  and width  $b$ , the modulus of this rectangle is defined by

$$\text{mod}(S) = \frac{a}{b}.$$

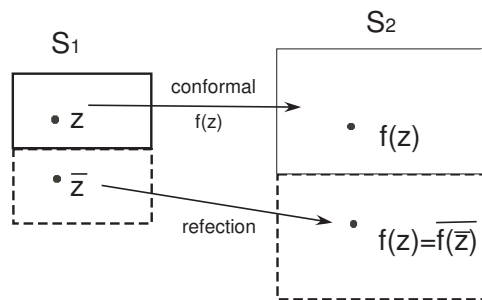


Figure 1.2: Schwarz reflection of conformal maps

An interesting property of modulus is:

**Proposition 2.** Moduli of rectangles are conformally invariant.

*Proof* Assume  $f$  is conformal from rectangle  $S_1$  to  $S_2$ . By Schwarz reflection principle, we can extend  $f$  into  $S'_1$  in the lower half plane by  $\bar{f}(\bar{z})$  (see Figure 1.2). Similarly we can reflect  $f(z)$  with respect to other three sides of  $S_1$ . Then we have a new conformal map from a much larger rectangle to another larger rectangle.

Apply the Schwarz reflection principle to all sides of new rectangles again and again, finally we have a conformal map from  $\bar{\mathbb{C}}$  to  $\bar{\mathbb{C}}$  fixing  $\infty$ . So  $f$  must be an affine map, i.e.,  $f(z) = az + b$ . The affine map does not change the modulus of a rectangle, so the modulus of a rectangle is a conformal invariant.

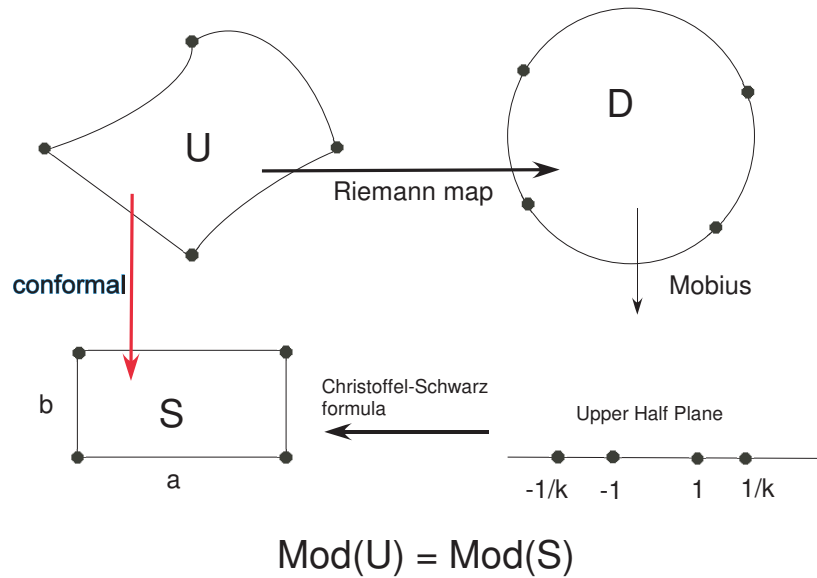


Figure 1.3: Topological triangle

In the Geometric Definition of a quasiconformal map, I consider the modulus of a topological rectangle, which is a simply connected region  $U \neq \mathbb{C}$  with 4 ordered marked boundary points. Next, I will use the Riemann mapping theorem to construct conformal maps from topological rectangles to real rectangles.

**Theorem 1** (Riemann mapping theorem). *Given any simply connected region  $U$  which is not the whole plane, there exists a conformal map  $f$  which maps  $U$  onto the unit disk  $\Delta$ .*

**Proposition 3.** *Any topological rectangle  $U$  can be conformally mapped to a rect-*

angle  $S$ . (see Figure 1.3)

The lower map in Figure 1.3 is the Christoffel-Schwarz formula

$$w = \int_{-\frac{1}{k}}^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}},$$

which maps the upper-half plane onto a rectangle. More precisely, it will map the real line to 4 sides of the rectangle, and  $\{-\frac{1}{k}, -1, 1, \frac{1}{k}\}$  are mapped to the vertices of the rectangle. At each corner of the rectangle, locally this map is  $\sqrt{z}$  which maps angles  $180^\circ$  to  $90^\circ$ .

Now, we can define the modulus for any topological rectangle  $U$  by

$$\text{mod}(U) = \text{mod}(S) = \frac{a}{b}.$$

Most quasiconformal maps do change the modulus, for example, the modulus of the image of the following horizontal stretch map  $f$  is only one half of the modulus of its pre-image.

$$f(x, y) = 2x + iy = (z + \bar{z}) + \frac{z - \bar{z}}{2}$$

is a 3-quasiconformal map,  $f_z = 3/2$  and  $f_{\bar{z}} = 1/2$ .

The map  $f(z) = e^z$  is a conformal gluing map which maps a rectangle with length  $2\pi$  onto an annulus. So we can also discuss the modulus of an annulus, and in Chapter 8, I will talk about extremal annulus with respect to modulus.

**Definition 6.** Suppose  $A$  is an annulus with two boundary circles with radius  $R_1$  and  $R_2$ , then the modulus of  $A$  is

$$\text{mod}(A) = \frac{1}{2\pi} \log \frac{R_2}{R_1}.$$

Let  $E_1$  and  $E_2$  be two disjoint, connected, simply connected and compact sets on the Riemann sphere  $\bar{\mathbb{C}}$ . Let  $A$  be the complement of  $E_1$  and  $E_2$ , which is called *topological annulus*, i.e.,  $A = \bar{\mathbb{C}} - \{E_1 \cup E_2\}$ .

**Theorem 2.** Any topological annulus  $A$  can be conformally mapped to an annulus.

*Proof* (following Zhong Li's book [65], page 49.)

Step1: By the Riemann mapping theorem, there is a Riemann map  $f$  which maps  $A \cup E_1$  to  $\Delta$ .

Step2: We can assume 0 is not in  $f(A)$ , then we can use map  $w = \log z$ , which maps  $f(A)$  to a vertical topological strip. The right side of the strip is imaginary axis and the left side is a Jordan curve which has period  $2\pi i$ .

Step3: Choose a  $K$ -periods topological rectangle in this topological strip, then it can be mapped to a real  $K$ -periods rectangle by Proposition 3.

Step4: Let  $K$  goes to  $\infty$ , then the image is a real strip. And the exponential map  $e^z$  will map it back to a real annulus.

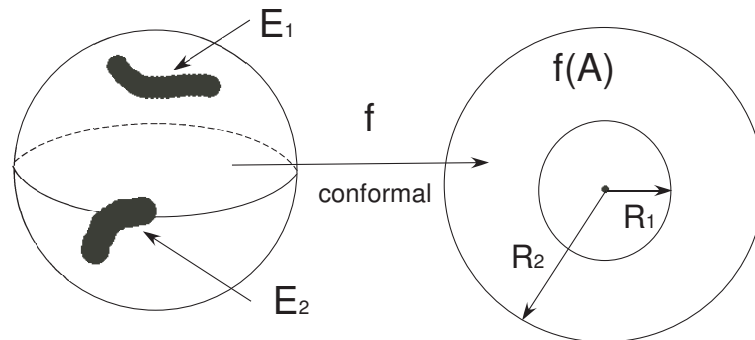


Figure 1.4: A topological annulus

The modulus of a topological annulus  $A = \overline{\mathbb{C}} - \{E_1 \cup E_2\}$  is defined as

$$\text{mod}(E_1, E_2) = \text{mod}(A) = \frac{1}{2\pi} \log \frac{R_2}{R_1}.$$

Just like the modulus of a rectangle, the modulus of an annulus is also conformally invariant.

**Proposition 4.** *Moduli of annuli are conformally invariant.*

Idea of the proof: Reflect the conformal map in two boundary circles again and again by Schwarz reflection principle. Then the resulting map is a conformal map from  $\overline{\mathbb{C}}$  to  $\overline{\mathbb{C}}$  fixing  $\infty$ . So it is an affine map which keeps the modulus of an annulus.

In Chapter 8, I will discuss the Teichmüller annulus and Mori annulus.

## 1.4 Quadratic differential

**Definition 7** (holomorphic quadratic differential). *A holomorphic quadratic differential  $\varphi$  on a Riemann surface is an assignment of a holomorphic function  $\varphi_1(z_1)$  to each local coordinate  $z_1$  such that if  $z_2$  is another local coordinate, then  $\varphi_1(z_1) = \varphi_2(z_2)(dz_2/dz_1)^2$ . If  $\|\varphi\| = \int \int_R |\varphi| < \infty$ ,  $\varphi$  is called an integrable holomorphic quadratic differential.*

It is elementary (by switching to polar coordinates) to see that integrable  $\varphi$  only have at most simple poles at the punctures of the Riemann surface  $R$ .

On the four punctured Riemann sphere  $\overline{\mathbb{C}} - \{0, 1, a, \infty\}$ , an integrable quadratic differential must have the following form

$$\varphi_a(z)(dz)^2 = \frac{c(dz)^2}{z(z-1)(z-a)}.$$

where  $c$  is a constant and  $\{0, 1, a, \infty\}$  are simple poles of  $\varphi_a$ .

If the Riemann surface  $R$  is  $\overline{\mathbb{C}} - \{0, 1, a_1, \dots, a_{n-3}, \infty\}$ , then the quadratic differentials  $\{\varphi_{a_i}\}$  for  $i = 1, 2, \dots, n-3$  are the basis of the space of all integrable holomorphic quadratic differentials on  $R$ .

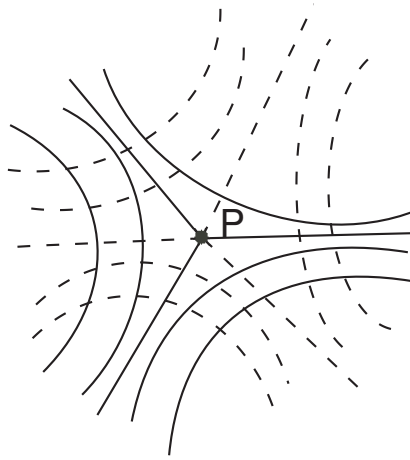
The natural parameter  $\zeta$  on  $R$  is defined by

$$\zeta(z) = \int_{z_0}^z \sqrt{\varphi(z)} dz.$$

It is clear that if  $\zeta_1(z_1)$  and  $\zeta_2(z_2)$  are two natural parameters coming from two different local coordinates of  $\varphi$  and defined on the overlapping coordinate patches  $U_1$  and  $U_2$ , then

$$\zeta_1 = \pm \zeta_2 + \text{constant}.$$

Notice that  $d\zeta^2 = \varphi dz^2$  for any natural parameter  $\zeta$  associated with  $\varphi$ . A parametric curve  $\gamma : I = [0, 1] \rightarrow R$  is called a horizontal trajectory of  $\varphi$  if, given any local coordinate  $z$  defined in a patch overlapping the image of  $\gamma$ , the function  $z(\gamma(t))$  satisfies  $\varphi(z(\gamma(t)))dz^2 > 0$ . It is called a vertical trajectory if  $\varphi(z(\gamma(t)))dz^2 < 0$ . This means in the  $\zeta$ -plane where  $\zeta$  is the natural parameter, the curve  $\gamma(t)$  is transformed into a horizontal line or a vertical line.



Horizontal and Vertical (dash-dot lines) trajectories for  $m=1$ .

Figure 1.5: Horizontal and Vertical trajectories

In an obvious sense, the horizontal and vertical trajectories give two transverse foliations. We say that two foliations are transversal at a singular point if they have  $C^1$ -topological structure equivalent to the horizontal and vertical trajectories of  $z^m dz^2$  for some integer  $m \geq -1$ .

Let  $\varphi$  have a zero of order  $m$  at  $p$  in  $R$ . At any such point there will exist a local coordinate  $z$  with  $z(p) = 0$  such that  $\varphi dz^2 = z^m dz^2$ . For the case where  $m=1$ , the trajectories in the  $z$ -plane have the appearance shown in the Figure 1.5.

By definition, if  $\gamma$  is a differentiable curve on  $R$ , its height with respect to  $\varphi$  is

given by

$$h_\varphi(\gamma) = \int_\gamma |\operatorname{Im} \sqrt{\varphi(z)} dz|.$$

Similarly, its width is given by

$$w_\varphi(\gamma) = \int_\gamma |\operatorname{Re} \sqrt{\varphi(z)} dz|.$$

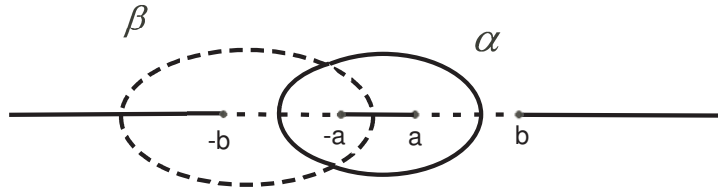
We call a trajectory critical if, when it is continued in either direction, it meets a singularity of  $\varphi$ .

**Theorem 3** (The minimal norm property). *Let  $\varphi$  be a holomorphic quadratic differential on  $R$  with*

$$\|\varphi\| < \infty$$

*and for which every noncritical vertical trajectory can be continued indefinitely in both directions. Let  $\psi$  be another quadratic (not necessarily holomorphic) differential which is continuous on  $R$ . Assume there exists a constant  $M > 0$  such that for every noncritical vertical segment  $\beta$ , one has  $h_\varphi(\beta) \leq h_\psi(\beta) + M$ . Then*

$$\|\varphi\| \leq \int \int_R |\sqrt{\varphi(z)}| |\sqrt{\psi(z)}| dx dy.$$



$\alpha$  and  $\beta$  are Horizontal and Vertical trajectories.

Figure 1.6: Trajectories for four punctured Riemann sphere

Remark. (1) Any holomorphic quadratic differential  $\varphi$  on a compact Riemann surface with finitely many punctures for which  $\|\varphi\| < \infty$  will satisfy the hypothesis on the trajectories of  $\varphi$ .

The curves in the Figure 1.6 are horizontal and vertical trajectories of

$$\varphi(z) = \frac{1}{(z+a)(z-a)(z+b)(z-b)}$$

for real numbers  $a$  and  $b$ .

(2) When we say the noncritical trajectories can be continued indefinitely in both directions, we do not exclude the possibility that they may be closed.

The following theorem is a easy corollary of Theorem 3 and Schwarz's inequality.

**Theorem 4.** *For  $\varphi$  and  $\psi$  satisfy the conditions in the Theorem 3, then*

$$\|\varphi\| \leq \|\psi\|,$$

*and if this inequality is an equality, then  $\varphi \equiv \psi$ .*

Now let's look at some relations between quasiconformal maps and holomorphic quadratic differentials.

**Lemma 1.** *Let  $\varphi$  be a holomorphic quadratic differential on  $R$  with  $\|\varphi\| < \infty$ . Let  $f$  be a quasiconformal self-mapping of  $R$  which is homotopy to the identity. Then there exists a constant  $M$  such that for every noncritical vertical segment  $\beta$ , one has*

$$h_\varphi(\beta) \leq h_\varphi(f(\beta)) + M.$$

*The constant  $M$  depends on  $\varphi$  and  $f$  but not on  $\beta$ .*

This Lemma will help us to construct a holomorphic quadratic differential  $\psi$  that satisfies the condition of Theorem 3.

**Lemma 2.** *Suppose  $f$  is a quasiconformal self-mapping of  $R$  with Beltrami coefficient  $\mu(z)$ , then*

$$\psi(z) = \varphi(f(z)) f_z^2 \left(1 - \mu \frac{\varphi(z)}{|\varphi(z)|}\right)^2$$

*is a holomorphic quadratic differential and*

$$\int \int_R |\varphi| dx dy \leq \int \int_R |\sqrt{\varphi}| |\sqrt{\psi}| dx dy.$$

*Proof*

An elementary calculation shows that  $\psi$  is a quadratic differential. From Lemma 1,  $h_\varphi(\beta) \leq h_\varphi(f(\beta)) + M$  for all non critical vertical segments  $\beta$ . From the definition of  $h_\varphi$ , we have

$$h_\varphi(f(\beta)) = \int_{f(\beta)} |Im \sqrt{\varphi(f)} df|.$$

Since  $df = f_z(1 + \mu(d\bar{z}/dz))dz$ , by introducing  $\sqrt{\psi}$ , this last integral becomes

$$h_\varphi(f(\beta)) = \int_\beta |Im \sqrt{\psi(z)}(1 + \mu \frac{d\bar{z}}{dz})(1 - \mu \frac{\varphi}{|\varphi|})^{-1} dz|.$$

Since  $\varphi dz^2 < 0$  along  $\beta$ , one easily sees that  $\varphi/|\varphi| = -d\bar{z}/dz$  along  $\beta$ . The final result is that

$$h_\varphi(f(\beta)) = h_\psi(\beta).$$

Hence  $h_\varphi(\beta) \leq h_\psi(\beta) + M$  for all vertical segments  $\beta$ . So we can use the minimal norm property, Theorem 3, to get

$$\int \int_R |\varphi| dx dy \leq \int \int_R |\sqrt{\varphi}| |\sqrt{\psi}| dx dy.$$

Now we are ready to prove the Reich and Strebel's inequality, also called Main inequality, for quasiconformal maps.

**Theorem 5** (Reich and Strebel's Main inequality). *Let  $f(z)$  be a quasiconformal map from  $R$  to  $f(R)$  with Beltrami coefficient  $\mu$  and  $\mu_1$  is the Beltrami coefficient of  $f_1$  from  $f(R)$  to  $R$  with the property that  $f_1 \circ f$  is homotopic to the identity relative to the boundary. Suppose  $\varphi$  is a holomorphic quadratic differential on  $R$  with norm 1. Then*

$$1 \leq \int \int_R \frac{|1 - \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} \cdot \frac{|1 + \mu_1(f) \theta \frac{\varphi}{|\varphi|}|^2}{1 - |\mu_1(f)|^2} |\varphi| dx dy$$

where

$$\theta = \frac{\overline{f_z}}{f_z} (1 - \overline{\mu \varphi}/|\varphi|)(1 - \mu \varphi/|\varphi|)^{-1}.$$

If we assume  $f$  is a quasiconformal self-mapping of  $R$  which is homotopic to the identity, then  $\mu_1 = 0$  and the Main inequality is

$$\|\varphi\| \leq \int \int_R |\varphi(z)| \frac{|1 - \mu(z) \varphi(z)/|\varphi(z)||^2}{1 - |\mu(z)|^2} dx dy.$$

Here, I only give the proof of the Main inequality for quasiconformal self-mapping of  $R$ .

*Proof* Let  $\psi$  defined by

$$\psi(z) = \varphi(f(z))f_z^2 \left(1 - \mu \frac{\varphi(z)}{|\varphi(z)|}\right)^2.$$

From Lemma 2,  $\int \int_R |\varphi| dx dy \leq \int \int_R |\sqrt{\varphi}| |\sqrt{\psi}| dx dy$ .

Substituting the formula of  $\psi$  into this inequality yields

$$\|\varphi\| \leq \int \int_R |\varphi(f(z))|^{1/2} |f_z| \left|1 - \mu \frac{\varphi(z)}{|\varphi(z)|}\right| \|\varphi\|^{1/2} dx dy.$$

Introducing a factor of  $(1 - |\mu|^2)^{1/2}$  into the numerator and denominator of this inequality and apply Schwarz's inequality yields

$$\|\varphi\| \leq \left( \int \int_R |\varphi(f(z))| |f_z|^2 (1 - |\mu|^2) dx dy \right)^{1/2} \left( \int \int_R |\varphi| \frac{|1 - \mu \varphi / |\varphi||^2}{1 - |\mu|^2} dx dy \right)^{1/2}.$$

The first integral on the right hand side of this expression is simply  $\|\varphi\|^{1/2}$ . Both side of this inequality divided by  $\|\varphi\|^{1/2}$ , then I have the Main inequality.

If  $f_1^{-1}$  from  $R$  to  $f(R)$  is a Teichmüller map with Beltrami coefficient  $\widetilde{\mu}_1 = k_0 \frac{|\varphi_0|}{\varphi_0}$  and Teichmüller equivalent to  $f$ , then

$$K_0 \leq \int \int_R \frac{|1 - \mu_f \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\mu_f|^2} |\varphi_0| dx dy.$$

This inequality implies the Teichmüller map is unique extremal (see Section 1.6).

## 1.5 Beltrami equation

The Beltrami equation is defined by

$$f_{\bar{z}} = \mu f_z$$

for any  $\mu$  in the open unit ball of  $L^\infty(\overline{\mathbb{C}})$ , i.e.,  $\|\mu\|_\infty = \sup_{z \in \mathbb{C}} |\mu(z)| < 1$ .

The existence and uniqueness of the solution of the Beltrami equation is given by the following theorem.

**Theorem 6** (Riemann Measurable Mapping Theorem). *The Beltrami equation gives a one-to-one correspondence between the set of quasiconformal homeomorphisms of  $\overline{\mathbb{C}}$  that fix  $0, 1, \infty$  and the set of measurable complex-valued functions  $\mu$  on  $\overline{\mathbb{C}}$  for which  $\|\mu\|_\infty < 1$ . Furthermore, the normalized solution  $f^\mu$  to the Beltrami equation depends holomorphically on  $\mu$ .*

For the Beltrami coefficients  $\mu$  with compact support, we can show that

$$f^\mu = z + P\mu + P\mu T\mu + P\mu T\mu T\mu + \dots$$

is a quasiconformal map for which the corresponding Beltrami coefficient  $\mu_f$  is equal to  $\mu$  almost everywhere. And it is also holomorphic on  $\mu$  for any fixed  $z$ , where

$$\mathcal{P}f(z) = -\frac{1}{\pi} \int \int_{\overline{\mathbb{C}}} \frac{f(\zeta)}{\zeta - z} d\xi d\eta, \quad \zeta = \xi + i\eta,$$

and

$$\mathcal{T}f(z) = \lim_{\varepsilon \rightarrow 0} -\frac{1}{\pi} \int \int_{|\zeta - z| > \varepsilon} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta, \quad \zeta = \xi + i\eta.$$

Three comments about this theorem:

1. The Beltrami coefficient  $\mu_f = \mu$  almost everywhere.
2. If  $\mu$  is defined on any sub-domain  $\Omega$ , we can extend  $\mu$  by defining  $\mu = 0$  on  $\overline{\mathbb{C}} - \Omega$ .
3.  $(\mathcal{P}f)_{\bar{z}} = f$  and  $(\mathcal{P}f)_z = \mathcal{T}f$ .

We will discuss the continuity of  $\mathcal{P}$ -operator in Chapter 2, and use this  $\mathcal{P}$ -operator to prove the Slodkowski's extension theorem of holomorphic motions. For  $t \in \Delta$ ,

$$f^{t\mu} = z + tP\mu + t^2P\mu T\mu + t^3P\mu T\mu T\mu + \dots$$

is an example of a holomorphic motion, which is a motion through quasiconformal mappings.

## 1.6 Universal Teichmuller space $T(\Delta)$ or $T(H)$

Let  $M(H)$  be the unit ball of  $L^\infty(H)$ , i.e.,  $M(H) = \{\mu \mid \|\mu(z)\|_\infty < 1, z \in H\}$ .

Let  $f^\mu(z)$  be the unique normalized solution of Beltrami equation  $f_{\bar{z}} = \mu f_z$  which is conformal in the lower half plane  $H^*$  i.e.  $\mu(z) = 0$  for  $z \in H^*$ . And let  $f_\mu(z)$  be the normalized quasiconformal self-map of  $H$  with Beltrami coefficient  $\mu$  where  $\mu(z) = \overline{\mu(\bar{z})}$  for  $z \in H^*$ . From the uniqueness of the solution of Beltrami equation, it is easy to show that  $f_\mu(z) = \overline{f_\mu(\bar{z})}$  and  $f_\mu$  maps the real line onto itself.

We say two Beltrami differentials  $\mu$  and  $\nu$  are Teichmuller equivalent,  $\mu \sim \nu$ , if and only if  $f^\mu(x) = f^\nu(x)$  (or  $f_\mu(x) = f_\nu(x)$ ) for all  $x \in R$ .

**Definition 8** (Universal Teichmüller's space). *The Universal Teichmüller's space  $T(H)$  is the space of all the Teichmuller equivalent classes of Beltrami differentials, i.e.  $T(H) = \{[\mu] \mid \mu \sim \nu \text{ if and only if } f^\mu(x) = f^\nu(x) \text{ for any real } x\}$ .*

Since there exists a Mobius map between the upper-half plane  $H$  and the unit disk  $\Delta$ , people sometimes also use the unit disk to define the Universal Teichmuller space.

Let  $M(\Delta)$  be the unit ball in  $L^\infty(\Delta)$ . And let  $f^\mu(z)$  be the normalized (fixing 1, -1 and  $i$ ) quasiconformal map which is conformal in the complement of the unit disk,  $\Delta^c$ , and  $\mu_f(z) = \mu(z)$  for any  $z \in \Delta$ . Then the Universal Teichmuller space is also can be defined by  $T(\Delta) = \{[\mu] \mid f^\mu(x) = f^\nu(x) \text{ for any } x \in S^1\}$ .

For the unit disk model,  $f_\mu(z)$  is defined to be the normalized quasiconformal map which is invariant under the conjugate of

$$h(z) = \frac{1}{\bar{z}}.$$

So it maps the unit circle onto itself.

The definition of Teichmuller space for any other Riemann surface  $R$  is much more complicated than the definition of the Universal Teichmuller space. There is a homotopy condition for the Teichmuller equivalent classes.

**Definition 9** (Teichmüller equivalence). *Suppose  $f_0 : R \rightarrow R_0$  and  $f_1 : R \rightarrow R_1$  are quasiconformal maps with Beltrami coefficients  $\mu_0$  and  $\mu_1$ . We define  $f_0$  to be Teichmüller equivalent to  $f_1$  (or  $\mu_0$  to be Teichmüller equivalent to  $\mu_1$ ) if there is a conformal map  $c$  from  $R_0$  onto  $R_1$  and a homotopy through quasiconformal self-maps  $g_t$  of  $R$  such that*

$$g_0 = \text{identity}$$

$$g_1 = (f_1)^{-1} \circ c \circ f_0, \text{ and}$$

$$g_t(p) = p \text{ for } 0 < t \leq 1 \text{ and for every } p \text{ in the boundary of } R.$$

In this dissertation, I will focus on Universal Teichmüller space  $T(H)$  or  $T(\Delta)$ , and I will study the Teichmüller's metric and Kobayashi's metric on the Universal Teichmüller space in Chapter 6 and on  $T_0$ , a subspace of  $T$ , in Chapter 7. Teichmüller space of closed set  $E$  will be introduced and discussed in Chapter 5.

**Definition 10** (Teichmüller's metric). *For two elements  $[\mu]$  and  $[\nu]$  of  $T(H)$ , Teichmüller's metric is equal to*

$$d_T([\mu], [\nu]) = \inf \log K(f^\mu \circ (f^\nu)^{-1}),$$

where the infimum is over all  $\mu$  and  $\nu$  in the equivalence classes  $[\mu]$  and  $[\nu]$ , respectively. In particular,

$$d_T(0, [\mu]) = \log \frac{1 + k_0}{1 - k_0}$$

where  $k_0$  is the minimal value of  $\|\mu\|_\infty$ , and  $\mu$  ranges over the Teichmüller class  $[\mu]$ .

**Definition 11** (extremal quasiconformal map). *A quasiconformal map  $f$  is said to be extremal in its Teichmüller equivalent class  $\tau \in T(H)$  if  $K(f) \leq K(\tilde{f})$  for any  $\tilde{f} \in \tau$ .*

Finding an extremal map is a very hard problem for arbitrary Teichmüller space  $T(R)$ . If a Riemann surface  $R$  is of finite analytic type, that is, conformally equivalent to a compact Riemann surface possibly with finitely many points removed, then the Teichmüller space  $T(R)$  is finite dimensional, and every  $\tau$  in  $T(R)$  is represented

by a uniquely extremal Beltrami coefficient  $\mu$  of the Teichmüller form (see definition 12). However, when  $T(R)$  is of infinite analytic type, the situation is more difficult and all of the following cases can occur; (i) a Beltrami coefficient not of Teichmüller form may be uniquely extremal, (ii) it may be extremal but not uniquely extremal, or (iii) it may be nonextremal. There are examples of these three cases in Chapter 9 of Gardiner and Lakic's book [30].

For the Universal Teichmüller space  $T(H)$ , the existence of extremal map is easy to show by Proposition 1. Suppose there is a sequence of quasiconformal maps  $f_n$  in a fixed point  $\tau \in T(\Delta)$  such that  $K(f_n) < K(f_{n-1})$  and  $K(f_n) \rightarrow \inf_{f \in \tau} K(f)$ . Then  $\{f_n\}$  is a normal family, so there is a limiting quasiconformal map  $f_0$  and  $f_0(x) = f_n(x)$  for any real number  $x$  and  $n > 0$ . So  $f_0(x)$  is in the Teichmüller equivalent class  $\tau$ .

**Definition 12** (Teichmüller map). *If a Beltrami coefficient  $\mu$  has the form  $k \frac{|\varphi|}{\varphi}$  where  $\varphi$  is a integrable holomorphic quadratic differential, then  $f^\mu$  is called Teichmüller map.*

If there is a Teichmüller map in some equivalent class, then it is unique extremal in its equivalent class. To show that Teichmüller map is uniquely extremal, we need the Reich and Strebel's Main inequality, Theorem 5 in Section 1.4.

**Theorem 7** (Teichmüller Uniqueness theorem). *Suppose  $f_0 \in \tau$  is a Teichmüller map with Beltrami coefficient  $\mu_0 = k_0 \frac{|\varphi_0|}{\varphi_0}$ , then for any  $f \in \tau$  with Beltrami coefficient  $\mu$ , either there exists a set of positive measure in  $R$  on which  $|\mu(z)| > k_0$  or  $\mu(z) = \mu_0(z)$  almost everywhere.*

*Proof* If  $|\mu(z)| < k_0$ , we have the following inequalities:

$$K_0 \leq \int \int_R \frac{|1 - \mu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\mu|^2} |\varphi_0| dx dy \leq \int \int_R \frac{1 + |\mu|}{1 - |\mu|} |\varphi_0| dx dy \leq \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} \leq K_0$$

Since the left and right ends of this string of inequalities are equal, each inequality must be equality. The only possibility is that  $\mu = k_0 \frac{|\varphi_0|}{\varphi_0} = \mu_0$  almost every.

In Chapter 7, I will introduce the Strebel's Frame mapping theorem. This theorem implies every point in  $T(R)$ , where  $R$  is finite analytic type, has a uniquely extremal Teichmüller map. And this theorem also shows that every point in  $T_0(\Delta)$  has a uniquely extremal Teichmüller map. I will use the Streble's Frame mapping theorem and Reich-Strebel's Main inequality for a Teichmüller map to give a direct proof of the theorem that Teichmüller's metric is equal to Kobayashi's metric on  $T_0(\Delta)$  [22] [38].

## 1.7 Beurling-Ahlfors Extension

In this section, I will talk about the boundary map of a quasiconformal map.

**Definition 13** (M-quasisymmetric map). *An orientation-preserving homeomorphism of the real line  $\mathbb{R}$  is said to be M-quasisymmetric if there exists a constant  $M \geq 1$  such that*

$$\frac{1}{M} \leq \frac{|f(x+t) - f(x)|}{|f(x) - f(x-t)|} \leq M \quad (1.1)$$

for all real  $x$  and  $t$ , where  $M$  is called a quasisymmetric constant for  $f$ .

**Proposition 5.** *Let  $h = f_\mu(z)|_{\mathbb{R}}$  for any  $\mu$ , then  $h$  is a M-quasisymmetric homeomorphism of the real axis where  $M$  is a constant only depend on  $K(f_\mu)$ . (see [3] for the proof)*

On the other side, the Beurling-Ahlfors extension (see [10] or [3]) provides a formula to construct a quasiconformal mapping  $F$  as a representative for any given point  $[f]$  in  $T$ .

**Definition 14** (Beurling-Ahlfors extension). *Assume  $f$  is a quasisymmetric homeomorphism of  $\mathbb{R}$ . The Beurling-Ahlfors extension  $F_r(z)$  of  $f$  is defined by  $F_r(z) = u(z) + iv(z)$  with*

$$u(x+iy) = \frac{1}{2} \int_0^1 [h(x+ty) + h(x-ty)] dt \quad (1.2)$$

and

$$v(x + iy) = \frac{r}{2} \int_0^1 [h(x + ty) - h(x - ty)] dt. \quad (1.3)$$

**Theorem 8** (Beurling-Ahlfors Theorem). *There exists a quasiconformal mapping of the upper half plane with the boundary correspondence  $x \rightarrow f(x)$  if and only if  $f(x)$  is a  $M$ -quasisymmetric map. More precisely, for some constant  $r > 0$ , the maximal complex dilatation of the Beurling-Ahlfors extension  $F$  of  $f$  does not exceed  $M^2$ . On the other hand, every quasiconformal mapping with boundary correspondence  $h$  must have a maximal dilatation  $\geq 1 + A \log M$  where  $A$  is 0.2284.*

If  $h(x) = x$ , then it is easy to see that  $F_2(z) = z$ . Now let us fix  $r = 2$  for any  $h(x)$ , then  $F_2(z) = u + iv$  with

$$u(x + iy) = \frac{1}{2y} \int_{x-y}^{x+y} h(t) dt \quad (1.4)$$

and

$$v(x + iy) = \frac{1}{y} \left[ \int_x^{x+y} h(t) dt - \int_{x-y}^x h(t) dt \right]. \quad (1.5)$$

It is clear that  $v(x, y) \geq 0$  and  $v(x, y) \rightarrow 0$  as  $y$  tends to 0 for  $y \rightarrow 0$ . Moreover,  $u(x, 0) = h(x)$ . In Ahlfors' book [3], he showed that if  $r = 1$  then

$$K(F_1(z)) < 2M(M + 1).$$

So

$$K(F_2(z)) = K(f \circ F_1(z)) \leq K(F_1) \cdot K(f) = 4M(M + 1)$$

where  $f(x, y) = x + i2y$ . This implies  $F_2(z)$  is a quasiconformal map.

From the formulas to construct the Beurling-Ahlfors extension  $F$  for  $f$ , one can see that for a point  $z$  near the boundary line  $\mathbb{R}$ ,  $F(z)$  is determined by the behavior of  $f$  near  $x$ , where  $z = x + iy$ . Therefore, if  $f$  satisfies certain smooth regularity then  $F$  may satisfy some regularity near the boundary. In chapter 7, I will study symmetric homeomorphisms.

**Definition 15** (symmetric homeomorphism). *By a symmetric homeomorphism we mean an orientation-preserving homeomorphism  $f$  of  $\mathbb{R}$  satisfying*

$$\frac{1}{1 + \varepsilon(t)} \leq \frac{|f(x+t) - f(x)|}{|f(x) - f(x-t)|} \leq 1 + \varepsilon(t) \quad (1.6)$$

for all real  $x$  and  $t$ , where  $\varepsilon(t)$  is a positive bounded function, independent of  $x$ , and converges to 0 as  $t$  converges to 0.

It has been proved that the Beurling-Ahlfors extension and the Douady-Extension of symmetric homeomorphism are asymptotically conformal [22] [30].

**Definition 16** (asymptotically conformal). *A quasiconformal map  $F$  from the upper half plane  $\mathbb{H}$  to itself is said to be asymptotically conformal if for any  $\varepsilon > 0$  there exists a compact subset  $\Omega$  in  $\mathbb{H}$  such that the maximal complex dilatation of  $F$  on  $\mathbb{H} \setminus \Omega$  is less than  $\varepsilon$ .*

The following proposition in [36] follows from estimating locally the complex dilatation of  $F$  in the Beurling-Ahlfors theorem.

**Proposition 6** ([36]). *If  $f$  is symmetric, then its Beurling-Ahlfors extension  $F$  is asymptotically conformal.*

## 1.8 Complex structure of Teichmüller space

### 1.8.1 Schwarzian derivative and Bers embedding

In this section, I will introduce the complex structure of  $T(\Delta)$ ,  $T_0(\Delta)$  and  $AT(\Delta)$ . In Chapter 6 and Chapter 7, I will talk about Kobayashi's metric on these complex Banach manifolds.

**Definition 17** (Schwarzian derivative). *if  $f(z)$  is holomorphic in  $\Omega \subset \overline{\mathbb{C}}$ . The Schwarzian derivative of  $f$  is*

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

**Lemma 3.**  $S(f) = 0$  if and only if  $f(z) = \frac{az+b}{cz+d}$ .

*Proof* If

$$S(f) = (-2)(f')^{\frac{1}{2}}((f')^{-\frac{1}{2}})'' \equiv 0,$$

then  $(f')^{\frac{1}{2}} = 0$  or  $((f')^{-\frac{1}{2}})'' = 0$ . Hence  $f$  is a constant map or  $f' = \frac{1}{(az+b)^2}$ , which implies  $f(z)$  is a mobius map.

It is easy to show that

$$S(f \circ g)(z) = S(f)(g(z))g'(z)^2 + S(g)(z).$$

And from this formula we have:

**Lemma 4.**  $S(f) = S(g)$  if and only if  $S(g \circ f^{-1}) = 0$  which means  $g = M \circ f$  for Mobius transformation  $M$ . If assume  $f$  and  $g$  are normalized, then  $S(f) = S(g)$  iff  $f|_{\mathbb{R}} = g|_{\mathbb{R}}$  i.e.  $f \sim g$ .

Bers realized that the Schwarzian derivative is a holomorphic quadratic differential, then he used the complex structure on the space of holomorphic quadratic differential to define the complex structure on a Teichmuller space [7].

**Lemma 5.**  $S(f^\mu)(z) = \varphi(z)$  is a holomorphic quadratic differential.

*Proof* Since  $g$  is a Mobius map,  $S(f \circ g)(z) = S(f)(g(z))g'(z)^2 + S(g)(z) = S(f)(g(z))g'(z)^2$ .

Let  $B$  be the Banach space of holomorphic quadratic differential with the norm  $\|\varphi\|_B = \sup_{z \in \Omega} \Omega|\varphi(z)|\rho^{-2}(z)$ , where  $\rho$  is the infinitesimal form of the Poincare metric for  $H$  or  $\Delta$ .

**Definition 18** (Bers embedding). *The Bers' map,  $\Phi : [\mu] \rightarrow S(f^\mu|_{H^*})$  for any  $f^\mu \in [\mu] = [\mu]$ , maps the Teichmuller space into the Banach space of Schwarzian derivatives with norm  $\|S(f^\mu)\|_B = \sup |S(f^\mu) \cdot \rho^{-2}(z)|$ .*

The following two theorems shows that the image of the Teichmuller space under the Bers' map is contained in a ball with radius  $3/2$  and it contains a ball with radius  $1/2$ .

**Theorem 9** (Nehari Theorem).  $\|S(f^\mu)\|_B = \sup |S(f^\mu) \cdot \rho^{-2}(z)| \leq \frac{3}{2}$ .

*Proof* (from Zhongli's old book [65]) Suppose  $f$  is holomorphic in  $\Delta$ . For any  $\zeta \in \Delta$ , let

$$\begin{aligned} h(z) &= \frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} \\ &= z + \left[\frac{1}{2}(1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - \bar{\zeta}\right]z^2 + \dots \end{aligned}$$

It is obvious that  $h(0) = 0$  and  $h'(0) = 1$ .

Let

$$\begin{aligned} g(z) &= \frac{1}{h\left(\frac{1}{z}\right)} + \left[\frac{1}{2}(1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - \bar{\zeta}\right] \\ &= z - \frac{1}{6}(1-|\zeta|^2)^2 S_f(\zeta) \frac{1}{z} + \dots, \end{aligned}$$

then it is a holomorphic function on  $\Delta^c$ . So

$$\frac{1}{6}(1-|\zeta|^2)^2 |S_f(\zeta)| \leq 1$$

by the Beiberbach Theorem.

The locally inverse map of the Bers embedding is given by the following theorem:

**Theorem 10.** (*Ahlfors-Weill Chart*) If  $\|\varphi\|_B < \frac{1}{2}$ , let  $\mu(z) = -2y^2\varphi(z)$  for  $z$  in  $H$  and  $\mu(z) = 0$  for  $z \in H^*$ , then  $S(f^\mu) = \varphi$ . Here  $\mu(z) = -2y^2\varphi(z)$  is called a *harmonic Beltrami differential*.

Idea of the proof: For two linearly independent solutions of

$$\eta'' = -\frac{1}{2}\varphi\eta$$

normalized by  $\eta'_1\eta_2 - \eta'_2\eta_1 = 1$ , let us form

$$\hat{f}(z) = \frac{\eta_1(\bar{z}) + (z - \bar{z})\eta'_1(\bar{z})}{\eta_2(\bar{z}) + (z - \bar{z})\eta'_2(\bar{z})}$$

for  $z \in H$  and

$$\hat{f} = \frac{\eta_1(z)}{\eta_2(z)}$$

for  $z \in H^*$ .

Then  $\hat{f}_{\bar{z}}/\hat{f}_z = -y^2\varphi(\bar{z})$  for  $z \in H$  and  $\hat{f}_{\bar{z}} = 0$  for  $z \in H^*$ . Since  $\hat{f}$  has Beltrami coefficient  $\|\mu_f\|_\infty < 1$ , we know  $\hat{f}$  is quasiconformal.

(To show that  $\hat{f}$  is a homeomorphism from  $\bar{\mathbb{C}}$  onto  $\bar{\mathbb{C}}$ , please read the book [30].)

**Theorem 11.** [30] *The maps  $\Phi$  and  $\Phi^{-1}$  are continuous in a neighborhood of  $[id]$ .*

Hence the Bers embedding is a homeomorphism in the neighborhood of  $[id]$ . Composition on the right with a quasiconformal map gives a homeomorphism in the neighborhood of any  $[f]$ .

Now I am ready to show the complex structure of a Teichmuller space, I will look at the transition map between two charts of Bers embedding, which is a holomorphic map in Banach space.

For the Banach Space, we use the following way to define derivative:

**Definition 19** (Frechet derivative). *Let  $E$  and  $F$  be Banach spaces over the complex numbers, and  $U \subset E$  an open set. A function  $f : U \rightarrow F$  has a derivative at a point  $x_0 \in U$  if there exists a continuous complex linear mapping  $Df(x_0) : E \rightarrow F$  such that*

$$\lim_{h \rightarrow 0} \frac{\|f(x_0 + h) - f(x_0) - Df(x_0)(h)\|_F}{\|h\|_E} = 0.$$

*The map  $Df(x_0)$  is called the derivative of  $f$  at  $x_0$ .*

Let  $\beta$  be the map from  $T(\Delta)$  with the base point  $[id]$  to  $T(\Delta)$  with the base point  $[\mu]$  induced by the right composition with  $w_\mu$  i.e.  $\beta([w_\nu]) = [w_\nu \circ w_\mu]$ .

**Theorem 12.** ([30]) *Let  $\varphi \in B$  and let  $\hat{\varphi} = j\varphi j(f^\mu)(f^\mu(\zeta)')^2$  where  $j(z) = \bar{z}$ . Let  $\hat{\mu} = \rho_\mu^{-2}(\zeta)\overline{\hat{\varphi}(\zeta)}$ , where  $\rho_\mu$  is the non-euclidean metric for the domain  $\Omega = w^\mu(H)$*

and  $f^\mu \circ w^\mu = w_\mu$ . Then

$$\Phi \circ \beta \circ \Phi^{-1} : \varphi \rightarrow S(w^{\hat{\mu}})(w^\mu)(w^\mu(z)')^2 + S(w^\mu).$$

Hence,  $\Phi \circ \beta \circ \Phi^{-1}$  is continuous and holomorphic in a neighborhood of the origin.

Moreover, the intermediate map  $\hat{\mu} \rightarrow S(\hat{\mu})$  is differentiable. We will follow Lipman Bers' [7] calculation to find the derivative of  $S(f^{t\mu})$ .

From the Riemann measurable mapping theorem, it is easy to see that

$$\frac{\partial f^{t\mu}(z)}{\partial t} \Big|_{t=0} = -\frac{z(z-1)}{\pi} \int \int \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)}$$

Let  $w = f^{t\mu}$ , then

$$\frac{\partial S(f^{t\mu})(z)}{\partial t} = \frac{\partial S(w)(z)}{\partial t} = \frac{(w')^3 \dot{w}''' - \dot{w}'(w')^2 w''' - 3\dot{w}''(w')^2 w'' + 6\dot{w}'w'w''}{(w')^4}$$

If  $t = 0$ , then  $w \equiv z$ . Hence  $w' \equiv 1, w'' = w''' = 0$  and

$$\begin{aligned} \frac{\partial S(w)(z)}{\partial t} &= \dot{w}''' = \frac{\partial^3}{\partial z^3} \left( -\frac{z(z-1)}{\pi} \int \int \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)} \right). \\ &= -\frac{6}{\pi} \int \int \frac{\mu(\zeta) d\xi d\eta}{(\zeta-z)^4} \end{aligned}$$

This derivative is called Bers'  $L$ -operator:

$$L(\mu) = -\frac{6}{\pi} \int \int \frac{\mu(\zeta) d\xi d\eta}{(\zeta-z)^4}.$$

### 1.8.2 Complex structure of $T_0(\Delta)$ and $AT(\Delta)$ .

$T_0$  is a subspace of  $T$  which is introduced by Gardiner and Sullivan in the paper [36].

Let  $M_0(\Delta) = \{\mu \mid \mu(z) \rightarrow 0 \text{ as } z \rightarrow S^1\}$ , and  $T_0(\Delta) = \{[\mu] \mid \text{there exists } \mu(z) \in [\mu] \text{ such that } \mu(z) \rightarrow 0, \text{ as } z \rightarrow S^1\}$ .

And we already know that the Beurling-Ahlfors extension of a symmetric maps is quasiconformal, so the space of symmetric homeomorphisms is equivalent to  $T_0$ .

Let  $B_0$  be the subspace of  $B$  consisting of those  $\varphi$  in  $B$  such that for every  $\varepsilon > 0$ , there is a compact subset  $A$  of  $H^*$  such that  $|y^2\varphi| < \varepsilon$  for  $z \in H^* - A$ .

**Theorem 13** (Gardiner and Sullivan [36]). *The Bers embedding is well-defined, one to one, complex analytic from  $T_0$  to  $B_0$ .*

*Proof* From the Ahlfors-Weill Chart, if  $\varphi \in B_0$  then the harmonic Beltrami differential  $\mu = -y^2\varphi(\bar{z}) \rightarrow 0$  as  $z \rightarrow R$  is in  $M_0$ .

On the other hand, suppose  $\mu = 0$  in a neighborhood  $U$  of  $H^*$ , then we can find  $S(f)$  in the  $U$ . Since  $S(f)$  is bounded on  $\overline{H^*}$ ,  $\mu = -y^2S(f) \rightarrow 0$  as  $y \rightarrow 0$ . For any  $\mu \in M_0$ , let  $\mu_n = \mu$  on  $D_n = \Delta_{1-1/n}$  and  $\mu_n = 0$  on  $\Delta - D_n$ , then  $\|\mu - \mu_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Such  $\mu_n$  are dense in  $M_0$ , and  $B_0$  is closed in  $B$  (note: if  $\varphi$  is not vanishing then there is a neighborhood of  $\varphi$  not vanishing.) and  $S(f)$  is continuous, then  $S(f)$  maps  $T_0$  to  $B_0$ .

For general Riemann surface  $R$ , the complex structure of a general  $T_0(R)$  space is not easy to obtain as  $T_0(\Delta)$ . Please read Gardiner and Lakic's book [30] for details. The complex structure of  $AT(\Delta) = T(\Delta)/T_0(\Delta)$  is also studied in this book.

**Theorem 14.** *Bers embedding of  $AT$ ,*

$$S([f]) : AT \rightarrow \frac{B}{B_0}$$

*is well defined.*

I will study the complex structure for  $T(E)$  in Chapter 5, where  $E$  is a closed subset of  $\mathbb{C}$ .

## 1.9 Measured foliation

In this section, I will assume that we are given two measured foliations  $|du|$  and  $|dv|$  on a Riemann surface  $R$  of finite analytic type. For a definition of measured

foliation see [24] or [26]. In our notation measured foliation  $|du|$  is made up of a family of  $C^1$ -real valued functions  $u_j$  each associated to an open subset  $U_j$  of  $R$ . If two of these subsets  $U_j$  and  $U_k$  intersect then on the overlap  $U_j \cap U_k$  there is a constant  $c_{jk}$  such that

$$u_j = \pm u_k + c_{jk}.$$

Moreover, the level sets

$$u = \text{constant}$$

are well defined on the union  $U = \bigcup_j U_j$  and determine continuous curves. For smooth curves  $\gamma$  contained in  $R$  we can form the line integrals

$$\int_{\gamma \cap U} |du|.$$

It is assumed that the union  $U = \bigcup_j U_j$  covers  $R$  except for a finite number punctures which can be points where the level curves  $u_j = \text{constant}$  have singularities.

In any case the heights of  $|du|$  along homotopy classes of closed curves contained in  $R$  are defined in the following way. For any particular smooth closed curve  $\gamma$  we define

$$ht(\gamma, |du|) = \int_{\gamma} |du|,$$

and for the free homotopy class  $[\gamma]$  of  $\gamma$ , we define

$$ht([\gamma], |du|) = \inf\{ht(\tilde{\gamma}, |du|)\},$$

where the infimum is taken over all  $\tilde{\gamma}$  in the same free homotopy class as  $\gamma$ .

We let  $\mathcal{S}$  denote the set of all essential simple closed curves on  $R$ . By definition a curve is essential if it is not homotopic to point and not homotopic to a puncture. By the correspondence

$$|du| \mapsto ([\gamma] \mapsto ht([\gamma], |du|))$$

the measured foliation  $|du|$  determines an element of the product space  $\mathbb{R}_+^{\mathcal{S}}$ . We say two measured foliations are height equivalent if they have the same image under this map.

**Theorem 15.** *Given a measured foliation  $|du|$  on  $R$  and a complex structure on  $R$ , there exists a unique holomorphic quadratic differential  $\varphi$  such that the foliation given by the horizontal trajectories of  $\varphi$  and the measure  $\text{Im}\sqrt{\varphi}dz$  is measure equivalent to  $|du|$ .*

In addition to its vector of heights, any measured foliation also has a Dirichlet norm. Because we are assuming the real valued functions  $u_j$  have continuous first partial derivatives, and because we are assuming  $R$  has a Riemann surface structure  $R_\tau$ , there is a star operator and so any measured foliation  $|du|$  has a well defined Dirichlet integral

$$\begin{aligned} \text{Dir}(|du|) &= \int \int_{R_\tau} du \wedge *du = \int \int_{R_\tau} (u_x dx + u_y dy) \wedge (-u_y dx + u_x dy) \\ &= \int \int_{R_\tau} (u_x^2 + u_y^2) dx dy. \end{aligned}$$

**Definition 20.**  $M_\tau(|du|)$  is the infimum of Dirichlet integrals  $\int \int_{R_\tau} (\tilde{u}_x^2 + \tilde{u}_y^2) dx dy$  where the infimum is taken over all  $|d\tilde{u}|$  in the same height equivalence class.

Let  $M[v] = \inf\{||\psi|| \text{ for all } \psi \text{ such that } h_\psi[\gamma] \geq h_v[\gamma]\}$ , then we can have an equivalent definition of  $M_\tau[|du|] = M[|u \circ f^-|]$  where  $\mu = f_{\bar{z}}/f_z$  and  $\tau = [\mu]$ .

It is easy to see that the infimum  $M[v]$  is achieved by a unique holomorphic quadratic differential  $\varphi$  in the previous theorem. More precisely,  $h_\psi[\gamma] \leq h_v[\gamma] = h_\varphi[\gamma]$  which implies

$$||\varphi|| \geq ||\psi||$$

by minimal norm property and Theorem 4 in section 1.4.

**Lemma 6.**

$$K^{-1}M[v] \leq M_\mu[v] \leq KM[v]$$

where  $K = (1 + ||\mu||_\infty)/(1 - ||\mu||_\infty)$ .

*Proof* Let  $\varphi_\mu$  be the unique holomorphic quadratic differential on  $R_\mu$  for which

$$h_v[\gamma] = h_{\varphi_\mu}[f(\gamma)].$$

We know that  $M_\mu[v] = \|\varphi_\mu\|$ . Let  $\gamma$  be any loop in  $R$ . Then

$$\int_{f(\gamma)} |Im\sqrt{\varphi_\mu(w)}dw| = \int_\gamma |Im\sqrt{\varphi_\mu(f(z))}f_z(1 + \mu(d\bar{z}/dz))dz| \leq (1+k)l_{\tilde{\varphi}}(\gamma)$$

where  $\tilde{\varphi} = \varphi_\mu(f(z))f_z^2$  and  $l_{\tilde{\varphi}}(\gamma)$  is the  $\tilde{\varphi}$  length of  $\gamma$ .

Since this inequality holds for every path  $\gamma$ , we see that  $h_\varphi \leq (1+k)l_{\tilde{\varphi}}(\gamma)$  for all  $\gamma$ . Therefore

$$\|\varphi\| \leq (1+k)^2 \int \int_R |\varphi_\mu(f(z))|^2 |f_z^2| dx dy \leq \frac{(1+k)^2}{(1-k)^2} \|\varphi_\mu\|.$$

and this yields  $M[v] \leq KM_\mu[v]$ . The opposite inequality follows by applying the same reasoning to the quasiconformal mapping  $f^{-1}$ .

Remark. This lemma shows that  $M_\mu[v]$  is a continuous function on  $T(R)$  since clearly

$$K^{-1}M_\sigma[v] \leq M_\mu[v] \leq KM_\sigma[v]$$

where  $K$  is the dilatation of the mapping  $f^\sigma \circ (f^\mu)^{-1}$ .

Moreover, this function is not only continuous but also differentiable.

**Theorem 16.** [26, 27, 32] *The Dirichlet norm  $M_\tau(|du|)$  of a height equivalence class on a Riemann surface  $R_\tau$  of finite analytic type is uniquely realizable by a measured foliation given by the horizontal trajectories and vertical measure of a holomorphic quadratic differential  $q$ .  $M_\tau(|du|)$  is differentiable and its derivative is given by*

$$\log M_{t\mu}(|du|) = \log M_0(|du|) + \frac{2Re t}{\|q\|} \int \int \mu q dx dy + o(t).$$

Idea of the proof: Let  $\varphi_\mu$  be the unique holomorphic quadratic differential on  $R_\mu$  with the same height as  $|du \circ f^{-1}|$  where  $f : R \rightarrow R_\mu$  and  $\mu = \mu_f$ . Form the quadratic differential

$$\tilde{\varphi} = \varphi_\mu(f(z))f_z^2 \left(1 - \mu \frac{\varphi(z)}{|\varphi(z)|}\right)^2.$$

By Lemma 2 in Section 4,  $\tilde{\varphi}$  is a quadratic differential on  $R$ . And since  $h_{\varphi_\mu}[f(\gamma)] = h_\varphi[\gamma]$ ,

$$\int \int_R |\varphi| dx dy \leq \int \int_R |\sqrt{\varphi}| |\sqrt{\tilde{\varphi}}| dx dy.$$

Upon multiplying the integrand on the right hand side of the previous inequality in the numerator and denominator by  $|f_z|(1 - |\mu|^2)^{1/2}$  and applying Schwarz's inequality (with the term  $\sqrt{\varphi_\mu(f)}|f_z|(1 - |\mu|^2)^{1/2}$  lumped together), we find that

$$\|\varphi\| \leq \|\varphi_\mu\|^{1/2} \left( \int \int_R |\varphi(z)| \frac{|1 - \mu(\varphi/|\varphi|)|^2}{1 - |\mu|^2} dx dy \right)^{1/2}$$

Square both sides and dividing by  $\|\varphi\|\|\varphi_\mu\|$ , we get

$$\frac{\|\varphi\|}{\|\varphi_\mu\|} \leq 1 - \frac{2}{\|\varphi\|} \operatorname{Re} \int \int \mu \varphi dx dy + O(\|\mu\|_\infty^2).$$

and so

$$\log \|\varphi_\mu\| \geq \|\varphi\| + 2 \operatorname{Re} \frac{1}{\|\varphi\|} \int \int \mu \varphi + O(\|\mu\|_\infty^2).$$

For the inverse inequality we can apply a similar argument to the inverse mapping  $f^{-1}$ .

In Chapter 8, we will use this formula to find extremal annulus under certain conditions and discuss Gardiner and Masur's minimal axis. For more details about the Lemmas and Theorems in Section 1.4 and 1.9, please read [27], [30].

# Holomorphic Motion Theorem

---

## Contents

2.1	The $\mathcal{P}$ -Operator and the Modulus of Continuity . . . . .	35
2.2	Extensions of holomorphic motions for $0 < r < 1$ . . . . .	39
2.3	Controlling quasiconformal dilatation . . . . .	49
2.4	Extension of holomorphic motions for $r = 1$ . . . . .	53

---

Chapter 2 and 3 are joint works with Fred Gardiner and Yunping Jiang. We study the Chirka's proof of Slodkowski's extension theorem of holomorphic motion and tangent vector field of Holomorphic motion.

**Definition 21.** *Suppose  $E \subset \overline{\mathbb{C}}$  is a subset. A map*

$$h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$$

*is called a holomorphic motion of  $E$  parametrized by  $\Delta$  and with base point 0 if*

1.  $h(0, z) = z$  for all  $z \in E$ ,
2. for every  $c \in \Delta$ ,  $z \mapsto h(c, z)$  is injective on  $\overline{\mathbb{C}}$ , and
3. for every  $z \in E$ ,  $c \mapsto h(c, z)$  is holomorphic for  $c$  in  $\Delta$

We think of  $h(c, z)$  as moving through injective mappings with the parameter  $c$ . It starts out at the identity when  $c$  is equal to the base point 0 and moves holomorphically as  $c$  varies in  $\Delta$ .

We always assume  $E$  contains at least three points,  $p_1, p_2$  and  $p_3$ . Then since the points  $h(c, p_1), h(c, p_2)$  and  $h(c, p_3)$  are distinct for each  $c \in \Delta$ , there is a unique

Möbius transformation  $B_c$  that carries these three points to  $0, 1$ , and  $\infty$ . Since  $B_c$  depends holomorphically on  $c$ ,  $\tilde{h}(c, z) = h(c, B_c(z))$  is also a holomorphic motion and it fixes the points  $0, 1, \infty$ . We shall call it a normalized holomorphic motion.

Holomorphic motions were introduced by Mañé, Sad and Sullivan in their study of the structural stability problem for the complex dynamical systems, [54]. They proved the first result in the topic which is called the  $\lambda$ -lemma and which says that any holomorphic motion  $h(c, z)$  of  $E$  parametrized by  $\Delta$  and with base point  $0$  can be extended uniquely to a holomorphic motion of the closure  $\overline{E}$  of  $E$  parametrized by  $\Delta$  and with the same base point. Moreover,  $h(c, z)$  is continuous on  $(c, z)$  and for any fixed  $c$ ,  $z \mapsto h(c, z)$  is quasiconformal on the interior of  $\overline{E}$ . Subsequently, holomorphic motions became an important topic with applications to quasiconformal mapping, Teichmüller theory and complex dynamics. After Mañé, Sad and Sullivan proved the  $\lambda$ -lemma, Sullivan and Thurston [62] proved an important extension result. Namely, they proved that any holomorphic motion of  $E$  parametrized by  $\Delta$  and with base point  $0$  can be extended to a holomorphic motion of  $\overline{\mathbb{C}}$ , but parametrized by a smaller disk, namely, by  $\Delta_r$  for some universal number  $0 < r < 1$ . They showed that  $r$  is independent  $E$  and independent of the motion. By a different method and published in the same journal with the Sullivan-Thurston paper, Bers and Royden [9] proved that  $r \geq 1/3$  for all motions of all closed sets  $E$  parameterized by  $\Delta$ . They also showed that on  $\overline{\mathbb{C}}$  the map  $z \mapsto h(c, z)$  is quasiconformal with dilatation no larger than  $(1 + |c|)/(1 - |c|)$ . All of these authors raised the question as to whether  $r = 1$  for any holomorphic motion of any subset of  $\overline{\mathbb{C}}$  parametrized by  $\Delta$  and with base point  $0$ . In [60] Slodkowski gave a positive answer by using results from the theory of polynomial hulls in several complex variables. Other authors [5] [15] have suggested alternative proofs.

In this chapter, we give an expository account of a recent proof of Slodkowski's theorem presented by Chirka in [12]. (See also Chirka and Rosay [13].) The method involves an application of Schauder's fixed point theorem [14] to an appropriate operator acting on holomorphic motions of a point and on showing that this operator

is compact. The compactness depends on the smoothing property of the Cauchy kernel acting on vector fields tangent to holomorphic motions. The main theorem of this chapter is the following.

**Theorem 17** (The Holomorphic Motion Theorem). *Suppose  $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion of a closed subset  $E$  of  $\overline{\mathbb{C}}$  parameterized by the unit disk. Then there is a holomorphic motion  $H(c, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  which extends  $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$ . Moreover, for any fixed  $c \in \Delta$ ,  $h(c, \cdot) : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is a quasiconformal homeomorphism whose quasiconformal dilatation*

$$K(h(c, \cdot)) \leq \frac{1 + |c|}{1 - |c|}.$$

The Beltrami coefficient of  $h(c, \cdot)$  given by

$$\mu(c, z) = \frac{\partial h(c, z)}{\partial \bar{z}} / \frac{\partial h(c, z)}{\partial z}$$

is a holomorphic function from  $\Delta$  into the unit ball of the Banach space  $\mathcal{L}^\infty(\mathbb{C})$  of all essentially bounded measurable functions on  $\mathbb{C}$ .

We will study the modulus of continuity of functions in the image of the Cauchy kernel operator. Then we apply the Schauder fixed point theorem to a non-linear operator given by Chirka in [12].

## 2.1 The $\mathcal{P}$ -Operator and the Modulus of Continuity

Let  $\mathcal{C} = \mathcal{C}(\mathbb{C})$  denote the Banach space of complex valued, bounded, continuous functions  $\phi$  on  $\mathbb{C}$  with the supremum norm

$$\|\phi\| = \sup_{c \in \mathbb{C}} |\phi(c)|.$$

We use  $\mathcal{L}^\infty$  to denote the Banach space of essentially bounded measurable functions  $\phi$  on  $\mathbb{C}$  with  $\mathcal{L}^\infty$ -norm

$$\|\phi\|_\infty = \text{ess sup}_{\mathbb{C}} |\phi(\zeta)|.$$

For the theory of quasiconformal mapping we are more concerned with the action of  $\mathcal{P}$  on  $\mathcal{L}^\infty$ . Here the  $\mathcal{P}$ -operator is defined by

$$\mathcal{P}f(c) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - c} d\xi d\eta, \quad \zeta = \xi + i\eta$$

where  $f \in \mathcal{L}^\infty$  and has a compact support in  $\mathbb{C}$ . Then

$$\mathcal{P}f(c) \longrightarrow 0 \quad \text{as } c \longrightarrow \infty.$$

Furthermore, if  $f$  is continuous and has compact support, one can show that

$$\frac{\partial(\mathcal{P}f)}{\partial\bar{c}}(c) = f(c), \quad c \in \mathbb{C}, \quad (2.1)$$

and by using the notion of generalized derivative [3] equation (2.1) is still true Lebesgue almost everywhere if we only know that  $f$  has compact support and is in  $\mathcal{L}^p$ , for  $p \geq 1$ .

We first show the classical result that  $\mathcal{P}$  transforms  $\mathcal{L}^\infty$  functions with compact support in  $\mathbb{C}$  to Hölder continuous functions with Hölder exponent  $1 - 2/p$  for every  $p > 2$ . See for example [3]. We also show that  $\mathcal{P}$  carries  $\mathcal{L}^\infty$  functions with compact supports to functions with an  $|\varepsilon \log \varepsilon|$  modulus of continuity.

**Lemma 7.** *Suppose  $p > 2$  and*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

*so that  $1 < q < 2$ . Then for any real number  $R > 0$ , there is a constant  $A_R > 0$  such that, for any  $f \in \mathcal{L}^\infty$  with a compact support contained in  $\Delta_R$ ,*

$$\|\mathcal{P}f\| \leq A_R \|f\|_\infty$$

*and*

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq A_R \|f\|_\infty |c - c'|^{1 - \frac{2}{p}}, \quad \forall c, c' \in \mathbb{C}.$$

*Proof* The norm

$$\|\mathcal{P}f\| = \sup_{c \in \mathbb{C}} \frac{1}{\pi} \left| \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - c} d\xi d\eta \right| \leq \sup_{c \in \mathbb{C}} \frac{1}{\pi} \int_{\Delta_R} \int_{\Delta_R} \frac{|f(\zeta)|}{|\zeta - c|} d\xi d\eta$$

So

$$\|\mathcal{P}f\| \leq \|f\|_\infty \sup_{c \in \mathbb{C}} \frac{1}{\pi} \int \int_{\Delta_R} \frac{1}{|\zeta - c|} d\xi d\eta \leq C_1 \|f\|_\infty$$

where

$$C_1 = \frac{1}{\pi} \int \int_{\Delta_R} \frac{1}{|\zeta|} d\xi d\eta = 2R < \infty.$$

Next

$$\begin{aligned} |\mathcal{P}f(c) - \mathcal{P}f(c')| &= \frac{1}{\pi} \left| \int \int_{\mathbb{C}} f(\zeta) \left( \frac{1}{\zeta - c} - \frac{1}{\zeta - c'} \right) d\xi d\eta \right| \\ &\leq \frac{|c - c'|}{\pi} \int \int_{\Delta_R} \frac{|f(\zeta)|}{|\zeta - c| |\zeta - c'|} d\xi d\eta \\ &\leq \frac{|c - c'|}{\pi} \left( \int \int_{\Delta_R} |f(\zeta)|^p d\xi d\eta \right)^{\frac{1}{p}} \left( \int \int_{\Delta_R} \left| \frac{1}{(\zeta - c)(\zeta - c')} \right|^q d\xi d\eta \right)^{\frac{1}{q}}. \\ &\leq \pi^{\frac{1}{p}-1} R^{\frac{2}{p}} |c - c'| \|f\|_\infty \left( \int \int_{\Delta_R} \left| \frac{1}{(\zeta - c)(\zeta - c')} \right|^q d\xi d\eta \right)^{\frac{1}{q}} \leq C_2 \|f\|_\infty |c - c'|^{\frac{2}{q}-1}. \end{aligned}$$

where

$$C_2 = \pi^{\frac{1}{p}-1} R^{\frac{2}{p}} \left( \int \int_{\mathbb{C}} \left( \frac{1}{|z||z-1|} \right)^q dx dy \right)^{\frac{1}{q}} < \infty, \quad z = x + iy.$$

Hence  $A_R = \max\{C_1, C_2\}$  satisfies the requirements of the lemma.

Next we prove a stronger form of continuity.

**Lemma 8.** *Suppose the compact support of  $f \in \mathcal{L}^\infty$  is contained in  $\Delta$ . Then  $\mathcal{P}f$  has an  $|\varepsilon \log \varepsilon|$  modulus of continuity. More precisely, there is a constant  $B$  depending on  $R$  such that*

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \|f\|_\infty B |c - c'| \log \frac{1}{|c - c'|}, \quad \forall c, c' \in \Delta_R, \quad |c - c'| < \frac{1}{2}.$$

*Proof* Since

$$\begin{aligned} |\mathcal{P}f(c) - \mathcal{P}f(c')| &= \frac{1}{\pi} \left| \int \int_{\mathbb{C}} f(\zeta) \left( \frac{1}{\zeta - c} - \frac{1}{\zeta - c'} \right) d\xi d\eta \right| \\ &\leq \frac{1}{\pi} \int \int_{\mathbb{C}} |f(\zeta)| \left| \frac{1}{\zeta - c} - \frac{1}{\zeta - c'} \right| d\xi d\eta \\ &\leq \frac{|c - c'| \|f\|_\infty}{\pi} \int \int_{\Delta} \frac{1}{|\zeta - c| |\zeta - c'|} d\xi d\eta, \end{aligned}$$

if we put  $\zeta' = \zeta - c = \xi' + i\eta'$ , then

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \frac{|c - c'| \|f\|_\infty}{\pi} \int \int_{\Delta_{1+R}} \frac{1}{|\zeta'| |\zeta' - (c' - c)|} d\xi' d\eta'.$$

The substitution  $\zeta'' = \zeta' / (c' - c) = \xi'' + i\eta''$  yields

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \frac{|c - c'| \|f\|_\infty}{\pi} \int \int_{\Delta_{\frac{1+R}{|c'-c|}}} \frac{1}{|\zeta''| |\zeta'' - 1|} d\xi'' d\eta''.$$

Since  $|c - c'| < 1/2$ , we have  $(1 + R)/|c' - c| > 2$ . This implies that

$$\begin{aligned} & |\mathcal{P}f(c) - \mathcal{P}f(c')| \\ & \leq \frac{|c - c'| \|f\|_\infty}{\pi} \left( \int \int_{\Delta_2} \frac{1}{|\zeta''| |\zeta'' - 1|} d\xi'' d\eta'' + \int \int_{\Delta_{\frac{1+R}{|c'-c|}} - \Delta_2} \frac{1}{|\zeta''| |\zeta'' - 1|} d\xi'' d\eta'' \right) \end{aligned}$$

Let

$$C_3 = \int \int_{\Delta_2} \frac{1}{|\zeta''| |\zeta'' - 1|} d\xi'' d\eta''.$$

Then

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \frac{|c - c'| C_3 \|f\|_\infty}{\pi} + \frac{|c - c'| \|f\|_\infty}{\pi} \int \int_{\Delta_{\frac{1+R}{|c'-c|}} - \Delta_2} \frac{1}{|\zeta''| |\zeta'' - 1|} d\xi'' d\eta''.$$

If  $|\zeta''| > 2$  then  $|\zeta'' - 1| > |\zeta''|/2$ , and so

$$\begin{aligned} \frac{1}{\pi} \int \int_{\Delta_{\frac{1+R}{|c'-c|}} - \Delta_2} \frac{1}{|\zeta''| |\zeta'' - 1|} d\xi'' d\eta'' & \leq \frac{1}{\pi} \int \int_{\Delta_{\frac{1+R}{|c'-c|}} - \Delta_2} \frac{2}{|\zeta''|^2} d\xi'' d\eta'' \\ & \leq \frac{1}{\pi} \int_0^{2\pi} \int_2^{\frac{1+R}{|c'-c|}} \frac{2}{r^2} r dr d\theta = 4 \int_2^{\frac{1+R}{|c'-c|}} \frac{1}{r} dr \\ & = 4 \left( \log \frac{1+R}{|c'-c|} - \log 2 \right) = 4(-\log |c - c'| + \log(1+R) - \log 2). \end{aligned}$$

Thus,

$$\begin{aligned} |\mathcal{P}f(c) - \mathcal{P}f(c')| & \leq \frac{|c - c'| C_3 \|f\|_\infty}{\pi} + 4|c - c'| \|f\|_\infty (-\log |c - c'| + \log(1+R) - \log 2) \\ & = -|c - c'| \log |c - c'| \left( \frac{4\pi \log(1+R) + C_3 \|f\|_\infty - 4\pi \log 2}{-\pi \log |c - c'|} + 4 \|f\|_\infty \right) \\ & \leq B \left( -|c - c'| \log |c - c'| \right) \end{aligned}$$

where

$$B = \frac{4\pi \log(1+R) + C_3 \|f\|_\infty - 4\pi \log 2}{\pi \log 2} + 4 \|f\|_\infty.$$

Now we have the following theorem.

**Theorem 18.** *For any  $f \in \mathcal{L}^\infty$  with a compact support in  $\mathbb{C}$ ,  $\mathcal{P}f$  has an  $|\varepsilon \log \varepsilon|$  modulus of continuity. More precisely, for any  $R > 0$ , there is a constant  $C > 0$  depending on  $R$  such that*

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq C \|f\|_\infty |c - c'| \log \frac{1}{|c - c'|}, \quad \forall c, c' \in \Delta_R, \quad |c - c'| < \frac{1}{2}.$$

*Proof* Suppose the compact support of  $f$  is contained in the disk  $\Delta_{R_0}$ . Then  $g(c) = f(R_0c)$  has the compact support which is contained in the unit disk  $\Delta$ .

Since

$$\mathcal{P}g(c) = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - c} d\xi d\eta = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{f(R_0\zeta)}{\zeta - c} d\xi d\eta = \frac{1}{R_0} \mathcal{P}f(R_0c).$$

This implies that

$$\mathcal{P}f(c) = R_0 \mathcal{P}g\left(\frac{c}{R_0}\right).$$

Thus

$$\begin{aligned} |\mathcal{P}f(c) - \mathcal{P}f(c')| &= R_0 \left| \mathcal{P}g\left(\frac{c}{R_0}\right) - \mathcal{P}g\left(\frac{c'}{R_0}\right) \right| \\ &\leq R_0 B \|f\|_\infty \left( - \left| \frac{c}{R_0} - \frac{c'}{R_0} \right| \log \left| \frac{c}{R_0} - \frac{c'}{R_0} \right| \right) \\ &= B \|f\|_\infty \left( - |c - c'| (\log |c - c'| - \log R_0) \right) \\ &= -|c - c'| \log |c - c'| B \|f\|_\infty \left( 1 - \frac{\log R_0}{\log |c - c'|} \right) \\ &\leq C \|f\|_\infty (-|c - c'| \log |c - c'|) \end{aligned}$$

where

$$C = B \left( 1 + \frac{\log R_0}{\log 2} \right).$$

## 2.2 Extensions of holomorphic motions for $0 < r < 1$ .

As an application of the modulus of continuity for the  $\mathcal{P}$ -operator, we first prove, for any  $r$  with  $0 < r < 1$ , that for any holomorphic motion of a set  $E$  parameterized

by  $\Delta$ , there is an extension to  $\Delta_r \times \overline{\mathbb{C}}$ . We take the idea of the proof from the recent papers of Chirka [12] and Chirka and Rosay [13].

**Theorem 19.** *Suppose  $E$  is a subset of  $\overline{\mathbb{C}}$  consisting of finite number of points. Suppose  $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion. Then for every  $0 < r < 1$ , there is a holomorphic motion  $H_r(c, z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  which extends  $h(c, z) : \Delta_r \times E \rightarrow \overline{\mathbb{C}}$ .*

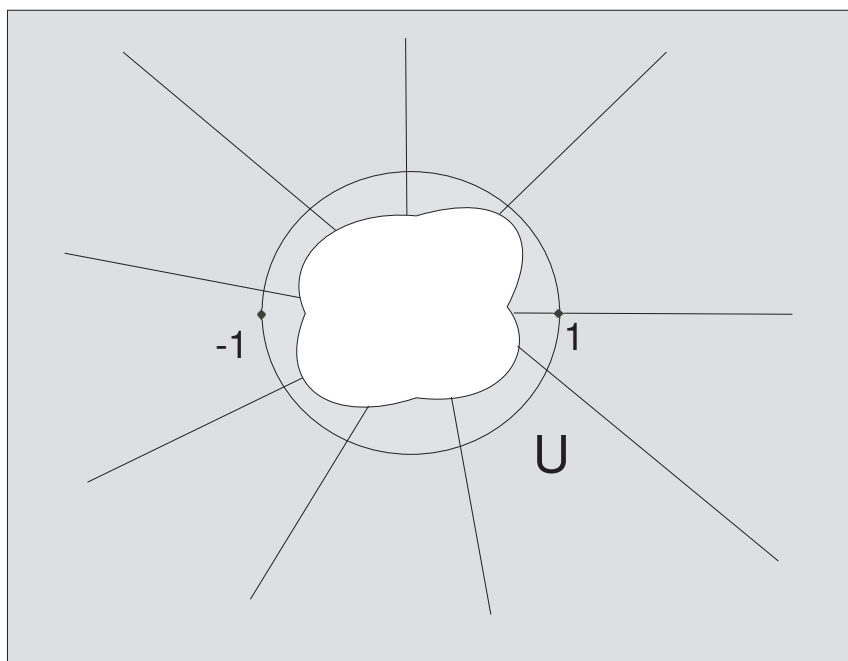


Figure 2.1: Holomorphic motion

Without loss of generality, suppose

$$E = \{z_0 = 0, z_1 = 1, z_\infty = \infty, z_2, \dots, z_n\}$$

is a subset of  $n + 2 > 3$  points in the Riemann sphere  $\overline{\mathbb{C}}$ . Let  $\Delta^c$  be the complement of the unit disk in the Riemann sphere  $\overline{\mathbb{C}}$ ,  $U$  be a neighborhood of  $\Delta^c$  in  $\overline{\mathbb{C}}$  and suppose

$$h(c, z) : U \times E \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion of  $E$  parametrized by  $U$  and with base point  $\infty$ . Define

$$f_i(c) = h(c, z_i) : U \rightarrow \overline{\mathbb{C}}$$

for  $i = 0, 1, 2, \dots, n, \infty$ . We assume the motion is normalized so

$$f_0(c) = 0, \quad f_1(c) = 1, \quad \text{and} \quad f_\infty(c) = \infty, \quad \forall c \in U.$$

Then we have that

- a)  $f_i(\infty) = z_i$ ,  $i = 2, \dots, n$ ;
- b) for any  $i = 2, \dots, n$ ,  $f_i(c)$  is holomorphic on  $U$ ;
- c) for any fixed  $c \in U$ ,  $f_i(c) \neq f_j(c)$  and  $f_i(c) \neq 0, 1$ , and  $\infty$  for  $2 \leq i \neq j \leq n$ .

Since  $\Delta^c$  is compact,  $f_i(c)$  is a bounded function on  $\Delta^c$  for every  $2 \leq i \leq n$  and so there is a constant  $C_4 > 0$  such that

$$|f_i(c)| \leq C_4, \quad \text{for all } c \in \Delta^c \text{ and all } i \text{ with } 2 \leq i \leq n.$$

Moreover, there is a number  $\delta > 0$  such that

$$|f_i(c) - f_j(c)| > \delta, \quad \text{for all } i \text{ and } j \text{ with } 2 \leq i \neq j \leq n, \text{ and for all } c \in \Delta^c.$$

We extend the functions  $f_i(c)$  on  $\Delta^c$  to continuous functions on the Riemann sphere  $\overline{\mathbb{C}}$  by defining

$$f_i(c) = f_i\left(\frac{1}{c}\right), \quad \text{for all } c \in \overline{\Delta}.$$

We still have

$$|f_i(c) - f_j(c)| > \delta, \quad \text{for all } i \text{ and } j \text{ with } 2 \leq i \neq j \leq n \text{ and for all } c \in \overline{\mathbb{C}}$$

and

$$|f_i(c)| \leq C_4 \text{ for all } i \text{ and } j \text{ with } 2 \leq i \neq j \leq n \text{ and for all } c \in \overline{\mathbb{C}}.$$

Since  $f_i(c)$  is holomorphic in  $\Delta^c$  and  $f_i(\infty) = z_i$ , the series expansion of  $f_i(c)$  at  $\infty$  is

$$f_i(c) = z_i + \frac{a_1}{c} + \frac{a_2}{c^2} + \dots + \frac{a_n}{c^n} + \dots, \quad c \in \Delta^c, \quad \forall c \in \Delta^c.$$

This implies that

$$f_i(c) = f_i\left(\frac{1}{\bar{c}}\right) = z_i + a_1\bar{c} + a_2(\bar{c})^2 + \cdots + a_n(\bar{c})^n + \cdots, \quad \forall c \in \bar{\Delta}.$$

We have that

$$\frac{\partial f_i}{\partial \bar{c}}(c) = a_1 + 2a_2\bar{c} + \cdots + na_n(\bar{c})^{n-1} + \cdots$$

exists at  $c = 0$  and is a continuous function on  $\bar{\Delta}$ . Furthermore,  $(\partial f_i / \partial \bar{c})(c) = 0$  for  $c \in \Delta^c$ . Since  $\bar{\Delta}$  is compact, there is a constant  $C_5 > 0$  such that

$$\left| \frac{\partial f_i}{\partial \bar{c}}(c) \right| \leq C_5, \quad \forall c \in \bar{\mathbb{C}}, \quad \forall 2 \leq i \leq n.$$

Pick a  $C^\infty$  function  $0 \leq \lambda(x) \leq 1$  on  $\mathbb{R}^+ = \{x \geq 0\}$  such that  $\lambda(0) = 1$  and  $\lambda(x) = 0$  for  $x \geq \delta/2$ . Define

$$\Phi(c, w) = \sum_{i=2}^n \lambda(|w - f_i(c)|) \frac{\partial f_i}{\partial \bar{c}}(c), \quad (c, w) \in \bar{\mathbb{C}} \times \mathbb{C}. \quad (2.2)$$

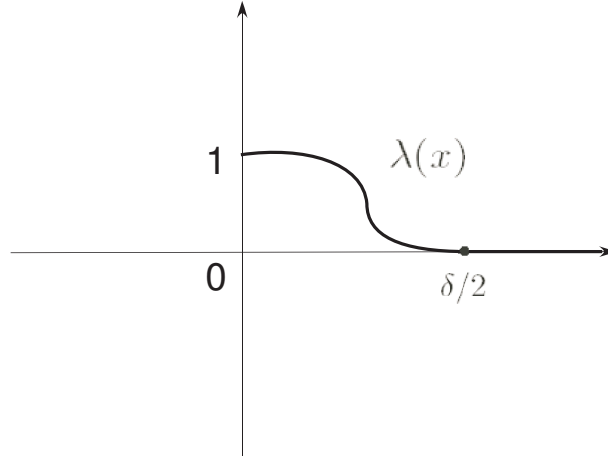


Figure 2.2: smooth function for Chirka's operator

**Lemma 9.** *The function  $\Phi(c, w)$  has the following properties:*

- i) only one term in the sum (2.2) defining  $\Phi(c, w)$  can be nonzero,*
- ii)  $\Phi(c, w)$  is uniformly bounded by  $C_5$  on  $\bar{\mathbb{C}} \times \mathbb{C}$ ,*

iii)  $\Phi(c, w) = 0$  for  $(c, w) \in \left( (\overline{\Delta})^c \times \mathbb{C} \right) \cup \left( \overline{\mathbb{C}} \times (\overline{\Delta}_R)^c \right)$  where  $R = C_4 + \delta/2$ ,

iv)  $\Phi(c, w)$  is a Lipschitz function in  $w$ -variable with a Lipschitz constant  $L$  independent of  $c \in \hat{\mathcal{C}}$ .

*Proof* Item i) follows because if a point  $w$  is within distance  $\delta/2$  of one of the values  $f_i(c)$  it must be at distance greater than  $\delta/2$  from any of the other values  $f_j(c)$ . Item ii) follows from item i) because there can be only one term in (2.2) which is nonzero and that term is bounded by the bound on  $\frac{\partial f_j(c)}{\partial \bar{c}}$ . Item iii) follows because if  $c \in (\overline{\Delta})^c$ , then  $(\partial f_i / \partial \bar{c})(c) = 0$ , and if  $w \in (\overline{\Delta}_R)^c$ ,  $\Phi(c, w) = 0$ . To prove item iv), we note that there is a constant  $C_6 > 0$  such that  $|\lambda(x) - \lambda(x')| \leq C_6|x - x'|$ . Since  $|(\partial f_i / \partial \bar{c})(c)| \leq C_5$ ,

$$|\Phi(c, w) - \Phi(c, w')| \leq C_6 C_5 \sum_{i=2}^n \left| |w - f_i(c)| - |w' - f_i(c)| \right|. \quad (2.3)$$

Since only one of the terms in the sum (2.2) for  $\Phi(c, w)$  is nonzero and possibly a different term is non-zero in the sum for  $\Phi(c, w')$ , we obtain

$$|\Phi(c, w) - \Phi(c, w')| \leq 2C_6 C_5 |w - w'|.$$

Thus  $L = 2C_5 C_6$  is a Lipschitz constant independent of  $c \in \hat{\mathcal{C}}$ .

Since  $\Phi(c, f(c))$  is an  $\mathcal{L}^\infty$  function with a compact support in  $\overline{\Delta}$  for any  $f \in \mathcal{C}$ , we can define an operator  $\mathcal{Q}$  mapping functions in  $\mathcal{C}$  to functions in  $\mathcal{L}^\infty$  with compact support by

$$\mathcal{Q}f(c) = \Phi(c, f(c)), \quad f(c) \in \mathcal{C}.$$

Since  $\Phi(c, w)$  is Lipschitz in the  $w$  variable with a Lipschitz constant  $L$  independent of  $c \in \overline{\mathbb{C}}$ , we have

$$|\mathcal{Q}f(c) - \mathcal{Q}g(c)| = |\Phi(c, f(c)) - \Phi(c, g(c))| \leq L|f(c) - g(c)|.$$

Thus

$$\|\mathcal{Q}f - \mathcal{Q}g\|_\infty \leq L\|f - g\|$$

and  $\mathcal{Q}: \mathcal{C} \rightarrow \mathcal{L}^\infty$  is a continuous operator.

From Lemma 7,

$$\|\mathcal{P}f\| \leq A_1 \|f\|_\infty$$

for any  $f \in \mathcal{L}^\infty$  whose compact support is contained in  $\Delta$ , and so the composition  $\mathcal{K} = \mathcal{P} \circ \mathcal{Q}$ , where

$$\mathcal{K}f(c) = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{\Phi(\zeta, f(\zeta))}{\zeta - c} d\xi d\eta, \quad \zeta = \xi + i\eta,$$

is a continuous operator from  $\mathcal{C}$  into itself.

**Lemma 10.** *There is a constant  $D > 0$  such that*

$$\|\mathcal{K}f\| \leq D, \quad \forall f \in \mathcal{C};$$

*Proof* Since  $\Phi(c, w) = 0$  for  $c \in \Delta^c$  and since  $\Phi(c, w)$  is bounded by  $C_5$ , we have that

$$\begin{aligned} |\mathcal{K}f(c)| &= \left| \frac{1}{\pi} \int \int_{\overline{\mathbb{C}}} \frac{\Phi(\zeta, f(\zeta))}{\zeta - c} d\xi d\eta \right| = \left| \frac{1}{\pi} \int \int_{\Delta} \frac{\Phi(\zeta, f(\zeta))}{\zeta - c} d\xi d\eta \right| \\ &\leq \frac{1}{\pi} \int \int_{\Delta} \frac{|\Phi(\zeta, f(\zeta))|}{|\zeta - c|} d\xi d\eta \\ &\leq \frac{C_5}{\pi} \int \int_{\Delta} \frac{1}{|\zeta - c|} d\xi d\eta \leq 2C_5 = D \end{aligned}$$

where  $\zeta = \xi + i\eta$ .

**Lemma 11.** *Suppose  $p > 2$  and  $q$  is the dual number between 1 and 2 satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

*Then for any  $f \in \mathcal{C}$ ,  $\mathcal{K}f$  is  $\alpha$ -Hölder continuous for*

$$0 < \alpha = \frac{2}{q} - 1 < 1$$

*with a Hölder constant  $H = A_1 C_5$  independent of  $f$ .*

*Proof* From Lemma 7,

$$\begin{aligned} |\mathcal{K}f(c) - \mathcal{K}f(c')| &= |\mathcal{P}(\mathcal{Q}f)(c) - \mathcal{P}(\mathcal{Q}f)(c')| \\ &\leq A_1 \|\mathcal{Q}f\|_\infty |c - c'|^\alpha \leq A_1 C_5 |c - c'|^\alpha = H |c - c'|^\alpha. \end{aligned}$$

**Lemma 12.** *For any  $\varepsilon > 0$ , there exists an  $R > 0$ , such that  $|\mathcal{K}f(z)| < \varepsilon$  for all  $f \in \mathcal{C}(\mathbb{C})$  and  $z \in \mathbb{C}$  with  $|z| \geq R$ .*

*Proof* From lemma 9, the numerator  $\Phi(\zeta, f(\zeta))$  in the formula of  $\mathcal{K}f(z)$  is bounded, but when  $z$  is very large, the denominator of this integral is very large.

The above three lemmas imply that  $\mathcal{K} : \mathcal{C} \rightarrow \mathcal{C}$  is a continuous compact operator. Now for any  $z \in \mathbb{C}$ , let

$$\mathcal{B}_z = \{f \in \mathcal{C} \mid \|f\| \leq |z| + D\}.$$

It is a bounded convex subset in  $\mathcal{C}$ . The continuous compact operator  $z + \mathcal{K}$  maps  $\mathcal{B}_z$  into itself. From the Schauder fixed point theorem [14],  $z + \mathcal{K}$  has a fixed point in  $\mathcal{B}_z$ . That is, there is a  $f_z \in \mathcal{B}_z$  such that

$$f_z(c) = z + \mathcal{K}f_z(c), \quad \forall c \in \mathbb{C}.$$

Since  $\mathcal{Q}f(c)$  has a compact support in  $\overline{\Delta}$  for any  $f \in \mathcal{C}$ ,  $\mathcal{K}f_z(c) \rightarrow 0$  as  $c \rightarrow \infty$ . So  $f_z$  can be extended continuously to  $\infty$  such that  $f_z(\infty) = z$ .

**Remark.** We know that  $|\mathcal{K}f(z)| \rightarrow 0$  if  $|z| \rightarrow \infty$ . However, to check compactness of the operator  $\mathcal{K}$ , we need a kind of uniformity around  $z = \infty$ , like the existence of  $R > 0$  independent of  $f \in \mathcal{C}(\mathbb{C})$  in the lemma 12.

In fact, from Lemma 10, we know that the family  $\{\mathcal{K}f\}_{f \in \mathcal{C}(\mathbb{C})}$  is uniformly bounded, and from Lemma 11, it follows that the family is equicontinuous. Therefore, from these lemmas, we merely conclude from the Ascoli-Arzelà theorem that the family is relatively compact with respect to *the topology of the uniform convergence on any compact sets* of  $\mathbb{C}$ , which is weaker than the topology of  $\mathcal{C}(\mathbb{C})$ . For example, let

$$f_n(z) = \min \left\{ \max \left\{ 1, \frac{|z|}{n} \right\}, 2e^{-|z|+2n} \right\}$$

for  $z \in \mathbb{C}$ . Then, each  $f_n$  is 2-Lipschitz and satisfies  $\|f_n\| \leq 2$  and  $f_n(z) \rightarrow 0$  as  $|z| \rightarrow \infty$  (and hence it is an  $\alpha$ -Hölder function for all  $\alpha \in (0, 1]$  whose Hölder norm depends only on  $\alpha$ ). However, the family  $\{f_n\}_{n \in \mathbb{N}}$  is not compact in  $\mathcal{C}(\mathbb{C})$ .

**Lemma 13.** *The solution  $f_z(c)$  is the unique fixed point of the operator  $z + \mathcal{K}$ .*

*Proof* Suppose  $f_z(c)$  and  $g_z(c)$  are two solutions. Take

$$\phi(c) = f_z(c) - g_z(c) = \mathcal{K}(f_z)(c) - \mathcal{K}(g_z)(c).$$

Then  $\phi(c) \rightarrow 0$  as  $c \rightarrow \infty$ . Now

$$\frac{\partial \phi}{\partial \bar{c}}(c) = \frac{\partial f_z}{\partial \bar{c}}(c) - \frac{\partial g_z}{\partial \bar{c}}(c) = \Phi(c, f_z(c)) - \Phi(c, g_z(c)).$$

So by Lemma 9

$$\frac{\partial \phi}{\partial \bar{c}}(c) = 0, \quad \forall c \in \Delta^c.$$

Since  $\Phi(c, w)$  is Lipschitz in  $w$ -variable with a Lipschitz constant  $L$ ,

$$\left| \frac{\partial \phi}{\partial \bar{c}}(c) \right| = |\Phi(c, f_z(c)) - \Phi(c, g_z(c))| \leq L|f_z(c) - g_z(c)| = L|\phi(c)|.$$

Assuming that  $\phi(c)$  is not equal to zero, define

$$\psi(c) = -\frac{\frac{\partial \phi}{\partial \bar{c}}(c)}{\phi(c)},$$

and otherwise, define  $\psi(c)$  to be equal to zero. Then  $\psi(c)$  is a function in  $\mathcal{L}^\infty$  with a compact support in  $\bar{\Delta}$ . So we have  $\mathcal{P}\psi$  in  $\mathcal{C}$  such that

$$\frac{\partial \mathcal{P}\psi}{\partial \bar{c}}(c) = \psi(c).$$

Consider  $e^{\mathcal{P}\psi} \cdot \phi$ . Then

$$\frac{\partial (e^{\mathcal{P}\psi} \cdot \phi)}{\partial \bar{c}} \equiv 0.$$

This means that  $e^{\mathcal{P}\psi} \cdot \phi$  is holomorphic on the complex plane  $\mathbb{C}$ .

When  $c \rightarrow \infty$ ,  $\mathcal{P}\psi \rightarrow 0$  and  $\phi(c) \rightarrow 0$ . This implies that  $e^{\mathcal{P}\psi} \cdot \phi$  is bounded on  $\mathbb{C}$ . So  $e^{\mathcal{P}\psi} \cdot \phi$  is a constant function. But  $\phi(\infty) = 0$ , so  $e^{\mathcal{P}\psi} \cdot \phi \equiv 0$ . Thus  $\phi(c) \equiv 0$  and  $f_z(c) = g_z(c)$  for all  $c \in \mathbb{C}$ .

For  $z_i \in E$ ,  $2 \leq i \leq n$ , consider

$$\mathcal{K}f_i(c) = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{\Phi(\zeta, f_i(\zeta))}{\zeta - c} d\xi d\eta,$$

where  $\zeta = \xi + i\eta$ . From the definition of  $\Phi(c, w)$ , we have that

$$\Phi(\zeta, f_i(\zeta)) = \frac{\partial f_i}{\partial \zeta}(\zeta).$$

So

$$\mathcal{K}f_i(c) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\frac{\partial f_i}{\partial \zeta}(\zeta)}{\zeta - c} d\xi d\eta.$$

This implies that

$$\frac{\partial \mathcal{K}f_i}{\partial \bar{c}}(c) = \frac{\partial f_i}{\partial \bar{c}}(c)$$

and that

$$\frac{\partial (f_i - \mathcal{K}f_i)}{\partial \bar{c}}(c) \equiv 0.$$

So  $f_i(c) - \mathcal{K}f_i(c)$  is holomorphic on  $\mathbb{C}$ . When  $c \rightarrow \infty$ ,  $f_i(c) \rightarrow z_i$  and  $\mathcal{K}f_i(c) \rightarrow 0$ .

So  $f_i(c) - \mathcal{K}f_i(c)$  is bounded. Therefore it is a constant function. We get that

$$f_i(c) = z_i + \mathcal{K}f_i(c).$$

Thus from Lemma 13,  $f_i(c) = f_{z_i}(c)$  for all  $c \in \bar{\mathbb{C}}$ .

By defining  $H(c, z) = f_z(c)$  for  $(c, z) \in \bar{\Delta}^c \times \mathbb{C} \setminus \{0, 1\}$  and  $H(c, 0) = 0$  and  $H(c, 1) = 1$  and  $H(c, \infty) = \infty$ , we get a map

$$H(c, z) = f_z(c) : \bar{\Delta}^c \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}},$$

which is an extension of

$$h(c, z) : \bar{\Delta}^c \times E \rightarrow \bar{\mathbb{C}}.$$

**Lemma 14.** *The map*

$$H(c, z) = f_z(c) : \bar{\Delta}^c \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}},$$

*is a holomorphic motion.*

*Proof* First  $H(\infty, z) = f_z(\infty) = z$  for all  $z \in \bar{\mathbb{C}}$ . From the fixed point equation

$$H(c, z) = z + \mathcal{K}H(c, z),$$

$$\frac{\partial H(c, z)}{\partial \bar{c}} = \Phi(c, H(c, z)).$$

Since  $\Phi(c, w) = 0$  for all  $c \in \overline{\Delta}^c$ ,

$$\frac{\partial H(c, z)}{\partial \bar{c}} = 0, \quad \forall c \in \overline{\Delta}^c.$$

Thus, for any fixed  $z \in \overline{\mathbb{C}}$ ,  $H(c, z) : \overline{\Delta}^c \rightarrow \overline{\mathbb{C}}$  is holomorphic.

For any two  $z \neq z' \in \overline{\mathbb{C}}$ , we claim that  $H(c, z) \neq H(c, z')$  for all  $c \in \mathbb{C}$ . This implies that for any fixed  $c \in \overline{\Delta}^c$ ,  $H(c, z)$  is an injective map on  $z \in \overline{\mathbb{C}}$  and that  $H(c, z)$  is a holomorphic motion. To prove the claim take any two  $z, z' \in \overline{\mathbb{C}}$ . Assume there is a point  $c_0 \in \overline{\mathbb{C}}$  such that  $H(c_0, z) = H(c_0, z')$ . If  $c_0 = \infty$ , then  $z = z'$ , because by assumption the holomorphic motion starts out at the identity. If  $c_0 \neq \infty$ , then

$$f_z(c_0) - f_{z'}(c_0) = (z - z') + \mathcal{K}f_z(c_0) - \mathcal{K}f_{z'}(c_0),$$

and we can repeat the same argument we have given in Lemma 13.

Let  $\phi(c) = f_z(c) - f_{z'}(c)$ . Then  $\phi(c_0) = 0$ . However,

$$\frac{\partial \phi}{\partial \bar{c}}(c) = \frac{\partial f_z}{\partial \bar{c}}(c) - \frac{\partial f_{z'}}{\partial \bar{c}}(c) = \Phi(c, f_z(c)) - \Phi(c, f_{z'}(c)).$$

This implies that

$$\frac{\partial \phi}{\partial \bar{c}}(c) = 0$$

for  $c \in \overline{\Delta}^c$ . Since  $\Phi(c, w)$  is Lipschitz in  $w$ -variable with a Lipschitz constant  $L$ ,

$$\left| \frac{\partial \phi}{\partial \bar{c}}(c) \right| = |\Phi(c, f_z(c)) - \Phi(c, f_{z'}(c))| \leq L|f_z(c) - f_{z'}(c)| = L|\phi(c)|.$$

If  $\phi(c) \neq 0$ , define

$$\psi(c) = -\frac{\frac{\partial \phi}{\partial \bar{c}}(c)}{\phi(c)},$$

otherwise, define  $\psi(c) = 0$ . Then

$$\frac{\partial e^{\mathcal{P}\psi} \cdot \phi}{\partial \bar{c}}(c) \equiv 0.$$

So  $e^{\mathcal{P}\psi} \cdot \phi$  is holomorphic on  $\mathbb{C}$ . When  $c \rightarrow \infty$ ,  $\mathcal{P}\psi(c) \rightarrow 0$  and  $\phi(c) \rightarrow z - z'$ . So  $e^{\mathcal{P}\psi(c)} \cdot \phi(c)$  is bounded on  $\mathbb{C}$ . This implies that  $e^{\mathcal{P}\psi(c)} \cdot \phi(c)$  is a constant function.

Since  $\phi(c_0) = 0$ ,  $e^{\mathcal{P}\psi(c)} \cdot \phi(c) \equiv 0$ . So  $z = z'$ .

*Proof* [Proof of Theorem 19] Suppose

$$h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion. For every  $0 < r < 1$ , consider  $\alpha_r(c) = r/c$ . Let  $U_r = \alpha_r(\Delta_r) \supset \overline{\Delta}^c$ . Then

$$h_r(\alpha_r^{-1}(c), z) : U_r \times E \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion. From Lemmas 13 and 14, it can be extended to a holomorphic motion

$$\tilde{H}_r(c, z) : \overline{\Delta}^c \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}.$$

Then

$$H_r(c, z) = \tilde{H}(\alpha_r(c), z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion which is an extension of  $h(c, z)$  on  $\Delta_r \times E$ .

## 2.3 Controlling quasiconformal dilatation

To control the quasiconformal dilatation of a holomorphic motion there are two methods available. One is given by the Bers-Royden paper [9] and the other is obtained by combining methods given in the Bers-Royden paper and in the Sullivan-Thurston paper [62]. We discuss the latter method first.

Consider a set of four points  $S = \{z_1, z_2, z_3, z_4\}$  in  $\overline{\mathbb{C}}$ . These points are distinct if and only if the cross ratio

$$Cr(S) = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

is not equal to 0, 1, or  $\infty$ . If one of these points is equal to  $\infty$ , say  $z_4$ , then this cross ratio becomes a ratio

$$Cr(S) = \frac{z_1 - z_3}{z_2 - z_3}.$$

Suppose  $H : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$  is an orientation-preserving homeomorphism such that  $H(\infty) = \infty$ . Then one of the definitions of quasiconformality [51] of  $H$  is that

$$\limsup_{r \rightarrow 0} \sup_{a \in \mathbb{C}} \frac{\sup_{|z-a|=r} |H(z) - H(a)|}{\inf_{|z-a|=r} |H(z) - H(a)|} < \infty.$$

In [62] Sullivan and Thurston used this definition to prove the following theorem.

**Theorem 20.** *Suppose  $H(c, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is a normalized holomorphic motion of  $\overline{\mathbb{C}}$  parametrized by  $\Delta$  and with base point 0. Then for each  $c_0 \in \Delta$ , the map  $h(c_0, \cdot) : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$  is quasiconformal.*

*Proof* Let  $a \in \mathbb{C}$  be any point. Let  $z_3 = a$ . Let  $z_1$  and  $z_2$  be two distinct points in  $\mathbb{C}$  not equal to  $a$  and  $z_4 = \infty$ . Then the cross ratio  $Cr(S) = (z_1 - z_3)/(z_2 - z_3)$ .

Now consider  $z_1(c) = H(c, z_1)$ ,  $z_2(c) = H(c, z_2)$ ,  $z_3(c) = H(c, z_3)$ , and  $z_4(c) = H(c, z_4) = \infty$  and  $S(c) = \{z_1(c), z_2(c), z_3(c)\}$ . The cross ratio

$$Cr(S(c)) = \frac{z_1(c) - z_3(c)}{z_2(c) - z_3(c)}.$$

Since  $H(c, z)$  is a holomorphic motion,  $Cr(S(c)) : \Delta \mapsto \mathbb{C} \setminus \{0, 1\}$  is a holomorphic function. Then it decreases the hyperbolic distances from  $\rho_\Delta$  to  $\rho_{0,1}$ . So

$$\rho_{0,1}(Cr(S(c_0)), Cr(S)) \leq \rho_\Delta(0, c_0) = \log \frac{1 + |c_0|}{1 - |c_0|}.$$

This implies that there is a constant  $K = K(c_0) > 0$  such that for any  $|Cr(S)| = 1$ ,

$$|Cr(S(c_0))| \leq K.$$

So we have that

$$\limsup_{r \rightarrow 0} \sup_{a \in \mathbb{C}} \frac{\sup_{|z-a|=r} |H(c_0, z) - H(c_0, a)|}{\inf_{|z-a|=r} |H(c_0, z) - H(c_0, a)|} < \infty,$$

that is,  $H(c_0, z)$  is quasiconformal.

Suppose  $\mathcal{L}^\infty(W)$  is the Banach space of all essentially bounded measurable functions on  $W$  equipped with  $\|\cdot\|_\infty$ -norm. Bers and Royden [9] proved the following theorem.

**Theorem 21.** *Suppose  $h(c, z) : \Delta \times E \rightarrow \hat{\mathbb{C}}$  is a holomorphic motion of  $E$  parametrized by  $\Delta$  and with base point 0 and  $E$  has nonempty interior  $W$ , then the Beltrami coefficient of  $h(c, \cdot)|_W$  given by*

$$\mu(c, z) = \frac{\partial h(c, z)|_W}{\partial \bar{z}} / \frac{\partial h(c, z)|_W}{\partial z}$$

is a holomorphic function mapping  $c \in \Delta$  into the unit ball of the Banach space  $\mathcal{L}^\infty(W)$ .

*Proof* Since the dual of the Banach space  $\mathcal{L}^1(W)$  of integrable functions on  $W$  is  $\mathcal{L}^\infty(W)$ , to prove  $\mu(c, \cdot)$  is a holomorphic map, it suffices to show that the function

$$c \mapsto \Psi(c) = \int \int_W \alpha(z) \mu_c(z) dx dy$$

is holomorphic in  $\Delta$  for every  $\alpha(z) \in \mathcal{L}^1(W)$ . Furthermore, it suffices to check this for every  $\alpha(z) \in \mathcal{L}^1(W)$  with a compact support in  $W$ .

Suppose  $\alpha(z) \in \mathcal{L}^1(W)$  has a compact support  $\text{supp}(\alpha)$  in  $W$ . There is an  $\varepsilon > 0$  such that the  $\varepsilon$ -neighborhood  $U_\varepsilon(\text{supp}(\alpha)) \subset W$ . From Theorem 20,  $h(c, \cdot)$  is quasiconformal, it is differentiable, a.e. in  $W$ . Thus

$$\Psi(c) = \int \int_{\text{supp}(\alpha)} \alpha(z) \frac{h_x(c, z) + ih_y(c, z)}{h_x(c, z) - ih_y(c, z)} dx dy$$

$$\Psi(c) = \int \int_{\text{supp}(\alpha)} \alpha(z) \frac{1 + i \frac{h_y(c, z)}{h_x(c, z)}}{1 - i \frac{h_y(c, z)}{h_x(c, z)}} dx dy$$

$$\Psi(c) = \int \int_{\text{supp}(\alpha)} \alpha(z) \lim_{\lambda \rightarrow 0} \frac{1 + i\sigma_c(z, \lambda)}{1 - i\sigma_c(z, \lambda)} dx dy$$

where

$$\sigma_c(z, \lambda) = \frac{h(c, z + i\lambda) - h(c, z)}{h(c, z + \lambda) - h(c, z)}.$$

For any fixed  $z \neq 0, 1, \infty$  and  $\lambda$  small,

$$\varrho(c) = \sigma_c(z) : \Delta \mapsto \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$$

is a holomorphic function of  $c \in \Delta$ . So it decreases the hyperbolic distances on  $\Delta$  and on  $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ . Since  $\varrho(0) = i$ , there is a number  $0 < r < 1$  such that for

$$|\sigma_c(z, \lambda) - i| \leq \frac{1}{2}, \quad |c| < r.$$

Therefore

$$\left| \frac{1 + i\sigma_c(z, \lambda)}{1 - i\sigma_c(z, \lambda)} \right| = \left| \frac{-i + \sigma_c(z, \lambda)}{i + \sigma_c(z, \lambda)} \right| \leq \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

By the Dominated Convergence Theorem, for  $|c| < r$ , the sequence of holomorphic functions

$$\Psi_n(c) = \int \int_{\text{supp}(\alpha)} \alpha(z) \frac{1 + i\sigma_c(z, \frac{1}{n})}{1 - i\sigma_c(z, \frac{1}{n})} dx dy$$

converges uniformly to  $\Psi(c)$  as  $n \rightarrow \infty$ . Thus  $\Psi(c)$  is holomorphic for  $|c| < r$  and this implies that

$$\mu(c, \cdot) : \{c \mid |c| < r\} \rightarrow \mathcal{L}^\infty(W)$$

is holomorphic.

Now consider arbitrary  $c_0 \in \Delta$ . Let  $s = 1 - |c_0|$  and let

$$E_0 = h(c_0, E) \quad \text{and} \quad W_0 = h(c_0, W)$$

and

$$g(\tau, \zeta) = h(c_0 + s\tau, z), \quad \zeta = h(c_0, z).$$

Then  $W_0$  is the interior of  $E_0$  since  $h(c, z)$  is a quasiconformal homeomorphism.

Also

$$g : \Delta \times E_0 \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion. So the Beltrami coefficient of  $g$  is a holomorphic function on  $\{\tau \mid |\tau| < r\}$ . Hence the Beltrami coefficient of  $h$  is a holomorphic function on  $\{c \mid |c - c_0| < sr\}$ . This concludes the proof.

**Theorem 22.** *Suppose  $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion of  $E$  parametrized by  $\Delta$  and with base point 0 and suppose  $E$  has a nonempty interior  $W$ . Then for each  $c \in \Delta$ , the map  $h(c, z)|_W$  is a  $K$ -quasiconformal homeomorphism of  $W$  into  $\overline{\mathbb{C}}$  with*

$$K \leq \frac{1 + |c|}{1 - |c|}.$$

*Proof* Since  $\mu(c, \cdot) : \Delta \mapsto \mathcal{L}^\infty(W)$  is a holomorphic map and since  $\mu(0, \cdot) = 0$ . From the Schwarz's lemma,  $\|\mu\|_\infty \leq |c|$ . This implies that the quasiconformal dilatation of  $h(c, \cdot)$  is less than or equation to  $K = \frac{1+|c|}{1-|c|}$ .

## 2.4 Extension of holomorphic motions for $r = 1$ .

**Theorem 23** (Słodkowski's Theorem). *Suppose  $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion. Then there is a holomorphic motion  $H(c, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  which extends  $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$ .*

*Proof* Suppose  $E$  is a subset of  $\overline{\mathbb{C}}$ . Suppose  $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion. Let  $E_1, E_2, \dots$  be a sequence of nested subsets consisting of finite number of points in  $E$ . Suppose

$$\{0, 1, \infty\} \subset E_1 \subset E_2 \subset \dots \subset E$$

and suppose  $\cup_{i=1}^{\infty} E_i$  is dense in  $E$ . Then  $h(c, z) : \Delta \times E_i \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion for every  $i = 1, 2, \dots$

From Theorem 19, for any  $0 < r < 1$  and  $i \geq 1$ , there is a holomorphic motion  $H_i(c, z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  such that  $H_i|_{\Delta_r \times E_i} = h|_{\Delta_r \times E_i}$ . From Theorem 22,  $z \mapsto H_i(c, z)$  is  $(1 + |c|/r)/(1 - |c|/r)$ -quasiconformal and fixes  $0, 1, \infty$  for all  $i > 0$ . So for any  $|c| \leq r$ , the functions  $z \mapsto H_i(c, z)$  form a normal family and there is a subsequence  $H_{i_k}(c, \cdot)$  converging uniformly (in the spherical metric) to a  $(1 + |c|/r)/(1 - |c|/r)$ -quasiconformal homeomorphism  $H_r(c, \cdot) : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  such that  $H_r(c, z) = h(c, z)$  for  $z \in \cup(E_{j_k})$ .

Let  $\zeta$  be a point in  $E$ . Replacing  $E_i$  by  $E_i \cup \{\zeta\}$  and repeating the previous construction we obtain a  $(1 + |c|/r)/(1 - |c|/r)$ -quasiconformal homeomorphism  $\tilde{H}_r$  which coincides with  $h(c, z)$  on  $\cup E_{i_k} \cup \{\zeta\}$ . But  $z \mapsto H_r(c, z)$  and  $z \mapsto \tilde{H}_r(c, z)$  are continuous everywhere and coincide on  $\cup E_{i_k}$ , hence on  $E$ . So  $H_r(c, \zeta) = \tilde{H}_r(c, \zeta) = h(c, \zeta)$  for any  $\zeta \in E$ .

Now for any  $z \neq 0, 1, \infty$ , since  $H_i(c, z) : \Delta \rightarrow \overline{\mathbb{C}}$  are holomorphic and omit three points  $0, 1, \infty$ . So the functions  $c \mapsto H_i(c, z)$  form a normal family. Any convergent subsequence  $H_{i_k}(c, z)$  still has a holomorphic limit  $H_r(c, z)$ , thus  $H_r(c, z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion which extends  $h(c, z)$  on  $\Delta_r \times \overline{\mathbb{C}}$ .

Now we are ready to take the limit as  $r \rightarrow 1$ . For each  $0 < r < 1$ , let  $H_r(c, z) :$

$\Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be a holomorphic motion such that  $H_r = h$  on  $\Delta_r \times E$ . From Theorem 22,  $H_r(c, \cdot)$  is  $(1+|c|/r)/(1-|c|/r)$ -quasiconformal for every  $c$  with  $|c| \leq r$ .

Take a sequence  $Z = \{z_i\}_{i=1}^\infty$  of points in  $\overline{\mathbb{C}}$  such that  $\overline{Z} = \overline{\mathbb{C}}$ , and assume  $0, 1, \infty$  are not elements of  $Z$ . For each  $i = 1, 2, \dots$ ,  $H_r(c, z_i) : \Delta_r \rightarrow \overline{\mathbb{C}}$  is holomorphic and omits  $0, 1, \infty$ . Thus  $\{H_r(c, z_i), c \in \Delta_r\}_{0 < r < 1}$  forms a normal family. We have a subsequence  $r_n \rightarrow 1$  such that  $H_{r_n}(c, z_i)$  tends to a holomorphic function  $\tilde{H}(c, z_i)$  defined on  $\Delta$  uniformly on the spherical metric for all  $i = 1, 2, \dots$ . For a fixed  $c \in \Delta$ ,  $H_{r_n}(c, \cdot)$  are  $(1+|c|/r_n)/(1-|c|/r_n)$ -quasiconformal for all  $r_n > |c|$ . So  $\{H_{r_n}(c, \cdot)\}_{r_n > |c|}$  is a normal family. Since  $H_{r_n}(c, \cdot)$  fixes  $0, 1, \infty$ , there is a subsequence of  $\{H_{r_n}(c, \cdot)\}$ , which we still denote by  $\{H_{r_n}(c, \cdot)\}$ , that converges uniformly in the spherical metric to a  $(1+|c|)/(1-|c|)$ -quasiconformal homeomorphism  $H(c, \cdot)$ . Since  $\tilde{H}(c, z_i) = H(c, z_i)$  for all  $i = 1, 2, \dots$ , this implies that for any fixed  $c \in \Delta$ ,  $H(c, z_i) \neq H(c, z_j)$  for  $i \neq j$ . Thus  $H(c, z) : \Delta \times Z \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion.

For any  $0 < r < 1$ ,  $H(c, z)$  is  $(1+r)/(1-r)$ -quasiconformal for all  $c$  with  $|c| \leq r$ , it is  $\alpha$ -Hölder continuous, that is,

$$d(H(c, z), H(c, z')) \leq A d(z, z')^\alpha \quad \text{for all } z, z' \in \overline{\mathbb{C}} \quad \text{and for all } |c| \leq r,$$

where  $d(\cdot, \cdot)$  is the spherical distance and where  $A$  and  $0 < \alpha < 1$  depend only on  $r$ .

For any  $z \in Z$  such that its spherical distances to  $0, 1, \infty$  are greater than  $\varepsilon > 0$ , the map  $H(c, z)$  is a holomorphic map on  $\Delta$ , which omits the values  $0, 1, \infty$ . So  $H(c, z)$  decreases the hyperbolic distance  $\rho_\Delta$  on  $\Delta$  and the hyperbolic distance  $\rho_{0,1}$  on  $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ . So we have a constant  $B > 0$  depending only on  $r$  and  $\varepsilon$  such that

$$d(H(c, z), H(c', z)) \leq B|c - c'|$$

for all  $|c|, |c'| \leq r$  and all  $z \in Z$  such that spherical distances between them and  $0, 1, \infty$  are greater than  $\varepsilon > 0$ . Thus we get that

$$d(H(c, z), h(c', z')) \leq A d(z, z')^\alpha + B|c - c'|.$$

for  $|c|, |c'| \leq r$  and  $z, z' \in Z$  such that their spherical distances from  $0, 1, \infty$  are greater than  $\varepsilon > 0$ . This implies that  $H(c, z)$  is uniformly equicontinuous on  $|c| \leq r$

and  $\{z \in Z \mid d(z, \{0, 1, \infty\}) \geq \varepsilon\}$ . Therefore, its continuous extension  $H(c, z)$  is holomorphic in  $c$  with  $|c| \leq r$  for any  $\{z \in \overline{\mathbb{C}} \mid d(z, \{0, 1, \infty\}) \geq \varepsilon\}$ . Letting  $r \rightarrow 1$  and  $\varepsilon \rightarrow 0$ , we get that  $H(c, z)$  is holomorphic in  $c \in \Delta$  for any  $z \in \overline{\mathbb{C}}$ . Thus  $H(c, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is a holomorphic motion such that  $H(c, z)|_{\Delta \times E} = h(c, z)$ . We completed the proof.

# The $|\varepsilon \log \varepsilon|$ continuity of a holomorphic motion

## Contents

3.1	Agard's formula for $\rho_{0,1}$ and its lower bound . . . . .	57
3.2	The $ \varepsilon \log \varepsilon $ continuity of a holomorphic motion . . . . .	61

### 3.1 Agard's formula for $\rho_{0,1}$ and its lower bound

As an application of the extension theorem of a holomorphic motion, I will prove the Agard's formula for the Poincare metric  $\rho_{0,1}$  on the three punctured Riemann sphere  $\overline{\mathbb{C}} - \{0, 1, \infty\}$ .

**Theorem 24** (Agard [1]).

$$(\rho_{0,1}(z_0))^{-1} = \frac{1}{2\pi} \int \int_{\mathbb{C}} \left| \frac{z_0(z_0 - 1)}{\zeta(\zeta - 1)(\zeta - z_0)} \right| d\xi d\eta.$$

*Proof* Suppose  $g(t)$  is a holomorphic map from  $\Delta \rightarrow \overline{\mathbb{C}} - \{0, 1, \infty\}$  and  $g(0) = z_0$ .

Let  $h(t, z_0) = g(t)$ ,  $h(t, 0) = 0$ ,  $h(t, 1) = 1$  and  $h(t, \infty) = \infty$ , then it is a holomorphic motion of four points  $0, 1, \infty$  and  $z_0$ .

By the Slodkowski's extension theorem,  $h(t, z)$  can be extended to a holomorphic motion  $H(t, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ . Let

$$\mu_t(z) = \frac{H_{\bar{z}}}{H_z},$$

then it is a holomorphic function of  $t$  from  $\Delta$  to the unit ball  $M$  in the space  $L^\infty$ .

From Schwarz's lemma, if  $\mu_t = t\nu + o(t)$ , then  $\|\nu\|_\infty \leq 1$ .

Hence

$$g'(0) = \frac{1}{2\pi} \int \int_{\mathbb{C}} \frac{z_0(z_0 - 1) \times \nu}{\zeta(\zeta - 1)(\zeta - z_0)} d\xi d\eta \leq \frac{1}{2\pi} \int \int_{\mathbb{C}} \left| \frac{z_0(z_0 - 1)}{\zeta(\zeta - 1)(\zeta - z_0)} \right| d\xi d\eta.$$

Let  $\varphi(\zeta) = \frac{z_0(z_0 - 1)}{\zeta(\zeta - 1)(\zeta - z_0)}$  and  $\mu = |\varphi|/\varphi$ . Then the derivative of  $f^{t\mu}$  at  $t = 0$  equals to the right hand side of the previous inequality which is the maximum of  $g'(0)$ .

So

$$\rho_{0,1}^{-1}(z_0) = \frac{1}{2\pi} \int \int_{\mathbb{C}} \left| \frac{z_0(z_0 - 1)}{\zeta(\zeta - 1)(\zeta - z_0)} \right| d\xi d\eta.$$

The following lemma, a form of which appeared in [65, Zhongli], is sufficient for the proof of the main theorem 25 of this chapter,  $|\varepsilon \log \varepsilon|$  continuity of the tangent vector of a holomorphic motion.

**Lemma 15.** *If  $0 < |z| < 1$ , then*

$$\rho_{0,1}(z) \geq \frac{1}{|z|(\log r + \log \frac{1}{|z|})},$$

where  $r$  is chosen so that

$$\log r > \max\left\{\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{d\xi d\eta}{|(\zeta + 1)\zeta(\zeta - 1)|}, 4 + \log 4\right\}$$

(Note that numerical calculation suggests that  $4 + \log 4$  is the larger of these two numbers.)

*Proof* From Agard's formula [1] (note that  $\rho_{0,1}$  has the curvature  $-1$ ),

$$\rho_{0,1}(z) = \left( \frac{1}{2\pi} \int \int_{\mathbb{C}} \left| \frac{z(z - 1)}{\zeta(\zeta - 1)(\zeta - z)} \right| d\xi d\eta \right)^{-1}.$$

Since the smallest value of  $\rho_{0,1}(z)$  on the circle  $|z| = 1$  occurs at  $z = -1$ , we see that

$$\frac{1}{\log r} \leq \min_{|z|=1} \rho_{0,1}(z) = \left( \frac{1}{\pi} \int \int_{\mathbb{C}} \left| \frac{1}{(\zeta - 1)(\zeta)(\zeta + 1)} \right| d\xi d\eta \right)^{-1}.$$

The infinitesimal form of the Poincaré metric  $\rho_r = \rho_{\Delta_r^*}$  with curvature constantly equal to  $-1$  for the punctured disk  $\Delta_r^* = \{z \in \mathbb{C} \mid 0 < |z| < r\}$  is

$$\rho_r(z) = \frac{1}{|z| \left[ \log r + \log \frac{1}{|z|} \right]}. \quad (3.1)$$

Note that  $\rho_r(z)$  takes the constant value  $\frac{1}{\log r}$  along  $|z| = 1$ . Then

$$\rho(z) \leq \rho_{0,1}(z) \text{ for all } z \text{ with } |z| = 1. \quad (3.2)$$

Our next objective is to show that the same inequality

$$\rho(z) \leq \rho_{0,1}(z) \quad (3.3)$$

holds for all  $z$  with  $|z| < \delta$  when  $\delta$  is sufficiently small. In [2] Ahlfors shows that

$$\rho_{0,1}(z) \geq \frac{|\zeta'(z)|}{|\zeta(z)|} \frac{1}{4 + \log \frac{1}{|\zeta(z)|}} \quad (3.4)$$

for  $|z| \leq 1$  and  $|z| \leq |z - 1|$ , where  $\zeta$  maps the complement of  $[1, +\infty]$  conformally onto the unit disk, origins corresponding to each other and symmetry with respect to the real axis being preserved.  $\zeta$  satisfies

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{z\sqrt{1-z}}, \quad (3.5)$$

$$\zeta(z) = \frac{\sqrt{1-z} - 1}{\sqrt{1-z} + 1} = \frac{z}{(\sqrt{1-z} + 1)^2} \quad (3.6)$$

with  $\operatorname{Re} \sqrt{1-z} > 0$ , and

$$|\zeta(z)| \rightarrow \frac{|z|}{4} \quad (3.7)$$

as  $z \rightarrow 0$ .

We now show that there is  $\delta > 0$  such that if  $|z| < \delta$ , then

$$\frac{|\zeta'|}{|\zeta|} \frac{1}{[4 + \log \frac{1}{|\zeta|}]} \geq \frac{1}{|z|[\log r + \frac{1}{|z|}]}$$

From (3.5) this is equivalent to showing that

$$|\sqrt{1-z}|(4 + \log \frac{1}{|\zeta|}) \leq \log r + \log \frac{1}{|z|},$$

which is equivalent to

$$|\sqrt{1-z}|(4 + \log 4) \leq \log r + \left\{ \left( \log \frac{1}{|z|} \right) \left( (1 - |\sqrt{1-z}|) \left( \frac{\log \frac{1}{|\zeta|} - \log 4}{\log \frac{1}{|z|}} \right) \right) \right\}. \quad (3.8)$$

From (3.7)

$$\left( \frac{\log \frac{1}{|\zeta|} - \log 4}{\log \frac{1}{|z|}} \right)$$

approaches 1 as  $z \rightarrow 0$  and the expression in the curly brackets on the right hand side of (3.8) approaches zero. Thus, in order to prove (3.3), it suffices to observe that

$$4 + \log 4 < \log r,$$

which is part of what we assumed.

We have so far established that  $\rho_{0,1}(z) \geq \rho_r(z)$  on the unit circle and on any circle  $|z| = \delta$  for sufficiently small  $\delta$ . To complete the proof of the lemma we observe that since both metrics  $\rho_{0,1}(z)$  and  $\rho_r(z)$  have constant curvatures equal to  $-1$ , if we denote the Laplacian by

$$\Delta = \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2,$$

then

$$-\rho_{0,1}^{-2} \Delta \log \rho_{0,1} = -1 \quad \text{and} \quad -\rho_r^{-2} \Delta \log \rho_r = -1.$$

Therefore,

$$\Delta(\log \rho_{0,1} - \log \rho_r) = \rho_{0,1}^2 - \rho_r^2 \quad (3.9)$$

throughout the annulus  $\{z : \delta \leq |z| \leq 1\}$ . The minimum of  $\rho_{0,1}/\rho_r$  in this annulus occurs either at a boundary point or in the interior. If it occurs at an interior point, then its Laplacian of  $\log(\rho_{0,1}/\rho_r) \geq 1$  at that point and if it occurs on the boundary then  $\rho_{0,1}/\rho_r \geq 1$  at that point. In either case

$$0 \leq \Delta(\log \rho_{0,1} - \log \rho_r) = \rho_{0,1}^2 - \rho_r^2$$

at that point, and therefore

$$\rho_{0,1} \geq \rho_r$$

throughout the annulus. This completes the proof of the lemma.

### 3.2 The $|\varepsilon \log \varepsilon|$ continuity of a holomorphic motion

In this section we show how the  $|\varepsilon \log \varepsilon|$  modulus of continuity for the tangent vector to a holomorphic motion can be derived directly from Schwarz's lemma. Then we go on to show how the Hölder continuity of the mapping  $z \mapsto w(z) = h(c, z)$  with Hölder exponent  $\frac{1-|c|}{1+|c|}$  follows from the  $|\varepsilon \log \varepsilon|$  continuity of the tangent vectors to the curve  $c \mapsto h(c, z)$ . In particular, since any  $K$ -quasiconformal map  $z \mapsto f(z)$  coincides with  $z \mapsto h(c, z)$  where  $K \leq \frac{1+|c|}{1-|c|}$ , we conclude that  $f$  satisfies a Hölder condition with exponent  $1/K$ .

**Lemma 16.** *Let  $h(c, z)$  be a normalized holomorphic motion parametrized by  $\Delta$  and with base point 0 and let  $V(z)$  be the tangent vector to this motion at  $c = 0$  defined by*

$$V(z) = \lim_{c \rightarrow 0} \frac{h(c, z) - z}{c}. \quad (3.10)$$

*Then  $V(0) = 0, V(1) = 0$  and  $|V(z)| = o(|z|^2)$  as  $z \rightarrow \infty$ .*

*Proof* Since  $h(c, z)$  is normalized,  $h(c, 0) = 0$  and  $h(c, 1) = 1$  for every  $c \in \Delta$ , and therefore  $V(0) = 0$  and  $V(1) = 0$ . Since  $h(c, \infty) = \infty$  for every  $c \in \Delta$  if we introduce the coordinate  $w = 1/z$  and consider the motion  $h_1(c, w) = 1/h(c, 1/w)$ , we see that  $h_1(c, 0) = 0$  for every  $c \in \Delta$ .

Put  $p(c) = h(c, z)$  and if we think of  $z$  as a local coordinate for the Riemann sphere,

$$z \circ p(c) = z + cV^z(z) + o(c^2)$$

and in terms of the local coordinate  $w = 1/z$ ,

$$w \circ p(c) = w + cV^w(w) + o(c^2).$$

Then  $V^w(0) = 0$ . Putting  $g = w \circ z^{-1}$ , the identity  $g(z(p(c))) = w(p(c))$  yields

$$g'(z(p(0)))z'(p(0)) = w'(p(0)). \quad (3.11)$$

Since  $g(z) = 1/z$ ,  $g'(z) = -(1/z)^2$  and since

$$V^w(0) = 0, \quad \frac{d}{dc}w(p(c))|_{c=0} = V^w(w(p(0)))$$

and  $V^w(w(p(c)))$  is a continuous function of  $c$ , the equation

$$V^z(z(p(c))) \frac{dw}{dz} = V^w(w(p(c)))$$

implies

$$\frac{V^z(z)}{z^2} \rightarrow 0$$

as  $z \rightarrow \infty$ .

Let  $\rho_{0,1}(z)$  be the infinitesimal form for the hyperbolic metric on  $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$  and let  $\rho_{\Delta}(z) = 2/(1 - |z|^2)$  be the infinitesimal form for the hyperbolic metric on  $\Delta$ . For any four distinct points  $a, b, c$  and  $d$ , the cross ratio

$$g(c) = cr(h_c(a), h_c(b), h_c(c), h_c(d))$$

is a holomorphic function of  $c \in \Delta$ , and omitting the values 0, 1 and  $\infty$ . Then by Schwarz's lemma,

$$\rho_{0,1}(g(c))|g'(c)| \leq \sigma_{\Delta}(c) = \frac{2}{1 - |c|^2}$$

and

$$\rho_{0,1}(g(0))|g'(0)| \leq 2. \quad (3.12)$$

But

$$|g'(0)| = |g(0)| \left| \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d} \right| \quad (3.13)$$

where  $g(0) = cr(a, b, c, d) = \frac{(b-a)(d-c)}{(c-b)(a-d)}$ .

**Lemma 17.** *If  $V(b) = o(b^2)$  as  $b \rightarrow \infty$ , then*

$$\left( \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} \right) \rightarrow 0 \quad \text{as } b \rightarrow \infty.$$

*Proof*

$$\left( \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} \right)$$

simplifies to

$$\frac{cV(b) - bV(c) - aV(b) - cV(a) + bV(a) + aV(c)}{(b - a)(c - b)}.$$

As  $b \rightarrow \infty$  the denominator grows like  $b^2$  but the numerator is  $o(b^2)$ .

**Theorem 25.** *For any vector field  $V$  tangent to a normalized holomorphic motion and defined by (3.10), there exists a number  $C$  depending on  $R$  such that for any two complex numbers  $z_1$  and  $z_2$  with  $|z_1| < R$  and  $|z_2| < R$  and  $|z_1 - z_2| < \delta$ ,*

$$|V(z_2) - V(z_1)| \leq |z_2 - z_1| \left( 2 + \frac{C}{\log \frac{1}{\delta}} \right) \left( \log \frac{1}{|z_2 - z_1|} \right).$$

*Proof* By applying Lemma 17, inequality (3.12) and equation (3.13) to  $a = z_1, b = z_2, c = 0, d = \infty$ , we obtain  $g(0) = \frac{z_2 - z_1}{z_2}$ ,

$$\begin{aligned} & \left| \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d} \right| \\ &= \left| \frac{V(z_2) - V(z_1)}{z_2 - z_1} - \frac{V(z_2)}{z_2} \right|, \end{aligned}$$

and

$$\rho_{0,1} \left( \frac{z_2 - z_1}{z_2} \right) \left| \frac{z_2 - z_1}{z_2} \right| \left| \frac{V(z_2) - V(z_1)}{z_2 - z_1} - \frac{V(z_2)}{z_2} \right| \leq 2$$

and so

$$\left| \frac{V(z_2) - V(z_1)}{z_2 - z_1} - \frac{V(z_2)}{z_2} \right| \leq \frac{2}{\rho_{0,1} \left( \frac{z_2 - z_1}{z_2} \right) \left| \frac{z_2 - z_1}{z_2} \right|}. \quad (3.14)$$

Applying (3.12) and (3.13) again with  $a = 0, b = 1, c = \infty, d = z_2$ , we obtain

$$\rho_{0,1}(z_2) |z_2| \left| \frac{V(z_2)}{z_2} \right| \leq 2,$$

and so

$$\frac{|V(z_2)|}{|z_2|} \leq \frac{2}{\rho_{0,1}(z_2) |z_2|} \quad (3.15)$$

and this together with (3.14) implies

$$\left| \frac{V(z_2) - V(z_1)}{z_2 - z_1} \right| \leq \frac{2}{\rho_{0,1} \left( \frac{z_2 - z_1}{z_2} \right) \left| \frac{z_2 - z_1}{z_2} \right|} + \frac{2}{\rho_{0,1}(z_2) |z_2|}. \quad (3.16)$$

From (3.16) and Lemma 15 we obtain

$$|V(z_2) - V(z_1)| \leq |z_2 - z_1| \left( \left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| + 2 \log \frac{1}{|z_2 - z_1|} \right).$$

Therefore to prove the theorem we must show that for  $\varepsilon = \frac{C}{\log(1/\delta)}$

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| + 2 \log \frac{1}{|z_2 - z_1|} \leq (2 + \varepsilon) \log \frac{1}{|z_2 - z_1|}.$$

This is equivalent to showing that

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| \leq \varepsilon \log \frac{1}{|z_2 - z_1|}.$$

If  $|z_2| < 1$ , from (3.15) and Lemma 15, we have

$$\rho_{0,1}(z_2) \geq \frac{1}{|z_2|(\log r + \log \frac{1}{|z_2|})},$$

and

$$\frac{|V(z_2)|}{|z_2|} \leq 2 \log r + 2 \log \frac{1}{|z_2|}.$$

Hence

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| \leq 4 \log r.$$

If  $1 \leq |z_2| \leq R$ , then since  $\left| \frac{V(z_2)}{z_2} \right| + 2 \log |z_2|$  is a continuous function, it is bounded by a number  $M_1$ , so

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| \leq M_1 + 2 \log r.$$

The constant  $C = M_1 + 2 \log r$  does not depend on  $\delta$  and  $\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| \leq C$  for any  $|z_2| \leq R$ . Thus, putting  $\varepsilon = C/\log(1/\delta)$ , we obtain

$$|V(z_2) - V(z_1)| \leq |z_2 - z_1|(2 + \varepsilon) \left( \log \frac{1}{|z_2 - z_1|} \right).$$

Applying the same argument at a variable value of  $c$  we obtain the following result.

**Theorem 26.** *Suppose  $0 < r < 1$  and  $R > 0$ . If  $|c| \leq r$ ,  $|z_1(c)| \leq R$ ,  $|z_2(c)| \leq R$  and  $|z_2(c) - z_1(c)| < \delta$ , then*

$$|V(z_2(c)) - V(z_1(c))| \leq \frac{2 + \varepsilon}{1 - |c|^2} |z_2(c) - z_1(c)| \log \frac{1}{|z_2(c) - z_1(c)|}, \quad (3.17)$$

where  $\varepsilon \leq \frac{M}{\log(1/\delta)}$  and  $\delta \geq |z_1(0) - z_2(0)|$ . Moreover, there is a constant  $C$  such that

$$|z_2(c) - z_1(c)| \leq C \cdot |z_2 - z_1|^{\frac{1-|c|}{1+|c|}}.$$

*Proof* Equation (3.17) follows by the same calculations we have just completed.

To prove the second inequality, put  $s(c) = |z_2(c) - z_1(c)|$  and assume  $0 < |c| < 1$ .

Then (3.17) yields

$$s'(c) \leq \frac{2 + \varepsilon}{1 - |c|^2} s(c) \log \frac{1}{s(c)}.$$

So

$$-(\log \frac{1}{s(c)})' \leq \frac{2 + \varepsilon}{1 - |c|^2} \log \frac{1}{s(c)}$$

and

$$-(\log(\log \frac{1}{s(c)}))' \leq \frac{2 + \varepsilon}{1 - |c|^2}.$$

By integration,

$$-\log(\log \frac{1}{s(c)}) \Big|_0^c \leq -\frac{2 + \varepsilon}{2} \log \frac{1 - |c|}{1 + |c|} \Big|_0^{|c|}$$

and

$$\log \log(\frac{1}{s(c)}) - \log \log(\frac{1}{s(0)}) \geq \log \left( \frac{1 - |c|}{1 + |c|} \right)^{1 + \frac{\varepsilon}{2}}.$$

Since  $\log x$  is increasing,

$$\frac{\log \frac{1}{s(c)}}{\log \frac{1}{s(0)}} \geq \left( \frac{1 - |c|}{1 + |c|} \right)^{1 + \frac{\varepsilon}{2}},$$

$$\log s(c) \leq \left( \frac{1 - |c|}{1 + |c|} \right)^{1 + \frac{\varepsilon}{2}} \log s(0)$$

and

$$s(c) \leq s(0) \left( \frac{1 - |c|}{1 + |c|} \right)^{1 + \frac{\varepsilon}{2}}.$$

Putting  $s = s(0)$  and  $\alpha = \frac{1 - |c|}{1 + |c|}$ , we wish to show that

$$s^{\alpha^{1+\varepsilon}} \leq C s^\alpha \text{ or equivalently that } s^{(\alpha^{1+\varepsilon} - \alpha)} \leq C. \quad (3.18)$$

This is equivalent to showing that

$$\alpha(\alpha^\varepsilon - 1) \log s \leq \log C$$

or that

$$\alpha(\exp(\frac{M}{\log(1/s)} \log \alpha) - 1) \log s \leq \log C.$$

Since  $0 < \alpha < 1$  and since we may assume  $s < e^{-1}$ , by using the inequality  $e^x - 1 \leq xe^{x_0}$  for  $0 \leq x \leq x_0$ , we see that it suffices to choose  $C$  so that

$$\alpha \frac{M}{\log(1/s)} \log(1/\alpha) e^{M \log \alpha} \log(1/s) = \alpha M \log(1/\alpha) e^{M \log \alpha} \leq \log C.$$

The idea for the proof of Theorem 26 is suggested but not worked out in [29].

# Different extensions of holomorphic motion

---

## Contents

4.1	From Green's Theorem to Chirka's proof . . . . .	68
4.2	Different ways to construct $\varphi(z, w)$ . . . . .	74
4.3	Examples of holomorphic motions . . . . .	76

---

This chapter is a joint work with Jun Hu. We try to study the connection between Chirka's proof and Slodkowski's proof of the extension theorem of a holomorphic motion. We use Chirka's operator to construct infinitely many extensions, and the uniqueness of extension is also discussed here.

Intuitively, a holomorphic motion of  $E$  in  $\overline{\mathbb{C}}$  over  $\Delta$  is a motion of the points in  $E$  under which all points in  $E$  move complex analytically in  $\overline{\mathbb{C}}$  from their initial positions and don't bump to each other at any time  $z \in \Delta$ . Based on our understanding, we think Slodkowski's proof is intuitive, that is to fill analytic disks into a polynomial convex hull, where the convex hull is constructed through the solution of an ordinary differential equation and the analytic disks are filled in by using harmonic functions and their conjugates determined by boundary values. It takes a great deal to show those analytic disks are mutually disjoint. By contrast, Chirka's proof is simple but not intuitive, which applies Schauder's fixed point theorem to an integral operator (defined by a Hilbert transformation) on a proper functional space. The following example shows the key idea and difference of these two extension methods.

**Example 2.** *Suppose holomorphic motion  $f(z, w)$  fixes  $0, 1$  and  $\infty$ . And suppose  $f(z, 3) = 3+z$   $f(z, 6) = 6+z$ . Then Slodkowski's extension of  $f$  at  $4$  is  $f(z, 4) = 4+z$  by filling in the disk. But the Chirka's extension of  $f$  at  $4$  is  $f(z, 4) = 4$  because the  $4$  is not in the image of the motion of point  $3$  and  $6$ .*

After learning these two extension methods, we are interested to know if there is a connection between them, or what is the motivation for Chirka to develop his proof. First, we will present a motivation from which we are able to develop a proof very similar to Chirka's, which will be given in the second section. After that, we apply our setting of Chirka's method in many different ways to construct the holomorphic extensions of a holomorphic motion  $f$  of  $E$  over  $\Delta$ . A natural question arises: are these extensions same? In the last section, we first summarize briefly the known sufficient conditions for the extensions to be unique; then give a naive necessary for the uniqueness of extension and some examples with non unique extensions; and finally raise two questions concerning the necessary conditions for the uniqueness of the extensions of the holomorphic motions of four points.

## 4.1 From Green's Theorem to Chirka's proof

The following two questions were posed by Sullivan and Thurston in [62], which now are called Slodkowski's theorems.

**Theorem 27** (Extendability of Holomorphic Motions). *Every holomorphic motion  $f : \Delta \times E \rightarrow \overline{\mathbb{C}}$  of an arbitrary subset  $E$  of  $\overline{\mathbb{C}}$  can be extended to a holomorphic motion  $F : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  of  $\overline{\mathbb{C}}$ , that is  $F$  is equal to  $f$  if restricted on  $\Delta \times E$  and is parameterized on the same time parameter space  $\Delta$ .*

**Theorem 28** (Holomorphic Axiom of Choice). *Let  $f(z, w) = f^z(w)$  be a holomorphic motion of a subset  $E$  in  $\mathbb{C}$ , parameterized by time variable  $z \in \Delta$ . Then for every point  $w$  outside  $E$ , there is a holomorphic map  $g : \Delta \rightarrow \overline{\mathbb{C}}$  such that (i)  $g(0) = w$  and (ii)  $g(z) \notin f^z(E)$  for any  $z \in \Delta$ .*

It was pointed out by Sullivan and Thurston in [62] that Theorem 28 implies Theorem 27. By using the limits of normal families and Hurwitz's theorem, Slodkowski reduces the proof of Theorem 28 to the following finite version.

**Theorem 29** (Finite Version of Holomorphic Axiom of Choice). *Let  $f(z, w) = f^z(w) = f_w(z)$  be a holomorphic motion of a finite subset  $E = \{w_j : 1 \leq j \leq n\}$  in  $\overline{\mathbb{C}}$ , parameterized by a complex time variable  $z$  in a neighborhood of the closed unit disk  $\overline{\Delta}$  in the complex plane. Then for every point  $w$  in  $\overline{\mathbb{C}} \setminus E$ , there is a holomorphic map  $f_w : \Delta \rightarrow \overline{\mathbb{C}}$  such that (i)  $f_w(0) = w$  and (ii)  $f_w(z) \neq f_{w_j}(z)$  for any  $z \in \Delta$  and any  $w_j \in E$ .*

Note: we always assume  $\infty$  is in the closed set  $E$  and  $f_\infty(z) \equiv \infty$ .

Chirka's proof of Theorem 29 is short and elegant, but the intuition leading to his proof is mysterious to us. After studying the main ideas in Chirka's and Slodkowski's proofs, a possible clue from Slodkowski's proof to Chirka's appears to us. Let us first briefly summarize the clue. Given a continuous function  $\varphi$  from the unit circle  $\mathbb{S}^1$  into  $\mathbb{C}$ , a naive way to construct a holomorphic map  $\tilde{\varphi}$  on  $\Delta$  from  $\varphi$  is to apply to  $\varphi$  the line integral in the Cauchy integral formula. Although the extension of  $\tilde{\varphi}$  to the boundary  $\mathbb{S}^1$  of  $\Delta$  may fail or be very different from  $\varphi$ , it defines an analytic function on  $\Delta$ . Fortunately, the map  $\varphi$  in our consideration is an analytic map defined on the neighborhood of  $\overline{\Delta}$  in  $\mathbb{C}$  and then  $\varphi$  can be expressed through the Cauchy integral formula. After applying Green's theorem to the line integral in the Cauchy integral expression of  $\varphi$ , we obtain so-called Pompeiu's formula and realize:

(i) the two non-analytic summands have their non-analytic parts canceled to form a holomorphic map, and

(ii) the double integral summand has holomorphic extension to the outside of the unit disk.

From there, we first take that double integral to define a functional operator  $\mathcal{R}$ . (It looks same to Chirka's but actually not.) The image under this operator is only

holomorphic outside  $\bar{\Delta}$ . By using the same scheme as Chirka's, we obtain  $\mathcal{R}$  has a unique fixed point  $\Phi$ . Then we work out the complex derivative of  $\Phi$  with respect to the time variable  $z$  for any  $z$  outside  $\bar{\Delta}$  and we find their values equal to the derivatives of the moves of the points in  $E$  under the given holomorphic motion at  $1/z$ . Therefore through pre-composing the time variable in  $\Phi$  by the reciprocal map, we obtain a holomorphic extension of the given holomorphic motion of the points in  $E$  for the time variable  $z$  in  $\Delta$ . Overall, by following this path, we work out a proof for Theorem 29. Since it is essentially as the same as Chirka's proof except that the setting at the beginning is different, we think it is probably the motivation to develop Chirka's proof. In the rest of this section, we lay out the details.

Let  $H(\Delta)$  be the set of holomorphic maps from  $\Delta$  to  $\mathbb{C}$  and  $H(\bar{\Delta})$  be the set of holomorphic maps from neighborhoods of  $\bar{\Delta}$  to  $\mathbb{C}$ .

Given a map  $f \in H(\bar{\Delta})$ , by Cauchy's integral formula

$$f(z) - f(0) = \frac{1}{2\pi i} \int_{|\xi|=1} f(\xi) \left( \frac{1}{\xi - z} - \frac{1}{\xi} \right) d\xi \quad (4.1)$$

where  $|z| < 1$ . We can also use the line integral in (4.1) to extend  $f$  to any point  $z$  with  $|z| > 1$ , for which  $f(z)$  is constantly equal to 0. Clearly,

$$f(z) - f(0) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)z}{(\xi - z)\xi} d\xi = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)}{\frac{1}{\xi} - \frac{1}{z}} d\left(\frac{1}{\xi}\right).$$

Now by substituting  $\xi$  by  $\frac{1}{\xi}$ , we obtain

$$f(z) - f(0) = -\frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\frac{1}{\xi})}{\xi - 1/z} d\xi. \quad (4.2)$$

Since  $|\xi| = 1$ ,  $\bar{\xi} = 1/\xi$  and then

$$f(z) - f(0) = -\frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\bar{\xi})}{\xi - 1/z} d\xi. \quad (4.3)$$

Now we define a map  $\tilde{f}$  from  $(\bar{\mathbb{C}} \setminus \bar{\Delta}) \cup \Delta$  to  $\mathbb{C}$  by letting  $\tilde{f}(\infty) = f(0)$  and

$$\tilde{f}(z) - \tilde{f}(\infty) = -\frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\bar{\xi})}{\xi - z} d\xi. \quad (4.4)$$

Clearly,  $\tilde{f}(z) = f(\frac{1}{z})$  if  $|z| > 1$ , and  $\tilde{f}(z) = 0$  if  $|z| < 1$ .

When  $|z| < 1$ , let us apply Green's Theorem to the right side of (4.4), then we obtain the following so-called Pompeiu's formula.

$$\tilde{f}(z) - \tilde{f}(\infty) = -\frac{1}{2\pi i} \int \int_{\bar{\Delta}} \frac{\frac{\partial}{\partial \bar{\xi}} f(\bar{\xi})}{\xi - z} d\bar{\xi} \wedge d\xi - f(\bar{z}). \quad (4.5)$$

Let  $\xi = \eta + i\zeta$ , then

$$\tilde{f}(z) - \tilde{f}(\infty) = -\frac{1}{\pi} \int \int_{\bar{\Delta}} \frac{\frac{\partial}{\partial \bar{\xi}} f(\bar{\xi})}{\xi - z} d\eta d\zeta - f(\bar{z}). \quad (4.6)$$

Furthermore, if  $|z| < 1$ , then  $\tilde{f}(z)$  is constantly equal to 0 and hence  $\frac{\partial}{\partial \bar{z}}$ -derivative of the double integral in (4.6) is equal equal to  $\frac{\partial}{\partial \bar{z}} f(\bar{z})$ . Now we separate this double integral as an operator acting on the functions in  $H(\bar{\Delta})$ , that is, we define

$$g(z) = \mathcal{R}(f)(z) = -\frac{1}{\pi} \int \int_{\bar{\Delta}} \frac{\frac{\partial}{\partial \bar{\xi}} f(\bar{\xi})}{\xi - z} d\eta d\zeta$$

for any  $z \in \mathbb{C}$ . Then  $g(z)$  is holomorphic when  $|z| > 1$  and  $\frac{\partial}{\partial \bar{z}} g(z) = \frac{\partial}{\partial \bar{z}} f(\bar{z})$  when  $|z| < 1$ .

Given any  $f \in H(\bar{\Delta})$  with  $f(0) = 0$ , let us extend  $f$  to a function  $\hat{f}$  on  $\mathbb{C}$  by defining  $\hat{f}(z) = f(\bar{z})$  for  $|z| \leq 1$  and  $\hat{f}(z) = f(1/z)$  when  $|z| > 1$ . Then  $\hat{f}$  is holomorphic when  $|z| > 1$  and  $\frac{\partial}{\partial \bar{z}} \hat{f}(z) = \frac{\partial}{\partial \bar{z}} f(\bar{z})$  when  $|z| < 1$ . Therefore,  $g - \hat{f}$  is continuous on  $\mathbb{C}$  and holomorphic on  $\mathbb{C} \setminus \mathbb{S}^1$ , and then holomorphic on  $\mathbb{C}$ . Furthermore,  $\lim_{z \rightarrow \infty} (g - \hat{f})(z) = 0$ , by Liouville's theorem  $g - \hat{f}$  is constantly equal to 0, that is,  $g = \hat{f}$ . So we obtain  $\hat{f}$  is a fixed point of the operator  $\mathcal{R}$ .

Similarly, given an arbitrary point  $w \in \mathbb{C}$ , if one modifies the operator  $\mathcal{R}$  as

$$\mathcal{R}_w(f)(z) = -\frac{1}{\pi} \int \int_{\bar{\Delta}} \frac{\frac{\partial}{\partial \bar{\xi}} f(\bar{\xi})}{\xi - z} d\eta d\zeta + w,$$

then for any function  $f \in H(\bar{\Delta})$  with  $f(0) = w$  the corresponding  $\hat{f}$  is a fixed point of  $\mathcal{R}_w$ .

The previous observations help us realize how the functional operator in Chirka's proof comes into play. In the following, we extend the operators in our observations to a functional operator similar to Chirka's (but not same) and give a proof of Theorem 29. As you have seen, through our operator, we will not obtain directly

holomorphic motions in  $\Delta$  and instead we first obtain holomorphic motions outside  $\bar{\Delta}$  and then we obtain holomorphic motions in  $\Delta$  by pre-composing with the reciprocal map  $z \mapsto \frac{1}{z}$ .

Since the following operator is similar to Chirka's operator which has been discussed a lot in chapter 2, we only write down the main propositions and theorems here. For details, please read chapter 2.

Let  $E = \{w_i : 1 \leq i \leq n\}$  be a finite set in  $\mathbb{C}$ . In this section, we assume  $f_i$ 's are  $n$  functions in  $H(\bar{\Delta})$  with  $f_i(0) = w_i$  for each  $i$  and for each  $z \in \bar{\Delta}$ ,  $f_i(z) \neq f_j(z)$  as soon as  $i \neq j$ . Then there exists  $\delta > 0$  such that  $|f_i(z) - f_j(z)| > \delta$  for any  $i \neq j$  and any  $z \in \bar{\Delta}$ . Now let  $\lambda$  be a real  $C^\infty$ -smooth function from  $[0, \infty)$  to  $[0, 1]$  with  $\lambda(0) = 1$  and  $\lambda(t) = 0$  for any  $t \geq \delta/2$ . For each  $1 \leq i \leq n$ , we extend  $f_i$  to a continuous function  $\hat{f}_i$  on  $\bar{\mathbb{C}}$  as follows:  $\hat{f}_i(z) = f_i(\bar{z})$  for  $|z| \leq 1$  and  $\hat{f}_i(z) = f_i(1/z)$  if  $|z| > 1$ . Since  $\bar{z} = 1/z$  when  $|z| = 1$ ,  $\hat{f}_i$  is continuous. Clearly,  $\hat{f}_i$  is anti-holomorphic in  $\Delta$  and holomorphic outside  $\bar{\Delta}$ . Define

$$\varphi(z, w) = \sum_{i=1}^n \lambda(|w - \hat{f}_i(z)|) \frac{\partial \hat{f}_i(z)}{\partial \bar{z}}, \quad (4.7)$$

where  $z \in \bar{\mathbb{C}}$  and  $w \in \mathbb{C}$ . Clearly, it has the following properties:

**Proposition 7.** (1)  $\varphi$  is bounded on  $\bar{\mathbb{C}} \times \mathbb{C}$ ;

(2)  $\varphi$  vanishes outside  $\Delta \times \mathbb{C}$ ;

(3) for any  $w \in \mathbb{C}$  and any time  $z \in \Delta$ , there exists exactly one  $1 \leq i \leq n$  such that

$$\varphi(z, w) = \lambda(|w - f_i(\bar{z})|) \frac{\partial f_i(z)}{\partial \bar{z}};$$

(4)  $\varphi(z, w)$  satisfies the Lipschitz condition in the  $w$  variable and the Lipschitz constant is independent of the  $z$  variable.

Let  $X$  be the space of all continuous functions from  $\hat{\mathbb{C}}$  to  $\mathbb{C}$ . Then  $X$  is a Banach space under  $\|\cdot\|_\infty$ -norm. Now define a functional operator  $K$  on  $X$  as

$$K(f)(z) = P(\varphi(\cdot, f(\cdot)))(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\varphi(\xi, f(\xi))}{\xi - z} d\eta d\zeta \quad (4.8)$$

where  $f \in X$ . From Propositions 7, one can see that the operator  $K$  has the following properties.

**Proposition 8.** (1)  $K$  is a continuous map from  $X$  to  $X$ .

(2)  $K(f)$  is bounded by a universal constant  $M > 0$  independent of  $f$ , and furthermore  $K(f)(\infty) = 0$ .

(3)  $K(f)$  is  $\alpha$ -Hölder continuous for a constant  $0 < \alpha < 1$  with the  $\alpha$ -Hölder constant independent of  $f$ .

(4)  $\frac{\partial}{\partial \bar{z}} K(f)(z) = \varphi(z, f(z))$ , and then  $K(f)$  is holomorphic outside  $\bar{\Delta}$ .

Given any point  $w \in \mathbb{C}$ , define a functional operator  $K_w$  on  $X$  as

$$K_w(f)(z) = K(f)(z) + w. \quad (4.9)$$

In our understanding, the operator  $\mathcal{R}_w$  in our previous discussion is a motivation to define the operator  $K_w$ . Therefore  $K_w$  has the following property as  $\mathcal{R}_w$ .

**Proposition 9.** For each  $1 \leq j \leq n$ ,  $\hat{f}_j$  is a fixed point of  $K_{w_j}$ , where  $w_j = \hat{f}_j(\infty) = f_j(0)$ .

Schauder's fixed point theorem (see [14] for a reference) shows the existence of fixed points for  $K_w$  for any  $w \in \mathbb{C}$ , and furthermore we know the fixed point is unique.

**Theorem 30** (Existence and uniqueness). For each  $w \in \mathbb{C}$ , the operator  $K_w$  has a unique fixed point  $f_w$  in  $X$ , where  $X$  denotes the space of all continuous functions from  $\hat{\mathbb{C}}$  to  $\mathbb{C}$ .

So far, we have seen that for each  $1 \leq j \leq n$ ,  $\hat{f}_j$  is the unique fixed point of  $K_{w_j}$ , is holomorphic outside  $\bar{\Delta}$ , and is equal to  $f_j(1/z)$  for any  $|z| > 1$ . In the meantime, for any  $w \neq w_j$  for any  $1 \leq j \leq n$ , the unique fixed point  $f_w$  of  $K_w$  is also holomorphic outside  $\bar{\Delta}$  and  $f_w(\infty) = w$ . Then  $f_w(1/z)$  is holomorphic in  $\Delta$  and takes the value  $w$  at  $z = 0$ . One can see  $f_w(1/z)$  provides a proof to Theorem 28 after the following theorem is proved.

**Theorem 31** (Injectivity). *For any two complex numbers  $w_1$  and  $w_2$ , let  $f_{w_1}$  and  $f_{w_2}$  be the fixed points of  $K_{w_1}$  and  $K_{w_2}$  respectively. Then  $f_{w_1}(z) \neq f_{w_2}(z)$  for any  $z \in \mathbb{C}$ .*

Through pre-composing the time variable  $z$  by the reciprocal map, Theorems 30 and 31 together imply Theorem 29.

We now finish this section by briefly recalling the setting in Chirka's paper [12] with comparisons to ours.

Define a continuous function  $\check{f}_i(z) : \overline{\mathbb{C}} \rightarrow \mathbb{C}$  as:  $\check{f}_i(z) = f_i(z)$  if  $z \in \overline{\Delta}$  and  $\check{f}_i(z) = f(1/\bar{z})$  otherwise, where  $1 \leq i \leq n$ . In order to obtain the holomorphic map  $f_{\check{w}}$  directly defined on  $\Delta$  (without pre-composing by the reciprocal map in our setting) for Theorem 28, in [12] Chirka used  $\check{f}_i$ 's, instead of  $\hat{f}_i$ 's, to define the  $K$ -operator and then applied Schauder's fixed point theorem.

## 4.2 Different ways to construct $\varphi(z, w)$

In this section, we apply the  $K$ -operators in infinitely many different ways to construct holomorphic maps  $f_w$  satisfying the two conditions in Theorem 29.

Recall that in the previous section we let  $X$  be the space of all continuous functions from  $\hat{\mathbb{C}}$  to  $\mathbb{C}$ , which is a Banach space under  $C_0$ -norm. Let  $f_j$  be the same as in the previous section with  $f_j(0) = w_j$  for  $1 \leq j \leq n$ . Now given any integer  $k$ , let  $g_j^{(k)}(z) = e^{kz} f_j(z)$  for  $1 \leq j \leq n$ . We define a function  $\hat{g}_j^{(k)}$  from  $g_j^{(k)}$  as the same as  $\hat{f}_j$ , that is,  $\hat{g}_j^{(k)}(z) = g_j^{(k)}(\bar{z})$  if  $|z| \leq 1$  and  $\hat{g}_j^{(k)}(z) = g_j^{(k)}(1/z)$ .

For each  $z \in \overline{\Delta}$ ,  $f_i(z) \neq f_j(z)$  as soon as  $i \neq j$ , so are  $g_i^{(k)}(z)$  and  $g_j^{(k)}(z)$  for each  $k$ . Then for each  $k$  there exists  $\delta_k > 0$  such that  $|g_i^{(k)}(z) - g_j^{(k)}(z)| > \delta_k$  for any  $i \neq j$  and any  $z \in \overline{\Delta}$ . Now let  $\lambda_k$  be a real  $C^\infty$ -smooth function from  $[0, \infty)$  to  $[0, 1]$  with  $\lambda_k(0) = 1$  and  $\lambda_k(t) = 0$  for any  $t \geq \delta_k/2$ . Similarly, we define

$$\varphi^{(k)}(z, w) = \sum_{i=1}^n \lambda_k(|w - \hat{g}_i^{(k)}(z)|) \frac{\partial \hat{g}_i^{(k)}(z)}{\partial \bar{z}}, \quad (4.10)$$

where  $z \in \overline{\mathbb{C}}$  and  $w \in \mathbb{C}$ . Again similarly, a functional operator  $K^{(k)}$  on  $X$  is defined

as: given  $g \in X$ ,

$$K^{(k)}(g)(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\varphi^{(k)}(\xi, g(\xi))}{\xi - z} d\zeta d\eta \quad (4.11)$$

and for any complex number  $w$ ,

$$K_w^{(k)}(f) = K^{(k)}(f) + w. \quad (4.12)$$

The exact work in the previous section implies:

**Proposition 10.** *For each integer  $k$  and each  $1 \leq j \leq n$ ,  $\hat{g}_j^{(k)}$  is a fixed point of  $K_{w_j}^{(k)}$ , where  $w_j = \hat{g}_j^{(k)}(\infty) = f_j(0)$ .*

**Theorem 32.** (1) *Each operator  $K_w^{(k)}$  has a unique fixed point  $\hat{g}_w^{(k)}$  in  $X$ .*

(2) *Each fixed point  $\hat{g}_w^{(k)}$  satisfies:  $\hat{g}_w^{(k)}(\infty) = w$  and  $\hat{g}_w^{(k)}(z) \neq \hat{g}_{w_j}^{(k)}(z)$  for any  $z \in \mathbb{C} \setminus \bar{\Delta}$  as soon as  $w \neq w_j$ .*

Clearly,  $\hat{g}_j^{(k)}(1/z) = g_j^{(k)}(z)$  if  $|z| < 1$ . For any complex number  $w \neq w_j$  for any  $1 \leq j \leq n$ , let  $g_w^{(k)}(z) = \hat{g}_w^{(k)}(1/z)$  for  $|z| < 1$ . Then  $g_w^{(k)}(0) = w$  and for any  $|z| < 1$ ,  $g_w^{(k)}(z) \neq g_j^{(k)}(z)$  for any integer  $k$  and  $1 \leq j \leq n$ . Therefore each  $f_w^{(k)}(z) = e^{-kz} g_w^{(k)}(z)$  satisfies the conditions required for the map  $f_w$  in Theorem 29, that is, each  $f_w^{(k)}$  provides an extension to the holomorphic motion  $\{f_j\}_{j=1}^n$  from the set  $E = \{w_j : 1 \leq j \leq n\}$  to the set  $E \cup \{w\}$ , where  $w \notin E$ .

A natural question arises: Assume  $k \neq k'$ , is  $f_w^{(k)}$  not equal to  $f_w^{(k')}$  for some point  $w$  outside the set  $E$ ? In the rest of this section we provide an example in which two different operators produce the same extensions. Then in the next and last section we briefly consider sufficient and/or necessary conditions for the uniqueness of extensions.

Given any point  $w \in \mathbb{C}$ , let  $K_w$  be the functional operator defined in (4.9), that is,

$$K_w(f)(z) = K(f)(z) + w = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\varphi(\xi, f(\xi))}{\xi - z} d\eta d\zeta + w,$$

where  $\xi = \eta + i\zeta$ .

Let  $f_w$  be the unique fixed point of  $K_w$ . Now define a continuous function on  $\hat{\Delta}$  by letting  $g(z) = 1 - |z|^2$  for any  $z \in \Delta$  and  $g(z) = 0$  for any  $z$  in the complement of  $\Delta$ . Let  $\tilde{f}_w(z) = f_w(z) + g(z)$ . Then

$$\frac{\partial(\tilde{f}_w(z))}{\partial \bar{z}} = \varphi(z, f_w(z)) - z$$

for any  $z \in \Delta$  and  $\frac{\partial(\tilde{f}_w(z))}{\partial \bar{z}}$  is equal to 0 for any  $z$  outside  $\Delta$ . Now we define a new operator  $\tilde{K}_w$  by substituting the  $\varphi(\xi, f(\xi))$  in  $K_w$  by  $\varphi(\xi, f(\xi)) - \xi$ . Then it has all the properties that the  $K_w$  operator has. Therefore  $\tilde{f}_w$  is the unique fixed point of the new operator  $\tilde{K}_w$ , which is continuous in  $\Delta$  and holomorphic outside  $\Delta$ . Since  $g(z) = 0$  for any  $z$  outside  $\Delta$ ,  $\tilde{f}_w = f_w$  outside  $\Delta$ . Hence by pre-composing with the reciprocal map, the two operators  $K_w$  and  $\tilde{K}_w$  produce the same holomorphic extensions.

### 4.3 Examples of holomorphic motions

There is a naive necessary condition for a holomorphic motion to have a unique extension.

**Proposition 11.** *If a holomorphic motion  $f$  of a proper subset  $E$  of  $\overline{\mathbb{C}}$  over  $\Delta$  has a unique extension, then  $\cup_{w \in E} f_w(\Delta)$  is dense in  $\overline{\mathbb{C}}$ .*

The next two examples show this condition is far from sufficient.

**Example 3.** *Let  $E$  be the set consisting of four points  $\infty$ ,  $-1$ ,  $0$  and  $1$ . A motion  $f$  of  $E$  in  $\overline{\mathbb{C}}$  over  $\Delta$  is defined as:  $f$  fixes  $0$ ,  $-1$  and  $\infty$  for all  $z \in \Delta$  and the move of  $1$  under  $f$  is defined by the holomorphic covering map from  $\Delta$  to  $\overline{\mathbb{C}} \setminus \{0, -1, -2, -3, \cdot, \infty\}$  with  $0$  mapped to  $1$ . Clearly, this motion satisfies the previous necessary condition. But  $f$  can be extended to  $E \cup \{2\}$  in following two different ways: one way is to define the motion to fix the point  $2$  and the other is define the motion on  $2$  as the shift of the motion of  $1$  to  $2$ . Therefore,  $f$  has two different extensions to  $\overline{\mathbb{C}}$ .*

More explicitly,

**Example 4.** Let  $E$  be the same in the previous example and a motion  $f$  of  $E$  fix three points  $\infty$ ,  $-1$  and  $0$ . The motion of the point  $1$  under  $f$  is given by  $f_1(z) = (\frac{1+z}{1-z})^2$ , where  $z \in \Delta$ . Then  $f_1$  maps  $\Delta$  onto  $\overline{\mathbb{C}}$  minus the closure of the negative half real axis. Then  $f$  can be extended to  $E \cup \{2\}$  in the same two different ways as in the previous example.

In [9], Bers and Royden include examples for which there are different extensions and examples for which the extensions are unique. And here the uniqueness of the extension of holomorphic motions is depending on the uniqueness of extremal quasiconformal map.

**Example 5.**

$$f(z, w) = w + \frac{z}{w}$$

for any  $w \in \overline{\mathbb{C}} - \Delta$ , and  $f(z, w) = w + z\overline{w}$  for any  $w \in \Delta$ .

For every fixed  $z$ ,  $f(z)$  maps the unit circle to an ellipse which is symmetric. So  $f(z) = [f(z, w)]$  for  $w \in S^1$  is a holomorphic map from  $\Delta$  into  $T_0(\Delta)$ . The holomorphic motion  $f(z, w)|_{\Delta \times (\overline{\mathbb{C}} - \Delta)}$  has unique extension into the unit disk, since  $f(z, w) = w + z\overline{w}$  is a Teichmüller map which is unique extremal in its Teichmüller equivalent class.

This example also shows that the holomorphic maps from  $\Delta$  to  $T_0(\Delta)$  do not always have the lifting property which will be discussed in Chapter 6. Suppose this map  $f(z) : \Delta \rightarrow T_0$  can be lifted to a map  $g(z) : \Delta \rightarrow M_0$ , then  $g(z, w)$  is a extension of the holomorphic motion  $f(z, w)|_{\Delta \times (\overline{\mathbb{C}} - \Delta)}$  and  $g(z, w)$  is asymptotically conformal in  $\Delta$  for any fixed  $z$ . But  $f(z, w)$  is Teichmüller in the unit disk, so  $|\mu_f(z, w)|$  is a constant in the unit disk for any foxed  $z$ . Hence  $f(z, w) \neq g(z, w)$  which is a contradiction.

# Lifting map of Teichmuller space

---

## Contents

---

5.1	Teichmüller space of a closed set . . . . .	79
5.2	The lifting theorem and Universal holomorphic motions . . . . .	81
5.2.1	Universal holomorphic motions . . . . .	82
5.2.2	A proposition . . . . .	84
5.3	Proof of the lifting theorem . . . . .	86
5.4	Some concluding remarks . . . . .	90

---

## 5.1 Teichmüller space of a closed set

This chapter is a joint work with Yunping Jiang and Sudeb Mitra. Let  $E$  be a closed subset in  $\overline{\mathbb{C}}$ ; we will always assume that  $E$  contains the points 0, 1, and  $\infty$ . A homeomorphism of  $\overline{\mathbb{C}}$  onto itself is called *normalized* if it fixes the points 0, 1, and  $\infty$ .

**Definition 22.** *Two normalized quasiconformal self-mappings  $f$  and  $g$  of  $\overline{\mathbb{C}}$  are said to be  $E$ -equivalent iff  $f^{-1} \circ g$  is isotopic to the identity rel  $E$ . The Teichmüller space  $T(E)$  is the set of  $E$ -equivalence classes of normalized quasiconformal self-mappings of  $\overline{\mathbb{C}}$ . The basepoint of  $T(E)$  is the  $E$ -equivalence class of the identity map.*

The following analytic description of  $T(E)$  will be more useful for our purposes.

Let  $M(\mathbb{C})$  denote the open unit ball of the complex Banach space  $L^\infty(\mathbb{C})$ . Each  $\mu$  in  $M(\mathbb{C})$  is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism  $w^\mu$  of  $\overline{\mathbb{C}}$  onto itself. The basepoint of  $M(\mathbb{C})$  is the zero function.

We define the quotient map  $P_E : M(\mathbb{C}) \rightarrow T(E)$  by setting  $P_E(\mu)$  equal to the  $E$ -equivalence class of  $w^\mu$ , written as  $[w^\mu]_E$ . Clearly,  $P_E$  maps the basepoint of  $M(\mathbb{C})$  to the basepoint of  $T(E)$ . In [53] Lieb proved that  $T(E)$  is a complex Banach manifold such that the map  $P_E$  from  $M(\mathbb{C})$  to  $T(E)$  is a holomorphic split submersion; see also [23] for a complete proof. The space  $T(E)$  is intimately related with holomorphic motions of the closed set  $E$ ; see §5.2 for more details.

**Two special cases** Let  $E$  be a finite set ( $0$ ,  $1$ , and  $\infty$  belong to  $E$ ). Its complement  $\Omega = \overline{\mathbb{C}} \setminus E$  is a sphere with punctures at the points of  $E$ , and there is a natural identification of  $T(E)$  with the classical Teichmüller space  $Teich(\Omega)$ . It is defined by setting  $\theta(P_E(\mu))$  equal to the Teichmüller class of the restriction of  $w^\mu$  to  $\Omega$ . It is clear that  $\theta : T(E) \rightarrow Teich(\Omega)$  is a well-defined map. It is easy to see that the map  $\theta$  is biholomorphic; see Example 3.1 in [57] for the details.

When  $E = \overline{\mathbb{C}}$ , the space  $T(\overline{\mathbb{C}})$  consists of all the normalized quasiconformal self-mappings of  $\overline{\mathbb{C}}$ , and the map  $P_{\overline{\mathbb{C}}}$  from  $M(\mathbb{C})$  to  $T(\overline{\mathbb{C}})$  is bijective. We use it to identify  $T(\overline{\mathbb{C}})$  biholomorphically with  $M(\mathbb{C})$ .

**Contractibility of  $T(E)$ :** The following fact was proved in §7.13 of [23].

**Proposition 12.** *There is a continuous basepoint preserving map  $s$  from  $T(E)$  to  $M(\mathbb{C})$  such that  $P_E \circ s$  is the identity map on  $T(E)$ .*

Since  $M(\mathbb{C})$  is contractible, it follows that the space  $T(E)$  is also contractible.

**Forgetful maps:** If  $E$  is a subset of the closed set  $\widehat{E}$  and  $\mu$  is in  $M(\mathbb{C})$ , then the  $\widehat{E}$ -equivalence class of  $w^\mu$  is contained in the  $E$ -equivalence class of  $w^\mu$ . Therefore, there is a well-defined ‘forgetful map’  $p_{\widehat{E},E}$  from  $T(\widehat{E})$  to  $T(E)$  such that  $P_E = p_{\widehat{E},E} \circ P_{\widehat{E}}$ . It is easy to see that this forgetful map is a basepoint preserving holomorphic split submersion.

**Changing the basepoint:** Let  $w$  be a normalized quasiconformal self-mapping

of  $\overline{\mathbb{C}}$ , and let  $\tilde{E} = w(E)$ . By definition, the *allowable map*  $g$  from  $T(\tilde{E})$  to  $T(E)$  maps the  $\tilde{E}$ -equivalence class of  $f$  to the  $E$ -equivalence class of  $f \circ w$  for every normalized quasiconformal self-mapping  $f$  of  $\overline{\mathbb{C}}$ .

**Lemma 18.** *The allowable map  $g : T(\tilde{E}) \rightarrow T(E)$  is biholomorphic. If  $\mu$  is the Beltrami coefficient of  $w$ , then  $g$  maps the basepoint of  $T(\tilde{E})$  to the point  $P_E(\mu)$  in  $T(E)$ .*

See §7.12 in [23] or §6.4 in [57] for a complete proof.

## 5.2 The lifting theorem and Universal holomorphic motions

The main purpose of this chapter is to give a self-contained proof of the following theorem.

**Theorem 33.** *Let  $E = \{0, 1, \infty, \zeta_1, \dots, \zeta_n\}$  where  $\zeta_i \neq \zeta_j$  for  $i \neq j$ , and  $\zeta_i \neq 0, 1, \infty$  for all  $i = 1, \dots, n$ . Let  $\hat{E} = E \cup \{\zeta_{n+1}\}$  where  $\zeta_{n+1}$  is any point in  $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$  distinct from  $\zeta_i$  for all  $i = 1, \dots, n$ . Then, given any holomorphic map  $f$  from  $\Delta$  into  $T(E)$ , there exists a holomorphic map  $\hat{f}$  from  $\Delta$  into  $T(\hat{E})$ , such that  $p_{\hat{E}, E} \circ \hat{f} = f$ .*

**Remark.** This “lifting problem” was mentioned in §7 of the classic paper [9], and the authors called it “a difficult open problem.” With the publication of [60], it became possible to give a quick solution of this problem, using Slodkowski’s theorem. We shall discuss this in more details in §5.4. More recently, Chirka (in [12]) published a new proof of Slodkowski’s theorem. See also [5], [15], and [41]. The novelty of our present paper is that we use some ideas of Chirka and a theorem of Nag ([58]) to give a direct proof of the above theorem. Our approach, therefore, also gives a new interpretation of Chirka’s methods.

### 5.2.1 Universal holomorphic motions

In this section we study the interesting relationship between "holomorphic lifts" and "universal holomorphic motions." The main purpose in this section is to prove a result of Bers-Royden (Proposition 4 in [9]) in its fullest generality.

**Definition 23.** *Let  $V$  be a connected complex manifold with a basepoint  $x_0$  and let  $E$  be any subset of  $\overline{\mathbb{C}}$ . A holomorphic motion of  $E$  over  $V$  is a map  $\phi : V \times E \rightarrow \overline{\mathbb{C}}$  that has the following three properties:*

- (i)  $\phi(x_0, z) = z$  for all  $z$  in  $E$ ,
- (ii) the map  $\phi(x, \cdot) : E \rightarrow \overline{\mathbb{C}}$  is injective for each  $x$  in  $V$ , and
- (iii) the map  $\phi(\cdot, z) : V \rightarrow \overline{\mathbb{C}}$  is holomorphic for each  $z$  in  $E$ .

We say that  $V$  is a *parameter space* of the holomorphic motion  $\phi$ . We will assume that  $\phi$  is a *normalized* holomorphic motion; i.e.  $0, 1$ , and  $\infty$  belong to  $E$  and are fixed points of the map  $\phi(x, \cdot)$  for every  $x$  in  $V$ .

**Definition 24.** *Let  $V$  and  $W$  be connected complex manifolds with basepoints, and  $f$  be a basepoint preserving holomorphic map of  $W$  into  $V$ . If  $\phi$  is a holomorphic motion of  $E$  over  $V$ , its pullback by  $f$  is the holomorphic motion*

$$f^*(\phi)(x, z) = \phi(f(x), z) \quad \text{for all } (x, z) \in W \times E$$

of  $E$  over  $W$ .

If  $E$  is a proper subset of  $\widehat{E}$  and  $\phi : V \times E \rightarrow \overline{\mathbb{C}}$ ,  $\widehat{\phi} : V \times \widehat{E} \rightarrow \overline{\mathbb{C}}$  are two holomorphic motions, we say that  $\widehat{\phi}$  *extends*  $\phi$  if  $\widehat{\phi}(x, z) = \phi(x, z)$  for all  $(x, z)$  in  $V \times E$ .

Henceforth, we shall always assume that  $E$  is a closed subset of  $\overline{\mathbb{C}}$  and that  $0, 1$ , and  $\infty$  belong to  $E$ .

**Definition 25.** *The universal holomorphic motion  $\Psi_E$  of  $E$  over  $T(E)$  is defined as follows:*

$$\Psi_E(P_E(\mu), z) = w^\mu(z) \quad \text{for } \mu \in M(\mathbb{C}) \text{ and } z \in E.$$

The definition of  $P_E$  in §5.1 guarantees that  $\Psi_E$  is well-defined. It is a holomorphic motion since  $P_E$  is a holomorphic split submersion and  $\mu \mapsto w^\mu(z)$  is a holomorphic map from  $M(\mathbb{C})$  to  $\overline{\mathbb{C}}$  for every fixed  $z$  in  $\overline{\mathbb{C}}$  (by Theorem 11 in [4]).

This holomorphic motion is “universal” in the following sense:

**Theorem 34.** *Let  $\phi : V \times E \rightarrow \overline{\mathbb{C}}$  be a holomorphic motion. If  $V$  is simply connected, then there exists a unique basepoint preserving holomorphic map  $f : V \rightarrow T(E)$  such that  $f^*(\Psi_E) = \phi$ .*

For a proof see §14 in [57].

In what follows,  $B$  is a path-connected topological space. Let  $\mathcal{H}(\overline{\mathbb{C}})$  denote the group of homeomorphisms of  $\overline{\mathbb{C}}$  onto itself, with the topology of uniform convergence in the spherical metric. As usual,  $E$  is a closed set in  $\overline{\mathbb{C}}$ , and  $0, 1$ , and  $\infty$  are in  $E$ . The following two lemmas were proved in [57].

**Lemma 19.** *Let  $h : B \rightarrow \mathcal{H}(\overline{\mathbb{C}})$  be a continuous map such that  $h(x)(e) = e$  for all  $x$  in  $B$  and for all  $e$  in  $E$ . If  $h(x_0)$  is isotopic to the identity rel  $E$  for some fixed  $x_0$  in  $B$ , then  $h(x)$  is isotopic to the identity rel  $E$  for all  $x$  in  $B$ .*

*Proof* Let  $x$  be any point in  $B$ . Choose a path  $\gamma : [0, 1] \rightarrow B$  such that  $\gamma(0) = x_0$  and  $\gamma(1) = x$ . It is clear that the map  $(t, z) \mapsto h(\gamma(t))(z)$  from  $[0, 1] \times \overline{\mathbb{C}}$  to  $\overline{\mathbb{C}}$  is an isotopy rel  $E$  between  $h(x_0)$  and  $h(x)$ .

**Lemma 20.** *Let  $f$  and  $g$  be two continuous maps from  $B$  to  $T(E)$ , satisfying:*

- (i)  $\Psi_E(f(x), z) = \Psi_E(g(x), z)$  for all  $z \in E$  and  $x \in B$ , and
- (ii)  $f(x_0) = g(x_0)$  for some  $x_0$  in  $B$ .

*Then,  $f(x) = g(x)$  for all  $x$  in  $B$ .*

*Proof* By Proposition 13, there exists a basepoint preserving continuous map  $s : T(E) \rightarrow M(\mathbb{C})$  such that  $P_E \circ s$  is the identity map on  $T(E)$ . For each  $x$  in  $B$ , define  $\mu(x) = s(f(x))$  and  $\nu(x) = s(g(x))$ . We will show that the quasiconformal

map  $h(x) = (w^{\mu(x)})^{-1} \circ w^{\nu(x)}$  is isotopic to the identity rel  $E$ . That will prove our lemma.

Since  $\mu$  and  $\nu$  are continuous maps of  $B$  into  $M(\mathbb{C})$  and  $\mathcal{H}(\overline{\mathbb{C}})$  is a topological group, Lemma 17 of [4] implies that  $h$  is a continuous map of  $B$  into  $\mathcal{H}(\overline{\mathbb{C}})$ .

By condition (i) and Definition 23, we have

$$w^{\mu(x)}(z) = \Psi_E(f(x), z) = \Psi_E(g(x), z) = w^{\nu(x)}(z)$$

for all  $x$  in  $B$  and  $z$  in  $E$ . Therefore,  $h(x)$  fixes the set  $E$  pointwise for each  $x$  in  $B$ . By condition (ii),  $h(x_0)$  is isotopic to the identity rel  $E$ . It follows by Lemma 20, that  $h(x)$  is isotopic to the identity rel  $E$  for all  $x$  in  $B$ .

Let  $E$  and  $\widehat{E}$  be any two closed subsets of  $\overline{\mathbb{C}}$  such that  $E \subset \widehat{E}$  (as in §5.1, we assume that  $0, 1,$  and  $\infty$  belong to both  $E$  and  $\widehat{E}$ ). Recall from §5.1, the forgetful map  $p_{\widehat{E}, E}$  from  $T(\widehat{E})$  to  $T(E)$  such that  $P_E = p_{\widehat{E}, E} \circ P_{\widehat{E}}$ . The following is a consequence of Lemma 21. Here,  $\Psi_E$  is the universal holomorphic motion of  $E$  and  $\Psi_{\widehat{E}}$  is the universal holomorphic motion of  $\widehat{E}$ .

**Lemma 21.** *Let  $V$  be a connected complex Banach manifold with basepoint, and let  $f$  and  $g$  be basepoint preserving holomorphic maps from  $V$  into  $T(E)$  and  $T(\widehat{E})$  respectively. Then  $p_{\widehat{E}, E} \circ g = f$  if and only if  $g^*(\Psi_{\widehat{E}})$  extends  $f^*(\Psi_E)$ .*

See §13 in [57] for the proof.

### 5.2.2 A proposition

We prove the following generalization of Proposition 4 in [9]. This is an easy consequence of Theorem 34 and Lemma 22, and shows the importance of universal holomorphic motions.

**Proposition 13.** *Let  $V$  be a simply connected complex Banach manifold with a basepoint. The following statements are equivalent:*

1. *Every holomorphic motion  $\phi : V \times E \rightarrow \overline{\mathbb{C}}$  extends to a holomorphic motion  $\widehat{\phi} : V \times \widehat{E} \rightarrow \overline{\mathbb{C}}$ .*

2. For every basepoint preserving holomorphic map  $f : V \rightarrow T(E)$ , there exists a basepoint preserving holomorphic map  $g : V \rightarrow T(\widehat{E})$  such that  $f = p_{\widehat{E},E} \circ g$ .

*Proof*

(1)  $\Rightarrow$  (2): Let  $f : V \rightarrow T(E)$  be a basepoint preserving holomorphic map. Then,  $f^*(\Psi_E) := \phi$  is a holomorphic motion of  $E$  over  $V$ . By (1) there exists a holomorphic motion  $\widehat{\phi} : V \times \widehat{E} \rightarrow \overline{\mathbb{C}}$  such that  $\widehat{\phi}$  extends  $\phi$ . By Theorem 34, there exists a basepoint preserving holomorphic map  $g : V \rightarrow T(\widehat{E})$  such that  $g^*(\Psi_{\widehat{E}}) = \widehat{\phi}$ . Since  $\widehat{\phi}$  extends  $\phi$ , it follows by Lemma 22 that  $p_{\widehat{E},E} \circ g = f$ .

(2)  $\Rightarrow$  (1): Let  $\phi : V \times E \rightarrow \overline{\mathbb{C}}$  be a holomorphic motion. By Theorem 29, there exists a basepoint preserving holomorphic map  $f : V \rightarrow T(E)$  such that  $f^*(\Psi_E) = \phi$ . By (2) there exists a basepoint preserving holomorphic map  $g : V \rightarrow T(\widehat{E})$  such that  $f = p_{\widehat{E},E} \circ g$ . Let  $g^*(\Psi_{\widehat{E}}) := \widehat{\phi}$ ; then,  $\widehat{\phi}$  is a holomorphic motion of  $\widehat{E}$  over  $V$ . It follows by Lemma 22 that  $\widehat{\phi}$  extends  $\phi$ .

Recall from §5.1, that when  $E = \overline{\mathbb{C}}$ , we can identify  $T(\overline{\mathbb{C}})$  biholomorphically with  $M(\mathbb{C})$ . The pullback  $\widetilde{\Psi}_{\overline{\mathbb{C}}}$  of  $\Psi_{\overline{\mathbb{C}}}$  to  $M(\mathbb{C})$  by  $P_{\overline{\mathbb{C}}}$  satisfies

$$\widetilde{\Psi}_{\overline{\mathbb{C}}}(\mu, z) = \Psi_{\overline{\mathbb{C}}}(P_{\overline{\mathbb{C}}}(\mu), z) = w^\mu(z)$$

for all  $(\mu, z) \in M(\mathbb{C}) \times \overline{\mathbb{C}}$ . So, when we use  $P_{\overline{\mathbb{C}}}$  to identify  $T(\overline{\mathbb{C}})$  with  $M(\mathbb{C})$ , the universal holomorphic motion of  $\overline{\mathbb{C}}$  becomes the map

$$\Psi_{\overline{\mathbb{C}}}(\mu, z) = w^\mu(z)$$

for  $(\mu, z) \in M(\mathbb{C}) \times \overline{\mathbb{C}}$ .

**Corollary 1.** *Let  $V$  be a simply connected complex Banach manifold with a basepoint. The following statements are equivalent:*

1. Every holomorphic motion  $\phi : V \times E \rightarrow \overline{\mathbb{C}}$  extends to a holomorphic motion  $\widehat{\phi} : V \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ .
2. For every basepoint preserving holomorphic map  $f : V \rightarrow T(E)$ , there exists a basepoint preserving holomorphic map  $g : V \rightarrow M(\mathbb{C})$  such that  $f = P_E \circ g$ .

### 5.3 Proof of the lifting theorem

Recall that  $E = \{0, 1, \infty, \zeta_1, \dots, \zeta_n\}$  where  $\zeta_i \neq \zeta_j$  for  $i \neq j$ , and  $\zeta_i \neq 0, 1, \infty$  for all  $i = 1, \dots, n$ . By Lemma 19, we may assume that  $f : \Delta \rightarrow T(E)$  is a basepoint preserving holomorphic map.

For a fixed  $0 < r < 1$ , let  $f_r(z) = f(rz) = [w^\mu]_E$ . Then we define  $n$  holomorphic functions  $f_{i,r}(z) = w^\mu(\zeta_i)$  for  $i = 1, \dots, n$ . Let  $D = \overline{\mathbb{C}} \setminus \Delta$  be the exterior of  $\Delta$ . We define  $n$  maps on  $D$ , which are holomorphic in a neighborhood of  $D$ , as

$$g_i(z) = f_{i,r}\left(\frac{1}{z}\right)$$

for  $|z| \geq 1$  and for all  $1 \leq i \leq n$ . Furthermore, we extend  $g_i$  to  $\overline{\mathbb{C}}$  as follows:

$$g_i(z) = g_i\left(\frac{1}{\bar{z}}\right)$$

for  $|z| \leq 1$  and for all  $1 \leq i \leq n$ . We have the following:

- (a)  $g_i(\infty) = g_i(0) = \zeta_i$  for  $i = 1, \dots, n$ ;
- (b) for any fixed  $z \in \overline{\mathbb{C}}$ ,  $g_i(z) \neq g_j(z)$  for  $1 \leq i \neq j \leq n$  and  $g_i(z) \neq 0, 1, \infty$  for all  $i = 1, \dots, n$ ;
- (c)  $g_i(z)$  is a bounded function on  $\overline{\mathbb{C}}$ .

Choose a  $C^\infty$  function  $0 \leq \lambda(x) \leq 1$  on  $\mathbb{R}^+ = \{x \geq 0\}$  such that  $\lambda(0) = 1$  and  $\lambda(x) = 0$  for  $x \geq \delta/2$ . Define

$$\Theta(z, w) = \sum_{i=1}^n \lambda(|w - g_i(z)|) \frac{\partial g_i}{\partial \bar{z}}(z), \quad (z, w) \in \overline{\mathbb{C}} \times \mathbb{C}. \quad (5.1)$$

Let  $\mathcal{C}(\mathbb{C})$  denote the complex Banach space of bounded, continuous functions  $\phi$  on  $\mathbb{C}$  with the norm

$$\|\phi\| = \sup_{z \in \mathbb{C}} |\phi(z)|.$$

As usual,  $L^\infty(\mathbb{C})$  denotes the complex Banach space of  $L^\infty$  functions on  $\mathbb{C}$  with the  $L^\infty$ -norm denoted by  $\|\phi\|_\infty$ .

Since  $\Theta(z, f(z))$  is an  $L^\infty$  function with a compact support in  $\overline{\Delta}$  for any  $f \in \mathcal{C}(\mathbb{C})$ , we can define an operator  $\mathcal{Q}$  mapping functions in  $\mathcal{C}(\mathbb{C})$  to functions in  $L^\infty(\mathbb{C})$

with compact support by

$$\mathcal{Q}f(z) = \Theta(z, f(z)), \quad f(z) \in \mathcal{C}(\mathbb{C}).$$

Since  $\Theta(z, w)$  is Lipschitz in the  $w$  variable with a Lipschitz constant  $L$  independent of  $z \in \overline{\mathbb{C}}$ , we have

$$|\mathcal{Q}f(z) - \mathcal{Q}g(z)| = |\Theta(z, f(z)) - \Theta(z, g(z))| \leq L|f(z) - g(z)|.$$

Thus,

$$\|\mathcal{Q}f - \mathcal{Q}g\|_\infty \leq L\|f - g\|$$

and  $\mathcal{Q} : \mathcal{C}(\mathbb{C}) \rightarrow L^\infty(\mathbb{C})$  is a continuous operator.

Now consider the operator

$$\mathcal{K} = \mathcal{P} \circ \mathcal{Q}.$$

Clearly, it is a continuous operator from  $\mathcal{C}(\mathbb{C})$  into itself.

**Lemma 22.** *There is a constant  $C_3 > 0$  such that*

$$\|\mathcal{K}f\| \leq C_3 \quad \text{for all } f \in \mathcal{C}(\mathbb{C}).$$

**Lemma 23.** *Let  $p > 2$  and*

$$0 < \alpha = 1 - \frac{2}{p} < 1.$$

*Then, for any  $f \in \mathcal{C}(\mathbb{C})$ ,  $\mathcal{K}f$  is  $\alpha$ -Hölder continuous with a Hölder constant*

$$H = \max \{A(1)C_1, 2^{1+\alpha}C_3\}$$

*where  $H$  is independent of  $f$ .*

**Lemma 24.** *For any  $\varepsilon > 0$ , there exists an  $R > 0$ , such that  $|\mathcal{K}f(z)| < \varepsilon$  for all  $f \in \mathcal{C}(\mathbb{C})$  and  $z \in \mathbb{C}$  with  $|z| \geq R$ .*

The above lemmas imply that  $\mathcal{K} : \mathcal{C}(\mathbb{C}) \rightarrow \mathcal{C}(\mathbb{C})$  is a continuous compact operator. Recall from the statement of the main theorem that  $\zeta_{n+1}$  is any point in  $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$  distinct from  $\zeta_1, \dots, \zeta_n$ . For  $\zeta_{n+1}$ , let

$$\mathcal{B} = \{f \in \mathcal{C}(\mathbb{C}) : \|f\| \leq |\zeta_{n+1}| + C_3\}.$$

It is a bounded convex subset in  $\mathcal{C}(\mathbb{C})$ . The continuous compact operator  $\zeta_{n+1} + \mathcal{K}$  maps  $\mathcal{B}$  into itself. By Schauder fixed point theorem (see Theorem 2A on page 56 of [64])  $\zeta_{n+1} + \mathcal{K}$  has a fixed point in  $\mathcal{B}$ . That is, there is a  $g_{n+1} \in \mathcal{B}$  such that

$$g_{n+1}(z) = \zeta_{n+1} + \mathcal{K}g_{n+1}(z) \quad \text{for all } z \in \mathbb{C}.$$

Since  $\mathcal{Q}f(z)$  has a compact support in  $\bar{\Delta}$  for any  $g \in \mathcal{C}(\mathbb{C})$ ,  $\mathcal{K}g_{n+1}(z) \rightarrow 0$  as  $z \rightarrow \infty$ . So,  $g_{n+1}$  can be extended continuously to  $\infty$  such that  $g_{n+1}(\infty) = \zeta_{n+1}$ .

**Lemma 25.** *The solution  $g_{n+1}(z)$  is the unique fixed point of the operator  $\zeta_{n+1} + \mathcal{K}$ . and  $g_i(z)$  is the unique solution of the operator  $\zeta_i + \mathcal{K}$  for all  $1 \leq i \leq n$ .*

We claim that  $g_{n+1}(z) \neq g_i(z)$  for all  $z \in \mathbb{C}$  and  $1 \leq i \leq n$ .

Now, let

$$f_{n+1,r}(z) = g_{n+1}\left(\frac{1}{z}\right) \quad \text{for } |z| < 1.$$

Let

$$M_{n+1} = \{w \in \mathbb{C}^{n+1} : w_i \neq w_j \text{ for } i \neq j \text{ and } w_i \neq 0, 1 \text{ for all } i = 1, \dots, n+1\}.$$

We can define a holomorphic function

$$F_r(z) = \left(f_{1,r}(z), \dots, f_{n,r}(z), f_{n+1,r}(z)\right) : \Delta \rightarrow M_{n+1}.$$

Recall that  $\widehat{E} = E \cup \{\zeta_{n+1}\}$ .

By a theorem of Nag (see [58]), there exists a holomorphic universal covering map  $\pi : T(\widehat{E}) \rightarrow M_{n+1}$  such that  $\pi$  maps the basepoint in  $T(\widehat{E})$  to the point  $(\zeta_1, \dots, \zeta_{n+1})$ . Since  $\Delta$  is simply connected, there exists a holomorphic map

$$\widehat{f}_r : \Delta \rightarrow T(\widehat{E})$$

such that  $\pi \circ \widehat{f}_r = F_r$ , and we can choose  $\widehat{f}_r$  to be basepoint preserving.

Recall from the beginning of §5.3, that  $f_r(z) = [w^\mu]_E$ . Suppose  $\widehat{f}_r(z) = [w^\nu]_{\widehat{E}}$ . Then, by §5.1, we have

$$p_{\widehat{E},E}\left([w^\nu]_{\widehat{E}}\right) = [w^\nu]_E.$$

Consider the two maps  $f_r : \Delta \rightarrow T(E)$  and  $p_{\widehat{E},E} \circ \widehat{f}_r : \Delta \rightarrow T(E)$ . They are both basepoint preserving. Furthermore, at each  $\zeta_i$ , for  $i = 1, \dots, n$ , we have  $w^\mu(\zeta_i) = w^\nu(\zeta_i)$ . Therefore, by Lemma 21, we conclude that  $p_{\widehat{E},E} \circ \widehat{f}_r = f_r$  on  $\Delta$ . This proves the lifting of the holomorphic map  $f_r$  on  $\Delta_r$ .

Since  $f_{n+1,r}$  misses the points 0, 1, and  $\infty$ , the family  $\{f_{n+1,r}\}_{0 < r < 1}$  forms a normal family. Therefore, there exists a convergent subsequence  $f_{n+1,r_k} \rightarrow f_{n+1}$  when  $r_k \rightarrow 1$ . It is clear that  $f_{i,r_k} \rightarrow f_i$  when  $r_k \rightarrow 1$ . We claim that

**Lemma 26.** *For all  $z \in \Delta$ ,  $f_{n+1}(z) \neq f_i(z)$ .*

See the proof at the end of this subsection.

For  $z$  in  $\Delta$ , define

$$F(z) = (f_1(z), \dots, f_{n+1}(z)).$$

By §1.3,  $T(\widehat{E})$  is identified with the classical Teichmüller space  $Teich(\overline{\mathbb{C}} \setminus \widehat{E})$ , which is finite dimensional. Since each  $\widehat{f}_r(0) = [id] \in T(\widehat{E})$  for all  $0 < r < 1$ , the family  $\{\widehat{f}_r\}_{0 < r < 1}$  is relatively compact, because of the completeness of the Kobayashi distance (which is the same as Teichmüller distance) on  $T(\widehat{E})$  (see Proposition 3 in [48], and also [63]). The holomorphy of the limit function  $\widehat{f}$  then follows from Weierstrass' theorem, since  $T(\widehat{E})$  is a bounded domain in  $\mathbb{C}^{n+1}$  via Bers embedding. Since  $\pi \circ \widehat{f}_r = F_r$ , we have  $\pi \circ \widehat{f} = F$ , by continuity.

Finally, suppose  $f(z) = [w^\mu]_E$  and  $\widehat{f}(z) = [w^\nu]_{\widehat{E}}$ . By §5.1, we have

$$p_{\widehat{E},E}([w^\nu]_{\widehat{E}}) = [w^\nu]_E.$$

Consider two maps  $f : \Delta \rightarrow T(E)$  and  $p_{\widehat{E},E} \circ \widehat{f} : \Delta \rightarrow T(E)$ . They are both basepoint preserving. Furthermore, at each  $\zeta_i$ , we have  $w^{\tilde{\mu}}(\zeta_i) = w^{\tilde{\nu}}(\zeta_i)$  (because  $\pi \circ \widehat{f} = F$ ). It follows by Lemma 21 that  $p_{\widehat{E},E} \circ \widehat{f} = f$ .

**Proof of Lemma 27** Consider a set of four points  $S = \{z_1, z_2, z_3, z_4\}$  in  $\mathbb{C}$ . These points are distinct if and only if the cross ratio

$$Cr(S) = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

is not equal to 0, 1, or  $\infty$ .

Consider  $S(z) = \{f_i(z), f_j(z), f_{n+1}(z), \infty\}$ . The cross ratio

$$Cr(S(z)) = \frac{f_i(z) - f_{n+1}(z)}{f_j(z) - f_{n+1}(z)}.$$

We only need to show that for any fixed  $0 < r_0 < 1$ ,  $Cr(S(z))$  is not equal to 0, 1, or  $\infty$  for any  $z \in \Delta_{r_0}$  where  $\Delta_{r_0}$  is the disk centered at zero with radius  $r_0$ .

For any  $0 < r < 1$ , let  $S_r(z) = \{f_{i,r}(z), f_{j,r}(z), f_{n+1,r}(z), \infty\}$ . Then

$$Cr(S_r(z)) = \frac{f_{i,r}(z) - f_{n+1,r}(z)}{f_{j,r}(z) - f_{n+1,r}(z)}.$$

Since  $\mathbb{C} \setminus \{0, 1\}$  is complete hyperbolic and

$$Cr(S_r(0)) = \frac{\zeta_i - \zeta_{n+1}}{\zeta_j - \zeta_{n+1}} \in \mathbb{C} \setminus \{0, 1\}$$

for all  $0 < r < 1$ , again by Proposition 3 in [48], the family  $\{Cr(S_r(z))\}_{0 < r < 1}$  is relatively compact in the space of holomorphic mappings from  $\Delta$  to  $\mathbb{C} \setminus \{0, 1\}$ . Thus, for any  $|z| < r_0$  and for any  $0 < r < 1$ , we obtain

$$|Cr(S_r(z))| \leq K$$

for some  $K > 0$ .

This implies that the cross ratio  $Cr(S(z))$  is bounded away from  $\infty$  by  $K$ , by letting  $r \rightarrow 1^-$ . Following a similar argument, we can show that the cross ratio  $Cr(S(z))$  is also bounded away from 0 and 1 for any  $|z| < r_0$ . So  $f_{n+1}(z) \neq f_i(z)$  for any  $1 \leq i \leq n$  on  $\Delta_{r_0}$ . Since  $0 < r_0 < 1$  is an arbitrary number, we conclude that  $f_{n+1}(z) \neq f_i(z)$  on  $\Delta$ , for any  $1 \leq i \leq n$ . This completes the proof.  $\square$

## 5.4 Some concluding remarks

In their papers [9], Bers and Royden showed the intimate relationship between Teichmüller spaces and holomorphic motions. They noted that the lifting problem in §5.1 is nicely connected with the question of extending holomorphic motions. In fact, in Proposition 14 of our paper, let  $V = \Delta$  and  $E$  and  $\widehat{E}$  be the two finite

sets given in the statement of our main theorem. Then, by our main theorem and Proposition 14, it follows that every holomorphic motion of  $E$  over  $\Delta$  extends to a holomorphic motion of  $\widehat{E}$  (over  $\Delta$ ). By Proposition 1 in [9], it then follows that given any holomorphic motion  $\phi : \Delta \times K \rightarrow \overline{\mathbb{C}}$ , where  $K$  is **any** set in  $\overline{\mathbb{C}}$  (not necessarily closed), there exists a holomorphic motion  $\widehat{\phi} : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  such that  $\widehat{\phi}$  extends  $\phi$ .

It is important to note that the lifting problem that we discuss in our main theorem does not work if  $\Delta$  is replaced by a domain in  $\mathbb{C}^n$  ( $n \geq 2$ ). In fact, let  $E$  and  $\widehat{E}$  be the two given finite sets in our main theorem, and  $n \geq 2$ . Then, by our discussion in §5.1,  $T(E)$  and  $T(\widehat{E})$  are the classical Teichmüller spaces of the sphere with punctures at  $E$  and  $\widehat{E}$  respectively. Consider the identity map  $i : T(E) \rightarrow T(E)$ ; if it has a holomorphic lift into  $T(\widehat{E})$ , i.e. if there exists a holomorphic map  $g : T(E) \rightarrow T(\widehat{E})$  such that  $p_{\widehat{E},E} \circ g = i$ , then the map  $g$  will be a holomorphic section of the map  $p_{\widehat{E},E}$ . This is impossible by a theorem of Earle and Kra; see [20] (also proved by Hubbard in [40]). By Proposition 14, that also means that the universal holomorphic motion  $\Psi_E : T(E) \times E \rightarrow \overline{\mathbb{C}}$  cannot be extended to a holomorphic motion of the set  $\widehat{E}$ .

# Kobayashi's and Teichmuller metrics on $T(R)$

## Contents

6.1 Kobayashi's and Teichmuller metrics . . . . .	93
6.2 The Lifting Problem . . . . .	97

## 6.1 Kobayashi's and Teichmuller metrics

Suppose  $\mathcal{N}$  is a connected complex manifold over a complex Banach space. Let  $\mathcal{H} = \mathcal{H}(\Delta, \mathcal{N})$  be the space of all holomorphic maps from  $\Delta$  into  $\mathcal{N}$ . For  $p$  and  $q$  in  $\mathcal{N}$ , let

$$d_1(p, q) = \log \frac{1+r}{1-r},$$

where  $r$  is the infimum of the nonnegative numbers  $s$  for which there exists  $f \in \mathcal{H}$  such that  $f(0) = p$  and  $f(s) = q$ . If no such  $f \in \mathcal{H}$  exists, then  $d_1(p, q) = \infty$ .

Let

$$d_n(p, q) = \inf \sum_{i=1}^n d_1(p_{i-1}, p_i)$$

where the infimum is taken over all chains of points  $p_0 = p, p_1, \dots, p_n = q$  in  $\mathcal{N}$ . Obviously,  $d_{n+1} \leq d_n$  for all  $n > 0$ .

**Definition 26** (Kobayashi's metric). *The Kobayashi pseudo-metric  $d_K = d_{K, \mathcal{N}}$  is*

defined as

$$d_K(p, q) = \lim_{n \rightarrow \infty} d_n(p, q), \quad p, q \in \mathcal{N}.$$

In general, it is possible that  $d_K$  is identically equal to 0, which is the case for example if  $\mathcal{N} = \mathbb{C}$ .

Another way to describe  $d_K$  is the following. Let the Poincaré metric on the unit disk  $\Delta$  be given by

$$\rho_\Delta(z, w) = \log \frac{1 + \frac{|z-w|}{|1-\bar{z}w|}}{1 - \frac{|z-w|}{|1-\bar{z}w|}}, \quad z, w \in \Delta.$$

Then  $d_K$  is the largest pseudo metric on  $\mathcal{N}$  such that

$$d_K(f(z), f(w)) \leq \rho_\Delta$$

for all  $z$  and  $w \in \Delta$  and for all holomorphic maps  $f$  from  $\Delta$  into  $\mathcal{N}$ . The following is a consequence of this property.

**Proposition 14.** *Suppose  $\mathcal{N}$  and  $\mathcal{N}'$  are two complex manifolds and  $F : \mathcal{N} \rightarrow \mathcal{N}'$  is holomorphic. Then*

$$d_{K, \mathcal{N}'}(F(p), F(q)) \leq d_{K, \mathcal{N}}(p, q).$$

**Lemma 27.** *Suppose  $\mathcal{B}$  is a complex Banach space with norm  $\|\cdot\|$ . Let  $\mathcal{N}$  be the unit ball of  $\mathcal{B}$  and let  $d_K$  be the Kobayashi's metric on  $\mathcal{N}$ . Then*

$$d_K(0, \mathbf{v}) = \log \frac{1 + \|\mathbf{v}\|}{1 - \|\mathbf{v}\|} = 2 \tanh^{-1} \|\mathbf{v}\|, \quad \forall \mathbf{v} \in \mathcal{N}.$$

*Proof* Pick a point  $\mathbf{v}$  in  $\mathcal{N}$ . The linear function  $f(c) = c\mathbf{v}/\|\mathbf{v}\|$  maps the unit disk  $\Delta$  into the unit ball  $\mathcal{N}$ , and takes  $\|\mathbf{v}\|$  into  $\mathbf{v}$ , and 0 into  $\mathbf{0}$ . Therefore

$$d_K(0, \mathbf{v}) \leq \rho_\Delta(0, \|\mathbf{v}\|),$$

where  $\rho_\Delta$  is the Kobayashi's metric on  $\Delta$  (it coincides with the Poincaré metric on  $\Delta$ ).

On the other hand, by the Hahn-Banach theorem, there exists a continuous linear function  $L$  on  $\mathcal{N}$  such that  $L(\mathbf{v}) = \|\mathbf{v}\|$  and  $\|L\| = 1$ . Thus,  $L$  maps  $\mathcal{N}$  into the unit disk  $\Delta$ , and so

$$d_K(0, \mathbf{v}) \geq \rho_\Delta(0, \|\mathbf{v}\|).$$

Therefore,

$$d_K(0, \mathbf{v}) = \rho_\Delta(0, \|\mathbf{v}\|) = \log \frac{1 + \|\mathbf{v}\|}{1 - \|\mathbf{v}\|} = 2 \tanh^{-1} \|\mathbf{v}\|.$$

Assume  $R$  is a Riemann surface conformal to  $\Delta/\Gamma$  where  $\Gamma$  is a discontinuous, fixed point free group of hyperbolic isometries of  $\Delta$ . Let  $\mathcal{M} = \mathcal{M}(\Gamma)$  be the unit ball of the complex Banach space of all  $\mathcal{L}^\infty$  functions defined on  $\Delta$  satisfying the  $\Gamma$ -invariance property:

$$\mu(\gamma(z)) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \quad (6.1)$$

for all  $z$  in  $\Delta$  and all  $\gamma$  in  $\Gamma$ . An element  $\mu \in \mathcal{M}$  is called a Beltrami coefficient on  $R$ . Points of the Teichmüller space  $T = T(R)$  are represented by equivalence classes of Beltrami coefficients  $\mu \in \mathcal{M}$ . Two Beltrami coefficients  $\mu, \nu \in \mathcal{M}$  are in the same Teichmüller equivalence class if the quasiconformal self maps  $f^\mu$  and  $f^\nu$  which preserve  $\Delta$  and which are normalized to fix  $0, i$  and  $-1$  on the boundary of the unit disk coincide at all boundary points of the unit disk.

**Definition 27** (Teichmüller's metric). *For two elements  $[\mu]$  and  $[\nu]$  of  $T(R)$ , Teichmüller's metric is equal to*

$$d_T([\mu], [\nu]) = \inf \log K(f^\mu \circ (f^\nu)^{-1}),$$

where the infimum is over all  $\mu$  and  $\nu$  in the equivalence classes  $[\mu]$  and  $[\nu]$ , respectively. In particular,

$$d_T(0, [\mu]) = \log \frac{1 + k_0}{1 - k_0}$$

where  $k_0$  is the minimal value of  $\|\mu\|_\infty$ , where  $\mu$  ranges over the Teichmüller class  $[\mu]$ .

**Lemma 28.** *Let  $d_K$  and  $d_T$  be Kobayashi's and Teichmüller's metrics of  $T(R)$ . Then  $d_K \leq d_T$ .*

*Proof* Let a Beltrami coefficient  $\mu$  satisfying (6.1) be extremal in its class and  $\|\mu\|_\infty = k$ . This is possible because by normal families argument every class possesses at least one extremal representative. By the definition of Teichmüller's metric

$$d_T(0, [\mu]) = \log \frac{1+k}{1-k}.$$

For such a  $\mu$ , let  $g(c) = [c\mu/k]$ . Then  $g(c)$  is a holomorphic function of  $c$  for  $|c| < 1$  with values in the Teichmüller space  $T(R)$ ,  $g(0) = 0$  and  $g(k) = [\mu]$ . Hence

$$d_K(0, [\mu]) \leq d_1(0, [\mu]) \leq d_T(0, [\mu]).$$

Now the right translation mapping  $\alpha([f^\mu]) = [f^\mu \circ (f^\nu)^{-1}]$  is biholomorphic, so it is an isometry in Kobayashi's metric. We also know that it is an isometry in Teichmüller's metric. Therefore, the inequality

$$d_K([\nu], [\mu]) \leq d_1([\nu], [\mu]) \leq d_T([\nu], [\mu])$$

holds for an arbitrary pair of points  $[\mu]$  and  $[\nu]$  in the Teichmüller space  $T(R)$ .

In order to describe holomorphic maps into  $T(R)$  we will use the Bers' embedding by which  $T(R)$  is realized as a bounded domain in the Banach space  $\mathcal{B}(R)$  of equivariant cusp forms. Here  $\mathcal{B}(R)$  consists of the functions  $\varphi$  holomorphic in  $\Delta^c$  for which

$$\sup_{z \in \Delta^c} \{(|z|^2 - 1)^2 |\varphi(z)|\} < \infty$$

and for which

$$\varphi(\gamma(z))(\gamma'(z))^2 = \varphi(z) \text{ for all } \gamma \in \Gamma.$$

We assume  $\Gamma$  is a Fuchsian covering group such that  $\Delta/\Gamma$  is conformal to  $R$ . For any Beltrami differential  $\mu$  supported on  $\Delta$ , we let  $w^\mu$  be the quasiconformal

self-mapping of  $\overline{\mathbb{C}}$  which fixes  $1, i$  and  $-1$  and which has Beltrami coefficient  $\mu$  in  $\Delta$  and Beltrami coefficient identically equal to zero in  $\Delta^c$ . Let  $w^\mu$  restricted to  $\Delta^c$  be equal to the Riemann mapping  $g^\mu$ . Then  $g^\mu$  has the following properties:

- a)  $g^\mu$  fixes the points  $1, i$  and  $-1$ ,
- b)  $g^\mu(\partial\Delta)$  is a quasiconformal image of the circle  $\partial\Delta$ ,
- c)  $g^\mu$  is univalent and holomorphic in  $\Delta^c$ .
- d)  $g^\mu \circ \gamma \circ (g^\mu)^{-1}$  is equal to a Möbius transformation  $\tilde{\gamma}$ , for all  $\gamma$  in  $\Gamma$ , and
- e)  $g^\mu$  determines and is determined uniquely by the corresponding point in  $T(R)$ .

The Bers' embedding maps the Teichmüller equivalence class of  $\mu$  to the Schwarzian derivative of  $g^\mu$  where the Schwarzian derivative of a  $C^3$  function  $g$  is defined by

$$S(g) = \left(\frac{g''}{g'}\right)' - \frac{1}{2} \left(\frac{g''}{g'}\right)^2.$$

For more details about Schwarzian derivatives and complex structures, please read Chapter 1. In the next section we use this realization of the complex structures to prove that  $d_T \leq d_K$ .

## 6.2 The Lifting Problem

Let  $\Phi$  be the natural map from the space  $\mathcal{M}$  of Beltrami differentials on  $R$  onto  $T(R)$  and let  $f$  be a holomorphic map from the unit disk into  $T(R)$  with  $f(0)$  equal to the base point of  $T(R)$ . The lifting problem is the problem of finding a holomorphic map  $\tilde{f}$  from  $\Delta$  into  $\mathcal{M}$ , such that  $\tilde{f}(0) = 0$  and  $\Phi \circ \tilde{f} = f$ .

In this section we prove the theorem of Earle, Kra and Krushkal [21] which says that the lifting problem always has a solution. We follow their technique which relies on proving an equivariant version of Ślodkowski's extension theorem and then going on to show that the positive solution to the lifting problem implies  $d_T \leq d_K$

for every Riemann surface that has a nontrivial Teichmüller space with complex structure.

**Theorem 35** (An equivariant version of Slodkowski's extension theorem of Earle Kra and Krushkal). *Let  $h(c, z)$  be a holomorphic motion of  $\Delta^c = \overline{\mathbb{C}} \setminus \Delta$  parametrized by  $\Delta$  and with base point 0 and let  $\Gamma$  be a torsion-free group of Möbius transformations mapping  $\Delta^c$  onto itself. Suppose for each  $\gamma \in \Gamma$  and  $c \in \Delta$  there is a Möbius transformation  $\tilde{\gamma}_c$  such that*

$$h(c, \gamma(z)) = \tilde{\gamma}_c(h(c, z)), \quad \forall z \in \Delta^c.$$

*Then  $h(c, z)$  can be extended to a holomorphic motion  $H(c, z)$  of  $\overline{\mathbb{C}}$  parametrized by  $\Delta$  and with base point 0 in such a way that*

$$H(c, \gamma(z)) = \tilde{\gamma}_c(H(c, z))$$

*holds for  $\gamma \in \Gamma$ ,  $c \in \Delta$  and  $z \in \overline{\mathbb{C}}$ .*

*Proof* Observe that  $\tilde{\gamma}_c$  is uniquely determined for all  $c \in \Delta$  because  $\Delta^c$  contains more than two points. To extend  $h(c, z)$  to  $\Delta$ , start with an point  $w \in \Delta$ . By Theorem 17, the motion  $h(c, z)$  can be extended to a holomorphic motion (still denote it as  $h(c, z)$ ) of the closed set  $\Delta^c \cup \{w\}$ . Furthermore, we may extend it to the orbit of  $w$  using the  $\Gamma$ -invariant property:

$$h(c, \gamma(w)) = \tilde{\gamma}_c(h(c, w)),$$

for all  $\gamma \in \Gamma$ . Since every  $\gamma \in \Gamma$  is fixed point free on  $\Delta$ , the motion  $h(c, z)$  is well defined and satisfies the  $\Gamma$ -invariant property for all  $c \in \Delta$  and all  $z$  in the set

$$E = \{\gamma(w) : \gamma \in \Gamma\} \cup (\overline{\mathbb{C}} \setminus \Delta).$$

So we only need to show that  $h(c, z)$  is a holomorphic motion of  $E$ . Observe first that  $h(0, z) = z$  since  $\tilde{\gamma}_0 = \gamma$  for all  $\gamma \in \Gamma$ . To show  $h(c, z)$  is injective for all

fixed  $c \in \Delta$ , suppose  $h(c, z_1) = h(c, z_2)$  for some  $c \in \Delta$ . Since  $h(c, z)$  is injective on  $\Delta^c \cup \{w\}$ , we may assume that  $z_1 = g(w)$  for some  $g \in \Gamma$ . By  $\Gamma$ -invariant property,

$$h(c, w) = (\tilde{g}_c)^{-1}(h(c, z_1)).$$

Thus,

$$h(c, w) = (\tilde{g}_c)^{-1}(h(c, z_2)) = h(c, g^{-1}(z_2)),$$

and we conclude that  $z_2$  belongs to the  $\Gamma$ -orbit of  $w$ . Let  $z_2 = \beta(w)$  for some  $\beta \in \Gamma$ .

Then

$$h(c, w) = \tilde{\gamma}_c(h(c, w)),$$

where  $\gamma = g^{-1} \circ \beta$ . Therefore  $h(c, w)$  is a fixed point of  $\tilde{\gamma}_c$ . On the other hand, since  $\gamma$  is a hyperbolic Möbius transformation,  $\tilde{\gamma}_c$  is also hyperbolic, so unless  $\tilde{\gamma}_c$  is identity, it can only fix points on the set  $h(c, \partial\Delta)$ . Hence  $\gamma$  is the identity map and  $z_1 = z_2$ .

Finally, we will show that  $l : c \rightarrow h(c, z)$  is holomorphic for any fixed  $z \in E$ . we may assume  $z = g(w)$ ,  $g \in \Gamma \setminus \{\text{identity}\}$ . Then  $l(c) = h(c, g(w)) = \tilde{g}_c(h(c, w))$ . Since  $c \rightarrow h(c, w)$  is holomorphic and  $\tilde{g}_c$  is a Möbius transformation, it is enough to prove the map  $k : c \rightarrow \tilde{g}_c(\zeta)$  is holomorphic for any fixed  $\zeta$ . Applying the  $\Gamma$ -invariant property to the three points  $0, 1, \infty$ , we obtain

$$\tilde{g}_c(0) = h(c, g(0)),$$

$$\tilde{g}_c(1) = h(c, g(1)),$$

$$\tilde{g}_c(\infty) = h(c, g(\infty)).$$

The right-hand sides of these three equations are holomorphic, so the maps  $c \mapsto \tilde{g}_c(0)$ ,  $c \mapsto \tilde{g}_c(1)$  and  $c \mapsto \tilde{g}_c(\infty)$  are holomorphic. Since  $\tilde{g}_c$  is a Möbius transformation,  $k : c \rightarrow \tilde{g}_c(\zeta)$  is holomorphic.

Therefore, we have extended  $h(c, z)$  to a holomorphic motion of

$$\Delta^c \cup \{\text{the } \Gamma \text{ orbit of } z\}.$$

By repeating this extension process to a countable set of points whose  $\Gamma$  orbits are dense in  $\Delta$ , we obtain the extension  $H(c, z)$  of  $h(c, z)$  with the property that

$$H(c, \gamma(z)) = \tilde{\gamma}_c(H(c, z))$$

for all  $\gamma \in \Gamma$ ,  $c \in \Delta$  and  $z \in \overline{\mathbb{C}}$ .

This equivariant version of Slodkowski's extension theorem leads almost immediately to the following lifting theorem.

**Theorem 36** (The lifting theorem). *If  $f : \Delta \rightarrow T(R)$  is holomorphic, then there exists a holomorphic map  $\tilde{f} : \Delta \rightarrow \mathcal{M}$  such that*

$$\Phi \circ \tilde{f} = f.$$

*If  $\mu_0 \in \mathcal{M}$  and  $\Phi(\mu_0) = f(0)$ , we can choose  $\tilde{f}$  such that  $\tilde{f}(0) = \mu_0$ .*

*Proof* By using the translation mapping  $\alpha$  of the Teichmüller space given by

$$\alpha([w^\mu]) = [w^\mu \circ (w^\nu)^{-1}],$$

we may assume  $f(0) = 0$ . For each  $c \in \Delta$ , let  $g(c, \cdot)$  be a meromorphic function whose Schwarzian derivative is  $f(c)$ . Then on  $\overline{\mathbb{C}} \setminus \Delta$  the map  $g(c, \cdot)$  is injective, and we can specify  $g(c, \cdot)$  uniquely by requiring that it fix  $1, i$  and  $-1$ . Thus  $g(0, z) = z$ . It is easy to verify that

$$g(c, z) : \Delta \times (\overline{\mathbb{C}} \setminus \Delta) \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion. For every  $\gamma \in \Gamma$  and  $c \in \Delta$ , there exists a Möbius transformation  $\tilde{\gamma}_c$  such that

$$g(c, \gamma(z)) = \tilde{\gamma}_c(g(c, z)).$$

Using the equivalent version of Slodkowski's extension theorem, we extend  $g$  to a  $\Gamma$ -invariant holomorphic motion (still denote it as  $g$ ) of  $\overline{\mathbb{C}}$ . For each  $c \in \Delta$ , let  $\tilde{f}(c)$  be the complex dilatation

$$\tilde{f}(c) = \frac{g_{\bar{z}}}{g_z}.$$

Then the  $\Gamma$ -invariant property of  $g$  implies that  $\tilde{f}(c) \in \mathcal{M}$ . From Theorem 17 in Section 4, we know that  $\tilde{f}(c)$  is a holomorphic function of  $c$ . By the definition of the Bers embedding,  $\Phi(\tilde{f}(c))$  is the Schwarzian derivative of  $g$ . So  $\Phi(\tilde{f}(c)) = g(c)$ .

Now we will use the lifting theorem to show that the Teichmüller metric and Kobayashi's metric of  $T(R)$  coincide.

**Lemma 29.** *Suppose  $\mathcal{M}$  is the unit ball in the space of essentially bounded Beltrami differentials on a Riemann surface  $R$ . Let  $d_K$  be the Kobayashi's metric on  $\mathcal{M}$ . Then*

$$d_K(\mu, \nu) = 2 \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \bar{\nu}\mu} \right\|_{\infty}$$

for all  $\mu$  and  $\nu$  in  $\mathcal{M}$ .

*Proof* From Lemma 27, for any  $\nu \in \mathcal{M}$ ,

$$d_K(0, \nu) = 2 \tanh^{-1} \|\nu\|_{\infty}$$

Observe the function defined by

$$\lambda \rightarrow \frac{\nu - \lambda}{1 - \bar{\nu}\lambda}$$

is a biholomorphic self map of  $\mathcal{M}$ . Therefore

$$d_K(\mu, \nu) = 2 \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \bar{\nu}\mu} \right\|_{\infty}.$$

**Theorem 37** (Gardiner [25], [26], and Royden [59]). *The Teichmüller's and Kobayashi's metrics of  $T(R)$  coincide.*

*Proof* In Lemma 28 we already showed that  $d_K \leq d_T$ . So we only need to prove  $d_K \geq d_T$ . Choose a holomorphic map  $f : \Delta \rightarrow T(R)$  so that  $f(0) = 0$  and  $f(c) = [\mu]$  for some  $c \in \Delta$ . Then the lifting theorem implies there exists a holomorphic map  $\tilde{f} : \Delta \rightarrow \mathcal{M}$  so that

$$\Phi(\tilde{f}(c)) = f(c) = [\mu].$$

So

$$d_K(0, \tilde{f}(c)) \leq \rho_\Delta(0, c).$$

By Lemma 29 and definition of Teichmüller metric,

$$d_T(0, [\mu]) \leq d_K(0, \tilde{f}(c)).$$

Therefore,

$$d_T(0, [\mu]) \leq \rho_\Delta(0, c).$$

Taking the infimum over all such  $f$ , we have

$$d_T(0, [\mu]) \leq d_K(0, [\mu]).$$

Hence  $d_T \leq d_K$ .

# Kobayashi's and Teichmüller's Metrics on $\mathcal{T}_0$

## Contents

7.1	Kobayashi's metric $\geq$ Teichmüller's metric on $\mathcal{T}_0$ . . . . .	103
7.2	Extremal maps of points in $\mathcal{T}_0$ . . . . .	105
7.3	The proof of Theorem 38, $d_K = d_T$ on $\mathcal{T}_0$ . . . . .	107
7.4	Approaching by frame mapping . . . . .	109

## 7.1 Kobayashi's metric $\geq$ Teichmüller's metric on $\mathcal{T}_0$

This chapter is a joint work with Jun Hu and Yunping Jiang. We study an interesting sub-space of  $\mathcal{T}$  which was introduced by Gardiner and Sullivan in [36]. It is the space of all symmetric orientation-preserving homeomorphisms  $g$  of the unit circle modulo the space of all Möbius transformations preserving the unit circle. Denote it by  $\mathcal{T}_0$ , it is also a complex Banach manifold modeled on another complex Banach space. So it has two natural metrics: one is the restriction of Teichmüller's metric from  $\mathcal{T}$  to  $\mathcal{T}_0$  and the other is Kobayashi's pseudo-metric. As an immediate consequence of Corollary 2 [30, pp.298], Earle, Gardiner and Lakic concluded in Theorem 1 of [19] that these two metrics coincide with each other, that is,

**Theorem 38** ([19]). *Teichmüller's metric coincides with Kobayashi's metric on  $\mathcal{T}_0$ .*

The purpose of this chapter is to give a direct proof of this theorem. The main idea of our proof is to construct a sequence of holomorphic quadratic differentials converging to a holomorphic quadratic differential  $\varphi$  for which the Beltrami coefficient of the extremal map can be expressed by  $k\frac{|\varphi|}{\varphi}$  for  $0 < k < 1$ . (see the forth paragraph in §7.4). By using the technique similar to the proof of Srebel's frame mappings theorem, we show that the limiting holomorphic quadratic differential is not identical zero. It seems to us that this method can be generalized to give a direct proof of Corollary 2 in [30, pp.298].

The space  $\mathcal{T}_0$  is a sub-manifold of  $\mathcal{T}$ . The restriction of Teichmüller's metric  $d_T$  to it is also a metric. We use  $d_{T_0}$  to denote  $d_T|_{\mathcal{T}_0}$  and  $d_{K_0}$  to denote Kobayashi's metric on  $\mathcal{T}_0$ . Proposition 12, in chapter 6, implies the following lemma,

**Lemma 30.** *For any  $\tau, \tau' \in \mathcal{T}_0$ ,*

$$d_{K_0}(\tau, \tau') \geq d_K(\tau, \tau') = d_T(\tau, \tau') = d_{T_0}(\tau, \tau').$$

And in chapter 1, we already showed the following proposition,

**Proposition 15** (Gardienr-Sullivan [36]). *If  $f$  is symmetric, then its Beurling-Ahlfors extension  $F$  is asymptotically conformal.*

More importantly, the Beltrami curve  $t\mu(F)$ ,  $t \in \Delta$ , induced by the Beltrami coefficient of  $F$ , stays in the space of asymptotically conformal maps and hence in the space of symmetric homeomorphisms. Furthermore, the space  $S$  of symmetric homeomorphism is a manifold isomorphic to  $PSL(2, \mathbb{R}) \times \mathcal{T}_0$ , that is,  $\mathcal{T}_0$  is the quotient space of  $S$  up to post-compositions by the isometries on  $\mathbb{H}$ .

In order to complete the proof of Theorem 38, it is left to show

$$d_{K_0}(\tau, \tau') \leq d_{T_0}(\tau, \tau').$$

for any  $\tau, \tau' \in \mathcal{T}_0$ .

Before we show it in Section 7.3, we prepare some background about Teichmüller maps in Section 7.2.

## 7.2 Extremal maps of points in $\mathcal{T}_0$

In our efforts of trying to prove that Kobayashi's metric is less than or equal to Teichmüller's metric on  $\mathcal{T}_0$ , we make a lot of use of the property that each point  $[f] \in \mathcal{T}_0$  has a unique extremal map of the Teichmüller form. Note that the property fails in the space  $\mathcal{T}$ . In this section, we recall why it is so. Our major reference is Reich's chapter on extremal quasiconformal mappings in [49].

Given any point  $[f] \in \mathcal{T}$ , let  $f_0$  be an extremal map in its class, an element  $f_1 \in [f]$  is called a frame mapping for  $f_0$  if there exists a compact subset  $\Omega$  in  $\Delta$  such that

$$\sup_{z \in \Delta \setminus \Omega} K(f_1)(z) < K(f_0),$$

where  $K(f_1)(z) = \frac{1+|\mu(f_1)(z)|}{1-|\mu(f_1)(z)|}$  and  $K(f_0)$  is the maximal value of  $K(f_0)(z)$  for all points  $z \in \Delta$ . By Proposition 15, we see that any non-base point  $[f]$  in  $\mathcal{T}_0$  has a frame mapping.

A sequence  $\{\varphi_n\}$  of quadratic differentials on  $\Delta$  is called a *Hamilton sequence* for  $\mu_0$  if  $\|\varphi_n\| = 1$  for each  $n$  and  $\lim_{n \rightarrow \infty} \sup \int_R \mu_0 \varphi_n dx dy = \|\mu_0\|_\infty$ .

**Theorem 39** (Hamilton-Krushkal Theorem). *Given any point  $[f] \in \mathcal{T}$ , if  $f^{\mu_0}$  is extremal in  $[f]$ , then  $\mu_0$  has a Hamilton sequence.*

**Theorem 40** (Strebel's Frame Mapping Theorem). *For any non-base point  $[f] \in \mathcal{T}$ , if an extremal map  $f_0$  for  $[f]$  has a frame mapping, then it has a unique extremal map whose Beltrami coefficient of the Teichmüller form  $\mu_0 = k_0 \frac{|\varphi_0|}{\varphi_0}$ , where  $0 < k_0 < 1$  and  $\varphi_0$  is a holomorphic quadratic differential with  $\|\varphi_0\| = 1$ .*

Now for each non-base point  $[f] \in \mathcal{T}_0$ , we obtain

**Theorem 41** (Teichmüller's Existence and Uniqueness Theorem). *Each non-base point  $[f]$  in  $\mathcal{T}_0$  has a unique extremal map whose Beltrami coefficient of the Teichmüller form.*

In the rest of this section, we prepare another proposition for the next section (see also [30] and [61]).

**Proposition 16.** *Assume that  $\{\varphi_n\}$  is a sequence of holomorphic quadratic differential on  $\Delta$  with  $\|\varphi_n\| = 1$ . Then there exists a holomorphic function  $\varphi_0$  on  $\Delta$  with  $\|\varphi_0\| \leq 1$  such that a subsequence of  $\{\varphi_n\}$  converges to  $\varphi_0$  uniformly on any compact subset of  $\Delta$ .*

*Proof* It suffices to show  $\{\varphi_n\}$  is uniformly bounded on any compact subset  $\Omega \subset \Delta$ . Suppose not, then there exists a compact subset  $\Omega$  and a sequence of points  $\{z_n\}$  in  $\Omega$  and a subsequence of  $\{\varphi_n\}$ , still denoted by  $\{\varphi_n\}$ , such that  $|\varphi_n(z_n)| \geq n$ . Since  $\Omega$  is compact,  $\{z_n\}$  has an accumulation point  $z_0$  in  $\Omega$ . Then there exists a subsequence of  $\{z_n\}$ , still denoted by  $\{z_n\}$ , such that  $z_n$  converges to  $z_0$ . Choose a small  $r > 0$  such that the closed disk  $B(z_0, r)$  centered at  $z_0$  and of radius  $r$  is contained in  $\Delta$ . Then  $z_n \in B(z_0, \frac{r}{4})$  when  $n$  is bigger than a large number  $N$ . For any  $n > N$ , one can apply the Cauchy integral formula for  $\varphi_n(z_n)$  to obtain

$$n \leq |\varphi_n(z_n)| \leq \frac{1}{2\pi} \int_{|z-z_0|=r'} \frac{|\varphi_n(z)|}{|z-z_n|} r' d\theta$$

for each  $\frac{r}{2} \leq r' \leq r$ . And then

$$n \leq \frac{1}{2\pi} \int_{|z-z_0|=r'} |\varphi_n(z)| \frac{4}{r} r d\theta = \frac{2}{\pi} \int_{|z-z_0|=r'} |\varphi_n(z)| d\theta.$$

Multiplying the previous inequality by  $r'$  and integrating both sides in radial direction from  $\frac{r}{2}$  to  $r$ , we obtain

$$\frac{3}{8}nr^2 = n \int_{\frac{r}{2}}^r r' dr' \leq \frac{2}{\pi} \int_{\frac{r}{2}}^r r' \int_{|z-z_0|=r'} |\varphi_n(z)| d\theta dr' \leq \frac{2}{\pi} \|\varphi_n\| = \frac{2}{\pi}.$$

Hence  $\frac{3}{8}nr^2 \leq \frac{2}{\pi}$  for any  $n > N$ . This is a contradiction when  $n$  is large enough. Therefore  $\{\varphi_n\}$  is uniformly bounded on any compact subset  $\Omega \subset \Delta$ . By Fatou's Lemma,  $\|\varphi_0\| \leq 1$ .

### 7.3 The proof of Theorem 38, $d_K = d_T$ on $\mathcal{T}_0$

To complete our proof of Theorem 38, it is left to show that

$$d_{K_0}(\tau, \tau') \leq d_{T_0}(\tau, \tau')$$

for any  $\tau, \tau' \in \mathcal{T}_0$ . Since the right translation map  $\alpha([f^\mu]) = [f^\mu \circ (f^\nu)^{-1}]$  is an isometry in the both Kobayashi's and Teichmüller's metrics, it suffices to show

$$d_{K_0}([0], \tau) \leq d_{T_0}([0], \tau)$$

for any point  $\tau \in \mathcal{T}_0$ . By Theorem 41,  $\tau$  has a unique extremal map  $f_0$  with Beltrami coefficient  $k_0 \frac{|\varphi_0|}{\varphi_0}$ , where  $0 < k_0 < 1$ , and  $\varphi_0$  is a holomorphic quadratic differential with  $\|\varphi_0\| = 1$ . Let  $K_0 = \frac{1}{2} \log \frac{1+k_0}{1-k_0}$ . In the meantime, by the Beurling-Ahlfors Theorem,  $\tau$  has an asymptotically conformal representative  $f$ , denote its Beltrami coefficient by  $\mu$ . Let  $K$  be the maximal complex dilatation of  $f$ .  $f_0$  is the unique extremal map in  $\tau$ , so  $1 < K_0 < K$ .

Let  $D_n$  denote the open disk centered at 0 and of radius  $1 - \frac{1}{n}$ . Then there exists a large  $N$  such that  $K(f)(z) < K_0$  for any point  $z$  in  $\Delta \setminus D_N$ .

Let  $n > N$  and  $h_n(z)$  be the restriction of  $f$  on the boundary of  $D_n$ . Observe first that the maximal complex dilation  $K_0(h_n)$  of any extremal map of  $h_n$  is greater than or equal to  $K_0$ ; otherwise the maximal dilation of the extremal map of  $\tau$  can be decreased. Hence the restriction of  $f$  to  $D_n$  is a frame map for  $h_n$ . Again by Theorem 41,  $h_n$  has a unique extremal representative  $\tilde{f}_n : D_n \rightarrow f(D_n)$  with boundary value  $h_n$  and with the Beltrami coefficient  $\tilde{\mu}_n = k_n \frac{|\varphi_n|}{\varphi_n}$ , where  $\int \int_{D_n} |\varphi_n| dx dy = 1$ .

Now we define  $f_n$  to be equal to  $\tilde{f}_n$  on  $D_n$  and  $f$  on  $\Delta \setminus D_n$ . Let  $\mu_n$  be the Beltrami coefficient of  $f_n$ , that is,  $\mu_n = \mu$  on  $\Delta - D_n$ , and  $\mu_n = \tilde{\mu}_n$  on  $D_n$ . Let  $K_n$  be the maximal dilation of  $f_n$ . Then for each  $n > N$ , (i)  $K_n > K_0$ , (ii)  $[\mu_n] = [\mu]$  and (iii)  $[t\mu_n] \in T_0$  for each  $t \in \Delta$ . The holomorphic map

$$g : \Delta \rightarrow T_0 : t \mapsto \left[ \frac{t\mu_n}{\|\mu_n\|_\infty} \right]$$

implies

$$d_{K_0}([0], \tau) \leq d_1([0], \tau) \leq \frac{1}{2} \log \frac{1 + \|\mu_n\|_\infty}{1 - \|\mu_n\|_\infty} = \frac{1}{2} \log K_n.$$

We are left to show  $K_n \rightarrow K_0$  as  $n \rightarrow \infty$  for a subsequence of  $n$ 's.

Similar to the proof of the previous Proposition 16, one can show that there exists a subsequence of  $\{\varphi_n\}$ , still denoted by  $\{\varphi_n\}$ , converging uniformly to a holomorphic function  $\varphi^*$  on any compact subset  $\Omega$  of  $\Delta$ . For any  $D_n$  with  $n > N$ ,

$$\|\varphi^*\|_{D_n} = \int \int_{D_n} |\varphi^*| dx dy \leq \lim_{n \rightarrow \infty} \int \int_{D_n} |\varphi_n| dx dy \leq 1.$$

Thus  $\|\varphi^*\| = \int \int_\Delta |\varphi^*| dx dy \leq 1$ .

In the next step, we show  $\|\varphi^*\| > 0$  by applying a special version of the Reich-Strebel main inequality and the idea to prove Strebel's Frame Mapping Theorem. Let us first recall the special version of the inequality for the maps in  $\mathcal{T}$ , which can be proved by the Grötzsch argument. If  $f_0^{\nu_0}$  is an extremal representative of the Teichmüller form for a point in  $\mathcal{T}$ , that is,  $\nu_0 = k_0 \frac{|\varphi_0|}{\varphi_0}$  for some  $0 < k_0 < 1$ , then for any  $f^\nu \in [f_0]$ ,

$$K_0 \leq \int \int_\Delta \frac{|1 + \nu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\nu|^2} |\varphi_0| dx dy.$$

Suppose  $\|\varphi^*\| = 0$ . Then  $\{\varphi_n\}$  has a subsequence converging uniformly to zero on any compact subset  $\Omega$  of  $\Delta$ , we still denote it by  $\{\varphi_n\}$ .

Now for any  $\varepsilon > 0$ , we first choose a compact subset  $\Omega$  of  $\Delta$  such that  $\|\mu\|_\infty < \varepsilon$  on  $\Delta - \Omega$ . Then there exists  $\tilde{N}$  such that  $\|\varphi_n\|_\Omega = \int \int_\Omega |\varphi_n| dx dy \leq \varepsilon$  for all  $n > \tilde{N}$  and  $\Omega \subset D_n$  for each  $n > \tilde{N}$ .

Now we assume that  $n$  is bigger than both  $N$  and  $\tilde{N}$ . Applying the previous inequality to each  $\tilde{f}_n$  and  $f$  on  $D_n$ , we obtain

$$K_n \leq \int \int_{D_n} \frac{|1 + \mu \frac{\varphi_n}{|\varphi_n|}|^2}{1 - |\mu|^2} |\varphi_n| dx dy.$$

Clearly,

$$K_n \leq \int \int_{D_n - \Omega} \frac{|1 + \mu \frac{\varphi_n}{|\varphi_n|}|^2}{1 - |\mu|^2} |\varphi_n| dx dy + \int \int_\Omega \frac{|1 + \mu \frac{\varphi_n}{|\varphi_n|}|^2}{1 - |\mu|^2} |\varphi_n| dx dy.$$

Then

$$K_n \leq \int \int_{D_n - \Omega} \frac{1 + \varepsilon}{1 - \varepsilon} |\varphi_n| dx dy + K \int \int_{\Omega} |\varphi_n| dx dy,$$

and hence

$$K_n \leq \frac{1 + \varepsilon}{1 - \varepsilon} \int \int_{D_n} |\varphi_n| dx dy + \left(K - \frac{1 + \varepsilon}{1 - \varepsilon}\right) \int \int_{\Omega} |\varphi_n| dx dy.$$

Therefore

$$1 < K_0 < K_n \leq \frac{1 + \varepsilon}{1 - \varepsilon} + \left(K - \frac{1 + \varepsilon}{1 - \varepsilon}\right) \cdot \varepsilon.$$

This is a contradiction when  $\varepsilon$  is sufficient small. Therefore  $\|\varphi^*\| > 0$ .

Now let  $\mu^* = k^* \frac{|\varphi^*|}{\varphi^*}$ , where  $k^* = \lim_{n \rightarrow \infty} k_n$  (a limit of a convergent subsequence). Then  $\mu_n \rightarrow \mu^*$  a.e. on  $\Delta$ . By the convergence theorem (see Theorem 4.6 in [?]) of families of quasiconformal maps, we know

$$\lim_{n \rightarrow \infty} f^{\mu_n}|_{\partial\Delta} = f^{\mu^*}|_{\partial\Delta} = f^{\mu}|_{\partial\Delta} = f^{\mu^*}|_{\partial\Delta}.$$

By the uniqueness of the extremal map for  $[f^{\mu_0}]$ ,  $k^* = k_0$ . Thus  $K_n \rightarrow K_0$  as  $n \rightarrow \infty$  for a subsequence of  $n$ 's. We complete the proof.

## 7.4 Approaching by frame mapping

**Definition 28.** If  $\mu$  is the Beltrami coefficient of a map  $f$ , then

$$h^*(f) = \inf_E \|\mu|_{\Delta - E}\|_{\infty}$$

where the infimum is over all compact sets  $E$  in  $\Delta$ .

Let  $h(f) = \inf \tilde{h}^*(\tilde{f})$  where the infimum is over all representative  $\tilde{f}$  of the Teichmüller class of  $f$ .

The following theorem is the theorem 35 in Chapter 15 of the book [30].

**Theorem 42.** every class in  $AT$  is represented by a Beltrami  $\mu$  such that  $h^*(\mu) = h(\mu)$

**Proposition 17.** *For every  $\tau$  in  $T$  and every  $\varepsilon > 0$ , there exist a representative  $\eta$  of  $\tau$  such that  $\|\eta\|_\infty < k_0(\tau) + \varepsilon$  and  $h^*(\eta) = h(\eta)$*

I think we also can use theorem 37 and frame mapping theorem to prove this proposition. It is like a more general result of using Beurling-Ahlfors extension and frame mapping theorem. So I think frame mapping theorem is the key point to get global smaller complex dilatation  $\mu$  and keep the boundary dilatation  $h(\mu)$ .

*Proof* For any fixed  $\tau \in T$ , there is an global extremal representative  $\mu_0$ . By theorem 37, there is also a  $\mu$  such  $h^*(\mu) = h(\mu)$ . From the definition,  $h^*(\mu) = h(\mu) \leq h^*(\mu_0) \leq k(\mu_0) = k_0$ .

If  $h^*(\mu) = h(\mu) = k_0$ , then  $\eta = \mu$ . we are done.

If  $h^*(\mu) < k_0$ , then there exist a compact set  $E$  such that  $\|\mu|_{\Delta-E}\|_\infty < k_0$ . So we use can frame mapping theorem. The unique global extremal element has the Teichmüller form  $\mu_0 = k_0 \frac{|\varphi_0|}{\varphi_0}$ .

For large enough  $n$ ,  $E$  is a subset of  $D_n$  where  $D_n$  is the disk centered at 0 and with radius  $1 - \frac{1}{n}$ .

Let  $h_n(z) = f^\mu(z)$  for  $z \in \partial D_n$ .

Let  $\alpha_E = \|\mu|_{\Delta-E}\|_\infty$ , then  $\|\mu|_{\Delta-D_n}\|_\infty \leq \alpha_E < k_0$

Then by Frame mapping theorem, there exists a unique global extremal element  $\widetilde{f_n}(z) : D_n \rightarrow f^{\mu_0}(D_n)$  with boundary value  $h_n(z)$  and with complex dilatation  $\widetilde{\mu}_n = k_n \frac{|\varphi_n|}{\varphi_n}$  where  $\int \int_{D_n} |\varphi_n| dx dy = 1$   
(note:  $|\varphi_n|$  may have pole on  $\partial D_n$ )

Let  $\mu_n(z) = \mu(z)$  if  $z \in \Delta - D_n$ , and  $\mu_n = \widetilde{\mu}_n$  if  $z \in D_n$ .

1.  $\mu_n \sim \mu$  i.e.  $\mu_n \in \tau$ .
2. It is clear  $K_n \geq K_{n+1} > K_0$ .

By normal convergence  $\varphi_n$  converges uniformly to  $\varphi^*$  on any compact subset of  $\Delta$ , so  $\varphi^*$  is holomorphic in  $\Delta$ . For any fixed  $D_N$ ,

$$\|\varphi^*\|_{D_N} = \int \int_{D_N} |\varphi^*| dx dy \leq \lim \int \int_{D_N} |\varphi_n| dx dy \leq 1,$$

then  $\|\varphi^*\| = \int \int_{\Delta} |\varphi^*| dx dy \leq 1$ .

Claim:  $\|\varphi^*\| \neq 0$ .

*Proof* :

If  $\|\varphi^*\| = 0$ , then  $\varphi_n$  convergence to zero on any compact subset of  $\Delta$  uniformly. So we can find  $N$ , such that  $E \subset D_n$  and  $\|\varphi_n\|_E = \int \int_E |\varphi_n| dx dy \leq \varepsilon$  for all  $n > N$ .

Now apply the main inequality to  $f^\mu$  and  $f^{\mu_n}$  on  $D_n$ .

Since  $f^{\mu_n}$  is the teichmuller map in  $D_n$  and since  $f^\mu \sim f^{\mu_n}$  on  $D_n$ ,

$$K_n \leq \int \int_{D_n} \frac{|1 + \mu_0 \frac{\varphi_n}{|\varphi_n|}|^2}{1 - |\mu_0|^2} |\varphi_n| dx dy.$$

Then

$$K_n \leq \int \int_{D_n - E} \frac{|1 + \mu \frac{\varphi_n}{|\varphi_n|}|^2}{1 - |\mu|^2} |\varphi_n| dx dy + \int \int_E \frac{|1 + \mu \frac{\varphi_n}{|\varphi_n|}|^2}{1 - |\mu|^2} |\varphi_n| dx dy.$$

For  $n > N$ ,

$$K_n \leq \int \int_{D_n - E} \frac{1 + \alpha_E}{1 - \alpha_E} |\varphi_n| dx dy + K(f^\mu) \int \int_E |\varphi_n| dx dy,$$

$$K_n \leq \frac{1 + \alpha_E}{1 - \alpha_E} \int \int_{D_n} |\varphi_n| dx dy + (K(f^\mu) - \frac{1 + \alpha_E}{1 - \alpha_E}) \int \int_E |\varphi_n| dx dy,$$

hence

$$K_0 < K_n \leq \frac{1 + \alpha_E}{1 - \alpha_E} + (K(f^\mu) - \frac{1 + \alpha_E}{1 - \alpha_E}) \cdot \varepsilon.$$

It is impossible for very small  $\varepsilon$ .

Since  $\varphi^*$  is not identically equal to zero, then  $\mu^* = k^* \frac{|\varphi^*|}{\varphi^*}$  is the teichmüller form where  $\lim k_n = k^*$ .

$\mu_n \rightarrow \mu^*$  a.e. , then

$$\lim f^{\mu_n}|_{\partial\Delta} = f^{\mu_n}|_{\partial\Delta} = f^{\mu_0}|_{\partial\Delta} = f^{\mu^*}|_{\partial\Delta}.$$

So  $\mu^* \in \tau$  i.e.  $\mu \sim \mu^*$ . Hence  $k_0 = k^*$  and  $\varphi_0 = \varphi^*$ . So  $k_n \rightarrow k_0$ . So we can just let  $\eta = \mu_{f_n}$ .

# Extremal Annuli on the Sphere

---

## 8.1 Extremal Length and Modulus of Annulus

An annulus in the Riemann sphere for which each complementary component contains two points a minimal distance apart can be extremal in different ways. In this chapter I want the modulus of the annulus to be as large as possible subject to geometrical constraints on the locations of the points. This part is a joint work with Fred Gardiner.

According to how one describes the constraints one can arrive at two types of annuli called Teichmüller and Mori annuli. We describe these constraints in a way similar to Ahlfors' description in [3] except that we use the chordal metric in place of the Euclidean metric. We find the minimal configurations in a new way by using variational techniques. Moreover, we show how the configurations relate to *Minsky's intersection inequality* [55, 56] and to two general principles of Teichmüller theory, namely, the *Dirichlet principle for measured foliations* [26, 32] and to the *minimal axis theorem* [35].

The Teichmüller space  $T(R)$  of a Riemann surface  $R$  measures deformations of its conformal structure. We assume  $S$  has the simplest possible, non-trivial form, namely, it is the sphere with four points removed. Points of  $T(S)$  parameterize homotopy classes of motions of four points up to postcomposition by Möbius transformations. Since Möbius transformations act transitively on triples of points, any such continuous motion can be postcomposed by a continuous curve of Möbius transformations so that three of the points remain fixed. Thus we can view  $T(S)$  as

homotopy equivalence classes of motions of one of the four points while the other three remain fixed. A small neighborhood of the terminal location of the fourth point determines a complex local coordinate for  $T(S)$ . Thus  $T(S)$  is a one dimensional complex manifold and by Teichmüller's theorem it is conformal to a disc.

The stereographic projection of the unit sphere centered at the origin in three dimensional space projects along rays from the north pole at  $(0, 0, 1)$  to points  $(x, y, 0)$  in the  $xy$ -plane. This projection projects a point  $p$  on the sphere along the line that passes through  $(0, 0, 1)$  and  $p$  to the point of intersection with the  $xy$ -plane. If  $z = x + iy$ , the chordal distance from  $p_1$  to  $p_2$  on the sphere is expressed by

$$d_C(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}.$$

In particular,

$$d_C(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

Note that for any two points  $z_1$  and  $z_2$ ,  $d_C(z_1, z_2) \leq 2$  and in terms of this metric  $z_1$  and  $z_2$  are antipodal only when  $d_C(z_1, z_2) = 2$ . The spherical distance  $d_S$  from  $z_1$  to  $z_2$ , which measures the angle of the sector on a great circle spanned by  $p_1$  and  $p_2$ , is related to the chordal distance  $d_C$  by

$$d_S(z_1, z_2) = 2 \arcsin(d_C(z_1, z_2)/2). \quad (8.1)$$

The spherical metric is the integrated form of the restriction to the sphere of the Riemannian metric  $dx^2 + dy^2 + dz^2$  in three dimensional space. Between any two points  $z_1$  and  $z_2$  with  $d_S(z_1, z_2) < \pi$  the unique geodesic joining  $z_1$  and  $z_2$  runs along the great circle that passes through these two points.

In order to formulate the extremal properties of the Mori and Teichmüller annuli, it is necessary to define the modulus of an annulus. We assume we are given two disjoint, connected, simply connected, compact and closed subsets  $E_1$  and  $E_2$  of  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . The region  $A = \bar{\mathbb{C}} \setminus (E_1 \cup E_2)$  is called an annulus and by definition its modulus  $\text{mod}(A)$  is equal to extremal length  $\Lambda(E_1, E_2)$  of the family of arcs in

$\overline{\mathbb{C}} \setminus (E_1 \cup E_2)$  that join  $E_1$  to  $E_2$ . In particular,

$$\text{mod}(A) = \Lambda(E_1, E_2) = \sup_{\rho} \frac{L(\rho)^2}{\iint_A \rho^2(z) dx dy}, \quad (8.2)$$

where the supremum is taken over all metrics  $\rho(z)|dz|$  and  $L(\rho)$  is the infimum of the arc lengths

$$\int_{\gamma} \rho(z) |dz|$$

where  $\gamma$  is any arc with initial point in  $E_1$  and terminal point in  $E_2$ .

This is a general definition that defines the modulus of a family of curves on any Riemann surface. When the family is the family of arcs that join the two boundary components of a topological annulus  $A$ , it is equivalent to the following definition. By uniformization, there is a conformal map  $f$  that carries  $A$  to a region in the complex plane bounded by two circles concentric to the origin, that is,  $f$  maps  $A$  to  $\{z : R_1 < |z| < R_2\}$ . Then the modulus of  $A$  is equal to

$$\text{mod}(A) = (1/2\pi) \ln(R_2/R_1).$$

Also, if we let  $\Lambda(A)$  be the extremal length of the family of closed curves in  $A$  that are homotopic to any curve  $\{|z| = r\}$  where  $R_1 < r < R_2$ , then  $\Lambda(A) = (\text{mod}(A))^{-1}$ .

For more properties about modulus, please read chapter 1.

Let us define an *annular configuration* in  $\overline{\mathbb{C}}$  to be three disjoint subsets  $E_1, E_2$  and  $A$  where both  $E_1$  and  $E_2$  are disjoint, connected, simply connected, compact, closed and contain at least two points and  $A = \overline{\mathbb{C}} \setminus (E_1 \cup E_2)$ . A core curve  $\alpha$  of  $A$  is any simple closed curve in  $A$  that separates its two boundary components. Given a curve  $\alpha$  separating  $z_1, z_2$  from  $z_3, z_4$  we can form a conjugate curve  $\beta_{2n}$ . It is any simple closed curve in  $\overline{\mathbb{C}} \setminus \{z_1, z_2, z_3, z_4\}$  that separates two pairs of points in  $\{z_1, z_2, z_3, z_4\}$  such that  $i(\alpha, \beta_{2n}) = 2n$ , where  $i(\alpha, \beta)$  is the smallest possible number of intersections of curves  $\tilde{\alpha}$  and  $\tilde{\beta}$  in the same homotopy classes as  $\alpha$  and  $\beta$  on  $\overline{\mathbb{C}} \setminus \{z_1, z_2, z_3, z_4\}$ .

By definition the chordal diameter of a closed set  $E$  is

$$\text{diam}(E, d_C) = \sup_{w, z \in E} d_C(w, z). \quad (8.3)$$

We now state the two extremal problems together so as to note their close similarity; we will see that they lead to different extremal annuli.

**The Mori extremal problem.** Assume  $\lambda_1$  and  $\lambda_2$  are two numbers between 0 and 2. Find an annular configuration such that

$$\text{diam}(E_1, d_C) \geq \lambda_1 \text{ and } \text{diam}(E_2, d_C) \geq \lambda_2$$

and such that  $\Lambda(E_1, E_2)$  is as large as possible.

**The Teichmüller extremal problem.** Assume  $z_1, z_2, z_3$  and  $z_4$  are four points such that  $z_1$  and  $z_2$  lie in  $E_1$ ,  $z_3$  and  $z_4$  lie in  $E_2$ , and  $d_C(z_1, z_2) \geq \lambda_1$ ,  $d_C(z_3, z_4) \geq \lambda_2$  and three of these four points lie on a great circle. Among all annular configurations  $(E_1, A, E_2)$  with these properties, find one such that  $\Lambda(E_1, E_2)$  as large as possible.

The following are the first two theorems of this chapter. Unlike the formulations given in [3], the constraints here are expressed in terms of the chordal metric.

**Theorem 43 (The Mori annulus).** *The Mori problem has a solution. Up to spherical isometry it is unique and takes the form  $E_1 = [-ib, ib]$ ,  $E_2 = [-\infty, a] \cup [a, \infty]$  where  $d_{\overline{C}}(-ib, ib) = \lambda_2$  and  $d_{\overline{C}}(-a, a) = \lambda_1$ .*

We also have a parallel statement for the Teichmüller annulus.

**Theorem 44 (The Teichmüller annulus).** *The Teichmüller problem has a solution. Up to spherical isometry it is unique and takes the form  $E_1 = [-a, a]$ ,  $E_2 = [b, \infty) \cup \{\infty\} \cup (-\infty, -b]$ , where  $a < b$  are positive numbers chosen so that*

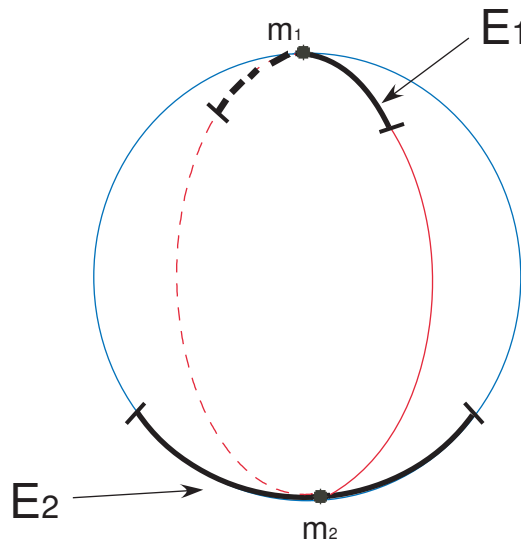
$$\lambda_1 = d_C(-a, a) \text{ and } \lambda_2 = d_C(-b, b).$$

For the proofs we will use two general principles of Teichmüller theory, namely, the *Dirichlet principle* [26, 27] and the *minimal axis theorem* [35]. These principles enable us to view Teichmüller and Mori annuli as special cases of one dimensional families of minimal annuli corresponding to pairs of transversely realizable cylindrical differentials on the four times punctured sphere. In particular we prove the following two results.

**Theorem 45 (The Mori minimal axis).** *Let four points  $z_1, z_2, z_3$  and  $z_4$  in  $\bar{\mathbb{C}}$  be given and let  $|du|$  and  $|dv|$  be measured foliations corresponding to simple closed curves  $\alpha$  and  $\beta$  both of which separate  $\{z_1, z_2\}$  from  $\{z_3, z_4\}$  and such that  $i(\alpha, \beta) = 4$ . Then up to pull back by Möbius transformations the minimal Mori axis corresponds to the Mori quadratic differentials:*

$$q(z)(dz)^2 = \frac{(dz)^2}{(z - ib)(z + ib)(z - a)(z + a)},$$

where  $a$  and  $b$  are positive numbers.



$m_1$  and  $m_2$  are midpoints of  $E_1$  and  $E_2$ .

Figure 8.1: Mori's annulus

**Theorem 46 (The Teichmüller minimal axis).** *With the same notation let  $|du|$  and  $|dv|$  be measured foliations corresponding to simple closed curves  $\alpha$  and  $\beta$  where  $\alpha$  separates  $\{z_1, z_2\}$  from  $\{z_3, z_4\}$ ,  $\beta$  separates  $\{z_2, z_3\}$  from  $\{z_1, z_4\}$  and  $i(\alpha, \beta) = 2$ . Then up to pull back by Möbius transformations the minimal Teichmüller axis corresponds to the Teichmüller quadratic differentials:*

$$q(z)(dz)^2 = \frac{(dz)^2}{(z - b)(z + b)(z - a)(z + a)},$$

where  $a$  and  $b$  are positive numbers with  $a < b$ . (see Figure 8.2)

We will also show how these extremal problems are special cases of extremal problems for pairs of conjugate extremal annular configurations associated with essential simple closed curves  $\alpha$  and  $\beta_{2n}$  where the homotopy type of  $\beta_{2n}$  is determined by its intersection number with  $\alpha$ , namely,  $i(\alpha, \beta_{2n}) = 2n$  on the four times punctured sphere. The cases  $n = 1$  and  $n = 2$  correspond to the Teichmüller and Mori annuli.

In the final section we use the same techniques to describe a modified Mori problem on the Riemann sphere for continua  $E_1$  and  $E_2$  that contain regular polygons of a given size.

## 8.2 The intersection inequality

In this section we prove Minsky's intersection inequality. First, we need one more definition. A closed curve is called essential if it is not homotopic to a point and not homotopic to a puncture of any Riemann surface  $R$ .

**Theorem 47. (Minsky's intersection inequality [55])** *Suppose  $\alpha$  and  $\beta$  are essential simple closed curves on any Riemann surface  $R$ . Then*

$$\Lambda(\alpha)\Lambda(\beta) \geq i(\alpha, \beta)^2.$$

*Proof* By the Dirichlet principle for measured foliations there is a unique quadratic differential  $q_\alpha$  holomorphic on  $R$  such that

a) all regular horizontal trajectories of  $q_\alpha$  are closed curves in the homotopy class of  $\alpha$ ,

b)  $\iint_R |q_\alpha| dx dy = 1$ , and

c)  $\Lambda(\alpha) = L(|q_\alpha|^{1/2})^2$ .

In particular, the metric  $|q_\alpha|^{1/2}$  realizes the maximum in the definition of the ex-

tremal length of the class  $\alpha$ , that is,

$$\Lambda(\alpha) = \frac{L(|q_\alpha|^{1/2})^2}{\int \int_R |q_\alpha| dx dy}.$$

Furthermore, any curve  $\beta$  with  $i(\alpha, \beta) = n$  must cross  $n$  times the cylinder determined by  $q_\alpha$ . If we assume this cylinder has height  $b$  and width  $a$ , then  $\Lambda(\alpha) = a^2/ab = a/b$ , and by plugging the same metric into the definition of the extremal length  $\Lambda(\beta)$  we obtain

$$\Lambda(\beta) \geq \frac{n^2 b^2}{ab} = n^2 \frac{b}{a}.$$

Thus,

$$\Lambda(\alpha)\Lambda(\beta) \geq n^2. \tag{8.4}$$

Note that the only way we could have equality in (8.4) is by having all of the regular horizontal trajectories of  $q_\beta$  intersect the regular horizontal trajectories of  $q_\alpha$  at right angles and by having  $q_\alpha$  equal to  $-q_\beta$ . If the Riemann surface is planar, by the Jordan curve theorem two homotopy classes of simple closed curve can intersect only an even number of times, so in this case the value of  $n$  must be even.

### 8.3 The minimal axis theorem

In this section we shall assume we are given two measured foliations  $|du|$  and  $|dv|$  on a Riemann surface  $R$  of finite analytic type. For a definition of measured foliation see [24] or [26]. In our notation measured foliation  $|du|$  is made up of a family of  $C^1$ -real valued functions  $u_j$  each associated to an open subset  $U_j$  of  $R$ . If two of these subsets  $U_j$  and  $U_k$  intersect then on the overlap  $U_j \cap U_k$  there is a constant  $c_{jk}$  such that

$$u_j = \pm u_k + c_{jk}.$$

Moreover, the level sets

$$u = \text{constant}$$

are well defined on the union  $U = \bigcup_j U_j$  and determine continuous curves. For smooth curves  $\gamma$  contained in  $R$  we can form the line integrals

$$\int_{\gamma \cap U} |du|.$$

It is assumed that the union  $U = \bigcup_j U_j$  covers  $R$  except for a finite number punctures which can be points the level curves  $u_j = \text{constant}$  have singularities.

In any case the heights of  $|du|$  along homotopy classes of closed curves contained in  $R$  are defined in the following way. For any particular smooth closed curve  $\gamma$  we define

$$ht(\gamma, |du|) = \int_{\gamma} |du|,$$

and for the free homotopy class  $[\gamma]$  of  $\gamma$ , we define

$$ht([\gamma], |du|) = \inf\{ht(\tilde{\gamma}, |du|)\},$$

where the infimum is taken over all  $\tilde{\gamma}$  in the same free homotopy class as  $\gamma$ .

We let  $\mathcal{S}$  denote the set of all essential simple closed curves on  $R$ . By definition a curve is essential if it is not homotopic to point and not homotopic to a puncture. By the correspondence

$$|du| \mapsto ([\gamma] \mapsto ht([\gamma], |du|))$$

the measured foliation  $|du|$  determines an element of the product space  $\mathbb{R}_+^{\mathcal{S}}$ . We say two measured foliations are height equivalent if they have the same image under this map.

In addition to its vector of heights, any measured foliation also has a Dirichlet norm. Because we are assuming the real valued functions  $u_j$  have continuous first partial derivatives, and because we are assuming  $R$  has a Riemann surface structure  $R_{\tau}$ , there is a star operator and so any measured foliation  $|du|$  has a well defined Dirichlet integral

$$\begin{aligned} Dir(|du|) &= \int \int_{R_{\tau}} du \wedge *du = \int \int_{R_{\tau}} (u_x dx + u_y dy) \wedge (-u_y dx + u_x dy) \\ &= \int \int_{R_{\tau}} (u_x^2 + u_y^2) dx dy. \end{aligned}$$

**Definition 29.**  $M_\tau(|du|)$  is the infimum of Dirichlet integrals  $\int \int_{R_\tau} (\tilde{u}_x^2 + \tilde{u}_y^2) dx dy$  where the infimum is taken over all  $|\tilde{d}\tilde{u}|$  in the same height equivalence class.

**Theorem 48.** [26, 27, 32] The Dirichlet norm  $M_\tau(|du|)$  of a height equivalence class on a Riemann surface  $R_\tau$  of finite analytic type is uniquely realizable by a measured foliation given by the horizontal trajectories and vertical measure of a holomorphic quadratic differential  $q$ .  $M_\tau(|du|)$  is differentiable and its derivative is given by

$$\log M_{t\mu}(|du|) = \log M_0(|du|) + \frac{2\operatorname{Re} t}{\|q\|} \int \int \mu q dx dy + o(t).$$

For the proof of this theorem, please read section 1.9.

**Definition 30.** We say two measured foliations  $|du|$  and  $|dv|$  are transversal if the following conditions are satisfied.

1. Away from singular points their horizontal leaves are transversal.
2. At singular points both  $|du|$  and  $|dv|$  have  $k$ -pronged singularities for the same value of  $k$  and the prongs are transversal.

**Definition 31.** Two measured foliations  $|du|$  and  $|dv|$  on a surface  $R$  satisfy the intersection hypothesis if there is a constant  $k > 0$  such that for every essential simple closed curve  $\gamma$  on  $R$ ,

$$\max\{ht(\gamma, |du|), ht(\gamma, |dv|)\} \geq k.$$

**Theorem 49. (Condition for transversality)** Two measured foliations on a surface of finite analytic type satisfying the intersection hypothesis are transversely realizable in their height equivalence classes.

*Proof* This theorem is a consequence of the following theorem which shows that there is a Riemann surface  $R_\tau$  on which the two height equivalence classes can be realized as the real and imaginary parts of the square root of a holomorphic quadratic differential on  $R_\tau$ .

**Theorem 50. (The minimal axis)** *Given any pair of measured foliations  $|du|$  and  $|dv|$  satisfying the intersection hypothesis, there is a unique Teichmüller line in the  $T(R)$  along which the product*

$$M_\tau(|du|)M_\tau(|dv|)$$

*is minimum. There is a point  $\tau_0$  on this line and a holomorphic quadratic differential  $q$  on  $R_{\tau_0}$  such that  $|du|$  and  $|dv|$  are height equivalent to the absolute value of the real and imaginary parts of  $\sqrt{q}$ . This minimal axis is spanned by the Beltrami line  $t\frac{|q|}{q}$  for  $-1 < t < 1$ .*

*Proof* This theorem is proved in [35]. Also see [32].

**Corollary 2.** *With the same hypotheses, if one holds  $M_\tau(|dv|)$  fixed there is a unique point  $\tau_0$  on the minimal axis for which  $M_{\tau_0}(|du|)$  is minimum.*

*Proof* The holomorphic quadratic  $q$  differential whose horizontal and vertical trajectories realize the height equivalence classes of  $|du|$  and  $|dv|$  at any point on the minimizing line is generated by the Beltrami line  $t\frac{\bar{q}}{|q|}$ ,  $-1 < t < 1$ , the product  $M_\tau(|du|)M_\tau(|dv|)$  is constant along this line and passes through the point  $\tau_0$ .

In the special case where the Riemann surface  $S_\tau$  is the four times punctured Riemann sphere, the space of holomorphic quadratic differentials is one dimensional. Any non zero differential in this space must have a one pronged singularity at each of the four punctures.

**Theorem 51.** *Let  $A_1$  and  $A_2$  be a pair of annuli on the four times punctured sphere  $S$  with essential, non homotopic core curves  $\alpha_1$  and  $\alpha_2$ . For a given complex structure  $\tau$  on  $S$  and a given essential closed curve  $\alpha$  let  $\text{mod}_\tau(A)$  be the maximal modulus of an annulus with core curve homotopic to  $\alpha$ . Then the locus of points  $\tau$  for which  $M_\tau(A_1)M_\tau(A_2)$  is minimum forms a Teichmüller line in  $T(S)$ .*

*Proof* Suppose two measured foliations  $|du|$  and  $|dv|$  are in the same measure class as two essential simple closed curves  $\alpha_1$  and  $\alpha_2$  on  $S$ . Then it is obvious that any

other essential simple closed curve on  $S$  must cross either  $\alpha_1$  or  $\alpha_2$  at least twice. Thus  $|du|$  and  $|dv|$  satisfy the intersection property with  $k = 2$ .

## 8.4 The Teichmüller annulus

In this section we prove Theorems 44 and 46. Recall that  $E_1, A, E_2$  is an annular configuration with  $E_1$  containing  $z_1$  and  $z_2$  and  $E_2$  containing  $z_3$  and  $z_4$ . Also assume  $d_C(z_1, z_2) \geq \lambda_1$  and  $d_C(z_3, z_4) \geq \lambda_2$  and three of the points  $z_1, z_2$ , and  $z_4$  lie on the same great circle. We begin with an existence lemma.

**Lemma 31.** *There exists an annular configuration  $E_1, A, E_2$  with four points  $z_1, z_2$  in  $E_1$  and  $z_3, z_4$  in  $E_2$  satisfying the conditions described above with*

$$d_C(z_1, z_2) = \lambda_1 \text{ and } d_C(z_3, z_4) = \lambda_2$$

*and with  $\Lambda(E_1, E_2)$  as large as possible.*

*Proof* We take as a standard annulus  $\Delta \setminus \Delta_\varepsilon$  where  $\Delta$  is the unit disc and  $\Delta_\varepsilon$  is the subdisc with the same center and radius  $\varepsilon < 1$ . For each annular configuration  $E_1, A, E_2$  satisfying the given conditions we form a univalent holomorphic function  $f_\varepsilon$  that maps  $\Delta \setminus \Delta_\varepsilon$  onto  $A$  where

$$\frac{2\pi}{\ln(1/\varepsilon)} = \Lambda(A).$$

The family  $\mathcal{F}$  is a normal family and the numbers  $\varepsilon$  satisfying these conditions have a positive greatest lower bound  $\varepsilon_0$ . Since  $\mathcal{F}$  is a normal family, the sequence  $f_{\varepsilon_n}$  with  $\varepsilon_n$  decreasing to  $\varepsilon_0$  will have a subsequence converging to a univalent function  $f_{\varepsilon_0}$  defined on  $\Delta \setminus \Delta_{\varepsilon_0}$  for which the annulus  $A = f_{\varepsilon_0}(\Delta \setminus \Delta_{\varepsilon_0})$  has maximum modulus among all annuli with two complementary components satisfying the described conditions.

**Lemma 32.** *Suppose  $E_1, A, E_2$  is an annular configuration with the property that  $\Lambda(\alpha)$  is as small as possible subject to the conditions described in the previous lemma. Then all four points  $z_1, z_2, z_3, z_4$  must lie on the same great circle.*

*Proof* We take a Möbius transformation that moves  $z_1$  to a point on the negative real axis and  $z_2$  to 0 so that  $d_C(z_1, z_2) = \lambda_1$ . Since by hypothesis, three of the four points lie on the same great circle, we are still free to move the point  $z_4$  to  $\infty$  with the same Möbius transformation that preserves the real axis. The condition that  $d_C(z_3, \infty) \geq \lambda_2$  is equivalent to

$$\frac{2}{\sqrt{1 + |z_3|^2}} \geq \lambda_2,$$

which implies

$$|z_3| \leq \left(\frac{2}{\lambda_2}\right)^2 - 1.$$

By the extremal property for the Teichmüller annulus [2, 3], making  $\Lambda(E_1, E_2)$  as large as possible (which makes  $\Lambda(\alpha)$  as small as possible) forces  $z_2$  to be a positive real number. In particular, all four numbers  $z_1, z_2, z_3, z_4$  lie on the extended real axis, which is a great circle.

Our goal now is to give a different proof of Lemma 32 which depends on understanding the variation of the extremal length  $\Lambda_t(\alpha)$  along a locus of points where the values of  $\lambda_1$  and  $\lambda_2$  do not change. For this purpose we pick  $z_3 = -1, z_2 = 0$  and  $z_4 = \infty$  and  $z_1(t) = Re^{it}$  where  $0 < 2\pi$ . We let  $\Lambda_t(\alpha)$  be the extremal length of the family  $\alpha$  of simple closed curves that are homotopic in  $\overline{\mathbb{C}} \setminus \{z_1, z_2, z_3, z_4\}$  to a curve that contains the interval  $[-1, 0]$  in its interior and  $z_3$  and  $z_4$  in its exterior.

**Theorem 52.**  $\Lambda_t(\alpha)$  is a continuous periodic function of  $t$ , monotone increasing for  $0 < t < \pi$ , monotone decreasing for  $\pi < t < 2\pi$ , attaining its maximum at  $t = \pi$  and its minimum at  $t = 0$ .

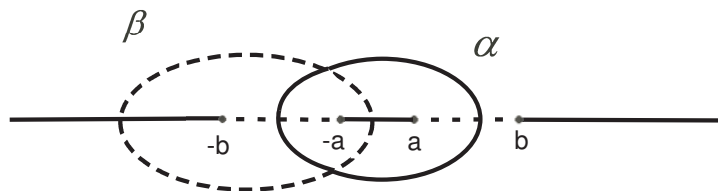


Figure 8.2: Teichmüller minimal axis

*Proof* The Weierstrass  $\mathcal{P}$ -function

$$w = \mathcal{P}_\tau(\zeta) = \frac{1}{\zeta^2} + \sum \left( \frac{1}{(\zeta - (m + n\tau))^2} - \frac{1}{(m + n\tau)^2} \right),$$

where the sum is over all integers  $(m, n) \in \mathbb{Z} \times \mathbb{Z}$  with  $(m, n) \neq (0, 0)$ , induces a two-to-one map from the period parallelogram with vertices at  $0, 1, \tau$  and  $\tau + 1$  onto the Riemann sphere with four branch points at the half periods and branch values at  $e_1, e_2, e_3$  and  $\infty$ .

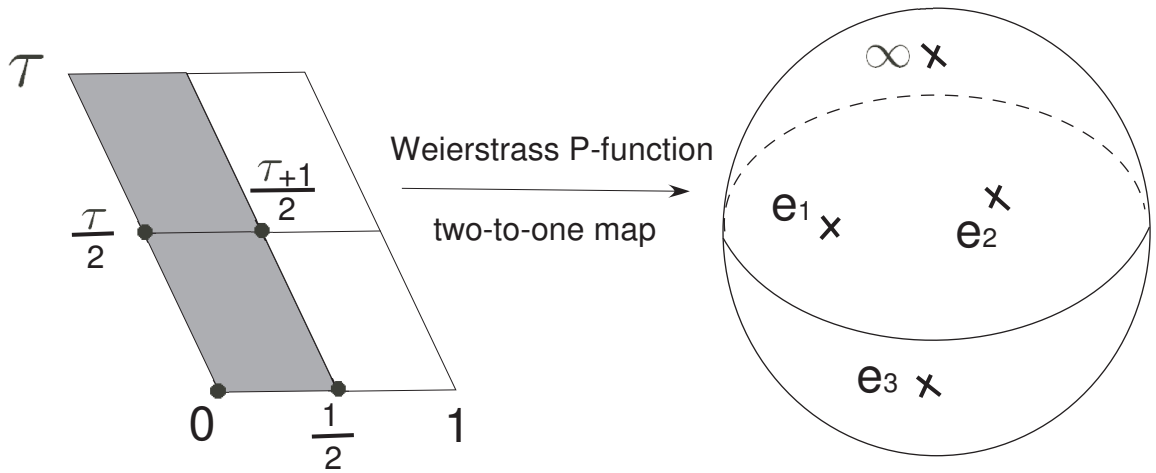


Figure 8.3: Weierstrass P function

$\mathcal{P}(\zeta)$  maps the interior of the half parallelogram with vertices at  $0, 1/2, \frac{1+\tau}{2}, \frac{\tau}{2}$  one-to-one to the sphere taking the vertices to four points  $\infty, e_1, e_3, e_2$  and taking the quadratic differential  $4(d\zeta)^2$  to the quadratic differential to the quadratic differential

$$4(d\zeta)^2 = q_\alpha^w(w)(dw)^2 = \frac{(dw)^2}{(w - e_1)(w - e_2)(w - e_3)}, \tag{8.5}$$

because of the identity

$$\mathcal{P}'^2(\zeta) = 4(\mathcal{P}(\zeta) - e_1)(\mathcal{P}(\zeta) - e_2)(\mathcal{P}(\zeta) - e_3).$$

The closed regular horizontal trajectories of  $(d\zeta)^2$  are mapped by  $\zeta \mapsto w$  to closed horizontal trajectories of  $q^w(w)(dw)^2$  on  $\overline{\mathbb{C}} \setminus \{e_1, e_2, e_3, \infty\}$  which are homotopic to

$\alpha$ . In particular, in the  $\zeta = \xi + i\eta$ -plane the closed regular trajectory  $\alpha$  is realized by any horizontal line segment that joins the left and right sides of period parallelogram with constant  $\eta$  between zero and  $\text{Im}\tau/2$ . In the  $w$ -plane  $\alpha$  is a simple closed curve that separates  $e_2$  and  $e_3$  from  $e_1$  and  $\infty$ .

The variation formula for the modulus of an annulus is

$$\log \Lambda_t(\alpha) = \Lambda_{t_0}(\alpha) + 2\text{Re} (t - t_0) \frac{1}{\|q_\alpha\|} \iint \mu_{t_0} q_\alpha dudv + o(t - t_0), \quad (8.6)$$

where  $q_\alpha$  is a holomorphic quadratic differential on the surface whose regular horizontal trajectories are closed and homotopic to  $\alpha$  and  $\mu_t$  is a Beltrami differential which expresses infinitesimally the motion of the point  $p_t = Re^{it}$ .

Now we change the coordinate on  $\mathbb{C}$  by the transformation  $w = (e_3 - e_2)z + e_3$ . It transforms  $(w_2, w_3, w_1, w_4) = (e_2, e_3, e_1, \infty)$  to

$$(z_2, z_3, z_1, z_4) = (-1, 0, \frac{e_1 - e_3}{e_3 - e_2}, \infty)$$

and the quadratic differential  $4(d\zeta)^2$  to

$$\frac{(dw)^2}{(w - e_1)(w - e_2)(w - e_3)} = \frac{(dz)^2}{(e_3 - e_2)(z + 1)z(z - z_1)}.$$

We wish to look at the first variation in the extremal length  $\Lambda_t(\alpha)$  along the curve  $z_1(t) = p(t) = R_0 e^{it}$ . Since  $z_2, z_3$  and  $z_4$  are fixed, the tangent vector  $V \frac{\partial}{\partial z}$  to the curve  $t \mapsto z_1(t)$  is represented by  $\mu_t = \bar{\partial}V$  where  $\|\mu_t\| < \infty$  and where  $(V(z_1), V(z_2), V(z_3), V(z_4)) = (iz_1, 0, 0, 0)$ . Therefore

$$\iint \mu_t q_\alpha dudv = \iint \bar{\partial}V^z \frac{d\bar{z}dz}{2i(e_3 - e_2)(z + 1)z(z - z_1)} = \quad (8.7)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{i}{2} \int_{|z-p|=\varepsilon} V^z(z_1 + \varepsilon e^{i\theta}) \frac{1}{(e_3 - e_2)(z_1 + 1)z_1(z - z_1)} dz, \quad (8.8)$$

where the line integral in (8.8) is taken in the counterclockwise direction. As  $\varepsilon \rightarrow 0$ ,  $V(z_1 + \varepsilon e^{i\theta})$  approaches  $iz_1$  and  $dz = i\varepsilon e^{i\theta} dt$  and  $(z - p) = \varepsilon e^{i\theta}$ . Thus, since

$$z_1 = \frac{e_1 - e_3}{e_3 - e_2},$$

$$z_1 + 1 = \frac{e_1 - e_2}{e_3 - e_2},$$

and (8.8) is equal to

$$\frac{-2\pi i}{2(e_3 - e_2)(z_1 + 1)} = \frac{-i\pi}{(e_1 - e_2)} = -i\pi \frac{\overline{e_1 - e_2}}{|e_1 - e_2|^2}. \quad (8.9)$$

Since the first variation must vanish at an extremal value, that means  $e_1 - e_2$  must be real valued. By carrying out the same calculation except for normalizing so that  $e_2$  and  $e_3$  correspond to the points 0 and  $-1$ , respectively, we find that  $e_1 - e_3$  must also be real valued. Since  $e_1 + e_2 + e_3 = 0$ , we see that  $(e_2 - e_1) + (e_3 - e_1) - (e_1 + e_2 + e_3) = -3e_1$  is real valued, which in turn implies that all three of the points  $e_1, e_2$  and  $e_3$  lie on the real axis.

We conclude there can only be two critical points on the circle  $Re^{it}$ , which occur when  $t = 0$  and  $\pi$ . Since  $z = (w - e_3)/(e_3 - e_2)$ ,  $z_1, z_2$  and  $z_3$  are also real valued and these values must occur at a maximum and a minimum. It is obvious that for  $R > 1$ ,  $\Lambda([-1, 0], [R, \infty]) > \Lambda([0, 1], [R, \infty])$ , so the maximum occurs when  $t = \pi$  and the minimum when  $t = 0$ .

**Lemma 33.** *Suppose  $0 < a < 1 < b$ ,  $E_1 = \overline{\mathbb{R}} \setminus [-b, b]$  and  $E_2 = [-a, a]$ . Let  $T$  be a Möbius transformation that fixes the real axis and the points  $-b$  and  $b$ . Then*

$$d_C(T(-a), T(a)) < d_C(-a, a)$$

*unless  $T$  is the identity.*

*Proof*  $T$  is an isometry for the hyperbolic metric on the disc of radius  $b$  centered at the origin. Since the segment  $[-a, a]$  is symmetrically placed about the origin, this implies  $|T(-a) - T(a)| < 2a$  unless  $T$  is the identity. But for line segments situated on lines that pass through the origin, chordal length is a monotone function of Euclidean length. In particular if the Euclidean length  $\ell_E$  and the chordal length  $\ell_C$  are related by

$$\ell_C = \frac{2\ell_E}{1 + |\ell_E/2|^2}.$$

The lemma follows since

$$\ell_E([T(-a), T(a)]) < \ell_E([-a, a]).$$

To begin the proof of Theorem 44, by a spherical isometry we normalize the three points  $z_1, z_2$  and  $z_4$  which lie on a great circle so that  $z_1$  and  $z_2$  lie on the real axis and  $z_4 = \infty$ . With the same normalization we can make  $z_1 = -b, z_2 = b$  where  $b$  is determined by the condition that  $\lambda_1 = d_C(z_1, z_2) = \frac{4b}{1+b^2}$ , with  $b > 1$ . From the hypothesis we can also assume that  $w_3$  lies on the real axis between  $-b$  and  $b$  with  $d_C(z_3, w_3) = \lambda_2$ . If  $\alpha$  is the homotopy class of simple closed curves on  $\overline{\mathbb{C}} \setminus ((-\infty, -b] \cup [b, \infty))$  with winding number  $+1$  around both points  $z_3$  and  $w_3$ , with the property that subject to these conditions  $\Lambda(\alpha)$  is as small as possible, then by Lemma 32,  $z_3$  must also be real. By Lemma 33 the points  $z_3$  and  $w_3$  must be situated symmetrically at  $-a$  and  $a$ , and this completes the proof of Theorem 44.

The next theorem shows that the family of Teichmüller annuli comprise a minimal axis for a pair of measured foliations. It contains Theorem 46 as a corollary.

**Theorem 53.** *Let  $S$  be the Riemann sphere with four points removed and  $\alpha$  and  $\beta$  be two essential simple closed curves on  $S$  with  $i(\alpha, \beta) = 2$ . Then*

$$\Lambda_\tau(\alpha)\Lambda_\tau(\beta) \geq 4 \quad (8.10)$$

and this product is equal to 4 along a unique Teichmüller line in  $T(S)$ . Up to pull back by a Möbius transformation the line is described by the locus of Teichmüller extremal annuli. One such point on this line corresponds to a surface conformal to  $\overline{\mathbb{C}} \setminus \{-b, -a, a, b\}$ , where  $0 < a < b$ , and all other points lie on this line are generated by the Beltrami coefficient  $t\frac{\bar{q}}{q}$  where  $-1 < t < 1$  and

$$q = \frac{(dz)^2}{(z+b)(z+a)(z-a)(z-b)}. \quad (8.11)$$

*Proof* Theorem 47 implies  $\Lambda_\tau(\alpha)\Lambda_\tau(\beta) \geq 4$  and Theorem 50 implies that this inequality is strict unless  $\tau$  lies along a unique line where this product is minimum. The line is generated by a Teichmüller Beltrami coefficient with quadratic differential  $q$ , such that the regular horizontal trajectories of  $q$  are homotopic to  $\alpha$  and the regular vertical trajectories of  $q$  are homotopic to  $\beta$ .

It is elementary to exhibit one Teichmüller line that has these properties. We let  $S = \overline{\mathbb{C}} \setminus \{-b, -a, a, b\}$  with  $0 < a < b$ ,  $\alpha$  be a simple closed curve that surrounds the interval  $[-a, a]$  and leaves  $b$  and  $-b$  in its exterior, and  $\beta$  be a simple closed curve that surrounds  $[a, b]$  and leaves  $-b$  and  $-a$  in its exterior. Note that these two curves satisfy  $i(\alpha, \beta) = 2$ . Up to conformal equivalence the Teichmüller stretch with Beltrami coefficient  $t \frac{\bar{q}}{q}$  where  $-1 < t < 1$  and  $q$  is given by (8.11) deforms  $S$  to  $S_t = \overline{\mathbb{C}} \setminus \{-b(t), -a(t), a(t), b(t)\}$  where  $0 < a(t) < b(t)$ .

By a conformal map  $S$  is mapped to rectangle  $S_0 = \{z = x + iy : 0 < x < 1, 0 < y < 2B\}$  such that  $S$  is reconstructed from  $S_0$  by certain side identifications. The bottom of  $S_0$  is identified with the top by the translation  $x \mapsto x + 2iB$ . The left hand vertical side is identified with itself by the rotation  $z \mapsto -z + 2iB$  and the right hand vertical side is also identified with itself by the rotation  $z \mapsto -z + 2iB + 2$ . Under this identification the curve made up of two horizontal segments  $[iy, iy + 1]$  and  $[2 - iy, 2 - iy + 1]$  forms a closed curve in the homotopy class of  $\alpha$  and the curve made up of the vertical segment  $[x, x + 2Bi]$  forms a closed curve in the homotopy class of  $\beta$ . In this presentation  $\Lambda_\tau(\alpha) = \frac{2^2}{2B}$  and  $\Lambda_\tau(\beta) = \frac{(2B)^2}{2B}$  and so  $\Lambda_\tau(\alpha)\Lambda_\tau(\beta) = 4$ .

## 8.5 The Mori annulus

In this section we prove Theorems 43 and 45 and also show that the Mori locus coincides with the minimal axis where  $\Lambda_\tau(\alpha)\Lambda_\tau(\beta) = 16$  when  $i(\alpha, \beta) = 4$ . For the proof of Theorem 43 we will use Möbius transformations that leave invariant extremal length problems and a special subclass of these that correspond to isometries of the sphere, which are isometries both with respect to the spherical metric and the chordal metric.

Since  $\Lambda(A) = \Lambda(E_1, E_2)^{-1}$ , we can use Lemma 31 to show there is an annular configuration  $E_1, A, E_2$  satisfying the conditions  $\text{diam}(E_1, d_C) = \lambda_1$  and  $\text{diam}(E_2, d_C) = \lambda_2$  for which  $\Lambda(E_1, E_2)$  is as large as possible and this configuration

makes  $\Lambda(A)$  as small possible. Let  $A_0$  be the annulus of such a minimizing configuration and  $\alpha$  be the homotopy class of a core curve of  $A_0$  in  $S_{\tau_0} = \overline{\mathbb{C}} \setminus \{z_1, z_2, z_3, z_4\}$ . Clearly,  $\Lambda_\tau(\alpha) \leq \Lambda_\tau(A_0)$  and so the ring domain for the quadratic differential that realizes the solution to the extremal problem for  $\Lambda(\alpha)$  contains  $\{z_1, z_2\}$  in one of its complementary components and  $\{z_3, z_4\}$  in its other complementary component. Moreover these four points are simple poles of the corresponding quadratic differential.

We begin by taking an extremal domain for the Mori problem with  $\lambda_1 = 2$  with  $z_2$  and  $z_3$  equal to  $i$  and  $-i$ ,  $z_1(t)$  equal to  $Re^{it}$  with  $R > 1$  and  $z_4 = \infty$ . Now we use the affine transformation

$$w = \frac{e_2 - e_3}{2i}z + \frac{e_2 + e_3}{2}. \quad (8.12)$$

The first variation in the formula (8.6) is

$$2\operatorname{Re} \frac{1}{\|q\|} \int \bar{\partial} V_{z_1}^z q^z \frac{dz}{2i} = 2\operatorname{Re} \left( \frac{1}{\|q\|} \int \bar{\partial} V_{e_1}^w q^w \frac{dw}{2i} = \right. \quad (8.13)$$

$$\left. -\pi \text{ times the residue at } e_1 \text{ of } V^w q^w. \right.$$

But  $V^w(e_1) = V^z(z_1) \frac{dw}{dz} = z_1 \frac{e_2 - e_3}{2}$  and also since  $e_1 + e_2 + e_3 = 0$ ,

$$z_1 = i \frac{2e_1 - (e_2 + e_3)}{e_2 - e_3} = \frac{3ie_1}{e_2 - e_3},$$

and we obtain  $V^w(e_1) = \frac{3}{2}ie_1$ . Therefore the residue of  $V^w q^w dw$  at  $e_1$  is equal to

$$\frac{3ie_1}{2(e_1 - e_2)(e_1 - e_3)},$$

and on keeping track of the three factors of  $i$  that enter into the calculation of the first variation, one finds that (8.6) is equal to zero precisely if the imaginary part of the fraction

$$\frac{e_1}{(e_1 - e_2)(e_1 - e_3)}, \quad (8.14)$$

is equal to zero, that is, precisely if

$$\frac{e_1}{(e_1 - e_2)(e_1 - e_3)} \text{ is real valued.} \quad (8.15)$$

We also have the trace condition, namely,

$$e_1 + e_2 + e_3 = 0. \quad (8.16)$$

Note that the conditions (8.15) and (8.16) are invariant under the two reflections  $j_1(z) = \bar{z}$  and  $j_2(z) = -\bar{z}$ . By applying the reflection  $j_2(z)$  if necessary, we may assume  $\operatorname{Re} e_1 \geq 0$  and that  $-\pi/2 \leq \arg e_1 \leq \pi/2$ .

From condition (8.16) the points  $0, v_3 = e_1 - e_3, v_2 = e_1 - e_2$  and  $v_1 = 3e_1$  are the vertices of a parallelogram and condition (8.15) implies that

$$\arg v_3 + \arg v_2 - \arg v_1 = 0.$$

In the case where  $\arg v_1 \geq 0$ , we rewrite this equation as

$$\arg v_3 - \arg v_1 = -\arg v_2$$

and we see that the angle between  $v_1$  and  $v_3$  is equal to the angle between  $v_2$  and the positive real axis. This implies that the angle between  $v_2$  and  $v_1$  is larger than or equal to the angle between  $v_1$  and  $v_3$ . By inspecting the triangle with vertices at  $0, v_2$  and  $3v_1$ , and observing that the side opposite the larger angle is longer than the side opposite the smaller angle we find that  $|v_3| \geq |v_2|$ .

To show the reverse inequality consider the reflection  $j$  around the great circle on the Riemann sphere that passes through  $\infty$  and the minimizing point  $p = Re^{it}$ . Since it is a spherical geodesic it coincides with the straight line passing through  $0$  and  $p$ . It preserves extremal length and chordal length and so realizes another another (possibly different) extremal point.  $j$  fixes  $e_1$  and carries  $e_2$  and  $e_3$  to  $\tilde{e}_2 = j(e_2)$  and  $j(\tilde{e}_3)$ . The same argument that showed that  $|\tilde{v}_3| \geq |\tilde{v}_2|$  now shows that  $|v_2| \geq |v_3|$ , and consequently  $|v_3| = |v_2|$ .

This equality is possible only if  $v_1$  and  $e_1$  are real-valued and there are two possible cases. Either

- a) **the Teichmüller case**, all three of numbers  $e_1, e_2$  and  $e_3$  are real valued, or
- b) **the Mori case**,  $e_1$  is real valued and  $e_2$  and  $e_3$  are complex conjugates.

Because in both cases the constants in (8.12) are real valued, the constants  $z_1, z_2$  and  $z_3$  fall into the same two cases. Since the Mori extremal problem involves fewer conditions, necessarily the Mori extremal value for  $\Lambda(E_1, E_2)$  cannot be less than the Teichmüller extremal value. In section 9.6 we show that this extremal value is actually larger.

By the same type of argument given in Lemma 33 we can show that the minimal chordal distances are realized by a configuration with  $E_1$  is equal to an arc of the unit circle passing through  $-1$  and with endpoints  $\omega$  and  $\bar{\omega}$  where  $\omega$  has negative real part and with  $E_2 = [a, 1/a]$ , where  $0 < a < 1$ . Here  $\lambda_1 = 2 \operatorname{Im} \omega$  and  $\lambda_2 = \frac{2(1-a^2)}{1+a^2}$ . This completes the proof of Theorem 1.

To go on to the proof of Theorem 45, let  $\beta_0$  be a simple closed curve in  $S$  such that  $i(\alpha_0, \beta_0) = 4$ . Note that  $\beta_0$  also separates the two sets  $\{z_1, z_2\}$  and  $\{z_3, w_3\}$  so that the bounds  $\lambda_1 > 0$  and  $\lambda_2 > 0$  provide positive lower bounds for  $\Lambda(\alpha_0)$  and  $\Lambda(\beta_0)$ . We know that the product  $\Lambda_\tau(\alpha_0)\Lambda_\tau(\beta_0)$  is minimum along a unique Teichmüller axis in  $T(S)$  and for points  $\tau$  on this unique axis in  $T(S)$ ,

$$16 = \Lambda_\tau(\alpha)\Lambda_\tau(\beta) = \Lambda_{\tau_0}(\alpha_0)\Lambda_{\tau_0}(\beta_0). \quad (8.17)$$

By applying an isometry of the sphere, which of course preserves extremal lengths as well as spherical lengths, we may assume the points  $z_1$  and  $z_2$  lie at  $-a$  and  $a$ , respectively, where  $a > 1$  and  $\lambda_1 = \frac{4a}{1+a^2} < 2$ . Now we consider the two reflections  $r_1$  and  $r_2$  around the real and imaginary axes;  $r_1(z) = \bar{z}$  and  $r_2(z) = -\bar{z}$ . The homotopy classes of  $r_2(\alpha)$  and  $r_2(\beta)$  in  $\bar{\mathbb{C}} \setminus \{-a, a, r_2(z_3), r_2(w_3)\}$  still satisfy  $i(r_2(\alpha), r_2(\beta)) = 4$  and  $\Lambda_{r_2(\tau_0)}(r_2(\alpha)) = \Lambda_{\tau_0}(\alpha)$ . Also  $r_2(E_1), r_2(A), r_2(E_2)$  is an annular configuration that maximizes  $\Lambda(E_1, E_2)$  subject to the conditions on the chordal diameters of  $E_1$  and  $E_2$ . Therefore, along the same line

$$16 = \Lambda_\tau(\alpha)\Lambda_\tau(\beta) = \Lambda_{r_2(\tau_0)}(r_2(\alpha_0))\Lambda_{r_2(\tau_0)}(r_2(\beta_0)). \quad (8.18)$$

The same argument applies to the reflection by  $r_1$  and therefore  $r_1$  and  $r_2$  leave  $\tau_0$

invariant. This implies  $\tau_0$  corresponds to the configuration where  $(z_1, z_2, z_3, w_3) = (-a, a, ib, -ib)$  where  $\lambda_2 = \frac{4b}{1+b^2}$  and  $\lambda_2 < 2$  and  $0 < b < 1$ .

Theorem 45 is a consequence of the minimal axis theorem applied to the measured foliations on  $R$  induced by the homotopy classes of simple closed curves  $\alpha$  and  $\beta$  with  $i(\alpha, \beta) = 4$ .

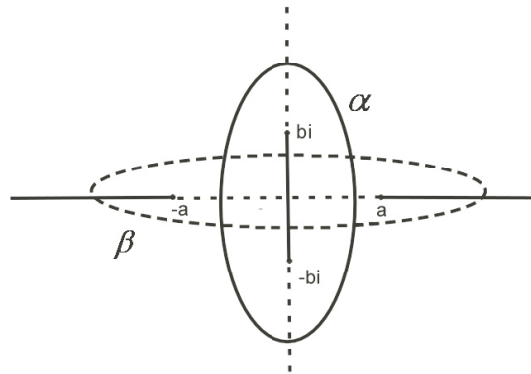


Figure 8.4: Mori's minimal axis

**Theorem 54.** *Let  $\lambda_1$  and  $\lambda_2$  be the chordal diameters of the extremal sets  $E_1$  and  $E_2$  for the Mori problem and assume that each of the arcs  $E_1$  and  $E_2$  has spherical length less than or equal to  $\pi$ . Also, suppose  $\lambda$  is the chordal diameter of the set  $E_1$  in the special case that  $E_2$  has chordal diameter equal to 2. Then*

$$\lambda \leq \lambda_1 \lambda_2 \leq 4\lambda. \quad (8.19)$$

*Moreover, for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that if the chordal diameters of  $E_1$  and  $E_2$  are less than  $\delta$ , then*

$$\frac{4}{1+\varepsilon} \leq \frac{\lambda_1 \lambda_2}{\lambda} \leq 4.$$

*Proof* We move the extremal configuration by the transformation  $w = \frac{z-1}{z+1}$  which carries the unit circle to imaginary axis which preserves the extended real axis. It

also is an isometry in the chordal metric. The sets  $E_2$  and  $E_1$  for the extremal configuration are carried to vertical intervals  $[-ib, ib]$  on the imaginary axis and an interval passing through  $\infty$  equal to  $\{\infty\} \cup (-\infty, -a] \cup [z, \infty)$ . Here  $a \geq 1$  and  $b \leq 1$ . The chordal length  $\lambda_1$  of  $E_1$  is  $\frac{4a}{1+a^2}$  and the chordal length  $\lambda_2$  of  $E_2$  is  $\frac{4b}{1+b^2}$ .

The transformation  $w \mapsto w/a$  moves  $E_1$  to a geodesic segment with chordal length 2 and contracts the geodesic  $E_2$  to the vertical line segment  $[-ib/a, ib/a]$ , which by definition has chordal length equal to

$$\frac{4b/a}{1+(b/a)^2} = \frac{4ab}{a^2+b^2}.$$

Therefore

$$\frac{\lambda_1\lambda_2}{\lambda} = \frac{4a}{1+a^2} \cdot \frac{4b}{1+b^2} \cdot \frac{a^2+b^2}{4ab} = 4 \frac{a^2+b^2}{(1+a^2)(1+b^2)} = 4 \frac{a^2+b^2}{1+a^2+b^2+a^2b^2} \leq 4.$$

Since we assume  $0 < b \leq 1$ , the fraction  $\frac{a^2+b^2}{(1+a^2)(1+b^2)}$  becomes smaller if we replace  $b^2$  in the numerator by 0 and replace it by 1 in the denominator. We obtain

$$\frac{\lambda_1\lambda_2}{\lambda} \geq 2 \frac{a^2}{1+a^2}.$$

But since  $a \geq 1$ ,  $2 \frac{a^2}{1+a^2} \geq 1$  and so

$$\frac{\lambda_1\lambda_2}{\lambda} \geq 1.$$

The second lower bound is a consequence of

$$\lim_{a \rightarrow \infty, b \rightarrow 0} 4 \frac{a^2+b^2}{1+a^2+b^2+a^2b^2} = 4.$$

## 8.6 Comparison of the Mori and Teichmüller annuli

In this section we compare the modulus of the Mori annulus to the modulus of the Teichmüller annulus. We use the notation in Ahlfors' book [2] which is adopted from Künzi [50]. There are three standard annular configurations. The the moduli of these annuli determine functions  $\Phi(R)$ ,  $\Psi(P)$  and  $X(\lambda)$  by the formulas:

1.  $\Lambda(\alpha_I) = \frac{1}{2\pi} \log \Phi(R),$     2.  $\Lambda(\alpha_{II}) = \frac{1}{2\pi} \log \Psi(P),$
3.  $\Lambda(\alpha_{III}) = \frac{1}{2\pi} \log X(\lambda).$

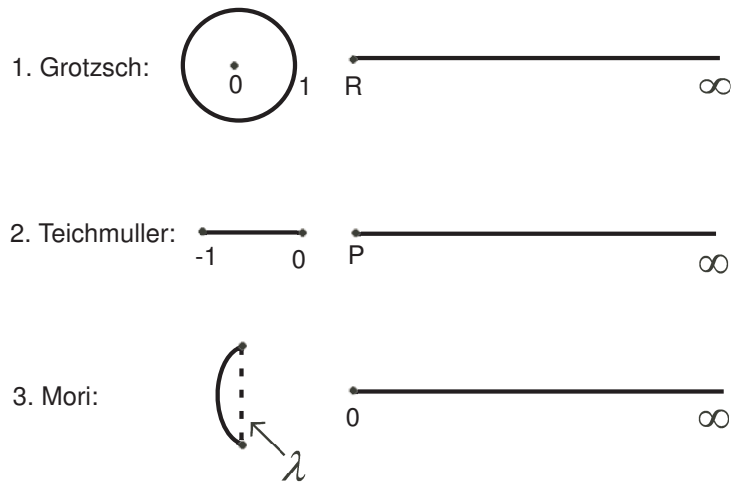


Figure 8.5: Extremal domains introduced by Ahlfors

Let

$$M(z) = \frac{z - b}{z + a} \cdot \frac{a - b}{2b},$$

so  $M(-a) = \infty, M(-b) = \infty, M(b) = 0$  and

$$M(a) = \frac{(a - b)^2}{4ab}.$$

Therefore, the extremal modulus for the Teichmüller configuration is

$$\frac{1}{2\pi} \log \Psi \left( \frac{(a - b)^2}{4ab} \right) = \frac{1}{2\pi} \log \Psi \left( \frac{1}{4} \left( \frac{a}{b} + \frac{b}{a} - 2 \right) \right). \tag{8.20}$$

Let

$$M(z) = \frac{z - b}{z + a} \cdot \frac{a - b}{2b},$$

so  $M(-a) = \infty, M(-b) = \infty, M(b) = 0$  and

$$M(a) = \frac{(a - b)^2}{4ab}.$$

Therefore, the extremal modulus for the Teichmüller configuration is

$$\frac{1}{2\pi} \log \Psi \left( \frac{(a-b)^2}{4ab} \right) = \frac{1}{2\pi} \log \Psi \left( \frac{1}{4} \left( \frac{a}{b} + \frac{b}{a} - 2 \right) \right). \quad (8.21)$$

If  $\tilde{M}(z) = \frac{z-a}{z+a}$ , it carries the annular configuration with  $E_1 = [\infty, -a] \cup [a, \infty]$  and  $E_2 = [-ib, ib]$  to  $\tilde{M}(E_1) = [0, \infty]$  and  $\tilde{M}(E_2)$  which is the arc on the unit circle that joins  $\tilde{M}(-ib)$  to  $\tilde{M}(ib)$  passing through  $-1$ . Thus

$$\tilde{M}(-ib) = \frac{-ib-a}{-ib+a} = \frac{-ib-a}{-ib+a} \cdot \frac{a+ib}{a+ib},$$

that is,

$$\operatorname{Im} \left( \tilde{M}(-ib) \right) = \frac{-2abi}{a^2 + b^2},$$

and

$$\tilde{M}(ib) = \frac{ib-a}{ib+a} = \frac{ib-a}{ib+a} \cdot \frac{a-ib}{a-ib},$$

$$\operatorname{Im} \left( \tilde{M}(ib) \right) = \frac{2abi}{a^2 + b^2},$$

so  $\lambda = \frac{4ab}{a^2+b^2}$  and the extremal modulus of the Mori configuration is

$$\frac{1}{2\pi} \log X \left( \frac{4ab}{a^2 + b^2} \right). \quad (8.22)$$

In order to compare (8.20) and (8.22) we use the relations

$$X(\lambda) = \Phi \left( \frac{\sqrt{4+2\lambda} + \sqrt{4-2\lambda}}{2\lambda} \right) \text{ and } \Phi(R) = \Psi \left( \frac{1}{4} (\sqrt{R} - 1/\sqrt{R})^2 \right)$$

to obtain

$$\begin{aligned} X \left( \frac{4ab}{a^2 + b^2} \right) &= \Phi \left( \frac{2(a+b)/\sqrt{a^2+b^2} + 2(a-b)/\sqrt{a^2+b^2}}{4ab/(a^2+b^2)} \right) = \\ &= \Phi \left( \frac{\sqrt{4 + \frac{8ab}{a^2+b^2}} + \sqrt{4 - \frac{8ab}{a^2+b^2}}}{4ab/(a^2+b^2)} \right) = \\ &= \Phi \left( \frac{\frac{2(a+b)}{\sqrt{a^2+b^2}} + \frac{2(a-b)}{\sqrt{a^2+b^2}}}{4ab/(a^2+b^2)} \right) = \end{aligned}$$

$$\Phi\left(\frac{\sqrt{a^2+b^2}}{b}\right) = \Psi\left(\frac{1}{4}\left(\frac{\sqrt{a^2+b^2}}{b} + \frac{b}{\sqrt{a^2+b^2}} - 2\right)\right).$$

But  $\Psi$  is an increasing function and since  $a > b$ ,  $1 < a/b < \sqrt{a^2+b^2}/b$  and

$$\frac{\sqrt{a^2+b^2}}{b} + \frac{b}{\sqrt{a^2+b^2}} > a/b + b/a,$$

by using (8.21) we see that the Mori configuration has larger modulus than the Teichmüller configuration.

## 8.7 Pairs of extremal of annuli on a four times punctured sphere

On a four times punctured sphere  $S$  with quasiconformal structure the only possibilities for the intersection number of any two non homotopic essential simple closed curves  $\alpha$  and  $\beta$  must be equal to  $2n$ . The cases  $n = 1$  and  $2$  correspond to the Teichmüller and Mori annuli. For any nonnegative integer  $n$  and Riemann surface structure  $S_\tau$  on  $S$  there are two integrable holomorphic quadratic differentials  $q_\alpha$  and  $q_\beta$  associated to  $\alpha$  and  $\beta$ .  $q_\alpha$  has the following properties:

- a) every regular trajectory of  $q_\alpha$  is homotopic to  $\alpha$ ,
- b) the set of all of these regular trajectories forms an annulus conformal to a Euclidean cylinder and each boundary of the cylinder splits into two segments of equal length that are isometrically identified on  $S_\tau$ ,
- c) The metric  $|q_\alpha|^{1/2}$  is extremal for the extremal problem  $\Lambda_\tau(\alpha)$ ,
- d)  $\int \int_{S_\tau} |q_\alpha| dx dy = 1$ .

$q_\beta$  has the same properties with  $\alpha$  replaced by  $\beta$ . Since the space of such differentials has dimension 1,  $q_\alpha = cq_\beta$  for some nonzero complex constant  $c$  with  $|c| = 1$ . We let  $|du| = \text{Re}(q_\alpha^{1/2} dz)$  and  $|dv| = \text{Re}(q_\beta^{1/2} dz)$ .

**Theorem 55.** *For every pair of essential simple closed curves  $\alpha$  and  $\beta$  on  $S$ , the locus of points in  $T(S)$  for which  $\Lambda_\tau(\alpha)\Lambda_\tau(\beta) = (2n)^2$  is a unique Teichmüller line in  $T(S)$  along which the leaves of  $|du|$  are orthogonal to the leaves of  $|dv|$ .*

*Proof* This is just the minimal axis theorem applied to the measured foliations determined by the simple curves  $\alpha$  and  $\beta$ . To see how  $\alpha$  and  $\beta$  determine such foliations, we realize  $S$  and  $\alpha$  in a special way. Construct  $S$  from a rectangle with vertical and horizontal sides joining the four points  $0, 1, 2i$  and  $1+2i$ . The bottom of the rectangle is identified with the top by  $z \mapsto z + 2i$ . The left side is identified with itself by  $z \mapsto 2i - z$  and the right side by  $z \mapsto 1 + 2i - z$ . The four punctures of  $S$  correspond to the four vertices at  $0, 1, i$  and  $1+i$ . The curve  $\alpha$  is realized by the union of the two horizontal line segments  $[i/2, 1 + i/2]$  and  $[3i/2, 1 + 3i/2]$ . The homotopy class of  $\beta$  is realized by the union of  $n$  translates by  $1/n$  of a line segment that slants upwards and to the right starting at a point  $x$  on the unit interval with slope  $2n$ . We choose the value  $x$  to be strictly between  $0$  and  $1/n$ . Note that  $i(\alpha, \beta) = 2n$  but  $\alpha$  is not orthogonal to  $\beta$ . On the other hand the shear

$$T = \begin{pmatrix} 1 & -1/n \\ 0 & 1 \end{pmatrix}$$

carries the rectangle  $S$  to a new rectangle  $S_\tau$ , where  $\tau = T(i)$  and the straight line segments that make up  $\alpha$  and  $\beta$  are carried to new straight line segments  $\alpha_\tau$  and  $\beta_\tau$ , which are perpendicular. If we break up  $\tau$  into its real and imaginary parts,  $\tau = \tau_1 + i\tau_2$ , then the minimal axis along which the horizontal trajectories of  $q_\alpha$  and  $q_\beta$  are realized perpendicularly along the vertical line in the upper half plane defined by  $\tau_1 = -1/n$ .

## 8.8 Mori type extremal problems

In cases with topological symmetry it is sometimes simple to identify the minimal axis for a pair of measured foliations corresponding to two simple curves with the intersection property. As an example consider the two simple closed curves  $\alpha$  and  $\beta$  shown in Figure 8.6.

**Theorem 56.** Consider the Riemann sphere with six points removed, namely,

$$(a_1, b_1, c_1) = r(1, -1/2 + i\sqrt{3}/2, -1/2 - i\sqrt{3}/2) \text{ where } 0 < r < 1,$$

and

$$(a_2, b_2, c_2) = R(-1, +1/2 + i\sqrt{3}/2, 1/2 - i\sqrt{3}/2) \text{ where } 1 < R.$$

Then  $i(\alpha, \beta) = 6$  and the quadratic differential

$$q(z)(dz)^2 = \frac{z(dz)^2}{(z - a_1)(z - b_1)(z - c_1)(z - a_2)(z - b_2)(z - c_2)} \quad (8.23)$$

generates a Teichmüller line which is the locus of points for which  $\Lambda_\tau(\alpha)\Lambda_\tau(\beta)$  takes its minimum value, which is 36.

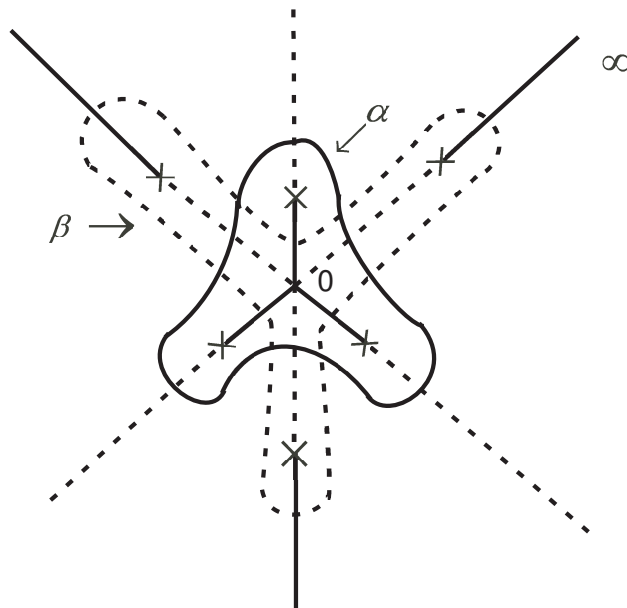


Figure 8.6: Six punctured Riemann sphere

*Proof* Because of the symmetry under reflections around the lines through the origin at angles in multiples of  $60^\circ$ , the regular horizontal trajectories of  $q$  comprise an annulus  $A_\alpha$  that fills the Riemann sphere except for the critical graph shown in the figure. Moreover, the regular vertical trajectories of  $q$  comprise another annulus  $A_\beta$  that also fills the Riemann sphere except for a similar critical graph.

Together with this description of the minimal axis there is a similar and more difficult Mori type problem. Consider annular configurations  $E_1, A, E_2$  in the Riemann sphere with the property that the continua  $E_1$  and  $E_2$  contain equilateral triangles of a prescribed size. That is, assume  $\lambda_1$  and  $\lambda_2$  are two positive numbers and each  $E_j$  contains three points  $a_j, b_j$  and  $c_j$  such that  $\min\{d_C(a_j, b_j), d_C(b_j, c_j), d_C(c_j, a_j)\} \geq \lambda_j$  for  $j = 1$  and  $2$ . Under these conditions make  $\Lambda(E_1, E_2)$  is as large as possible.

**Conjecture:** Under the conditions described above, up to spherical isometry there is a unique annular configuration for which  $\Lambda(E_1, E_2)$  is as large as possible.

We can take

$$(a_1, b_1, c_1) = r(1, -1/2 + i\sqrt{3}/2, -1/2 - i\sqrt{3}/2) \text{ where } 0 < r < 1,$$

$$(a_2, b_2, c_2) = R(-1, +1/2 + i\sqrt{3}/2, 1/2 - i\sqrt{3}/2) \text{ where } 1 < R,$$

and the sets  $E_1$  and  $E_2$  form the critical horizontal trajectory of the quadratic differential

$$q(z)(dz)^2 = \frac{z(dz)^2}{(z - a_1)(z - b_1)(z - c_1)(z - a_2)(z - b_2)(z - c_2)}. \quad (8.24)$$

There are a pair of essential simple closed curves  $\alpha$  and  $\beta$  on  $R$ , the Riemann sphere minus six points for which  $i(\alpha, \beta) = 6$  and a unique line in the Teichmüller space along which  $\Lambda(\alpha)\Lambda(\beta) = 36$  and along which the extremal configurations lie for variable  $\lambda_1$  and  $\lambda_2$ .

# Papers

---

## A.1 Papers

1. Kobayashi's and Teichmuller metrics on the Teichmuller space of symmetric circle homeomorphisms (with Jun Hu and Yunping Jiang), *Acta Mathematica Sinica*, Vol. 26(9), 2010.
2. Holomorphic Motions and Related Topics (with Fred Gardiner and Yunping Jiang), *Geometry of Riemann Surfaces*, 166-203, London Math Society Lecture Notes series, 368, Cambridge Univ. Press, Cambridge, Jan. 2010.
3. Remarks on Chirka's Proof of Slodkowski's Theorem (with Jun Hu), to appear in *Adv. Math. (China)*.
4. Liftings of holomorphic maps into Teichmuller (with Yunping Jiang and Sudeb Mitra), *KODAI MATH. J.* Vol.32 (2009), 547-563
5. The Distance Between Different Components of The Universal Teichmuller Space, *Chin. Ann. Math.* 26B: 4(2005),537-542, Master Thesis
6. The Uniqueness and the Solution of the Inverse Sturm-Liouville Problem, *Journal of Fudan Univ. (Natural Science)* 2004 Vol.43 No.3 P.361-365 (In Chinese), Undergraduate Thesis

## A.2 Preprints

1. Extremal Annuli on the sphere (with Fred Gardiner), Preprint, 2011
2. Martingales for Quasisymmetric Systems and Complex Manifold Structures (Yunchun Hu and Yunping Jiang), Preprint, 2011
3. Douady-Earle extension and symmetric rigidity theorem, Preprint 2011.
4. Geodesics in Two Models of Universal Teichmüller Space, Preprint.

# Bibliography

- [1] S. Agard, Distortion of quasiconformal mappings. *Ann. Acad. Sci. Fenn.*, 413:1-7, 1968. [3](#), [57](#), [58](#)
- [2] L. V. Ahlfors, *Conformal Invariant: Topics in Geometric Function Theory*. McGraw-Hill, New York, 1973. [59](#), [124](#), [134](#)
- [3] L. V. Ahlfors, *Lectures on Quasiconformal Mapping*. Volume 38 of *University Lecture Series*. Amer. Math. Soc, 2006. [5](#), [20](#), [21](#), [36](#), [113](#), [116](#), [124](#)
- [4] L. V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*. *Ann. of Math.* **72** (1960), 385-404. [83](#), [84](#)
- [5] K. Astala and G. J. Martin, *Holomorphic motions*. *Papers on Analysis; Report. Univ. Jyväskylä* **83** (2001), 27-40. [34](#), [81](#)
- [6] K. Astala, T. Iwaniec, and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*. Princeton University Press, Princeton and Oxford, 2009.
- [7] L. Bers, A non-standard integral equation with applications to quasiconformal mappings. *Acta Math.* 116 (1966), 113-134. [23](#), [26](#)
- [8] L. Bers, Fibre spaces over Teichmüller spaces. *Acta Math.*, **130** (1973), 89-126.
- [9] L. Bers and H. Royden, Holomorphic families of injections. *Acta Math.*, **157** (1986), 259–286. [34](#), [49](#), [50](#), [77](#), [81](#), [82](#), [84](#), [90](#), [91](#)
- [10] A. Beurling and L. Ahlfors, The boundary correspondence under quasiconformal mappings. *Journal d'Analyse Mathématique*, 125-142, Vol 7.1, 1953/54. [20](#)
- [11] V. Bozin, N. Lakic, V. Markovic, and M. Mateljevic, Unique extremality. *J. Analyse Math.* 75:299-238, 1998.

- 
- [12] E. M. Chirka, On the propagation of holomorphic motions. *Doklady Akademii Nauk*, **397** (1) (2004), 37–40. [34](#), [35](#), [40](#), [74](#), [81](#)
- [13] E. M. Chirka and J.-P. Rosay, Remarks on the proof of a generalized Hartogs lemma. *Ann. Polon. Math.*, **70** (1998), 43–47. [34](#), [40](#)
- [14] R. Courant and D. Hilbert, *Methods of Mathematical Physics*. Wiley-Interscience, New York, 1963 [34](#), [45](#), [73](#)
- [15] A. Douady, Prolongements de mouvements holomorphes (d’après slodkowski et autres). *Asterisque*, **227** (1995), 7–20. [34](#), [81](#)
- [16] A. Douady and C. J. Earle, Conformal natural extension of homeomorphism of the circle. *Acta. Math.*, 157:23-48, 1986.
- [17] C. J. Earle, *Teichmüller spaces as complex manifolds: Warwick, 1992*. Teichmüller Theory and Moduli Problems, no. 10, (2009), 5-33, Harish Chandra Research Institute, Allahabad.
- [18] C. J. Earle, F. P. Gardiner, and N. Lakic, Asymptotic Teichmüller space, Part I: The complex structure. *Contemp. Math.*, 256:17-38, 2000.
- [19] C. J. Earle, F. P. Gardiner, and N. Lakic, Asymptotic Teichmüller space, Part II: The metric structure. *Contemp. Math.*, 355:187-220, 2004. [103](#)
- [20] C. J. Earle and I. Kra, *On holomorphic mappings between Teichmüller spaces*. Contributions to Analysis, Academic Press, New York, (1974), 107-124. [91](#)
- [21] C. J. Earle, I. Kra, and S. L. Krushkal, Holomorphic motions and Teichmüller spaces. *Trans. Am. Math. Soc.*, 343(2):927–948, 1994. [97](#)
- [22] C. J. Earle, V. Markovic, and D. Saric, Barycentric extension and the Bers embedding for asymptotic Teichmüller space. *Contemp. Math.*, 311:87-105, 2002. [20](#), [22](#)

- 
- [23] C. J. Earle and S. Mitra, *Variation of moduli under holomorphic motions*. Contemporary Mathematics, **256**, (2000) 39–67. [80](#), [81](#)
- [24] A. Fathi, F. Laudenbach, and V. Poenaru, Travaux de Thurston sur les Surfaces. Vol 66-67 of Asterisque. Soc. math. de France, Seminaire Orsay, 1979. [28](#), [119](#)
- [25] F. P. Gardiner, Approximation of infinite dimensional Teichmüller spaces. *Trans. Am. Math. Soc.*, 282(1):367-383, 1984. [101](#)
- [26] F. P. Gardiner, *Teichmüller Theory and Quadratic Differentials*. John Wiley & Sons, New York, 1987. [28](#), [30](#), [101](#), [113](#), [116](#), [119](#), [121](#)
- [27] F. P. Gardiner, Measured foliations and the minimal norm property for quadratic differentials. *Acta math.*, 152:57-76, 1984 [30](#), [31](#), [116](#), [121](#)
- [28] F. P. Gardiner and William J. Harvey, *Universal Teichmüller space*. Handbook Of Complex Analysis: Geometric Function Theory, Vol. 1, 457–492, North-Holland, Amsterdam, 2002.
- [29] F. P. Gardiner and L. Keen, Holomorphic motions and quasi-Fuchsian manifolds. *Contemp. Math.*, 240:159–174, 1999. [66](#)
- [30] F. P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*. AMS, Providence, Rhode Island, 2000. [19](#), [22](#), [25](#), [27](#), [31](#), [103](#), [104](#), [106](#), [109](#)
- [31] F. P. Gardiner and N. Lakic, Tracking a moving point on a Riemann surface. Preprint.
- [32] F. P. Gardiner and N. Lakic, A synopsis of the Dirichlet principle for measured foliation. *Kokyuroku Bessatsu*, 4:53-62, 2010. [30](#), [113](#), [121](#), [122](#)
- [33] F. P. Gardiner and Y. Jiang, Asymptotically affine and asymptotically conformal circle endomorphisms. RIMS Kokyuroku Bessatsu B17, Infinite Dimensional Teichmüller Spaces and Moduli Spaces, Ed. Ege Fujikawa, June, 2010, 37-53.

- 
- [34] F. P. Gardiner, Y. Jiang, and Z. Wang, Holomorphic motions and related topics. *Geometry of Riemann Surfaces*, London Mathematical Society Lecture Note Series, No. 368, 156-193.
- [35] F. P. Gardiner, H. Masur Extremal length geometry of Teichmüller Space. *Complex variables*, Vol 16:209-237, 1991. [113](#), [116](#), [122](#)
- [36] F. P. Gardiner and D. P. Sullivan, Symmetric structures on a closed curve. *Amer. J. of Math.*, 114:683-736, 1992. [22](#), [26](#), [27](#), [103](#), [104](#)
- [37] F. P. Gardiner and Z. Wang, Extremal annuli on the sphere. pre-print
- [38] J. Hu, Y. Jiang, and Z. Wang, Kobayashi's and Teichmüller metrics on the Teichmüller space of symmetric circle homeomorphisms. *Acta Mathematica Sinica*, Vol. 26(9), 2010. [20](#)
- [39] J. Hu and Z. Wang, Remarks on Chirka's Proof of Slodkowski's Theorem. to appear in *Adv. Math. (China)*.
- [40] J. H. Hubbard, *Sur les sections analytiques de la courbe universelle de Teichmüller*. *Memoirs Amer. Math. Soc.* **166** (1976), 1-137. [91](#)
- [41] J. H. Hubbard, *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics – Volume I: Teichmüller Theory*. Matrix Editions, Ithaca, NY, 2006. [81](#)
- [42] Y. Jiang, Function model of the Teichmüller space of a closed hyperbolic Riemann surface. Preprint, 2008.
- [43] Y. Jiang, Holomorphic motions and normal forms in complex analysis. *Studies in Advanced Mathematics*, ICCM2004@2008, AMS/IP, Vol. **42** (2008), Part 2, 457-466.
- [44] Y. Jiang, Asymptotically conformal fixed points and holomorphic motions. *Complex Analysis and Applications*, Proceedings of the 13th International Con-

- ference on Finite or Infinite Dimensional Complex Analysis and Applications, World Scientific, 2006, 109-129.
- [45] Y. Jiang, Holomorphic motions, Fatou linearization, and quasiconformal rigidity for parabolic germs. *Michigan Mathematical Journal*, **58** (2009), 397-414.
- [46] Yunping Jiang, Sudeb Mitra, and Zhe Wang, Liftings of holomorphic maps into Teichmuller. *KODAI MATH. J. Vol.32* (2009), 547-563.
- [47] L. Keen and N. Lakic, *Hyperbolic geometry from a local viewpoint*. London Mathematical Society Student Texts, 68. Cambridge University Press, Cambridge, 2007. 3
- [48] P. Kiernan, *On the relations between taut, tight and hyperbolic manifolds*. Bull. Amer. Math. Soc. **76** (1970), 49-51. 89, 90
- [49] R. Kuknau, *Handbook of Complex Analysis: Geometric Function Theory*. Elsevier Science, New York, 2002. 105
- [50] H. P. Kunzi, *Quasikonforme Abbildungen*. Springer-Verlag, Berlin, 1960 134
- [51] O. Letho and K. Virtanen, *Quasiconformal Mappings in the plane*. Springer-Verlag, New York, 1973. 5, 6, 49
- [52] O. Letho, *Univalent Functions and Teichmuller Spaces*. Springer-Verlag, New York, 1987. 5, 6
- [53] G. Lieb, *Holomorphic motions and Teichmüller space*. Ph.D. dissertation, Cornell University, 1990. 80
- [54] R. Mãñé, P. Sad, and D. Sullivan, On the dynamics of rational maps. *Ann. Èc. Norm. Sup.*, **96** (1983), 193–217. 34
- [55] Y. Minski, Teichmuller geodesics and ends of hyperbolic 3-manifolds. *Topology*, 32:625-647, 1993. 113, 118

- 
- [56] Y. Minski, Quasi-projections in Teichmüller space. *J. Reine Angew. Math.*, 473:121-136, 1996. 113
- [57] S. Mitra, *Teichmüller spaces and holomorphic motions*. *J. d'Analyse Math.* **81** (2000) 1-33. 80, 81, 83, 84
- [58] S. Nag, *The Torelli spaces of punctured tori and spheres*. *Duke Math. J.* **48** (1981), 359-388. 81, 88
- [59] H. Royden, Automorphisms and isometries of Teichmüller space. In *Advances in the Theory of Riemann Surfaces*, 1969 Stony Brook Conference, *Ann. of Math. Studies*, 66: 369-383, Princeton Univ. Press, 1971. 101
- [60] Z. Slodkowski, Holomorphic motions and polynomial hulls. *Proc. Amer. Math. Soc.*, 111:347-355, 1991. 34, 81
- [61] K. Strebel, *Quadratic Differentials*. Springer-Verlag, Berlin & New York, 1984. 106
- [62] D. P. Sullivan and W. P. Thurston, Extending holomorphic motions. *Acta Math.*, **157** (1986), 243-257. 5, 34, 49, 50, 68, 69
- [63] H. Wu, *Normal families of holomorphic mappings*. *Acta Math.* **119** (1967), 193-233. 89
- [64] E. Zeidler, *Nonlinear functional analysis and its applications, I, Fixed-point theorems*. Springer-Verlag, New York, 1986. 88
- [65] Li Zhong, *Introduction to Complex Analysis (in chinese)*. Peking University Press, 2004 9, 24, 58