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Ad-c.r. geometries in dimension ≤ 4

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City University of New York, 1989

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Ad-c.r. GEOMETRIES IN DIMENSION ≤ 4

by

MING WANG

A dissertation submitted to the Graduate Faculty
in Mathematics in partial fulfillment of the requirements
for the degree of Doctor of Philosophy, The City
University of New York.

1989

This manuscript has been read and accepted for the Graduate Faculty In Mathematics in satisfaction of the dissertation requirment for the degree of Doctor of Philosophy.

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Abstract

Ad-c.r. GEOMETRIES IN DIMENSION ≤ 4

by

Ming Wang

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In this dissertation, I am studying a special kind of homogeneous geometries in dimension ≤ 4. A geometry means a pair (G,H) where G is a Lie group, H its closed subgroup, both connected, such that G acts effectively on $S = G/H$, and S is simply connected. The dimension of the geometry is the dimension of S. The linear isotropy representation $\rho : H \rightarrow Aut(T_{x_\circ}(S))$, $x_\circ = [H]$ being the basepoint of S, is closely related to the adjoint representation of Hon the Lie algebra $L(G)$ of G. We say that (G,H) is $Ad - c.r.$ if this representation is completely reducible and if $\rho(H)$ is closed in $Aut(T_{x_\circ}(S)) \simeq GL_n(R)$, $n = dim\ S$. In §1 - §4 of this dissertation, $Ad-c.r.$ geometries in dimension ≤ 4 are classified. The topology of each geometry is given in §5. In §6, for the most part of the classification list, their boundedness are determined (we call a geometry (G,H) bounded, if there exists a subgroup Γ of G acting properly discontinuously on S such that $\Gamma \backslash S$ is compact).In §7, the flat complet compact pseudo- Riemannian space-forms with signature (2,2) is classified upto finite covers (This is the study of co-compact properly discontinuous subgroups in the geometry $(SO_\circ(2,2) \times R^4, SO_\circ(2,2))$). Our main theorem is: *Suppose X is a flat compact complete space-form with fundamental group $\Gamma \subseteq SO_\circ(2,2) \times R^4$, then there is a uniquely determined subgroup H of $SO_\circ(2,2) \times R^4$ that acts simply transitively on R^4 and $H \cap \Gamma$ has finite index in Γ.* Then we need only to find the simply transitive subgroups and their uniform lattices. To prove the main theorem, we first proved that Γ is virtually solvable. This result confirms a conjecture by J. Milnor in a special case: *The fundamental group of a complete affinely flat manifold is virtually polycyclic.* In §8, G-invariant pseudo-Riemannian, contact, symplectic and complex structures on G/H in $dim. \leq 4$ are classified.

ACKNOWLEDGMENT

The author wishes to thank Professor Ravi S. Kulkarni whose guidance and encouragment made this work possible.

CONTENTS

INTRODUCTION

In this dissertation, I am studying a special kind of homogeneous geometries in dimension ≤ 4.

The first concern, of course, is to specify **what constitutes a geometry**? Undoubtedly, the topology of the underlying space and the various possible constructions of metrics, measures, geodesics . . . consistent with the topology are the salient features of a geometry. In addition, however, in all parts of mathematics where one adopts a geometric viewpoint–explicity or implicity–the symmetry considerations are also usually present–for, any explicit calculations beyond 1- or 2-dimensional situations, without symmetry considerations are tedious if not impossible. In particular certain geometries e.g., the **Euclidean, projective, hyperbolic, Lorentz** . . . geometries stand out and their remarkable properties are to be traced back to the rich symmetry they possess. Historically, the symmetry was formulated only implicitly, such as e.g., in the synthetic axiom system of Euclid, or the more sophisticated axioms such as the axiom of free mobility of Helmholtz,the axiom of the infinitesimal isotropy of F. Schur [26], or the axiom of geodesic planes of E. Cartan [3], ch. 5, cf. Lie [22] §21 and Freudenthal [7] for historical comments. Of the same nature and on a much more sophisticated level is E. Cartan's characterization of a locally symmetric space by the condition $\nabla R = 0$. The idea to systemalize these symmetry considerations based only on a general notion of a group with finitely many parameters is due to Lie cf. the preface of [22]. With this hindsight, and reversing the process we can now use the Lie theory to classify the geometries in a rigorous way.

In this paper a **geometry** means a pair (G,H) where G is a Lie group, H its closed subgroup, both connected, such that G acts effectively on $S = G/H$, and S is simply connected. The dimension of the geometry is the dimension of S. The first achievement of Lie was to classify 1-dimensional geometries. In dimensions ≥ 2 however the list of geometries in this broad sense increases very rapidly making it imperative to be selective–and even if the lists are compiled it is important to record in addition the relevant geometric information such as e.g., whether the geometry is **unimodular** i.e., whether S admits a G-invariant measure, or whether the geometry is **semi-Riemannian** or **symplectic** i.e, whether S admits an appropriate G-invariant bilinear form. It is also of great interest to see whether the geometry has models with some finiteness properties, e.g., we call a geometry (G,H) **bounded**, if there exists a subgroup Γ of G acting properly discontinuously on S such that $\Gamma\backslash S$ is compact. A basic invariant of a geometry which is useful in deciding such questions is the linear isotropy representation $\rho : H \rightarrow Aut(T_{x_0}(S))$, $x_0 = [H]$ being the basepoint of S. A closely related and better manageable invariant is the adjoint representation of Hon the Lie algebra $L(G)$ of G. We say that (G,H) is $Ad - c.r.$ if this representation is completely reducible and if $\rho(H)$ is closed in $Aut(T_{x_0}(S)) \simeq GL_n(R)$, $n = dim\ S$. For example, if H is compact or semisimple the geometry is $Ad - c.r.$.

When (G,H) is an $Ad - c.r.$ geometry H is reductive, the isotropy representation is injective and the weights of tori in H in the isotropy representation gives a good hold on the geometry, allowing classifications. The **Riemannian** geometries are special cases in which the weights are compact.

In §1 - §4 of this dissertation, I shall classify $Ad - c.r.$ geometries in dimension ≤ 4. A good number of geometries are decomposable i.e., they are products of low-dimensional ones. Usually while compiling lists we record only the indecomposable ones. The topology of each geometry is given in §5.

In §6 I shall decide, for the most part of the classification list, their boundedness. When $H$is compact, (G,H) is bounded iff Γ is a uniform lattice in G. A.I. Malcev, G.D. Mostow, H.C. Wang, L. Anslander, A. Borel, G.A. Margulis and many others obtained important results in this direction during 1950s to 1970s (cf. Raghunathan [25]). On the other hand, when H is non-compact, the only important work we know is about $G/H = O(p+1,q)/O(p,q)$ by R.S. Kulkarni and F. Raymond [21] and by R.S. Kulkarni [20]. Using a theory about compact complete affine manifolds by Fired, Goldman and Hirsh [10], [11], I am able to solve the boundeness problem for all the solvable and most of the mixed $Ad - c.r.$ geometries.

In §7, I shall study the following example of flat compact pseudo- Riemannian space-forms with signature (2,2) in complete detail (This is the study of co-compact properly discontinuous subgroups in the geometry $(SO_o(2,2) \ltimes R^4, SO_o(2,2)))$. The case of signature (3,1) was studied by D. Fried [8]. Also W.M. Goldman and Y. Kamishima [13] contains some general results in the case of signature (n, 1). Our main theorem is

Theorem. *Suppose X is a flat compact complete space-form with fundamental group* $\Gamma \subseteq SO_o(2,2) \ltimes R^4$, *then there is a uniquely determined subgroup* H *of* $SO_o(2,2) \ltimes R^4$ *that acts simply transitively on* R^4 *and* $H \cap \Gamma$ *has finite index in* Γ .

Then we need only to find the simply transitive subgroups and their uniform Lattices. The case of signature (2,2) involves some new aspects but curiously the diffeomorphism types of these space-forms is the same as those occurring in the case (3,1) studied by D. Fried. To prove the main theorem, we first prove that Γ is virtually solvable. This result confirms a conjecture by J. Milnor [23] in a special case:

Conjecture. *The fundamental group of a complete affinely flat manifold is virtually polycyclic.*

My result, combined with Fried's result, shows that this conjecture is true for compact pseudo-Riemannian 4-manifolds.

In §8 I shall classify G-invariant pseudo-Riemannian, contact, symplectic and complex structures on G/H in *dim.* ≤ 4. This involves quite a few Lie algebra calculations. A part of it carries over to higher dimensions. The interesting among these geometries are the Riemannian geometries E^n, S^n, H^n , the Hermitian Kahler geometries of constant holomorphic curvature CP^2, CH^2 , the analogies in indifinite metric, $E^{p,q}, S^{p,q}, p+q \leq 4, CP^{1,1}, CH^{1,1}$, the nilpotent geometries Nil^3 and Nil^4 , certain solvable geometries and affine geometries, some total spaces of rank-2 vector bundles over S^2 with Euler number 1 or 2. Following W. Thurston's revolutionary work in 3-dimensional topology, G.P. Scott (c.f. [27]) gave a lot of information about the 3-dimensional Riemannian geometries and showed the importance of topology in studying those geometries. In this thesis, Scott's classification shall be obtained with a different proof based on standard Lie theories. In Filipkiewicz's thesis at Warwick (1984), a classification of 4-dimensional Riemannian geometries was obtained. C.T.C. Wall [34], [35] has studied it further and applied it in the classification of complex algebraic surfaces. Our $Ad - c.r.$ geometries contain the above results as special cases, i.e., (G,H) with Hcompact.

§1.BASIC NOTATIONS

(1.1) Definition. A **geometry** means a pair (G, H) where G is a Lie group, H its closed subgroup, both connected, such that G acts effectively on $S = G/H$, and S is simply connected. The dimension of the geometry is the dimension of S.

Let $x_o = [H]$ be the basepoint of S, $\rho : H \to Aut(T_{x_o}(S))$ be the linear isotropy representation and $Ad : H \to Aut(L(G))$ be the adjoint representation of H on the Lie algebra $L(G)$ of G.

(1.2) Definition. A geometry (G, H) is called $Ad - c.r.$ if

(i) $Ad : H \to Aut(L(G))$ is completely reducible and

(ii) $\rho(H)$ is closed in $Aut(T_{x_o}(S)) \simeq GL_n(R)$, where $n = dim\ S$.

(1.3) Definition. A geometry (G,H) is called decomposable if

(i) there are two geometries (G_1, H_1) and (G_2, H_2) such that neither G_1 nor G_2 is trivial and

(ii) G_1 and G_2 are Lie subgroups of Gsuch that $G= G_1 \times G_2$; H_1 and H_2 are Lie subgroups of H such that $H = H_1 \times H_2$.

If (G, H) is not decomposable, it is called indecomposable.

If (G, H) is decomposable, G_i, H_i are as in the above definition and $S_i = G_i/H_i$, $i = 1,2$, then $S = S_1 \times S_2$.

(1.4) Definition. A morphism $(G_1, H_1) \xrightarrow{\theta} (G_2, H_2)$ means a Lie group homomorphism $\theta : G_1 \to G_2$ such that $\theta(H_1) = H_2$. Such a morphism is called an equivalence if θ is an isomorphism and $\theta(H_1) = H_2$.

In the first part of this thesis, we will classify, upto equivalence , $Ad - c.r.$ geometries in $dim \leq 4$. We will list only the indecomposable ones.

(1.5) Definition. A geometry (G, H) is abelian, resp. nilpotent, resp. solvable, resp. semisimple, resp. reductive if $L(G)$ is such respectively.

If $L(G)$ is not of any of these types then the geometry is said to be mixed.

It is naturally to divide our classification list into three groups : solvable, reductive and mixed. Since abelian geometries are both solvable and reductive, there are overlaps between solvable and reductive geometries. To avoid these overlaps, we will divide our classification list into three basic types: nonabelian reductive, solvable and mixed.

§2. Reduction of the problem.

(2.1) Definition. An infinitesimal geometry is a pair $(\mathcal{G}, \mathcal{H})$ where $\mathcal{G}$ is a real Lie algebra and $\mathcal{H}$ its subalgebra such that $\mathcal{H}$ does not contain a nonzero subalgebra which is an ideal in $\mathcal{G}$.

The notations of dimensions, morphism, equivalence of infinitesimal geometries are defined similar to those for geometries.

(2.2) Proposition. The map $(G, H) \to (L(G), L(H))$ induces an injective map from the equivalence classes of geometries to those of infinitesimal geometries. The image of this map consists of equivalence classes of the pairs $(\mathcal{G}, \mathcal{H})$ such that if $\tilde{G}$ is the connected, simply connected Lie group with Lie algebra $\mathcal{G}$ then the connected Lie subgroup of G corresponding to $\mathcal{H}$ is closed.

Proof (Ravi Kulkarni):

Effectiveness of the G-action on G/H amounts to the fact that $\bigcap_{g \in G} gHg^{-1} = \{e\}$, i.e. equivalently H does not contain a subgroup $\neq \{e\}$ which is normal in G. Linearizing this condition we see that the pair $(L(G), L(H))$ satisfies (2.1). So the map $(G, H) \to (L(G), L(H))$ induces a map from the equivalence classes of geometries to those of infinitesimal geometries. If (G, H) is a geometry $\tilde{G}$ is the universal covering of G, π is the covering map , then $L(\tilde{G}) = L(G)$ and $L((\pi^{-1}(H))_{\circ}) = L(H)$,where $(\pi^{-1}(H))_{\circ}$ is the identical component of $\pi^{-1}(H)$ and is closed. Conversely let $(\mathcal{G}, \mathcal{H})$ be an infinitesimal geometry,$\tilde{G}$ a connected, simply connected Lie group with Lie algebra $\mathcal{G}$ and assume that the connected Lie group $\tilde{H}$ of $\tilde{G}$ with Lie algebra $\mathcal{H}$ is closed. The long exact homotopy sequence associated to the fibration $\tilde{H} \hookrightarrow \tilde{G} \twoheadrightarrow$ S shows that S is connected and simply connected. Let Z be the kernel of the $\tilde{G}$-action on S. Clearly Z is a closed normal subgroup, hence a normal Lie subgroup of $\tilde{G}$. So $L(Z)$ is an ideal in $\mathcal{G}$ and $L(Z) \subseteq \mathcal{H}$. By the condition in (2.1) $L(Z) = \{0\}$, i.e. Z is a discrete normal subgroup of $\tilde{G}$, and $L(Z) \subseteq \mathcal{H}$. Set $G = \tilde{G}/Z, H = \tilde{H}/Z$. Clearly $G/H = S$ and G acts effectively on S. Moreover if $\tilde{G} \xrightarrow{\pi} G$ is the canonical projection then $\pi^{-1}(H) = \tilde{H}$, so H is a closed subgroup. Hence (G, H) is a geometry. Q.E.D.

(2.3) Corollary. Notations as in (2.2). Then $Z = Z(\tilde{G}) \cap \tilde{H}$.

Proof. C.f. Chevalley [4], Chap. II., §VII, Proposition 2. Q.E.D.

(2.4) The following example shows that the map $(G, H) \rightarrow (L(G), L(H))$ from the equivalence classes of geometries to those of infinitesimal geometries is not surjective.

Let $\mathcal{G} = L_1 \oplus L_2$, where $L_i = so(3), i = 1, 2$, and $\mathcal{H}_i \subseteq L_i$, s.t.

$$\mathcal{H}_i = R \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i = 1,2. Let

$$\mathcal{H} = R \left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t & 0 \\ -t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

where t is irrational. Then $(\mathcal{G}, \mathcal{H})$ is an infinitestimal geometry. We can choose $G = SU(2) \times SU(2) \supseteq H_1 \times H_2 \simeq SO(2) \times SO(2), H \subset SO(2) \times SO(2)$ but the closure of H is $H_1 \times H_2$.

(2.5) Let $(\mathcal{G}, \mathcal{H})$ be an infinitesimal geometry, then the restriction of $ad : \mathcal{G} \rightarrow End(\mathcal{G})$ on $\mathcal{H}$ induces a representation $\overline{ad} : \mathcal{H} \rightarrow End(\mathcal{G}/\mathcal{H})$. If (G, H) is a geometry, the adjoint representation $Ad : H \rightarrow Aut(T_e(G)) \simeq L(G)$ also induces $\overline{Ad} : H \rightarrow Aut(L(G)/L(H))$. It's linearizing is $\overline{ad} : L(H) \rightarrow Ent(L(G)/L(H))$. Let $x_o = [H]$ be the basepoint of $S = G/H$, ρ the isotropy representation and $\overline{\rho} : L(H) \rightarrow Ent(T_{x_o}(S))$ its linearizing. Then we have

(2.6) Lemma. $\overline{Ad}$ is equivalent to ρ .

Proof: (R. Kulkarni). Consider the map $\phi : L(G) \rightarrow T_{x_o}(S)$ given by

$$\phi(X)f = \frac{d}{dt} f((exptX)x_o)|_{t=o}$$

for all C^∞ - functions defined near x_o. For $h \in H$

$$\begin{aligned}(h_{\circ}\phi(X))f &= \frac{d}{dt}f(h((exptX)x_{\circ}))|_{t=0}\\ &= \frac{d}{dt}f(h((exptX)h^{-1}x_{\circ}))|_{t=o}\\ &= \frac{d}{dt}f((expt(AdhX))x_{\circ}))|_{t=0}\\ &= \phi(Adh(X))f.\end{aligned}$$

So ϕ is H-equivariant. One sees immediately that $Ker\phi = L(H)$ Q.E.D.

(2.7) Corollary. $\overline{ad}$ is equivalent to $\bar{\rho}$.

(2.8) Definition. An infinitesimal geometry is called $ad_r - c.r.$ if $ad : \mathcal{H} \rightarrow End(\mathcal{G})$ is completely reducible.

(2.9) Proposition. Let (G, H) be an $Ad - c.r.$ geometry. The map $(G, H) \rightarrow (L(G), L(H))$ induces an injective map from the equivalence classes of $Ad - c.r.$ geometries to those of $ad_r - c.r.$ infinitesimal geometries. The image of this map consists of equivalence classes of the pairs $(\mathcal{G}, \mathcal{H})$ satisfying the following condition:

(i) if $\tilde{G}$ is the connected, simply connected Lie group with Lie algebra $\mathcal{G}$ then the connected Lie subgroup of $\tilde{G}$ corresponding to $\mathcal{H}$ is closed;

(ii) the connected subgroup of $Aut(\mathcal{G}/\mathcal{H})$ corresponding to $\overline{ad}\,(\mathcal{H})$ is closed.

Proof. By (2.2) , (2.6) and (2.8). Q.E.D.

(2.10) Remark. The example given in (2.3) shows that there is such an $ad_r - c.r >$ infinitesimal geometry which satisfies (2.9) (ii) but not (i).

(2.11) Based on Proposition (2.9), we will classify $ad_r - c.r.$ infinitesimal geometries first , then complete the classification of $Ad - c.r.$ geometries by checking conditions (i) and (ii) in (2.9).

(2.12) Definition. $\mathcal{G}^C \overset{\text{def}}{=} \mathcal{G} \otimes_R C$, $\mathcal{H}^C \overset{\text{def}}{=} \mathcal{H} \otimes_R C$, where $\mathcal{G}$, $\mathcal{H}$ are real Lie algebras.

(2.13) Definition. A pair of compex Lie algebras (L, K) is called an $ad_c - c.r..$ pair if

(i) L is a complex Lie algebra and K is its subalgebra such that K does not contain a nonzero ideal of L.

(ii) $ad : K \to End(L)$ is completely reducible.

(2.14) Definition. A real form of an $ad_c - c.r.$ pair (K, L) is a pair of real Lie algebras $(\mathcal{G}, \mathcal{H})$ such that $\mathcal{G}^C = K$ and $\mathcal{H}^C = L$.

The notations of dimensions and equivalence of $ad_c - c.r.$ pairs are defined similarly to those of infinitesimal geometries.

(2.15) Proposition. An infinitesimal geometry $(\mathcal{G}\ ,\ \mathcal{H})$ is $ad_r - c.r.$ iff $(\mathcal{G}^C,\ \mathcal{H}^C)$ is $ad_c - c.r..$ If $(\mathcal{G}_1,\ \mathcal{H}_1)$ and $(\mathcal{G}_2\ ,\ \mathcal{H}_2)$ are equivalent then $(\mathcal{G}_1^C\ ,\ \mathcal{H}_1^C)$ and $(\mathcal{G}_2^C\ ,\ \mathcal{H}_2^C)$ are equivalent.

Proof. We need the following Lemmas.

(2.15.1) Lemma. (i) If $(\mathcal{G},\ \mathcal{H})$ is $ad_r - c.r.$, then $\mathcal{H}$ is reductive. If (L, K) is $ad_c - c.r.$, then K is reductive.

(ii) $\mathcal{H} \subseteq \mathcal{G}$ such that $\mathcal{H}$ contains a non-zero ideal of $\mathcal{G}$ iff $\mathcal{H}^C$ contains a nonzero ideal of $\mathcal{G}^C$.

The proofs of the above lemmas are simple. Now we only have to prove that $ad : \mathcal{H} \to End(\mathcal{G})$ is completely reducible iff $ad : \mathcal{H}^C \to End(\mathcal{G}^C)$ is completely reducible.

First assume $ad : \mathcal{H} \to End(\mathcal{G})$ is completely reducible, then $\mathcal{H}$ is reductive by Lemma (2.15.1) and $\mathcal{H} = Z(\mathcal{H}) \oplus [\mathcal{H}\ ,\ \mathcal{H}]$ such that $Z(\mathcal{H})$ is the centre of $\mathcal{H}$ and $[\mathcal{H}\ ,\ \mathcal{H}]$ is semisimple. By [33] $ad(Z(\mathcal{H}))$ is completely reducible, i.e. $ad(Z(\mathcal{H}))|_{\mathcal{G}}$ is diagonalizable. So $ad(Z(\mathcal{H}^C))|_{\mathcal{G}^C}$ is diagonalizable , since $Z(\mathcal{H}^C) = (Z(\mathcal{H}))^C$. Since $\mathcal{H}^C = Z(\mathcal{H}^C) \oplus [\mathcal{H}^C, \mathcal{H}^C]$ where $[\mathcal{H}^C, \mathcal{H}^C]$ is semisimple. We know that a representation of a reductive Lie algebra is completely reducible iff the restriction of this representation on the center of the Lie algebra is completely reducible. So $ad : \mathcal{H}^C \to Ent(\mathcal{H}^C)$ is completely reducible.

The inverse can be proved similarly. Q.E.D.

(2.16) Proposition. If $(\mathcal{G}, \mathcal{H})$ is $ad_r - c.r.$, then $\overline{ad} : \mathcal{H} \to End(\mathcal{G}/\mathcal{H})$ is injective. Also $\overline{ad}^C : \mathcal{H}_C \to End(\mathcal{G}^C/\mathcal{H}^C\ )$ is injective.

Proof. Let $\mathcal{G} = \mathcal{H} + \mathcal{M}$, where $\mathcal{M}$ is an $ad\mathcal{H}$-invariant complement in $\mathcal{G}$. Let $\mathcal{H}_\circ = Ker\ ad|_{\mathcal{M}}$. It's clear that $\overline{ad} : \mathcal{H} \to End(\mathcal{G}/\mathcal{H})$ is equivalent to $ad|_{\mathcal{M}} : \mathcal{H} \to$ End $(\mathcal{M})$. Every $x \in \mathcal{G}$ can be written in the form $x = y + z$ such that $y \in \mathcal{H}$, $z \in \mathcal{M}$. If $h \in \mathcal{H}_\circ$, $w \in \mathcal{M}$, then $[[h, x], w] = [[h, y], w] = [h, [y, w]] + [y, [w, h]] = 0$, since $[y, w] \in \mathcal{M}$. So $[h, x] \in Kerad|_{\mathcal{M}}$, i.e. $\mathcal{H}_\circ$ is an ideal of $\mathcal{G}$. Since $(\mathcal{G}, \mathcal{H})$ is $ad_r - c.r.$, $\mathcal{H}_\circ \subseteq \mathcal{H}$, so $\mathcal{H}_\circ = 0$. The second part of the Proposition can be proved similarly. Q.E.D.

(2.17) Corollary. If $(\mathcal{G}, \mathcal{H})$ is $ad_r - c.r.$, n = $dim\ \mathcal{G} - dim\ \mathcal{H}$, then

$dim\ \mathcal{H} \leq n^2$, $dim\ \mathcal{G} \leq n^2 + n$.

Proof. By (2.16), $\mathcal{H}$ can embedded as a subalgebra of $End(\mathcal{G}/\mathcal{H}) \simeq End(R^n)$. Q.E.D.

(2.18) By (2.9) and (2.15), we will complete our classification in 3 steps:

Step 1. Classify, up to equivalence, the $ad_c - c.r.$ pairs (L, K) in $dim \leq 4$;

Step 2. Find all real forms of each (L, K) and classify them upto equivalence;

Step 3. Use (2.9) (i) and (ii) to obtain the final list of those $ad_r - c.r.$ infinitesimal geometries which are in the image of the map $(G, H) \to (L(G), L(H))$, where the domain of this map is the equivalence classes of $Ad - c.r.$ geometries.

Since the map in Step 3 is bijective from its domain onto its image, we can write down our classification list by describing $(L(G), L(H))$. Then we will try to describe the corresponding (G, H), especially, if a linear model is available.

The list of our classification table will use the following symbols:

(2.19) Some notations

Type Sol^n : indecomposable, solvable $ad_r - c.r.$ infinitesimal geometries in dimension n

Type Red^n : indecomposable, non abelian reductive $ad_r - c.r.$ infinitesimal geometries in dimension n

Type Mix^n : indecomposable mixed $ad_r - c.r.$ infinitesimal geometries in dimension

n

Id_n :the unit matix

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

$S_{n,A} = R^{n-1} \rtimes \langle A\rangle, \ \langle A\rangle = R[A] \subseteq gl_{n-1}(R)$

$T_{4,A} = N_3 \rtimes \langle A\rangle$

$\sigma_n : sl_2(R) \rightarrow gl_n(R)$, the n-dimensional irreducible representation of $sl_2(R)$.

(2.20) Basic Types

(2.20.1) nonabelian reductive:

$Red^2 : (\mathcal{G}^C, \mathcal{H}^C) = (A_1, C)$

Red^3 - (i) : $(\mathcal{G}^C, \mathcal{H}^C) = (A_1, 0)$

Red^3 - (ii) : $(\mathcal{G}^C, \mathcal{H}^C) = (A_1 \oplus C, C)$

Red^3 - (iii): $(\mathcal{G}^C, \mathcal{H}^C) = (A_1 \oplus A_1, A_1)$

Red^4 - (i) : $(\mathcal{G}^C, \mathcal{H}^C) = (B_2, A_1 \oplus A_1)$

Red^4 - (ii) : $(\mathcal{G}^C, \mathcal{H}^C) = (A_2, A_1 \oplus C)$

Red^4 - (iii) : $(\mathcal{G}, \mathcal{H}) = (A_1, C)_R$

(2.20.2) Solvable:

Sol^1 - (i) : $(R, 0)$

Sol^1 - (ii) : $(R \rtimes \mathcal{H}, \mathcal{H})$

Sol^2 - (i) : $(\mathcal{G}, 0)$

Sol^2 - (ii) : $(R^2 \rtimes \mathcal{H}, \mathcal{H})$

Sol^3 - (i) : $(\mathcal{G}, 0)$

Sol^3 - (ii) : $(R^3 \rtimes \mathcal{H}, \mathcal{H})$

Sol^3 - (iii) : $(N_3 \rtimes \mathcal{H}, \mathcal{H})$

Sol^3 - (iv) : $(S_{3,A} \rtimes \mathcal{H}, \mathcal{H})$

Sol^4 - (i) : $(\mathcal{G}, 0)$

Sol^4 - (ii) : $(R^4 \rtimes \mathcal{H}, \mathcal{H})$

Sol^4 - (iii) : $(N_4 \rtimes \mathcal{H}, \mathcal{H})$

Sol^4 - (iv) : $((N_3 \oplus R) \rtimes \mathcal{H}, \mathcal{H})$

Sol^4 - (v) : $(S_{4,A} \rtimes \mathcal{H}, \mathcal{H}) = (R^3 \rtimes \mathcal{H}_o, \mathcal{H})$, $\mathcal{H} \subset \mathcal{H}_o$

Sol^4 - (vi) : $(T_{4,A} \rtimes \mathcal{H}, \mathcal{H}) = (N_3 \rtimes \mathcal{H}_o, \mathcal{H})$, $\mathcal{H} \subset \mathcal{H}_o$

(2.20.3) Mixed :

Mix^2 - (i) : $(R^2 \rtimes \mathcal{H}, \mathcal{H})$

Mix^3 - (i) : $(R^3 \rtimes \mathcal{H}, \mathcal{H})$

Mix^3 - (ii) : $(R^2 \rtimes \mathcal{H}_o, \mathcal{H})$, $\mathcal{H} \subset \mathcal{H}_o$

Mix^3 - (iii) : $(N_3 \rtimes \mathcal{H}, \mathcal{H})$

Mix^4 - (i) : $(R^4 \rtimes \mathcal{H}, \mathcal{H})$

Mix^4 - (ii) : $(R^3 \rtimes \mathcal{H}_o , \mathcal{H})$, $\mathcal{H} \subset \mathcal{H}_o$

Mix^4 - (iii) : $(R^2 \rtimes \mathcal{H}_o, \mathcal{H})$, $\mathcal{H} \subset \mathcal{H}_o$

Mix^4 - (iv) : $((N_3 \oplus R) \rtimes \mathcal{H}, \mathcal{H})$,

Mix^4 - (v) : $(\ell \oplus affine(1, R), \mathcal{H})$, ℓ semisimple.

§3. $ad_c - c.r.$ Pairs in $dim \leq 4$

(3.1) Definition. We call an $ad_c - c.r.$ pair (L, K) decomposable if $L = L_1 \oplus L_2, K = K_1 \oplus K_2$ and $(L_i, K_i), i = 1, 2$ are $ad_c - c.r.$. If an $ad_c - c.r.$ is not decomposable, we call it indecomposable.

In this section, we only study indecomposable $ad_c - c.r.$ pairs. Then we can obtain the decomposable ones easily.

We will call (L, K) nonabelian reductive, resp., solvable, resp. mixed if L is nonabelian reductive, resp., solvable, resp., neither reductive nor solvable.

(3.2) Nonabelian reductive $ad_c - c.r.$ pairs in $dim \leq 4$.

(3.2.1) Theorem. The indecomposable nonabelian reductive $ad_c - c.r.$ pairs in $dim \leq 4$ are given in the following table

No.	*dim*	(L, K)
i	2	(A_1, C)
ii	3	$(A_1, 0)$
iii	3	$(A_1 \oplus C, C)$
iv	3	$(A_1 \oplus A_1, A_1)$
v	4	$(B_2, A_1 \oplus A_1)$
vi	4	$(A_2, A_1 \oplus C)$

In each of the above six cases, there is exactly one equivalence class.

Proof. First, let us prove the uniqueness of the equivalence class in each case.

Case i. $L = A_1, K = C$, i.e. $dim K = 1$, $ad K$ acts on A_1 completely reducibly, so $K = CSA A_1$, i.e. K is one of the Cartan subalgebra of A_1. We know that all Cartan subalgebras of a semisimple Lie algebra are conjugate under the inner automorphisms of A_1.

Case ii. This is trivial.

Case iii. $L = [L,L] \oplus Z(L)$, where $[L,L] = A_1$, $dimZ(L) = 1$. Since (L,K) is indecomposable , we write $K = Cx$ such that $x = x_1 + x_2, x_1 \in A_1, x_2 \in Z(L), x_1 \neq 0, x_2 \neq 0$. Then it's clear that Cx_1 is an CSA of A_1 and we can extend any inner automorphism of A_1 to an automorphism of L which keeps $[L,L]$ and $Z(L)$ invariant. If (L,K') is another pair such that $K' = Cy$, $y = y_1 + y_2, y_1 \in A_1 = [L,L], y_2 \in Z(L)$. Then there is $\theta \in Int(A_1)$ such that $\theta(Cx_1) = Cy_1$. Let $\theta(x_1) = \lambda y_1, y_1 \neq 0$, and extend θ linearly to $\bar{\theta}$ on L such $\bar{\theta}(x_1) = \lambda y_1, \bar{\theta}(x_2) = \lambda y_2$ then $\bar{\theta}(K) = K'$ and $\bar{\theta}$ is clearly an automorphism of L.

Case iv. It's well known that if $K = A_1$ is a subalgebra of $L = A_1 \oplus A_1$ then either K is one of the ideal of L or K is conjugate under $Aut(L)$ to $\{(x,x), x \in A_1\}$.

Case v and vi. $Rank(L) = Rank(K) = 2$. Since $ad : K \to End(L)$ is completely reducible, any CSA of K is a CSA of L. Using the conjugation of CSA's of L, we may choose a fixed CSA of L, say, T which is also a CSA of K. Then we have the root-space decomposation of L w.r.t T :

$L = T \oplus \sum_{\alpha \in \Phi_L} L_\alpha$

where Φ_L is the root system of L, and the root-space decomposation of K w.r.t T

$K = T \oplus \sum_{\alpha \in \Phi_K} K_\alpha = T \oplus \sum_{\alpha \in \Phi_K} L_\alpha$

where Φ_k is the root system of K, and we must have $\Phi_k \subseteq \Phi_L$. .

$\Phi_L = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2)\}$

Since $K = A_1 \oplus A_1$, $\Phi_k = \{\beta_1, -\beta_1, \beta_2, -\beta_2\}$ such that $[K_{\pm\beta_1}, K_{\pm\beta_2}] = 0$. So either $\Phi_K = \{\pm\alpha_1, \pm(\alpha_1 + \alpha_2)\}$ or $\Phi_K = \{\pm\alpha_2, \pm(2\alpha_1 + \alpha_2)\}$ .If $\Phi_K = \{\pm\alpha_1, \pm(\alpha_1 + \alpha_2)\}$, then $[K,K] = L$. Contradiction! So $\Phi_K = \{\pm\alpha_2, \pm(2\alpha_1 + \alpha_2)\}$, i.e. Φ_K is the long root set of Φ_L which is independent of the choice of the simple roots $\{\alpha_1, \alpha_2\}$. So K is uniquely determined by T.

In Case vi the root system of A_2 is

$\Phi_L = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$

$K = A_1 \oplus C$, $\Phi_K = \{\beta, -\beta\}$ can be any pair of $\{\pm\alpha_1\}, \{\pm\alpha_2\}$,or $\{\pm(\alpha_1 + \alpha_2)\}$. We know that there is an automorphism of A_2 which keeps T invariant and sends $\{L_\alpha + L_{-\alpha}\}$ to $\{L_\gamma + L_{-\gamma}\}$ for any given two pairs of roots in Φ_L.

Now let us prove that the above six cases are all the possible cases. Corollary (2.16) shows if $dimL - dimK = n$, then $dimK \leq n^2$, $dimL \leq n^2 + n$. For $n \leq 4$, $dimL \leq 20$. Since $L = [L, L] \oplus Z(L), K = [K, K] \oplus Z(K)$, where $[L, L]$ is semisimple, $[K, K]$ is semisimple or zero. By Levi' Lemma, if $[K, K]$ is not zero then $[K, K]$ is conjugate under $Aut(L)$ to a semisimple subalgebra of $[L, L]$. So we may assume that $[K, K] \subseteq [L, L]$. We need the following definition.

(3.2.1.1) Definition. If L is reductive, $RankL \overset{\text{def}}{=} Rank[L, L] + dimZ(L)$

The following Lemma is easy to prove.

(3.2.1.2) Lemma. If (L, K) is indecomposable, then

i) $dimZ(L) \geq dimZ(K)$;

ii) $dim[L, L] \geq dim[K, K] + n$, where $n = dimL - dimK$;

iii) K semisimple $\Longrightarrow$ L semisimple;

iv) $RankL \geq RankK$;

v) If $RankL = RankK$, then L is semisimple. and $dimL - dimK = n$ is even.

The following Lemmas can be obtained by discussing the root systems of semisimple Lie algebras .

(3.2.1.3) Lemma. If (L, L) is indecomposable $ad_c - c.r.$, $RanKL = RankK$ and K is semisimple, then

i) $RankL \neq 3$.

ii) $RankL = 2 \Longrightarrow (L, K)$ is $(B_2, A_1 \oplus A_2), (G_2, A_2)$ or $(G_2, A_1 \oplus A_1)$.

(3.2.1.4) Lemma.

i) $B_2 \not\subseteq A_1 \oplus A_1 \oplus \ldots \oplus A_1 \oplus A_2 \oplus A_2 \oplus \ldots \oplus A_2$;

ii) $A_2 \not\subseteq A_1 \oplus A_1 \oplus \ldots \oplus A_1$;

iii) $A_3 \not\subseteq A_1 \oplus \ldots \oplus A_1 \oplus A_2 \oplus A_2 \oplus A_2 \ldots \oplus A_2 \oplus B_2 \oplus \ldots \oplus B_2 \oplus G_2 \oplus \ldots \oplus G_2$;

iv) $G_2 \not\subseteq A_3$.

From (2.15) and (2.16), we know that if (L, L) is $ad_c - c.r.$, then K is isomorphic to a subalgebra of $End(C^n)$, where $n = \dim L - \dim K$. So for any given n, K, upto isomorphism, has only limited choices. In (3.2.1.5) we will list, upto isomorphism all reductive subalgebra of $End(C^n)$ for $n \leq 4$. Then for every such a reductive subalgebra K, we will check if there is an L such that (L, K) is indecomposable $ad_c - c.r.$.

(3.2.1.5) Table of non trivial reductive subalgebras of $End(C^n), n \leq 4$.

n	$Rank K$	K (upto isomorphism)
1	1	C
2	1	C, A_1
2	2	$C^2, A_1 \oplus C$
3	1	C, A_1
3	2	$C^2, A_2, A_1 \oplus C$
3	3	$C^3, A_2 \oplus C, A_1 \oplus C^2$
4	1	C, A_1
4	2	$C^2, A_1 \oplus A_1, A_1 \oplus C, A_2, B_2$
4	3	$C^3, A_1 \oplus C^2, A_1 \oplus_1 \oplus C, A_2 \oplus C, B_2 \oplus C, A_3$
4	4	$C^4, A_1 \oplus C^3, A_1 \oplus A_1 \oplus C^2, A_2 \oplus C^2, A_3 \oplus C$

To determine the above table, we decompose $End(C^n) = A_{n-1} \oplus C$, for $n = 2, 3, 4$, then use Lemma (3.2.1.3) and (3.2.1.4) to find all K's.

The final step of this proof is to check if there is an L for each given n and K such that (L, K) is indecomposable $ad_c - c.r.$. Here we will only give an example to show this method.

Let $n = 4$, $Rank K = 4, dim L = dim K + 4$.

Case i) $K = C^4$. So $dim L = 8, dim Z(L) \leq dim Z(K) = 4$ by (3.2.1.2)-i), i.e. $4 \leq dim[L, L] \leq 8$. So $[L, L]$ can be $A_1 \oplus A_1$ or A_2, i.e. $L = A_1 \oplus A_1 \oplus C^2$ or A_2. Since $Rank K = 4 > Rank A_2$, so L cannot be A_2. So L can only be $A_1 \oplus A_1 \oplus C^2$, i.e. $Rank L = Rank K = 4$, then L must be semisimple by (3.2.1.2) - v) and we have a contradiction.

Case ii) $K = A_1 \oplus C^3$. So $dim\ L = 10$, $7 \leq dim\ [L, L] \leq 10$ by (3.2.1.2) - i), i.e., $[L, L]$ must be A_2, $A_1 \oplus A_1 \oplus A_1$ or B_2 and L must be $A_2 \oplus C^2$, $A_1 \oplus A_1 \oplus A_1 \oplus C$ or B_2. Since $Rank L \geq Rank\ K = 4$, L can not be B_2. In the other two cases $Rank\ L = Rank\ K$, L must be semisimple which leads to a contradiction.

Case iii). $K = A_1 \oplus A_1 \oplus C^2$, $dim\ L = 12$, $dim\ Z(L) \leq 2$ i.e. $10 \leq dim\ [L, L] \leq 12$. So $[L, L]$ must be B_2, $A_2 \oplus A_1$ or $A_1 \oplus A_1 \oplus A_1 \oplus A_1$. So L must be $B_2 \oplus C^2$, $A_2 \oplus A_1 \oplus C$ or $A_1 \oplus A_1 \oplus A_1 \oplus A_1$, i.e. $Rank\ L = Rank\ K$, L must be semisimple. So L can only be $A_1 \oplus A_1 \oplus A_1 \oplus A_1$. Let T4 be the common CSA of L and K, we can write root-space decomposition of L and K:

$$L = T + \sum_{i=1}^{4} (L_{\alpha_i} + L_{-\alpha_i}),$$

$$K = T + \sum_{j=1}^{2} (K_{\beta_j} + K_{-\beta_j}),$$

where $dim\ L_{\pm\alpha_i} = dim\ K_{\pm\beta_j} = 1$, $i = 1,2,3,4$, $j=1,2$. Since $K \subsetneq L$ and the root-space decomposation of L is unique, there must be some i, j such that $K_{\beta_j} + K_{-\beta_j} = L_{\alpha_i} + L_{-\alpha_i}$, i.e. K contains a non-zero ideal of L, so (L, K) is not an $ad_c - c.r.$ pair.

Case iv). $K = A_2 \oplus C^2$, $dim\ L = 14$, $dim\ Z(L) \leq 2$, $12 \leq dim\ [L, L] \leq 14$. $[L, L]$ must be G_2, $B_2 \oplus A_1$, $A_2 \oplus A_1 \oplus A_1$ or $A_1 \oplus A_1 \oplus A_1 \oplus A_1$. So L must be G_2, $A_2 \oplus A_1 \oplus A_1$, $B_2 \oplus A_1 \oplus C$ or $A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus C^2$. Since $Rank\ G_2 < Rank\ K$, L cannot be G_2. $Rank\ (B_2 \oplus A_1 \oplus C) = Rank\ K$, so $B_2 \oplus A_1 \oplus C$ is semisimple by (3.2.1.2) then we have a contradiction. By (3.2.1.4) $A_2 \not\subseteq A_1 \oplus A_1 \oplus A_1 \oplus A_1$ so $L \neq A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus C^2$. Finally if $K \subsetneq A_2 \oplus A_1 \oplus A_1$ we can show that K contains a nonzero ideal of L as we did in Case iii).

Case v). $K = A_3 \oplus C$, $dim\ L = 20$, $dim\ Z(L) \leq 1$, $dim\ [L, L] = 13$ or 20. By (3.2.1.4) - iii), $[L, L]$ must have a simple ideal whose rank is at least 3, i.e. $[L, L] = A_3 \oplus W$, $dim\ W = 4$ or 5. But there is no semisimple Lie algebra in dim 4 or 5.

Q.E.D.

(3.3) Mixed $ad_c - c.r.$ pairs in dimension ≤ 4.

In this section let $\sigma_i : sl_2(C) \to gl_2(C)$ be the i-dimensional irreducible representation of $sl_2(C)$.

(3.3.1) Theorem. The indecomposable mixed $ad_c - c.r.$ pairs, upto equivalence, are given in the following table:

$dim = 2$, $(L, K) = (C^2 \rtimes K, K)$, where $K =$

i) $sl_2(C)$,

ii) $gl_2(C)$,

$dim = 3$, Type -i, $(L, K) = (C^3 \rtimes K, K)$, where $K =$

i) $\sigma_3(sl_2(C))$,

ii) $\sigma_3(sl_2(C)) \oplus CId_3$,

iii) $sl_3(C)$,

iv) $gl_3(C)$,

v) $\begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix} \oplus C \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & t \end{pmatrix}, t \neq 0,$

$dim = 3$, Type -ii, $(L, K) = (C^2 \rtimes gl_2(C), sl_2(C))$,

$dim = 3$, Type -iii , $(L, K) = (N_3^C \rtimes K, K)$, where $K =$

i) $\begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix}$,

ii) $\begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix} \oplus C \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

In Type iii , $N_3^C = Ce_1 + Ce_2 + Ce_3$ such that $[e_1, e_2] = e_3, [e_3, e_i] = 0, i = 1, 2.$

$dim = 4$, Type-i , $(L, K) = (C^4 \rtimes K, K)$, where $K =$

i) $gl_4(C)$,

ii) $sl_4(C)$,

iii) $so_4(C)$,

iv) $so_4(C) \oplus CId_4$,

v) $\begin{pmatrix} sl_3(C) & 0 \\ 0 & 0 \end{pmatrix} \oplus C \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & t \end{pmatrix}, t \neq 0,$

vi) $\begin{pmatrix} sl_2(C) & 0 \\ 0 & sl_2(C) \end{pmatrix} \oplus C \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & t & \\ 0 & & & t \end{pmatrix}, t \neq 0,$

vii) $\sigma_4(sl_2(C)),$

viii) $\sigma_4(sl_2(C)) \oplus CId_4,$

ix) $\begin{pmatrix} \sigma_3(sl_2(C)) & 0 \\ 0 & 0 \end{pmatrix} \oplus C \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & t \end{pmatrix}, t \neq 0,$

x) $C \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} \oplus C \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \oplus C \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix},$

xi) $C \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} \oplus C \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \oplus C \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix},$

$\oplus \mathrm{C} \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & t & \\ 0 & & & t \end{pmatrix}$

xii) $\begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix} \oplus C \begin{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \oplus C \begin{pmatrix} \begin{pmatrix} t_2 & 0 \\ 0 & t_2 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix},$

$t_1 t_2 \neq 0$

xiii) $\begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix} \oplus C \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & t_1 & \\ 0 & & & t_2 \end{pmatrix}, t_1 t_2 \neq 0,$

xiv) $\sigma(sl_2(C) \oplus sl_2(C)),$

xv) $\sigma(sl_2(C) \oplus sl_2(C)) \oplus CId_4,$

In xiv) and xv) σ is the product representation.

$dim = 4$, Type-ii , $(L, K) = (C^3 \rtimes K', K)$

i) $K' = gl_3(C), K = sl_3(C),$

ii) $K' = \sigma_3(sl_2(C)) \oplus CId_3, K = \sigma_3(sl_2(C))$,

iii) $K = \begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix}, K' = K \oplus C \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & t \end{pmatrix}, t \neq 0$,

iv) $K = \begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix} \oplus C \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & t \end{pmatrix}, t \neq 0$,

$$K' = \begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix} \oplus \left\{ \begin{pmatrix} a & & 0 \\ & a & \\ 0 & & b \end{pmatrix} ; a, b \in C \right\}$$

dim 4 , Type-iii , $(L, K) = (C^2 \rtimes K', K)$

i) $K' = sl_2(C), K = C \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

ii) $K' = gl_2(C), K = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} ; a, b \in C \right\}$

dim 4 , Type-iv , $(L, K) = ((N_3^C \oplus C) \rtimes K, K)$, where $N_3^C \oplus C = Ce_1 + Ce_2 + Ce_3 + Ce_4$, s.t. $[e_1, e_2] = e_3, [e_3, e_i] = 0, i = 1, 2, [e_4, e_j] = 0, j = 1, 2, 3$, then

$$K = \begin{pmatrix} sl_2(C) & 0 \\ 0 & 0 \end{pmatrix} \oplus C \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 2 & \\ 0 & & & t \end{pmatrix}, t \neq 0.$$

dim 4 , Type-v, $(L, K) = (affin(1, C) \oplus sl_2(C), C)$, where $affin(1, C) = Ce_1 + Ce_2$, such that $[e_1, e_2] = e_2$ and

$$K = C\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_1\right).$$

Proof. Let $L = R \rtimes S$ be a Levi's decomposation of L. By Malcev-Harish-Chandra Theorum, we may assume that $[K, K] \subseteq S$ if $[K, K] \neq 0$. Let $P_R : K \to R, P_S : K \to S$ be the projection map from K to R, S respectively. $Z_1 = Z(K) \cap R$, $Z_2 = P_S(Z(K))$. The following lemma will play the key role in this proof.

(3.3.1.1) Lemma. If (L, K) is a mixed $ad_c - c.r.$ pair in dimension n, then

i) $P_S(K)$ is reductive, $Z(P_S(K)) = P_S(Z(K)) = Z_2$, $[P_S(K), P_S(K)] = [K, K]$, $P_S(K) = Z_2 \oplus [K, K]$.

ii) $P_S(K)$ acts (as the adjoint representation on L) completely reducibly.

iii) $n = (dim\ R - dim\ Z_1) + (dim\ S - dim\ Z_2 - dim\ [K,K])$.

The first part of the Lemma is based on the fact that $P_S : K \to S$ is a Lie algebra homomorphism. To prove the second part, we only have to show that S is invariant under adZ_2 and adZ_2 is diagnolizable, so $Z_2|_S$ can be contained in a CSA of S and any representation of S (which is semisimple) restricted on its CSA is diagonolizable, this implies the complete reducibility of $ad(P_S(K))$ on L. The last part of this Lemma is trivial.

(3.3.1.2) Corollary. $P_S(K)$ is isomorphic to a reductive subalgebra of $gl_n(C)$.

For $n \leq 4$, all such reductive subalgebras are given in (3.2). Our first step is to find all such pairs (S,P) that S semisimple, $dim\ S \leq n^2 + n$, P is a reductive subalgebra of S and $adP|_S$ is completely reducible, i.e. any CSA of P is a subspace of a CSA of S. The second step is to find all indecomposable mixed $ad_c - c.r.$ pairs (L,K) such that $L = R \times S$ and $P_S(K) = P$. The following Lemma shall be needed in step 2.

(3.3.1.3) Lemma. If (L,K) is an indecomposable mixed $ad_c - c.r.$ pair in dimension $n, L = R \times S$ as before, $Z_1 = Z(K) \cap R$, then

i) $dim\ S$ - $dim\ P_S(K) \leq n - 1$;

ii) $2\ dim\ Z_1 \leq dim\ R$;

iii) R abelian $\Longrightarrow Z_1 = \{0\}$;

iv) $n \leq 4 \Longrightarrow dim\ S - dim\ P_S(K) \neq 3$;

Part i) in this Lemma is based on the requirement of indecomposability. Part ii) and iii) are based on the fact that K can not contain a non zero ideal of L, i.e. we can prove that otherwise $K \cap Z(L) \neq 0$ by finding an element $z \in Z$, such that $adz|L$ is both nilpotent and diagonalizable. Part iv) is a corollary of ii) and iii).

Using (3.3.1.3) and the list obtained in Step 1, plus the well known results about the dimensions of irreducible representations of simple Lie algebras, we obtain a table of all possible pairs (S,P) which will be used to build indecomposable mixed $ad_c - c.r.$ pairs.

(3.3.1.4) Table of all (S, P), where S semisimple, P is a reductive subalgebra of S such that there may exist an indecomposable mixed $ad_c - c.r.$ pairs (L, K) in $dim \leq 4$ with the Levi's decomposation $L = R \times S$ and $P_S(K) = P$.

	dim S - dim P	(S, P)
i)	0	$(A_1 \ , A_1 \ )$
ii)	0	$(A_1 \oplus A_1 \ , A_1 \oplus A_1)$
iii)	0	$(A_2 \ , A_2 \ )$
iv)	0	$(B_2 \ , B_2 \ )$
v)	0	$(A_3 \ , A_3 \ )$
vi)	2	$(A_1 \ , C \ )$, where $P = CSA(A_1) \simeq C$

In Table (3.3.1.4) case i) - v) , we have $P_S(K) = P = S = [K, K]$ semisimple, so $Z_2 = P_S(Z(K)) = Z(P_S((k)) = 0$ and $Z(K) = Z_1 \overset{\text{def}}{=} K \cap R$. So we can write $(L, K) = (R \rtimes S, Z(K) \oplus S) = ((V + Z(K))) \times S, Z(K) \oplus S)$, where V is a complement of $Z(K)$ in R such that $[K, V] \subseteq V$, $dim\ V = n \leq 4$, $dim\ Z(K) \leq dim\ V$ by (3.3.1.3) - ii). On the other hand, K has to be isomorphic to a reductive subalgebra, so by Table (3.2.1.5) if K is not abelian, $dim\ Z(K) \leq 1$ if S = [K , K] = A_3 or B_2, dim Z(K) $\leq$ 2 if [K ,K] is A_2 or $A_1 \oplus A_2$ and $dim\ Z(K) \leq 3$ if $[K, K]$ is A_1. So in each case, the dimension of R has very limited choices.The semisimple Lie algebra $S = [K, K]$ acts on V as a derivation as well as a representation of S on a space whose dimension is ≤ 4. All such representations of S in Case i) - v) are well known and are finite many (upto equivalence). So to complete step 2, we need only to consider all possible representations of S on V, then consider all possible $Z(K) = \{0\}, C, C^2$ or C^3 (depends on S), such that $V + Z(K)$ forms a Lie algebra $[V + Z(K)]$ acts on $V + Z(K)$ as a derivation. Here we will only give some examples to show this method.

Case iii) $(S, P) = (A_2 \ , A_2)$

By Table (3.2.1.5) n can be 3 or 4. If $n = 3$, $Z(K) = \{0\}$ or C, if $n = 4$, $Z(K) = \{0\}, C$ or C^2.

If $L = R \rtimes S$ such that $[R, S] = 0$, then S is an ideal of L contained in K. So we only consider the case when $[R, S] \neq 0$.

First let us recall some facts about A_2. Let $\{\alpha_1 , \alpha_2\}$ be the simple root-system of A_2. Let λ_1 , λ_2 be the fundamental weights s.t. $\lambda_1 = (1/3)\,(2\alpha_1 + \alpha_2)$, $\lambda_2 = (1/3)\,(\alpha_1 + 2\alpha_2)$. Let $V(\lambda)$ be the irreducible module of A_2 with the highest weight $\lambda = \mathrm{m}_1\lambda_1 + \mathrm{m}_2\lambda_2$, where m_1 , m_2 are nonnegative integers, then *dim* $V(\lambda) = (1/2)(\mathrm{m}_1 + 1)\,(\mathrm{m}_2 + 1)\,(\mathrm{m}_1 + \mathrm{m}_2 + 2)$. So *dim* $V(0) = 1$, *dim* $V(\lambda_1) =$ *dim* $V(\lambda_2) = 3$, *dim* $V(2\lambda_i) = 6$, $i = 1,2$, and *dim* $V(\lambda_1 + 2\lambda_2) = 8$, ...

It's clear that if $L = R \rtimes S$, $K = S \oplus Z(K)$, $Z(K) \subseteq R$ and $[R, S] \neq 0$, then there is a vector space $V \subseteq R$ such that $V + Z(K) = R$, $[Z(K) + S, V] \subseteq V, [S, V] \neq 0$. So V, as a A_2-module, must contain an irreducible A_2-module with dimension at least 3. Since *dim* $V \leq 4$, this irreducible module must be $V(\lambda_i)$, $i = 1$ or 2. We may assume $i = 1$, then $V(\lambda_1) \subseteq V$, *dim* V - *dim* $V(4\lambda_1) = 0$ or 1. We have $V = V(\lambda_1)$ if $n = 3$ and $V = V(0) + V(\lambda_1)$ if $n = 4$.

We know the weights set of $V(\lambda_1)$ is $\Pi(\lambda_1) = \{\lambda_1, \lambda_1 - \alpha_1, \lambda_1 - \alpha_1 - \alpha_2\}$, i.e. $\{(1/3)\,(2\alpha_1 + \alpha_2),\ (1/3)\,(-\alpha_1 + \alpha_2),\ 1/3\,(-\alpha_1 - 2\alpha_2)\}$, each of these weights has a single multiplilicity. Since $[S, Z(K)] = 0, [S, V(0)] = 0$, we can write

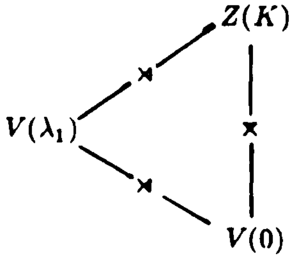
$$R = R_0 + \textstyle\sum_{\alpha\in\Pi(\lambda_1)} R_\alpha,$$

where R_α is the weight space of weight α for $\alpha \in \Pi(\lambda_1)$, R_0 is the weight space of weight zero and $R_0 = Z(K) + V(0)$. Since S acts on R as the inner derivation , i.e. $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$, we must have

$$[R_\alpha, R_\beta] \subseteq \begin{cases} R_{\alpha+\beta}, & \text{if } \alpha + \beta \in \Pi(\lambda_1) \\ \{0\}, & \text{otherwise.} \end{cases}$$

We see immediately that $[R_0, V(\lambda_1)] \subseteq V(\lambda_1)$, $[V(\lambda_1), V\lambda_1)] = 0$, i.e. $V(\lambda_1)$ is abelian and $[R_0, R_0] \subseteq R_0$. Since $[Z(K) + V(0)] \subseteq V(0) + V(\lambda_1)$, we must have $[Z(K), V(0)] \subseteq V(0)$ in case $n = 4$. Notice that $[R_0, S] = 0$ and $V(\lambda_1)$ is an irreducible module,

$adR_0|_{V(\lambda_1)}$ must be scalers, also $adZ(K)|_{V(0)}$ is scalers since $dim\ V(0) = 1$, i.e. when $n = 4$, we must have the following relations



where all " × "'s are scalers.

Claim. If $Z(K) \neq 0$, then $[V(0)\ ,\ V(\lambda_1)] = 0$

Proof of the Claim. Let $0 \neq X \in Z(K), 0 \neq Y \in V(0)\ , 0 \neq Z \in V(\lambda_1)$ s.t.

$adX|_{V(\lambda_1)} = a, adX|_{V(0)} = b, adY|_{V(\lambda_1)} = c,$

then $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$ implies

$[X, cZ] = [by, Z] + [Y, aZ],$

$acZ = bcZ + acZ,$

$bc = 0,$

So $b = 0$ or $c = 0$. If $b = 0$, then $a \neq 0$, otherwise $[Z(K), S + V(0) + V(\lambda_1)] = 0$, $K \cap Z(L) \neq 0$, i.e. K contains a non-zero ideal of L. Then

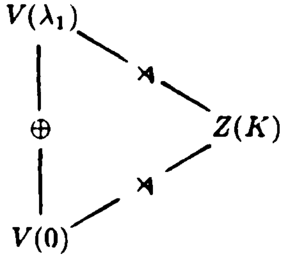
$ad(-\frac{c}{a}X + Y)|_{V(\lambda_1)} = 0,$

i.e.

$(L, K) = ((V(\lambda_1) + Z(K)) \times S, S) + (C(-\frac{c}{a}X + Y), 0)$

is decomposable. So c must be zero.

So we will have



if $Z(K) \neq 0$. In this case, if $dim\ Z(K) = 2$, then it's easy to show that $Z(K) = C \oplus C$ such that

$$\begin{tikzcd}
V(\lambda_1) \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \rtimes \arrow[r, dash] \arrow[d, dash] & C \arrow[d, dash] \arrow[dl, dash] \\
\oplus \arrow[d, dash] & \oplus \arrow[d, dash] \arrow[dl, dash] \arrow[dr, dash] & \oplus \arrow[d, dash] \\
V(0) \arrow[r, dash] & \rtimes \arrow[r, dash] & C
\end{tikzcd}$$

and again we have a decomposable case

$(L, K) = (affin(1, C), 0) + (C^3 \rtimes gl_3(C), gl_3(C))$.

So $dim\ Z(K) \leq 1$.

Similarly we can prove that if $n = 3$, $dim\ Z(K) \leq 1$.

Summarizing our results in matrix forms , we get

$n = 3$:

i) $(C^3 \rtimes sl_3(C), sl_3(C))$,

ii) $(C^3 \rtimes gl_3(C), gl_3(C))$,

n = 4 :

i) $$(C^4 \rtimes \left\{\begin{pmatrix} sl_3(C) & 0 \\ 0 & 0 \end{pmatrix} + C\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & t \end{pmatrix}\right\}, \begin{pmatrix} sl_3(C) & 0 \\ 0 & 0 \end{pmatrix} + C\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & t \end{pmatrix}), t \neq 0$$

ii) $(C^3 \rtimes gl_3(C), sl_3(C))$.

It's easy to show for different $0 \neq t \in C$, we will get unequivalent pairs in Case $n =$ 4, i).

The rest cases in Case i) - Case v) can be solved in the same way. The proof of Case i) is much more complicated than that of Case iii) since $dim\ Z(K)$ is higher and consequently the structures of V and R are much more complicated and more interesting. But the discussion is too long and we are not going to write it here.

Finally, we will give a brief discussion about Case iv) which is the only case that $P \neq S$ in (S, p).

Case vi). $(S, P) = (A_1\ , C)$, where $C = P =$ a CSA of A_1. We can choose a basis of A_1 :$\{Z, X, Y\}$, such that $[X, Y] = Z, [Z, X] = 2X, [Z, Y] = -2Y$ and $P = CZ$. Suppose that (L, K) is an indecomposable mixed $ad_c - c.r.$ pair such that $L = R \rtimes S, S = A_1$, $[K, K] = 0$, $P_S(K) = Z_2 = C$. Now $K = Z(K)$ is abelian. By (3.3.1.1) - iii),

$n = (dim\ R$ - $dim\ Z_1) + (dim\ S$ - $dim\ Z_2$ - $dim\ [K, K])$,

Now $Z_1 = K \cap R$, $[K, K] = 0$, $dim\ S - dim\ Z_2 = 2$, so $dim\ R$ - $dim\ Z_1 = n - 2$. Then by (3.3.1.3) - ii) $dim\ Z_1 \leq dim\ R - dim\ Z_1 = n - 2$. By (3.3.1.3) - i), $n - 1 \geq 2$, $n = 3$ or 4. So $dim\ Z_1 \leq 1$, $dim\ K \leq 2$ for $n = 3$, $dim\ Z_1 \leq 2$, $dim\ K \leq 3$ for $n = 4$. The rest discussion is similar to that of Case i) - Case v).

Q.E.D.

(3.4) Solvable ad_c – *c.r.* pairs.

(3.4.1) Definition. An $ad_c - c.r.$ pair (L, K) is called **nil-affine** if there is a nilpotent ideal N of L such that $L = N \rtimes K$.

(3.4.2) Definition. An $ad_c - c.r.$ pair (L, K) is called **truncated-nil-affine** if there is a nilpotent ideal N of L and a subalgebra $K1$ of L such that $L = N \rtimes K$ and $K \subseteq K_1$.

(3.4.3) Proposition. A solvable $ad_c - c.r.$ pair (L, K) in dimension ≤ 4 is either nil-affine or truncated-nil-affine.

To prove (3.4.3) we need the following Lemma due to Ravi Kulkarni:

(3.4.3.1) Lemma. (L, K) as in (3.4.3), then

i) K is abelian;

ii) if L is nilpotent then $K = \{0\}$;

iii) let $L = L_0 + \sum_{\alpha \in \Phi} L_\alpha$ be the root space decomposation of L w.r.t. K, where Φ is the set of nonzero roots. Let $L_+ = \sum_{\alpha \in \Phi} L_\alpha$ and $n_\circ$ be the nilradical of L, then the geometry is nil-affine under any of the following conditions.

a) $dim\ K + dim\ n_\circ \leq dim\ L$;

b)$K = L_0$;

c) $dim\ L_0$ - $dim\ K = 1$ and L_+ is not an ideal in L.

Proof of the Lemma.

Consider i). Since $K \simeq \overline{ad}|_{L/K}(\mathrm{K})$ is solvable and acts completely reducibly, K must be abelian.

Consider ii). adK acts on L both nilpotently and completely reducibly. So $adK = 0$. Hence $K = \{0\}$.

Now consider iii). Since $K \cap n_\circ$ acts both nilpotently and completely reducibly, we have $K \cap n_\circ = 0$. So by a) $dim\ K + dim\ n_\circ = dim\ L$. Since $[K, n_\circ] \subseteq [L, L]$ which is nilpotent hence must be contained in $n_\circ$, so $[K, n_\circ] \subseteq n_\circ$ and $(L, K) = (n_\circ \times K, K)$. By b), we have $L_+ = [K, L_+] \subseteq [L, L] \subseteq n_\circ$, so L_+ is nilpotent and we can write L $= L_+ \times L_0 = L_+ \times K$. In this case $L_+ = n_\circ$. Finally consider c). Since $L_+ \subseteq n_\circ$ and L_+ is not an ideal by assumption, we must have $L_+ \neq n_\circ$. So

$$dim\ L = dim\ L_0 + dim\ L_+ = dim\ K + 1 + dim\ L+ \leq dim\ K + dim\ n_\circ\ .$$

Again the conclusion follows by a).

Q.E.D.

(3.4.3.2) Corollary. *dim* K = *dim* $\{span\{\alpha;\ \alpha\ \ in\ \Phi\ \}\}$.

Proof. Since $\alpha\ \in K^*$ and *dim* K^* = *dim* K, we have

dim $K \geq$ *dim* $\{span\{\alpha;\ \alpha\ \in \Phi\ \}\}$.

But if dim $K >$ *dim* $\{span\{\ \alpha\in \Phi\ \}\}$, then there is $z \in K, z \neq 0$, s.t. $\alpha(z) = 0\ \forall\, \alpha\in \Phi$. So $[z, L] = 0$, namely $K \cap Z(L) \neq 0$, contradicts that K can not contain a nonzero ideal of L.

Q.E.D.

Proof of Proposition (3.4.3). We only have to consider the case when (L, K) is indecomposable. We also assume that $K \neq 0$ since if $K = 0$, the Proposition is obviously true . Let n = *dim* $L-$ *dim* K be the dimension of the pair (L, K). We will consider n = 1, 2, 3 and 4 separately.

(3.4.3.3) $n = 1$.

Since $L = L_0 + L_+$, $K \subseteq L$, we must have *dim* $L_+ = 1$ and $K = L_0$. By Lemma (3.4.3.1)-iii)-b), the Proposition is true. By (3.4.3.2) *dim* $K = 1$ since there can only be one root. So $(L, K) = (C \rtimes C, C)$.

(3.4.3.4) n = 2

We have two cases : *dim* $L_+ = 2$ or 1. If *dim* $L_+ = 2$ then $L_0 = K$. By Lemma (3.4.4.1)-iii)-b) the pair is nil-affine, namely $L = L_+ \rtimes K$, where L_+ is nilpotent. Sicne *dim* $L_+ = 2$, L_+ must be abelian. So $(L, K) = (C^2 \rtimes K, K)$. Since (L, K) is indecomposable *dim* K must be 1. So (L, K) has the form $(C^2 \rtimes C, C)$.

If *dim* $L_+ = 1$, then *dim* L_0 - *dim* K = 1. Since we have only one non- zero root, *dim* K = 1 by (3.4.3.2). So *dim* $L_0 = 2$. Since $[L_0, L_+] \subseteq L_+$, *dim* $L_+ = 1$, adL_0 acts on L_+ as scalars. Since *dim* $L_0 = 2$, there is $z \in L_0$, $z \neq 0$, $z \in K, [z, L_+] = 0$. So $(L, K) = (L_+ \rtimes K, K) + (Cz, 0)$ which is decomposable.

(3.4.3.5) $n = 3$.

We have 3 cases according to *dim* L_+ = 1, 2 or 3.

i) $dim\ L_+ = 3$. Then we must have $K = L_0$, $\mathrm{n}_\circ = L_+$ and $(L, K) = (L_+ \rtimes K, K)$ by (3.4.3.2). As 3-dimensional nilpotent Lie algebra , $L_+ \simeq N_3^C$ or C^3, K has 3 nonzero roots on L_+. We will determine (L, K) later.

ii) $dim\ L_+ = 2$. dim L_0 - dim K = 1. Consider $[L, L]$ which is a nilpotent ideal of L and contained in $n_\circ$. We have

$L_+ \subseteq [L, L] \subseteq n_\circ$.

If L_+ is not an ideal, $L_+ \subseteq n_\circ$, but $L_+ \neq \mathrm{n}_\circ$. By (3.4.3.2), the Proposion is true. Since $[L_0,\ L_+] \subseteq L_+$, we must have $[L_+,\ L_+] \subseteq L_+$. So $L_+ \subseteq [L, L] = n_\circ$, $dim\ [L, L] = 3$. Hence $[L, L] = n_\circ \simeq N_3^C$, $(L, K) = (N_3^C \rtimes K, K)$ and K has two nonzero roots and one zero root on N_3^C. We will determine K later. If L_+ is an ideal, then $[L_+,\ L_+] = 0$, i.e., $L_+ \simeq C^2, (L, K) = (C^2 \rtimes L_0,\ K)$ so the pair is trucated-nil-affine. Since $dim\ L_+ = 2$, we can write $L_+ = L_\alpha + L_\beta$ where α and β are independent, $L_+ = L_\alpha + L_{t\alpha}$ where $t \neq 1$, $t \neq 0$, or $L_+ = L_\alpha$. In the first case $dim\ K = 2$; in the rest two cases $dim\ K = 1$. Since $[L_0,\ L_\alpha] \subseteq L_\alpha$, $[L_0,\ L_{t\alpha}] \subseteq L_{t\alpha}$, adL_0 acts on L_+ completely reducibly. dim L = 3 implies that there is $0 \neq z \in L_0$ s.t.$[z, L_+] = 0$. So the pair is decomposable: $(L, K) = (L_\alpha,\ K_1) + (L_\beta,\ K_2) + (Cz,\ 0)$ where $K_1 + K_2 = K$. If $L_+ = L_\alpha + L_{t\alpha}$, $t \neq 1$, $t \neq 0$ we also have $[L_0,\ L_{t\alpha}] \subseteq L_{t\alpha}$, i.e., adL_0 acts on L_+ completely reducibly and $[L_0,\ L_0] = 0$. adL_0 must act on L_+ with two indpendent roots, otherwise there is $0 \neq z \in L_0$ s.t. $[z, L_+] = 0$ and again it leads to a decomposable case

$(L, K) = (L_+, K) + (Cz, 0)$.

So the indecomposable pair is

$$(L, K) = (C^2 \rtimes \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in C\right\}, C\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}),\ t \neq 0, t \neq 1.$$

Finally consider $L_+ = L_\alpha$, $L_+ \times K = C^2 \rtimes C\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If adL_0 acts on L_+ completely reducibly, then

$$(L, K) = (C^2 \rtimes \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in C\right\}, C\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}).$$

If adL_0 doesn't act on L_+ completely reducibly, we can write, w.r.t. a suitable basis

$$(L,K) = (C^2 \rtimes \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}; a,b \in C \right\}, C \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}).$$

It's easy to see that (L,K) is equivalent to

$$(N_3^C \rtimes C \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, C \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}),$$

where the matrix is w.r.t. the basis { X , Y , Z } of N_3^C s.t. $[X,Y] = Z, Z \in Z(N_3^C)$. So this is a nil-affine pair.

iii) $dim\ L_+ = 1$. We have only one nonzero root, so $dim\ K = 1$, $dim\ L_0 - dim\ K = 2$, $dim\ L_0 = 3$. Since adL_0 acts on L_+ completely reducibly, $[L_0, L_0] = 0$ will lead to a decomposable pair

$(L,K) = (L_+ \rtimes K, K) + (C, 0) + (C, 0)$.

If $[L_0, L_0] \neq 0$, we have $L_0 = K + K', K' \simeq C \rtimes C, [K, K'] = 0$ since $[L_0, K] = 0$, $[L_0, L_0] \subseteq L_0$. Hence $dim\ [L,L] = 2, [L,L]$ is abelian since it is nilpotent. If we write $K' = CX + CY$ s.t. $[X,Y] = Y$. Then $[Y, L_+] = 0$, $CY + L_+ = [L,L] = C^2$, $ad(K + CX)$ acts on $[L,L]$ completely reducibly. So the pair is decomposable

$(L_+ \rtimes K, K) + (C \rtimes C, 0)$.

(3.4.3.6)$n = 4$.

Since $[L,L]$ is nilpotent, $[L,L] \cap K = \{0\}, [L,L] + K$ is a direct sum. adK keeps $[L,L] + K$ invariant so $L = [L,L] + K + K_1$, s.t. $[K, K_1] \subseteq K_1$. Hence $K_1 \subseteq [L,L] \cap K_1 = \{0\}$, namely $K_1 \subseteq L_0$. Since $L_+ \subseteq [L,L]$, we can decompose $[L,L]$ into

$[L,L] = L_+ + K_2$

s.t. $K_2 \subseteq L_0$. Since $[K, K_2] = \{0\}$, we must have

$[K_1 , K_2] \subseteq [L_0 , L_0] \cap [L,L] \subseteq L_0 \cap [L,L] = K_2$,

$[K_1 , L_+] \subseteq [L_0 , L_+] \subseteq L_+$,

$[L_0 , L_0] = [K + K_1 + K_2, K + K_1 + K_2] = [K_1 + K_2, K_1 + K_2]$.

Since $[L,L] = [L_+ + L_0, L_+ + L_0] = L_+ + [L_0, L_0]$, we have

$[K_1 + K_2 , K_1 + K_2] = K_2$.

So L has the following structure

$$\begin{tikzcd}
 & & K_1 \\
L = L_+ \arrow[urr, dash, "\rtimes"] \arrow[r, dash] \arrow[drr, dash, "\rtimes"'] & K_2 \arrow[ur, dash, "\rtimes"] \arrow[dr, dash, "\oplus"'] & \\
 & & K \arrow[uu, dash, "\oplus"']
\end{tikzcd}$$

Since $dim\ L-\ dim\ K = 4$, $dim\ L_+ + dim\ K_2 + dim\ K_1 = 4$, $0 \leq dim\ K_1 \leq 3$. We will discuss $dim\ K_1 = 0, 1, 2$, and 3 separately.

i) $dim\ K_1 = 0$. $L = [L, L] \rtimes K$. The pair is nil-affine. $[L, L]$ is C^4, N_4^C or $N_3^C \oplus C$. We will discuss the derivation of K on [L , L] later.

ii) $dim\ K_1 = 3$. $dim\ L_+ = 1, K_2 = 0$. So $[K_1, K_1] = 0$. $K_1 \simeq C^3, adL_0$ acts on L_+ completely reducibly and (L, K) is decomposable.

$(L, K) = (L_+ \rtimes K, K) + (C, 0) + (C, 0) + (C\ , 0)$.

iii) $dim\ K_1 = 1$. Then $dim\ [L, L] = 3$, so $[L, L]$ is N_3^C or C^3 and the pair is truncated-nil-affine

$(L, K) = (C^3 \rtimes (C \oplus K), K)$ or $(N_3^C \rtimes (C \oplus K), K)$.

We will classify them later.

iv) $dim\ K_1 = 2$. Then $dim\ [L, L] = 2, [L, L] \simeq C^2$ is abelian. We have two subcases: $dim\ L_+ = 2$, $dim\ K_2 = 0$ and $dim\ L_+ = dim\ K_2 = 1$.

Consider the first one: $[L, L] = L_+, K_2 = 0$. Then $[K_1, K_1] \subseteq K_2 = 0$, namely K_1 is abelian. If $L_+ = L_\alpha + L_\beta$ where $\alpha \neq \beta$, then $[L_0, L_\alpha] \subseteq L_\alpha$, $[L_0\ , L_\beta] \subseteq L_\beta$, $L_0 \simeq C^k$ acts on $L_+ \simeq C^2$ completely reducibly with $k \geq 3$. So the pair is decomposable

$(L, K) = (L_+ \rtimes L'_0, K) + (Cz, 0)$,

where $Cz + L'_0 = L_0$ s.t. $[z, L_+] = 0$. So $L_+ = L_\alpha$, namely $dim\ K = 1$,

$$adK|_{L_+} = C \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $x \in K_2$, then w.r.t. a suitable basis,

$$adx|_{L_+} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

and there is $x' \in K$ s.t.

$$adx'|_{L_+} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

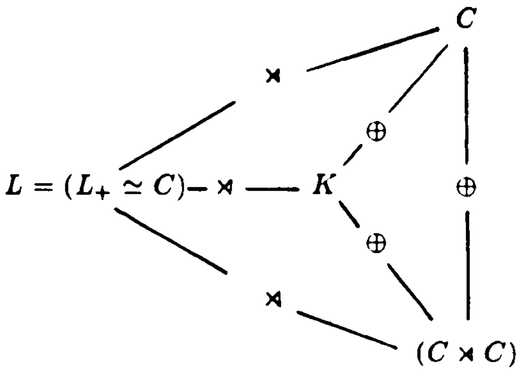
then $ad(x - x')|_{L_+}$ is nilpotent. Similarly if$\{ e_1, e_2\}$ is a basis of K_2, then $ad(e_1 - e_1')|_{L_+}$ and $ad(e_2 - e_2')|_{L_+}$ are nilpotent. Replace K_2 by $K_2{}' = C(e_1 - e_1') + C(e_2 - e_2')$ then for every $x \in K_2{}'$, $adx|_{L_+}$ is nilpotent since $K_2{}' \subseteq L_0$ which is abelian. So $adK_2{}'|_{L_+}$ is a nilpotent representation. By a corollary of the well known Engel's Theorem, there is a basis of L_+ s.t. with respect to which the matrix of each $adx|_{L_+}, x \in K_2{}'$, has the form $\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$. Since $dim\ K_2{}' = 2$, there is $x \in K_2{}'$ s.t. $[x, L_+] = 0$, namely $[x, L] = 0$ Hence the pair is decomposable

$$(L, K) = (L_+ \rtimes L_0', K) + (Cx, 0),$$

where $Cx + L_0' = L_0$.

Now consider the second case : $dim\ L_+ = dim\ K_2 = 1$. We get $dim\ K = 1$. Then $[K_1 + K_2, K_1 + K_2] = K_2$. So $K_1 + K_2$ is a 3-dimensional solvable subalgebra of L, s.t. $dim\ [K_1 + K_2, K_1 + K_2] = 1$. If $K_1 + K_2$ is decomposable, then $K_1 + K_2 \simeq C \oplus C \rtimes C$; if $K_1 + K_2$ is indecomposable, then $K_1 + K_2 \simeq N_3^C$.

Subcase i) $K_1 + K_2 \simeq C \oplus C \rtimes C$, then



Hence $ad(K \oplus C)$ acts on L_+ completely reducibly, since $dim\ L_+ = 1$. So there is $0 \neq z \in K \oplus C \subseteq L_0$ s.t. $[z, L_+] = [z, L] = 0$. Hence the pair is decomposable

$(L, K) = (L_+ \rtimes (C \rtimes C \oplus K), K) + (Cz, 0)$.

Subcase ii) is also decomposable since L has the following structure

$$\begin{tikzcd}
& C \simeq K \arrow[dd, dash, "\oplus"] \\
L = (L_+ \simeq C) \arrow[ur, dash, "\rtimes"] \arrow[dr, dash, "\rtimes"'] & \\
& N_3^C
\end{tikzcd}$$

But 1-dimensional representation of N_3^C is trivial, so

$[L_+, N_3^C] = 0$ and

$(L, K) = (L_+, K) + (N_3^C, 0)$.

We have completed the proof of (3.4.3)

Q.E.D.

(3.4.4) Corollary. An indecomposable $ad_c - c.r.$ pair (L, K) in *dim* ≤ 4 must, upto equivalence, have one of the following forms:

dim 1 : i) $(C \times C, C)$,

dim 2 : i) $(C^2 \rtimes C\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}, C\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}), t \neq 0$,

dim 3 : i) $(C^3 \rtimes K, K)$, *dim* $L_+ = 3$,

ii) $(N_3^C \rtimes K, K)$, *dim* $L_+ = 2$ or 3,

iii) $(C^2 \rtimes \left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}; a, b \in C\right\}, C\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}), t \neq 0$,

dim 4 : i) $(C^4 \rtimes K, K)$, $L_+ = [L, L] = C^4$,

ii) $(N_4^C \rtimes K, K), L_+ = [L, L] = N_4^C$,

iii) $((N_3^C \oplus C) \rtimes K, K), L_+ = [L, L] = N_3^C \oplus C$,

iv) $(C^3 \rtimes (C \oplus K), K), [L, L] = C^3$,

v) $(N_3^C \rtimes (C \oplus K), K), [L,L] = N_3^C$.

(3.4.5) Classification of indecomposable $ad_c - c.r.$ pairs in dimension ≤ 4.

The list in (3.4.4) is not a classification list (upto equivalence). We need to determine K as derivations in the nil-affine case. In the truncated-nil-affine case, we need to find one more derivation which commutes with the action of K. The following Lemma is due to Ravi Kulkarni.

(3.4.5.1) Lemma. Let (L, K) be an $ad_c - c.r.$ pair.

i) If $L = N_3^C \rtimes K$, then the root set of K on N_3^C must have the form $\{\alpha, \beta, \alpha + \beta\}$;

ii) If $L = N_4^C \rtimes K$, then the root set of Kon N_4^C must have the form $\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$;

iii) If L = $(N_3^C \oplus C) \rtimes K$, then the root set of K on $N_3^C \oplus C$ must have the form $\{\alpha, \beta, \alpha + \beta, \gamma\}$.

Proof. Let $k \in K, N$ be a nilpotent Lie algebra s.t. adk acts on N as a derivation. Then adk keeps the centre $Z(N)$ invariant. adk also keeps $[N, N]$ invarient. N_3^C has a basis{ x, y, z} s.t. $Z(N_3^C) = Cz, [x,y] = z. N_4^C$ has a basis { x, y, z, w} s.t. $[x,y] = z, [x,z] = w, y, z, w$ commutes and $Z(N_4^C) = Cw$. $N_3^C \oplus C$ has a basis {x, y, z, w} s.t. $[x,y] = z, Z(N_3^C \oplus C) = Cz \oplus Cw$. Then $[N_4^C, N_4^C] = Cz + Cw. [N_3^C \oplus C, N_3^C \oplus C] = [N_3^C, N_3^C] = Cz$.

Now consider i). Since adK keeps $Z(N_3^C) = Cz$ invariant, let N_α be a root space of adK with root α, then we can write $N_3^C = N_\gamma + N'$, s.t. $N_\gamma = Cz$ and $adK(N') = N'$. So $N' = N_\alpha + N_\beta$. Then $[N_\alpha, N_\beta] = [N_\alpha + N_\beta, N_\alpha + N_\beta] = [N', N'] = [N_3^C, N_3^C] = Cz = N_\gamma$, i.e., $\gamma = \alpha + \beta$.

Consider iii). adK keeps $Z(N_3^C \oplus C)$ and $[N_3^C \oplus C, N_3^C \oplus C]$ invariant and we have $[N_3^C \oplus C, N_3^C \oplus C] \subseteq Z(N_3^C \oplus C)$. By the complete reducibility of adK, we can write $N_3^C \oplus C = N' + N" + N'''$ s.t. adK keeps $N', N"$ and N''' invariant, $N' + N" = Z(N_3^C \oplus C), N" = [N_3^C \oplus C, N_3^C \oplus C]$. So $[N''', N'''] = [N_3^C \oplus C, N_3^C \oplus C] = N"$. Let $N' =$

N_γ, $N'' = N_\lambda$, $N''' = N_\alpha + N_\beta$. Then $[N_\alpha, N_\beta] = [N''', N'''] = [N_3^C \oplus C] = N''$, so $\lambda = \alpha+\beta$.

Finally consider ii). We have $Z(N_4^C) \subseteq [N_4^C, N_4^C]$. Let

$N' = \{x \in N_4^C; [x, [N_4^C, N_4^C]] = 0\}$.

It's easy to show that $dim\ N' = 3$, $adK(N') \subseteq N'$. So we have

$Z(N_4^C) \subseteq [N_4^C, N_4^C] \subseteq N'$,

and we can assume that $N_4^C = N_\gamma + N_\lambda + N_\beta + N_\alpha$ s.t. $N_\gamma = Z(N_4^C)$, $N_\gamma + N_\lambda = [N_4^C, N_4^C]$, $N_\gamma + N_\lambda + N_\beta = N'$. It's clear that $0 \neq [N_\alpha, N_\beta] \subseteq [N_4^C, N_4^C] = N_\gamma + N_\lambda$, and $0 \neq [N_\alpha, N_\lambda] \subseteq [N_4^C, [N_4^C, N_4^C]] = N_\gamma$, namely $\gamma = \alpha + \lambda$. We only have to prove that $[N_\alpha, N_\beta] = N_\lambda$. If $\gamma \neq \lambda$, then $[N_\alpha, N_\beta] = N_\lambda$ or N_γ. But if $[N_\alpha, N_\beta] = N_\gamma$, then $[N_4^C, N_4^C] = [N_\alpha, N_\beta + N_\gamma + N_\lambda] = N_\gamma$, we have a contradiction. If $\gamma = \lambda \neq 0$, $\alpha + \lambda = \gamma$ implies $\alpha = 0$, $\alpha + \beta = \lambda$ implies $\beta = \lambda$. So the root set is $\{0, \beta, \beta, \beta\}$, a special case of $\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$. If $\lambda = \gamma = 0$, then we have $\alpha = \beta = \lambda = \gamma = 0$. In all cases, we can assume that $N_\alpha = Cx$, $N_\beta = Cy$, $N_{\alpha+\beta} = Cz$, $N_{2\alpha+\beta} = Cw$.

Q.E.D.

(3.4.5.2) Corollary. Upto equivalence, indecomposable nil-affine $ad_c - c.r.$ pairs in *dim* 3 and *dim* 4 are

dim 3 : i) $(C^3 \rtimes K, K)$, dim K = 2, $\Phi = \{\alpha, \beta, t_1\alpha + t_2\beta\}$, where

α, β are independent, $(t_1, t_2) \neq (0, 0)$;

ii) $(C^3 \rtimes K, K)$, $dim\ K = 1$, $\Phi = \{\alpha, t_1\alpha, t_2\alpha\}$, where

$1 \geq t_1 \geq t_2$, $t_1 t_2 \neq 0$;

iii) $(N_3^C \rtimes K, K)$, $dim\ K = 2$, $\Phi = \{\alpha, \beta, \alpha + \beta\}$, where

α, β are independent;

iv) $(N_3^C \rtimes K, K)$, $dim\ K = 1$, $\Phi = \{\alpha, t\alpha, (1 + t\alpha)\}$;

dim 4 : i) $(C^4 \rtimes K, K)$, $dim\ K = 3$, $\Phi = \{\alpha, \beta, \gamma, t_1\alpha + t_2\beta + t_3\gamma\}$, where

$(t_1, t_2, t_3) \neq (0, 0, 0)$, α, β, γ are independent;

ii) $(C^4 \rtimes K, K)$, dim $K = 2$, $\Phi = \{\alpha, \beta, t_1\alpha + t_2\beta, t_3\alpha + t_4\beta\}$, where

α, β are independent, $(t_1, t_2) \neq (0, 0)$, $(t_3, t_4) \neq (0, 0)$;

iii) $(C^4 \rtimes K, K)$, $dim\ K = 1$, $\Phi = \{\alpha, t_1\alpha, t_2\alpha, t_3\alpha\}$, where

$1 \geq t_1 \geq t_2 \geq t_3$, $t_1t_2t_3 \neq 0$;

iv) $(N_4^C \rtimes K, K)$, $dim\ K = 2$, $\Phi = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$ where α, β are independent;

v) $(N_4^C \rtimes K, K)$, $dim\ K = 1, \Phi = \{0, \beta, \beta, \beta\}$;

vi) $(N_4^C \rtimes K, K)$, $dim\ K = 1, \Phi = \{\alpha, t\alpha, (1+t)\alpha, (2+t)\alpha\}$;

vii) $((N_3^C \oplus C) \rtimes K, K)$, $dim\ K = 2$, $\Phi = \{\alpha, \beta, \alpha+\beta, t_1\alpha+t_2\beta\}$, where $(t_1, t_2) \neq (0, 0)$, α, β are independent;

viii) $((N_3^C \oplus C) \rtimes K, K)$, $dim\ K = 1, \Phi = \{\alpha, t_1\alpha, (1 + t_1\alpha), t_2\alpha\}$,where $t_2 \neq 0$.

Proof. By (3.4.5.1) and (3.4.3.2).

Q.E.D.

(3.4.5.3) Corollary. If (L, K) is a truncated-nil-affine indecomposable $ad_c - c.r.$ pair of dim 4 which is not nil-affine and the nil-radical of L is N_3^C, then $(L, K) = (N_3^C \rtimes K_1, K)$ s.t.

i) $dim\ K = 1, dim\ K_1 = 2$,

ii) K_1 is abelian, $K \subseteq K_1$, adK_1 acts on N_3^C completely reducibly with roots $\{\beta, \gamma, \beta + \gamma\}$,

iii) adK acts on N_3^C with roots $\{\alpha, t\alpha, (1+t)\alpha\}$, $t \neq 0$.

Proof. In (3.4.5.1) we have proved that $L = N_3^C \rtimes K_1$, $K \subseteq K_1$, K_1 is abelian and $N_3^C = N_\alpha + N_\beta + N_\gamma$ s.t. α, β, γ are roots of adK on N_3^C, N_α, N_β, N_γ are the corresponding root spaces and $\alpha + \beta = \gamma$, $[N_\alpha, N_\beta] = N_\gamma$, $N_\gamma = Z(N_3^C)$. By (3.4.3.2), $dim\ K \leq 2$. If $dim\ K = 2$, then α, β, $\alpha + \beta$ are three different roots. Since $K_1 \subseteq L_0$, L_0 is the root space of adK on L, we must have $[K_1, N_\alpha] \subseteq N_\alpha$, $[K_1, N_\beta] \subseteq N_\beta$, $[K_1, N_\gamma] \subseteq N_\gamma$, namely, adK_1 acts on N_3^C completely reducibly. Since $dim\ K_1 = 3$ but $dim\ adK_1 = 2$, there is an $x \in K_1$ s.t. $adx|_L = 0$, then (L, K) is decomposable

$$(L, K) = (N_3^C \rtimes K, K) + (Cx, 0).$$

So we must have $dim\ K = 1$ and we can rewrite N_3^C as $N_\alpha + N_{t\alpha} + N_{(1+t\alpha)}$ with $t\alpha = \beta$, $(1+t)\alpha = \gamma$. If α, $t\alpha$ and $(1+t)\alpha$ are three different roots, we can repeat the above argument to show that adK_1 acts on N_3^C completely reducibly, so we only have to conasider two special cases: $t = 0$ or $t = 1$.

Subcase 1: $t = 1$. $N_3^C = N_\alpha + N_{2\alpha}$, where $dim\ N_\alpha = 2$. Now $[K_1, N_{2\alpha}] \subseteq N_{2\alpha}$, $[K_1, N_\alpha] \subseteq N_\alpha$. So adK_1 acts on N_3^C completely reducibly iff adK_1 acts on N_α completely reducibly. If adK_1 doesn't act on N_α completely reducibly, then we can find $x \in K_1$ s.t. $adx|_{N_\alpha}$ has the form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ w.r.t. a suitable basis. then $N_3^C + Cx \simeq N_4^C, (LK) = (N_4^C \rtimes K, K)$ which is nil-affine.

Subcase 2: $t = 0$. $N_3^C = N_\alpha + N_0 + Z(N_3^C)$, $[N_\alpha, N_0] = Z(N_3^C)$. $adK|_{N_\alpha + Z(N_3^C)}$ are scalers. As derivations, adK fixes $Z(N_3^C)$. If $\{e_1, e_2, e_3\}$ is a basis of N_3^C s.t. $Ce_1 = N_0, Ce_2 = N_\alpha$, $Ce_3 = Z(N_3^C)$, then there is an $x \in K_1$, s.t. $adx|_{N_3^C}$ has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

namely $[x, e_1] = [x, e_3] = 0$, $[x, e_2] = e_3$. Then $ad(x - e_1)|_{N_3^C} = ad(x - e_1)|_L = 0$, namely (L, K) is decomposable

$(L, K) = (N_3^C \rtimes K, K) + (Cx, 0)$.

The remaining part is trivial.

Q.E.D.

(3.4.5.4) Corollary. If (L, K) is a truncated-nil-affine indecomposable $ad_c - c.r.$ pair of dim 4 s.t. the nil-radical of L is C^3, then $dim\ K$=1 or 2, $L = C^3 \rtimes K_1$, $K \subseteq K_1$, $dim\ K_1 = dim\ K$+1. K_1 is abelian. We have three subcases.

(3.4.5.4..1) $dim\ K = 2$ and K_1 acts on C^3 completely reducibly with three independent weight α, β, γ and any two of the following

$\alpha|_K, \beta|_K, \gamma|_K$

are independent. In matrix form, we can always write

$$K_1 = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} ; a, b, c, \in C \right\},$$

$$K = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & t_1 a + t_2 b \end{pmatrix} ; a, b, c \in C \right\}, t_1 t_2 \neq 0.$$

(3.4.5.4.2) *dim* $K = 1$ and K_1 acts on C^3 completely reducibly with weights $\{\alpha, \beta, t_1\alpha + t_2\beta\}$, $(t_1, t_2) \neq (0,0)$, α and β are independent. The restriction of these weights on K gives a classification.

(3.4.5.4.3) *dim* $K = 1$ but K_1 doesn't act on C^3 completely reducibly. Using the Jordan form, we have two cases:

i) $$K' = \left\{ \begin{pmatrix} a + t_2 b & 0 \\ 0 & \begin{pmatrix} t_1 a & t_2 b \\ 0 & t_1 a \end{pmatrix} \end{pmatrix} ; a, b \in C \right\},$$

$$K = \left\{ \begin{pmatrix} a & & \\ & t_1 a & \\ & & t_1 a \end{pmatrix} ; a, b \in C \right\}$$

where $t_1 \neq 0$;

ii) $$K_1 = \left\{ \begin{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} & 0 \\ 0 & tb \end{pmatrix} ; a, b \in C \right\}$$

$$K = \left\{ \begin{pmatrix} a & & \\ & a & \\ & & 0 \end{pmatrix} ; a \in C \right\},$$

where $t \neq 0$.

§4. Real Forms of $ad_c - c.r.$ Pairs

(4.1) Definition. Let $\mathcal{T}$ be an abelian real Lie algebra, V a vector space, $\rho : \mathcal{T} \to End(V)$ be a completely reducible representation of $\mathcal{T}$. Then V decomposes into a sum of 1- or 2-dimentional subspaces. let α's, resp. $(\beta,\overline{\beta})$'s the corresponding weights and V_α, resp. $V_{(\beta,\overline{\beta})}$the sum of all 1-, resp. 2-dimentional irreducible subspaces with weights α, resp. $(\beta, \overline{\beta})$. Then

$$V = (\oplus_\alpha V_\alpha) \oplus (\oplus_{(\beta,\overline{\beta})} V_{(\beta,\overline{\beta})}).$$

The multiplicity of α, resp. β is $dim\ V_\alpha$, resp. $\frac{1}{2}dim\ V_{(\beta,\overline{\beta})}$.

We call α a real weight. We know that there is a basis $\{e_1, e_2\}$ of $V_{(\beta,\overline{\beta})}$ s.t.

$$\rho(z)|_{V_{\beta,\overline{\beta}}} = \begin{pmatrix} \gamma(z) & -\delta(z) \\ \delta(z) & \gamma(z) \end{pmatrix}, z \in \mathcal{T},$$

where $\gamma : \mathcal{T} \to R, \delta : \mathcal{T} \to R$ are linear functions, $\delta \neq 0$. If $\gamma = 0$, we call $(\beta, \overline{\beta})$ a compact weight. If $\gamma \neq 0$, we call $(\beta, \overline{\beta})$ a mixed weight.

When V is a real semisimple algebra and $\mathcal{T}$ is its CSA, these weights are its roots.

If two representations are equivalent, then the set of weights and their mulitiplicities are exactly the same.

(4.2) Real forms of nonabelian reductive $ad_c - c.r.$ pairs.

In [30], the CSA's of real semisimple Lie algebras are classified upto congugacy. We will use these results.

(4.2.1)$(\mathcal{G}^C, \mathcal{H}^C) = (B_2, A_1 \oplus A_1)$.

It's claerly that $\mathcal{G}$ has to be $so(5), so(1,4)$ or $so(2,3)$, and $\mathcal{H}$ has to be $so(3) \oplus so(3) \simeq so(4)$, $so(3) \oplus sl_2(R) \simeq so^*(4)$, $sl_2(R) \oplus sl_2(R) \simeq so(2,2)$ and $sl_2(C) \simeq so(1,3)$. Always $Rank\ \mathcal{G} = Rank\ \mathcal{H}$.

(4.2.1.1) Lemma. Let $(\mathcal{G}, \mathcal{H}_1)$ and $(\mathcal{G}, \mathcal{H}_2)$ be two ereal forms of $(B_2, A_1 \oplus A_1)$. If $CSA\mathcal{H}_1 = CSA\mathcal{H}_2$, then $\mathcal{H}_1 = \mathcal{H}_2$.

Proof. Let $\mathcal{G}_0 = CSA\mathcal{H}_1 = CSA\mathcal{H}_2$, then $\mathcal{G}_0^C = CSAB_2$. W.r.t. $\mathcal{G}_0^C$ we have a root-space decomposation

$$B_2 = \mathcal{G}_0^C + \textstyle\sum_{\alpha\in\Phi} \mathcal{G}_\alpha^C,$$

where $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2)\}$ with α_1 as the short simple root and α_2 as the long simple root of B_2. In (3.2.1) we proved that the root-spaces of $\mathcal{H}^C$ w.r.t. $\mathcal{G}_0^C$ must be $\sum_\beta \mathcal{G}_\beta$, where β runs through all long roots, i.e. $\mathcal{H}^C$ is uniquely determined by the common CSA of $A_1 \oplus A_1$ and B_2. So $\mathcal{H}_1^C = \mathcal{H}_2^C$, $\mathcal{H}_1 = \mathcal{H}_2$.

(4.2.1.2) Corollary. Let $\mathcal{G}_0$ be a CSA of $\mathcal{G}$, then there is at most one $\mathcal{H}$such that $\mathcal{G}_0$ is also a CSA of $\mathcal{H}$ and $(\mathcal{G}^C, \mathcal{H}^C)) = (B_2, A_1 \oplus A_1)$.

(4.2.1.3) Corollary. Let $(\mathcal{G}, \mathcal{H})$ be a real form of $(B_2, A_1 \oplus A_1)$,then

No.{equivalent classes of $(\mathcal{G}, \mathcal{H})$ } $\leq No.\{$ conjugate classes of CSA's in $\mathcal{G}\}$.

(4.2.11.4) $\mathcal{G} = so(5)$. By [30], all CSA's in $so(5)$ are conjugate, so there is just one real form $(so(5), so(4))$.

(4.2.1.5) $\mathcal{G} = so(1,4)$. We know that $so(1,3)$ and $so(4)$ can be embedded in $\mathcal{G}$; $(so(1,4), so(4))$ and $(so(1,4), so(1,3))$ are not equivalent since $so(4)$ and $so(1,3)$ are not isomorphic. By [30], there are only two conjugate classes of CSA in $so(1,4)$, so by (4.2.1.3) we have exactly two real forms.

(4.2.1.6) $\mathcal{G} = so(2,3)$.

By [30], there are four conjugate classes of CSA, their representitives w.r.t. a given basis are

$$\text{i) } CSA_1 = \left\{ \begin{pmatrix} h_1 & & & & 0 \\ & h_2 & & & \\ & & -h_1 & & \\ & & & -h_2 & \\ 0 & & & & 0 \end{pmatrix} ; h_i \in R \right\},$$

$$\text{ii) } CSA_2 = \left\{ \begin{pmatrix} 0 & h_1 & 0 & -h_2 & 0 \\ h_1 & 0 & h_2 & 0 & 0 \\ 0 & -h_2 & 0 & -h_1 & 0 \\ h_2 & 0 & -h_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; h_i \in R \right\},$$

$$\text{iii) } CSA_3 = \left\{ \begin{pmatrix} 0 & & & 0 & -h_1 \\ & h_2 & & & 0 \\ & & 0 & & h_1 \\ 0 & & & -h_2 & 0 \\ h_1 & 0 & -h_1 & 0 & 0 \end{pmatrix} ; h_i \in R \right\},$$

$$\text{iv) } CSA_4 = \left\{ \begin{pmatrix} 0 & h_1 & 0 & -h_2 & 0 \\ -h_1 & 0 & h_2 & 0 & 0 \\ 0 & -h_2 & 0 & h_1 & 0 \\ h_2 & 0 & -h_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; h_i \in R \right\},$$

and

$$so(2,3) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} & a_{15} \\ a_{21} & a_{22} & -a_{14} & 0 & a_{25} \\ 0 & a_{32} & -a_{11} & -a_{21} & a_{35} \\ -a_{32} & 0 & -a_{12} & -a_{22} & a_{45} \\ a_{35} & a_{45} & a_{15} & a_{25} & 0 \end{pmatrix} ; a_{ij} \in R \right\}.$$

It's clear that CSA_1, CSA_2 and CSA_4 are contained in

$$so(2,2) \hookrightarrow \begin{pmatrix} so(2,2) & 0 \\ 0 & 0 \end{pmatrix},$$

so CSA_1, CSA_2 and CSA_4 correspond to the same equivalence class $(so(2,3), so(2,2))$.

CSA_3 is contained in

$$\mathcal{H} = \left\{ \begin{pmatrix} 0 & -d & 0 & -a & -h_1 \\ -a & h_2 & a & 0 & b \\ 0 & d & 0 & a & h_1 \\ -d & 0 & d & -h_2 & c \\ h_1 & c & -h_1 & b & 0 \end{pmatrix} ; a, b, c, d, h_1, h_2 \in R \right\}$$

and $\mathcal{H} \simeq so(1,3)$ (see (4.2.1.7) for a proof). We get the second equivalent class

$(so(2,3), so(1,3))$.

(4.2.1.7). To show $\mathcal{H} \simeq so(1,3)$, we first show $[\mathcal{H}, \mathcal{H}] = \mathcal{H}$ by direct computation; so $\mathcal{H}$ has to be one of $so(4), so(1,3), so(2,2), so^*(4)$. W.r.t. their own CSA's, their roots are

$so(4)$		2 pair of compact roots,
$so(1,3)$		2 pair of mixed roots,
$so(2,2)$	i)	4 real roots,
	ii)	2 real roots and a pair of compact roots,
	iii)	2 pair of compact roots,
$so^*(4)$	i)	2 pair of compact roots,
	ii)	2 real roots and a pair of compact roots.

so we only have to show that $\mathcal{H}$ has mixed roots only. And it is easy to show that the roots of $\mathcal{H}$ have the form $\{(\beta, \overline{\beta}), (-\beta, -\overline{\beta})\}$, two pair of mixed roots.

Q.E.D.

(4.2.2) $(\mathcal{G}^C, \mathcal{H}^C) = (A_2, A_1 \oplus C)$.

$\mathcal{G}$ must be one of $su(3), su(1,2)$ and $sl_3(R)$, $\mathcal{H}$ must be isomorphic to either $su(2) \oplus R$ or $su(1,1) \oplus R \simeq sl_2(R) \oplus R$.

Notice that *Rank* $\mathcal{G}$ = *Rank* $\mathcal{H}$, so if $(\mathcal{G}, \mathcal{H}_1)$ is equivalent to $(\mathcal{G}, \mathcal{H}_2)$, any CSA of $\mathcal{H}_1$ must be conjugate under $Aut(\mathcal{G})$ to a CSA of $\mathcal{H}_2$. We will first find all conjugate classes of CSA's in $\mathcal{G}$ from [30], then for a given representation of each conjugate class decompose $\mathcal{G}$ into root-spaces w.r.t. the same CSA, finally we figure out all the possible $\mathcal{H}$'s.

(4.2.2.1) $\mathcal{G} = sl_3(R)$.

$\mathcal{G}$ has two CSA's upto conjugacy. They are

$$\text{i) } CSA_1 = \begin{pmatrix} h_1 & & 0 \\ & h_2 & \\ 0 & & -h_1 - h_2 \end{pmatrix}, \text{ ii) } CSA_2 = \begin{pmatrix} h_1 & -h_2 & 0 \\ -h_2 & h_1 & 0 \\ 0 & 0 & -2h_1 \end{pmatrix}.$$

Case i) $sl_3(R) = CSA_1 + \sum_{\alpha\in\Phi} \mathcal{G}_\alpha$, where Φ has real roots only. If $\mathcal{H} = CSA_1 + \sum_\beta \mathcal{G}_\beta$, then $\beta \in \Phi$ has to be a real root, i.e. $\mathcal{H} = CSA_1 + \mathcal{G}_\beta + \mathcal{G}_{-\beta}$. Since $su(2) \oplus R$ has a pair of compact roots, $sl_2(R)$ has two real roots, or a pair of compact roots, it is clear that $\mathcal{H}$ can only be $sl_2(R) \oplus R$. The root-space decomposition tells us that there are three pairs of root-spaces: $\{\mathcal{G}_{\alpha_i} + \mathcal{G}_{-\alpha_i}\}$, i = 1,2,3, and $\mathcal{H} = \mathcal{H}_i = CSA_1 + \{\mathcal{G}_{\alpha_i} + \mathcal{G}_{-\alpha_i}\}$

for some i. But it's easy to show that all three $\mathcal{H}_i$'s are conjugate under $Aut(sl_3(R))$: we only need to change the order of the basis in R^3.

Case ii) $\mathcal{G} = CSA_2 + \mathcal{G}_{\alpha,\overline{\alpha}} + \mathcal{G}_{\beta,\overline{\beta}} + \mathcal{G}_{-\beta,-\overline{\beta}}$. The root-system has a pair of compact roots $\{\alpha, \overline{\alpha}\}$ and two pairs of mixed roots. So if $\mathcal{H}$ contains CSA_2, $\mathcal{H} = CSA_2 + \mathcal{G}_{\alpha,\overline{\alpha}}$. Since

$$\mathcal{G}_{\alpha,\overline{\alpha}} = \left\{ \left(\begin{array}{cc} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} & 0 \\ 0 & 0 \end{array} \right) ; a, b \in R \right\},$$

$$\text{so } \mathcal{H} = \left\{ \left(\begin{array}{cc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & 0 \\ 0 & -a-d \end{array} \right) ; a, b, c, d \in R \right\}$$

and we see that this is a special case of Case i). So

$$(sl_3(R), \begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix} + R \left(\begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & -2 \end{array} \right))$$

is the unique real form of $(A_2, A_1 \oplus C)$ such that $\mathcal{G} = sl_3(R)$.

(4.2.2.2) $\mathcal{G} = su(3)$. It has only one CSA upto conjugacy and it's easy to show there is only one $(\mathcal{G}, \mathcal{H})$ upto equivalence:

$$(su(3), \begin{pmatrix} su(2) & 0 \\ 0 & 0 \end{pmatrix} + R \begin{pmatrix} \sqrt{-1} & & 0 \\ & \sqrt{-1} & \\ 0 & & -2\sqrt{-1} \end{pmatrix}).$$

(4.2.2.3) $\mathcal{G} = su(2,1)$. It has two CSA's upto conjugacy. They are

$$\text{i) } CSA_1 = \left\{ \begin{pmatrix} u & h & 0 \\ h & u & 0 \\ 0 & 0 & -2u \end{pmatrix} ; u \in \sqrt{-1}R, h \in R \right\},$$

$$\text{ii) } CSA_2 = \left\{ \begin{pmatrix} u_1 & & 0 \\ & u_2 & \\ 0 & & -u_1 - u_2 \end{pmatrix} ; u_i \in \sqrt{-1}R \right\}.$$

By an argument similar to (4.2.2.1), we can prove that we get

$$(su(2,1), \begin{pmatrix} su(2) & 0 \\ 0 & 0 \end{pmatrix} + R \begin{pmatrix} \sqrt{-1} & & 0 \\ & \sqrt{-1} & \\ 0 & & -2\sqrt{-1} \end{pmatrix})$$

from i) and

$$(su(2,1), \begin{pmatrix} 0 & 0 \\ 0 & su(1,1) \end{pmatrix} + R \begin{pmatrix} -2\sqrt{-1} & & 0 \\ & \sqrt{-1} & \\ 0 & & \sqrt{-1} \end{pmatrix})$$

from ii). Upto equivalence, they are the only real forms.

(4.2.2.4) 2-dimentional and three dimentional cases can be solved similarly.

(4.2.2.5) We have discussed the real forms of indecomposable nonabelian reductive $ad_c - c.r.$ pairs. But a very important fact is that if $(\mathcal{G},\mathcal{H})$ is an indecomposable infinitesimal geometry, $(\mathcal{G}^C, \mathcal{H}^C)$ may be decomposable. For example, $(\mathcal{G}, \mathcal{H}) = (sl_2(C), C\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ where $sl_2(C)$, resp. $C\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is considered as a 6-dimentional, resp. 2-dimetional real Lie algebra. $(\mathcal{G}, \mathcal{H})$ is indecomposable, but $(\mathcal{G}^C, \mathcal{H}^C) = (A_1, C) + (A_1, C)$. So we have to be careful to find out such cases. In the nonabelian reduvtive $ad_r - c.r.$ case, this example is the only such case in $dim \leq 4$. The general method is to find also the real forms of decomposable $ad_c - c.r.$ pairs and then to find the indecomposable ones among them.

(4.3) Real forms of mixed $ad_c - c.r.$ pairs.

The following fact play the key role in (4.3).

(4.3.1) Lemma. Let $\mathcal{G} = r \rtimes s$ and $\mathcal{G}^C = R \rtimes S$ be the Levy's decomposation of $\mathcal{G}$ and $\mathcal{G}^C$ respectively, then $r^C = R, s^C \simeq S$.

By this lemma , we need only to find all "real form" r and s of R and S respectively, then check wether there is a real representation of r on s (as derivation), such that the complixification of this representation is equivalent to the representation of (as derivation) of S on R. The following example shows how to do it.

Let $(L, K) = (C^2 \rtimes sl_2(C), sl_2(C))$. By Lemma(4.3.1), the real form of C^2 is R^2 and the real form of S is $sl_2(R)$ or $su(2)$. But it's well skown that $su(2)$ has no 2-dimentional irreducible real representation, so we only have to consider $s = sl_2(R)$. There is a unique

2-dimentional irreducible real representation of $sl_2(R)$ on R^2, so $(R^2 \times sl_2(R), sl_2(R))$ is the only real form of $(C^2 \times sl_2(C), sl_2(C)))$.

Of course, the above example is the simpliest acase, other cases needs much longer discussion. Using the result in [32], we can determine whether an irreducible complex representation of a simple Lie algebra is of the **Real Type**, i.e. whether there is a real representation whose complexification is this complex representation. Let us show another example.

Let $(\mathcal{G}^C, \mathcal{H}^C) = (C^3 \times gl_3(C), sl_3(C))$. $\mathcal{G} = r \times s$, $s^C = A_2$. If $[s, r] = r$, then $[s^C, r^C] = r^C$, i.e. $[S, R] = R$, contradicts that $[S, R] = C^C \neq R$. So $[s, r] = r_1 \neq r$. Since ads acts on r completely reducibly, there is a complement r_1 such that $r = r_1 + r_2$, $[s, r_1] \subseteq r_1$, $[s, r_2] = 0$. Since $[s^C, r^C] = {r_1}^C = C^3$, so $dim\ r_1 = 3$, $dim\ r_2 = 1$. So s must have a 3-dimentional representation which is of Real Type. We know from (3.3.1.4) Case iii) that the A_2-module C^3 has the highest weight λ_1. A_2 has three real forms: $sl_3(R), su(3)$ and $su(2,1)$. by Tits's table([32]), the irreduvible representation with the highest weight λ_1 is of Real Type iff $s = sl_3(R)$.

(4.4) Real forms of solvable $ad_c - c.r.$ pairs.

In [19], R. Kulkarni classified indecomposable real Lie algebras in $dim \leq 4$. He proved that every indecomposable solvable $ad_r - c.r.$ geometry is either nil-affine (including affine) or truncated-nil-affine. He used a method which is different from mine to classify affine and nil-affine geometry in $dim \leq 4$. His results , with miner refinments and being rewritten in matrix form, are included in Table 11, 12, 13, 15 and 16. He didn't classify truncated-nil-affine cases. My method is similar to that we used in (4,3). After classifing indecomposable $ad_C - c.r.$ pairs in $dim \leq 4$, it's easy to find their real forms and we omit the detail.

§5. Indecomposable $Ad - c.r.$ geometries of $dim \leq 4$ and their topology.

In this section we will solve the following problems

i) If an $ad_r - c.r.$ infinitesimal geometry $(\mathcal{G}, \mathcal{H})$ is given, is there an $Ad - c.r.$ geometry (G, H) s.t. $L(G) = \mathcal{G}$ and $L(H) = \mathcal{H}$?

ii) If the answer to question i) is 'yes', give a representitive of the equivalence class containing (G, H).

iii) Describe the quotient space $S = G/H$, especially the topology of $S = G/H$.

Of course we will only deal with indecomposable $ad_r - c.r.$ infinitesimal geometries of $dim \leq 4$.

(5.1). Solvable geometry. We have such a well known result

(5.1.1). Theorem. Let G be a solvable analytic group. If G is simply connected, then every analytic subgroup of G is closed and simply connected.

Proof. C.f. Varadarajan [33], Theorem 3.18.12.

So by (2.8), for a solvable $ad_r - c.r.$ infinitesimal geometry $(\mathcal{G}, \mathcal{H})$, we only have to check if the connected subgroup of $Aut(\mathcal{G}/\mathcal{H})$ corresponding to $\overline{ad}(\mathcal{H})$ is closed; or equivalently we can find a geometry (G, H) first (by (2.2) and (2.8) such a (G, H) exists), then we only have to show that $\rho(H)$ is closed in $Aut(T_{x_0}(S)) \simeq GL_n(R)$ (c.f. (1.1) and (1.2) for notations). We will solve the problem case by case.

First we prove a Lemma which is needed for both solvable and mixed cases.

(5.1.2) Lemma. Assume an infinitesimal geometry has the form $(\mathcal{G}, \mathcal{H}) = (R^k \rtimes_\sigma \mathcal{H}, \mathcal{H})$, where $\mathcal{H}$ acts on R^k through the derivation $\sigma : \mathcal{H} \to gl_k(R)$. Let $\tilde{H}$ be the connected, simply connected Lie group with Lie algebra $\mathcal{H}$. Let $\tilde{\sigma}$ be the representation $\tilde{\sigma} : \tilde{H} \to GL_k(R)$ s.t. $d\tilde{\sigma} = \sigma$, and $Z = Ker\tilde{\sigma}$. Let $H = \tilde{H}/Z$, $\overline{\sigma}$ be the induced representation of $\tilde{\sigma}$ on H, and $G = R^k \rtimes_{\overline{\sigma}} H$, Then

(i) (G, H) is a geometry ,

(ii) $G/H \simeq R^4$,

(iii) if $(\mathcal{G}, \mathcal{H})$ is $ad_r - c.r.$, then (G, H) is $Ad - c.r.$ iff $\overline{\sigma}(H)$ is closed in $GL_k(R)$.

Proof. The proof of part (i) is exactly the same as the proof of (2.2) and the only thing we have to do is to show that $\tilde{H}$, embedded in $R^k \rtimes_{\tilde\sigma} \tilde{H}$, is closed; but since the topology of $R^k \rtimes_{\tilde\sigma} \tilde{H}$ is the product topology, this is clearly true.

To prove part(ii), we will establish a model. Let $E = R^k$ and $Aff(E)$ be the group of affine motions. Then $G = R^k \rtimes_{\bar\sigma} H$ acts on E as a subgroup of $Aff(E)$ s.t. R^k acts on E as translations, H acts on E via $\bar\sigma$. The action of G on E is transitive since all translations are in G and effective since the kernal of this action must be in H which cannot contain any normal subgroup of G except $\{ e\}$. Choose $X =(0, 0, \ldots , 0)$ as a base point in E, then the isotropy subgroup is H. So $E = G/H$.

To prove part (iii) , we only have to prove that $\overline{Ad} : H \rightarrow Aut(\mathcal{G}/ \mathcal{H})$ is equivalent to $\bar\sigma$, or to prove that $\overline{ad}$: $\mathcal{H} \rightarrow End(\mathcal{G}/ \mathcal{H})$ is equivalent to $d\bar\sigma$. But we know $\overline{ad}$ is equivalent to σ, so we only have to show that σ is equivalent to $d\bar\sigma$. By the definition of σ, $\tilde\sigma$ and $\bar\sigma$, this is clear.

Q.E.D.

In our classification list of $ad_r - c.r.$ infinitesimal geometries, we always express ($R^k \rtimes_\sigma \mathcal{H}, \mathcal{H}$) as ($R^k \rtimes \sigma(\mathcal{H}), (\mathcal{H})$) , where $\sigma(\mathcal{H}) \subseteq gl_k(R)$. Since when $(\mathcal{G}, \mathcal{H})$ is $ad_r - c.r.$, $\sigma \sim \overline{ad}$: $\mathcal{H} \rightarrow \mathrm{End}\ (\mathcal{G}/ \mathcal{H}) = gl_k(R)$ is injective, we can identify $\mathcal{H}$ with $\sigma(\mathcal{H})$. So $\mathcal{G} \subseteq aff(E), E = R^k$. Then there is a natural way to find (G, H), namely , taking G as the connected subgroup in $Aff(E)$ with Lie algebra $\mathcal{G}$ and H the connected subgroup of G with Lie algebra $\mathcal{H}$. Since G contains all translations, the action of G on E is transitive, and the action is clearly effective with the isotropy group H at $(0, 0, \ldots, 0) \in E$. So $E = G/H$ and (G, H) is a geometry. since $(\mathcal{G}, \mathcal{H})$ is $ad_r - c.r.$, we only have to check the closeness of H in $GL(E)$. H is exactly the connected subgroup of $Aut(R^k \simeq E)$ with Lie algebra $\overline{ad}(\mathcal{H}) \simeq \mathcal{H}$.

We use the following formula to compute the exponential map from $aff(E)$ into $Aff(E)$. We identify $aff(E)$ with

$$\left\{ \begin{pmatrix} L & c \\ 0 & 0 \end{pmatrix} ; L \in gl(E), c \in E \right\}$$

and identify $Aff(E)$ with

$$\left\{ \begin{pmatrix} l & c \\ 0 & 1 \end{pmatrix} ; l \in GL(E), c \in E \right\}.$$

Then

$$\textbf{(5.1.3).}\ (\exp\begin{pmatrix} L & c \\ 0 & 0 \end{pmatrix})(x) = (\exp L)(x) + \Big(\sum_{n=1}^{\infty} \frac{1}{n!} L^{n-1}\Big)(c),$$

for $x \in E$.

Using this formula , we first obtain (G, H) from $(\mathcal{G}, \mathcal{H})$, then check if H is closed in $GL(E)$. This method can be applied to the case Sol^1-i, Sol^3-ii and Sol^4-ii and we see that in most of these cases, H is closed, exceptions happen only in Sol^4-ii-4, Sol^4-ii-9, sol^4-ii-10 and Sol^4-ii-13.

(5.1.4). In Sol^4-ii-4, we have

$$(\mathcal{G}, \mathcal{H}) = (\ R^4 \rtimes_A R, R), \text{ where } A = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} \end{pmatrix}, t \neq 0.$$

By (5.12) - (5.1.3) , the corresponding (G, H) is $(R^4 \rtimes\ H, H)$, where

$$H = \left\{ \begin{pmatrix} \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} cost\theta & sint\theta \\ -sint\theta & cost\theta \end{pmatrix} \end{pmatrix} ; \theta \in R, t \neq 0 \right\}.$$

If t is an irrational number, then the set $\{tk2\pi\ mod(2\pi); k = 0, 1, 2...\}$ is dense in $[0, 2\pi]$. So the closure of

$$\left\{ \begin{pmatrix} \begin{pmatrix} cos2k\pi & sin2k\pi \\ -sin2k\pi & cos2k\pi \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} cos2kt\pi & sin2kt\pi \\ -sin2kt\pi & cos2kt\pi \end{pmatrix} \end{pmatrix} ; k = 0, 1, 2, ... \right\}$$

is

$$\left\{ \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & B \end{pmatrix} ; B \in SO(2) \right\}.$$

Similarly the closure of

$$\left\{ \begin{pmatrix} \begin{pmatrix} cos(2k\pi/t) & sin(2k\pi/t) \\ -sin(2k\pi/t) & cos(2k\pi/t) \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} cos2k\pi & sin2k\pi \\ -sin2k\pi & cos2k\pi \end{pmatrix} \end{pmatrix} ; k = 0, 1, 2, ... \right\}$$

is

$$\left\{ \begin{pmatrix} B & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} ; B \in SO(2) \right\}.$$

So $SO(2) \times SO(2) \subseteq \overline{H}$. Since $dim\ H=1$, H is not closed in $GL^4(R)$. It's easy to show that if t is rational, then H is closed in $GL_4(R)$.

A similar argument can be applied to Sol^4-ii-9, Sol^4-ii-10, and Sol^4-ii-13.

(5.1.5). Type $(\mathcal{N} \rtimes_\sigma \mathcal{H}, \mathcal{H})$ and $(\mathcal{S} \rtimes_\sigma \mathcal{H}, \mathcal{H})$, where $\mathcal{N}$ and $\mathcal{S}$ are nilpotent and solvable respectively. Following an argument similar to that of (2.2), we obtain

(5.1.5.1). Propostition. If $(\mathcal{G}, \mathcal{H}) = (\mathcal{M} \rtimes_\sigma \mathcal{H}, \mathcal{H})$ is an infinitesimal geometry where $\mathcal{H}$ acts on $\mathcal{M}$ via derivation ; M and $\tilde{H}$ are connected, simply connected Lie groups with Lie algebra $\mathcal{M}$ and $\mathcal{H}$ respectively; $\tilde{\sigma}$ is the induced representation of $\tilde{H}$ in $\mathcal{M}$ whose differential is σ . Let $\tilde{\tau} : \tilde{H} \to Aut(M)$ be the homomorphism s.t. $d(\tilde{\tau}(t)) = \tilde{\sigma}(t)$ for every $t \in \tilde{H}$. (It's well known the existence and uniqueness of such $\tilde{\tau}$(t)). Let $Z \subseteq H$ be the kernal of $\tilde{\tau}$, then $(G, H) = (M \rtimes_\tau H, H)$, where$\tau$ is the induced homomorphism of $\tilde{\tau}$ on $\tilde{H}/Z = H$.

When $\mathcal{H} = 0$, i.e., Type Sol^2-i , Sol^3-i and Sol^4-i, there is nothing to do. If $\mathcal{H} \neq 0$ we need not only to find Z, but also to check if (G, H) is $Ad - c.r.$.

(5.1.5.2) Corollary. Notations as in (5.1.5.1.). If $(\mathcal{G}, \mathcal{H})$ is a solvable $ad_r - c.r.$ infinitesimal geometry, let $A = \{X \in \sigma(\mathcal{H})$; the eigenvalues of X are of the form $2k\pi\sqrt{-1}$, $k = 0, \pm 1, \pm 2, ...\}$, then $Z \approx A, A$ is a free abelian group of finite $Rank \leq dim\ \mathcal{H}$.

Proof. We have proved that if $(\mathcal{G}, \mathcal{H})$ is solvable $ad_r - c.r.$ then $\mathcal{H}$ is abelian. Since $\tilde{H}$ is connected, simply connected, the exponential map $exp : \mathcal{H} \to \tilde{H}$ is a diffeomorphism. Assume $g \in Z$, then there is $X \in \mathcal{H}$, s.t. $expX = g$. Since $\tilde{\tau}(g) : M \to M$ is an identity map, so is $d(\tilde{\tau}(g)) : \mathcal{M} \to \mathcal{M}$, i.e. $\tilde{\sigma}(g)$: $\mathcal{M} \to \mathcal{M}$ is an identity map. The following diagram is commutative (see Helgason [15]):

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\sigma"] \arrow[d, "exp"'] & \partial(\mathcal{M}) \arrow[r, phantom, "\subseteq"] \arrow[d, "exp"'] & gl(\mathcal{M}) \\
\tilde{H} \arrow[r, "\tilde{\sigma}"] & Aut(\mathcal{M}) \arrow[r, phantom, "\subseteq"] & GL(\mathcal{M})
\end{tikzcd}$$

where $\partial(\mathcal{M})$ is the set of all derivations of $\mathcal{M}$. The image of $\mathcal{H}$ in $\partial(\mathcal{M})$ under σ is abelian. The exponetial map $exp : \partial(\mathcal{M}) \rightarrow Aut(\mathcal{M})$ is defined by $expD = e^D$ for $D \in \partial(\mathcal{M})$. We know that σ is 1 - 1 and $exp : \mathcal{H} \rightarrow \tilde{H}$ is also 1 - 1. So the kernal of $\tilde{\sigma}$ is isomorphic to the kernal of $exp : \sigma(\mathcal{H}) \rightarrow Aut(\mathcal{M})$. We have proved that, up to conjugation, $\sigma(\mathcal{H})$ is diagonalizable in C. We can prove that the restriction of exp: $gl(\mathcal{M}) \rightarrow GL(\mathcal{M})$ to any one dimensional subspace of $\sigma(\mathcal{H})$ is injective if and only if this space has a real or complex non-zero root. Assume an subspace is of the form $RX, X \in gl(\mathcal{M})$, then, w.r.t. a suitable basis, X has the form

$$\begin{pmatrix}
A_1 &&&&&&&&&&&&&& 0 \\
& A_2 &&&&&&&&&&&&& \\
&& \ddots &&&&&&&&&&&& \\
&&& A_l &&&&&&&&&&& \\
&&&& B_1 &&&&&&&&&& \\
&&&&& B_2 &&&&&&&&& \\
&&&&&& \ddots &&&&&&&& \\
&&&&&&& B_u &&&&&&& \\
&&&&&&&& C_1 &&&&&& \\
&&&&&&&&& C_2 &&&&& \\
&&&&&&&&&& \ddots &&&& \\
&&&&&&&&&&& C_s &&& \\
&&&&&&&&&&&& 0 && \\
&&&&&&&&&&&&& \ddots & \\
0 &&&&&&&&&&&&&& 0
\end{pmatrix}$$

where A_i, $1 \leq i \leq l$, is of the form $\begin{pmatrix} a_i & a_i' \\ -a_i' & a_i \end{pmatrix}$ s.t. $a_i \neq 0, a_i' \neq 0$; B_i, $1 \leq i \leq u$ is of the form $\begin{pmatrix} 0 & b_i \\ -b_i & 0 \end{pmatrix}$ s.t. $b_i \neq 0$; C_i, $1 \leq i \leq s$, is of the form(C_i), s.t. $C_i \neq 0$, and $l + u + s \neq 0$ since $\mathcal{H}$ contains no non-zero ideal of $\mathcal{G}$. It's clear that $exp : tX \rightarrow e^{tX}, t \in R$, is injective if l or s is not zero. If $l = s = 0$, $u \neq 0$, then $e^{tX} = Id_2$ if and only if the system of equations

$$\begin{cases} cos tb_i = 1 \\ sin tb_i = 0, \quad i = 1, 2, ..., u, \end{cases}$$

has a solution. For each i, the solution of

$$\begin{cases} cos tb_i = 1 \\ sin tb_i = 0 \end{cases}$$

is the set $\{2k\pi/b_i; k \in Z\}$. So the solution of this system is the set

$\Phi = \cap_{i=1}^{u} \{2k\pi/b_i; k \in Z\}$

which is isomorphic to Z if there is $t \in R$ s.t. $\{tb_i\}_{1 \leq i \leq u} \subseteq \mathrm{Z}$ or $\{0\}$ if such a t doesn't exist. The kernal of exp: $\sigma(\mathcal{H}) \rightarrow Aut(\mathcal{M})$ consists of exactly those matrices in $\sigma(\mathcal{H})$ such that (i) all nonzero entries of them are of the form $2k\pi$ for some $k \in Z$, (ii) all entries on the diagonal are zero. Equivalently, we can say that the kernal consists of those elements of $\sigma(\mathcal{H})$ that their eigenvalues are either zero or $2k\pi\sqrt{-1}$, for some $k \in Z$. This kernal is a free abelian group under addition.

(5.1.5.3.). Corollary. Notations as in (5.1.5.2). For indecomposable solvable $Ad - c.r.$ geometry, $Z \neq 0$ happens only when $(\mathcal{G}, \mathcal{H})$ is of Type Sol^2-ii-(2) and (4), Sol^3-ii-(5) if $t_2 = 0$, Sol^3-iii-(3) and (5), Sol^3-iv-(3), Sol^4-ii-(4) if t is rational, Sol^4-ii-(9) if $t_2 = 0$, t_4 is rational, Sol^4-ii-(12) if $t_2 = 0$, Sol^4-ii-(13) if $t_2 = 0$ or $t_3 = 0$, Sol^4-iv-(6) if $t_2 = 0$, Sol^4-v-(7), Sol^4-v-(9) if $t_2 = 0$, Sol^4-v-(4). If H is nontrivial, then H is compact only if $(\mathcal{G}, \mathcal{H})$ is of Type Sol^2-ii-(2), Sol^3-iv-(3), Sol^4-ii-(4) if t is rational, Sol^4-v-(7), and Sol^4-vi-(4).

(5.1.5.4) Corollary. Notations as in (5.1.5.1). If $(\mathcal{G}, \mathcal{H}) = (\mathcal{M} \rtimes \mathcal{H}, \mathcal{H})$ is an $ad_\tau - c.r.$ infinitesimal geometry, then $(M \rtimes_\tau \tilde{H}/Z, \tilde{H}/Z)$ is an $Ad - c.r.$ geometry iff $\tilde{\sigma}(\tilde{H}) = e^{\sigma(\mathcal{H})}$ is closed in $GL(\mathcal{M})$.

The checking of closeness of $e^{\sigma(\mathcal{H})}$ in $GL(\mathcal{M})$ for the remaining solvable cases is easy and similar to (5.1.4), and we omit the detail.

(5.1.5.5) Proposition. If (G, H) is a solvable $Ad - c.r.$ geometry, then $S = G/H = R^k, k = dim\ S$, topologically.

Proof. Notice that every solvable $Ad - c.r.$ geometry has the form (MH, H) where the topology of $G = M \rtimes_r H$ is the product topology of M and H, so the quotient topology of MH over H is homeomorphic to M. It's well known that a connected, simply connected solvable real Lie group of $dim\ k$ is diffeomorphic to R^k.

Q.E.D.

(5.2) Mixed $Ad - c.r.$ Geometry.

We know that Proposition (5.1.5.1) applies not only to solvable cases, but also to mixed cases. So if $(\mathcal{G}, \mathcal{H}) = (\mathcal{M} \rtimes_\sigma \mathcal{H}, \mathcal{H})$, then the corresponding geometry is $(M \rtimes_r \tilde{H}/Z, \tilde{H}/Z)$ (for notations, see (5.1.5.1)). Most of our indecomposable mixed $Ad - c.r.$ geometries in $dim \leq 4$ are of this type. So we need to determine Z. As we did in (5.1.5.2), we have the following commutative map

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\sigma"] \arrow[d, "\exp"'] & \partial(\mathcal{M}) \arrow[r, phantom, "\subseteq"] \arrow[d, "\exp"'] & gl(\mathcal{M}) \\
\tilde{H} \arrow[r, "\tilde{\sigma}"] & Aut(\mathcal{M}) \arrow[r, phantom, "\subseteq"] & GL(\mathcal{M})
\end{tikzcd}$$

where $\partial(\mathcal{M})$ is the set of all derivations of $\mathcal{M}$. Then $Z = Ker\tilde{\sigma}$, and $\tilde{\sigma}$ induces an injective homomorphism $\overline{\sigma} : \tilde{H}/Z \to Aut(\mathcal{M})$. It's clearly $\overline{\sigma}(\tilde{H}/Z)$ is a connected subgroup of $Aut(\mathcal{M})$ with Lie algebra $\sigma(\mathcal{H})$. So $H = \tilde{H}/Z$ is isomorphic to $\tilde{\sigma}(\tilde{H})$, the connected subgroup of $Aut(\mathcal{M})$ generated by$\{e^D; D \in \sigma(\mathcal{H})\ \}$ and the geometry is $Ad - c.r.$ if and only if this subgroup is closed. $\tilde{H} \to H$ is a covering map. Since $\tilde{H}$ is simply connected, Z can be determined by finding the fundamental group of $H \simeq \tilde{\sigma}(\tilde{H})$. Since $\tilde{\sigma}(\tilde{H})$ is a connected subgroup of $GL(\mathcal{M})$, there are some well known facts (cf. Helgason [15]).

(5.2.1) Lemma. Let A be a connected Lie group and B an analytic subgroup. Let $\mathcal{A}$ and $\mathcal{B}$ denote the corresponding Lie algebras.

(i) Assume $A = GL_n(C)$. If $\mathcal{B}$ is semisimple, then B is closed in A.

(ii) B is closed if $exp\mathcal{B}$ is closed.

(5.2.2.) Corollary. Notations as in (5.1.5.1), If $(\mathcal{G}, \mathcal{H})$ is $ad_r - c.r.$, then $(M \rtimes_r H, H)$ is $Ad - c.r.$ if $\sigma(\mathcal{H})$ is semisimple or $e^{\sigma(\mathcal{H})}$ is closed in $GL(\mathcal{M})$.

Most of our mixed $ad_r - c.r.$ geometries in $dim \leq 4$ are of the following type: $(\mathcal{M} \rtimes_\sigma \mathcal{H}, \mathcal{H})$, where $\mathcal{H}$ is either semisimple or $\mathcal{H} = [\mathcal{H}, \mathcal{H}] \oplus Z(\mathcal{H})$, $dim\ Z(\mathcal{H}) = 1$ or 2, $\sigma(Z(\mathcal{H}))$ is diagonalizable. Then we have the following Lemma.

(5.2.3.) Lemma. Let (G, H) be the geometry corresponding to a mixed $ad_r - c.r.$ geometry of the form $(\mathcal{M} \rtimes_\sigma \mathcal{H}, \mathcal{H})$ as mentioned above. Then (G, H) is $Ad - c.r.$ if and only if $e^{\sigma(Z(\mathcal{H}))}$ is closed in $GL(\mathcal{M})$.

Proof. Let H^* be the connected subgroup of $GL(\mathcal{M})$ with Lie algebra $\sigma(\mathcal{H})$. Then $e^{\sigma(Z(\mathcal{H}))}$ is a closed subgroup of H^* . So if H^* is closed in $GL(\mathcal{M})$, so is $e^{\sigma(Z(\mathcal{H})}$. Let $H^* = A \rtimes S$ be the Levi's decomposition of H^* . Then $A = e^{\sigma(Z(\mathcal{H}))}$; S is semisimple. So S is closed in $GL(\mathcal{M})$. We can assume that $S \subseteq SL(\mathcal{M})$. Since $GL(\mathcal{M}) = D \rtimes SL(\mathcal{M})$ where D is the solvable radical of $GL(\mathcal{M})$, $GL(\mathcal{M})$ is the semi-direct product of D and $SL(\mathcal{M})$. Now A is closed in $GL(\mathcal{M})$, $A \subseteq D$, so A is closed in D; S is closed in $GL(\mathcal{M})$, $S \subseteq SL(\mathcal{M})$, so S is closed in $SL(\mathcal{M})$, so $A \rtimes S$ is closed in $D \rtimes SL(\mathcal{M}) = GL(\mathcal{M})$.

Q.E.D.

(5.2.4). Corollary. Notations as in (5.2.3). Then if $dim\ \mathcal{M} \leq 4$, then $(\mathcal{G}, \mathcal{H}) = (\mathcal{M} \times_\sigma \mathcal{H}, \mathcal{H})$ is $ad_r - c.r.$ if and only if (G, H) is $Ad - c.r.$

Proof. Using (5.2.3), we can check the list of mixed $ad_r - c.r.$ geometries case by case easily.

Q.E.D.

(5.2.5). Corollary. Notations as in (5.2.3). Then G/H is homeomorphic to the simply connected, connected Lie group whose Lie algebra is $\mathcal{M}$. Since $\mathcal{M}$ is a solvable ideal of $\mathcal{G}$, $G/H = R^k$, $k = dim\ G/H$.

There are a few indecomposable mixed $ad_r - c.r.$ geometries which is not of the form $(\mathcal{G}, \mathcal{H}) = (\mathcal{M} \rtimes_\sigma \mathcal{H}, \mathcal{H})$, where $\mathcal{M}$ is a solvable ideal of $\mathcal{G}$. They are Mix^4-iv, (1)-(4) and Mix^4-vi,(1)-(3).

(5.2.6). Type Mix^4-iv is of the form $(R^2 \rtimes \mathcal{H}_o, \mathcal{H})$ s.t. $\mathcal{H} \subseteq \mathcal{H}_o$, $\mathcal{H}_o = sl_2(R)$ or $gl_2(R)$; $\mathcal{H}$ is the maximal Cartan subalgebra of $\mathcal{H}_o$. The simply connected, connected

Lie group with Lie algebra $R^2 \rtimes sl_2(R)$ is $R^2 \rtimes_{\tilde{\sigma}} \widetilde{SL}_2(R)$ whereσ is the induced representation of the standard representation of $\widetilde{SL}_2(R)$. Let $\widetilde{SL}_2(R) = KAN$ be the Iwasawa decomposation. Then the connected Lie subgroup corresponding to a Cartan subalgebra is conjugate to either K or A. Both K and A are closed. The quotient space is $S \simeq R^2 \rtimes_{\tilde{\sigma}} \widetilde{SL}_2(R)/A$ or $S = R^2 \rtimes_{\tilde{\sigma}} \widetilde{SL}_2(R)/K$. The center Z of $\widetilde{SL}_2(R)$ is contained in K; The covering map $\rho : K \to SO(2)$ satisfies $\rho^{-1}(e) = Z' \subsetneq Z$, s.t. $Z/Z' \approx \{\pm 1\}$. The action of an element $z \in K$ on R^2 is trivial if and only if $z \in Z'$. So $(G,H) = ((R^2 \rtimes_{\tilde{\sigma}} \widetilde{SL}_2(R))/Z', K/Z') \sim (R^2 \rtimes SL_2(R), SO(2))$ is the corresponding geometry. It's easy to check that this geometry is $Ad-c.r.$ This is the Type Mix^4-iv-(2). In Type Mix^4-iv-(1), the quotient space is $S = R^2 \rtimes_{\tilde{\sigma}} \widetilde{SL}_2(R)/A$, where $A = exp\{R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$. Let $x \in A$, s.t. x acts on S trivially. Then x acts on $\widetilde{SL}_2(R)/A$ trivially, i.e., $xyA = yA$ for every $y \in \widetilde{SL}_2(R)$. So $y^{-1}xyA = A$, i.e., $y^{-1}xy \in A$. Since $A = exp\{R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$, we can assume $x = expX$ for some $X \in R\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. We know $y^{-1}(expX)y = exp(Ad(y^{-1})X)$. Then $Ad(y)X \in R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for every $y \in \widetilde{SL}_2(R)$. But this is impossible, except for $X = 0$, since if $y = expY$, then $Ad(expY)X = e^{adY}(X)$ and we can choose a suitable $Y \in sl_2(R)$ so that $e^{adY}(X) \in RX$ if $X \neq 0$. So the corresponding geometry is $(G,H) = (R^2 \rtimes_{\tilde{\sigma}} \widetilde{SL}_2(R), A)$. The Iwasawa decompositions of $SL_2(R)$ and $\widetilde{SL}_2(R)$ tell us that the topology of the above two geometries is R^4.

Now let us turn to Type Mix^4-iv-(3) and (4). Similar discussions show that the corresponding geometries are $(R^2 \rtimes (R_+ \times \widetilde{SL}_2(R)), R_+ \times A)$ and $(R^2 \rtimes GL_2^+(R), R_+ \times SO(2))$. Their topology is R^4. All the geometries are $Ad-c.r.$. This can be proved by checking the closeness of $e^{\overline{ad(\mathcal{H})}}$ in $GL_4(R)$.

(5.2.7). Type Mix^4-(v). The $ad_r - c.r.$ infinitesimal geometry is of the form $(l \oplus aff(1,R), \mathcal{H})$, where $l \simeq su(2)$ or $sl_2(R)$. $dim\ \mathcal{H} = 1$. The projection of $\mathcal{H}$ in l is a Cartan subalgebra; the projection of $\mathcal{H}$ in $aff(1,R)$ is Re_1 if $aff(1,R) = Re_1 \oplus Re_2$, $[e_1, e_2] = e_2$. The simply connected, connected Lie group G with Lie algebra $\mathcal{G}$ is $SU(2) \times Aff(1,R)$ or $\widetilde{SL}_2(R) \times Aff(1,R)$ where $Aff(1,R) = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; a > 0, b \in R\}$. The connected subgroup $\tilde{H}$ with Lie algebra $\mathcal{H}$ is of the form $\{(x,y)\}$, where y assumes all

$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a \in R$. And such y acts on $Aff(1,R)\ /\{y\}$ trivially if and only if $y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So if (x,y) acts on G/H trivially y must be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This implies $x = e$, the identity element of $SU(2)$ or $\widetilde{SL}_2(R)$. So $(\tilde{G}, \tilde{H})$ is the corresponding geometry and it is $Ad-c.r.$ (by computing $e^{\overline{ad}(\mathcal{H})}$ directly).

(5.3). Nonabelian reductive cases.

(5.3.1). Red^2-i : $(\mathcal{G}, \mathcal{H}) = (so(1,2), so(1,1))$.

The connected simply connected Lie group $\widetilde{SL}_2(R)$ has Lie algebra $sl_2(R) \approx so(1,2)$. Let H be the connected subgroup of $\widetilde{SL}_2(R)$ with Lie algebra $R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \simeq so(1,1)$. If $\widetilde{SL}_2(R) = KAN$ is its Iwasawa decomposation, then $H = A$. As we have studied in (5.2.6), the corresponding geometry is $(\widetilde{SL}_2(R), A)$. Its topology is R^2.

(5.3.2). Red^2-ii : $(\mathcal{G}, \mathcal{H}) = (so(1,2), so(2))$.

The corresponding geometry $(SO_o(1,2), SO(2))$ is well known. The quotient space is $S^{0,2} \simeq R^2$.

(5.3.3). Red^2-iii; $(\mathcal{G}, \mathcal{H}) = (so(3), so(2))$.

The corresponding geometry is $(SO(3), SO(2)), S \simeq S^3$.

(5.3.4). Red^3-i-(1) and (2) are trivial.

(5.3.5). Red^3-ii-(1) : $(\mathcal{G}, \mathcal{H}) = (gl_2(R), R\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$.

The simply connected Lie group with Lie algebra $gl_2(R)$ is $\tilde{G} = R_+ \times \widetilde{SL}_2(R)$. Since $R\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = R\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the connected subgroup with Lie algebra $\mathcal{H}$ is $\tilde{H} = \{expt(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}); t \in R\} = \{(e^t, expt\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}); t \in R\}$, where $\{expt\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; t \in R\} = A$. Since the only element in A which acts on $SL^2(R)/A$ trivially is the identity, $(\tilde{G}, \tilde{H})$ is the corresponding geometry.

(5.3.6). Red^3-(iii)-(3) $(\mathcal{G}, \mathcal{H}) = (u(2), R\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix})$. It is clearly that $(G,H) = (U(2), \{\begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & 1 \end{pmatrix}; \theta \in R\})$ is the corresponding geometry. There is an one to one correspondance between G/H and $SU(2) : gH \to gH \cap SU(2)$. Then it is clearly that $G/H \simeq SU(2) \simeq S^3$ topologically.

(5.3.7). Red^3-iii-(1) : $(so(4), so(3))$, Red^3-iii-(4) : $(so(1,3), so(3))$ and Red^3-(iii)-(3) : $((so(3,1), so(2,1))$ lead to well known geometries $(SO(4), SO(3)), (SO_o(1,3), SO(3))$ and $(SO_o(3,1), SO_o(2,1))$, with topology $S^3, R^3, S^2 \times R$ respectively (c.f. [20]).

(5.3.8). Red^3-(iii)-(2) : $(\mathcal{G}, \mathcal{H}) = (so(2,2), so(1,2))$.

We know that $(SO_o(2,2), SO_o(1,2))$ is not a geometry since $SO_o(2,2)/SO_o(1,2) = S^{1,2}$ and $\pi_1(S^{1,2}) = \mathbf{Z}$. We have to go to the simply connected, connected Lie group $\tilde{G} = \widetilde{SL}_2(R) \times \widetilde{SL}_2(R)$ whose Lie algebra is $sl_2(R) \oplus sl_2(R) \simeq so(2,2)$. $sl_2(R)$ ($\simeq so(1,2)$) is embedded in $\mathcal{G}$ in the following way: $\mathcal{H} = sl_2(R) \simeq \{(X,X); X \in sl_2(R)\}$. The connected subgroup of $\widetilde{SL}_2(R) \times \widetilde{SL}_2(R)$ with this subalgebra is $\tilde{H} = \{(x,x); x \in \widetilde{SL}_2(R)\} \overset{def}{=} \widetilde{SL}_2(R)^*$. Then $\tilde{G}/\tilde{H} \simeq \widetilde{SL}_2(R)$ under the diffeomorphism:

$$(g_1, g_2)\tilde{H} \mapsto g_1 g_2^{-1}.$$

If $(f,f) \in \tilde{H}$ acts on $\tilde{G}/\tilde{H}$ trivially, then for $g \in \tilde{G}, (f,f)(g,e)\tilde{H} = (g,e)\tilde{H}$, i.e., $(fg,f)\tilde{H} = (g,e)\tilde{H}$, or $(fgf^{-1},e)\tilde{H} = (g,e)\tilde{H}$. But under the diffeomorphism

$$(fgf^{-1},e)\tilde{H} \mapsto fgf^{-1},$$

$$(g,e)\tilde{H} \mapsto g.$$

So $fgf^{-1} = g$ for every $g \in \tilde{G}$, i.e., $f \in Z(SL_2(R))$. Let $Z^{**} = \{(f,f); f \in Z(\widetilde{SL}_2(R))\}$, then the corresponding geometry is $(G,H) = (\widetilde{SL}_2(R) \times \widetilde{SL}_2(R)/Z^{**}, \widetilde{SL}_2(R)^*/Z^{**})$. We know that $\widetilde{SL}_2(R)^*/Z^{**} \simeq PSL_2(R)$.

(5.3.9). Red^3-(ii)-(2) $(\mathcal{G}, \mathcal{H}) = (gl_2(R), R\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix})$.

The simply connected, connected Lie group with Lie algebra $gl_2(R)$ is $\tilde{G} = R_+ \times \widetilde{SL}_2(R)$. Let $\widetilde{SL}_2(R) = KAN$ be the Iwazawa decomposation. $L(K) = R\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $L(A) = R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $L(R_+ \times K) = RX_1 \oplus RX_2$. Let $X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then $RX = \mathcal{H}$. Let $\tilde{H}$ be the connected subgroup of $\tilde{G}$ with Lie algebra$\mathcal{H}$. Then $\tilde{H} = \{(exptX_1, exptX_2); t \in R\}$ which is closed in $R_+ \times K$, so it is also closed in $\tilde{G}$. If for some $t, (exptX_1, exptX_2) \in \tilde{H}$ acts on $\tilde{G}/\tilde{H}$ trivially, then for any $x \in R_+$ and $y \in SL_2(R)$, we have

$$(exptX_1, exptX_2)(x, y)\tilde{H} = (x, y)\tilde{H},$$

i.e.

$$(exptX_1, exptX_2)(x, y)(exp(-tX_1), (exp(-tX_2)) = (x, y)\tilde{H},$$

i.e.

$$(x, (exptX_2)y(exp(-tX_2)))\tilde{H} = (x, y)\tilde{H}.$$

Let $x = 1$, then

$$(1, (exptX_2)y(exptX_2)^{-1}\tilde{H} = (1, y)\tilde{H}.$$

Since there is an one to one correspondance between $\tilde{G}/\tilde{H}$ and $\widetilde{SL_2}(R)$ given by $(x, y)\tilde{H} \mapsto (x, y)\tilde{H} \cap (1, \widetilde{SL_2}(R))$, we have $(exptX_2)y(exptX_2)^{-1} = y$, i.e., $exptX_2 \in Z(\widetilde{SL_2}(R)$. So it's clearly that the corresponding geometry is $(R_+ \times \widetilde{SL_2}(R)/Z^*, \tilde{H}/Z^*)$, where $Z^* = \{(exptX_1, exptX_2)\}; exptX_2 \in Z(\widetilde{SL_2}(R))$. Then $\tilde{H}/Z^* \simeq SO(2)/\{\pm 1\}$.

(5.3.10). In Type Red^4 we have many well known geometries:

$(SO(5), SO(4)), (SO_o(1,4), SO(4)), (SO_o(4,1), SO_o(3,1)), (SO_o(3,2), SO_o(2,2)),$

with quotient spaces

$S^4, H^4, S^{3,1}, S^{2,2},$

and topology

$S^4, R^4, S^3 \times R, S^2 \times R^2$

respectively (c.f. [20]). But for Riemannien symmetric spaces CP^2 and CH^2, the usual pairs (c.f. [15]) $(SU(3), S(U(2) \times U(1))$ and $(SU(2,1), S(U(2) \times U(1)))$ are not geometries since the isotropy subgroup contains the center of the group in each case. By Corollary (2.3) the corresponding geometry is $(G/Z, H/Z)$, where $Z \approx \mathbf{Z}_3$.

(5.3.11). Type Red^4-i-(4): $(so(2,3), so(1,3))$.

First we notice that $(SO_o(2,3), SO_o(1,3))$ is not a geometry since

$$SO_o(2,3)/SO_o(1,3) = S^{1,3} \simeq S^1 \times R^3$$

which is not simply connected ($\pi_1(S^{1,3}) \approx \mathbf{Z}$). Let $\rho : \tilde{G} \to G = SO_o(2,3)$ be the covering map from the simply connected Lie group $\tilde{G} = SO_o(2,3)$ with Lie algebra $so(2,3)$ to

$G = SO_o(2,3)$. Let $\tilde{G} = \tilde{K}\tilde{A}\tilde{N}$ and $G = KAN$ be the Iwasawa decomposation of $\tilde{G}$ and G respectively. Then $\tilde{A} \simeq A, \tilde{N} \simeq N$ and both A and N are simply connected. $\pi_1(G) \approx \pi_1(K) \approx \pi_1(SO(2) \times SO(3)) = \mathbf{Z} \times \mathbf{Z}_2$. We notice that $SO_o(1,3)$ has finite center ($\approx \mathbf{Z}_2$). As before, let $\tilde{H}$ be the connected subgroup of $\tilde{G}$ with Lie algebra $so(1,3)$ and $\tilde{H} = K_1A_1N_1$ be the Iwasawa decomposation s.t. $K_1 \subseteq \tilde{K}$. $L(K_1) = so(3) \subseteq so(2) \oplus so(3) = L(K)$. $\tilde{K} \simeq R \times SU(2)$. We must have $K_1 = SU(2)$. So $Z(\tilde{G}) \cap \tilde{H} = Z(SU(2)) \approx \mathbf{Z}_2$ and the corresponding geometry is $(SO_o(2,3)/\mathbf{Z}_2,\ SO_o(1,3)/\mathbf{Z}_2) = (SO_o(2,3)/\mathbf{Z}_2,\ SO_o(1,3))$. It's easy to show that the quotient space is $S^{1,3}$ with topology R^4.

(5.3.12). Type Red^4-ii-(4):

$$(\mathcal{G}, \mathcal{H}) = (\mathrm{su}(2,1)\ , \begin{pmatrix} 0 & 0 \\ 0 & su(1,1) \end{pmatrix} \oplus R \begin{pmatrix} -2\sqrt{-1} & & 0 \\ & \sqrt{-1} & \\ 0 & & \sqrt{-1} \end{pmatrix}).$$

It's easy to see that $G = SU(2,1)$ has a connected subgroup

$$\tilde{H} = S(U(1) \times U(1,1)) = \{\begin{pmatrix} detB^{-1} & 0 \\ 0 & B \end{pmatrix}; B \in U(1,1)\}$$

with Lie algebrea $\mathcal{H}$. $\tilde{H}$ contains the center Z of $\tilde{G}(Z \approx \mathbf{Z}_3)$. Let $PSU(2,1) = SU(2,1)/\mathbf{Z}_3$, then the corresponding geometry is $(PSU(2,1), S(U(1) \times U(1,1))/\mathbf{Z}_3)$.

To find out the topology of this geometry, we set up the following geometric model. Let

$$E = \{-|z_1|^2 - |z_2|^2 + |z_3|^2 = -1; z_i, i = 1,2,3, \in C\}.$$

Under the mapping

$$(z_1, z_2, z_3) \mapsto (\frac{z_1}{|z_1|^2 + |z_2|^2}, \frac{z_2}{|z_1|^2 + |z_2|^2}, z_3)$$

We see that $E \simeq S^3 \times R$. Group $U(2,1)$ acts on E transitively. At the point (1, 0, 0) the isotropy subgroup is $\{\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}; u \in U(1,1)\}$. The following equivalent relation

$$(z_1, z_2, z_3) \sim (z_1e^{i\theta}, z_2e^{i\theta}z_3e^{i\theta}), \theta \in R$$

introduces a principle fiber bundle

$$S^1 \to E \to M.$$

Since the following diagram

$$\begin{tikzcd}
(z_1, z_2, z_3) \arrow[r] \arrow[d] & g(z_1, z_2, z_3) \arrow[d] \\
(z_1 e^{i\theta}, z_2 e^{i\theta}, z_3 e^{i\theta}) \arrow[r] & g(z_1 e^{i\theta}, z_2 e^{i\theta}, z_3 e^{i\theta})
\end{tikzcd}$$

is commutative for any $g \in U(2,1)$, the action of $U(2,1)$ on E introduces an action of $U(2,1)$ on M. The isotropy subgroup of this introduced action of $U(2,1)$ at $\pi(1,0,0)$ is $\{\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}; v \in U(1), u \in U(1,1)\}$. Now

$M \simeq U(2,1)/U(1) \times U(1,1)$

$\simeq SU(2,1)/S(U(1) \times U(1,1))$

$\simeq PSU(2,1)/S(U(1) \times U(1,1) \ / \ \mathbf{Z}_3$.

Let us consider the topology of M.

Claim: $M \simeq CP^2 - D^4$, where $D^4 = \{x \in R^4; \|x\| \leq 1\}$.

Proof of the Claim.

By the definition

$CP^2 = C^3 - \{0\}\}/z \sim \lambda z$, where $z \in C^3 - 0$, $\lambda \in \{C - \{0\}\}$.

$C^3 - \{0\} = \{|z_1|^2 + |z_2|^2 - |z_3|^2 > 0\}$

$\cup\{|z_1|^2 + |z_2|^2 - |z_3^2 \leq 0; (z_1, z_2, z_3) \neq 0\}$.

Then it's clear that

$\{|z_1|^2 + |z_2|^2 - |z_3|^2 > 0\}/z \sim \lambda z, \lambda \in C - \{0\}$

$= \{|z_1|^2 + |z_2|^2 - |z_3|^2 = 1\}/z \sim \lambda z$ ($|\lambda|=1$). The right hand side is M.

Since $|z_1|^2 + |z_2|^2 - |z_3|^2 \leq 0$, and $z \neq 0$, we must have $z_3 \neq 0$, so

$$\left|\frac{z_1}{z_3}\right|^2 + \left|\frac{z_2}{z_3}\right|^2 \leq 1,$$

this lead to D^4 under "$\sim$".

Q.E.D.

(5.3.13). Type Red^4-ii-(1):

$$(\mathcal{G}, \mathcal{H}) = (sl_3(R), \begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix} + R \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -2 \end{pmatrix})$$

Let $S = \{(\pi, \rho)\}$, where π is an oriented plane in R^2 through the origin and ρ is an oriented ray through the origin in R^3, $\rho \not\subseteq \pi$ so that the orientation of π plus the orientation of ρ is the standard orientation of R^3. It's easy to see that $G = SL_3(R)$ acts transitively on S.

At $\pi = <X, Y>$ - plane, $\rho =$ the positive Z-axis, the isotropy group is

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & detA^{-1} \end{pmatrix}; A \in GL_2^+(R) \right\} \simeq S(GL_2^+(R) \times R_+).$$

Every π corresponds to a unique ray $\lambda(\pi)$ through the origin which is perpendicular to π so that the direction of π plus the direction of $\lambda(\pi)$ is the standard direction of R^3. $\lambda(\pi)$ has an intersection with S^2, say, L. ρ has an intersection with S^2, say, P. Then we can identify π with L, ρ with P and (L, P) is an ordered pair of points in S^2. Since $\rho \not\subseteq \pi$, the angle between $\lambda(\pi)$ and ρ is less than $\frac{1}{2}\pi$. So by the sterographic projection. we can identify point P with a unique vector on the tangent plane $T(S^2)$ of S^2 at point L.

So we have set up an one to one correspondance between S and $T(S^2)$. It's easy to show that they are diffeomorphic. It's well known that $T(S^2)$ is simply connected. So

$$T(S^2) \simeq S \simeq SL_3(R)/S(GL_2^+(R) \times R_+).$$

It can be proved directly that the action of $SL_3(R)$ on S is effective.

(5.3.14). Type Red^4-iii-(1):

$$(\mathcal{G}, \mathcal{H}) = (sl_2(C), C\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \simeq (so(3,1),\ so(2) \oplus\ so(1,1)),$$

where $so(2) \oplus so(1,1)$ forms a Cartan subalgebra of $so(1,3)$

$$\left\{ \begin{pmatrix} \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \end{pmatrix}; a, b \in R \right\}.$$

$SL_2(C)$, as a real Lie group, is simply connected, the connected Lie subgroup with Lie algebra $C\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a \in C - \{0\} \right\}$$

So we have $\tilde{G} = SL_2(C), \tilde{H} \simeq C^*$, $Z(\tilde{G}) \subseteq \tilde{H}$, and

$$Z(\tilde{G}) \sim \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\} \sim \mathbf{Z}_2.$$

So the corresponding geometry is $(SL_2(C) \ / \ \mathbf{Z}_2, C^* \ / \ \mathbf{Z}_2) = (PSL_2(C), C^* \ / \ \pm 1)$. It's well known that topologically, this geometry is $T(S^2)$.

§6. Boundedness of a geometry.

(6.1) Definition A geometry (G, H) is called bounded if there is a subgroup $\Gamma \subseteq G$ which acts on G/H properly discontinuously with compact quotiet.

There are many bounded geometries in our classification list of indecomposable $Ad-$ $c.r.$ geometries in $dim \leq 4$. Besides the trivial case that G/H is itself compact, the following proposition covers the majority of our list.

(6.1.1) Proposition. If $(L(G), L(H)) = (\mathcal{M} \rtimes_\sigma \mathcal{H}, \mathcal{H})$ s.t. the connected Lie subgroup $M \subseteq G$ with Lie algebra $\mathcal{M}$ has a uniform lattice, then G/H is bounded.

The proof of (6.1.1) is easy, since $G/H \simeq M$ diffeomorphically. Since $R^k, \mathcal{N}il^3, \mathcal{N}il^4$, $\mathcal{N}il^3 \times R$ have uniform lattices, we have

(6.1.2) Corollary. The following geometries are bounded: Mix^2, Mix^3-(i) and (ii), Mix^4 - (i), Sol^1, Sol^2-(ii), Sol^3-(ii) and (iii), Sol^4-(ii), (iii) and (iv).

If H is compact, then G/H is bounded iff G has a uniform lattice. So we have

(6.1.3) Corollary. The following geomtries are bounded:

(i) symmetric spaces: $S^2, S^3, S^4, H^2, H^3, H^4, CP^2$ and CH^2.

(ii) those (G, H) that H is $\{e\}$ or compact and G is semisimple.

The proof of (6.1.3) is based on the fact that a connected non-compact semisimple Lie group has unifrom lattices. (c.f. Rughunathan [25]).

In Wall's paper [35], we find a list which tells when $(G, \{e\})$ has a uniform lattice (c.f. also Fried [8]) for $dim\ G \leq 4$. It solves the following cases: Type Sol^2-(i), Sol^3-(i), Sol^4-(i). We need the following list for (6.2).

(6.1.4.) The bounded geometrie among Sol^2-(i), Sol^3-(i) and Sol^4-(i) are (given in the infinitesimal form $(\mathcal{G}, 0)$

Type Sol^2-(i) : none;

Type Sol^3-(i) :

No. 1. $(N_3\ , 0)$;

No. 2. when $t=-1$, i.e. $(R^2 \rtimes\ R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0)$;

No. 3. when $\theta = (k+\frac{1}{2})\pi$, i.e. $(R^2 \rtimes\ R\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0)$;

Type Sol^4-(i) :

No. 1. $(N_4, 0)$;

No. 3. $(R^3 \rtimes\ R\begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix}, 0)$, where a, b, c are real, $a+b+c=0$, $abc \neq 0$ and a, b, c satisfy either

(1): $a > b > c$ and e^a, e^b, e^c are the roots of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$ with $m \neq n$ positive integers or

(2) : $a = b \neq c$, i.e. $(R^3 \rtimes\ R\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -2 \end{pmatrix}, 0)$;

No. 4 : only if $a = -2cos\theta$, i.e. $(R^3 \rtimes\ R\begin{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} & 0 \\ 0 & -2 \end{pmatrix}, 0)$;

No. 8 : only if $a=-1$. This corresponds to the matrix group

$$G = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} ; \alpha, a, b, c \in R, \alpha > 0 \right\}.$$

We will study the remaining cases in (6.2) , (6.3) and (6.4).

(6.2) Other bounded solvable geometries.

The following proposition will play the key role.

(6.2.1). Proposition [Goldman and Hirsch]. Let M be a compact complete affine manifold. Suppose that $\pi = \pi_1(M)$ is virtually solvable. Then M has parallel volume.

For a proof, c.f. [11].

So if $E = R^n, G \subseteq Affine(E)$ with G acting on E transitively and H is an isotropy subgroup of G at some point of E, then $E \simeq G/H$. If G is solvable, $\Gamma\ \subseteq G$ is solvable discrete and Γ acts on E properly discontinuously with compact quotient, then by a lemma of Selberg [10], Γ contains a normal subgroup Γ_1 with finite index s.t. Γ_1 acts on E freely. If $\Gamma\backslash E$ is compact, so is $\Gamma_1\backslash E$. Then By (6.2.1), E has parallel volume and the linear part of Γ_1 must be contained in $SL(E)$. So if a solvable geometry (G,H) is bounded, G/H can be identified with $E = R^n$ with $G \subseteq Affine(E)$ acting on E transitively and H is an isotropy subgroup of G at some point of E, then the intersection of $SL(E)$ and the linear part of G must contain the linear part of a discret subgroup Γ of G s.t. Γ acts properly continuously and freely on E with compact quotient.

Based on the above observation, we first try to express a solvable geometry into the above "affine" form, then study the linear part of G. Let $P_l(A)$ be the linear part of $A \subseteq Affine(E)$.

We also try to use other methods, if the above method doesn't work or there is a better way to do it.

(6.2.2) Type Sol^3 -(iv). $(\mathcal{G}, \mathcal{H}) = (S_{3,A} \rtimes \mathcal{H}, \mathcal{H}) = (R^2 \rtimes \sigma(RA + \mathcal{H}), \mathcal{H})$.

(6.2.2.1) No.1. $\sigma(\mathcal{H}) = R\begin{pmatrix} 1 & \\ & t \end{pmatrix}, t \neq 0, A = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$. $\mathcal{G}$ is

$R^2 \rtimes \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}; a, b \in R\}$

If $t \neq -1$, then $R^2 \rtimes R\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \subseteq \mathcal{G}$, and

$G/H \simeq R^2 \rtimes \{\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}; t \in R\}$(diffeomorphism)

Since $R^2 \rtimes \{\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}; t \in R\}$ has a uniform lattice Γ, so

$\Gamma\backslash G/H \simeq \Gamma\backslash R^2 \rtimes \{\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}; t \in R\}$

is compact. If $t = -1$, then it is easy to show that$(\mathcal{G}, \mathcal{H}) \simeq$

$$(\left\{\begin{pmatrix} \begin{pmatrix} c & & 0 \\ & -c+d & \\ 0 & & 0 \end{pmatrix} & \begin{pmatrix} a \\ b \\ d \end{pmatrix} \\ 0 & 0 \end{pmatrix}; a, b, c, d \in R\right\}, \left\{\begin{pmatrix} \begin{pmatrix} c & & 0 \\ & -c & \\ 0 & & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \\ 0 & 0 \end{pmatrix}; c \in R\right\}).$$

Then the corresponding geometry is $(G, H) =$

$$(\left\{\begin{pmatrix} \begin{pmatrix} e^c & & \\ & e^{-c+d} & \\ & & 1 \end{pmatrix} & \begin{pmatrix} a \\ b \\ d \end{pmatrix} \\ 0 & 1 \end{pmatrix}; a, b, c, d \in R\right\}, \left\{\begin{pmatrix} \begin{pmatrix} e^c & & \\ & e^{-c} & \\ & & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ 0 & 1 \end{pmatrix}; c \in R\right\})$$

and $G \subseteq Affine(E), E = R^3$. The action of G on R^3 is transitive. At (0,0,0) the isotropy subgroup is H. If (G, H) is bounded, then there is a $\Gamma \subseteq G$ s.t. $\Gamma\backslash$E is compact, the liner part $\mathrm{P}_l(\Gamma) \subseteq SL(E)$, i.e.,

$$P_l(\Gamma) \subseteq \left\{ \begin{pmatrix} e^c & & \\ & e^{-c} & \\ & & 1 \end{pmatrix} ; c \in R \right\} = P_l(H).$$

So we must have

$$\Gamma \subseteq \left\{ \left(\begin{pmatrix} e^c & & \\ & e^{-c} & \\ & & 1 \\ & 0 & \end{pmatrix} \quad \begin{pmatrix} a \\ b \\ 0 \\ 1 \end{pmatrix} \right) ; a, b, c \in R \right\}$$

i.e. $\Gamma\backslash$E $\simeq (\Gamma\backslash R^2) \times R$ which is not compact. We have a contradiction, so if $t = -1$, the geometry is not bounded.

(6.2.2.2) No.2. We have

$$G = \left\{ \begin{pmatrix} e^c cos(tc+\theta) & -e^c sin(tc+\theta) & 0 & a \\ e^c sin(tc+\theta) & e^c cos(tc+\theta) & 0 & b \\ 0 & 0 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{pmatrix} ; a, b, c, \theta \in R \right\}.$$

At $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $H = \left\{ \begin{pmatrix} e^c & & & 0 \\ & e^c & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} ; c \in R \right\}$ is the isotropy subgroup.

$P_l(G) = \left\{ \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \theta \in R \right\}$. So if (G, H) is bounded,

then there is $\Gamma \subseteq G$, s.t. $\Gamma\backslash E$ is compact and $P_l(\Gamma) \subseteq P_l(G)$.So

$$\Gamma \subseteq \left\{ \begin{pmatrix} cos\theta & -sin\theta & 0 & a \\ sin\theta & cos\theta & 0 & b \\ 0 & 0 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{pmatrix} ; a, b, \theta \in R \right\} = G'.$$

We also have

$\Gamma\backslash G/H \simeq \Gamma\backslash G'$.

But G' has uniform lattices according to (6.1.4) $L(G') \simeq R^2 \rtimes R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We can take

$$\Gamma = \left\{\begin{pmatrix} 1 & & & i \\ & 1 & & j \\ & & 1 & 2k\pi \\ 0 & & & 1 \end{pmatrix} ; i, j, k \in \mathbf{Z}\right\}.)$$

(6.2.2.3) No. 3. It's easy to show that

$$(G, H) = (\left\{(R^2 \rtimes \{\begin{pmatrix} e^t & \\ & e^t \end{pmatrix} ; t \in R\}) \rtimes SO(2)\right\}/\{\pm Id_2\}, SO(2)/\{\pm Id_2\}),$$

$H = PSO(2)$ is compact. So we only need to find if G has a uniform lattice. Since G is not unimodular, $dim\ G = 4$, according to our list in (6.1.4), G has no uniform lattices. So (G, H) is not bounded.

(6.2.2.4) No. 4. It is easy to show that $(G, H) =$

$$(\left\{\begin{pmatrix} e^c cos(tc+\theta) & -e^c sin(tc+\theta) & 0 & a \\ e^c sin(tc+\theta) & e^c cos(tc+\theta) & 0 & b \\ 0 & 0 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{pmatrix} ; a, b, c, \theta \in R\right\},$$

$$\left\{\begin{pmatrix} e^c cost c & -e^c sint c & & 0 \\ e^c sint c & e^c cost c & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} ; c \in R\right\})$$

G acts transitively on $R^4 = E, H$ is the isotropy subgroup at $\{0, 0, 0\}$. Again, the maximal subgroup contained in $SL_4(R)$ is

$$G' = \left\{\begin{pmatrix} cos\theta & -sin\theta & 0 & a \\ sin\theta & cos\theta & 0 & b \\ 0 & 0 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{pmatrix} ; a, b, c, \theta \in R\right\},$$

as we know from (6.2.2.2), the geometry is bounded.

(6.2.3) Type Sol^4-v.

6.2.3.1) No.1. $(\mathcal{G}, \mathcal{H}) \sim$

$$(R^3 \rtimes (R \begin{pmatrix} 0 & & \\ & 1 & \\ & & t_3 \end{pmatrix} + R \begin{pmatrix} 1 & & \\ & t_1 & \\ & & t_2 \end{pmatrix}), R \begin{pmatrix} 1 & & \\ & t_1 & \\ & & t_2 \end{pmatrix}) \sim$$

$$(\left\{\begin{pmatrix} f & & & & a \\ & d+t_1 f & & & b \\ & & t_3 d + t_2 f & & c \\ & & & 0 & d \\ 0 & & & & 0 \end{pmatrix} ; a,b,c,d,f \in R\right\},$$

$$\left\{\begin{pmatrix} f & & & & 0 \\ & t_1 f & & & \\ & & t_2 f & & \\ & & & 0 & \\ 0 & & & & 0 \end{pmatrix} ; f \in R\right\}).$$

$(G,H) \sim$

$$(\left\{\begin{pmatrix} e^f & & & & a \\ & e^{d+t_1 f} & & & b \\ & & e^{t_3 d + t_2 f} & & c \\ & & & 1 & d \\ 0 & & & & 1 \end{pmatrix} ; a,b,c,d,f \in R\right\},$$

$$\left\{\begin{pmatrix} e^f & & & & 0 \\ & e^{t_1 f} & & & \\ & & e^{t_2 f} & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} ; f \in R\right\}).$$

G acts transtively on R^4 with isotropy subgroup H at (0,0,0,0).

We have two cases

Case 1: $t_3 = -1$ then G has subgroup G' s.t.

$G' \simeq R \times (R^2 \rtimes \{\begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}; s \in R\})$ (Lie group isomorphism)

and G' has a uniform lattice Γ. So $\Gamma \backslash G' \simeq \Gamma \backslash G/H$ (diffeomorphism), the geometry is bounded.

Case 2: $t_3 \neq -1, 1 + t_1 + t_2 = 0$. Then if the geometry is bounded, there is a Γ s.t. $\Gamma \backslash G/H$ is compact and Γ must be contained in

$$\left\{\begin{pmatrix} e^f & & & & a \\ & e^{t_1 f} & & & b \\ & & e^{t_2 f} & & c \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} ; a,b,c,f \in R\right\}$$

and $\Gamma \backslash R^4 = (\Gamma \backslash R^3) \times R$, it's a contradiction. So the geometry is not bounded.

Case 3: $t_3 \neq -1, 1 + t_1 + t_2 \neq 0$.

As we can see from Case 2, Γ can not be contained in a subgroup of G, s.t. d $\equiv 0$. And Γ can only be contained in a subgroup s.t.

$$f + (d + t_1 f) + t_3 d + t_2 f = 0 \ ,$$

i.e.

$$(1 + t_1 + t_2)f + (1 + t_3)d = 0.$$

and

$$f = -\tfrac{1+t_3}{1+t_1+t_2} d,$$

i.e. Γ , if exists, should be contained in

$$G' = \left\{ \begin{pmatrix} e^{-\frac{1+t_3}{1+t_1+t_2} d} & & & & a \\ & e^{\frac{1+t_2-t_1t_3}{1+t_1+t_2} d} & & & b \\ & & e^{\frac{-t_2+t_3+t_1t_3}{1+t_1+t_2} d} & & c \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; a, b, c, d \in R \right\}$$

$$\simeq R^3 \rtimes \left\{ \begin{pmatrix} e^{-(1+t_3)d} & & 0 \\ & e^{(1+t_2-t_1t_3)d} & \\ 0 & & e^{(-t_2+t_3+t_1t_3)d} \end{pmatrix} ; d \in R \right\}$$

and $\Gamma \backslash G/H \simeq G'$ (diffeomorphism), i.e. $\Gamma \backslash G/H$ is bounded iff Γ is a uniform lattice of G'.

Claim: The matrix

$$(*) \begin{pmatrix} e^{-(1+t_3)d} & & \\ & e^{(1+t_2-t_1t_3)d} & \\ & & e^{(-t_2+t_3+t_1t_3)d} \end{pmatrix}$$

has eigenvalues $\{\lambda, \lambda, -2\lambda\}$ iff $t_3 = -1, t + t_1 + t_2 = 0$, and $\lambda = 0$.

Proof of the Claim:

Case $1. -(1 + t_3) = (1 + t_2 - t_1t_3) = -2(-t_2 + t_3 + t_1t_3)$.

Then

$$t_2 = t_1t_3 - t_3 - 2,$$

$$-(1 + t_3) = -2(-t_1t_3 + t_3 + 2 + t_3 + t_1t_3).$$

$$\Rightarrow -1 - t_3 = -2(2 + 2t_3)$$

$$\Rightarrow t_3 = -1$$

$\Rightarrow 1 + t_1 + t_2 = 0.$

Case 2. $-(1 + t_3) = (-t_2 + t_3 + t_1t_3) = -2(1 + t_2 - t_1t_3).$

Then

$t_2 = 1 + 2t_3 + t_1t_3$

$-(1 + t_3) = -2(1 + 1 + 2t_3)$

$\Rightarrow t_3 = -1$

$\Rightarrow 1 + t_1 + t_2 = 0.$

Case 3. $-2\{-(1 + t_3)\} = (1 + t_2 - t_1t_3) = (-t_2 + t_3 + t_1t_3)$

Then $2(t_2 - t_1t_3) = -1 + t_3$

$\Rightarrow 2(1 + t_3) = 1 + \frac{1}{2}(-1 + t_3) = \frac{1}{2}(1 + t_3)$

$\Rightarrow t_3 = -1$

$\Rightarrow 1 + t_1 + t_2 = 0.$

Q.E.D.

So under the condition $t_3 \neq -1$, the matrix $(*)$ cannot have eigenvalues as

$\{e^{\lambda}, e^{\lambda}, e^{-2\lambda}\}$

and we must have three eigen values $\{e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3}\}$ s.t. $\lambda_i \neq \lambda_j$ for $i \neq j$. Then by (6.1.4), G has a uniform lattices iff $\{e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3}\}$ are solutions of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$ with $m \neq n$ positive integers. We can find t_1, t_2, t_3 by setting

$-(1 + t_3) = \lambda_1,$

$1 + t_2 - t_1t_3 = \lambda_2,$

and

$-t_2 + t_3 + t_1t_3 = \lambda_3.$

$\Rightarrow t_3 = -\lambda_1 - 1,\ t_2 = \lambda_2 - 2 + t_1(-\lambda_1 - 1)$

Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$, the solutions of the system of equations are not unique. So if $t_3 \neq -1$, the geometry is bounded iff t_1, t_2, t_3 satisfy:

$t_3 = -\lambda_1 - 1$

$t_2 = \lambda_2 - 1 + t_1(-\lambda_1 - 1)$

where $\lambda_1 \neq 0$, $e^{\lambda_1}, e^{\lambda_2}, e^{1-\lambda_1-\lambda_2}$ are solutions of

$\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$ with $m \neq n$ positive integers.

(6.2.3.2) No. 2. and No. 11. In both cases, G has a normal subgroup isomorphic to

$$G' = \left\{ \begin{pmatrix} cos\theta & -sin\theta & & & a \\ sin\theta & cos\theta & & & b \\ & & 1 & & c \\ & & & 1 & \theta \\ & & & & 1 \end{pmatrix} \right.$$

which has a uniform lattice Γ, and

$$\Gamma \backslash G' \simeq \Gamma \backslash G/H$$

so the geometry is bounded.

(6.2.3.3) No. 3. If $t_3 = -2$, then G has a normal subgroup isomorphic to

$$G' = R^3 \rtimes \left\{ \begin{pmatrix} e^{s} & & \\ & e^{s} & \\ & & e^{-2s} \end{pmatrix} ; s \in R \right\}$$

which has a uniform lattice Γ so the geometry is bounded.

If $t_3 = 1/t_1$, then G has a normal subgroup isomorphic to G' as in (6.2.3.2) so the geometry is bounded.

If $t_3 \neq -2$, $t_3 \neq 1/t_1$, we have $(G, H) =$

$$(\left\{ \begin{pmatrix} e^{dt_3+f} & & & & a \\ & e^{d+t_1 f} cost_2 f & -e^{d+t_1 f} sint_2 f & & b \\ & e^{d+t_1 f} sint_2 f & e^{d+t_1 f} cost_2 f & & c \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; a, b, c, d, f \in R \right\},$$

$$\left\{ \begin{pmatrix} e^{f} & & & & \\ & e^{t_1 f} cost_2 f & -e^{t_1 f} sint_2 f & & \\ & e^{t_1 f} sint_2 f & e^{t_2 f} cost_2 f & & \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; f \in R \right\}.$$

If the geometry is bounded, then there is a Γ contained in $G \subsetneq SL_5(R)$, i.e. in $G' =$

$$\left\{ \begin{pmatrix} e^{(2-2t_1 t_3)f} & & & & a \\ & e^{(t_1 t_3 - 1)f} cost_2 f & -e^{(t_1 t_3 - 1)f} sint_2 f & & b \\ & e^{(t_1 t_3 - 1)f} sint_2 f & e^{(t_1 t_3 - 1)f} cost_2 f & & c \\ & & & 1 & -(1 + 2t_1)f \\ & & & & 1 \end{pmatrix} ; a, b, c, f \in R \right\}.$$

Since $\Gamma\backslash G/H \simeq \Gamma\backslash G'$. If $t_1 \neq -1/2$, G' must be in the list of (6.1.4). We know from (6.1.4) that G' has no uniform lattices. If $t_1 = -1/2$, $\Gamma\backslash R^4 = (\Gamma\backslash R^3) \times R$ which is not compact.

(6.2.3.4) No. 4. As in (6.2.3.3) , if the geometry is bounded, we can find Γ, s.t. Γ is contained in

$$G' = \left\{ \begin{pmatrix} e^{-2t_1 f} & & & & a \\ & e^{t_1 f} cos t_2 f & -e^{t_1 f} sin t_2 f & & b \\ & e^{t_1 f} sin t_2 f & e^{t_1 f} cos t_2 f & & c \\ & & & 1 & -(1+2t_2)f \\ & & & & 1 \end{pmatrix} ; a, b, c, f \in R \right\}$$

also $\Gamma\backslash G/H \simeq \Gamma\backslash G' \simeq \Gamma\backslash R^4$. If $t_2 = -1/2$, then $\Gamma\backslash R^4 = (\Gamma\backslash R^3) \times R$ which is not compact. If $t_2 \neq -1/2$, G' is indecomposable solvable Lie group of *dim* 4, but G' is not in the list (6.1.4), so the geometry is not bounded.

(6.2.3.5) No. 6. As in (6.2.3.4), if the geometry is bounded, Γ should be contained in

$$G' = \left\{ \begin{pmatrix} e^f cos(-2tf) & -e^f sin(-2tf) & & & a \\ e^f sin(-2tf) & e^f cos(-2tf) & & & b \\ & & e^{-2f} & c & \\ & & & 1 & -2f \\ & & & & 1 \end{pmatrix} ; a, b, c, f \in R \right\},$$

$\Gamma\backslash R^4 \simeq G'$. Since G' has no uniform lattices, the geometry is not bounded.

(6.2.3.6) No. 7. The geometry is

$$(R^3 \rtimes \left\{ \begin{pmatrix} e^{ta} & & 0 \\ & e^{ta} & \\ & & e^a \end{pmatrix} ; a \in R \right\} \rtimes \begin{pmatrix} SO(2) & 0 \\ 0 & 1 \end{pmatrix}) / \begin{pmatrix} \pm 1 & & 0 \\ & \pm 1 & \\ 0 & & 1 \end{pmatrix} ; PSO(2))$$

where H is compact. So the geometry is bounded iff G has uniform lattices. If G has uniform lattices, G must be unimordular, i.e. $t = -1/2$. If $t = -1/2$,

$$G' = R^3 \rtimes \left\{ \begin{pmatrix} e^{-\frac{1}{2}a} & & \\ & e^{-\frac{1}{2}a} & \\ & & e^a \end{pmatrix} ; a \in R \right\}$$ has uniform lattices Γ,

$\Gamma\backslash G' \simeq \Gamma'\backslash G/H$, where Γ' is the image of Γ in G. So the geometry is bounded iff $t = -1/2$.

(6.2.3.7) No. 8. If $t_1 = -1/2$, then G has a normal subgroup isomorphic to

$$R^3 \rtimes \left\{ \begin{pmatrix} e^a & & \\ & e^a & \\ & & e^{-2a} \end{pmatrix} ; a \in R \right\}$$

which has a uniform lattice so the geometry is bounded. If $t_1 \neq -1/2$, and if the geometry is bounded, Γ must be in

$$G' = \left\{ \begin{pmatrix} e^{\frac{f}{2t_1+1}} cost_2 f & -e^{\frac{f}{2t_1+1}} sint_2 f & & & a \\ e^{\frac{f}{2t_1+1}} sint_2 f & e^{\frac{f}{2t_1+1}} cost_2 f & & & b \\ & & e^{\frac{-2f}{2t_1+1}} & & c \\ & & & 1 & \frac{-2f}{2t_1+1} \\ & & & & 1 \end{pmatrix} ; a, b, c, f \in R \right\}.$$

$\Gamma\backslash G' \simeq \Gamma\backslash G/H$. Since G' has no uniform lattices, the geometry is not bounded.

(6.2.3.8) No. 10. Since

$$(\mathrm{G}, \mathrm{H}) \sim (\mathrm{R}^3 \rtimes \left\{ \begin{pmatrix} e^a & & \\ & e^b & \\ & & e^c \end{pmatrix} ; a, b, c \in R \right\}, \left\{ \begin{pmatrix} e^a & & \\ & e^b & \\ & & e^{t_1 a + t_2 b} \end{pmatrix} ; a, b \in R \right\})$$

so if $t_1 \neq -1$,

$$G' = R^3 \rtimes \left\{ \begin{pmatrix} e^{-c} & & \\ & 1 & \\ & & e^c \end{pmatrix} ; c \in R \right\} \subseteq G,$$

if $t_2 \neq -1$,

$$G' = R^3 \rtimes \left\{ \begin{pmatrix} 1 & & \\ & e^{-c} & \\ & & e^c \end{pmatrix} ; c \in R \right\} \subseteq G.$$

In both cases G' is normal in G, G' has a uniform lattice and $\Gamma\backslash G/H \simeq \Gamma\backslash G'$ so the geometry is bounded if $t_1 \neq -1$ or $t_2 \neq -1$.

If $t_1 = t_2 = -1$, $(G, H) \sim$

$$(\left\{ \begin{pmatrix} e^a & & & & d \\ & e^b & & & f \\ & & e^{c-a-b} & & g \\ & & & 1 & c \\ & & & & 1 \end{pmatrix} ; a, b, c, d, f, g \in R \right\},$$

$$\left\{\begin{pmatrix} e^{a} & & & & \\ & e^{b} & & & \\ & & e^{-a-b} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} ; a, b \in R \right\}).$$

G acts transitively on R^4 with isotropy subgroup H at (0,0,0,0). If there is $\Gamma \subseteq G$, s.t. $\Gamma \backslash R^4$ is compact, then Γ must be contained in

$$G' = \left\{\begin{pmatrix} e^{a} & & & & d \\ & e^{b} & & & f \\ & & e^{-a-b} & & g \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} ; a, b, d, f, g \in R \right\}.$$

Then $\Gamma \backslash R^4 \simeq (\Gamma \backslash R^3) \times R$ can not be compact. The contradiction means the geometry is not bounded.

(6.2.3.9) No. 5. $(G, H) \sim$

$$(\left\{\begin{pmatrix} e^{t_2 d + f} & & & & a \\ & e^{t_1 f} & d e^{t_1 f} & & b \\ & & e^{t_1 f} & & c \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; a, b, c, d, f \in R \right\},$$

$$\left\{\begin{pmatrix} e^{f} & & & & \\ & e^{t_1 f} & & & \\ & & e^{t_1 f} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} ; f \in R \right\}).$$

If the geometry is bounded, then Γ should be contained in G' s.t.

$$\text{Case 1: } G' = \left\{\begin{pmatrix} e^{-2t_1 f} & & & & a \\ & e^{t_1 f} & -\frac{1+2t_1}{t_2} f e^{t_1 f} & & b \\ & & e^{t_1 f} & & c \\ & & & 1 & -\frac{1+2t_1}{t_2} f \\ & & & & 1 \end{pmatrix} ; a, b, c, d \in R \right\}$$

if $t_2 \neq 0$; or

$$\text{Case 2: } G' = \left\{\begin{pmatrix} e^{f} & & & & a \\ & e^{-f/2} & d e^{-f/2} & & b \\ & & e^{-f/2} & & c \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; a, b, c, d, f \in R \right\}$$

if $t_2 = 0$ and $t_1 = -1/2$; or

$$\text{Case 3: } G' = \left\{ \begin{pmatrix} 1 & & & & a \\ & 1 & d & & b \\ & & 1 & & c \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; a, b, c, d \in R \right\}$$

if $t_2 = 0$, and $t_1 \neq -1/2$.

In Case 1, $\Gamma\backslash G' \simeq \Gamma\backslash G/H$, but G' has no uniform lattices, so the geometry is not bounded.

In Case 2 and Case 3, G has a normal unipotent subgroup

$$U = \left\{ \begin{pmatrix} 1 & & & & a \\ & 1 & d & & b \\ & & 1 & & c \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; a, b, c, d \in R \right\} \simeq Nil^3 \times R.$$

U has a uniform lattice Γ and $\Gamma\backslash U \simeq \Gamma\backslash G/H$. The geometry is bounded.

(6.2.3.10) No. 9. We have $(G, H) \sim$

$$(\left\{ \begin{pmatrix} e^f & de^f & & & a \\ & e^f & & & b \\ & & e^{td} & & c \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; a, b, c, d, f \in R \right\},$$

$$\left\{ \begin{pmatrix} e^f & & & & \\ & e^f & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} ; f \in R \right\})$$

Since $t \neq 0$, if the geometry is bounded, Γ should be in

$$G' = \left\{ \begin{pmatrix} e^f & -\frac{2}{t} f e^f & & & a \\ & e^f & & & b \\ & & e^{-2f} & & c \\ & & & 1 & -\frac{2}{t} f \\ & & & & 1 \end{pmatrix} ; a, b, c, f \in R \right\}$$

and $\Gamma\backslash G' \simeq \Gamma\backslash G/H$. But G' has no uniform lattices, so the geometry is not bounded.

(6.2.4) Type Sol^4-iv.

(6.2.4.1) No. 1 and No. 5. It's easy to show that there is a normal subgroup G' s.t.

$$G' = Nil^3 \rtimes \left\{ \begin{pmatrix} e^s & & \\ & e^{-s} & \\ & & 1 \end{pmatrix} ; s \in R \right\}.$$

By (6.1.4), G' is Sol^4-i- No. 8, G' has a uniform lattice Γ, then $\Gamma \backslash G' \simeq \Gamma \backslash G/H$, so the geometry is bounded.

(6.2.4.2) No. 4. Since H is compact, the geometry is bounded iff G has a uniform lattice. But G is not unimodular, so it has no unifrom lattice. The geometry is not bounded.

(6.2.4.3) No. 2. It's easy to show that $\mathcal{G}$ can be identified with a subalgebra of $affine(E)$, $E = R^4$, i.e.

$$\mathcal{G} \simeq \left\{ \begin{pmatrix} d & c & & & a \\ & d+f & & & b \\ & & -f & & c \\ & & & 0 & d \\ & & & & 0 \end{pmatrix} ; a, b, c, d, f \in R \right\},$$

and

$$G \simeq \left\{ \begin{pmatrix} e^d & \frac{c}{f}(e^{d+f} - e^d) & & & a \\ & e^{d+f} & & & b \\ & & e^{-f} & & -\frac{c}{f}(e^{-f} - 1) \\ & & & 1 & d \\ & & & & 1 \end{pmatrix} ; a, b, c, d, f \in R \right\},$$

where if $f = 0$, replace $\frac{c}{f}(e^{d+f} - e^d)$, resp. $-\frac{c}{f}(e^{-f} - 1)$ by ce^d, resp. c. The action of G is clearly transitive on $R^4 = E$. The isotropy subgroup at $(0, 0, 0, 0)$ is

$$H = \left\{ \begin{pmatrix} 1 & & & & \\ & e^f & & & \\ & & e^{-f} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} ; f \in R \right\}.$$

So (G, H) is the corrosponding geometry of $(\mathcal{G}, \mathcal{H})$.$G/H \simeq E = R^4$. Now we can use the fact that if (G, H) is bounded then there is $\Gamma \subseteq G$, s.t. Γ acts on E properly continuously and freely with compact quotient. By (6.2.1) Γ must be contained in

$$G' = \left\{ \begin{pmatrix} 1 & \frac{c}{f}(e^f - 1) & & & a \\ & e^f & & & b \\ & & e^{-f} & & -\frac{c}{f}(e^{-f} - 1) \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} ; a, b, c, f \in R \right\}.$$

So $\Gamma \backslash E \simeq (\Gamma \backslash R^3) \times R$ which is not compact. The contradiction means that (G, H) is not bounded.

(6.2.4.4) No. 3. It's easy to show that if $N = Re_1 + Re_2 + Re_3$ s.t. $[e_1, e_2] = e_3 \in Z(N_3), t \neq 0$ is a fixed real number, then

$$(N_3 \rtimes \left\{ R\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} + R\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \right\}, R\begin{pmatrix} 1 & -t & \\ t & 1 & \\ & & 2 \end{pmatrix}) \sim$$

$$(\left\{ \begin{pmatrix} 2d & c & 0 & -b & a \\ 0 & d & 0 & -ft & b \\ 0 & 0 & 0 & 0 & d-f \\ 0 & ft & 0 & d & c \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; a, b, c, d, f \in R \right\}, R\begin{pmatrix} 2 & & & & 0 \\ & 1 & & -t & \\ & & 0 & & \\ & t & & 1 & \\ 0 & & & & 0 \end{pmatrix}).$$

We know that

(i) $e^d cos(tf) = \sum_{k=0}^{\infty} \sum_{l+2s=k} (-1)^s \frac{d^l}{l!} \cdot \frac{(ft)^{2s}}{(2s)!}$,

(ii) $e^d sin(tf) = \sum_{k=0}^{\infty} \sum_{l+2s-1=k} (-1)^{s+1} \frac{d^l}{l!} \cdot \frac{(ft)^{2s-1}}{(2s-1)!}$.

Claim.

(i) $F_1(d) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l+2s=k-1} (-1)^s \frac{d^l}{l!} \cdot \frac{(dt)^{2s}}{(2s)!}$

$$= \begin{cases} 1, & \text{if d} = 0, \\ \frac{d^{-1}}{1+t^2}\{e^d(cos(dt)) + tsin(dt)) - 1\}, & \text{otherwise;} \end{cases}$$

(ii) $F_2(d) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l+2s-1=k-1} (-1)^s \frac{d^l}{l!} \cdot \frac{(dt)^{2s-1}}{(2s-1)!}$

$$= \begin{cases} 1, & \text{if d} = 0, \\ \frac{d^{-1}}{1+t^2}\{e^d(-tcos(dt) + sin(dt)) + t\}, & \text{otherwise.} \end{cases}$$

Proof of Claim.

Since

$$dF_1(d) = \sum_{k=1}^{\infty} \frac{d^k}{k} \sum_{l+2s=k-1} (-1)^s \frac{1}{l!} \cdot \frac{t^{2s}}{(2s)!},$$

so

$$(dF_1(d))' = \sum_{k=1}^{\infty} d^{k-1} \sum_{l+2s=k-1} (-1)^s \frac{1}{l!} \cdot \frac{t^{(2s)}}{(2s)!}$$
$$= \sum_{k=1}^{\infty} \sum_{l+2s=k-1} (-1)^s \frac{d^l}{l!} \cdot \frac{(dt)^{2s}}{(2s)!}$$
$$= e^d cos(dt).$$

Similarly we have the derivative of $dF_2(d)$:

$$(dF_2(d)' = e^d sin(dt).$$

Solving

$$\begin{cases} (dF_1(d))' = e^d cos(dt) \\ (dF_2(d))' = e^d sin(dt) \end{cases},$$

we have

$$\begin{cases} dF_1(d) = \frac{1}{1+t^2}\{e^d(cos(dt) + tsin(dt))\} + C_1, C_1 = -\frac{1}{1+t^2}, \\ dF_2(d) = \frac{1}{1+t^2}\{e^d(-tcos(dt) + sin(dt))\} + C_2, C_2 = \frac{t}{1+t^2}. \end{cases}$$

Then using the formula (5.1.3) we have

$$G = \left\{ \begin{pmatrix} e^{2d} & A(b,c,d,ft) & 0 & B(b,c,d,ft) & a \\ 0 & e^d cos(ft) & 0 & -e^d sin(ft) & C(b,c,d,ft) \\ 0 & 0 & 1 & 0 & d-f \\ 0 & e^d sin(ft) & 0 & e^d cos(ft) & D(b,c,d,ft) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; a,b,c,d,f \in R \right\},$$

where

$$C(b,c,d,ft) = bX(d,ft) - cY(d,ft),$$

$$D(b,c,d,ft) = bY(d,ft) + cX(d,ft),$$

$$X(d,ft) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l+2s=k-1} (-1)^s \frac{d^l}{l!} \cdot \frac{(ft)^{2S}}{(2s)!},$$

$$Y(d,ft) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l+2s-1=k-1} (-1)^{s+1} \frac{d^l}{l!} \cdot \frac{(ft)^{2s-1}}{(2s-1)!}.$$

Since if $f \equiv 0$, G contains a subgroup

$$G' = \left\{ \begin{pmatrix} e^{2d} & A(b,c,d,0) & 0 & B(b,c,d,0) & a \\ & e^d & & & \frac{b}{d}(e^d-1) \\ & & 1 & & d \\ & & & e^d & \frac{c}{d}(e^d-1) \\ & & & & 1 \end{pmatrix} ; a,b,c,d \in R \right\}$$

(if $d = 0$, replace $\frac{b}{d}(e^d - 1)$ by b, and replace $\frac{c}{d}(e^d - 1)$ by c), the action of G on $E = R^4$ is transitive. Now let us compute the isotropy subgroup H of G at (0,0,0,0). Since all the translation parts must be trivial, we must have

$$\begin{cases} a = 0 & (1) \\ C(b,c,d,ft) = 0 & (2) \\ d - f = 0 & (3) \\ D(b,c,d,ft) = 0 & (4) \end{cases}$$

From (3) we get $d = f$, so

$$\begin{cases} C(b,c,d,dt) = 0 \\ D(b,c,d,dt) = 0 \end{cases}$$

i.e.

$$\begin{cases} bX(d,dt) - cY(d,dt) = 0 \\ bY(d,dt) + cX(d,dt) = 0 \end{cases} \qquad (5)$$

The system of equations (5) will have nontrivial solution iff

$$\begin{vmatrix} X(d,dt) & -Y(d,dt) \\ Y(d,dt) & X(d,dt) \end{vmatrix} = 0$$

i.e.

$$X^2(d,dt) + Y^2(d,dt) = 0,$$

so

$$X(d,dt) = Y(d,dt) = 0.$$

Since $X(d,dt) = F_1(d), Y(d,dt) = F_2(d)$, we must have $d \neq 0$ and

$$\begin{cases} \frac{d^{-1}}{1+t^2}\{e^d(cos(dt) + tsin(dt)) - 1\} = 0 \\ \frac{d^{-1}}{1+t^2}\{e^d(-tcos(dt) + sin(dt)) - 1\} = 0 \end{cases} \qquad (6)$$

If we assume that $sin\theta = \frac{1}{1+t^2}, cos\theta = \frac{t}{1+t^2}, 0 < \theta < \pi$, then

$$\begin{cases} e^d\{sin\theta cos(dt) + cos\theta sin(dt)\} = sin\theta \\ e^d\{-cos\theta cos(dt) + sin\theta sin(dt)\} = cos\theta \end{cases} \qquad (7)$$

$$\text{or} \begin{cases} e^d sin(\theta + dt) = sin\theta \\ -e^d cos(\theta + dt) = -cos\theta \end{cases} \qquad (8)$$

Since $cos\theta \neq 0$, we have from (8)

$$tan(\theta + dt) = tan\theta$$

So $\theta + dt = \theta + k\pi$, k is an integer. and $dt = k\pi$ for some integer k. Then (8) becomes

$$\begin{cases} e^{k\pi} sin\theta = (-1)^k sin\theta \\ e^k cos\theta = (-1)^k cos\theta \end{cases}$$

So $e^{k\pi} = \pm 1$ for some integer k. Since $d \neq 0$, we have a contradiction. So the isotropy subgroup is

$$H = \left\{ \begin{pmatrix} e^{2d} & & & & \\ & e^{d}cos(dt) & & -e^{d}sin(dt) & \\ & & 1 & & \\ & e^{d}sin(dt) & & e^{d}cos(dt) & \\ & & & & 1 \end{pmatrix} ; d \in R \right\}$$

So (G, H) is the corresponding geometry of $(\mathcal{G}, \mathcal{H})$. If this geometry is bounded, there is a discrete $\Gamma \subseteq G$ with compact quotien $\Gamma \backslash E$ and Γ must be contained in

$$G'' = \left\{ \begin{pmatrix} 1 & A(b,c,0,ft) & 0 & B(b,c,0,ft) & a \\ 0 & cos(ft) & 0 & -sin(ft) & C(b,c,0,ft) \\ 0 & 0 & 1 & 0 & -f \\ 0 & sin(ft) & 0 & cos(ft) & D(b,c,0,ft) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; a, b, c, f \in R \right\},$$

where

$A(b,c,0,ft) = -b\frac{1-cos(ft)}{ft} + c\frac{sin(ft)}{ft}$,

$B(b,c,0,ft) = -b\frac{sin(ft)}{ft} - c\frac{1-cos(ft)}{ft}$,

$C(b,c,0,ft) = b\frac{sin(ft)}{ft} - c\frac{1-cos(ft)}{ft}$,

$D(b,c,0,ft) = b\frac{1-cos(ft)}{ft} + c\frac{sin(ft)}{ft}$,

if $f \neq 0$, and

$A(b,c,0,0) = c$,

$B(b,c,0,0) = -b$,

$C(b,c,0,0) = b$,

$D(b,c,0,0) = c$.

It's easy to show that $\Gamma \backslash G'' \simeq \Gamma \backslash E$, but

$$L(G'') \simeq N^3 \rtimes R \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (see Type } Sol^4\text{-(i)-(9)),}$$

and G'' has no uniform Lattices. So the geometry is not bounded.

(6.3) Other bounded mixed geometries.

We want to use Proposition (6.2.1) (Goldman and Hirsh) without assuming that G is solvable. Then we must prove that Γ is virtually solvable. We first prove the following Lemma.

(6.3.1) Lemma. **Let $\Gamma \subseteq Affine(R^4)$ s.t. Γ acts on $E = R^4$ properly discontinuously with compact quotient. If $P_l(\Gamma)$, the linear part of Γ, fixes a non zero vector of E, then Γ is virtually solvable.**

Proof. (Fried [8]) Let $v \neq 0$ be the vector fixed by $P_l(\Gamma)$. Let Γ_1 be the normal subgroup of Γ with finite index s.t. Γ_1 acts on E freely. Then $\Gamma_1 \backslash E$ is a complete compact affine space-form. $P_l(\Gamma_1)$ also fixes v. Let $\tilde{Y}$ be the parallel vector field on E determined by v and let Y be corresponding vector field on $\Gamma_1 \backslash E$. The 1-form ω on $\Gamma_1 \backslash E$ dual to Y is parallel and hence closed.

Perturb ω to a closed 1-form ω_1 with rational periods P, where P is the set of real number obtained by integrating ω_1 around closed loops in $\Gamma_1 \backslash E$. As Γ_1 is finitely generated, P is discrete and R/P is a circle. Also $\omega_1(Y)$ never vanishes, assuming ω_1 is close enough to ω.

Let $b \in \Gamma_1 \backslash E$ be a basepoint and define $\theta : \Gamma_1 \backslash E \rightarrow R/P$ as the definite integral $\theta(y) = \int_b^y \omega_1$. Then θ is a fibration of $\Gamma_1 \backslash E$ over the circle (c.f. Tishler [31]). Let K be a connected component of a fiber of θ. Then K is a connected cross-section to the flow ϕ on $\Gamma_1 \backslash E$ generated by Y.

Since Y is parallel and $\Gamma_1 \backslash E$ is flat (in the meaning of [23]), the flow ϕ has a transverse affine structure that induces an affine structure on K. Lifting ϕ to the universal cover E one obtains the one parameter group $\tilde{\phi}$ of translations of E with velocity v. So K is naturally identified with the orbit space of this flow, namele E/Rv, c.f. Fried [10]. This orbit space is just an affine 3-space so K is complete in its induced affine structure.

A complete compact affine 3-manifold has solvable fundamental group by Fried and Goldman [10]. Thus, $\pi_1 K$ is solvable. K is the fiber of a fibration of $\Gamma_1 \backslash E$ over the circle so the homotopy exact sequence of this fibration shows that Γ_1 is an extention of $\pi_1 S^1 =$ Z. Hence Γ_1 is solvable, i.e. Γ is virtually solvable.

Q.E.D.

(6.3.2) Type Mix^4-(ii). $(\mathcal{G}, \mathcal{H}) = (R^3 \rtimes (RId_3 + \mathcal{H}) , \mathcal{H})$, where $\mathcal{H} \subseteq sl_3(R)$. We notice that $(\mathcal{G}, \mathcal{H}) \sim$

$$(\left\{\begin{pmatrix} A & & & a \\ & & & b \\ & & & c \\ & & 0 & \frac{1}{3}TrA \\ & & & 0 \end{pmatrix}; a,b,c \in R, A \in RId_3 + \mathcal{H}\right\}, \left\{\begin{pmatrix} B & & \\ & 0 & \\ & & 0 \end{pmatrix}; B \in \mathcal{H}\right\})$$

and $(G, H) \sim$

$$(\left\{\begin{pmatrix} A & & & a \\ & & & b \\ & & & c \\ & & 1 & \frac{1}{3}ln(detA) \\ & & & 1 \end{pmatrix}; a,b,c \in R, A \in \mathcal{A}\right\}, \left\{\begin{pmatrix} B & & \\ & 1 & \\ & & 1 \end{pmatrix}; B \in \mathcal{B}\right\}),$$

where $\mathcal{A}$, resp. $\mathcal{B}$, is the connected subgroup of $GL_3^+(R)$ with Lie algebra $RId_3 + \mathcal{H}$,resp. $\mathcal{H}$. It's clearly $P_l(G)$ fixes $v = e_4$, if we write $R^4 = \sum_{i=1}^4 Re_i$. G acts transitively on R^4 with an isotropy subgroup H at (0,0,0,0). $G/H \simeq R^4 = E$. If there is $\Gamma \subseteq G$, s.t. $\Gamma\backslash E \simeq \Gamma\backslash G/H$ is compact , $P_l(\Gamma)$ fixes v, so by (6.3.1), Γ is virtually solvable , then by the Proposition of Goldman and Hirsh, $P_l(\Gamma) \subseteq SL(E)$. In our case, we must have

$$\Gamma \subseteq \left\{\begin{pmatrix} B & & & a \\ & & & b \\ & & & c \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}; a,b,c \in R\right\}.$$

So $\Gamma\backslash E \simeq (\Gamma\backslash R^3) \times R$ which is not compact. So in Type Mix^4-(ii), No. 1, No.2 and No. 3 are not bounded.

(6.3.2.1) Type Mix^4-(ii) - No. 4. Using the same method as we used in No. 1 - No. 3, we can show that Γ is virtually solvable. Here $(G, H) \sim$

$$(\left\{\begin{pmatrix} A \times \begin{pmatrix} e^f & 0 \\ 0 & e^f \end{pmatrix} & & & a \\ & & & b \\ & e^{d+tf} & & c \\ & & 1 & d \\ & & & 1 \end{pmatrix}; a,b,c,d,f \in R, A \in SL_2(R)\right\},$$

$$\left\{\begin{pmatrix} A \times \begin{pmatrix} e^f & 0 \\ 0 & e^f \end{pmatrix} & & & \\ & e^{tf} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; f \in R, A \in SL_2(R)\right\}).$$

If $t = -2$, then Γ must be contained in

$$G'' = \left\{ \begin{pmatrix} A \times \begin{pmatrix} e^f & 0 \\ 0 & e^f \end{pmatrix} & & & a \\ & & & b \\ & e^{-2f} & & c \\ & & 1 & c \\ & & & 1 \end{pmatrix} ; a, b, c, f \in R, A \in SL_2(R) \right\}.$$

So $\Gamma \backslash R^4 \simeq (\Gamma \backslash R^3) \times R$ which is not compact.

If $t \neq -2$, there is a subgroup

$$G'' = \left\{ \begin{pmatrix} e^f & & & & a \\ & e^f & & & b \\ & & e^{-2f} & & c \\ & & & 1 & (-2-t)f \\ & & & & 1 \end{pmatrix} ; a, b, c, f \in R \right\}$$

which has a uniform lattice Γ, s.t. $\Gamma \backslash G'' \simeq \Gamma \backslash G/H$.

So the geometry is bounded iff $t \neq -2$.

(6.3.2.2) Type Mix^4-(ii)- No. 5. By the same method as in (6.3.2.1), Γ must be virtually solvable, then Γ must be contained in $SL_5(R)$. Then it's easy to show that the geometry is bounded iff $t = -2$.

(6.3.3) Type Mix^4-(v). Since we can always find a normal subgroup G' of G s.t. G' has a uniform lattice Γ and $\Gamma \backslash G' \simeq \Gamma \backslash G/H$, all three geometries are bounded.

In No. 1: $G' = \widetilde{SL}_2(R) \times \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; b \in R\}$,

In No. 2: $G' = \widetilde{SL}_2(R) \times \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; b \in R\}$,

In No. 3: $G' = SU(2) \times \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; b \in R\}$.

(6.4) Other bounded reductive geometries.

(6.4.1) Pseudo-Riemannion Space Forms.

Let V be a real vector space of dimension n+1 equipped with a nonsingular quadratic form Q of type $(p+1, q)$; i.e., in appropriate coordinates

$Q(x_1, ..., x_{p+1}, y_1, ..., y_q) = \sum_{i=1}^{p+1} x_i^2 - \sum_{j=1}^{q} y_j^2, n = p + q, p \geq 0.$

Consider the quadric $\{v \in V; Q(v) = 1\}$. The quadric has two components if $p = 0$, otherwise it has one component. Let $S^{p,q}$ denote the component containing $(1, 0, ..., 0)$ and

$SO_o(p+1, q)$ the orietation preserving subgroup of the full group of Q-othogonal transformations which preserve $S^{p,q}$. Then $S^{p,q} \simeq SO_o(p+1,q)/SO_o(p,q)$, where $SO_o(p,q)$ is the isotropy subgroup of $SO_o(p+1,q)$ at $(1, 0, ..., 0)$. $S^{p,q} \simeq S^p \times R^q$ if $p > 0$, and $S^{0,q} \simeq R^q$ via the map

$(x,y) \mapsto (x/|x|, y)$,

where $x = (x_1, ..., x_{p+1}), y = (y_1, ..., y_q)$, so $\pi_1(S^{p,q}) = 0$ or $\mathbf{Z}$ accordingly as $p \neq 1$ or $p = 1$.

R. Kulkarni proved (c.f. [20]) the following Theorem.

(6.4.1.1) Theorem. If $p \geq q$, then only finite subgroups of $O(p+1,q))$ can act properly discontinuous on $S^{p,q}$.

(6.4.1.2) Corollary. Type Red^3-(iii) No. 3, Red^4-(i) No. 3, No. 5 are not bounded.

(6.4.1.3) By a further argument in [20], Type Red^4-(i) No.4 is not bounded.

(6.4.2) Type Red^4-(ii) No.4. We use those notations in (5.3.12). Let $\Gamma \subseteq G$ be a discrete subgroup.

Claim 1. If Γ acts on M properly, then Γ acts on E properly.

Proof. Let $K \subseteq E$ be a compact set. Then since

$S^1 \to E \xrightarrow{P} M$

is a principle fiber boundle, $P(K) \subseteq M$ is compact. Since Γ acts on M properly, $\{\gamma \in \Gamma; P(K) \cap \gamma P(K) \neq \emptyset\}$ is finite. The action of G on M is the reduced action of G on E, i.e., the following diagram is commutative:

$$\begin{tikzcd}
E \arrow[r, "P"] \arrow[d, "g"'] & M \arrow[d, "g"] \\
E \arrow[r, "P"] & M
\end{tikzcd}$$

So $P \circ \gamma = \gamma \circ P$. Since $P(K \cap \gamma K) \subseteq P(K) \cap P(\gamma K)$, so $K \cap \gamma K \neq \emptyset$ implies $P(K) \cap P(\gamma K) = P(K) \cap \gamma P(K) \neq \emptyset$. So $\{\gamma \in \Gamma; K \cap \gamma K \neq \emptyset\} \subseteq \{\gamma \in \Gamma; P(K) \cap \gamma P(K) \neq \emptyset\}$, i.e. $\{\gamma \in \Gamma : K \cap \gamma k \neq \emptyset\}$ is finite.

Q.E.D.

Claim 2. Γ is finite, if Γ acts on M properly.

Proof. By Claim 1, Γ acts on E properly. Consider the set

$\{z \in \mathrm{E}; z_3 = 0\} = \{|z_1|^2 + |z_2|^2 = 1\} \simeq S^3$.

It's easy to show that if $g \in SU(2,1)$, then $gS^3 \cap S^3 \neq \emptyset$. So $\Gamma = \{\gamma \in \Gamma; \gamma S^3 \cap S^3 \neq \emptyset\}$ which is finite.

Q.E.D.

So the geometry is not *Bdd*.

(6.4.3) Type *Red*^{3}-ii-1.

We identify G/H with $\widetilde{SL}_2(R)$ by the mapping i:

$i : (e^t, y)H \mapsto y exp(-t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$.

Let Γ be a uniform lattice of $\widetilde{SL}_2(R)$, $g \in \Gamma$. Then

$\Gamma \backslash G/H \simeq \Gamma \backslash \widetilde{SL}_2(R)$.

Let $g \in \Gamma$, then the following diagram is commutative:

$$\begin{tikzcd}
G/H \arrow[r, "i"] \arrow[d, "g"'] & \widetilde{SL}_2(R) \arrow[d, "g"] \\
G/H \arrow[r, "i"] & \widetilde{SL}_2(R)
\end{tikzcd}$$

So Γ acts on G/H properly discontinuously and the geometry is *Bdd*.

(6.4.4) Type *Red*^{3}-ii-2.

Since H is compact, we only have to show that G has a uniform lattice. Let

$\Gamma_1 = \{e^t; exp(t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})) \in Z(\widetilde{SL}_2(R))\}$.

Then

$$R_+ \times \widetilde{SL}_2(R)/\Gamma_1 \times Z(\widetilde{SL}_2(R) = S^1 \times PSL_2(R), Z^* \subsetneq \Gamma_1 \times Z(\widetilde{SL}_2(R)).$$

Let

$$\pi : \mathrm{G} \to S^1 \times PSL_2(R)$$

be the covering map, Γ_2 be a uniform lattice of G. Then

$$\Gamma = \pi^{-1}\{(1, \Gamma_2)\}$$

is a uniform lattice of G. So the geometry is *Bdd*.

(6.4.5) *Red^3*-iii-2.

We have three covering maps:

$$\pi_1 : \widetilde{SL_2}(R) \times \widetilde{SL_2}(R) \to \widetilde{SL_2}(R) \times \widetilde{SL_2}(R)/Z^{**},$$

$$\pi_2 : \widetilde{SL_2}(R) \times \widetilde{SL_2}(R) \to PSL_2(R) \times PSL_2(R),$$

$$\pi_3 : \widetilde{SL_2}(R) \times \widetilde{SL_2}(R)/Z^{**} \to PSL_2(R) \times PSL_2(R).$$

And

$$\pi_2 = \pi_3 \circ \pi_1.$$

Let Γ_1 be a uniform lattice of $PSL_2(R)$, e be the identity element of $PSL_2(R)$, $\pi_2^{-1} = \{Z(\widetilde{SL_2}(R)) \times \Gamma_2\}, \pi_3^{-1} = \Gamma$. Then, it's easy to show that

$$\Gamma \simeq \Gamma_2,\ \Gamma \backslash G/H \simeq \Gamma_2\backslash\widetilde{SL_2}(R).$$

Since Γ_2 is a uniform lattice of $\widetilde{SL_2}(R)$, the geometry is *Bdd*.

(6.4.6) Cases remain unsolved. There are four geometries in our classification list, we don't know if they are bounded or not. They are:

$Red^2 - 1:$ $(so(1,2), so(1,1))$

$Red^4 - \text{iii} - 1:$ $\left(R^2 \times sl_2(R), R\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right)$

$Red^4 - \text{iii} - 2:$ $\left(R^2 \times gl_2(R), \left\{\begin{pmatrix} a & \\ & b \end{pmatrix}; a, b\in R\right\}\right)$

$Red^4 - \text{iii} - 3:$ $\left(R^2 \times gl_2(R), \left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}; a, b\in R\right\}\right)$

§7. The classification of flat compact complete space-forms with metric of signature (2.2)

(7.1) In §6 we tried to determine if a given $Ad - c.r.$ geometry (G, H) is bounded, i.e., to determine the existence or nonexistence of such a discrete subgroup $\Gamma \subsetneq G$ that Γ acts on G/H properly discontinuously with compact quotient. If such Γ's exist, we naturally want to find all of them and to classify them up to some conjugacy. So far, only a few special cases in dimension 4 have been solved, cf. [20], [21] . Recently D. Fried [8] has classified those flat compact complete space-forms with metric of signature (1,3) upto finite covers, i.e., he has classified those Γ's that are in $R^4 \rtimes SO(1,3)$ and Γ act on R^4 freely and properly discontinuously with compact quotient. Since the classification is upto finite covers, the condition of free action of Γ is not necessary (see (6.2.1)). So D.Fried's work solved the case $(G, H) = (R^4 \rtimes SO_o(1,3), SO_o(1,3))$. Fried's method can be applied to the case of signature (2,2) which corresponds to the geometry $(G, H) = (R^4 \rtimes SO_o(2,2), SO_o(2,2))$. The basic idea of Fried's method is in the following Theorem.

(7.1.1) Theorem. Suppose X is a flat compact complete space-form with fundamental group $\Gamma \subseteq R^4 \rtimes SO(2,2)$, then there is a uniquely determined subgroup H of $R^4 \rtimes SO(2,2)$ that acts simply transitively on R^4 and $H \cap \Gamma = \pi$ has finite index in Γ.

Since our classification is up to finite covers, we need only to find the corresponding simply transitive subgroups and their uniform lattices.

(7.1.2) In (7.2) we classify those subgroups of $R^4 \rtimes$ SO(2,2) that act on R^4 simply transitively, up to the conjugacy of $R^4 \rtimes O(2,2)$. Every such a subgroup, as a Lie group, is isomorphic to one of the following:

$$R^4,\ R \times Nil^3,\ Nil^4,\ R \times \{R^2 \rtimes \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}; t \in \mathrm{R}\}, \mathrm{R} \times \{R^2 \rtimes \mathrm{SO}(2)\}$$

All of them, except the last one, correspond to Γ's. Their uniform lattices are known, cf. [8] and [23].

(7.1.3.) To prove Theorem (7.1.2), we first prove in (7.3) that Γ is virtually solvable. This result confirms a conjecture by Milnor in a special case. In [23], it is conjectured that the fundamental group of a complete affinely flat manifold is virtually polycyclic. Our result, combined with Fried's result, shows that this conjecture is true for compact pseudo-Riemannian 4-manifolds.

(7.1.4) In (7.4) we complete the proof of Theorem (7.1.1), using the theory of crystallographic hull developed by Fried and Goldman, cf. [10]. In (7.5), we give our classification. By comparing our list with Fried's, we obtain an interesting fact: as differential manifolds, they are the same coset spaces of the form H/Γ, where H is a Lie group isomorphic to R^4, $R \times Nil^3, Nil^4$ or $R \times \{R^2 \rtimes \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}; t \in R\}$ and Γ is a uniform lattice of H. These Lie groups have simply transitive representations as affine motions and when the signature is (2,2) (resp. (3,1)), the images of the representations are in $R^4 \rtimes SO(2,2)$ (resp. $R^4 \rtimes SO(3,1)$).

(7.1.5) Notations and some properties of $SO(2,2)$ and $so(2,2)$. Throughout §7 we will call $\{e_i\}, 1 \leq i \leq 4$, a standard basis s.t. the metric Q, w.r.t. this basis, has the form

$$Q(v,v) = v_1 v_3 + v_2 v_4,$$

where $v = \sum_{i=1}^{4} v_i e_i$. The full group of orientation-preserving isometries is $R^4 \rtimes SO(2,2)$ and

$$SO(2,2) = \{g \in SL_4(R); {}^t g \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\},$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The infinitesimal isometries are $R^4 \rtimes so(2,2)$ and

(7.1.5.1) $so(2,2) = \{X \in gl_4(R); {}^t X \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X = 0\} =$

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & b \\ a_{21} & a_{22} & -b & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix}; a_{ij}, b, c \in R \right\}.$$

(7.1.5.1) $so(2,2) = L_1 \oplus L_2$, where $L_i \simeq sl_2(R)$, i = 1,2; and

$$L_1 = \left\{ \begin{pmatrix} a & b & & \\ c & -a & & \\ & & -a & -c \\ & & -b & a \end{pmatrix}; a, b, c \in R \right\}$$

$$L_2 = \left\{ \begin{pmatrix} a' & 0 & 0 & b' \\ 0 & a' & -b' & 0 \\ 0 & c' & -a' & 0 \\ -c' & 0 & 0 & -a' \end{pmatrix} ; a', b', c' \in R \right\}$$

(7.1.5.3) It is easy to show that any Cartan subalgebra of $so(2,2)$ is conjugate under $O(2,2)$ to one of the following:

$$(1) \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -b \end{pmatrix} ; a, b \in R \right\},$$

$$(2) \left\{ \begin{pmatrix} 0 & a & 0 & b \\ -a & 0 & -b & 0 \\ 0 & b & 0 & a \\ -b & 0 & -a & 0 \end{pmatrix} ; a, b \in R \right\}$$

$$(3) \left\{ \begin{pmatrix} a & b & & \\ -b & a & & \\ & & -a & b \\ & & -b & -a \end{pmatrix} ; a, b \in R \right\}$$

An immediate corollary is

(7.1.5.4) If X is in a Cartan subalgebra of $so(2,2)$ and $detX = 0$, then X must conjugate under $O(2,2)$ to

$$(4) \left\{ \begin{pmatrix} a & & & \\ & 0 & & \\ & & -a & \\ & & & 0 \end{pmatrix} \right\}$$

or

$$(5) \left\{ \begin{pmatrix} 0 & a & 0 & a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ -a & 0 & -a & 0 \end{pmatrix} \right\}.$$

(7.1.6) We identify $Aff(n)$, resp. $aff(n)$, with

$\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} ; A \in GL_4(R), v \in R^4 \}$,

resp.

$\{ \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} ; X \in gl_4(R), v \in R^4 \}$

w.r.t. a given basis. Let P_l be the natural homomorphism taking an affine transformation (or an infinitesimal affine transformation) to its linear part. Let $L(G)$ be the Lie algebra of a Lie group G and $A(G)$ be the algebraic hull of G. We will need the following well known Lemma.

(7.1.6.1) Lemma. If $G \subseteq Aff(n)$ s.t. G acts freely on R^n, then every $A \in P_l(G)$ has 1 as an eigenvalue.

(7.1.6.2) Lemma. (Kostant and Sullivan, cf. [18]). If G is as in (7.1.6.1), then every $A \in P_l(A(G))$ has 1 as an eigenvalue.

(7.1.6.3) Corollary. If G is as In (7.1.6.1), then every $X \in P_l(L(A(G)))$ or $X \in L(A(P_l(G)))$ has 0 as an eigenvalue.

(7.2) Simply transitive subgroups.

We will classify subgroups of $R^4 \rtimes SO(2,2)$ that act simply transitively on R^4. Our classification is up to the conjugation under $R^4 \rtimes O(2,2)$. It is well known that a simply transitive group of affine motions must be solvable, connected, simply connected and of dimension 4, cf. [1]. We will start from a special case when the groups are unipotent. The following Lemma from Auslander and Scheuneman plays the key role in this section.

(7.2.1) Lemma. Let U be a nilpotent Lie group which has a faithful repre- sentation $\rho : U \to Aff(n)$, let ρ_* be the induced monomorphism of Lie algebras

$$\rho_* : L(U) \to \{\begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix}; X \in gl_n(R), v \in R^n\} = aff(n),$$

and let P_l be as in (7.1.6), let P_t be the projection from an element in $aff(n)$ to its translation part. Then $\rho(U)$ acts on R^n simply transitively if and only if

(1) $P_l \circ \rho_*(L(U))$ is nilpotent , and

(2) $P_t \circ \rho_*(L(U))$ is a linear isomorphism of $L(U)$ onto R^n.

For a proof, cf. [1]. So unipotent simply transitive subgroups are exactly the following U's s.t.

(7.2.2) $L(U) = \{\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix}; v \in R^n\}$,

where $X(v)$ is a linear function of v and $P_l(L(U)) = \{X(v); v \in R^n\}$ is nilpotent.

Using Engel's theorem, we can show

(7.2.3) Lemma. There is a vector $v_o \in R^4$ such that

(i) $P_l(L(U))(v_o) = 0$,

(ii) $Q(v_o, v_o) = 0$.

Let $\{e_i\}$ be our standard basis. Then we can choose $v_o = e_1$ since 0(2,2) is transitive

on

$\{v; Q(v,v) = 0\}/v \sim tv$,

where $t \in R - \{0\}$.

(7.2.4) Corollary. W.r.t. the above standard basis, $X \in P_l(L(U))$ has the form

$$X(v) = \left\{ \begin{pmatrix} 0 & a & 0 & b \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \end{pmatrix} \right\},$$

where $a = a(v)$ and $b = b(v)$ are linear functions of v.

To find $a(v)$ and $b(v)$, we compute the commutator of $L(U)$.

(7.2.5) $[\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix}] = \begin{pmatrix} X(v'') & v'' \\ 0 & 0 \end{pmatrix}$,

where $v'' = X(v)v' - X(v')v, X(v'') = X(v)X(v') - X(v')X(v) = 0$. So

$a(v'') = b(v'') \equiv 0$.

Write

(7.2.6) $a(v) = \sum_{i=1}^4 a_i v_i, b(v) = \sum_{i=1}^4 b_i v_i$.

Then we have

(7.2.7) $0 = \sum_{i=1}^4 a_i v_i'', 0 = \sum_{i=1}^4 b_i v_i''$,

where v_i'''s are linear functions of a_i, b_i and $v_i v_j'$, $1 \leq i, j \leq 4$, and all coefficients of $v_i v_j'$ must be zero. We obtain

(7.2.8) Lemma.

(i) $a_1 = b_1 = 0$,

(ii) $a_2 b_4 + a_4^2 = 0$,

(iii) $a_2 b_2 + a_4 a_2 = 0$,

(iv) $b_2 b_4 + b_4 a_4 = 0$,

(v) $b_2^2 + b_4 a_2 = 0$.

(7.2.9) Corollary.

(i) $b_4(b_2 + a_4) = 0$,

(ii) $a_2(b_2 + a_4) = 0$,

(iii) $(b_2 - a_4)(b_2 + a_4) = 0$.

(7.2.10) Now we can get some necessary conditions for the nonabelion unipotent simply transitive subgroups. If $b_2 + a_4 \neq 0$, then $b_4 = a_2 = 0$. By (7.2.8) (i) and (v), $b_2^2 = a_4^2 = 0$ and we get a contradiction. So $b_2 + a_4 = 0$, and we have three subcases:

(7.2.10.1) $b_2 = a_4 = b_4 = a_2 = 0$, but $(a_3, b_3) \neq (0,0)$, i.e.,

$$\begin{cases} a(v) = a_3 v_3 \\ b(v) = b_3 v_3. \end{cases}$$

(7.2.10.2) $b_2 + a_4 = 0$ but $b_2 \neq 0, a_4 \neq 0$. Then by (7.2.8) $b_4 \neq 0, a_2 \neq 0$, i.e.

$$\begin{cases} a(v) = a_2 v_2 + a_3 v_3 + a_4 v_4 \\ b(v) = b_2 v_2 + b_3 v_3 + b_4 v_4. \end{cases}$$

(7.2.10.3) $b_2 = 0, a_4 = 0, (a_2, b_4) \neq (0,0)$. By (7.2.8), $b_4 a_2 = 0$, so

$$\begin{cases} a(v) = a_2 v_2 + a_3 v_3 \\ b(v) = b_3 v_3, \end{cases}$$

or

$$\begin{cases} a(v) = a_3 v_3 \\ b(v) = b_3 v_3 + b_4 v_4. \end{cases}$$

(7.2.11) Theorem. Up to conjugacy under $R^4 \rtimes O(2,2)$, the nonabelian unipotent simply transitive subgroups U of $R^4 \rtimes SO(2,2)$ have the following Lie algebras:

$$L(U) = \{\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix}; v \in R^4\},$$

where

$$X(v) = \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix},$$

$a(v)$ and $b(v)$ are listed in the following table:

Type of $L(U)$	$a(v)$	$b(v)$	isomorphim type as an abstract Lie algebra
I-1	v_3	v_3	$N_3 \oplus R$
I-2	v_3	$-v_3$	$N_3 \oplus R$
I-3	v_3	0	$N_3 \oplus R$
II-1	$v_2+v_4+tv_3$, $(t \geq 0)$	$-v_2-v_4$	N_4
II-2	$-v_2+v_4+tv_3$, $(t \geq 0)$	$-v_2+v_3$	N_4
II-3	v_2	v_3	N_4

The equivalence classes are uniquely determined by the type of $L(U)$ and the parameter t (in Type II).

Proof: The discussion of the conjugacy under $R^4 \rtimes O(2,2)$ is long and tedious. We will only write down a brief one for subcase (7.2.10.2). We give the following lemma without proof.

(7.2.11.1) Lemma. If $a(v) \neq 0, b(v) \neq 0, a'(v') \neq 0, b'(v') \neq 0$, and if there is a matrix $A = (a_{ij}) \in O(2,2)$ such that

$$A^{-1} \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} A = \begin{pmatrix} 0 & a'(v') & 0 & -b'(v') \\ 0 & 0 & -b'(v') & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a'(v') & 0 \end{pmatrix},$$

then either

$$(1) \begin{cases} a'(v') = \frac{a_2 a_{22}^2}{a_{11}} v_2' + \{\frac{a_{22}a_{23}}{a_{11}} a_2 + \frac{a_{22}}{a_{11}^2} a_3 + \frac{a_{22}a_{43}}{a_{11}} a_4\} v_3' + \frac{a_4}{a_{11}} v_4' \\ b'(v') = \frac{b_2}{a_{11}} v_2' + \{\frac{a_{23}}{a_{11}a_{22}} b_2 + \frac{1}{a_{11}^2 a_{22}} b_3 + \frac{a_{43}}{a_{11}a_{22}} b_4\} v_3' + \frac{b_4}{a_{11}a_{22}^2} v_4' \end{cases}$$

where $a_{11}a_{22} \neq 0$; or

$$(2) \begin{cases} a'(v') = \frac{b_4a_{42}^2}{a_{11}}v'_2 + \{\frac{a_{42}a_{23}}{a_{11}}b_2 + \frac{a_{42}}{a_{11}^2}b_3 + \frac{a_{42}a_{43}}{a_{11}}b_4\}v'_3 + \frac{b_2}{a_{11}}v'_4 \\ b'(v') = \frac{a_4}{a_{11}}v'_2 + \{\frac{a_{23}}{a_{11}a_{42}}a_2 + \frac{1}{a_{11}^2a_{42}}a_3 + \frac{a_{43}}{a_{11}a_{42}}a_4\}v'_3 + \frac{a_4}{a_{11}a_{41}^2}v'_4 \end{cases}$$

where $a_{11}a_{42} \neq 0$¿

Write $a'(v') = \sum_{i=2}^4 a'_iv'_i$ and $b'(v') = \sum_{i=2}^4 b'_iv'_i$, then from (7.2.11.1)

$$a'_2b'_4 = a'_4b'_2 = \frac{a_4b_2}{a_{11}^2} = -\frac{a_4^2}{a_{11}^2} < 0,$$

since $a_4 = -b_2 \neq 0$. So we can choose a_{11} such that $a'_2b'_4 = a'_4b'_2 = -1$, i.e. $\frac{a_4}{a_{11}} = \pm 1$. Next we use (1) (resp. (2)) if $\frac{a_4}{a_{11}} = 1$ (resp. -1), and choose a_{22} (resp. a_{42}) to reduce

$$\begin{pmatrix} a'_2 & a'_4 \\ b'_2 & b'_4 \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \text{ if } a_2a_4 > 0,$$

or

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \text{ if } a_2a_4 < 0.$$

Now a'_3, b'_3 have the form

$$\begin{cases} a'_3 = z_1 + \frac{a_{22}}{a_{11}^2}a_3 \\ b'_3 = \pm z_1 + \frac{1}{a_{11}a_{22}}b_3 \end{cases},$$

$$\text{or } \begin{cases} a'_3 = z_2 + \frac{a_{42}}{a_{11}^2}b_3 \\ b'_3 = \pm z_2 + \frac{1}{a_{11}^2a_{42}}a_3 \end{cases},$$

where z_1 (resp. z_2) depends on a_{23}, a_{43} (resp. a_{23}, a_{43}) and $z_i, i = 1, 2$ can assume any real number. We can choose z_i so that $b'_3 = 0$ and we can choose the sign of a_{22} (resp. a_{42}) so that $a'_3 \geq 0$. So we can find an $A \in O(2,2)$ such that

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

is of Type II-1 or Type II-2. We can replace $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ by $\begin{pmatrix} A & w \\ 0 & 1 \end{pmatrix}$ and show that the translation part doesn't contribute to the classification.

We omit the rest of the proof.

Q.E.D.

(7.2.12) To handle the general case, namely when the simply transitive group of affine motion is non-unipotent solvable, we need the following lemma from Auslander, cf. [1].

(7.2.12.1) Lemma. Let H be an ndimensional, connected, simply connected, solvable Lie group acting simply transitively as affine motions on R^n. Let $A(H)$ be the algebraic hull of H and let U be the unipotent radical of $A(H)$. Then U operates simply transitively as affine motions on R^n.

Now all such non-abelion U's are known from (7.2,11), and we'll study them first.

(7.2.12.2) Lemma. Let H, U be as in (7.2.12.1) and assume that U is not the translation group T. Then $H = U$.

Proof: W.r.t. the standard basis $\{e_i\}$, $1 \leq i \leq 4$, we know

$$L(P_l(U) = \left\{ \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} ; v \in R^4 \right\}.$$

So $L(P_l(U))(e_1^{\perp}) \subseteq e_1^{\perp}$, since $e_1^{\perp} = Re_1 + Re_2 + Re_4$. Notice $A(H)$ is contained in the normalizer of U. Also

$[L(P_l(A(H))), L(P_l(A(H)))] \subseteq L(P_l(U))$, so we have $L(P_l(A(H)))(e_1^{\perp}) \subseteq e_1^{\perp}$, i.e. every $X \in L(P_{\perp}(A(H)))$ must have the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & b \\ a_{21} & a_{22} & -b & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & b \\ 0 & a_{22} & -b & 0 \\ 0 & 0 & -a_{11} & 0 \\ 0 & 0 & -a_{12} & -a_{22} \end{pmatrix}$$

By (7.1.7.3) $det X = 0$, so $a_{11}a_{22} = 0$. By computing the commutator

$$[\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix}] \in L(U),$$

where $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in L(A(H))$, $\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} \in L(U)$, and by using an argument similar to the one we did in (7.2.5)-(7.2.7), we have $a_{11} = a_{22} = 0$, i.e. $A(H)$ is unipotent; so H is unipotent. But any unipotent connected Lie group is Zariski closed, so $H = A(H)$. U, as the unipotent radical of H, must be H itself.

Q.E.D.

(7.2.12.3) Now consider the case when the unipotent radical of $A(H)$ is precisely the group T of translations of R^4. Suppose $H \neq T$, i.e. H is not unipotent.

(7.2.12.3.1) Lemma. $P_l(H)$ is abelian.

Proof: $P_l(H) \simeq H/Ker.(P_l|_H)) = H/(H \cap T) \subseteq A(H)/T$, but $A(H)/T$ is abelian (cf. [2]).

Q.E.D.

(7.2.12.3.2) Lemma. $dim\ P_l(H) = 1; L(P_l(H))$ is diagnolizable in $\mathbf{C}$.

Proof: $P_l(H)$ is a connected abelian subgroup of $SO_o(2,2)$, so

$dim\ P_l(H) \leq 2$. By (7.1.7.3) $det X = 0$ for every $X \in L(P_l(H))$, i.e. 0 is an eigenvalue of X. Since $X \in so(2,2)$, so

$$X = \begin{pmatrix} a_{11} & a_{12} & 0 & b \\ a_{21} & a_{22} & -b & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix},$$

and $det(X - \lambda I) = \lambda^4 + (2bc - 2a_{12}a_{21} - a_{11}^2 - a_{22}^2)\lambda^2 + (-a_{11}a_{22} + a_{12}a_{21} + +bc)^2 = \lambda^4 + \{-4a_{12}a_{21} - (a_{11} - a_{12})^2\}\lambda^2$, since 0 is an eigenvalue. So the eigenvalues of X are $\{0,0,0,0\}$ or $\{0,0,\lambda,-\lambda\}$, $\lambda \neq 0$, $\lambda \in R$ or $\sqrt{-1}R$. If $dim\ P_l(H) = 2$, then by (7.1.5.2) $so(2,2) = L_1 \oplus L_2$, $L_i \simeq sl_2(R)$. So $L(P_l(H)) = RX_1 + RX_2$ where $X_i \in L_i, i = 1,2$. But by (7.1.5.2)

$$det(X_1 - \lambda I) = \lambda^4 + 2(a^2 + bc)\lambda^2 + (a^2 + bc)^2,$$

and

$$det(X_2 - \lambda I) = \lambda^4 + 2(b'c' - a'^2)\lambda^2 + (b'c' - a'^2)^2.$$

So zero is an eigenvalue of $X_i, i = 1,2$, if and only if all the eigenvalues of X_i are zero. This means $P_l(H)$ is unipotent and leads to a contradiction. So $dim\ P_l(H) = 1, L(P_l(H) = RX$ and X has eigenvalues $\{0,0,\lambda,-\lambda\}$, $\lambda \neq 0$, $\lambda \in R$ or $\sqrt{-1}R$. Since X is an infinitesimal isometry, it is diagnolizable.

Q.E.D.

(7.2.12.3.3) Corollary. $L(P_l(H))$ is contained in a Carton subalgrbra of $so(2,2)$ and is conjugate under $O(2,2)$ to

$$(1) \quad \begin{pmatrix} a & & & 0 \\ & 0 & & \\ & & -a & \\ 0 & & & 0 \end{pmatrix}$$

$$(2) \quad \begin{pmatrix} 0 & a & 0 & a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ -a & 0 & -a & 0 \end{pmatrix}.$$

Proof: By (7.1.5.4).

Q.E.D.

Since H is simply transitive, the map $P_t : L(H) \to R^4$ is a linear isomorphism, so in (7.2.12.3.3) we have $a = \sum_{i=1}^{4} a_i v_i$, where

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

is the corresponding translation part. Since T is the unipotent radical of $A(H)$, we have $[L(H), L(H)] \subseteq L(T) = R^4$. By computing the commutator and using the fact that H is simply transitive, we must have $a(v) = a_2 v_2 + a_4 v_4, (a_2, a_4) \neq (0,0)$ in Case (1) and $a(v) = a_1(v_1 - v_3) + a_2(v_2 - v_4), (a_1, a_2) \neq (0,0)$ in Case (2). Finally, by considering the conjugation under $R^4 \rtimes O(2,2)$, we get

(7.2.12.4) Theorem. If $H \subseteq R^4 \rtimes SO(2,2)$ acts simply transitively on R^4 and H is not unipotent, then H is conjugate under $R^4 \rtimes O(2,2)$ to one of the following:

$$\text{i) Type III-1:} \quad \begin{pmatrix} a(v) & & & 0 \\ & 0 & & \\ & & -a(v) & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} \\ v \\ \\ \end{pmatrix},$$

where $a(v) = t(v_1 - v_3), t > 0$ and $L(H) \simeq R \oplus \{R^2 \rtimes R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$.

ii) Type III-2: $\begin{pmatrix} 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \\ 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \end{pmatrix} \begin{pmatrix} v \end{pmatrix}$,

where $a(v) = t(v_1 - v_3), t > 0$ and $L(H) = R \oplus \{R^2 \rtimes R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$.

The type and the parameter t determine the equivalence classes uniquely.

(7.2.13) Combining (7.2.11) with (7.2.12.4) and denoting $H = T_4$ as Type 0, we complete the classification of simply transitive subgroups of $R^4 \rtimes SO(2,2)$. We summarize our result in the following table. We denote

$$A(a,b,v) = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix} ; v \in R^4 \right\},$$

$$B(a,v) = \left\{ \begin{pmatrix} \begin{pmatrix} a(v) & & & 0 \\ & 0 & & \\ & & -a(v) & \\ 0 & & & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix} ; v \in R^4 \right\},$$

$$C(a,v) = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \\ 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix} ; v \in R^4 \right\}.$$

Table of equivalence classes of simly trasitive subgroups of $R^4 \rtimes SO(2,2)$ (given in the form of subalgebras of $aff(n)$ w.r.t. a standard basis)

type of $L(H)$	affine form of L(H)	isomorphism type as abstract Lie algebra
0	$\{\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}; v \in R^4\}$	R^4
I-1	$A(a,b,v), \begin{cases} a(v) = v_3 \\ b(v) = v_3 \end{cases}$	$R \oplus N_3$
I-2	$A(a,b,v), \begin{cases} a(v) = v_3 \\ b(v) = -v_3 \end{cases}$	$R \oplus N_3$
I-3	$A(a,b,v), \begin{cases} a(v) = v_3 \\ b(v) = 0 \end{cases}$	$R \oplus N_3$
II-1	$A(a,b,v), \begin{cases} a(v) = v_2 + v_4 + tv_3 \\ b(v) = -v_2 - v_4, t \geq 0 \end{cases}$	N_4
II-2	$A(a,b,v), \begin{cases} a(v) = -v_2 + v_4 + tv_3 \\ b(v) = -v_2 + v_4, t \geq 0 \end{cases}$	N_4
II-3	$A(a,b,v), \begin{cases} a(v) = v_2 \\ b(v) = v_3 \end{cases}$	N_4
III-1	$B(a,v), a(v) = tv_2 + v_4, t \in R$	$R \oplus \{R^2 \rtimes R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$
III-2	$C(a,v), a(v) = t(v_1 - v_3), t > 1$	$R \oplus \{R^2 \rtimes R\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$

The type of $L(H)$ and the parameter t determine the equivalence classes uniquely.

(7.3) Γ is virtually solvable.

A group with a solvable subgroup of finite index is called virtually solvable.

(7.3.1) Theorem. If $\Gamma \subseteq R^4 \rtimes SO(2,2)$ and Γ acts freely and properly discontinuously on R^4 with compact quotient, then Γ is virtually solvable.

Proof: Let $\pi = P_l(\Gamma)$ and $A(\pi)$ be the algebraic hull of Γ. The identity component A_o is of finite index in $A(\pi)$. We will show A_o is solvable. The following lemma is due to D. Fried.

(7.3.2) Lemma. If A_o fixes a vector $v \in R^4$ of nonzero length, then A_o is solvable. For a proof, cf. [8].

Assume that A_o is not solvable. As in (7.1.6.2), for every $g \in A(\pi)$, $det\ (g - I) = 0$. This shows $dim\ A_o < dim\ SO(2,2)$. So A_o contains a semisimple connected subgroup

S such that $dim\ S = 3$ and $L(S) \simeq sl_2(R)$. By (7.1.5.2) $so(2,2) = L_1 \oplus L_2$, $L_i = sl_2(R)$, i=1,2. Let $P_i : L(S) \to L_i$ be the projection map: $X = X_1 + X_2 \mapsto X_i$, where $X \in L(S), X_i \in L_i, i = 1,2$. Then we have two cases: $L(S) = L_i$ for some i or $P_i(L(S)) = L_i$, i=1,2. But $det(e^{-}I) = 0$ for $X = diag.(a, \mp a, \mp a, a), a \neq 0$. So $L(S) = L_i, i = 1,2$. So we must have $P_i(L(S)) = L_i, i$=1,2, and this means that $L(S)$ is a maximal subalgebra of $so(2,2)$, so $A_o = S$.

(7.3.3) Claim: There is a nonzero vector $v \in R^4$ such that

i) $Q(v, v) = 0$;

ii) $A_o(v) = v$.

To prove the claim, let $0 \neq X \in L(A_o)$ such that RX is a split Cartan subalgebra of $L(A_o)$. Then $h = P_1(RX) \oplus P_2(RX)$ is a split Cartan subalgebra of $so(2,2)$. By (7.1.5.3) h is conjugate under $O(2,2)$ to $\{diag.(a, b, -a, -b); a, b \in R\}$, so we may assume that $X = diag.(a, b, -a, -b)$. Since $det(e^X - I) = 0$, we have $ab = 0$ and $X = diag.(a, 0, -a, 0)$ or $diag.(0, a, 0, -a)$. Let$\{X, Y, Z\}$ be the basis of $L(A)$ such that $[X,Y] = 2Y, [X, Z] = -2Z, [Y, Z] = X$ and $X = diag.(a, 0, -a, 0)$ for some $0 \neq a \in R$. Then adX has three real eigenvalues on $so(2,2)$: $\{2, 0, -2\}$. Let E_λ be the corresponding eigenspaces, then

$$E_2 = \left\{ \begin{pmatrix} 0 & c & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{pmatrix} ; c, e \in R \right\},$$

$$E_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & f & 0 & -d \\ -f & 0 & 0 & 0 \end{pmatrix} ; d, f \in R \right\},$$

and $$[E_2, E_{-2}] = \left\{ \begin{pmatrix} cd - ef & & & 0 \\ & -cd - ef & & \\ & & -cd + ef & \\ 0 & & & cd + ef \end{pmatrix} ; c, d, e, f \in R \right\}.$$

So there are $c, e, d, f \in R$ such that

$$Y = \begin{pmatrix} 0 & c & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & f & 0 & -d \\ -f & 0 & 0 & 0 \end{pmatrix}$$

and $[Y, Z] = X$ implies

$$\begin{cases} cd - ef = a \\ cd + ef = 0 \end{cases}$$'

i.e $cd = -ef = \frac{a}{2}, cdef \neq 0$. Let $v = \frac{1}{c}e_2 - \frac{1}{e}e_4$. It's easy to check that $Q(v, v) = \frac{1}{ce} \neq 0$, $A_\circ(v) = v$. When $X = diag.(0, a, 0, -a)$, we can prove it in a similar way.

Combining (7.3.3) with Lemma (7.3.2), we have a contradiction, so A must be solvable.

Q.E.D.

(7.4) Proof of Theorem (7.1.1)

The principal tool is the following theorem from [10].

(7.4.1) Theorem (Fried and Goldman). Let $\Gamma \subseteq Aff(n)$ be virtually polycyclic and suppose that Γ act properly discontinuously on R^n. Then there exists at least one subgroup $H \subseteq Aff(n)$ containing Γ such that:

(a) H has finitely many components and each component meets Γ;

(b) H/Γ is compact;

(c) H and Γ have the same algebraic hull $in Aff(n)$;

(d) if Γ has a subgroup Γ_1 of finite index such that every element of $P_l(\Gamma_1)$ has all real eigenvalues, then H is uniquely determined by the above conditions;

(e) the identity component $H_\circ$ of H acts simply transitively on R^n and $H_\circ \cap \Gamma$ is a discrete cocompact subgroup of $H_\circ$ and is of finite index in Γ.

Such a subgroup H in (7.4.1) is called a crystallographic hull for Γ. Since a discrete solvable subgroup of a Lie group with finitely many components is polycyclic and we proved in (7.3) that Γ in (7.1.1) is virtually solvable, by (7.4.1) we need only to check for the uniqueness of H. By (7.4.1)-(d), we need only to show that $P_l(\Gamma)$ has a subgroup of finite index with real eigenvalues only. Since $H_\circ$ must occur in our table of simply transitive motions and all these simply transitive motions, except Type III-2, have linear parts with only real eigenvalues, we need only to check Type III-2. By Bieberbach's theorem (cf. [36]), any discrete subgroup of Type III-2 meets T in a subgroup of finite index.

Q. E. D.

(7.5) Classification of Γ.

(7.5.1) Lemma. Let Γ be a uniform lattice in a simply transitive group $H \subseteq R^4 \rtimes SO(2,2)$. Then H is the identity component of the crystallographic hull of Γ if and only if H is not of Type III-2.

Proof: If H is of Type III-2, then Γ has a subgroup of finite index, say Γ_1, such that $\Gamma_1 \subseteq T$. So Γ is virtually abelian. By [10], the crystallographic hull of a virtually abelian affine polycyclic group is itself virtually abelian, so H doesn't arise from any Γ.

In the unipotent cases, the algebraic hull of H is H itself. So $A(\Gamma)$, the algebraic hull of Γ, is contained in H. Since $H'_\circ$, the identity component of the crystallographic hull H' of Γ, acts simply transitively on R^4, the dimension of $H'_\circ$ must be four, then by (7.4.1)-(C) we have

$H'_\circ \subseteq H' \subseteq A(H') = A(\Gamma) \subseteq H.$

So $H = H'_\circ$, then $H' = H$.

The only remaining case is Type III-1. Since Γ is not unipotent, $H'_\circ$, the identity component of the crystallographic hull H' of Γ, must be nonunipotent solvable, i.e. $H'_\circ$ is of Type III-1 and $\Gamma \subseteq H \cap H'_\circ$. Then it's easy to show that $H'_\circ = H$.

Q. E. D.

(7.5.2) Corollary. Up to finite covers, every flat compact complete space-form with metric of signature (2,2) is of the form H/Γ, where H is a simply transitive subgroup of $R^4 \rtimes SO(2,2)$ of Type 0, Type I, Type II or Type III-1 and Γ is a uniform lattice of H.

(7.5.3) Uniform lattices.

The uniform lattices depend only on the structure of H as a Lie group and do not depend on its embedding in $R^4 \rtimes SO(2,2)$. Since Type 0 $\simeq$ R^4 Type I $\simeq$ $R \times Nil^3$, Type II $\simeq$ Nil^4 and Type III-1 $\simeq$ $R \times \{R^2 \rtimes \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} ; t \in R\}$, as Lie groups, they are exactly the same groups as that listed in [8], and D. Fried gave a list of their uniform lattices there. C.T.C. Wall also studied them, cf. [35]. Here we only write them down to complete our classification.

(7.5.3.1) The uniform lattices of H are semidirect products $Z^3 \rtimes Z_A$, where
$A \in SL_3(Z)$ has characteristic polynomial
$det(t - A) = (t - 1)(t^2 - bt + 1)$,
where $b \geq 2$ is an integer, and A and b satisfy:

i) Type 0 : $A = I, b = 2$;
ii) Type I : $(A - I)^2 = 0, A \neq I, b = 2$;
iii) Type II : $(A - I)^2 \neq 0, (A - I)^3 = 0, b = 2$;
iv) Type III-1: $b \geq 3$.

(cf. [8] and [35] for a proof)

§8. Invariant Structures

(8.1) Unimodular $Ad - c.r.$ geometries.

(8.1.1) Definition. A geometry (G, H) is unimodular if $S = G/H$ admits a G-invariant measure which is positive on open sets and finite on compact sets.

We have the following propositions (cf. (2.5) for notations)

(8.1.2) Proposition. (G, H) is unimodular iff $det\rho(H) = 1$, or equivalently $trace\ \overline{\rho}(L(H)) = 0$.

Proof. See R. Kulkarni [19].

Q.E.D.

(8.1.3) Corollary. If (G, H) is $Ad - c.r.$, then (G, H) is unimodular iff $trace\ \overline{\rho}|_{(Z(L(H)))} = 0$.

Proof. If (G, H) is $Ad - c.r.$, then H is reductive $L(H) = Z(L(H)) + [L(H), L(H)]$ where $[L(H), L(H)]$ is semisimple. We know $trace\ \overline{\rho}|_{[L(H), L(H)]} = 0$.

Q.E.D.

(8.1.4) Recall that a locally compact group G is said to be unimodular if the left- and right-Haar measures coincide upto a positive multiple. For a Lie group there is a simple criterion to decide unimodularity.

(8.1.5) Proposition. Let G be a connected Lie group. Then G is unimodular iff $|detAd| = 1$ or equivalently $trace\ ad = 0$.

For a proof, see Milnor [24] or Kulkarni [19].

With (8.1.2), (8.1.3), (8.1.4), it is easy to determine the unimodularity of $Ad - c.r.$ geometries in $dim \leq 4$.

(8.2) Semi-Riemannian $Ad - c.r.$ geometries.

(8.2.1) Definition. A geometry (G, H) is said to be semi-Riemannian if $S = G/H$ admits a G-invariant semi-Riemannian metric, i.e. a continuous family of non-singular symmetric bilinear forms on $T_x(S), x \in S$. If the forms are positive definite then the metric is Riemannian, otherwise they have a fixed signature (p, q) since S is connected.

(8.2.2) Proposition.

i) (G, H) is semi-Riemannian of signature (p, q) iff $T_x(S)$ admits $\rho(H)$-invariant or $\bar{\rho}(L(H))$-invariant nonsingular symmetric bilinear form of signature (p, q).

ii) Let (G, H) be a geometry. If H is compact then (G, H) is Riemannian. Conversely if (G, H) is an $Ad - c.r.$ Riemannian geometry then H is compact.

For a proof, see R. Kulkarni [19].

(8.2.3) Remark. In the converse part of the proof of (8.2.2) , we used the fact that (G, H) $Ad - c.r.$ implies $H \simeq \rho(H)$ is a closed subgroup of $Aut(T_x(S))$ (see (2.16)). It is curious that (G, H) can be a Riemannian geometry without H being compact. This happens if $\rho(H)$ is not closed in $Aut(T_x(S))$, equivalently, the compact-open topology on G as a transformation on S, differs from its standard topology.

In §5, all H's are determined. So it's easy to find all Riemannian geometries by checking whether H is compact. We can also do this on Lie algebra level.

(8.2.4) Proposition. Let (G, H) be $Ad - c.r.$. Then (G, H) is semi-Riemannian with signature (p, q) iff $\overline{ad}(\mathcal{H}) \simeq so(p, q)$ w.r.t. a suitable basis.

The proof of (8.2.4) is trivial. By using (8.2.4), it's easy to find all $Ad - c.r.$ semi-Riemannian geometries in $dim \leq 4$.

(8.3) Symplectic $Ad - c.r.$ geometries.

(8.3.1) Definition. A geometry (G, H) is said to be symplectic if $S = G/H$ admits a G-invariant symplectic structure, i.e. a G-invariant nondegenerate 2-form Ω. In this case $dim\ S = n$ is necessarily even, $n = 2k$, and nondegeneracy of Ω is equivalent to the fact that $\Omega^k \neq 0$. We call such a Ω symplectic form.

(8.3.2) Definition. An $Ad - c.r.$ geometry (G, H) is said to be almost symplectic if $\overline{ad}(\mathcal{H}) \simeq sp(k, R)$, where $k = \frac{1}{2}n, n$ is even $= dim\ G/H$.

(8.3.3) Proposition. An $Ad-c.r.$ symplectic geometry is also almost symplectic.

The proof (8.3.3) is directly from the definition. By this Proposition, we shall first find all almost symplectic geometries then select those which are also symplectic. We will use the symbol IAS , resp. IS to denote (G-invariant) almost symplectic, resp. (G-invariant) symplectic geometry. W.r.t. a standard basis

$$sp(m,R)=\left\{\begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{pmatrix};\ \begin{matrix} X_1,X_2,X_3\ real\ m\times m, \\ X_2,X_3\ symmetric \end{matrix}\right\}.$$

When $dim\ G/H=2$, it's easy to find all IAS. When $dim\ G/H=4$, $m=2$, we need the following well known facts.

(8.3.4) Lemma.

i) $sp(2,R)\simeq so(3,2), sp(2,R)^C=B_2$;

ii) $sp(2,R)$ has no subalgebra isomorphic to $so(4)$ or $so(3)\oplus sl_2(R)$;

iii) The 4-dimensional nontrivial representation of $sp(2,R)$ is irreducible, the 4-dimensional irreducible representation of B_2 has wrights $\{\pm\alpha,\pm\beta\}$, α, β independent.

(8.3.5) Corollary.

i) Let $\overline{ad}\mathcal{H}\subseteq sp(2,R)$, $Z\in\mathcal{H}$, s.t. $\overline{ad}Z$ diagonalizable, then $\overline{ad}Z$ has the form $\{\lambda_1, -\lambda_1, \lambda_2, -\lambda_2\}$, $(\lambda_1,\lambda_2)\neq(0,0)$, w.r.t. a suitable basis.

ii) If (G,H) is IAS, then $dim\ H\leq 10$;

iii) If (G,H) is IAS and H is abelian, then $dim\ H\leq 2$.

The above Corollary enables us to reduce our list sharply. In existance case, we have to find a suitable basis s.t. w.r.t. which $\overline{ad}\subseteq sp(2,R)$. For $X\in\mathcal{G}$, let $\overline{X}$ be its equivalence class in $\mathcal{G}/\mathcal{H}$; also let e_{ij} be the entry at the i-th row and the j-th colume of a given matrix. In (8.3.6), we record $IAS\ Ad-c.r.$ geometries in Lie algebra form ($\mathcal{G}$, $\mathcal{H}$) and corresponding basises of $\mathcal{G}/\mathcal{H}$ s.t. $\overline{ad}\subseteq sp(2,R)$ w.r.t. the given basis.

(8.3.6) Existence of IAS.

Red^4-ii-1 : $\{e_{13},e_{23},e_{31},e_{32}\}$;

Red^4-ii-2 : by (8.3.14) and (8.3.3);

Red^4-ii-3 : by (8.3.14) and (8.3.3);

Red^4-ii-4 :

$$\left\{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} & 0 \\ \sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \sqrt{-1} \\ 0 & 0 & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix}\right\};$$

Red^4-iii-1 : let $e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then the basis is $\{e_1, \sqrt{-1}e_1, -e_2, \sqrt{-1}e_2\}$;

Mix^4-i-3 : trivial;

Mix^4-i-11 : there is a basis of $sl_2(R)$, say $\{X, Y, Z\}$, s.t. $[Z, X] = 2X, [Z, Y] = -2Y, [X, Y] = Z$ and a basis $\{v_o, v_1, v_2, v_3\}$ of R^4 s.t.

$$\overline{ad}Z = diag.\{3, 1, -1, -3\}, \overline{ad}X = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \overline{ad}Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Then we have a new basis $\{\sqrt{3}v_o, -v_1, \sqrt{3}v_3, v_2\}$ to show that $\overline{ad}\mathcal{H} \subseteq sp(2, R)$;

Mix^4-i-15 : replace $\{e_1, e_2, e_3, e_4\}$ in Table 5 by $\{e_1, -e_2, e_4, e_3\}$;

Mix^4-i-16 : same as in Mix^4-i-15;

Mix^4-i-17 : replace $\{e_1, e_2, e_3, e_4\}$ in Table 5 by $\{e_1, e_2, e_3, e_4\}$;

Mix^4-i-25: let $C^2 = Ce_1 + Ce_2, sl_2(C) =$

$$\left\{\begin{pmatrix} a_1 + \sqrt{-1}a_2 & b_1 + \sqrt{-1}b_2 \\ c_1 + \sqrt{-1}c_2 & -a_1 - \sqrt{-1}a_2 \end{pmatrix}; a, b, c \in R\right\}$$

then $\{e_1, \sqrt{-1}e_1, -e_2, \sqrt{-1}e_2\}$ is a required basis;

Mix^4-i-27 : let $C^2 = Ce_1 + Ce_2, su(2) = \left\{\begin{pmatrix} \sqrt{-1}a & b + \sqrt{-1}c \\ -b + \sqrt{-1}c & -\sqrt{-1}a \end{pmatrix}; a, b, c, \in R\right\}$

then $\{e_1, e_2, \sqrt{-1}e_1, \sqrt{-1}e_2\}$ is a required basis.

Mix^4-ii-5 : it's easy to see that $\overline{ad}\mathcal{H} = \begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix}$ w.r.t. the basis $\{e_1, e_2, e_3, e_4\}$. Then we can choose a new basis $\{e_1, e_3, e_2, e_4\}$;

Mix^4-iii-1 : let $R^2 = Re_1 + Re_2$ and choose $\{e_1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_2, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$ as a new basis;

Mix^4-iii-2 : let $R^2 = Re_1 + Re_2$ and choose $\{e_1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$;
Solvable cases are trivial, since we can use (8.3.4)-iii and (8.3.5)-i,iii.

(8.3.7) The study of G-invariant symplectic structures on homogeneous spaces was initiated by Kostant [17] and Souriau [28] and developed from a more general point of view by Chu [5]. S. Sternberg [29] generalized their results and applied his results to compute symplectic forms of some low dimensional geometries ($dim\ G \leq 3$). In general situation, the computation of symplectic forms is still a problem. In my dissertation, I shall either prove or construct a symplectic form for each "existence" case among $Ad-c.r.$ geometries in $dim \leq 4$. When G is semisimple, I shall give a more general result.

Let (G, H) be a geometry and $\pi : G \to G/H = S$ be the projection. IfΩ is an invariant form on S then it is clear that $\sigma = \pi^*\Omega$ is a left invariant form on G which satisfies

(i) $i(X)\sigma = 0$ for all $X \in \mathcal{H} = L(H)$;

(ii) σ is invariant under right multiplication by elements of H, and hence invariant under Ad for elements of H.

Conversely, it is clear that any left invariant form σ on G satisfying (i) and (ii) arises from G/H. So to find a symplectic form on G/H, we only have to find a left invariant 2-form σ on G which satisfies (i), (ii) and

(iii) $d\sigma = 0$;

(iv) $\sigma^k \neq 0$ everywhere, $k = \frac{1}{2}dim$ G/H (When the dimension is 4, we need to show $\sigma \wedge \sigma \neq 0$).

Let (G, H) be $Ad-c.r.$, $L(G) = \mathcal{G}$, $L(H) = \mathcal{H}$. Then $\mathcal{G} = \mathcal{H} + \mathcal{M}$ s.t. $[\mathcal{H}, \mathcal{M}] \subseteq \mathcal{M}$. For $X \in \mathcal{G}$, let $X = X_{\mathcal{H}} + X_{\mathcal{M}}$ where $X_{\mathcal{H}} \in \mathcal{H}$, $X_{\mathcal{M}} \in \mathcal{M}$. It is clear that condition (i) and (ii) plus σ nonsingular is equivalent to that (G, H) is IAS. The following lemma helps us to check condition (iii).

(8.3.8) Lemma. Notations as in (8.3.7) and assume (i) and (ii). Then $d\sigma = 0$ iff for every $X, Y, Z \in \mathcal{G}$,

$$([X, Y]_{\mathcal{M}}, Z_{\mathcal{M}} + ([Y, Z]_{\mathcal{M}}, X_{\mathcal{M}}) + ([Z, X]_{\mathcal{M}}, Y_{\mathcal{M}}) = 0.$$

The proof is simple. The following corollary helps us to show that Red^4-ii-1–4 are IS.

(8.3.9) Corollary. If $[\mathcal{M},\mathcal{M}] \subseteq \mathcal{H}$, then $d\sigma = 0$.

(8.3.10) Corollary. $d\sigma = 0$ iff for every $X, Y, Z \in \mathcal{M}$,

$\sigma([X,Y]_{\mathcal{M}}, Z_{\mathcal{M}}) + \sigma([Y,Z]_{\mathcal{M}}, X_{\mathcal{M}}) + \sigma([Z,X]_{\mathcal{M}}, Y_{\mathcal{M}}) = 0.$

The last Corollary enables us to check if $d\sigma = 0$ by computing no more than four times since $dim\ \mathcal{M} \leq 4$. We shall rely on the basis in (8.3.6) to construct σ, i.e., we shall first select a basis for $\mathcal{M}$, then use the corresponding dual left invariant forms to construct σ. Let's see two examples.

(8.3.11) Existence for Sol^4-vi-3 : $(\mathcal{G}, \mathcal{H}) = (\mathcal{N}^3 \times \mathcal{H}_\circ, \mathcal{H})$, where

$$\mathcal{H}_\circ = RA + \mathcal{H},\ A = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 2 \end{pmatrix},\ \mathcal{H} = R\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrices are w.r.t. the basis $\{e_1, e_2, e_3\}$ of $\mathcal{N}_3$, s.t. $[e_1, e_2] = e_3 \in Z(\mathcal{N}_3)$. Let $e_4 = A$. Then $[e_4, e_1] = e_1, [e_4, e_2] = e_2, [e_4, e_3] = 2e_3$. Now $\{e_1, e_2, e_3, e_4\}$ forms a basis of $\mathcal{M}$, and $0 \neq [\mathcal{M}, \mathcal{M}] \subseteq \mathcal{M}$. Let $\{W_i\}$ be the dual to $\{e_i\}$ s.t. $W_i(e_j) = \delta_{ij}$ and $W_i(\mathcal{H}) = 0$ for $1 \leq i, j \leq 4$.

Let $W = \sum_{i<j} a_{ij} w_i \wedge w_j$. W is $ad\mathcal{H}$-invariant iff

$W([Z,X],Y) + W(X,[Z,Y]) = 0$

for every $X, Y \in \mathcal{G}$, $Z \in \mathcal{H}$. Let $X = e_1, Y = e_3, Z \in \mathcal{H}$, then since $[Z, e_3] = 0$, we have $W([Z,e_1], e_3) = 0$ which implies $a_{23} = 0$. Similarly we get $a_{24} = a_{13} = a_{14} = 0$. So we must have

$W = a_{12} w_1 \wedge w2 + a_{34} w_3 \wedge w_4,$

$W^2 \neq 0$ iff $a_{12} a_{34} \neq 0$.

Now (8.3.10) becomes : $dW \neq 0$ iff for every e_i, e_j, e_k,

$W([e_i, e_j], e_k) + W([e_k, e_i], e_j) + W([e_j, e_k], e_i) = 0.$

The above equality is an identity for $1 \leq i, j, k \leq 4$ except for $i = 1, j = 2, k = 4$:

LHS $= W(e_3, e_4) + W(e_1, e_2) + W(-e_2, e_1) = a_{34} + 2a_{12}$

which is zero iff $a_{34} = -2a_{12}$. So we can choose

$$W = w_1 \wedge w_2 - 2w_3 \wedge w_4$$

and the geometry is IS.

(8.3.12) Existence and Non-existence for Sol^4-i-9 :

$(\mathcal{G}, \mathcal{H}) = (T_{4,\{\theta\}}, 0)$ where $T_{4,\{\theta\}} = N_3 \rtimes R \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 2c \end{pmatrix}, c = cos\theta, s = sin\theta, \theta \neq k\pi$. Since $\mathcal{H} = 0$, we only need $W^2 \neq 0$ and $dW = 0$. Again the matrix is w.r.t. the same basis as that in (8.3.11) and we add

$$e_4 = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 2c \end{pmatrix}.$$

Let $\{w_i\}$ be the dual to $\{e_i\}$. Then using the formula

$$dw_i = -\tfrac{1}{2} \textstyle\sum_{k<j} c^i_{kj} w_k \wedge w_j, \quad ([e_i, e_j] = \sum c^k_{ij} e_k),$$

we have

$$dw_1 = -cw_4 \wedge w_1 + sw_4 \wedge w_2,$$

$$dw_2 = -sw_4 \wedge w_1 - cw_4 \wedge w_2,$$

$$dw_3 = -w_1 \wedge w_2 - 2cw_4 \wedge w_3,$$

$$dw_4 = 0.$$

Let $W = \sum_{i<j} a_{ij} w_i \wedge w_j$. Then

$$dW = \{-2ca_{12} - a_{34}\} w_1 \wedge w_2 \wedge w_4 + \{-3ca_{13} - sa_{23}\} w_1 \wedge w_3 \wedge w_4 + \{sa_{13} - 3ca_{23}\} w_2 \wedge w_3 \wedge w_4.$$

$$dW = 0 \text{ iff } \begin{cases} 2ca_{12} + a_{34} = 0 & (1) \\ 3ca_{13} + sa_{23} = 0 & (2) \\ sa_{13} - 3ca_{23} = 0 & (3). \end{cases}$$

Since $s \neq 0$, from (2), (3) we must have $a_{13}^2 + a_{23}^2 = 0$, i.e. $a_{13} = a_{23} = 0$. We have two subcases:

Subcase 1 : $c \neq 0$. Let $a_{12} = 1, a_{34} = -2c$, and $W = w_1 \wedge w_2 - 2cw_3 \wedge w_4$. Then $dW = 0, W^2 \neq 0$. The geometry is IS.

Subcase 2 : $c = 0$. Then (i) implies $a_{34} = 0$ and $W = a_{12} w_1 \wedge w_2 + a_{14} w_1 \wedge w_4 + a_{24} w_2 \wedge w_4$. Since $W^2 = 0$, we have non-existence.

The other cases can be treated similarly and we omit the detail.

Finally, we study the following special cases : G is semisimple. Again let $L(G) = \mathcal{G}$. $B(\cdot,\cdot)$ the killing form on $\mathcal{G}$. Let $0 \neq X \in \mathcal{G}$. Define $\theta : \mathcal{G} \to R$ s.t. $\theta(Y) = B(X,Y)$. Then $\theta \in \mathcal{G}^*$, $d\theta \in \wedge^2\mathcal{G}^*$. Let $\mathcal{H} = \{Y \in \mathcal{G};\ [Y,X] = 0\} = C_X(\mathcal{G})$ which is a subalgebra of $\mathcal{G}$. Let H_θ be the analytic subgroup of G with Lie algebra $\mathcal{H}$. Then we have

(8.3.13) Proposition.

i) H_θ is closed;

ii) $d\theta$ induces a symplectic form on G/H_θ.

Proof of i). Let $H = \{g \in G; Ad^*(g)\theta = \theta\}$. Then H is clearly closed. Let $H_\circ$ be the identity component of H. We only have to show that $L(H_\circ) = \mathcal{H}$.

Let $Y \in \mathcal{H}$. Then $[Y,X] = 0$, so $B([X,Y],Z) = 0$ for every $Z \in \mathcal{G}$, i.e., $B(X,[Y,Z]) = 0$ or $\theta([Y,Z]) = 0$ for every $Z \in \mathcal{G}$, i.e., $(ad^*Y(\theta))(Z) = 0$ or $ad^*Y(\theta) = 0$. So $Ad^*(expY)\theta = \theta$, $expY \in H$. similarly $exptY \in H_\circ$ for $Y \in \mathcal{H}$, $t \in R$. This implies $Y \in L(H_\circ)$.

Conversely, let $exptY$ be an one-parameter subgroup of $H_\circ$, then $Ad^*(exptY)\theta = \theta$ or $\theta(Ad(exptY)Z) = \theta(Z)$ for every $Z \in \mathcal{G}$, i.e.,

$\theta(\sum_{n=1}^{\infty} ad^n(tY)Z) = 0,$

i.e.,

$\theta([tY,Z]) + \sum_{n=2}^{\infty} t^n ad^n Y(Z) = 0,$

$\theta([Y,Z]) + \sum_{n=2}^{\infty} t^{n-1} ad^n Y(Z) = 0.$

Let $t \to 0$, we have $\theta([Y,Z]) = 0$ for every $Z \in \mathcal{G}$. So we have $[X,Y] = 0$ and $Y \in \mathcal{H}$.

Proof of ii). Now $d\theta$ is closed, $i_Y d\theta = 0$ iff $Y \in \mathcal{H}$. For the rest of the proof, cf. Chu [5].

Q.E.D.

(8.3.14) Corollary. Red^4-ii-1,2,3 and 4 are IS.

So we gave a different proof to (8.3.8).

(8.4) G-invariant contact structure.

(8.4.1) Definition. (G, H) has a G-invariant contact structure iff there is a left invariant 1-form θ of G s.t.

i) $i_X d\theta = 0$ for every $X \in \mathcal{H}= L(H)$,

ii) $i_X\theta = 0$ for every $X \in \mathcal{H}$,

iii) $\theta \wedge (d\theta)^k \neq 0$ everywhere, $dim\ G/H = 2k+1$.

Such a θ is called a contact form and the geometry is said to be *ICT*.

We shall study G-invariant contact structures on 3-*dim* $Ad - c.r.$ geometries. For higher dimensions, we only study a special case: when G is semisimple with real rank 1.

(8.4.2) Existence.

(8.4.2.1) Red^3-i-1 : $(\mathcal{G}, \mathcal{H}) = (sl_2(R), 0)$. Let $\{X_1, X_2, X_3\}$ be a basis of $sl_2(R)$ s.t. $[X_1, X_2] = 2X_2, [X_1, X_3] = -2X_3, [X_2, X_3] = X_1$, let $\{w_1, w_2, w_3\}$ be the dual forms to $\{X_i\}$. Then w_1 is a contact form.

(8.4.2.2) Red^3-i-2 : $(\mathcal{G}, \mathcal{H}) = (so(3), 0)$. Let $\{X_i\}, i \leq 3$, be a basis of $\mathcal{G}$ s.t. $[X_1, X_2] = -X_3, [X_1, X_3] = X_2, [X_2, X_3] = -X_1$, and $\{w_i\}, i \leq 3$, be the dual to $\{X_i\}$. Then w_1 is a contact form.

(8.4.2.3) Red^3-ii-1. $(\mathcal{G}, \mathcal{H}) = (gl_2(R), R\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$. Let $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\{w_i\}$ is the dual to $\{X_i\}$. Then w_2 is a contact form.

(8.4.2.4) Red^3-ii-2 : $(\mathcal{G}, \mathcal{H}) = (gl_2(R), R\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix})$. $\{X_i\}$ and $\{w_i\}$ are the same as that in (8.4.2.3). Then $\theta = w_1 + w_2 - w_3 + w_4$ is a contact form.

(8.4.2.5) Red^3-ii-3 : $(\mathcal{G}, \mathcal{H}) \sim (so(3) + R\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}, R\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix})$.

Let

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, X_4 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

and $\{w_i\}$ be the dual to $\{X_i\}$. Then $\theta = w_3 - w_4$ is a contact form.

(8.4.2.6) Mix^3-ii-1 : $(\mathcal{G}, \mathcal{H}) = (N_3 \rtimes \mathcal{H}, \mathcal{H})$, $\mathcal{H} = \begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix}$,

the matrix is w.r.t. the basis $\{X_1, X_2, X_3\}$ s.t. $[X_1, X_2] = X_3, X_3 \in Z(N_3)$. Let $\{w_i\}, i \leq 3$ be the dual to $\{X_i\}$ s.t. $w_i(\mathcal{H}) = 0$. Then w_3 is a contact form.

(8.4.2.7) Sol^3-i-1, sol^3-ii-1 and Sol^3-ii-3. We have the same result as in (8.4.2.6).

(8.4.2.8) Sol^3-i-2 : $(S_{3,\{a,b\}}, 0)$. Let $\{X_i\}$ be a basis of $\mathcal{G}$ s.t. $[X_1, X_2] = 0, [X_3, X_1] = aX_1, [X_3, X_2] = bX_2$, let $\{w_i\}$ be the dual to $\{X_i\}$. Then the geometry is ICT iff $a \neq b$. If $a \neq b, w_1 + w_2$ is a contact form.

(8.4.2.9) Sol^3-i-3 : $(\mathcal{G}, \mathcal{H}) = (S_{3,\{\theta\}}, 0)$. Let $\{X_i\}$ be a basis of $\mathcal{G}$ s.t. $[X_1, X_2] = 0, [X_3, X_1] = (cos\theta)X_1 + (sin\theta)X_2, [X_3, X_2] = -(sin\theta)X_1 + (cos\theta)X_2$. Then $a_1 w_1 + a_2 w_2, (a_1, a_2) \neq (0, 0)$ is a contact form.

(8.4.2.10) Sol^3-i-4 : $(\mathcal{G}, \mathcal{H}) = (S_{3,\bigcirc}, 0)$. Let $\{X_i\}$ be a basis of $\mathcal{G}$, s.t. $[X_1, X_2] = 0, [X_3, X_1] = X_1 + X_2, [X_3, X_2] = X_2$, let $\{w_i\}$ be the dual forms. Then $a_1 w_1 + a_2 w_2 + a_3 w_3, a_2 \neq 0$ is a contact form.

(8.4.3) Non-existence.

Let $L(G) = \mathcal{G} = \mathcal{M} + \mathcal{H}$ s.t. $[\mathcal{H}, \mathcal{M}] \subseteq \mathcal{M}$. Let $\{X_1, X_2, X_3\}$ be a basis of $\mathcal{M}$ and $\{X_4, ..., X_k\}$ be a basis of $\mathcal{H}$, w_i be the daul form to X_i. Then if $\theta \in \mathcal{G}^*$, then

i) $i_X \theta = 0$ for every $X \in \mathcal{H}$ iff $\theta = a_1 w_1 + a_2 w_2 + a_3 w_3$.

Let C^k_{ij} be the coefficients in $[X_i, X_j] = \sum C^k_{ij} X_k$. Then it's easy to show that if $j \geq 4$, then

$i_{X_j} d\theta = \sum_{i \leq 3} (\sum_{1 \leq k \leq 3} a_k C^k_{ij}) w_i$.

So we have

ii) $i_X d\theta = 0$ for every $X \in \mathcal{H}$ iff $\sum_{1 \leq k \leq 3} a_k C^k_{ij} = 0$ for $4 \leq j$ and $i \leq 3$.

(8.4.3.1) Corollary. If $det(C^k_{ij})_{1 \leq i,k \leq 3} \neq 0$ for some $j \geq 4$, then $a_1 = a_2 = a_3 = 0$, i.e. we have non-existence.

This simple computation covers many of our non-existence cases. Let's see an example.

(8.4.3.2) Non-existence for Mix^3-i-2 : $(R^3 \rtimes sl_3(R), sl_3(R))$. We may assume $\mathcal{M} = R^3$, $\mathcal{H} = sl_3(R)$. Then let $X_4 = diag.(1,1,2)$. It's easy to see that

$$(C^k_{i4})_{1 \le i,k \le 3} = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 2 \end{pmatrix}.$$

So $det(C^k_{i4}) \neq 0$ implies non-existence.

(8.4.3.3) In many truncated-nil-affine cases, we can find some $X_j \in \mathcal{H}$, s.t. adX_j has the form

$$\begin{pmatrix} -C^1_{1j} & -C^1_{2j} & 0 \\ -C^2_{1j} & -C^2_{2j} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $det(C^k_{ij})_{1 \le i,k \le 3} = 0$. But then $i_{X_j} d\theta = 0$ iff

$$\begin{cases} a_1 C^1_{1j} + a_2 C^1_{2j} = 0 \\ a_1 C^2_{1j} + a_2 C^2_{2j} = 0 \end{cases}.$$

So if $det(C^k_{ij})_{1 \le i,k \le 2} \neq 0$, then $a_1 = a_2 = 0$ and we only have to check if $\theta = w_3$ is a contact form. For example, if $(\mathcal{G}, \mathcal{H}) = (N_3 \rtimes RA, RA)$, where

$$A = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we try $\theta = w_3$ to obtain exastence. If a geometry is of type Sol^3-iv, we can similarly consider $\theta = w_3 = A^*$ (see the classification list for notation). But $C^3_{ij} = 0$ for any i, j, i.e. $dw_3 = 0$ so condition iii) is not satisfied and we get non-existence.

The above method can be applied to geometries in higher dimensions.

(8.4.4) Semisimple Lie groups of real rank one.

Let G be a semisimple connected Lie group of real rank one (with finite center). $\mathcal{G} = L(G)$, $B(\cdot,\cdot)$ the Killing form on $\mathcal{G}$, $\mathcal{G} = \mathbf{t} + \mathbf{p}$ a Cartan decomposition. Let $\mathbf{a} \subseteq \mathbf{p}$ be a maximal abelian subspace of $\mathbf{p}$. G is of real rank one if $dim\ \mathbf{a} = 1$. Let $0 \neq W \in$

a. Define θ: $\mathcal{G} \to R$ by $\theta(X) = B(W,X), X \in \mathcal{G}$. Then $\theta \in \mathcal{G}^*$, $d\theta \in \wedge^2\mathcal{G}^*$. Let $\mathcal{H} = \{X \in \mathcal{G}; [X,W] = 0\}$, $\mathcal{K} = \{X \in \mathcal{H}; B(W,X) = 0\}$. $\mathcal{H}$, $\mathcal{K}$ are subalgebras of $\mathcal{G}$. Let K_θ, resp. H_θ, be the analytic subgroup of G with Lie algebra $\mathcal{K}$, resp. $\mathcal{H}$. By (8.3.13), H_θ is closed in G. We shall show that K_θ is compact and θ induces a contact form on G/K_θ.

(8.4.4.1) Lemma. K_θ is compact.

Proof. First we show that $\mathcal{K} = \mathbf{t}\cap\mathcal{H}$.

Since $\mathbf{a} \subseteq \mathbf{p}$ is maximal abelian in $\mathbf{p}$, if $[X,W] = 0$ for some $X \in \mathcal{G}$, we write $X = X_\mathbf{t}+X_\mathbf{p}$ with $X_\mathbf{t} \in \mathbf{t}$, $X_\mathbf{p} \in \mathbf{a}$. Then $B(X,W) = B(X_\mathbf{t},W)+B(X_\mathbf{p},W) = B(X_\mathbf{p},W)$ since $B(\mathbf{t},W) = 0$. Since $B(X_\mathbf{p},W) > 0$ if $X_\mathbf{p} \neq 0$, so $B(X,W) = 0$ means $X \in \mathbf{t}$, i.e. $\mathcal{K} = \mathbf{t}\cap\mathcal{H}$.

Now let $K' = K \cap H_\theta$, where K is the connected maximal compact subgroup of G, $L(K) = \mathbf{t}$. Let $K'_\circ$ be the identity component of K'. Since K is compact, K is closed, so is $K \cap H_\theta$. So $K'_\circ$ is closed. Since $L(K'_\circ) = \mathbf{t}\cap\mathcal{H}$, $K'_\circ = K_\theta$.

Q.E.D.

(8.4.4.2) Proposition. θ introduces an invariant contact form on G/K_θ.

Proof. $K_\theta \subseteq H_\theta \subseteq H = \{g \in G; Ad^*(g)\theta = \theta\}$, so θ is well defined on G/K_θ and is invariant by (8.3.13). We need only to show that $\theta \wedge (d\theta)^t \neq 0$, where $2t + 1$ is the dimension of G/K_θ.

Sinse $B(\cdot,\cdot) < 0$ on $\mathbf{t}$, $B(\cdot,\cdot) > 0$ on $\mathbf{p}$, we can define

$\mathcal{K}^\perp = \{X \in \mathbf{t};\ B(X,\mathcal{K}) = 0\}$,

$\mathbf{a}^\perp = \{X \in \mathbf{p};\ B(X,\mathbf{a}) = 0\}$.

Then $\mathcal{G} = (\mathcal{K} + \mathcal{K}^\perp) + (\mathbf{a} + \mathbf{a}^\perp)$.

(8.4.4.2.1) Claim. Let $0 \neq W \in \mathbf{a}$. Then $[W,\mathbf{a}^\perp] = \mathcal{K}$, $[W,\mathcal{K}^\perp] = \mathbf{a}^\perp$.

Proof. We know that $[\mathbf{t},\ \mathbf{p}] \subseteq \mathbf{p}$, $[\mathbf{p},\ \mathbf{p}] \subseteq \mathbf{t}$. So $[W,\ \mathbf{a}^\perp] \subseteq \mathbf{t}$, $[W,\mathcal{K}^\perp] \subseteq \mathbf{p}$. Since $B(W,[W,X]) = B([W,W],X) = 0$ for every $X \in \mathcal{G}$, $[W,\mathcal{K}^\perp] \subseteq \mathbf{a}^\perp$. Let $Y \in \mathcal{G}$, $B(\mathcal{K},[W,Y]) = B([\mathcal{K},W],Y) = 0$ since $\mathcal{K} \subseteq \mathcal{H} = C_W(\mathcal{G})$, so $[W,\mathbf{a}^\perp] \in \mathcal{K}^\perp$.

If $dim\ [W,\mathbf{a}^{\perp}] < dim\ \mathbf{a}^{\perp}$, then $Ker.adW|_{\mathbf{a}^{\perp}} \neq 0$, but **a** is maximal abelian in **p**, i.e. $Ker.adW|_{\mathbf{a}^{\perp}} = 0$. We have a contradiction. So $dim\ [W,\mathbf{a}^{\perp}] = dim\ \mathbf{a}^{\perp}$, i.e. $dim\ \mathbf{a}^{\perp} \leq dim\ \mathcal{K}^{\perp}$.

If $dim\ [W,\mathcal{K}^{\perp}] < dim\ \mathcal{K}^{\perp}$, then $Ker.adW|_{\mathcal{K}^{\perp}} \neq 0$, i.e. there is $X \in \mathcal{K}^{\perp}$ s.t. $[W,X] = 0$. So $X \in \mathcal{H}$, then $X \in \mathcal{K}$. We have a cotradiction. So $dim\ [W,\mathcal{K}^{\perp}] = dim\ \mathcal{K}^{\perp}$, i.e. $dim\ \mathcal{K}^{\perp} \leq dim\ \mathbf{a}^{\perp}$.

Combining above results, we get $dim\ \mathcal{K}^{\perp} = dim\ \mathbf{a}^{\perp}$ and $adW(\mathcal{K}^{\perp}) = \mathbf{a}^{\perp}$, $adW(\mathbf{a}^{\perp}) = \mathcal{K}^{\perp}$.

Let $\{X_i\}, 1 \leq i \leq t$, be a basis of $\mathcal{K}^{\perp}$, s.t. $-B(X_i, X_j) = \delta_{ij}$. Then $\{(adW|_{\mathbf{a}^{\perp}})^{-1}X_i\}$ form a basis of $\mathbf{a}^{\perp}$ and

$$\begin{aligned}
& d\theta(X_i, (adW|_{\mathbf{a}^{\perp}})^{-1}X_j) \\
&= -\theta([X_i, (adW|_{\mathbf{a}^{\perp}})^{-1}X_j]) \\
&= B(W, [(adW|_{\mathbf{a}^{\perp}})^{-1}X_j, X_i]) \\
&= B([W, (adW|_{\mathbf{a}^{\perp}})^{-1}X_j], X_i) \\
&= B(X_j, X_i) \\
&= \delta_{ij}.
\end{aligned}$$

Let $Y_i = (adW|_{\mathbf{a}^{\perp}})^{-1}X_i$, then it's easy to check that

$$d\theta(X_i, X_j) = d\theta(W, X_i) = d\theta(Y_i, Y_j) = d\theta(W, Y_i) = \theta(X_i) = \theta(Y_i) = 0,$$

and

$$\theta \wedge (d\theta)^t(W, X_1, Y_1, X_2, Y_2, ..., X_t, Y_t) = cB(W, W) \neq 0,$$

where c is a nonzero constant.

Q.E.D.

(8.4.4.3) Examples.

i) $G = SO_o(n,1), K_\theta = SO(n-1)$;

ii) $G = SU(n,1), K_\theta = S(U(n-1) \times U(1))$;

iii) $G = SP(n,1), K_\theta = SP(n-1) \times SP(1)$;

iv) $L(G) = f_{4,(-20)}, K_\theta = SO(7)$.

(8.5) Invariant complex structure.

Let (G, H) be an $Ad-c.r.$ geometry. $\mathcal{G} = \mathcal{H} + \mathcal{M}$, s.t. $[\mathcal{H}, \mathcal{M}] \subseteq \mathcal{M}$. For $X \in \mathcal{G}$, let X_m be the image of X under the projection from $\mathcal{H} + \mathcal{M}$ to $\mathcal{M}$.

(8.5.1) Defination. An invariant almost complex structure (IAC) on G/H is a linear map

$J : \mathcal{M} \to \mathcal{M}$

s.t.

i) $J^2 = -1$;

ii) $J \circ Ad(g) = Ad(g) \circ J$, for every $g \in H$.

Condition ii) can be replaced by

ii)' $J \circ ad(X) = ad(X) \circ J$ for every $X \in \mathcal{H}$.

An invariant almost complex structure is said to be integrable if

$[JX, JY]_m - [X, Y]_m - J[X, JY]_m - J[JX, Y]_m = 0$

for every $X, Y \in \mathcal{M}$. Then we say G/H has an invariant complex structure (IC).

By Kobayashi and Nomizu [16] Vol.I, Chap.X, §6.5, our definations are equivalent to the usual definations of almost complex and complex structures invariant under H.

(8.5.2) Corollary.

i) (G, H) is IAC iff there is a basis of $\mathcal{M}$ w.r.t. which $ad\mathcal{H}|_{\mathcal{M}} \subseteq (gl_k(C))_R$, where $k = \frac{1}{2}dim\mathcal{M}$, and $gl_k(C) = A + \sqrt{-1}B, (gl_k(C))_R = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where A, B are $k \times k$ real matrices;

ii) If (G, H) is IAC, then for any $X \in \mathcal{H}$, any real eigenvalue of $adX|_{\mathcal{M}}$ must have even multiplicity;

iii) If (G, H) is IAC, $dim\ \mathcal{H} < 2k^2$, where $k = \frac{1}{2}dim\ \mathcal{M}$;

iv) If $[\mathcal{M},\mathcal{M}] = 0$ or $[\mathcal{M},\mathcal{M}] \subseteq \mathcal{H}$, then every IAC is IC.

We shall use the above corollary to determine the existence or non-existence of IAC. The following examples shall show how to do it.

(8.5.2.1) Red^4-1 to -5. Every $\mathcal{H}$ is semisimple. $ad\mathcal{H}|_{\mathcal{M}} \subseteq (gl_2(C))_R$ implies $ad\mathcal{H}|_{\mathcal{M}} \subseteq (sl_2(C))_R \simeq so(1,3)$. So we know that we have non-existence for $\mathcal{H} = so(4)$ or

$so(2,2)$. When $(\mathcal{G}, \mathcal{H}) = (so(1,4), so(1,3))$, we can decompose $so(1,4)$ into root-spaces w.r.t a Cartan subalgebra of $so(1,3)$ and write:

$$so(1,4) = so(1,3) + L_\alpha + L_\beta + L_{\gamma,\bar{\gamma}},$$

where α, β are real roots of multiplicity 1, $\{\gamma, \bar{\gamma}\}$ is a pair of compact roots. We can find $X \in so(1,3)$ s.t. $adX|_{\mathcal{M}} = diag.(1,-1,0,0)$. Then by (8.5.2)-ii), we have non-existence. Similarly we have non-existence for $(so(2,3), so(1,3))$.

(8.5.2.2) Non-existence for $(sl_3(R), \begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix} + R\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix})$.

We can choose $\{e_{13}, e_{32}, e_{31}, e_{23}\}$ as the basis of $\mathcal{M}$. $e_{11} - e_{22} \in \mathcal{H}$, $ad(e_{11} - e_{22})|_{\mathcal{M}} = diag.(1,1,-1,-1)$. Since $adX \circ J = J \circ adX$ for every $X \in \mathcal{H}$, we must have

$$J = \begin{pmatrix} a & b & & \\ c & d & \multicolumn{2}{c}{0} \\ & & a' & b' \\ \multicolumn{2}{c}{0} & c' & d' \end{pmatrix}.$$

But $ad(e_{21})|_{\mathcal{M}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $ad(e_{12})|_{\mathcal{M}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

so $ad(e_{21}) \circ J = J \circ ad(e_{21}), ad(e_{12}) \circ J = J \circ ad(e_{12})$ iff

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Also

$$ad(\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix})|_{\mathcal{M}} = diag.(3,-3,-3,3),$$

which is commutitive with J, so J must has the form

$$\begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix}.$$

So J is diagnolizable in R, $J^2 \neq -1$. (We can also use corollary (8.5.2) - ii to show non-existence).

(8.5.2.3) Existence for $(su(2,1), su(1,1) + R\begin{pmatrix} -2\sqrt{-1} & & \\ & \sqrt{-1} & \\ & & \sqrt{-1} \end{pmatrix})$, where

$$su(1,1) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & w_1 & w_2 \\ 0 & w_2 & -w_1 \end{pmatrix} ; w_1 \in \sqrt{-1}R, w_2 \in C \right\}.$$

Let $\{e_i\}$ be a basis of $\mathcal{M}$ s.t.

$$e_1 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & -\sqrt{-1} & -\sqrt{-1} \\ -\sqrt{-1} & 0 & 0 \\ \sqrt{-1} & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & \sqrt{-1} & -\sqrt{-1} \\ \sqrt{-1} & 0 & 0 \\ \sqrt{-1} & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$ad\mathcal{H}|_{\mathcal{M}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} ; A \in sl_2(R) \right\} + R \begin{pmatrix} 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} & 0 \end{pmatrix} \subsetneq (gl_2(C))_R.$$

The remaining three Red^4 cases are CP^2, CH^2 and $(PSL_2(C), C^*/\{\pm 1\})$ and they are IC (then IAC). We omit the details of mixed and solvable IAC cases. Then we check the integralbility of each J to find IC among IAC. The following method, which was developed in C.T.C. Wall [34] for Riemanian geometries, works well for our $Ad - c.r.$ geometries.

(8.5.3) Let $\mathcal{G} = \mathcal{H} + \mathcal{M}$ as at the beginning of (8.5). Assume that $\mathcal{M}$ is a subalgebra of $\mathcal{G}$ (then $\mathcal{M}$ is an ideal of $\mathcal{G}$). It is well known that there is an one to one correspondence between the invariant complex structure on G/H and the almost complex structure J on $\mathcal{M}$ which satisfies

i) $J \circ adX = adX \circ J$ for every $X \in \mathcal{H}$,

ii) the $\sqrt{-1}$-eigenspace of J on $\mathcal{M}^C = \mathcal{M} \otimes_R C$ is a Lie subalgebra.

Let $W = \{X \in \mathcal{M}^C; JX = \sqrt{-1}X\}$. If $dim\ \mathcal{M} = 4$, then $dim\ _C W = 2$. If W is a subalgebra of $\mathcal{M}^C$, then W is abelian or (nonabelian) solvable. So we can find a basis of W, say $\{U, V\}$, s.t.

(a) $[U, V] = 0$, or

(b) $[U, V] = V$.

Then we call W is of type (a) or (b) respectively. We have the following propositions similar to those in Wall [34].

(8.5.4) Proposition. If W is of type (a), then

(i) $dim_R Z(\mathcal{M})$ is even;

(ii) $Rank\ ad(JX) = Rank\ adX$ for every $X \in \mathcal{M}$.

(8.5.5) Proposition. If W is of type (b), then $dim\ [\mathcal{M},\mathcal{M}] \geq 2$.

When studying IC in dim 4, we first apply (8.5.4) and (8.5.5) to $Ad-c.r.$ geometries $(\mathcal{G}, 0)$. Then we apply obtained results to those $(\mathcal{G}, \mathcal{H})$ with $\mathcal{G} = \mathcal{M} \rtimes \mathcal{H}$, where $\mathcal{M}$ is an ideal of $\mathcal{G}$. All solvable and most of the indecomposable mixed cases are of this type. The only exceptions are Mix^4-iii and Mix^4-v. But using (8.5.2), we can easily show that Mix^4-v and Mix^4-iii-1 and 3 are not IAC (so are not IC). If we only ask that $\mathcal{M}$ is a subalegbra but not necesserely an ideal, i.e. without the condition $[\mathcal{H}, \mathcal{M}] \subseteq \mathcal{M}$, then (8.5.3) also holds and can be applied to Mix^4-iii-2 and 4 as Wall did in [34]. In Wall [34], Wall studied invariant complex structures of Riemannian geometries (G, H) s.t. G contains a discrete subgroup Γ with $\Gamma \backslash G/H$ of finite volume (this is not our **Boundedness** though in many cases Γ can be chosen so that $\Gamma \backslash G/H$ is also compact). Wall's result covers a small part of our $Ad-c.r.$ geometries, but his method can be applied to all of our mixed and solvable geometries. Here we only record a few cases which are not covered by Wall's paper.

(8.5.6) Existence and non-existence for some $(\mathcal{G}, 0)$'s.

(8.5.6.1) Existence.

Sol^4-i-2 : $\mathcal{G} = R^2 \rtimes \{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}; a, b \in R\}$. $R^2 = Re_1 + Re_2$.

Let $\{e_i\} = \{e_1, e_2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$ be a basis. Let $U = e_1 + \sqrt{-1}e_3, V = e_2 + \sqrt{-1}e_4$. Then we have type (a).

Sol^4-i-3 : Let $R^3 = Re_1 + Re_2 + Re_3, \{e_i\} = \{e_1, e_2, e_3, e_4 = diag.(a,b,c)\}$ be a basis. If $(a,b,c) = (a,a,c)$, then let $U = e_1 + \sqrt{-1}e_2, V = \frac{1}{a}e_4 + \sqrt{-1}e_3$, we get type (b) (Wall studied the case $a + b + c = 0$).

$$Sol^4\text{-i-5} : R^3 = Re_1 + Re_2 + Re_3, e_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}.$$

If $a = 1$, let $U = e_4 + \sqrt{-1}e_1, V = e_2 + \sqrt{-1}e_3$, we get type (b).

Sol^4-i-8 : $N_3 = Re_1 + Re_2 + R_3, [e_1, e_2] = e_3 \in Z(N_3)$. Let $e_4 = diag.(1, a, 1 + a), a \neq 0$. Then let $U = \frac{1}{a}(e_4 + \sqrt{-1}e_1), V = e_3 + \sqrt{-1}e_2$, we get type (b) (Wall studied the case $a = -1$).

$$Sol^4\text{-i-10} : e_1, e_2, e_3 \text{ as in } Sol^4\text{-i-8}, e_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let $U = e_4 - \sqrt{-1}e_1, V = e_2 + \sqrt{-1}e_3$, we get type (b).

(8.5.6.2) Non-existence.

$$Sol^4\text{-i-6} : R^3 = Re_1 + Re_2 + Re_3, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

V_4 has a center Re_2, so by (8.5.4), it is not of type (a). If it's of type (b), we should have $[U, V] = V \subseteq [\mathcal{G},\mathcal{G}]^C = Ce_2 + Ce_3$. Since $[\mathcal{G}, Re_2 + Re_3] = Re_3$, we must have $V = ae_3, a \in C$, i.e. $Je_3 = \sqrt{-1}e_3$. We have a contradiction.

Sol^4-i-7 : W_4 has a basis $\{e_i\}$ where $Re_1 + Re_2 + Re_3 = R^3$ and

$$e_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let $0 \neq X = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$. It's easy to see that adX has rank 3 if $a_4 \neq 0$ and rank 1 otherwise. If this is type (a), then adX and $adJX$ must have the same rank, i.e. J has an 1-dimensional invariant subspace. Since $J^2 = -1$, this is impossable. Assume that this is of type (b). Since $[\mathcal{G},\mathcal{G}] = R^3$, we must have $V = a_1e_1 + a_2e_2 + a_3e_3$. Let $U = be_4 + U', U' \in Ce_1 + Ce_2 + Ce_3$. Then $[U, V] = V$ leads to

$$ba_1(e_1 + e_2) + ba_2(e_2 + e_3) + ba_3e_3 = a_1e_1 + a_2e_2 + a_3e_3,$$

i.e.

$$\begin{cases} (b-1)a_1 = 0 \\ ba_1 + (b-1)a_2 = 0 \\ ba_2 + (b-1)a_3 = 0. \end{cases}$$

If $b \neq 1$, we have $a_1 = a_2 = a_3 = 0$. If $b = 1$, then $a_1 = a_2 = 0$, i.e. $V \in Ce_3$, we must have $Je_3 = \sqrt{-1}e_3$. Again we have a contradiction.

Table 1. Type Red^2

No	$(\mathcal{G},\mathcal{H})$	(G,H)	topology of G/H	metric signature $(p,q), p \geq q$	UM	IAC	IC	Bdd
1	$(so(1,2), so(1,1))$	$(SL_2(R), A)^*$	R^2	(1,1)	✓	-	-	
2	$(so(1,2), so(2))$	$(SO_0(1,2), SO(2))$	R^2	(2,0)	✓	✓	✓	✓
3	$(so(3), so(2))$	$(SO(3), SO(2))$	S^2	(2,0)	✓	✓	✓	✓

$(*)$: $SL_2(R) = KAN$ is the Iwasawa decomposation.

Table 2. Type Red^3

No	$(\mathcal{G},\mathcal{H})$	(G,H)	topology of G/H	metric signature $(p,q), p\geq q$	UM	ICT	Bdd
i - 1	$(sl_2(R),0)$	$(\widetilde{SL}_2(R),\{e\})$	R^3		✓	✓	✓
i - 2	$(su(2),0)$	$(SU(2),\{e\})$	S^3		✓	✓	✓
ii-1*	$(gl_2(R),R\begin{pmatrix}1&0\\0&0\end{pmatrix})$	$R_+\times\widetilde{SL}_2(R), H\simeq R)$	R^3	(2,1)	✓	✓	✓
ii-2**	$(gl_2(R),R\begin{pmatrix}1&-1\\1&1\end{pmatrix})$	$(R_+\times\widetilde{SL}_2(R)/Z^*,$ $H\simeq SO(2)/\{\pm Id_2\})$	R^3	(3,0) (2,1)	✓	✓	✓
ii-3	$(u(2),R\begin{pmatrix}\sqrt{-1}&0\\0&0\end{pmatrix})$	$(U(2),\{\begin{pmatrix}e^{i\theta}&0\\0&1\end{pmatrix};\theta\in R\})$	S^3	(3,0) (2,1)	✓	✓	✓
iii - 1	$(so(4),so(3))$	$(SO(4),SO(3))$	S^3	(3,0)	✓	-	✓
iii-2***	$(so(2,2),so(2,1))$	$(\{\widetilde{SL}_2(R)\times\widetilde{SL}_2(R)\}/Z^{**},$ $H\simeq PSL_2(R))$	R^3	(2,1)	✓	-	✓
iii - 3	$(so(3,1),so(2,1))$	$(SO_o(3,1),SO_o(2,1))$	R^3	(2,1)	✓	-	-
iii - 4	$(so(1,3),so(3))$	$(SO_o(1,3),SO(3))$	R^3	(3,0)	✓	-	✓

$*$: $H=\{(e^t,exp(t\begin{pmatrix}1&0\\0&-1\end{pmatrix}));t\in R\}$, where

$\{exp(t\begin{pmatrix}1&0\\0&-1\end{pmatrix};t\in R\}=A$,

$\widetilde{SL}_2(R)=KAN$ is the Iwasawa decomposation.

$**$: Let $\widetilde{SL}_2(R)=KAN$ as in $*$,

$K=\{exp(t\begin{pmatrix}0&-1\\1&0\end{pmatrix});t\in R\}$,

$R_+=\{e^t;t\in R\}$,

$Z^*=\{(e^t,exp(t\begin{pmatrix}0&-1\\1&0\end{pmatrix}));exp(t\begin{pmatrix}0&-1\\1&0\end{pmatrix})\in Z(SL_2(R))\}$,

$H=\{(e^t,exp(t\begin{pmatrix}0&-1\\1&0\end{pmatrix});t\in R\}/Z^*$.

$***$: $\widetilde{SL}_2(R)^*=\{(X,X);X\in\widetilde{SL}_2(R)\}, Z^{**}=\{(X,X);X\in Z(\widetilde{SL}_2(R))\}, H=\widetilde{SL}_2(R)^*/Z^{**}$.

Table 4. Type Red^4

No	$(\mathfrak{g},\mathfrak{h})$	(G,H)	topology of G/H	metric signature $(p,q), p\geq q$	*UM*	*IAC*	*IC*	*IAS*	*IS*	*Bdd*
i-1	$(\mathfrak{so}(5),\mathfrak{so}(4))$	$(SO(5),SO(4))$	S^4	$(4,0)$	✓	-	-	-	-	✓
i-2	$(\mathfrak{so}(1,4),\mathfrak{so}(4))$	$(SO_0(1,4),SO(4))$	R^4	$(4,0)$	✓	-	-	-	-	✓
i-3	$(\mathfrak{so}(4,1),\mathfrak{so}(3,1))$	$(SO_0(4,1),SO_0(3,1))$	$S^3\times R$	$(3,1)$	✓	-	-	-	-	-
i-4	$(\mathfrak{so}(2,3),\mathfrak{so}(1,3))$	$(\breve{SO}_0(2,3)/Z_2,$ $\breve{SO}_0(1,3)/Z_2)$	R^4	$(3,1)$	✓	-	-	-	-	-
i-5	$(\mathfrak{so}(2,3),\mathfrak{so}(2,2))$	$(SO_0(2,3),SO_0(2,2))$	$S^2\times R^2$	$(2,2)$	✓	-	-	-	-	-
ii-1	$(\mathfrak{sl}_3(R),$ $\begin{pmatrix}\mathfrak{sl}_2(R) & 0\\ 0 & 0\end{pmatrix}+R\begin{pmatrix}1 & & \\ & 1 & \\ & & -2\end{pmatrix})$	$(SL_3(R),$ $S(GL_2^+(R)\times R^*)$	$T(S^2)$	$(2,2)$	✓	-	-	✓	✓	-
ii-2	$(\mathfrak{su}(3),$ $\begin{pmatrix}\mathfrak{su}(2) & 0\\ 0 & 0\end{pmatrix}+R\begin{pmatrix}\sqrt{-1} & & \\ & \sqrt{-1} & \\ & & -2\sqrt{-1}\end{pmatrix})$	$(PSU(3),$ $S(U(2)\times U(1))/Z_3)$	CP^2	$(4,0)$	✓	✓	✓	✓	✓	✓
ii-3	$(\mathfrak{su}(2,1),$ $\begin{pmatrix}\mathfrak{su}(2) & 0\\ 0 & 0\end{pmatrix}+R\begin{pmatrix}\sqrt{-1} & & \\ & \sqrt{-1} & \\ & & -2\sqrt{-1}\end{pmatrix})$	$(PSU(2,1),$ $S(U(2)\times U(1))/Z_3)$	R^4	$(4,0)$	✓	✓	✓	✓	✓	✓
ii-4	$(\mathfrak{su}(2,1),$ $\begin{pmatrix}0 & 0\\ 0 & \mathfrak{su}(1,1)\end{pmatrix}+R\begin{pmatrix}-2\sqrt{-1} & & \\ & \sqrt{-1} & \\ & & \sqrt{-1}\end{pmatrix})$	$(PSU(2,1),$ $S(U(1)\times U(1,1))/Z_3)$	$CP^2 - D^4$	$(2,2)$	✓	✓	✓	✓	✓	-
iii	$(\mathfrak{sl}_2(C),C\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix})$	$(PSL_2(C),C^*/\{\pm 1\})$	$T(S^2)$	$(2,2)$	✓	✓	✓	✓	✓	-

Table 4. Type Mix^2

$(\mathcal{G}, \mathcal{H})=(R^2 \rtimes_\sigma \mathcal{H}, \mathcal{H})$

No	$\sigma(\mathcal{H})$	topology of G/H	metric signature $(p,q), p\geq q$	UM	IAC	IC	Bdd
1	$sl_2(R)$	R^2	-	✓	-	-	✓
2	$gl_2(R)$	R^2	-	-	-	-	✓

Table 5. Type Mix^3-i and -iii

i : $(\mathcal{G}, \mathcal{H})=(R^3 \rtimes_\sigma \mathcal{H}, \mathcal{H})$; iii : $(\mathcal{G}, \mathcal{H})=(N_3 \rtimes_\sigma \mathcal{H}, \mathcal{H})^*$

No	$\sigma(\mathcal{H})$	topology of G/H	metric signature $(p,q), p\geq q$	UM	ICT	Bdd
i-1	$so(3)$	R^3	(3,0)	✓	-	✓
i-2	$sl_3(R)$	R^3	-	✓	-	✓
i-3	$gl_3(R)$	R^3	-	-	-	✓
i-4	$\begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix} + R\begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix}, t\neq 0$	R^3	-	iff $t=-2$	-	✓
i-5	$so(3)+R(Id_3)$	R^3	-	-	-	✓
i-6	$so(2,1)$	R^3	(2,1)	✓	-	✓
i-7	$so(2,1)+R(Id_3)$	R^3	-	-	-	✓
iii-1	$\begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix}$	R^3	-	✓	✓	✓
iii-2	$\begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix} + R\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$	R^3	-	-	-	✓

$*$: $\sigma(\mathcal{H})$ is w.r.t. the basis $\{e_1, e_2, e_3\}$ s.t. $[e_1.e_2]=e_3 \in Z(N_3)$.

Table 6. Type Mix^3-ii

$(\mathcal{G}, \mathcal{H})=(R^2 \rtimes_\sigma \mathcal{H}_o, \mathcal{H})$

No	$\sigma(\mathcal{H})$	$\sigma(\mathcal{H}_o$	topology of G/H	metric signature $(p,q), p\geq q$	UM	ICT	Bdd
1	$sl_2(R)$	$gl_2(R)$	R^3	-	✓	-	-

Table 7. Type Mix^4-i

$(\mathcal{G}, \mathcal{H}) = (R^4 \times_\sigma \mathcal{H}, \mathcal{H})$; every geometry is *Bdd* with topology R^4

No	$\sigma(\mathcal{H})$	metric signature $(p,q), p \geq q$	*UM*	*IAC*	*IC*	*IAS*	*IS*
1	$gl_4(R)$	-	-	-	-	-	-
2	$sl_4(R)$	-	✓	-	-	-	-
3	$sp_2(R)$	-	✓	-	-	✓	✓
4	$sp_2(R) + R(Id_4)$	-	-	-	-	-	-
5	$\begin{pmatrix} sl_3(R) & 0 \\ 0 & 0 \end{pmatrix} + R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t \end{pmatrix}, t \neq 0$	-	iff $t = 3$	-	-	-	-
6	$so(4)$	(4,0)	✓	-	-	-	-
7	$so(4) + R(Id_4)$	-	-	-	-	-	-
8	$so(2,2)$	(2,2)	✓	-	-	-	-
9	$so(2,2) + R(Id_4)$	-	-	-	-	-	-
10	$\begin{pmatrix} sl_2(R) & 0 \\ 0 & sl_2(R) \end{pmatrix} + R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & t & \\ & & & t \end{pmatrix}, t \neq 0$	-	iff $t = -1$	-	-	-	-
11	$\sigma_4(sl_2(R))$	-	✓	-	-	-	-
12	$\sigma_4(sl_2(R)) + R(Id_4)$	-	-	-	-	-	-
13	$\begin{pmatrix} so(3) & 0 \\ 0 & 0 \end{pmatrix} + R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t \end{pmatrix}, t \neq 0$	-	iff $t = -3$	-	-	-	-
14	$\begin{pmatrix} \sigma_3(sl_2(R)) & 0 \\ 0 & 0 \end{pmatrix} + R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t \end{pmatrix}, t \neq 0$	-	iff $t = -3$	-	-	-	-
15	$(*)$	(2,2)	✓	✓	✓	✓	✓
16	$(*) + R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & t & \\ & & & t \end{pmatrix}$	(2,2) iff $t = -1$	iff $t = -1$	iff $t = 1$	iff $t = 1$	iff $t = -1$	iff $t = -1$
17	$(*) + R\begin{pmatrix} 0 & & 1 & \\ & 0 & & 1 \\ -1 & & 0 & \\ & -1 & & 0 \end{pmatrix}$	(2,2)	✓	✓	✓	✓	✓
18	$(*) + R\begin{pmatrix} 1 & & t & \\ & 1 & & t \\ -t & & 1 & \\ & -t & & 1 \end{pmatrix}, t \neq 0$	-	-	✓	✓	-	-

(continued on next page)

Table 7. (from previous page)

No	$g(\mathcal{H})$	metric signature $(p,q), p \geq q$	UM	IAC	IC	IAS	IS
19	$(**) + R\begin{pmatrix} t_1 & & & \\ & t_1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} + R\begin{pmatrix} t_2 & & & \\ & t_2 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, t_1 t_2 \neq 0$	-	iff $t_1 = -\frac{1}{2}$ $t_2 = -\frac{1}{2}$	-	-	-	-
20	$(**) + \begin{pmatrix} t_1 & & & \\ & t_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + R\begin{pmatrix} t_2 & & & \\ & t_2 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}, (t_1, t_2) \neq (0,0)$	-	iff $t_1 = -1$ $t_2 = 0$	-	-	-	-
21	$(**) + \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & t_1 & \\ & & & t_2 \end{pmatrix}, t_1 t_2 \neq 0$	-	iff $t_1 + t_2$ $= 0$	-	-	-	-
22	$so(3,1)$	$(3,1)$	✓	-	-	-	-
23	$so(3,1) + R(Id_4)$	-	-	-	-	-	-
24	$(gl_2(C))_R$	-	-	✓	✓	-	-
25	$(sl_2(C))_R$	-	✓	✓	✓	✓	✓
26	$(sl_2(C))_R + R(Id_4)$	-	-	✓	✓	-	-
27	$\rho_4(su(2))^{***}$	$(4,0)$	✓	✓	✓	✓	✓
28	$\rho_4(su(2)) + R(Id_4)^{***}$	-	-	✓	✓	-	-

$$(*) = R\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} + R\begin{pmatrix} & 0 & & 0 \\ 1 & & 0 & \\ & 0 & & 0 \\ 0 & & 1 & \end{pmatrix} + R\begin{pmatrix} & 1 & & 0 \\ 0 & & 0 & \\ & 0 & & 1 \\ 0 & & 0 & \end{pmatrix};$$

$$(**) = \begin{pmatrix} sl_2(R) & & \\ & 0 & \\ & & 0 \end{pmatrix};$$

$***$: ρ_4 is the 4-dimensional irreducible real representation of $sp_2(R)$.

Table 8.Type Mix^4-ii,iii and iv

$$(\mathcal{G},\,\mathcal{H}) = \begin{cases} (R^3 \times_\sigma \mathcal{H}_\circ, \mathcal{H}), \mathcal{H} \subseteq \mathcal{H}_\circ & \text{in Type ii,} \\ (R^2 \times_\sigma \mathcal{H}_\circ, \mathcal{H}), \mathcal{H} \subseteq \mathcal{H}_\circ & \text{in Type iii,} \\ ((\mathrm{N}_3 \oplus R) \times_\sigma \mathcal{H}, \mathcal{H}) & \text{in Type iv.} \end{cases}$$

No	$\sigma(\mathcal{H})$	$\sigma(\mathcal{H}_\circ)$	metric signature $(p,q), p \geq q$	UM	IAC	IC	IAS	IS	Bdd
ii-1	$sl_3(R)$	$gl_3(R)$	-	✓	-	-	-	-	-
ii-2	$so(3)$	$\sigma(\mathcal{H})+(Id_3)$	(4,0) or(3,1)	✓	-	—	-	-	-
ii-3	$\sigma_3(sl_2(R))$	$\sigma(\mathcal{H})+R(Id_3)$	(3,1) or(2,2)	✓	-	-	-	-	-
ii-4	$\begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix} + R\begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix}, t \neq 0$	$\sigma(\mathcal{H})+R\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$	-	iff $t=-2$	-	-	-	-	iff $t \neq -2$
ii-5	$\begin{pmatrix} sl_2(R) & 0 \\ 0 & 0 \end{pmatrix}$	$\sigma(\mathcal{H})+R\begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix}, t \neq 0$	-	✓	-	-	✓	-	iff $t=-2$
iii-1	$R\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	$sl_2(R)$	(2,2)	✓	-	-	✓	✓	
iii-2	$R\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$sl_2(R)$	(4,0)	✓	✓	✓	✓	✓	-
iii-3	$\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a,b \in R\}$	$gl_2(R)$	-	-	-	-	-	-	
iii-4	$\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a,b \in R\}$	$gl_2(R)$	-	-	✓	✓	-	-	
iv	$\begin{pmatrix} sl_2(R) & & \\ & 0 & \\ & & 0 \end{pmatrix} + R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & t \end{pmatrix}, t \neq 0$	-	-	iff $t=-4$	-	-	-	-	✓

$*$: W.r.t. the basis $\{e_i\}_{1 \leq i \leq 4}$ s.t. $[e_1, e_2] = e_3, [e_i, e_j] = 0$ for $\{i,j\} \neq \{1,2\}$.

Table 9. Type Mix^4-v

$$(\mathcal{G},\,\mathcal{H}) = (l \oplus aff(1,R),\,\mathcal{H})$$

No	l	$\mathcal{H}$	topology of G/H	metric signature $(p,q), p \geq q$	UM	IAC	IC	IAS	IS	Bdd
1	$sl_2(R)$	$R(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_1)^{**}$	R^4	-	-	-	-	-	-	✓
2	$sl_2(R)$	$R(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_1)^{**}$	R^4	-	-	-	-	-	-	✓
3	$su(2)$	$R(\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, e_1)^{**}$	$S^3 \times R$	-	-	-	-	-	-	✓

$**$: $aff(1,R) = Re_1 + Re_2$ s.t. $[e_1, e_2] = e_2$.

Table 10. Type Sol^1

No	$(\mathcal{G},\mathcal{H})$	$\sigma(\mathcal{H})$	topology of G/H	metric signature $(p,q), p \geq q$	UM	Bdd
i	$(R,0)$	-	R	(1,0)	✓	✓
ii	$(R \times_\sigma \mathcal{H}, \mathcal{H})$	R	R	-	-	✓

Table 11. Type Sol^2

No	$(\mathcal{G}, \mathcal{H})$	$\sigma(\mathcal{H})$	topology of G/H	metric signature $(p,q), p\geq q$	UM	IAC	IC	Bdd
i	$(aff(1,R),0)$	-	R^2		-	-	-	-
ii-1	$(R^2 \rtimes_\sigma \mathcal{H}, \mathcal{H})$	$R\begin{pmatrix}1 & \\ 0 & t\end{pmatrix}, t\neq 0$	R^2	(1,1) iff $t=-1$	iff $t=-1$	iff $t=1$	iff $t=1$	✓
ii-2	$(R^2 \rtimes_\sigma \mathcal{H}, \mathcal{H})$	$R\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$	R^2	(2,0)	✓	✓	✓	✓
ii-3	$(R^2 \rtimes_\sigma \mathcal{H}, \mathcal{H})$	$R\begin{pmatrix}1 & -t \\ t & 1\end{pmatrix}, t\neq 0$	R^2	-	-	✓	✓	✓
ii-3	$(R^2 \rtimes_\sigma \mathcal{H}, \mathcal{H})$	$\{\begin{pmatrix}a & -b \\ b & a\end{pmatrix}; a,b\in R\}$	R^2	-	-	✓	✓	✓

Table 12. Type Sol^3-i

No	$(\mathcal{G}, \mathcal{H})$	$\mathcal{G}$	topology of G/H	metric signature $(p,q), p\geq q$	UM	ICT	Bdd
1	$(\mathcal{G}, 0)$	N_3	R^3		✓	✓	✓
2	$(\mathcal{G}, 0)$	$S_{3,\{a,b\}}{}^{(*)}$	R^3		iff $t=-1$	iff $t\neq 1$	iff $t=-1$
3	$(\mathcal{G}, 0)$	$S_{3,\{\theta\}}{}^{(**)}$	R^3		iff $\theta=(k+\frac{1}{2})\pi$	✓	iff $\theta=(k+\frac{1}{2})\pi$
4	$(\mathcal{G}, 0)$	$S_{3,\bigcirc}{}^{(***)}$	R^3		-	✓	-

$$(*) : S_{3,\{a,b\}} = R^2 \rtimes_A R, A = \begin{pmatrix}1 & \\ & t\end{pmatrix}, t\neq 0;$$

$$(**) : S_{3,\{\theta\}} = R^2 \rtimes_A R, A = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix};$$

$$(***) : S_{3,\bigcirc} = R^3 \rtimes_A R, A = \begin{pmatrix}1 & 0 \\ 1 & 1\end{pmatrix}.$$

Table 13. Type Sol^3-ii and iii

$$(\mathcal{G},\, \mathcal{H}) = \begin{cases} (R^3 \rtimes_\sigma \mathcal{H}, \mathcal{H}) & \text{in Type-ii,} \\ (\mathrm{N}_3 \rtimes_\sigma \mathcal{H}, \mathcal{H})^{(*)} & \text{in Type-iii.} \end{cases}$$

No	$\sigma(\mathcal{H})$	topology of G/H	metric signature $(p,q), p\geq q$	UM	ICT	Bdd
ii-1	$R\begin{pmatrix} 1 & & \\ & t_1 & \\ & & t_2 \end{pmatrix}, t_1t_2\neq 0$	R^3	-	iff $t_1+t_2=-1$	-	✓
ii-2	$R\begin{pmatrix} 1 & & \\ & 0 & -t \\ & t & 0 \end{pmatrix}, t\neq 0$	R^3	-	-	-	✓
ii-3	$R\begin{pmatrix} 1 & & \\ & t_1 & -t_2 \\ & t_2 & t_1 \end{pmatrix}, t_1t_2\neq 0$	R^3	-	iff $t_1=-\frac{1}{2}$	-	✓
ii-4	$\left\{\begin{pmatrix} a & & \\ & b & \\ & & t_1a+t_2b \end{pmatrix}; a,b\in R\right\}, t_1t_2\neq 0$	R^3	-	iff $t_1=t_2=-1$	-	✓
ii-5	$\left\{\begin{pmatrix} t_1a+t_2b & & \\ & a & -b \\ & b & a \end{pmatrix}; a,b\in R\right\}, (t_1,t_2)\neq(0,0)$	R^3	-	iff $t_1=-2$ $t_2=0$	-	✓
iii-1	$R\begin{pmatrix} 1 & & \\ & t & \\ & & 1+t \end{pmatrix}$	R^3	$(2,1)$ iff $t=-1$	iff $t=-1$	iff $t=-1$	✓
iii-2	$R\begin{pmatrix} 1 & -t & \\ t & 1 & \\ & & 2 \end{pmatrix}$	R^3	-	-	-	✓
iii-3	$R\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$	R^3	$(3,0)$ or $(2,1)$	✓	✓	✓
ii-4	$\left\{\begin{pmatrix} a & & \\ & b & \\ & & a+b \end{pmatrix}; a,b\in R\right\}$	R^3	-	-	-	✓
ii-5	$\left\{\begin{pmatrix} a & -b & \\ b & a & \\ & & 2a \end{pmatrix}; a,b\in R\right\}$	R^3	-	-	-	✓

$(*)$: $\sigma(\mathcal{H})$ is w.r.t. the basis $\{e_1, e_2, e_3\}$ s.t. $[e_1, e_2] = e_3 \in Z(\mathrm{N}_3)$.

Table 14. Type Sol^3-iv

$$(\mathcal{G},\, \mathcal{H}) = (S_{3,A} \rtimes \mathcal{H}, \mathcal{H}) = (R^2 \rtimes_\sigma (RA + \mathcal{H}), \mathcal{H})$$

No	$\sigma(\mathcal{H})$	$\sigma(A)$	topology of G/H	metric signature $(p,q), p\geq q$	UM	ICT	Bdd
1	$R\begin{pmatrix} 1 & \\ & t \end{pmatrix}, t\neq 0$	$\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$	R^3	$(2,1)$ iff $t=-1$	iff $t=-1$	-	iff $t\neq -1$
2	$R(Id_2)$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	R^3	-	-	-	✓
3	$R\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Id_2	R^3	$(3,0)$ or $(2,1)$	✓	-	-
4	$R\begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}, t\neq 0$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	R^3	-	-	-	✓

Table 15. Type Sol^4-i

$$(\mathcal{G}, \mathcal{H}) = (\mathcal{G}, 0)$$

The *topology* of every geometry is R^4

No	$\mathcal{G}$	UM	IC	IS	Bdd
1	N_4	✓	-	✓	✓
2	$R^2\times B, B = \{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}; a, b\in R\}$	-	✓	✓	-
3	$S_{4,(a,b,c)} = R^3\times R\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}, abc\neq 0$	iff $a+b+c = 0$	iff $a=b$ or $b=c$ or $a=c$	iff $a+b=0$ or $b+c=0$ or $a+c=0$	(**)
4	$S_{4,(\theta,a)} = R^2\times R\begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & a \end{pmatrix}, \begin{matrix} a\neq 0 \\ \theta\neq k\pi \end{matrix}$	iff $a = -2\cos\theta$	-	iff $\theta = (k+\frac{1}{2})\pi$	-
5	$S_{4,(\bigcirc,a)} = R^3\times R\begin{pmatrix} 1 & 0 & \\ 1 & 1 & \\ & & a \end{pmatrix}, a\neq 0$	iff $a = -2$	iff $a = 1$	iff $a = -1$	-
6	$V_4 = R^3\times R\begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$	-	-	✓	-
7	$W_4 = R^3\times R\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}$	-	-	-	-
8	$T_{4,(1,a)} = N_3\times R\begin{pmatrix} 1 & & \\ & a & \\ & & 1+a \end{pmatrix}^{(*)}, a\neq 0$	iff $a = -1$	✓	iff $a\neq -1$	iff $a = -1$
9	$T_{4,(\theta)} = N_3\times R\begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 2\cos\theta \end{pmatrix}^{(*)}, \theta\neq k\pi$	iff $\theta = (k+\frac{1}{2})\pi$	-	iff $\theta\neq(k+\frac{1}{2})\pi$	-
10	$T_{4,\bigcirc} = N_3\times R\begin{pmatrix} 1 & 0 & \\ 1 & 1 & \\ & & 2 \end{pmatrix}^{(*)}$	-	✓	-	-
11	$aff(1,C)_R = R^2\times\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}; a, b\in R\}$	-	✓	✓	-

(*): w.r.t. the basis $\{e_1, e_2, e_3\}$ s.t. $[e_1, e_2] = e_3 \in Z(N_3)$;

(**) : The geometry is Bdd iff $R\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = R\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$ or $a > b > c$ and $\{e^a, e^b, e^c\}$ are the roots of $\lambda^3 - m\lambda^2 + n\lambda = 0$ with m and n integers, $m \neq n$.

Table 16. Type Sol^4-ii and iii

$$(\mathcal{G}, \mathcal{H}) = \begin{cases} (R^4 \times_\sigma \mathcal{H}, \mathcal{H}), & \text{in Type ii,} \\ (\mathrm{N}_4 \times_\sigma \mathcal{H}, \mathcal{H}), & \text{in Type iii.} \end{cases}$$

Every geometry is *Bdd* with *topology* R^4.

No	$\sigma(\mathcal{H})$	metric signature $(p,q), p \geq q$	*UM*	*IAC*	*IC*	*IAS*	*IS*
ii-1	$R\begin{pmatrix} 1 & & & \\ & t_1 & & \\ & & t_2 & \\ & & & t_3 \end{pmatrix}, t_1t_2t_3 \neq 0$	$(2,2)$ iff $t_1 = -1$ $t_2 + t_3 = 0$	iff $t_1 + t_2 + t_3$ $= -1$	iff $t_1 = 1$ $t_2 = t_3$	iff $t_1 = 1$ $t_2 = t_3$	iff $t_1 = -1$ $t_2 = -t_3$	iff $t_1 = -1$ $t_2 = -t_3$
ii-2	$R\begin{pmatrix} 1 & & & \\ & t_1 & & \\ & & 0 & -t_2 \\ & & t_2 & 0 \end{pmatrix}, t_1t_2 \neq 0$	$(3,1)$ or $(2,2)$ iff $t_1 = -1$	iff $t_1 = -1$	iff $t_1 = 1$	iff $t_1 = 1$	iff $t_1 = -1$	iff $t_1 = -1$
ii-3	$R\begin{pmatrix} 1 & & & \\ & t_1 & & \\ & & t_2 & -t_3 \\ & & t_3 & t_2 \end{pmatrix}, t_1t_2t_3 \neq 0$	-	iff $t_1 + 2t_2$ $= -1$	iff $t_1 = 1$	iff $t_1 = 1$	-	-
ii-4	$R\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -t \\ & & t & 0 \end{pmatrix}^{(*)}, t \neq 0$	$(3,1)$ or $(2,2)$ or $(4,0)$	✓	✓	✓	✓	✓
ii-5	$R\begin{pmatrix} 1 & -t_1 & & \\ t_1 & 1 & & \\ & & t_2 & -t_3 \\ & & t_3 & t_2 \end{pmatrix}, t_1t_2t_3 \neq 0$	$(2,2)$ iff $t_2 = -1$ $t_1 = t_3$	iff $t_2 = -1$	✓	✓	iff $t_2 = -1$ $t_3 = \pm t_1$	iff $t_2 = -1$ $t_3 = \pm t_1$
ii-6	$R\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & t_1 & -t_2 \\ & & t_2 & t_1 \end{pmatrix}, t_1t_2 \neq 0$	-	-	✓	✓	-	-
ii-7	$\left\{\begin{pmatrix} a & & & \\ & b & & \\ & & t_1a + t_2b & \\ & & & t_3a + t_4b \end{pmatrix}; \begin{matrix} t_1t_2 \neq 0 \\ (t_3, t_4) \neq (0,0) \\ a, b \in R \end{matrix}\right\}$	-	iff $t_1 + t_2$ $= t_3 + t_4$ $= -1$	-	-	-	-
ii-8	$R\begin{pmatrix} 1 & & & \\ & t_1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + R\begin{pmatrix} 0 & & & \\ & t_2 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, t_1t_2 \neq 0$	-	-	-	-	-	-
ii-9	$R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & t_1 & -t_2 \\ & & t_2 & t_1 \end{pmatrix} + R\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & t_3 & -t_4 \\ & & t_4 & t_3 \end{pmatrix}^{(*)}, \begin{matrix} (t_1, t_2) \neq (0,0) \\ (t_3, t_4) \neq (0,0) \end{matrix}$	$(2,2)$ iff $t_1 = -1$ $t_2 = 0$ $t_3 = 0$ $t_4 = 1$	iff $t_1 = -1$ $t_3 = 0$	✓	✓	-	-
ii-10	$R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & -t_1 \\ & & t_1 & 0 \end{pmatrix} + R\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -t_2 \\ & & t_2 & 0 \end{pmatrix}^{(*)}, (t_1, t_2) \neq (0,0)$	-	-	✓	✓	-	-
ii-11	$\left\{\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & t_1a + t_2b + t_3c \end{pmatrix}; \begin{matrix} a, b, c \in R \\ t_1t_2t_3 \neq 0 \end{matrix}\right\}$	-	iff $t_1 = t_2$ $= t_3 = -1$	-	-	-	-
ii-12	$R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & t_1 \end{pmatrix} + R\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & t_2 \end{pmatrix} + R\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & t_3 \end{pmatrix}, \begin{matrix} t_3 \neq 0 \\ (t_1, t_2) \\ \neq (0,0) \end{matrix}$	-	iff $t_1 = -2$ $t_2 = 0$ $t_3 = -1$	-	-	-	-
ii-13	$R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & t_1 & \\ & & & t_1 \end{pmatrix} + R\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & t_2 & \\ & & & t_2 \end{pmatrix} + R\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & t_3 & -1 \\ & & 1 & t_3 \end{pmatrix}^{(*)}, \begin{matrix} (t_1, t_3) \\ \neq (0,0) \end{matrix}$	-	iff $t_1 = -1$ $t_2 = 0$ $t_3 = 0$	✓	✓	-	-
iii-1	$R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 2t - 1 & \\ & & & 1 - t \end{pmatrix}^{(**)}, t \neq 1, t \neq 2$	-	$t = -\frac{1}{2}$	-	-	-	-
iii-2	$R\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & -1 \end{pmatrix}^{(**)}$	-	-	-	-	-	-
iii-3	$R\begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} + R\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}^{(**)}$	-	-	-	-	-	-

(*): See (5.1) to find when (G, H) is $Ad - c.r.$;

(**): σ is w.r.t. the basis $\{e_1, e_2, e_3, e_4\}$ s.t. $[e_4, e_3] = e_2, [e_4, e_2] = e_1 \in Z(\mathrm{N}_4)$.

Table 17. Type Sol^4-iv

$(\mathcal{G}, \mathcal{H}) = ((\mathrm{N}_3 \oplus R) \times_\sigma \mathcal{H}, \mathcal{H})$

Every geometry is *Bdd* with *topology* R^4

No	$\sigma(\mathcal{H})^{(*)}$	metric signature $(p,q), p \geq q$	*UM*	*IAC*	*IC*	*IAS*	*IS*
1	$R\begin{pmatrix} t_1 & & & \\ & 1 & & \\ & & t_2 & \\ & & & 1-t_2 \end{pmatrix}, t_1 t_2 \neq 0, t_2 \neq 1$	(2,2) iff $t_1 = -2$ $t_1 = -1$ or 2	iff $t_1 = -2$	iff $t_1 = 1$ $t_2 = \frac{1}{2}$	iff $t_1 = 1$ $t_2 = \frac{1}{2}$	iff $t_1 = -2$ $t_2 = 2$ or -1	iff $t_1 = -2$ $t_2 = 2$ or -1
2	$R\begin{pmatrix} 1 & & & \\ & 2t_1 & & \\ & & t_1 & -t_2 \\ & & t_2 & t_1 \end{pmatrix}, t_1 t_2 \neq 0$	-	iff $t_1 = -\frac{1}{2}$	iff $t_1 = \frac{1}{2}$	iff $t_1 = \frac{1}{2}$	-	-
3	$\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & t & \\ & & & -t \end{pmatrix}, t \neq 0$	-	-	-	-	-	-
4	$\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & -t \\ & & t & 0 \end{pmatrix}, t \neq 0$	-	-	-	-	-	-
5	$R\begin{pmatrix} t_1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} + R\begin{pmatrix} t_2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \begin{matrix}(t_1, t_2) \\ \neq (0,0)\end{matrix}$	-	iff $t_1 = -2$ $t_2 = -2$	-	-	-	-
6	$R\begin{pmatrix} t_1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} t_2 & & & \\ & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, \begin{matrix}(t_1, t_2) \\ \neq (0,0)\end{matrix}$	-	iff $t_1 = -4$ $t_2 = 0$	iff $t_1 = 2$ $t_2 = 0$	iff $t_1 = 2$ $t_2 = 0$	-	-

(*): w.r.t. the basis $\{e_i\}_{1 \leq i \leq 4}$, s.t. $e_1 \in R, [e_3, e_4] = e_2 \in Z(\mathrm{N}_3)$.

Table 18. Type Sol^4-v

$$(\mathcal{G}, \mathcal{H}) = (\mathrm{S}_{4,A} \times \mathcal{H}, \mathcal{H}) = (R_3 \times_\sigma \mathcal{H}_\circ, \mathcal{H}), \mathcal{H}_\circ = \mathrm{RA} + \mathcal{H}$$

The *topology* of every geometry is R^4

No	$\sigma(A)$	$\sigma(\mathcal{H})$	metric signature $(p,q), p\geq q$	*UM*	*IAC*	*IC*	*IAS*	*IS*	*Bdd*
1	$\begin{pmatrix} 0 & & \\ & 1 & \\ & & t_3 \end{pmatrix}$	$R\begin{pmatrix} 1 & & \\ & t_1 & \\ & & t_2 \end{pmatrix}, t_2\neq 0$	$(3,1)$ or $(2,2)$ iff $t_1=0$ $t_2=-1$	iff t_1+t_2 $=-1$	iff $t_2=1$ $t_1=0$	iff $t_1=0$ $t_2=1$ $t_3=0$	iff $t_1=0$ $t_2=-1$	iff $t_1=0$ $t_2=-1$ $t_3=0$	$(*)$
2	$\begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$	$R\begin{pmatrix} 1 & & \\ & 0 & -t \\ & t & 0 \end{pmatrix}, t\neq 0$	-	-	-	-	-	-	✓
3	$\begin{pmatrix} t_3 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$R\begin{pmatrix} 1 & & \\ & t_1 & -t_2 \\ & t_2 & t_1 \end{pmatrix}, t_1t_2\neq 0$	-	iff $t_1=-\frac{1}{2}$	-	-	-	-	iff $t_3=-2$ or $t_3=\frac{1}{2}$
4	$\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$	$R\begin{pmatrix} 1 & & \\ & t_1 & -t_2 \\ & t_2 & t_1 \end{pmatrix}, t_1t_2\neq 0$	-	iff $t_1=-\frac{1}{2}$	-	-	-	-	-
5	$\begin{pmatrix} t_3 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$	$R\begin{pmatrix} 1 & & \\ & t_1 & \\ & & t_1 \end{pmatrix}, t_1\neq 0$	-	iff $t_1=-\frac{1}{2}$	-	-	-	-	iff $t_3=0$
6	$\begin{pmatrix} 0 & -t & \\ t & 0 & \\ & & 1 \end{pmatrix}, t\neq 0$	$R\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$	-	-	✓	-	-	-	-
7	$\begin{pmatrix} t & & \\ & t & \\ & & 1 \end{pmatrix}, t\neq 0$	$R\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$	$(4,0)$ or $(2,2)$	✓	✓	✓	✓	-	iff $t=-\frac{1}{2}$
8	$\begin{pmatrix} t_1 & & \\ & t_1 & \\ & & 0 \end{pmatrix}, t_1\neq 0$	$R\begin{pmatrix} 1 & -t_2 & \\ t_2 & 1 & \\ & & 0 \end{pmatrix}, t_2\neq 0$	-	-	✓	✓	-	-	iff $t_1=-\frac{1}{2}$
9	$\begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & t \end{pmatrix}, t\neq 0$	$R\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$	-	-	✓	-	-	-	-
10	$\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$	$R\begin{pmatrix} 1 & & \\ & 0 & \\ & & t_1 \end{pmatrix} + R\begin{pmatrix} 0 & & \\ & 1 & \\ & & t_2 \end{pmatrix}, t_1t_2\neq 0$	-	iff $t_1=-1$ $t_2=-1$	-	-	-	-	iff $t_1\neq -1$ or $t_2\neq -1$
11	$\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$	$\left\{\begin{pmatrix} a & -b & \\ b & a & \\ & & t_1a+t_2b \end{pmatrix}, \begin{matrix} a,b\in R \\ t_1t_2\neq 0 \end{matrix}\right\}$	-	-	-	-	-	-	✓

$(*)$: iff t_1, t_2, t_3 satisfy:

$t_3 = -\lambda_1 - 1$, $t_2 = \lambda_2 - 1 + t_1(-\lambda_1 - 1)$, where $\lambda_1 \neq 0$, $e^{\lambda_1}, e^{\lambda_2}, e^{1-\lambda_1-\lambda_2}$ are solutions of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$ with m, n positive integers, $m \neq n$.

Table 19. Type Sol^4-vi

$$(\mathcal{G}, \mathcal{H}) = (\mathrm{T}_{4,A} \times \mathcal{H}, \mathcal{H}) = (\mathrm{N}_3 \times_\sigma \mathcal{H}_\circ, \mathcal{H}), \mathcal{H}_\circ = \mathrm{R}A + \mathcal{H}$$

The *topology* of every geometry is R^4

No	$\sigma(A)^{(*)}$	$\sigma(\mathcal{H})^{(*)}$	metric signature $(p,q), p \geq q$	*UM*	*IAC*	*IC*	*IAS*	*IS*	*Bdd*
1	$\begin{pmatrix} t & & \\ & -1 & \\ & & 0 \end{pmatrix}$	$R\begin{pmatrix} 1 & & \\ & t & \\ & & 1+t \end{pmatrix}, \begin{matrix} t \neq 0 \\ t \neq -1 \end{matrix}$	-	-	-	-	-	-	✓
2	$\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$R\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$	(2,2) or (3,1)	✓	-	-	-	-	-
3	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$	$R\begin{pmatrix} 1 & -t & \\ t & 1 & \\ & & 2 \end{pmatrix}, t \neq 0$	-	-	-	-	-	-	-
4	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$	$R\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$	(4,0) (2,2) (3,1)	✓	✓	✓	✓	✓	-
5	$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$	$R\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	-	-	✓	✓	-	-	✓

(∗): w.r.t. the basis $\{e_1, e_2, e_3\}$ s.t.$[e_1, e_2] = e_3 \in Z(\mathrm{N}_3)$.

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