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ON THE LINE GRAPHS OF THE COMPLETE MULTIPARTITE GRAPHS

by

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SECTION 1

INTRODUCTION

Let G be a graph. By the line graph of G denoted by $L(G)$, we mean the graph whose points are the lines of G with two points of $L(G)$ adjacent if and only if the corresponding lines of G are incident, (our terminology and definitions are from Harary [6]). We shall be concerned in this thesis with the characterizations of certain families of line graphs.

In the past, two different kind of characterizations have been considered. The first is by properties of the spectrum of the adjacency matrix of the graph (see [4], [5], [8], [9], [10], [11], [12], [13], [15] and [16]). In general, these results were of the following form: each family of graphs was defined by a few parameters, and for all but at most finitely many values of the parameters the line graph was characterized by its spectrum. By way of illustration [12] let π be a (v,k,λ) design. Consider the graph whose points are the totality of blocks and varieties (2v altogether) with two points adjacent if and only if one corresponds to a block, the other to a variety and the variety is contained in the block. Let $G(v,k,\lambda)$ be the graph so defined and $L(v,k,\lambda)$ its line graph is a regular connected graph on vk points whose adjacency matrix has $2k-2$, $k-2 \pm \sqrt{k-\lambda}$ and -2 as its distinct eigenvalues. The result is that any regular connected graph on vk points whose adjacency matrix has the given spectral properties is a $L(v,k,\lambda)$ unless $v = 4$, $k = 3$, $\lambda = 2$, where there is exactly one exception.

The other kind of characterization deals with "incidence" properties of graphs, of which the following is a typical and early example [17].

Let K_{nn} be the complete bipartite graph on $n+n$ points. By this we mean that K_{nn} has $2n$ points partitioned into two sets S_1, S_2 with $|S_1| = |S_2| = n$. Two points are adjacent if and only if they belong to different S's. Then $L(K_{nn})$ has the following properties: it is a regular connected graph of degree $2n-2$ on n^2 points any two non-adjacent points are mutually adjacent to two points, any two adjacent points are mutually adjacent to $n-2$ points. Then $L(K_{nn})$ is characterized by these properties unless $n = 4$, where there is exactly one exception.

Another result is contained in [7] and [14]. Let K_{mn} be a graph whose points are partitioned into two sets S_1 and S_2 with $|S_1| = m$, $|S_2| = n$, $m > n \geq 2$ and two points adjacent if and only if they are in different S's. Then $L(K_{mn})$ satisfies the following conditions: it is a regular connected graph of degree $m + n - 2$ on mn points, any two non-adjacent points are mutually adjacent to two points, $n\binom{m}{2}$ of the pairs of adjacent points are mutually adjacent to $m-2$ other points and the remaining $m\binom{n}{2}$ pairs of adjacent points are mutually adjacent to $n-2$ other points. The result is that $L(K_{mn})$ is characterized by these conditions with no exceptional values of m and n. We shall in §4 establish this result under slightly weaker hypotheses without involving the use of spectral methods as was done in [7] to handle the possible exceptional cases left over from [14].

In [1], [2] and [3], $L(K_n)$ is treated where K_n is the complete graph on n points, i.e., every pair of points in K_n is adjacent. Then $L(K_n)$ satisfies the following properties: it is a regular connected graph with degree $2(n-2)$ on $\binom{n}{2}$ points. Any two non-adjacent points are mutually adjacent to exactly four points. Any two adjacent points are mutually adjacent to exactly $n-2$ points. The result is that $L(K_n)$ is characterized by these properties, unless $n = 8$ where there are exactly three exceptional graphs satisfying these properties.

The main part of this thesis will be concerned with the line graphs of complete multipartite graphs. Let $K_{n,n,n}$ (for $n \geq 2$) be the graph on $3n$ points, partioned into subsets S_1, S_2, S_3 with $|S_i| = n$, $i = 1,2,3$ and two points are adjacent if and only if they are in different S's. Then $L(K_{n,n,n})$ can be easily seen to satisfy the following conditions: it is a regular connected graph with degree $4n-2$ on $3n^2$ points, any two non-adjacent points are mutually adjacent to either 2 or 3 points. Any two adjacent points are mutually adjacent to either $2n-2$ or $2n-1$ points. We shall prove in §2 that for $n \geq 4$, these conditions characterize $L(K_{n,n,n})$. For $n = 2$ and 3, we do not know.

Before stating the other results of the thesis, we make a remark about spectral characterization of $L(K_{n,n,n})$. In an unpublished manuscript with Professor A.J. Hoffman, we have shown that for $n \geq 2$, $L(K_{n,n,n})$ may be characterized by the fact that it is a regular connected graph on $3n^2$ points whose adjacency matrix has $4n-2$, $2n-2$, $n-2$ and -2 as its distinct eigenvalues. This manuscript is included here as a supplement in §5.

Next, let $K_{n,n,n,\ldots,n\ (t\text{-times})}$ $(= K_n(t))$, be a graph on nt points, partitioned into subsets $S_1, S_2, \ldots, S_t$ such that $|S_i| = n$, $i = 1,2,\ldots,t$, $t \geq 4$ and $n \geq 2$ with two points adjacent if and only if they belong to different S's. Then $L(K_n(t))$ can be easily seen to satisfy the following conditions: it is a regular connected graph with degree

$2(t-1)n-2$ with $\binom{t}{2}n^2$ points, any two non-adjacent points are mutually adjacent to either 2, 3 or 4 points, any two adjacent points are mutually adjacent to $(t-1)n-2$ or $(t-1)n-1$ points. We shall prove in §3 that these conditions characterize $L(K_n(t))$, except for the following four cases: $(t = 4; n = 2$ and $3)$ and $(t = 5$ and $6; n = 2)$. In all of these cases, we do not know what the situation is.

SECTION 2

THE CHARACTERIZATION OF $L(K_{n,n,n})$

Theorem (2.1): For an integer $n \geq 2$, a regular connected graph G with the valence $4n-2$, satisfying the following conditions is the line graph of $K_{n,n,n}$ except for $n = 2$ and $n = 3$, if and only if

(1) $|V(G)| = 3n^2$

(2) Any pair of adjacent points in G is mutually adjacent to either $2n-2$ or $2n-1$ distinct points in G,

(3) Any pair of non-adjacent points in G is mutually adjacent to 2 or 3 more points in G.

For $n = 2$ and 3 we do not know what the situation is.

Definition (2.1): A claw in a graph is an induced subgraph with 4 points such that one of the points is adjacent to all the remaining 3 mutually non-adjacent points.

Lemma (2.1): The graph G in Theorem (2.1) does not contain a claw.

Proof: Suppose the contrary and let the graph G have a claw with a point 0 adjacent to the three mutually non-adjacent points 1,2 and 3. Let S_i, $i = 1,2,3$ be the number of points in G adjacent to the points 0 and i, and let S_{ij} $(i \neq j)$, $i = 1,2,3$ and $j = 1,2,3$ be the number of points in G adjacent to the points $0, i$ and j and S_{123} be the number of points in G adjacent to the points 0,1,2 and 3. Clearly by the hypothesis of the Theorem (2.1), we have

$$S_{12} + S_{123} \leq 2, \quad S_{13} + S_{123} \leq 2 \quad \text{and} \quad S_{23} + S_{123} \leq 2$$

which gives $S_{12} + S_{13} + S_{23} + 3S_{123} \geq 6$ and since $S_{123} \geq 0$ therefore we have $S_{12} + S_{13} + S_{23} + 2S_{123} \geq 6$. Again by the conditions of the Theorem (2.1) we have

$$S_1 + S_{12} + S_{13} + S_{123} \leq 2n-2$$

$$S_2 + S_{23} + S_{12} + S_{123} \leq 2n-2$$

$$S_3 + S_{13} + S_{23} + S_{123} \leq 2n-2$$

and by adding we get

$$S_1 + S_2 + S_3 + 2(S_{12} + S_{13} + S_{23}) + 3S_{123} \leq 6n-6 .$$

But we also have $-S_{12} - S_{13} - S_{23} - 2S_{123} \leq -6$. Therefore

$$S_1 + S_2 + S_3 + S_{12} + S_{13} + S_{23} + S_{123} \leq 6n-12 .$$

Since the valence of 0 is 4n-2,

$$S_1 + S_2 + S_3 + S_{12} + S_{13} + S_{23} + S_{123} \geq (4n-2)-3.$$

So we have

$$4n - 5 \leq 6n - 12 , \quad \text{or} \quad n \geq \frac{7}{2} .$$

This implies that the lemma holds for $n \geq 4$.

<u>Definition (2.2)</u>: A line in G is called a <u>(2n-2) line</u>, if both the end points are mutually adjacent to exactly 2n-2 distinct points in G. Similarly we can define a <u>(2n-1) line</u> in G.

<u>Lemma (2.2)</u>: If p is a point in G such that every line on p is a (2n-1) line, then p is common to exactly 2 line disjoint complete subgraphs in G, each equal to K_{2n} .

<u>Proof</u>: Let $A = \{p_1, p_2, \ldots, p_{2n-1} , q_1, q_2, \ldots, q_{2n-1}\}$ be the set of 4n-2 points in G adjacent to p. Let $A_1 = \{p_1, p_2, \ldots, p_{2n-2}, q_{2n-1}\}$ be the subset of 2n-1 points that are adjacent to both p and p_{2n-1} and let $A_2 = \{q_1, q_2, \ldots, q_{2n-2}\}$ be the subset consisting of the points

in G adjacent to p but not to p_{2n-1} .

First we shall prove that two non-adjacent points in A must be mutually adjacent to exactly 2 points in A. Suppose not and let p_1 not be adjacent to q_{2n-1} . By the hypothesis of the Theorem (2.1) p_1 and q_{2n-1} are not mutually adjacent to more than 2 points in A since both are already adjacent to p . So we suppose that p_1 and q_{2n-1} are mutually adjacent to exactly one point, namely p_{2n-1} in A . In this case note that by the hypothesis p and p_1 are mutually adjacent to 2n-1 points in A and p and q_{2n-1} are mutually adjacent to 2n-1 points in A . But other than p_1, q_{2n-1} and p_{2n-1} there are 4n-5 points in A . But on the other hand, we need 4n-4 points in A other than p_{2n-1} to be adjacent to p_1 and q_{2n-1} which is impossible. So any two non-adjacent points in A are mutually adjacent to exactly 2 points in A .

Now since p_{2n-1} is not adjacent to any point in A_2 therefore every point in A_2 must be adjacent to exactly 2 points in A_1 and by the hypothesis each point in A_2 must be adjacent to exactly 2n-3 more points in A and clearly all of these points must be in A_2 therefore this implies that all the points in A_2 are mutually adjacent. This also implies that all the points in A_1 are not mutually adjacent. Without loss of generality let p_1 not be adjacent to q_{2n-1}, since both of these points are already mutually adjacent to p_{2n-1} therefore both are mutually adjacent to one more point in A .

Suppose this point (adjacent to both p_1 and q_{2n-1}) is in A_1, say p_2 . Since each one of p_1 and q_{2n-1} is adjacent to 2n-3 more distinct points in A other than p_2 and p_{2n-1} and there are 2n-2 total number of points in A_2, therefore p_1 is not adjacent to at least one point, say q_1 in A_2 and q_{2n-1} is not adjacent to at least one point in A_2 other than q_1, say q_2. Since all the points in A_2 are mutually adjacent and two non-adjacent points in A are mutually adjacent to exactly 2 points in A therefore each one of p_1 and q_{2n-1} is adjacent to at most 2 distinct points in A_2. But other than p_2 and p_{2n-1} each one of p_1 and q_{2n-1} must be adjacent to exactly $2n-3$ more distinct points in A. Therefore we have

$$(2n-3) + (2n-3) \leq (2n-1)-3 + 4$$

which gives $n \leq 3$. So this is possible only if $n = 2$ and $n = 3$ which are the exceptional cases.

The other possibility is that p_1 and q_{2n-1} are mutually adjacent to one point in A_2 namely q_1. We shall prove that this case is valid only if any one of these two non-adjacent points is adjacent to all the points in A_2 because otherwise suppose that q_{2n-1} is not adjacent to m points in A_2 where $1 \leq m < 2n-3$. In this case note that all the points in A that are not adjacent to q_{2n-1} must be adjacent to p_1 and vice versa, since p_1 and q_{2n-1} are not mutually adjacent to any point in A other than q_1 and p_{2n-1} and other than these two points p_1 and q_{2n-1} are adjacent to $4n-6$ total number of points in A and there are the same number of points in A other than p_1, p_{2n-1}, q_1 and q_{2n-1}. This implies that p_1 is adjacent to the same $m \geq 1$ points in A_2 that are not adjacent to q_{2n-1} and since $m < 2n-3$ therefore p_1 is not adjacent to at least one point in A_2 namely q_2. So we have p_1 not adjacent to q_2 and both are mutually adjacent to $m+1$ points in A_2 because all the points in A_2 are mutually adjacent therefore q_2 is adjacent to all the m points in A_2 that are adjacent to p_1 and p_1 is also adjacent to q_1 in A_2. But since for $m > 1$ we have $m+1 > 2$, therefore this is a contradiction, and for $m = 1$ we have q_{2n-1} not adjacent to one point say q_3 in A_2 and both are mutually adjacent to $2n-3$ points in A_2 which is also a contradiction. This implies that q_{2n-1} is either adjacent to no point in A_2 other than q_1 or q_{2n-1} is adjacent to all the points in A_2. Without loss of generality suppose q_{2n-1} is adjacent to all the points in A_2 then this implies that p_1 must be adjacent to all the points in A_1 other than q_{2n-1}.

Now since q_{2n-1} is not adjacent to any point in A_1 and all the points in A_1 are adjacent to p_{2n-1} which is adjacent to q_{2n-1} and two non-adjacent points in A are mutually adjacent to exactly two points in A therefore this implies that each one of $p_2, p_3, \ldots, p_{2n-2}$ is adjacent to exactly one point in A_2 and since each one of these points is adjacent to $2n-1$ total number of points in A therefore all of these points must be mutually adjacent. Note that each one of these points is adjacent to a distinct point in A_2 since otherwise if, say p_2 and p_3 are adjacent to the same point, say q_2 in A_2 then we have q_2 adjacent to more than $2n-1$ points in A which is contrary to the hypothesis. Hence p is common to two complete subgraphs each equal to K_{2n}. Clearly these two subgraphs are line disjoint since they are point disjoint other than the point p.

Remark (2.2): It is interesting to note that although Lemma (2.2) plays a vital role in the proof of our theorem, the case it considers does not happen when $G = L(K_{n,n,n})$. Since then we have $K_{n,n,n}$ with $3n$ points in the sets S_1, S_2 and S_3 where $|S_1| = |S_2| = |S_3| = n$ and two points in $K_{n,n,n}$ are adjacent if and only if they belong to the distinct S's. Now let p be a point in G corresponding to a line uv in $K_{n,n,n}$ joining a point u in S_1 to a point v in S_2. Now since the point

u is adjacent to all the points in S_2 and S_3 and the point v is adjacent to all the points in S_1 and S_3 therefore by the definition of a line graph this implies that p is common to 2 complete subgraphs in G each equal to K_{2n}. Since the $n-1$ lines joining the point u to $n-1$ points in S_2 are not incident with the $2n-1$ lines joining the point v to $n-1$ points in S_1 and n points in S_3 and the $n-1$ lines joining the point v to $n-1$ points in S_1 are not incident with $2n-1$ lines joining the point u to $n-1$ points in S_2 and n points in S_3 therefore this implies that $n-1$ points in one of the two complete subgraphs on the point p in G are not adjacent to $n-1$ points in the other complete subgraph and each one of the remaining n points in one of the two complete subgraphs in G must be adjacent to exactly one distinct point out of the n remaining points in the other complete subgraph in G. This implies that out of the $4n-2$ lines in G with p as one of their end point, there are exactly $2n$ of $(2n-1)$ lines and the remaining $2n-2$ lines are $(2n-2)$ lines.

Lemma (2.3): If p is a point in G such that at least one line on p is a $(2n-2)$ line then p is common to exactly 2 line disjoint complete subgraphs in G each equal to K_{2n}.

Proof: Let $B = \{p_1, p_2, \ldots, p_{2n-1}, q_1, q_2, \ldots, q_{2n-1}\}$ be the set of $4n-2$ points in G adjacent to p. Let $B_1 = \{p_2, p_3, \ldots, p_{2n-1}\}$ be the subset of $2n-2$ points adjacent to both p and p_1 and let $B_2 = \{q_1, q_2, \ldots, q_{2n-1}\}$ be the subset of $2n-1$ points adjacent to p but not to p_1.

First we shall prove that all the points in B_1 are mutually adjacent. Suppose not and let p_2 not be adjacent to p_3. Now since p_2 being adjacent to p must be adjacent to at least $2n-2$ points in B and other than the points p_2 and p_3 there are only 2n-4 points in B_1 and p_2 is already adjacent to the point p_1 therefore there is at least one point in B_2 which is adjacent to p_2 and let this point be q_1. Now since p_1 is not adjacent to q_1 and both are adjacent to p therefore there are the following 4 possibilities:

(i) p_1 and q_1 are mutually adjacent to one point in B and q_1 is adjacent to 2n-2 points in B.

(ii) p_1 and q_1 are mutually adjacent to 2 points in B and q_1 is adjacent to 2n-2 points in B.

(iii) p_1 and q_1 are mutually adjacent to 2 points in B and q_1 is adjacent to 2n-1 points in B.

(iv) p_1 and q_1 are mutually adjacent to one point in B and q_1 is adjacent to 2n-1 points in B.

We have 2n-2 points in B_2 other than q_1 and since in (i) q_1 is adjacent to one point in B_1 (since p_1 is only adjacent to the points in B_1) therefore there is exactly one point, say q_2 in B_2 not adjacent to q_1 and hence to p_1. Similarly in (ii) there are 2 points, namely q_2 and q_3 in B_2 not adjacent to both q_1 and p_1 and in (iii) there is exactly one point, say q_2 in B_2 not adjacent to both p_1 and q_1. This implies that in (i), (ii), and (iii) we have a claw consisting of 4 points, namely p, p_1, q_1 and q_2 and therefore by Lemma (2.1) (i), (ii), and (iii) are not valid for $n \geq 4$.

In case (iv) first notice that all the points in B_2 must be mutually adjacent since otherwise if q_2 is not adjacent to q_3 then we have a claw consisting of 4 points, namely p, p_1, q_2 and q_3 which is not possible for $n \geq 4$ by Lemma (2.1). Similarly all the points in B_1 other than the point p_2 are mutually adjacent since otherwise if p_3

is not adjacent to p_4 then we have a claw consisting of 4 points, namely p, q_1, p_3 and p_4 which again contradicts Lemma (2.1) for $n \geq 4$. This implies that p_2 is adjacent to at most one point in B_2 and at most one point in B_1 other than the point p_3, because otherwise if p_2 is adjacent to say q_2 and q_3 in B_2 then since there are 2n-2 points in B_2 other than q_1 and moreover p_2 is already adjacent to q_1 and p_1 therefore there exists at least one point q_4 in B_2 not adjacent to p_2, so we have q_4 not adjacent to p_2 and both are mutually adjacent to 4 points, namely p, q_1, q_2 and q_3 which contradicts the hypothesis of Theorem (2.1). On the other hand, if p_2 is adjacent to say p_4 and p_5 in B_1, then we have p_2 not adjacent to p_3 and both are mutually adjacent to 4 points, namely p, p_1, p_4 and p_5 which contradicts the hypothesis of the Theorem (2.1). So p_2 is adjacent to at most 4 points in B. But since p is adjacent to p_2 therefore there are at least 2n-2 points in B adjacent to p_2. So this implies that $2n-2 \leq 4$ or $n \leq 3$. Hence case (iv) is also not valid for $n \geq 4$. So all the points in B_1 must be mutually adjacent for $n \geq 4$.

Now it is easy to check that if all the points in B_1 are mutually adjacent then all the points in B_2 must also be mutually adjacent since otherwise if q_1 is not adjacent to q_2 then we have a claw consisting of 4 points, namely p, p_1, q_1 and q_2 which contradicts Lemma (2.1). So this implies that the point p is common to exactly 2 complete subgraphs K_{2n}, one is on the points in $B_2 \cup \{p\}$ and the other one is on the points in $B_1 \cup \{p, p_1\}$. Since both of these K_{2n} are point disjoint other than the point p therefore they are also line disjoint.

Lemma (2.4): G is a line graph.

Proof: By Lemmas (2.2) and (2.3) we have proved that every point in G is common to exactly two complete subgraphs each equal to K_{2n} and since these complete subgraphs are line disjoint and G is regular of degree $4n-2$ therefore this implies that every line in G must be on one and only one K_{2n} . This implies that all the lines in G can be partitioned into complete subgraphs K_{2n} such that every point is common to exactly two of the complete subgraphs. Hence by [6] G is a line graph.

Proof of Theorem (2.1): It is easy to check that the conditions in the theorem are necessary, since, if $G = L(K_{n,n,n})$ then by the definition of a line graph the number of lines $3n^2$ in $K_{n,n,n}$ is the number of points in G, therefore $|V(G)| = 3n^2$.

Next, let S_1, S_2, S_3 be the distinct subsets of points in $K_{n,n,n}$ such that $|S_1| = |S_2| = |S_3| = n$ and let L_{12} be the subset of lines in $K_{n,n,n}$ consisting of n^2 lines joining the points in S_1 to the points in S_2, L_{13} and L_{23} can be defined accordingly. Now since two incident lines from the same subset of lines, say L_{12} , are mutually incident with $2n-2$ more lines in $K_{n,n,n}$ and two incident lines from two distinct subsets of lines are mutually incident with $2n-1$ more lines in $K_{n,n,n}$ and since $G = L(K_{n,n,n})$ therefore this implies that two adjacent points in G are mutually adjacent to either $2n-2$ or $2n-1$ more points in G . Since two non-incident lines in the same subset of lines, say L_{12} , are mutually incident with exactly two more lines, and two non-incident lines from two distinct subsets of lines are mutually incident with exactly 3 more lines and $G = L(K_{n,n,n})$ therefore this implies that two non-adjacent points in G are mutually adjacent to 2 or 3 more points in G .

To prove that the conditions are sufficient we proceed as follows: By Lemma (2.4) G is a line graph. Let G be the line graph of a graph H . We have to prove that $H = K_{n,n,n}$. Since $G = L(H)$ and G is regular and connected therefore by [5] H is either regular or H is bipartite and not regular. We shall prove that H is not bipartite and irregular. Suppose the contrary and let H be bipartite and irregular and let $V(H) = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$ and $|V_1| = n_1$ and $|V_2| = n_2$ where $n_1 \neq n_2$ and let $H = K_{n_1 n_2}$. Now since the valence of G is 4n-2 therefore every line in H is incident with 4n-2 lines in H and since $H = K_{n_1 n_2}$ therefore we have

$$(2.5) \qquad n_1 + n_2 - 2 = 4n - 2 .$$

But in this case note that two incident lines in H are mutually incident with either n_1-2 or n_2-2 more lines in H. Since $G = L(H)$ therefore this implies that two adjacent points in G are mutually adjacent to either n_1-2 or n_2-2 more points in G and by the hypothesis of the Theorem (2.1) we have $n_1-2 = 2n-2$ or $n_1-2 = 2n-1$ and $n_2-2 = 2n-2$ or $n_2-2 = 2n-1$. Since $n_1 \neq n_2$ therefore if $n_1-2 = 2n-2$ then $n_2-2 = 2n-1$ or if $n_1-2 = 2n-1$ then $n_2-2 = 2n-2$. But in any case, we have $n_1 + n_2 - 4 = 4n-3$ or $n_1 + n_2 - 2 = 4n-1$ which contradicts (2.5). So H must be regular.

Now since H is regular and every line in H is incident with 4n-2 more lines in H because the degree of G is 4n-2 this implies that each end point of a line in H is adjacent to exactly 2n-1 more points in H and this implies that H must be of degree 2n. Since $|V(G)| = 3n^2$ therefore total number of lines in $H = 3n^2$. Let $|V(H)|$ be the total number of points in H, therefore we have

$$\frac{|V(H)| \times 2n}{2} = 3n^2$$

or

$$|V(H)| = 3n .$$

So H is regular of degree 2n with 3n points.

Now let P be the set of $2n$ points in $V(H)$ that are adjacent to a point p in H and let Q be the set of $n-1$ points in $V(H)$ that are not adjacent to p. We shall prove that all the points in Q must be adjacent to all the points in P. Suppose the contrary and let a point $q \in Q$ not be adjacent to a point $r \in P$. Since the degree of q is $2n$ and there are $2n$ points in P and q is not adjacent to r in P therefore there exists at least one point $s \in Q$ and adjacent to q. Now since $G = L(H)$ therefore let p_1 be the point in G corresponding to the line joining the points p and r in H and p_2 be the point in G corresponding to the line joining the points q and s in H. Clearly the point p_1 is not adjacent to the point p_2 in G and since p is not adjacent to q and s in H and q is not adjacent to r in H therefore this implies that there is at most one point in G (only if s is adjacent to r in H) adjacent to both p_1 and p_2 which contradicts the hypothesis of the Theorem (2.1). So all the points in Q must be adjacent to all the points in P. Hence $H = K_{n,n,n}$, and this establishes the Theorem (2.1).

SECTION 3

THE CHARACTERIZATION OF $L(K_n(t))$

Theorem (3.1): If t and n are integers such that $t \geq 4$ and $n \geq 2$, then a regular connected graph G with the valence $2(t-1)n-2$, satisfying the following conditions is the line graph of $K_n(t)$, except for the cases when $(t = 4; n = 2$ and $3)$ and $(t = 5$ and $6; n = 2)$, if and only if

(1) $|V(G)| = \binom{t}{2} n^2$,

(2) Any pair of adjacent points in G is mutually adjacent to either $(t-1)n-2$ or $(t-1)n-1$ distinct points in G ,

(3) Any pair of non-adjacent points in G is mutually adjacent to 2,3 or 4 points in G.

We do not know what the situation is for $(t = 4; n = 2$ and $3)$ and $(t = 5$ and $6; n = 2)$.

Lemma (3.1): The graph G in Theorem (3.1) does not contain a claw.

Proof: Suppose the contrary and let the graph G have a claw with a point 0 adjacent to the three mutually non-adjacent points 1,2 and 3. Using the notations of Lemma (2.1) and by the hypothesis of Theorem (3.1) we have

$$S_{12} + S_{123} \leq 3, \; S_{13} + S_{123} \leq 3 \text{ and } S_{23} + S_{123} \leq 3$$

which gives $S_{12} + S_{13} + S_{23} + 3S_{123} \leq 9$ and since $S_{123} \geq 0$ therefore we have $S_{12} + S_{13} + S_{23} + 2S_{123} \leq 9$. Again by the conditions of the Theorem (3.1) we have

$$S_1 + S_{12} + S_{13} + S_{123} \leq (t-1)n-2$$

$$S_2 + S_{23} + S_{12} + S_{123} \leq (t-1)n-2$$

$$S_3 + S_{13} + S_{23} + S_{123} \leq (t-1)n-2$$

and by adding we get

$$S_1 + S_2 + S_3 + 2(S_{12} + S_{13} + S_{23}) + 3S_{123} \le 3(t-1)n-6 \ .$$

But we also have $-S_{12} - S_{13} - S_{23} - 2S_{123} \le -9$ therefore

$$S_1 + S_2 + S_3 + S_{12} + S_{13} + S_{23} + S_{123} \le 3(t-1)n-15.$$

Since the valence of 0 is $2(t-1)n-2$,

$$S_1 + S_2 + S_3 + S_{12} + S_{13} + S_{23} + S_{123} \le [2(t-1)n-2]-3 \ .$$

So we have

$$3(t-1)n-15 \le 2(t-1)n-5 \quad \text{or} \quad n \le \frac{10}{t-1} \ .$$

This implies that the lemma (3.1) holds except for $(t = 4;\ n = 2$ and $3)$ and $(t = 5$ and $6;\ n = 2)$.

<u>Definition (3.1)</u>: A line in G is called a <u>$[(t-1)n-2]$ line</u>, if both the end points are mutually adjacent to exactly $(t-1)n-2$ distinct points in G and similarly we can define a <u>$[(t-1)n-1]$ line</u> in G .

<u>Lemma (3.2)</u>: If p is a point in G such that every line on p is a $[(t-1)n-1]$ line then p is common to exactly 2 line disjoint complete subgraphs in G each equal to $K_{(t-1)n}$.

<u>Proof</u>: Let $A = \{p_1, p_2, \ldots, p_{(t-1)n-1}, q_1, q_2, \ldots, q_{(t-1)n-1}\}$ be the set of $2(t-1)n-2$ points in G adjacent to p . Let $A_1 = \{p_1, p_2, \ldots, p_{(t-1)n-2},$ $q_{(t-1)n-1}\}$ be the subset of $(t-1)n-1$ points that are adjacent to both p and $p_{(t-1)n-1}$ and let $A_2 = \{q_1, q_2, \ldots, q_{(t-1)n-2}\}$ be the subset of points in G adjacent to p but not to $p_{(t-1)n-1}$.

First we shall prove that the two non-adjacent points in A must be mutually adjacent to exactly **2** points in A. Suppose not and let p_1 not be adjacent to $q_{(t-1)n-1}$ and both be mutually adjacent to 3 points in A namely q_1, q_2 and $p_{(t-1)n-1}$. By the hypothesis each one of the points

p_1 and $q_{(t-1)n-1}$ must be adjacent to $(t-1)n-4$ more distinct points in A other than the three points q_1, q_2 and $p_{(t-1)n-1}$. This implies that there are $2(t-1)n-8$ total numbers of points in A that are adjacent to either p_1 or to $q_{(t-1)n-1}$ but not to both. But other than the 5 points $p_1, p_{(t-1)n-1}, q_1, q_2$ and $q_{(t-1)n-1}$ there are exactly $2(t-1)n-7$ points in A, this implies that there is exactly one point in A namely q_3 which is not adjacent to both p_1 and $q_{(t-1)n-1}$. So we have a claw in G consisting of 4 points p, p_1, q_3 and $q_{(t-1)n-1}$ which is a contradiction to Lemma (3.1).

Now suppose that p_1 and $q_{(t-1)n-1}$ the two non-adjacent points are mutually adjacent to exactly one point namely $p_{(t-1)n-1}$ in A. In this case we have $2(t-1)n-5$ points in A other than the three points p_1, $p_{(t-1)n-1}$ and $q_{(t-1)n-1}$. But each one of p_1 and $q_{(t-1)n-1}$ must be adjacent to $(t-1)n-2$ more distinct points in A, therefore we have

$$2(t-1)n-5 \geq (t-1)n-2 + (t-1)n-2$$

which is not possible. So any two non-adjacent points in A must be mutually adjacent to exactly 2 points in A.

Now since $p_{(t-1)n-1}$ is not adjacent to any point in A_2 therefore every point in A_2 must be adjacent to exactly 2 points in A_1 and since all the points in A are adjacent to $(t-1)n-1$ points in A therefore this implies that all the points in A_1 are not mutually adjacent and all the points in A_2 must be mutually adjacent. Without loss of generality let p_1 not be adjacent to $q_{(t-1)n-1}$ since both the points are already adjacent to $p_{(t-1)n-1}$ therefore both are mutually adjacent to one more point in A. Suppose this point is in A_1 namely p_2. Since each one of p_1 and $q_{(t-1)n-1}$ is adjacent to $(t-1)n-3$ more distinct points in A other than p_2 and $p_{(t-1)n-1}$ and there are $(t-1)n-2$ total number of points in A_2 therefore p_1 is not adjacent to at least one point say q_1 in A_2 and $q_{(t-1)n-1}$ is not adjacent to at least one point in A_2 other than q_1 say q_2. Since all the points in A_2 are mutually adjacent and two non-adjacent points in A are mutually adjacent to exactly 2 points in A therefore each one of p_1 and $q_{(t-1)n-1}$ is adjacent to at most 2 distinct points in A_2. But other than p_2 and $p_{(t-1)n-1}$ each one of p_1 and $q_{(t-1)n-1}$ must be adjacent to exactly $(t-1)n-3$ more distinct points in A. Therefore we have

$$(t-1)n-3 + (t-1)n-3 \le [(t-1)n-1] - 3 + 4$$

or $(t-1)n \le 6$ or $n \le \frac{6}{t-1}$ so it is impossible except for $n = 2$ when $t = 4$ which is an exceptional case.

The other possibility is that p_1 and $q_{(t-1)n-1}$ are mutually adjacent to a point in A_2 namely q_1. This case is valid only if any one of these two non-adjacent points is adjacent to all the points in A_2 because otherwise suppose that $q_{(t-1)n-1}$ is not adjacent to m points in A_2 where $1 \le m < (t-1)n-3$. In this case note that all the points in A that are not adjacent to $q_{(t-1)n-1}$ must be adjacent to p_1 and vice versa, since p_1 and $q_{(t-1)n-1}$ are not mutually adjacent to any point in A other than q_1 and $p_{(t-1)n-1}$ and other than these two points p_1 and $q_{(t-1)n-1}$ are adjacent to $2(t-1)n-6$ total number of points in A and there are the same number of points in A other than p_1, $p_{(t-1)n-1}$, q_1 and $q_{(t-1)n-1}$. This implies that p_1 is adjacent to the same $m \ge 1$ points in A_2 that are not adjacent to $q_{(t-1)n-1}$ and since $m < (t-1)n-3$ therefore p_1 is not adjacent to at least one point in A_2 namely q_2. So we have p_1 not adjacent to q_2 in A and both are mutually adjacent to $m + 1$ points in A_2 because all the points in A_2

are mutually adjacent, therefore q_2 is adjacent to all the m points in A_2 that are adjacent to p_1 and p_1 is also adjacent to q_1 in A_2. But this is a contradiction since for $m > 1$, $m + 1 > 2$ and for $m = 1$ we have $q_{(t-1)n-1}$ not adjacent to one point say q_3 and both are mutually adjacent to $(t-1)n-3$ points in A_2 which is again a contradiction. This implies that $q_{(t-1)n-1}$ is either adjacent to no point in A_2 other than q_1 or $q_{(t-1)n-1}$ is adjacent to all the points in A_2. Without loss of generality suppose that $q_{(t-1)n-1}$ is adjacent to all the points in A_2 then this implies that p_1 must be adjacent to all the points in A_1 other than $q_{(t-1)n-1}$.

Now since $q_{(t-1)n-1}$ is not adjacent to any point in A_1 and all the points in A_1 are adjacent to $p_{(t-1)n-1}$ and two non-adjacent points in A are mutually adjacent to exactly 2 points in A therefore this implies that each one of $p_2, p_3, \ldots, p_{(t-1)n-2}$ is adjacent to exactly one point in A_2 and since each one of these points is adjacent to $(t-1)n-1$ total number of points in A therefore all of these points must be mutually adjacent. Note that each one of these points is adjacent to a distinct point in A_2 since otherwise if say p_2 and p_3 are adjacent to the same point say q_2 in A_2 then we have q_2 adjacent to more than $(t-1)n-1$ points in A which is a contradiction. Hence p is common to two complete subgraphs each equal to $K_{(t-1)n}$. Clearly these two subgraphs are line disjoint since they are point disjoint other than the point p.

Remark (3,2): It is interesting to note that although Lemma (3.2) plays a vital role in the proof of our theorem, the case it considers does not happen when $G = L(K_n(t))$. Since then we have $K_n(t)$ with tn points in the sets $S_1, S_2, \ldots, S_t$ where $|S_1| = |S_2| = \ldots = |S_t| = n$ and two points in $K_n(t)$ are adjacent if and only if they belong to the distinct S's. Now let p be a point in G corresponding to a line uv in $K_n(t)$ joining a point u in S_1 to a point v in S_2. Now since point u is adjacent to all the points in $S_2, S_3, \ldots, S_t$ and the point v is adjacent to all the points in $S_1, S_3, \ldots, S_t$ therefore by the definition of a line graph this implies that p is common to two complete subgraphs in G each equal to $K_{(t-1)n}$. Since the n-1 lines joining the point u to n-1 points in S_2 are not incident with the (t-1)n-1 lines joining the point v to n-1 points in S_1 and (t-2)n points in $S_3, \ldots, S_t$ and the n-1 lines joining the point v to n-1 points in S_1 are not incident with (t-1)n-1 lines joining the point u to n-1 points in S_2 and (t-2)n points in $S_3, \ldots, S_t$, therefore this implies that n-1 points in one of the complete subgraphs on the point p in G are not adjacent to n-1 points in the other complete subgraph and each one of the remaining (t-2)n points in one of the two complete subgraphs in G must be adjacent to exactly one distinct point out of the (t-2)n remaining points in the other complete subgraph in G. This implies that out of 2(t-1)n-2 lines in G with p as one of their end point, there are exactly 2(t-2)n of [(t-1)n-1] lines and the remaining 2n-2 lines are [(t-1)n-2] lines.

Lemma (3.3): If p is a point in G such that at least one line on p is a (t-1)n-2 line then p is common to exactly 2 line disjoint complete subgraphs in G each equal to $K_{(t-1)n}$.

Proof: Let $B = \{p_1, p_2, \ldots, p_{(t-1)n-1}, q_1, \ldots, q_{(t-1)n-1}\}$ be the set of 2(t-1)n-2 points in G adjacent to p. Let $B_1 = \{p_2, p_3, \ldots, p_{(t-1)n-1}\}$ be the subset of (t-1)n-2 points adjacent to both p and p_1 and let

$B_2 = \{q_1, q_2, \ldots, q_{(t-1)n-1}\}$ be the subset of $(t-1)n-1$ points adjacent to p but not to p_1.

First we shall prove that all the points in B_1 are mutually adjacent. Suppose not and let p_2 not be adjacent to p_3. Now since p_2 being adjacent to p must be adjacent to at least $(t-1)n-2$ points in B and other than the points p_2 and p_3 there are only $(t-1)n-4$ points in B_1 and p_2 is already adjacent to p_1 therefore there is at least one point in B_2 which is adjacent to p_2 and let this point be q_1. Now since q_1 is not adjacent to p_1 and both are adjacent to p therefore there are the following 6 possibilities:

(i) p_1 and q_1 are mutually adjacent to one point in B and q_1 is adjacent to $(t-1)n-2$ points in B.

(ii) p_1 and q_1 are mutually adjacent to 2 points in B and q_1 is adjacent to $(t-1)n-2$ points in B.

(iii) p_1 and q_1 are mutually adjacent to 3 points in B and q_1 is adjacent to $(t-1)n-2$ points in B.

(iv) p_1 and q_1 are mutually adjacent to one point in B, and q_1 is adjacent to $(t-1)n-1$ points in B.

(v) p_1 and q_1 are mutually adjacent to 2 points in B and q_1 is adjacent to $(t-1)n-1$ points in B.

(vi) p_1 and q_1 are mutually adjacent to 3 points in B and q_1 is adjacent to $(t-1)n-1$ points in B.

Since we have $(t-1)n-2$ points in B_2 other than q_1 and in (i) q_1 is already adjacent to one point in B_1 (since p_1 is only adjacent to the points in B_1) therefore there is exactly one point say q_2 in B_2 not adjacent to q_1 and hence to p_1. Similarly in (v) there is exactly one point in B_2 say q_2 which is not adjacent to q_1 and p_1. In (ii) and (vi) there are exactly 2 points in B_2 namely q_2 and q_3 which are not adjacent to q_1 and p_1 and similarly in (iii) there are exactly 3 points in B_2 which are not adjacent to q_1 and p_1. So this implies that in (i), (ii), (iii), (v) and (vi) there is at least one point in B_2 namely q_2 not adjacent to both p_1 and q_1, this implies that in these cases there is a claw in G consisting of the 4 points namely p, p_1, q_1 and q_2 which contradicts Lemma (3.1) therefore these cases are not possible.

In case (iv) where all the points in B_2 are adjacent to q_1, first notice that all the points in B_2 must be mutually adjacent since otherwise if q_2 is not adjacent to say q_3 then we have a claw consisting of 4 points namely p, p_1, q_2 and q_3 which contradicts Lemma (3.1). Similarly all the points in B_1 other than the point p_2 are mutually adjacent since otherwise if p_3 is not adjacent to p_4 then we have a claw consisting of 4 points namely p, q_1, p_3 and p_4 which contradicts Lemma (3.1). This implies that p_2 is adjacent to at most 2 points in B_2 and at most 2 points in B_1 other than the point p_3, because otherwise if p_2 is adjacent to more than 2 points in B_2 say q_2, q_3 and q_4 then since there are $(t-1)n-2$ points in B_2 other than q_1, and p_2 is already adjacent to q_1 and p_1 in B therefore there exists at least one point in B_2 not adjacent to p_2 and let it be q_5. So we have p_2 not adjacent to q_5 and both are mutually adjacent to 5 points in B namely p, q_1, q_2, q_3 and q_4 which contradicts the hypothesis of Theorem (3.1). On the other hand, if p_2 is adjacent to more than 2 points in B_1 say p_4, p_5 and p_6 then we have p_2 not adjacent to p_3 and both are mutually adjacent to 5 points in B namely p, p_1, p_4, p_5 and p_6 which contradicts the hypothesis of Theorem (3.1). So P_2 is adjacent to at most 6 points in B. But since p is adjacent to p_2 therefore there are at least $(t-1)n-2$ points in B adjacent to p_2. So we have $(t-1)n-2 \leq 6$ or $n \leq \frac{8}{t-1}$. Hence case (iv) is also valid except when $(t = 4; n = 2)$ and $(t = 5; n = 2)$ which are the exceptional cases. So all the points in B_1 must be mutually adjacent.

Now it is easy to check that if all the points in B_1 are mutually adjacent then all the points in B_2 must also be mutually adjacent since otherwise if q_1 is not adjacent to q_2 then we have a claw consisting of 4 points namely p, p_1, q_1 and q_2 which contradicts Lemma (3.1). This implies that the point p is common to exactly 2 complete subgraphs one is on the points in $B_2 \cup \{p\}$ and the other one is on the points in $B_1 \cup \{p, p_1\}$. Since both of these $K_{(t-1)n}$ are point disjoint other than the point p therefore they are also line disjoint.

Lemma (3.4): G is a line graph.

Proof: By Lemma (3.2) and (3.3) we have proved that every point in G is common to exactly two complete subgraphs each equal to $K_{(t-1)n}$ and since these complete subgraphs are line disjoint and G is regular of degree $2(t-1)n-2$ therefore this implies that every line in G is on one and only one $K_{(t-1)n}$. This implies that all the lines in G can be partitioned into complete subgraphs $K_{(t-1)n}$ such that every point is common to exactly two of the complete subgraphs, therefore by [6] G is a line graph.

Proof of Theorem (3.1): It is easy to check that the conditions of the theorem are necessary, since if $G = L(K_n(t))$ then, by the definition of a line graph, the number of lines $\binom{t}{2}n^2$ in $K_n(t)$ is the number of points in G, therefore $|V(G)| = \binom{t}{2}n^2$.

Next, let $S_1, S_2, \ldots, S_t$ be the subsets of points in $K_n(t)$ such that $|S_1| = |S_2| = \ldots = |S_t| = n$ and let L_{ij} where $i < j$ and $i = 1,2,\ldots,t$, $j = 1,2,\ldots,t$ be the subsets of lines in $K_n(t)$ such that L_{ij} consists of n^2 lines joining the points in S_i to the points in S_j. Clearly in $K_n(t)$ two incident lines from the same subset of lines L_{ij} are mutually incident with $(t-1)n-2$ more lines in $K_n(t)$ and two incident lines from distinct subsets of lines are mutually incident with $(t-1)n-1$ more lines in $K_n(t)$ and since $G = L(K_n(t))$ therefore this implies that in G, two adjacent points are mutually adjacent to either $(t-1)n-2$ or $(t-1)n-1$ more points.

Now since two non-incident lines in the same subset of lines say L_{12} are mutually incident with exactly two more lines and since two non-incident lines in two distinct subsets of lines L_{ij} and L_{jk} where $i < j < k$ are mutually incident with 3 more lines and two non-incident lines in two distinct subsets of lines L_{ij} and $L_{k\ell}$ where $i = k$ or ℓ or $j = k$ or ℓ are mutually incident with exactly 4 more lines and since $G = L(K_n(t))$ therefore this implies that in G, two non-adjacent points are mutually adjacent to 2, 3 or 4 more points.

To prove that the conditions are sufficient, we proceed as follows: By Lemma (3.4) G is a line graph. Let G be the line graph of a graph H. We have to prove that $H = K_n(t)$.

Since $G = L(H)$ and G is regular and connected therefore by [5] H is either regular or H is bipartite and not regular. We shall prove that H is not bipartite and irregular. Suppose not and let H be bipartite and irregular and let $V(H) = V_1 \cup V_2$ such that $|V_1| = n_1$ and $|V_2| = n_2$ and $V_1 \cap V_2 = \emptyset$ where $n_1 \neq n_2$ and let $H = K_{n_1 n_2}$. Now since the valence of G is $2(t-1)n-2$ therefore every line in H is incident with $2(t-1)n-2$ more lines in H and since $H = K_{n_1 n_2}$ therefore we have

$$n_1 + n_2 - 2 = 2(t-1)n-2 \quad . \tag{3.5}$$

But in this case note that two incident lines in H are mutually incident with either n_1-2 or n_2-2 more lines in H. Since $G = L(H)$ therefore this implies that two adjacent points in G are mutually adjacent to either n_1-2 or n_2-2 more points in G and by the hypothesis of Theorem (3.1) we have $n_1-2 = (t-1)n-2$ or $n_1-2 = (t-1)n-1$ and $n_2-2 = (t-1)n-2$ or $n_2-2 = (t-1)n-1$ since $n_1 \neq n_2$ therefore if $n_1-2 = (t-1)n-2$ then $n_2-2 = (t-1)n-1$ or if $n_1-2 = (t-1)n-1$ then $n_2-2 = (t-1)n-2$. But in any case we have $n_1+n_2-4 = 2(t-1)n-3$ or $n_1+n_2-2 = 2(t-1)n-1$ which contradicts (3.5). So H must be regular.

Now since H is regular and every line in H is incident with $2(t-1)n-2$ more lines in H since the degree of G is $2(t-1)n-2$, this implies that each end point of a line in H is adjacent to exactly $(t-1)n-1$ more points in H and this implies that H must be of degree $(t-1)n$. Since $|V(G)| = \frac{t(t-1)}{2} n^2$ therefore total number of lines in H is $\frac{t(t-1)}{2} n^2$. Let $|V(H)|$ be the total number of points in H then we have

$$\frac{|V(H)| \times (t-1)n}{2} = \frac{t(t-1)n^2}{2}$$

which gives $|V(H)| = tn$. So H is regular of degree $(t-1)n$ with tn points.

Now let P be the set of $(t-1)n$ points in $V(H)$ that are adjacent to a point p in H, and let Q be the set of $n-1$ points in $V(H)$ that are not adjacent to p. We shall prove that all the points in Q must be adjacent to all the points in P. Suppose not and let a point $q \in Q$ not be adjacent to a point $r \in P$. Since the degree of q is $(t-1)n$ and there are $(t-1)n$ points in P and q is not adjacent to r in P therefore there exists at least one point $s \in Q$ adjacent to q. Now since $G = L(H)$ therefore let p_1 be the point in G corresponding to the line joining the points p and r in H and p_2 be the point in G corresponding to the line joining the points q and s in H. Clearly p_1 is not adjacent to p_2 in G and since p is not adjacent to q and s in H and q is not adjacent to r in H therefore this implies that there is at most one point in G (only if s is adjacent to r in H) adjacent to both p_1 and p_2 which contradicts the hypothesis of the Theorem (3.1). So all the points in Q must be adjacent to all the points in P. Hence $H = K_n(t)$ and this establishes the Theorem (3.1).

SECTION 4

THE CHARACTERIZATION OF $L(K_{mn})$

Theorem (4.1): A graph G with the following properties is isomorphic to $L(K_{mn})$ where m and n are integers such that $m \geq n \geq 2$, if and only if

(1) G is regular of degree $m+n-2$, with mn points.

(2) Two non-adjacent points are mutually adjacent to exactly two points in G.

(3) Two adjacent points are mutually adjacent to either $m-2$ or $n-2$ points in G.

For $m = n = 4$ the theorem does not hold and there is a counter-example by Shrikhande [17].

Proof: It is easy to check that the conditions of the theorem are necessary. To prove that the conditions are sufficient, we shall first prove that G is a line graph of a graph say H by [6] showing that all the lines in G can be partitioned into a family of cliques such that any two cliques of the family have at most one point in common. Then we shall prove that $H = K_{mn}$.

To prove that G is a line graph, let p be a point in G adjacent to $m+n-2$ points in a subset A of $V(G)$. Let B be the set of $(m-1)(n-1)$ points in G which are not adjacent to p. By (2) every point in B is adjacent to exactly two points in A. By pairing p with each one of the points in A, we get $m+n-2$ pairs of adjacent points. Let, for some integer k, $m+k$ of these pairs be mutually adjacent to $m-2$ points in A and rest of $n-k-2$ pairs be mutually adjacent to $n-2$ points in A. Since the degree of each point in G is $m+n-2$ therefore each one of the $m+k$ points in A is adjacent to $n-1$ points in B and each one of the $n-k-2$ points in A is adjacent to $m-1$ points in B. Since each point in B is adjacent to two points in A, therefore we have

$$(m+k)(n-1) + (n-k-2)(m-1) = 2(m-1)(n-1)$$

which gives $k = -1$. So for $m > n$, we have a subset A_1 of A consisting of $m-1$ points such that each point in A_1 is adjacent to $m-2$ points in A and $n-1$ points in B, and we have a subset A_2 of A consisting of the rest of $n-1$ points such that each point in A_2 is adjacent to $n-2$ points in A and $m-1$ points in B. For $m = n$, every point in A is adjacent to $n-2$ points in A and $n-1$ points in B.

For $m > n$ there exists at least one point namely $q \in A_2$ which is not adjacent to at least one point, namely $r \in A_1$ (since q is adjacent to exactly $n-2$ points in A and there are $m-1$ points in A_1). For $m = n$, (there are no subsets A_1 and A_2 of A) there are at least two points in A which are not adjacent to each other and without loss of generality let these points also be q and r.

Now we claim that for $m \geq n > 4$, q and r are not mutually adjacent to any point in A.

To prove our claim, suppose the contrary and let a point say s in A be adjacent to both q and r. By (2) q and r are not mutually adjacent to any other point in G. This implies that there is exactly $(m+n-2) - 2 - 1-[(m-3) + (n-3)] = 1$ point say t in A which is neither adjacent to q nor to r and all other points in A are either adjacent to q or to r but not to both except s. Now t is either adjacent to $n-2$ points in A or to $m-2$ points in A. Since $m > n$ therefore t is adjacent to at least $n-2$ points in A. Since $n-2 > 2$ therefore t is adjacent to at least 3 points in A. Since all the points in A other than s are either adjacent to q or to r, this implies that at least 2 of the 3 points in A which are adjacent to t must also be adjacent to either q or to r which along with the point p contradicts (2). So q and r are not mutually adjacent to any point in A. Let A_{11} be the set of $m-2$ points from A adjacent to r and let A_{22} be the set of $n-2$ points from A adjacent to q. For $m = n$, A_{11} contains $n-2$ points of A adjacent to r and A_{22} contains the remaining $n-2$ points of A adjacent to q.

Now we claim that for $m \geq n > 4$, no point in A_{11} is adjacent to a point in A_{22}.

To prove this, suppose the contrary and let a point say $a \in A_{11}$ be adjacent to a point say $b \in A_{22}$. Now b is adjacent to either $n-4$ or $m-4$ more points in A other than the points q and a and since $m \geq n$ therefore this implies that b is adjacent to at least $n-4 > 0$ more points in A. Now if any one of these $n-4$ points is in A_{11} say c then we have r not adjacent to b and both are adjacent to 3 points p, a and c which contradicts (2). This implies that all of the $n-4$ points adjacent to b must be in A_{22}. Since there are $n-3$ points in A_{22} other than b and b is adjacent to $n-4$ points in A_{22} therefore there is exactly one point in A_{22} which is not adjacent to b and without loss of generality, let this point be d. Now d is adjacent to either $n-3$ points or to $m-3$ points in A other than q and since $m \geq n$ therefore this implies that d is adjacent to at least $n-3$ more points in A. Since $n-3 > 1$, therefore d is adjacent to at least two more points in A other than q. Now if any one of these two points is in A_{22} say e then we have b not adjacent to d and both are adjacent to 3 points p, q and e which contradicts (2), and if both the points are in A_{11} then these two points along with the point p are 3 points adjacent to two non-adjacent points r and d, which contradicts (2). This proves our claim.

Now for $m \geq n$, since no point in A_{22} is adjacent to any point in $\{r\} \cup A_{11}$ therefore this implies that each point in A_{22} must be adjacent to the remaining n-3 points in A_{22} and since all the points in A_{22} are adjacent to q therefore this implies that $A_2 = \{q\} \cup A_{22}$ and the induced subgraph on n points in $\{p\} \cup A_2$ is K_n and this in turn implies that $A_1 = \{r\} \cup A_{11}$ and the induced subgraph on m points in $\{p\} \cup A_1$ is K_m.

Now we have to prove that for $m > n$ and $n \leq 4$ the induced subgraph on m-1 points in A_1 is K_{m-1} and the induced subgraph on n-1 points in A_2 is K_{n-1}. Suppose not and let a point say r in A_1 not be adjacent to a point say u in A_1. Since r and u are already adjacent to p therefore by (2) r and u are mutually adjacent to at most one point in A. So there are two cases:

(i) r and u are mutually adjacent to no point in A.

(ii) r and u are mutually adjacent to one point in A.

In case (i) since each one of r and u is adjacent to distinct m-2 points in A and A has (m-1)+(n-1)-2 points other than r and u therefore we have

$$(m-2) + (m-2) \leq (m-1) + (n-1) - 2$$

which gives $m \leq n$, which is contrary to the hypothesis $m > n$. In case (ii) we have

$$(m-3) + (m-3) + 1 \leq (m-1) + (n-1) - 2$$

which gives $m \leq n + 1$ which is contrary to the hypothesis $m > n$ except for the case when $m = n + 1$ and $n \leq 4$ which we shall consider later. This implies that the induced subgraph on m-1 points in A_1 is K_{m-1} and this further implies that the induced subgraph on n-1 points in A_2 must be K_{n-1}. Since p is adjacent to all the points in A therefore the induced subgraph on m points in $\{p\} \cup A_1$ is K_m and the induced subgraph on n points in $\{p\} \cup A_2$ is K_n.

Now we shall consider the exceptional cases when $n \leq 4$ and $m = n + 1$ and when $n \leq 4$ and $m = n$. So there are the following possibilities other than $m = n = 4$.

(4.2) $m = n = 2$ (4.3) $m = n = 3$ (4.4) $m = 3, n = 2$

(4.5) $m = 4, n = 3$ (4.6) $m = 5, n = 4$

The cases (4.2), and (4.4) are trivial therefore we shall only consider the cases (4.3), (4.5) and (4.6).

In case (4.3), let $A = \{p_{12}, p_{13}, p_{21}, p_{31}\}$ be the set of points adjacent to a point p_{11} . Now each point in A is adjacent to exactly one point in A and without loss of generality, let p_{12} be adjacent to p_{13} then p_{21} must be adjacent to p_{31} . This implies that $A_1 = \{p_{12}, p_{13}\}$ and $A_2 = \{p_{21}, p_{31}\}$. So the subgraph on the points in $A_1 \cup \{p_{11}\}$ is K_3 and the subgraph on the points $A_2 \cup \{p_{11}\}$ is also K_3 .

In case (4.5), let $A_1 = \{p_{12}, p_{13}, p_{14}\}$ and $A_2 = \{p_{21}, p_{31}\}$ and $A = A_1 \cup A_2$. Let p_{12} be adjacent to p_{21} and not to p_{14} . Since each point in A_2 is adjacent to only one point in A therefore the point to which both p_{12} and p_{14} are adjacent must be p_{13}, so p_{31} must be adjacent to p_{14} since p_{21} and p_{31} are not mutually adjacent to a single point in A therefore both are mutually adjacent to one point in B (B is the set of points in G which are not adjacent to p_{11}) and let it be p_{22} and let p_{23} and p_{24} be adjacent to p_{21} and not to p_{31} and let p_{33} and p_{34} be adjacent to p_{31} and not to p_{21} and let p_{32} be the point not adjacent to both p_{21} and p_{31}. Since p_{13} and p_{21} are mutually adjacent to p_{11} and p_{12} therefore p_{13} is not adjacent to any point in B which is also adjacent to p_{21} and similarly p_{13} is not adjacent to any point in B which is also adjacent to p_{31}

and this is impossible because there is only one point in B namely p_{32} which is not adjacent to any one of the points p_{21} and p_{31}. But p_{13} must be adjacent to 2 points in B. So the induced subgraphs on the points in $A_1 \cup \{p_{11}\}$ is K_4 and on the points in $A_2 \cup \{p_{11}\}$ is K_3.

In case (4.6), let $A = A_1 \cup A_2$ where $A_1 = \{p_{12}, p_{13}, p_{14}, p_{15}\}$ and $A_2 = \{p_{21}, p_{31}, p_{41}\}$ and let p_{12} be adjacent to p_{21} but not to p_{15}. There are twocases.

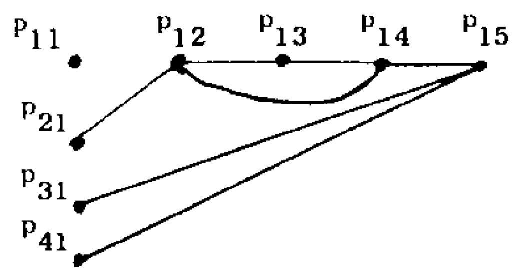
(1) The point which is adjacent to both p_{12} and p_{15} is in A_1.

(2) The point which is adjacent to both p_{12} and p_{15} is in A_2.

In case (1) without loss of generality, let p_{14} be adjacent to both p_{12} and p_{15}. Now p_{14} is adjacent to 3 points in A and two are already in A_1 and for the third one there are two cases:

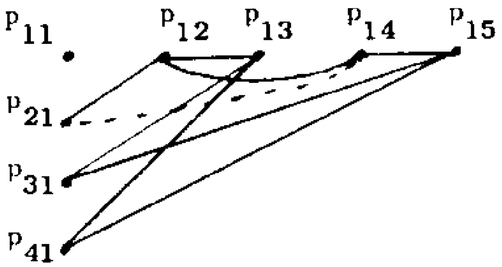
(i) The third point adjacent to p_{14} is in A_1.

(ii) The third point adjacent to p_{14} is in A_2.

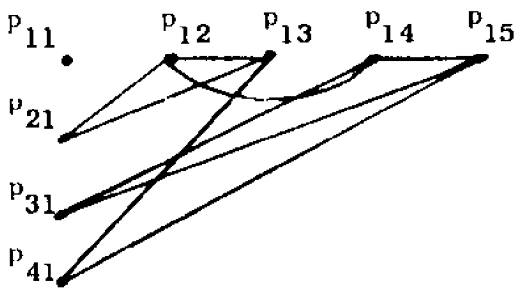


In case (i) the third point adjacent to p_{14} must be p_{13}. Let p_{12} be adjacent to p_{13}, and p_{15} be adjacent to p_{31} and p_{41}. Now the third point adjacent to p_{13} must be in A_2 since each one of the rest of the points in A_1 is already adjacent to 3 points in A. Now if p_{13} is adjacent to p_{21} then we have p_{21} not adjacent to p_{14} and both are adjacent to 3 points p_{11}, p_{12} and p_{13}. If p_{13} is adjacent to p_{31} then we have p_{31} not adjacent to p_{14} and both are adjacent to 3 points p_{11}, p_{13} and p_{15} and if p_{13} is adjacent to p_{41} then we have p_{14} not adjacent to p_{41} and both are adjacent to 3 points p_{11}, p_{13} and p_{15} which contradict (2).

In case (ii) first notice that the third point in A_2 adjacent to p_{14} cannot be p_{21} because otherwise p_{13} must have to be adjacent to both p_{31} and p_{41} and we have p_{31} not adjacent to p_{41} and both are adjacent to 3 points p_{11}, p_{13} and p_{15} a contradiction.



So p_{14} is adjacent to any one of p_{31} and p_{41}. Without loss of generality, let p_{14} be adjacent to p_{31}, then p_{13} is adjacent to p_{21} and p_{41}.



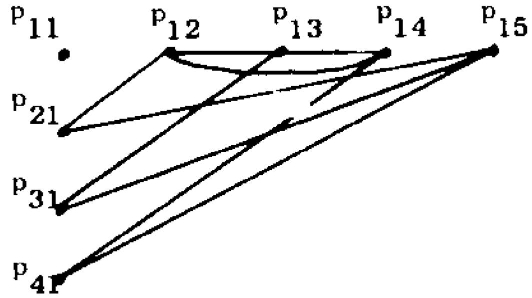
Now consider the points p_{14}, p_{21}, p_{31} and p_{41}. Since p_{41} is not adjacent to p_{31} and both are already adjacent to two points in A, therefore each one of them is adjacent to 4 distinct points in B. So we have 8 distinct points in B adjacent to p_{41} and p_{31}. Similarly the 4 points in B adjacent to p_{21} must be all distinct from the 4 points in B adjacent to p_{41}. Since p_{21} and p_{31} are not mutually adjacent to any point in A and p_{21} is not adjacent to p_{31} therefore p_{31} and p_{21} are mutually adjacent to exactly one point in B. This implies that there are exactly 11 points in B which are adjacent to p_{21}, p_{31} and p_{41} and only one of the 11 points in B is adjacent to both p_{21} and p_{31}. Now since p_{14} is not adjacent to p_{21} and both are adjacent to p_{11} and p_{12}. Moreover p_{14} is not adjacent to p_{41} and both are adjacent to p_{11} and p_{15} therefore p_{14} could be adjacent to either 3 points in B which are only adjacent to p_{31} but not to p_{21} or p_{41}; or else p_{14} could be adjacent to the one remaining point in B which is not adjacent to any one of p_{21}, p_{31} and p_{41} and 2 out of 3 points in B which are adjacent to only p_{31}. But in either case we have two adjacent points p_{14} and p_{31} both adjacent to more than 3 points which contradicts condition (3) of the hypothesis.

In case (2) without loss of generality, let p_{21} be the point in A_2 adjacent to both p_{12} and p_{15}. Now we have two cases.

(i) p_{12} and p_{15} are adjacent to two more distinct points in the same subset A_1 or A_2.

(ii) Among the other two points adjacent to p_{12} one is in A_1 and the other is in A_2 and from the two points adjacent to p_{15} one is in A_1 and the other is in A_2.

In case (i) either p_{12} is adjacent to p_{31} and p_{41}, and p_{15} is adjacent to p_{13} and p_{14} or vice versa. Since the proof is the same in either case therefore we shall consider the case where p_{12} is adjacent to p_{13} and p_{14}, and p_{15} is adjacent to p_{31} and p_{41}.



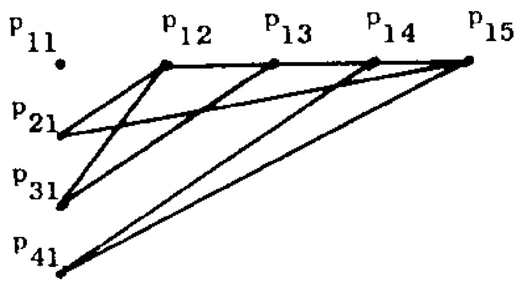
Now it is easy to see that p_{31} cannot be adjacent to p_{41} because otherwise we have p_{13} and p_{14} both adjacent to only 2 points in A (p_{13} is adjacent to p_{12} and p_{14} and p_{14} is adjacent to p_{12} and p_{13})

which is contrary to the fact that each point in A_1 is adjacent to 3 points in A. So p_{31} is not adjacent to p_{41} and without loss of generality, let p_{31} be adjacent to p_{13}, and p_{41} be adjacent to p_{14}, and p_{13} be adjacent to p_{14}. Now both the points in the pairs (p_{21},p_{31}), (p_{21},p_{41}) and (p_{31},p_{41}) are not adjacent to each other and both are adjacent to a point p_{15} in A. This implies that each one of the points p_{21},p_{31} and p_{41} is adjacent to a distinct set of 4 points in B.

Let p_{21} be adjacent to 4 points in $A_3 = \{p_{22},p_{23},p_{24},p_{25}\}$,
p_{31} be adjacent to 4 points in $A_4 = \{p_{32},p_{33},p_{34},p_{35}\}$,
p_{41} be adjacent to 4 points in $A_5 = \{p_{42},p_{43},p_{44},p_{45}\}$.

Now p_{13} is not adjacent to p_{21} and both are adjacent to p_{12} therefore p_{13} is not adjacent to any point in A_3. Similarly p_{13} is not adjacent to p_{41} and both are adjacent to p_{14} therefore p_{13} is not adjacent to any point in A_5. This implies that all the 3 points in B which are adjacent to p_{13} are in A_4 and this implies that p_{13} and p_{31} two adjacent points are both adjacent to 3 points in B and the fourth point is p_{11} which contradicts (3). Similarly we can prove that p_{14} must be adjacent to 3 points in A_5 to prove that p_{14} and p_{41} are both adjacent to 4 points, another contradiction.

In case (ii), without loss of generality, let p_{12} be adjacent to p_{31} and p_{13}; and p_{15} be adjacent to p_{14} and p_{41}.



It is easy to see that p_{31} cannot be adjacent to p_{41} therefore let

p_{31} be adjacent to p_{13} and p_{41} be adjacent to p_{14}. Again p_{21} is not adjacent to p_{31} and both are adjacent to p_{12} therefore p_{21} and p_{31} are adjacent to distinct subsets of points in B each consisting of 4 points. Let p_{21} be adjacent to the points in A_3 and p_{31} be adjacent to the points in A_4. Similarly p_{21} and p_{41} are adjacent to distinct points in B and let p_{41} be adjacent to all the points in $A_6 = \{p_{32}, p_{43}, p_{44}, p_{45}\}$. The reason we have $A_4 \cap A_6 = \{p_{32}\}$ is that p_{31} and p_{41} both are not mutually adjacent to any point in A therefore both must be adjacent to one point in B. Now p_{13} is not adjacent to p_{21} and both are adjacent to p_{12} therefore p_{13} is not adjacent to any point in A_3 and similarly p_{13} is not adjacent to any point in A_6. Now one out of the 3 points in B to which p_{13} is adjacent could be p_{42} (since p_{42} is not adjacent to p_{21}, p_{31} and p_{41}). So at least 2 of the 3 points to which p_{13} is adjacent in B must be in $A_4 - \{p_{32}\}$. This implies that p_{13} and p_{31} two adjacent points in G are both adjacent to at least 4 points namely p_{11}, p_{12} and at least two points in A_4 other than p_{32} which contradicts (3). Similarly p_{14} is not adjacent to p_{31} and both are adjacent to p_{13} therefore p_{14} is not adjacent to any point in A_4 and similarly p_{14} is not adjacent to any points in A_3 and moreover p_{14} could be adjacent to p_{42}. So if p_{14} is adjacent to p_{42} then this implies that p_{14} is adjacent to at least 2 points in $A_6 - \{p_{32}\}$ which in turn implies that p_{14} and p_{41} both are adjacent to at least 4 points and two of the 4 points are p_{11} and p_{15} which contradicts (3). Hence the induced subgraphs on the points in $A_1 \cup \{p_{11}\}$ is K_5 and on the points in $A_2 \cup \{p_{11}\}$ is K_4.

We have now seen that, in all cases that concern us, G contains a family of cliques and that

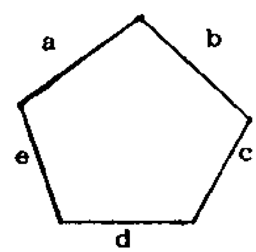
(i) for each point p of G, there exist exactly two cliques of the family whose union contains all points adjacent to p.

(ii) any two cliques of the family have at most one point of G in common.

It follows [6] that G is a line graph. We further know that

(iii) if K^1 and K^2 are the two cliques containing p, and if $p \neq q$, $q \in K^1$, $p \neq r$, $r \in K^2$, then q and r are not adjacent.

This means that, if $G = L(H)$, H contains no triangle. It follows that H must be bipartite. If not, let C be the smallest (odd) cycle contained in H. C cannot be of order 3, since H contains no triangle. If C is of order 5, say

C =


then in G, point a and point d are mutually adjacent to exactly one point of $L(H) = G$, contradicting (2). Clearly, the same argument works if order of $C = 7, 9, \ldots$ so H is bipartite. Indeed, H must be a complete bipartite graph, otherwise H would contain a cycle of (even) order ≥ 4 with no chords, again contradicting (2). Hence, H is a complete bipartite graph, say K_{ab}, $a \geq b$. Now since $|V(G)| = mn$ therefore $ab = mn$, moreover the valence of G is $m+n-2$ therefore $a+b-2 = m+n-2$. From these two equations it follows that $a = m$ and $b = n$, therefore $H = K_{mn}$.

Supplement: Spectral Characterization of $L(K_{n,n,n})$

1. INTRODUCTION

If G is a graph, its line graph $L(G)$ has for its vertices the edges of G, with two vertices of $L(G)$ adjacent if the corresponding edges of G have a common vertex. If G is a graph, its adjacency matrix $A(G) = (a_{ij})$ is defined by the rule: $a_{ij} = 1$ if i and j are adjacent vertices of G, 0 otherwise. The symbol $K_{n,n}$ stands for a graph with $2n$ vertices, partitioned into two sets of n vertices each, such that two vertices are adjacent if and only if they are in different parts. The symbol $K_{n,n,n}$ denotes a graph with $3n$ vertices, partitioned into three sets of n vertices each, such that two vertices are adjacent if and only if they are in different parts. The matrix which has all entries unity is denoted by J.

In [6], S. Shrikhande proves the following theorem about $L(K_{n,n})$. Let

$$(1.1) \qquad Q(x) = \frac{1}{2}(x^2-(n-4)x-(2n-4)).$$

If G is a graph with $|V(G)| = n^2$, $n \geq 2$ and if $A = A(G)$, $Q(A) = J$, where $Q(x)$ is defined by (1.1), then $G = L(K_{n,n})$ unless $n = 4$ (where there is exactly one exception). We shall discuss the analogous question for $L(K_{n,n,n})$.

<u>Theorem</u>. Let

$$(1.2) \qquad P(x) = \frac{1}{8}(x-(2n-2))(x-(n-2))(x+2).$$

If G is a graph with $|V(G)| = 3n^2$, $n \geq 2$, and if $P(A(G)) = J$, where $P(x)$ is defined by (1.2), then $G = L(K_{n,n,n})$.

Note that, by contrast with [6], there are no exceptional values of n.

2. POLYNOMIAL OF A GRAPH

In this section, we first show that $P(A(L(K_{n,n,n}))) = J$. Then we prove that, if $P(A(G)) = J$, where $|V(G)| = 3n^2$, then G is a regular connected graph with valency $4n-2$, whose adjacency matrix has eigenvalues $4n-2$ (of multiplicity 1), $2n-2$ (of multiplicity $3n-3$), $n-2$ (of multiplicity 2), and -2 (of multiplicity $3n^2-3n$).

We begin by explaining the concept of polynomial of a graph [4]. If H is any graph such that there is some polynomial $R(x)$ with $R(A(H)) = J$, then H is regular and connected. Conversely, assume H regular and connected of valence d, $|V(H)| = m$, and $A(H)$ has $d>\beta_1>\ldots>\beta_r$ as its distinct eigenvalues. Define $Q(x)$ by

$$(2.1)\qquad Q(x) = m\,\frac{\prod(x-\beta_i)}{\prod(d-\beta_i)} \quad .$$

Then $Q(x)$ is the unique polynomial $R(x)$ of least degree satisfying $R(A(H)) = J$; if $R(x)$ is any polynomial such that $R(A(H)) = J$, then $R(x) = Q(x) + S(x)\,(x-d)\prod(x-\beta_i)$, where $S(x)$ is an arbitrary polynomial.

Let us therefore find the eigenvalues of $A(L(K_{n,n,n}))$. As explained in [2], if G is a regular connected graph on m vertices of valence d, with distinct eigenvalues $d>\beta_1>\ldots>\beta_r$ of respective multiplicities $1, m_1, \ldots, m_r$, and if $\beta_r>-d$ and $(md)/2>m$, then $L(G)$ is a regular connected graph on $(md)/2$ vertices of valence $2d-2$, with distinct eigenvalues $2d-2>\beta_1+d-2>\ldots>\beta_r+d-2>-2$, of respective multiplicities $1, m_1, \ldots, m_r, (md)/2-m$. Now $K_{n,n,n}$ has $3n$ vertices and distinct eigenvalues $2n$ of multiplicity 1, 0 of multiplicity $3n-3$, $-n$ of multiplicity 2. It follows that

(2.2) $\quad A(L(K_{n,n,n}))$ has eigenvalues

$4n-2$	of multiplicity 1
$2n-2$	of multiplicity $3n-3$
$n-2$	of multiplicity 2
-2	of multiplicity $3n^2-3n$.

From (2.1), we see that (1.2) is the polynomial of $L(K_{n,n,n})$.

Now suppose that G is any graph on $3n^2$ vertices such that $P(A(G))=J$, where $P(x)$ is given by (1.2). We first prove that its distinct eigenvalues are given by the numbers in (2.2), deferring consideration of multiplicities until later. By the second paragraph of this section, either $P(x)$ is the polynomial of G or $P(x)$ is the sum of the polynomial of G and a nonzero multiple of the minimum polynomial of G. We want to show that the latter is impossible. The first case is that G has only two distinct eigenvalues. By [4] G is a clique on $3n^2$ vertices, one eigenvalue is -1, and $(x+1)$ divides $P(x)$. But this is impossible (remember $n>1$). The second case is where G has 3 distinct eigenvalues $d>\epsilon_1>\epsilon_2$. Then $P(x)$ must be

$$(2.3)\quad \frac{1}{8n}(x-d)(x-\epsilon_1)(x-\epsilon_2) + 3n^2\frac{(x-\epsilon_1)(x-\epsilon_2)}{(d-\epsilon_1)(d-\epsilon_2)}\ .$$

This implies that ϵ_1 and ϵ_2 are two of the three numbers $2n-2$, $n-2$, -2. One of them must be -2, since $A(G)$ must have a negative eigenvalue. From (2.3) and (1.2) we have either

$$(2.4)\quad -d+\frac{24n^3}{(d+2)(d-(n-2))} = -2n+2$$

or

$$(2.5)\quad -d+\frac{24n^3}{(d+2)(d-(2n-2))} = -n+2.$$

In both (2.4) and (2.5), this implies $d = 4n-2$. But there is no regular connected graph on $3n^2$ vertices with distinct eigenvalues $4n-2$, $n-2$, -2 or with distinct eigenvalues $4n-2$, $2n-2$, -2. The reason is that, in both cases, one can calculate the multiplicities of the remaining eigenvalues, using $\mathrm{Tr}A = 0$ and the fact that the multiplicity of $4n-2$ is 1. Then $\mathrm{Tr}A^2 = 3n^2(4n-2)$, since the diagonal entries of A^2 are all $4n-2$. But the other calculation of $\mathrm{Tr}A^2$ is the sum of the squares of the eigenvalues of A reveals a contradiction.

This proves that the numbers appearing in (2.2) are the complete list of distinct eigenvalues of G. To verify that the multiplicities are correct, let m_1 be the multiplicity of $2n-2$, m_2 the multiplicity of $n-2$, m_3 the multiplicity of -2. We know

$$m_1+m_2+m_3 = 3n^2-1$$

$$(2n-2)\,m_1+(n-2)m_2+(-2)m_3 = -(4n-2), \text{ since } \mathrm{Tr}A = 0$$

$$(2n-2)^2m_1+(n-2)^2m_2 + (-2)^2m_3=3n^2(4n-2)-(4n-2)^2,$$

since $\mathrm{Tr}A^2 = 3n^2(4n-2)$.

This system of equations can be solved, giving the desired answers. But we can avoid the calculation by appealing to a principle of sufficient information. Since we know the answer is unique, because the Vandermonde matrix is nonsingular, and the only information needed is that (1.2) is the polynomial of G, we can know the unique answer if we know it for *any* graph G satisfying the condition, namely $L(K_{n,n,n})$.

3. PROOF OF THEOREM

Our object is to establish (3.15) and (3.16) below. Very recent results of Cameron, Goethals, Seidel and Shult [1] enable one to go directly from the preceding material to (3.15) if n>2. The following more pedestrian argument (covering also n=2) may nevertheless be of some interest.

We begin by considering the matrix

$$(3.1)\quad B = 3(J-I-A) + (2n-1)A - A^2 + (4n-1)I,$$

where G satisfies (1.2) and $A = A(G)$. Observe that every diagonal entry of B is 1. We first prove that B has eigenvalues n^2 of multiplicity 3, and 0 of multiplicity $3n^2-3$. The proof is to use the results of the preceding section, and the fact that J and A commute, with the eigenvalue $3n^2$ of J corresponding to the eigenvalue $4n-2$ of A. One can then calculate. Alternatively, one can use the principle of sufficient information to describe B if $A = A(L(K_{n,n,n}))$.

Next, we prove that every entry in B is 0 or 1. To do this, it is sufficient to show that, for each i, $\sum_j b_{ij} = \sum_j b^2_{ij}$. But we can calculate the former sum by using the fact that $Je=3n^2e$, $Ae=(4n-2)e$, where e is the vector of 1's. We can calculate the latter sum by observing it $(B^2)_{ii}$. Since we know $0 = A_{ii}$, $(4n-2) = (A^2)_{ii}$, and $(A^3)_{ii}$ and $(A^4)_{ii}$ can be calculated from (1.2), and $(J^2)_{ii} = 3n^2$, $(AJ)_{ii} = (JA)_{ii} = 4n-2$, $(A^2J)_{ii} = (JA^2)_{ii} = (4n-2)^2$ it follows that we can calculate $(B^2)_{ii}$. The desired result will follow from the calculation, and we invoke the principle of sufficient information and avoid the calculation.

Since each $b_{ii} = 1$, and B is positive semi-definite, it follows that $b_{ij} = 1$, $b_{jk} = 1$ implies $b_{ik} = 1$. Otherwise

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

which has $1-\sqrt{2} < 0$ for one of its eigenvalues, would be a principal submatrix of B, contradicting the fact that B is positive semi-definite. So B is a matrix with blocks of J's on the diagonal, 0 everywhere else. But since B has exactly 3 nonzero eigenvalues, each n^2, B in fact has exactly 3 J's on the diagonal, each of order n^2. We call the corresponding three sets of vertices V_1, V_2, V_3, and they partition $V = V(G)$. The corresponding induced subgraphs we call G_1, G_2, G_3. Now the definition of B in (3.1) tells us

(3.2) If $v, w \in V_i$ $(i=1,2,3)$, v and w not adjacent there are exactly two vertices in V adjacent to both.

(3.3) If $v, w \in V_i$ $(i=1,2,3)$, v and w adjacent, there are exactly $2n-2$ vertices in V adjacent to both.

(3.4) If $v_i \in V_i$, $v_j \in V_j$ $(i \neq j)$, v_i and v_j not adjacent, there are exactly 3 vertices in V adjacent to both.

(3.5) If $v_i \in V_i$, $v_j \in V_j$ $(i \neq j)$, v_i and v_j adjacent, there are exactly $2n-1$ vertices in V adjacent to both.

Next we show

(3.6) every vertex in V_i $(i=1,2,3)$ has valence $2n-2$ in G_i. This follows from observing that the given valence is a diagonal entry in AB.

(3.7) Every vertex in V_i is adjacent to exactly n vertices in V_j and exactly n vertices in V_k, i, j, k distinct.

To prove (3.7), first partition $A(G)$ as

$$A(G) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where $A_{ii} = A(G_i)$, i 1,2,3. Let $C_1 = [A_{11}, A_{12}, A_{13}]$, $D_1 = [A_{12}, A_{13}]$. We calculate the sum of the square of the entries of $C_1C_1^T$ in two ways, namely $e'(C_1C_1^T)e = (e'C_1)(C_1^Te)$. The matrix $C_1C_1^T$ is the first diagonal block of A^2. By (3.2), (3.3), and the fact that each diagonal entry of A^2 is $4n-2$, $e'(C_1C_1^T)e$ can be calculated. On the other hand $(e'C_1)(C_1^Te)$ is the sum of the squares of the sums of the columns of C_1. The column sums in A_{11} are all $2n-2$ by (3.6). The remaining column sums add up to $n^2(2n)$. The minimum possible value for the sum of the squares of the column sums in D_1 occurs if and only if they are all equal, namely n. But that minimum is what happens (by the principle of sufficient information). Hence, each vertex in V_2 and each vertex in V_3 is adjacent to exactly n vertices in V_1. Clearly this argument proves (3.7) for all i,j,k.

(3.8) If $v,w \in V_i$, (i=1,2,3) and are not adjacent, there are exactly two vertices in V_i adjacent to v and w.

To prove (3.8), assume $v,w \in V_1$, not adjacent, $v \in V_2$ adjacent to both v and w (recall (3.2)). Let y be the other vertex adjacent to v and w. Suppose first that v is adjacent to x. Then by (3.5) there are exactly $2n-2$ other vertices adjacent to x and v, but not to w, and exactly $2n-2$ other vertices adjacent to x and w, but not to v. Counting v, w and y, this gives $4n-1$ vertices adjacent to x, which has valence $4n-2$, a contradiction.

Suppose y is not adjacent to x. Then we get 4n vertices adjacent to x, also a contradiction.

(3.9) If $u \in V_i$, the n vertices in $V_j (j \neq i)$ adjacent to u in (3.7) form a complete subgraph in V_j.

Otherwise, we would have two vertices in V_j not adjacent, and a vertex in V_i adjacent to both, contradicting (3.2) and (3.8).

(3.10) If u, $v \in V_i$, u,v adjacent, then there are exactly n-2 vertices in V_i adjacent to both.

Since the valence of u in G_i is 2n-2, and there are 2n-2 vertices in V adjacent to both, there must be at least one vertex x in V_j, $j \neq i$, adjacent to both u and v. By (3.9), there are at least n-2 other vertices in V_i which are adjacent to u and v. Now consider any row of the matrix $(A_{ii})^2$. The sum of the entries in that row is $(2n-2)^2$. On the other hand, the diagonal entry in that row is 2n-2, the off diagonal entries are 2 in case the corresponding entry of A_{ii} is 0, and <u>at least</u> n-2 if the corresponding entry in A_{ii} is 1. But the second way of calculating the sum of the entries yields the same number as the first only if there are <u>exactly</u> n-2 vertices in V_i adjacent to u and v.

(3.11) Each G_i is isomorphic to $L(K_{n,n})$.

Each G_i has n^2 vertices, each vertex has valence 2n-2, and (3.8) and (3.10) show that the hypotheses of Shrikhande's theorem [6] hold, so (3.11) is true except possibly if n = 4. But in that exceptional case, it is false that the vertices adjacent to two adjacent vertices are adjacent to each other. But the proof of (3.10) shows that to be true. Hence we do not have the exceptional case even if n = 4.

(3.12) If $u_i \in V_i, u_j \in V_j$, u_i adjacent to u_j, $i \neq j$, there is exactly one vertex $u_k \in V_k$, $k \neq i, j$, which is adjacent to u_i and u_j.

By (3.9), there are exactly n-1 vertices in V_i adjacent to u_i and u_j, exactly n-1 vertices in V_j adjacent to u_i and u_j. By (3.5), there is exactly one other vertex in V adjacent to u_i and u_j, which must be in V_k.

(3.13) If $u_i \in V_i$ and $u_j \in V_j$, $i \neq j$, are adjacent, the 2n-2 vertices in V_i and V_j adjacent to both, together with u_i and u_j, form a clique in G.

Let $u_1 \in V_1$ and $u_2 \in V_2$ be adjacent, $v_1 \in V_1$ adjacent to u_1 and u_2, $v_2 \in V_2$ adjacent to u_1 and u_2. We must show v_1 and v_2 are adjacent, so we assume otherwise. Now by (3.12) there is a $u_3 \in V_3$ adjacent to both u_1 and u_2. If u_3 is adjacent to neither v_1 nor v_2, then these 5 vertices $(u_1, u_2, u_3, v_1, v_2)$ determine an induced subgraph which cannot be an induced subgraph of a regular connected graph which has -2 for its least eigenvalue, by [5]. Therefore, u_3 is adjacent to at least one of v_1 and v_2. Suppose u_3 is adjacent to v_1. Then the vertices u_2, u_3, u_1, v_1 violate (3.12).

(3.14) Each point in V is common to exactly two distinct complete subgraphs on 2n points in G.

Proof. Let $v_i \in V_i$. By (3.7) and (3.9) there exist two complete subgraphs on n vertices each, namely $K_n^{(j)}$ and $K_n^{(k)}$ in G_j and G_k respectively (i,j and k are all distinct) such that v_i is adjacent to all the n vertices in $K_n^{(j)}$ and n vertices in $K_n^{(k)}$. Now by (3.6) v_i is adjacent to 2n-2 points in V_i and by (3.13) v_i is on a complete subgraph namely $K_{2n}^{(ij)}$ of G which has the n vertices of $K_n^{(j)}$ and n-1 vertices in V_i other than v_i, and similarly v_i is also on a complete subgraph $K_{2n}^{(ik)}$ of G which has the n vertices of $K_n^{(k)}$ and n-1 vertices in V_i other than v_i.

We claim that v_i is the only vertex common to both $K_{2n}^{(ij)}$ and $K_{2n}^{(ik)}$. Suppose not and let u be a vertex in V_i which is also common to $K_{2n}^{(ij)}$ and $K_{2n}^{(ik)}$. Then we have v_i adjacent to u in V_i and both are adjacent to more than 2n vertices in V which contradicts (3.3). Hence each point in V is common to exactly two distinct complete subgraphs K_{2n} in G.

(3.15) G is a line graph.

Since G is regular of degree 4n-2 and $|V(G)| = 3n^2$ therefore total number of edges in G is $3n^2(4n-2)/2=3n^2(2n-1)$ and total number of edges in K_{2n} is $\frac{2n(2n-1)}{2} = n(2n-1)$. So $3n^2(2n-1)$ edges can be partitioned into $\frac{3n^2(2n-1)}{n(2n-1)} = 3n$ edge disjoint K_{2n}'s such that each vertex is common to exactly two of the distinct K_{2n}'s. This implies [3] that G is a line graph.

(3.16) G is $L(K_{n,n,n})$.

Since G = L(H) for some graph H, and G is regular and connected, then H is regular, or H is bipartite and not regular [2]. The results of [2] show that H is regular and not bipartite. It follows from Section 2 that H is a regular graph on 3n vertices of valence 2n-2 with distinct eigenvalues 2n, 0, -n of respective multiplicities 1,3 n-3 and 2. Therefore J-A(H) is a (0,1) matrix with eigenvalues n and 0 of multiplicities 3 and 3n-3 respectively. But this means H is $K_{n,n,n}$.

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