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Multiplicities of Galois Representations in the Higher Weight Sheaf Cohomology Associated to Shimura Curves

by

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Abstract

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by

Lei Yang

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Let $\mathbf{T}_N$ be the Hecke algebra attached to $S_k(\Gamma_0(N), \varepsilon)$ where ε is a Dirichlet character. For a maximal ideal $\mathbf{m}$ of $\mathbf{T}_N$ with residue characteristic ℓ, assume that the modular Galois representation $\rho_{\mathbf{m}}$ of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ attached to $\mathbf{m}$ is irreducible. We generalize Ribet's theorem regarding the multiplicities of $\rho_{\mathbf{m}}$ in jacobians of Shimura curves to higher even weights. The two main results of this paper are as follows:

(1) Let p, q be two distinct primes not equal to ℓ, and M a positive integer prime to $pq\ell$. We show that for a Shimura curve $V_B(M)$ arising from an Eichler order of conductor M in the indefinite quaternion algebra B of discriminant pq over $\mathbf{Q}$, if $\rho_{\mathbf{m}}$ is ramified at at least one of the primes dividing the discriminant, say p, the multiplicity of $\rho_{\mathbf{m}}$ in $H^1(V_B(M) \times \bar{\mathbf{Q}}, \bar{\vartheta}_\ell)[\mathbf{m}]$ is 1, unless $\rho_{\mathbf{m}}$ is unramified at q and Frob_q acts as a scalar in $\rho_{\mathbf{m}}$ whose square is $\varepsilon(q)$. In this exceptional case, the multiplicity is 2. We show that this exceptional case does occur.

(2) We give an upper bound for the multiplicity in the case when the discriminant of B is arbitrary and $\rho_{\mathbf{m}}$ is ramified at at least half of the primes dividing the discriminant. We prove that higher multiplicities also exist in this case.

in memory of my mother

Preface

Let $k \geq 2$ be a positive integer. For each positive integer N, set

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

For a Dirichlet character $\varepsilon : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, define the space $S_k(\Gamma_0(N), \varepsilon)$ of weight k cusp forms of level N and Nebentypus ε to be the set of all holomorphic functions f on the Poincaré upper half plane, such that

(i) For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have $f\left(\frac{az+b}{cz+d}\right) = \varepsilon(d)(cz+d)^k f(z)$;

(ii) f is holomorphic at the cusps.

The second condition above means that the Fourier expansion of f at ∞ is of the form $\sum_{n\geq 1} a_n(f) e^{2\pi i n}$. Let $\mathbf{T} = \mathbf{T}_N$ be the standard Hecke algebra generated by the Hecke operators $\{T_r\}_{r\geq 1}$ on the space $S_k(\Gamma_0(N), \varepsilon)$. If we let $\mathbf{Z}[\varepsilon]$ be the ring of rational integers joined with the values of ε, $\mathbf{T}$ is then a projective $\mathbf{Z}[\varepsilon]$-module of finite rank. Let $\mathbf{F}$ be a finite field of characteristic $\ell \geq 3$. Fix an algebraic closure $\bar{\mathbf{F}}$ of $\mathbf{F}$. Let

$$\rho : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}(2, \mathbf{F}),$$

be a continuous representation. Let χ be the mod ℓ cyclotomic character, and identify ε with a character of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ via the surjective homomorphism $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to (\mathbf{Z}/N\mathbf{Z})^\times$. The representation ρ is said to be *modular of level N and weight k with Nebentypus ε* (of type (N,k,ε) for short) if its determinant is $\chi \cdot \varepsilon^{k-1}$, and there is a homomorphism

$$\omega : \mathbf{T} \to \bar{\mathbf{F}},$$

such that

$$\operatorname{trace}(\rho(\mathrm{Frob}_r)) = \omega(T_r)$$

for all prime numbers $r \nmid \ell N$, where Frob_r is a Frobenius element for r in the Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}_r/\mathbf{Q}_r)$.

The problem of this paper stems from the so-called *Multiplicity One Principle*, first considered by Mazur in [21] in the case of jacobians of modular curves. To be more specific, let $X_0(N) = Y_0 \cup \{\text{cusps}\}$ be the complete modular curve of level N, and $J_0(N)$ its jacobian. For a maximal ideal $\mathbf{m}$ of the Hecke algebra $\mathbf{T}$, let $\rho_{\mathbf{m}}$ be the modular Galois representation attached to $\mathbf{m}$ (see [25], Chapter 5 for more detail). Consider the $\mathbf{T}/\mathbf{m}[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$-module $J_0(N)[\mathbf{m}]$, the kernel of the action of $\mathbf{m}$ on $J_0(N)$. If $\rho_{\mathbf{m}}$ is absolutely irreducible and $(\ell, 2N) = 1$, Mazur ([21], Proposition 14.2) shows that $J_0(N)[\mathbf{m}]$ is isomorphic to the representation $\rho_{\mathbf{m}}$. It is interesting to see if the Multiplicity One Principle holds for the jacobians of other curves. One immediate choice is to look at the Shimura curves. Let J be the jacobian of a Shimura curve. When $\rho_{\mathbf{m}}$ is irreducible and ℓ is odd, a general result proved in [1] shows that $J[\mathbf{m}]$ is semisimple. Mazur's standard argument then shows that $J[\mathbf{m}]$ is a direct sum of copies of $\rho_{\mathbf{m}}$. The number of copies in the direct sum is the *multiplicity* of $\rho_{\mathbf{m}}$ in $J[\mathbf{m}]$. In [26], Ribet studies the case when $\rho_{\mathbf{m}}$ is of type $(N, 2, \mathrm{id})$, and the Shimura curve V_B arises from an indefinite quaternion algebra B over $\mathbf{Q}$ with Disc $B = pq$, a product of two distinct primes. It turns out that the Multiplicity One Principle is true under most circumstances but fails to hold in one particular case. Namely, when ρ_{m} is unramified at exactly one of the two primes in the discriminant of B and the Frobenius element at that prime acts as ± 1 in ρ_{m}, the multiplicity becomes two.

In this paper we generalize Ribet's multiplicity theorem cited above to higher even weight with arbitrary Nebentypus in two stages: first we consider Shimura curves V_B when Disc B is a product of two distinct primes; then we remove this restriction and consider the case when Disc B is arbitrary. The generalization is, however, not without its limitations. The assumption on the weight (namely, even) is probably not essential. One should be able to work out similar results when the weight is odd, using the method provided here with some adjustments on the construction of the Symm^{k-2} representation of $B^\times$ and the ℓ-adic sheaf ϑ_ℓ. But we do need the condition that the representation ρ_{m} is *dominantly ramified at the discriminant* (Definition 4.4), so that we can equate the weight k dual character groups of two Shimura curves (localized at a maximal ideal of the Hecke algebra), and relay the multiplicity information encoded in the first Shimura curve arising from a quaternion algebra with (two) fewer prime factors in its discriminant to the second curve. So we begin by observing modular curves. And here we also inherit the restriction of $\ell > k$ from Faltings-Jordan's Multiplicity One Principle for higher weight modular curves (Theorem

3.1) obtained via crystalline cohomology theory. Under these conditions, we will be able to completely classify the multiplicity in the first stage of the generalization (Disc $B = pq$). As for the second stage (arbitrary discriminant), we provide an upper-bound of the multiplicity. However, we do not claim that it is effective. But we prove that higher multiplicities do exist. We also wish to remove the dominant ramification condition on the representation ρ_{m} in the future. In the case when Disc $B = pq$, Fred Diamond, in a conversation with the author, first raised the question about the multiplicity of ρ_{m} in $J_0[\mathrm{m}]$ in the case that ρ_{m} is unramified at both p and q. This was never an issue in Ribet's case: if ρ_{m} were unramified at both p and q, by applying conjecture "epsilon," one would conclude that ρ_{m} would be modular of level 1. But there is no weight two cusp form in $S_2(\Gamma_0(1))$. In higher weight, however, it is entirely possible that ρ_{m} is unramified at both primes.

Let us now describe a little more about our method in this paper. In Chapter 1 we construct the weight k analogue of the weight two (dual) character groups of the special fibers of Shimura curves in two different ways. First in Section 1.1 we describe Jordan-Livné's theory on local systems attached to models of Symm^{k-2} representations of Eichler orders in definite quaternion algebras over $\mathbf{Q}$. Here the language of Jacquet-Langlands provides a natural connection between the space of automorphic forms on the multiplicative group of a quaternion algebra and the groups of cochains of the dual graph of the special fiber of the curve. And that makes it possible to transport the Hecke structure of the former to the latter. In Section 1.2 we use étale cohomology of a certain weight k sheaf ϑ_ℓ defined by Carayol to give another construction of the character group. The Hecke-compatibility of the two constructions will be provided in a forthcoming joint work by Jordan and Livné. In Chapter 2 we study the structure of the group $H^1(C, \vartheta_\ell)(k-1)$, the higher weight equivalent to the jacobian of the curve C in weight two. In Section 2.1 we prove a formula (Theorem 2.5) relating the fixed part and the toric part of $H^1(C, \vartheta_\ell)(k-1)$ with the locally constant sheaf ϑ_ℓ. For semistable curves with constant sheaves, the theorem was established in [13] and called Orthogonality Theorem by Grothendieck. It was reformulated by Illusie in [14] to explicitely realize Grothendieck's monodromy pairing on character groups. Our method used here is a close study of Illusie's. In Section 2.2 we prove several important formulas concerning the bad reductions of a Shimura curve C. Then we make a little adjustment in Section 2.3 to extend our results in the previous sections to the case when C is a modular curve. In Chapter 3 we give a proof of conjecture "epsilon" for higher weight. Even though this result was given in [16], [28] and [9], we still find it beneficial to include its proof here in order to provie the necessary language to be used later. The method used here, however, is based on a proof used in [26]. Chapter 4 deals with the generalization of the multiplicity problem. The proof of the two-prime case requires Faltings-Jordan's Multiplicity One Principle, conjecture "epsilon," as well as various results on Hecke actions proved in the previous chapter; the arbitrary discriminant case is obtained by two recursive inequalities. See Theorems 4.3 and 4.5 for the results. In the end we show that there are infinitely many kernels with multiplicities at least two. For this, in addition to the tools we mentioned above, one also needs a result of "raising levels" from Diamond-Taylor [10].

This paper is the author's doctoral dissertation at the Graduate School and University Center of the City University of New York. It is the author's great pleasure here to acknowledge his indebtedness to his advisor Bruce Jordan for introducing him into this field and laying the foundation of this paper in his joint works with Gerd Faltings ([12]) and Ron Livné ([16], [17] and [18]). Many ideas in the first three chapters of this paper were originated in [16](construction of local systems, Theorems 2.8, 2.9, 2.13 and 2.15 to be specific). Without his generous support through the past several years the completion of this paper would not be possible. The author also would like to take this opportunity to express his gratitude to Ron Livné, for his continuous interest in this paper and valuable suggestions made during his several visits to the States; to Ken Ribet, for his ubiquitous influence through his works, especially [25] and [26], and for insightful discussions and correspondences; and to Fred Diamond, for helpful conversations and posing the question of multiplicities without the dominant ramification condition.

Contents

Multiplicities of Galois Representations in the Higher Weight Sheaf Cohomology Associated to Shimura Curves

1 Constructions of high weight character groups

1.1 Local system à la Jordan-Livné

In this section, we describe the results established in [17] on structures of local systems induced by Symm^{k-2} representations arising from definite quaternion algebras, and their relationship with certain spaces of automorphic forms. We also explain how Hecke algebras are attached to these spaces. In presenting Jordan-Livné's theory here, we will basically follow their notation in [17].

We begin by recalling Serre's definition of an oriented graph in [29]. Hence a graph $\mathfrak{G}$ consists of two sets $\mathrm{Ver}(\mathfrak{G})$, $\mathrm{Ed}(\mathfrak{G})$, and two maps

$$\mathrm{Ed}(\mathfrak{G}) \to \mathrm{Ver}(\mathfrak{G}) \times \mathrm{Ver}(\mathfrak{G}), \qquad e \mapsto (o(e), t(e))$$

and

$$\mathrm{Ed}(\mathfrak{G}) \to \mathrm{Ed}(\mathfrak{G}), \qquad e \mapsto \bar{e}$$

satisfying the following conditions: for each $e \in \mathrm{Ed}(\mathfrak{G})$, we have $\bar{\bar{e}} = e$, $\bar{e} \neq e$ and $o(e) = t(\bar{e})$. An element in $\mathrm{Ver}(\mathfrak{G})$ is called a vertex, while a member of $\mathrm{Ed}(\mathfrak{G})$ is called an edge. The edge $\bar{e}$ is the inverse of e. By an orientation of the graph $\mathfrak{G}$ we mean a partition of the set of edges $\mathfrak{G} : \mathrm{Ed}(\mathfrak{G}) = E_1 \cup E_2$ such that $E_2 = \{\bar{e} | e \in E_1\}$.

Let $\mathcal{A}$ be a commutative ring with identity. We now define the notion of a local system on a graph $\mathfrak{G}$.

Definition 1.1 *An $\mathcal{A}$-local system on $\mathfrak{G}$ is a collection*

$$\mathcal{L} = \{\mathcal{L}(v), v \in \mathrm{Ver}(\mathfrak{G});\ \mathcal{L}_e, e \in \mathrm{Ed}(\mathfrak{G})\},$$

such that each $\mathcal{L}(v)$ is an $\mathcal{A}$-module, and each $\mathcal{L}_e$ is an $\mathcal{A}$-isomorphism from $\mathcal{L}(o(e))$ to $\mathcal{L}(t(e))$. Furthermore, $\mathcal{L}_e = \mathcal{L}_{\bar{e}}^{-1}$ for all $e \in \mathrm{Ed}(\mathfrak{G})$.

For a graph $\mathfrak{G}$ endowed with a local system $\mathcal{L}$, we define the group of 0-cochains to be the group

$$C^0(\mathfrak{G},\mathcal{L}) = \{s :\ \mathrm{Ver}(\mathfrak{G}) \to \prod_{v \in \mathrm{Ver}(\mathfrak{G})} \mathcal{L}(v) \mid s(v) \in \mathcal{L}(v),\ \forall v\}$$

and the group of 1-cochains as the group

$$C^1(\mathfrak{G},\mathcal{L}) \;=\; \{f :\ \mathrm{Ed}(\mathfrak{G}) \to \prod_{e \in \mathrm{Ed}(\mathfrak{G})} \mathcal{L}(o(e)) \mid f(e) \in \mathcal{L}(o(e)), f(\bar{e}) = -\mathcal{L}_e(f(e)),\ \forall e\}.$$

The coboundary operator $d :\ C^0(\mathfrak{G},\mathcal{L}) \to C^1(\mathfrak{G},\mathcal{L})$ sends $\{s(v)\}_{v \in \mathrm{Ver}(\mathfrak{G})}$ to $\{f(e)\}_{e \in \mathrm{Ed}(\mathfrak{G})}$ such that

$$f(e) = \mathcal{L}_e^{-1}(s(t(e))) - s(o(e)).$$

Let $B/\mathbf{Q}$ be a definite quaternion algebra of discriminant Disc B. Let $\mathbf{B}^\times$ be the algebraic group over $\mathbf{Q}$ associated with $B^\times$, the multiplicative group of B. The adèles and finite idèles of $\mathbf{B}$ are denoted by $\mathbf{B}^\times(\mathbf{A})$ and $\mathbf{B}^\times(\mathbf{A}^f)$, respectively. Let $\mathcal{M}$ be an Eichler order of conductor M in B with $(M, \text{Disc } B) = 1$. Suppose p is a prime such that $(p, M\text{Disc } B) = 1$. Let Δ be the well-known tree attached to $\mathbf{SL}(2, \mathbf{Q}_p)$ (cf. [29], Chapter II, §1), and $\varepsilon :\ (\mathbf{Z}/M\mathbf{Z})^\times \to \mathbf{C}^\times$ a Dirichlet character. For any positive even integer k, Jordan and Livné have constructed in [17] a $\mathbf{Z}[\frac{1}{p}]$-model $\mathcal{L}_{k-2}$ for the Symm^{k-2} representation of $\mathcal{M}[\frac{1}{p}]^\times$. The group $\mathcal{M}[\frac{1}{p}]^\times$ then acts on $\mathcal{L}_{k-2}$ naturally via this representation. Set

$$\Gamma = \{\gamma \in \mathcal{M}[\frac{1}{p}]^\times \mid \mathrm{Nm}_{B/\mathbf{Q}}(\gamma) \text{ has even } p\text{-adic valuation}\}.$$

Fix an embedding of $\mathcal{M}$ into $\mathrm{M}(2, \mathbf{Z}_p)$, so that $\mathcal{M}^\times$ is mapped into the subgroup K_p of matrices in $\mathbf{GL}(2, \mathbf{Z}_p)$ with the element on the lower left corner divisible by M. The group Γ is then viewed as a discrete co-compact subgroup of $\mathbf{GL}(2, \mathbf{Q}_p)$. It therefore acts on the tree Δ via this embedding since it is a standard fact that Δ is a quotient of $\mathbf{GL}(2, \mathbf{Q}_p)$. Denote the ring of values of ε by $\mathbf{Z}[\varepsilon]$, and let Γ act on it via multiplication by $\varepsilon(\bar{d}(\gamma_0))^{-1}$ where, for $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $\bar{d}(\gamma_0) = d \,(\mathrm{mod}\, M)$. Put $\mathcal{L}_{k-2}[\varepsilon] \overset{\mathrm{def}}{=} \mathcal{L}_{k-2} \otimes \mathbf{Z}[\varepsilon]$. Then the group Γ also acts on the product by applying the diagonal action. In case Γ has no elliptic elements, it acts freely on Δ. According to [15], Section 1.7, the above data will then induce a local system $\mathcal{L}$ on the finite graph $G = \Gamma\backslash\Delta$ which can be calculated by *equivariant cochains* on Δ. Namely, for $i = 1, 2$, if we let $C^i_\Gamma(\_,\_)$ be the subgroup of elements of $C^i(\_,\_)$ fixed by the action of Γ, then there are natural isomorphisms

$$\begin{aligned} C^0(G, \mathcal{L}) &\cong C^0_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)) \\ C^1(G, \mathcal{L}) &\cong C^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)) \end{aligned}$$

By calculating the action of $-\mathrm{id} \in \Gamma_0$ on these groups, we find that they vanish unless $\varepsilon(-1) = (-1)^k = 1$.

To define Hecke actions on these groups, we need to find their connections with certain spaces of automorphic forms on $B^{\times}$. Let ω be a Hecke character of $\mathbf{Q}$ with cond $(\omega) \mid M$. Fix an isomorphism: $B \otimes \mathbf{C} \cong \mathrm{M}(2, \mathbf{C})$. The natural representation of $\mathbf{GL}(2, \mathbf{C})$ on $\mathbf{C}^2$ gives the Symm^1 representation of $B^{\times}$. Let $(\mathrm{Symm}^{k-2}, \mathbf{C}^{k-1})$ be its $(k-2)^{\mathrm{nd}}$ symmetric power.

Definition 1.2 *For an open $U \subseteq \mathbf{B}^{\times}(\mathbf{A}^f)$, the space $\mathcal{B}_{k-2}(U, \omega)$ of automorphic forms on U induced by* Symm^{k-2} *is the set of all the functions φ: $\mathbf{B}^{\times}(\mathbf{A}^f) \to \mathbf{C}^{k-1}$, satisfying the following condition*

$$\varphi(\gamma^f z g u) = \omega(z) \mathrm{Symm}^{k-2}(\gamma_\infty) \varphi(g) \tag{1}$$

for all $\gamma \in B^{\times}, z \in$ Center$(\mathbf{B}^{\times}(\mathbf{A}^f)), g \in \mathbf{B}^{\times}(\mathbf{A}^f)$, *and $u \in U$.*

In general, let G be a locally profinite group, and $U \subset G$ is a open compact subgroup. Let μ_U be the unique Haar measure on G satisfying $\mu_U(U) = 1$.

Definition 1.3 *The Hecke algebra $\mathcal{H}(G, U)$ is the U-bi-invariant convolution* **C***-algebra of compactly supported distributions on G. The set $U \backslash G / U$ forms a basis for $\mathcal{H}(G, U)$. For a double class $UxU \in U \backslash G / U$, the corresponding distribution T_{UxU} is called a standard generator. It sends a test function f on G to*

$$T_{UxU} f(g) = \int_{UxU} f(x^{-1} g)\, d\mu_U(x).$$

The group $\mathbf{B}^{\times}(\mathbf{A}^f)$ acts on the finite-dimensional space $\mathcal{B}_{k-2}(U, \omega)$ by right translation, and the action is U-invariant. Let the Dirichlet character ε be determined by $\omega^{-1}|_{\hat{\mathbf{Z}}^{\times}}$ via the isomorphism $\hat{\mathbf{Z}}^\times(1+M\hat{\mathbf{Z}}^\times) \cong (\mathbf{Z}/M\mathbf{Z})^\times$, where $\hat{\mathbf{Z}}$ is the inverse limit of all cyclic groups. If we choose the open subset U to be the following

$$U = U^f = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}^\times \mid d \equiv 1 (\operatorname{mod} M) \right\},$$

the space $\mathcal{B}_{k-2}(U,\omega)$ is then equipped with *integral* Hecke structure. To be more precise, decompose U^f componentwise into a product $U^f = \prod_p U_p$. The Hecke algebra $\mathcal{H}(\mathbf{B}^\times(\mathbf{A}^f), U^f)$ is then the restricted tensor product of $\mathcal{H}(B_p^\times, U_p)$, where $B_p^\times$ is the p-component of $\mathbf{B}^\times(\mathbf{A}^f)$. Therefore, it is generated as an $\mathbf{Z}[\varepsilon]$-module by the $U^f x_p U^f$, with $x_p \in B_p^\times$ for p prime. These generators are called *standard generators at p* by Jordan-Livné. The integral Hecke algebra $\mathbf{T}_{\mathcal{M},\omega}$ is then defined as the $\mathbf{Z}[\varepsilon]$-submodule with 1 of $\mathcal{H}(\mathbf{B}^\times(\mathbf{A}^f), U^f)$ generated by specific standard generators at primes called Hecke operators. For the detailed construction of these Hecke operators, the reader is referred to [17], Definition 1.4. For a representation V of $\mathbf{B}^\times(\mathbf{A}^f)$, the group U^f acts on the space V^{U^f} of U^f-invariants of V(*loc. cit.*). The integral Hecke algebra $\mathbf{T}_{\mathcal{M},\omega}(V)$ attached to V is then the image of $\mathbf{T}_{\mathcal{M},\omega}$ in $\operatorname{End}(V^{U^f})$. Recall that we have fixed an isomorphism $\mathcal{M}^\times \cong K_p$, and the Iwahori subgroup of K_p is the group

$$I_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p \mid c \equiv 0 (\operatorname{mod} p) \right\}.$$

Then there is a unique Eichler order $\mathcal{M}' \subset \mathcal{M}$ of conductor Mp corresponding to I_p under this isomorphism.

Proposition 1.4 *Let* $U^p = \ker(\bar{d} : \hat{\mathcal{M}}^{\times,p} \to (\mathbf{Z}/M\mathbf{Z})^\times)$, *and let* $|\ |^{-k+2}$ *be the real character* $t \mapsto t^{-k+2}$. *Assume further that* ω *is unitary. Then there are isomorphisms of finite dimensional* $\mathbf{C}$*-vector spaces*

$$\begin{aligned} \mathcal{B}_{k-2}(U^p\mathcal{M}_p^\times, \omega\,|\ |^{-k+2})^2 &\xrightarrow{\cong} C^0_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)) \otimes_{\mathbf{Z}[\varepsilon]} \mathbf{C}, \\ \mathcal{B}_{k-2}(U^p\mathcal{M}_p'^\times, \omega\,|\ |^{-k+2}) &\xrightarrow{\cong} C^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)) \otimes_{\mathbf{Z}[\varepsilon]} \mathbf{C}. \end{aligned} \tag{2}$$

Proof. This is [17], Proposition 3.6. □

The above proposition allows us to transport Hecke structures from the spaces of automorphic forms to the groups of 0- and 1-cochains. Let $\varepsilon' : (\mathbf{Z}/Mp\mathbf{Z})^\times \to \mathbf{C}^\times$ be the composition of ε with the natural map $(\mathbf{Z}/Mp\mathbf{Z})^\times \to (\mathbf{Z}/M\mathbf{Z})^\times$. Set

$$\begin{aligned} \mathbf{T}^0 &= \mathbf{T}_{\mathcal{M},\varepsilon}(\mathcal{B}_{k-2}(U^p\mathcal{M}_p^\times, \omega\,|\ |^{-k+2})^2), \\ \mathbf{T}^1 &= \mathbf{T}_{\mathcal{M}',\varepsilon'}(\mathcal{B}_{k-2}(U^p\mathcal{M}_p'^\times, \omega\,|\ |^{-k+2})). \end{aligned}$$

Proposition 1.5 *Each* $C^i_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon))$ *is a projective* $\mathbf{Z}[\varepsilon]$*-module of finite rank, preserved by the Hecke algebra* $\mathbf{T}^i$.

Proof. See [17], Proposition 3.11. □

For a prime $r|pM$, we can also define the operator w_r on $\mathcal{B}_{k-2}(U^p\mathcal{M}_p'^\times, \omega\,|\ |^{-k+2})$. For a test function f on $\mathbf{B}^\times(\mathbf{A}^f)$ (or on $B_p^\times$), w_r sends f to

$$w_r f(g) = f\left(g\begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}\right).$$

Theorem 1.6 (i) *There is an exact sequence of* $\mathbf{Z}[\varepsilon]$*-modules*

$$0 \to H^0_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)) \to C^0_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)) \xrightarrow{d} C^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)) \to H^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)) \to 0$$

It is equivariant for all the standard generators away from p*. Furthermore, the standard generator* T_p *at* p *acts as* $-w_p$ *on the group* $H^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon))$;

(ii) *Replace* $\mathcal{L}_{k-2}(\varepsilon)$ *by* $\mathcal{L}_{k-2}(\varepsilon)_\ell = \mathcal{L}_{k-2}(\varepsilon) \otimes \mathbf{Z}_\ell$ *with any prime* ℓ *not dividing* Disc B*. There is then an exact sequence of* $\mathbf{Z}_\ell$*-modules analogous to the one in* (i), *and the other claims remain true.*

Proof. (i) The exactness of the sequence is tautological. As for the Hecke action part, see [17], Theorem 3.19.

(ii) This can be proved similarly. □

1.2 Local system attached to the dual graph of an admissible curve

Another way to construct the group $H^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon))$ in § 1.1 is to use étale cohomology with a suitably defined locally constant sheaf of $\mathbf{Z}_\ell[\varepsilon]$-modules on a Shimura curve arising from an Eichler order of an indefinite quaternion algebra over $\mathbf{Q}$. In fact, our construction given here applies to the extent of *admissible curves* (in the sense of [15]).

Let $S = \text{Spec } R$ be the spectrum of a strictly Henselian ring with residue characteristic of p. Let C be an admissible curve over S, i. e., a proper and flat S-scheme, whose generic fiber is a nonsingular curve, and whose special fiber is smooth outside a finite set Σ and has rational irreducible components. For each $x \in \Sigma$, C is locally, for the étale cohomology, S-isomorphic to a subscheme of $S[X_1, X_2]$ defined by the equation $X_1X_2 = a_x$, where a_x is a nonzero element in the maximal ideal of R. We put $e_x = \mathrm{val}_p(a_x)$ the valuation of a_x. The generic and special fibers of C are denoted by C_{gen} and Y, respectively. We also let $Y_{(\bar{x})}$ be the strict localization of Y at a geometric point $\bar{x}$ of x. The normalization of Y is $\tilde{Y}$, with π the canonical morphism $\tilde{Y} \to Y$. Let I be the inertia group of Gal $(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ at p. For a Dirichlet character ε and a prime ℓ distinct from p, let ϑ_ℓ be a locally constant sheaf of $\mathbf{Z}_\ell[\varepsilon]$-modules on C. Finally, since the largest pro-p quotient $I(\ell)$ of I is canonically isomorphic to $\mathbf{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}$, we have a canonical projection t_ℓ: $I \to \mathbf{Z}_\ell(1)$.

For an admissible curve C, one can attach to its special fiber Y an oriented graph, called the *dual graph*.

Definition 1.7 (*cf.* [15], 3.2) *The dual graph $G(Y)$ of the special fiber Y of the curve C is the following graph:*

(i) *The set* $\mathrm{Ver}(G(Y))$ *consists of the irreducible components of Y;*

(ii) *The set* $\mathrm{Ed}(G(Y))$ *consists of the irreducible components of $\tilde{Y}$ contributed by the double points of Y. The contribution from each double point $x \in \Sigma$ is the two components, inverse of each other in* $\mathrm{Ed}(G(Y))$, *whose images under π are the two components of Y passing through x;*

(iii) *For $e \in \mathrm{Ed}(G(Y))$, $o(e)$ is the component $\pi(e)$ of Y, and $t(e) = o(\bar{e})$.*

To simplify our notation, from now on we will just write G for the dual graph $G(Y)$ when there is no ambiguity about the curve Y in the context. The sheaf ϑ_ℓ induces a sheaf on Y which, by abuse of notation, will also be denoted by ϑ_ℓ. We would like to use this sheaf to attach a $\mathbf{Z}_\ell[\varepsilon]$-local system $\mathcal{L}$ on G. Each $x \in \Sigma$ may be considered as an (unoriented) edge of G. Let x_1 and x_2 be the two points of $\tilde{Y}$ above x, and let Y_1 and Y_2 be the two components of $\tilde{Y}$ containing x_1 and x_2, respectively. With the choice of $o(x) = \pi(Y_1)$, $t(x) = \pi(Y_2)$, the double point x becomes an edge of G. Define

$$\begin{aligned} \mathcal{L}(o(x)) &= \Gamma(Y_1, \pi^*\vartheta_\ell), \\ \mathcal{L}(t(x)) &= \Gamma(Y_2, \pi^*\vartheta_\ell). \end{aligned}$$

To define the transition map between the fibers of $o(x)$ and $t(x)$, first let us observe that for α=1 or 2, since Y_α is rational, the sheaf ϑ_ℓ induces a constant sheaf on Y_i, hence the natural map

$$i_\alpha : \Gamma(Y_\alpha, \pi^*\vartheta_\ell) \to (\pi^*\vartheta_\ell)_{x_\alpha}$$

is an isomorphism. Also it is a standard fact (see, e. g., [22], Theorem 3.2 (a)) that, for $\alpha = 1$ or 2, there is a canonical isomorphism

$$j_\alpha : (\pi^*\vartheta_\ell)_{x_\alpha} \to (\vartheta_\ell)_x$$

The transition map $\mathcal{L}_x$ is then defined as the composition map

$$\Gamma(Y_1, \pi^*\vartheta_\ell) \xrightarrow{i_1} (\pi^*\vartheta_\ell)_{x_1} \xrightarrow{j_1} (\vartheta_\ell)_x \xrightarrow{j_2^{-1}} (\pi^*\vartheta_\ell)_{x_2} \xrightarrow{i_2^{-1}} \Gamma(Y_2, \pi^*\vartheta_\ell),$$

We call $\mathcal{L}$ so defined the local system of G induced by ϑ_ℓ.

Now let $C^1(x, \mathcal{L})$ be the cokernel of the injection $(\vartheta_\ell)_x \to \mathcal{L}(o(x)) \times \mathcal{L}(t(x))$ defined by $a \mapsto (i_1^{-1} \circ j_1^{-1}(a), i_2^{-1} \circ j_2^{-1}(a))$. So we have an exact sequence

$$0 \to (\vartheta_\ell)_x \longrightarrow \mathcal{L}(o(x)) \times \mathcal{L}(t(x)) \xrightarrow{p_x} C^1(x, \mathcal{L}) \to 0 \tag{3}$$

Definition 1.8 *The weight k dual character group $\hat{X}_\ell(C)$ derived from C is defined by the following exact sequence*

$$C^0(G, \mathcal{L}) \xrightarrow{\partial} \oplus_{x\in\Sigma} C^1(x, \mathcal{L}) \longrightarrow \hat{X}_\ell(C) \longrightarrow 0, \tag{4}$$

where the map ∂ is the sum of compositions $C^0(G, \mathcal{L}) \to \mathcal{L}(o(x)) \times \mathcal{L}(t(x)) \xrightarrow{p_x} C^1(x, \mathcal{L})$ over the set Σ. The weight k character group $X_\ell(C)$ is defined as the group $\mathrm{Hom}(\hat{X}_\ell(C), \mathbf{Z}_\ell[\varepsilon])$.

Let us define a map

$$\iota : \; C^1(G, \mathcal{L}) \to \oplus_{x\in\Sigma} C^1(x, \mathcal{L}),$$

$$\{s(x)\}_{x\in\Sigma} \mapsto \oplus_{x\in\Sigma}\, p_x(s(x), -\mathcal{L}_x(s(x))).$$

Lemma 1.9 *The map ι is an isomorphism.*

Proof. To see that ι is an injection, let $\{s(x)\}_{x\in\Sigma}$ in $C^1(G, \mathcal{L})$ be such that it is mapped to 0 by ι. Then for each $x \in \Sigma$, there exists an element $a_x \in \vartheta_\ell$ satisfying the following two identities:

$$s(x) = i_1^{-1} j_1^{-1}(a_x),$$

$$-\mathcal{L}_x(s(x)) = i_2^{-1} j_2^{-1}(a_x).$$

By our definition of the transition map $\mathcal{L}_x$, we find that the second identity is equivalent to the following identity

$$-s(x) = i_1^{-1} j_1^{-1}(a_x).$$

Hence $s(x)$ must be 0 for all x in Σ. To see that ι is also surjective, we will show that ι is surjective on the components, i. e., that the restriction $\iota_x = \iota|_{\mathcal{L}(o(x))}$:

$$\mathcal{L}(o(x)) \xrightarrow{\iota_x} C^1(x,\, \mathcal{L})$$

is a surjection for each $x \in \Sigma$. Observe that for $a \in \mathcal{L}(o(x))$,

$$\iota_x(a) = p_x(a, -\mathcal{L}_x(a)) = p_x(2a, 0).$$

Also, for any $p_x(\xi, \eta) \in C^1(x,\, \mathcal{L})$, we have

$$p_x(\xi, \eta) = p_x(\xi - \mathcal{L}_x^{-1}(\eta), 0).$$

Therefore, we may choose a to be the element $\frac{\xi - \mathcal{L}_x^{-1}(\eta)}{2}$, and it will satisfy the desired identity

$$\iota_x(a) = p_x(\xi, \eta).$$

The denominator is justified since ϑ_ℓ is a sheaf of $\mathbf{Z}_\ell[\varepsilon]$-modules, and 2 acts as an invertible scalar on any $\mathbf{Z}_\ell$-module since $(2, \ell) = 1$. $\square$

Lemma 1.10 *Let d be the coboundary operator of the local system $\mathcal{L}$. Then $\iota \circ d = -2\partial$.*

Proof. Let $s = (s(v))_{v \in \mathrm{Ver}(G)}$ be any element in $C^0(G,\, \mathcal{L})$. By the definition of the coboundary operator d, for each x in Σ we have

$$\begin{aligned}
\iota \circ d(s)(x) &= \iota(\mathcal{L}_x^{-1}(s(t(x))) - s(o(x))) \\
&= \oplus_{x\in\Sigma}\, p_x(\mathcal{L}_x^{-1}(s(t(x))) - s(o(x)), \mathcal{L}_x(s(o(x))) - s(t(x))) \\
&= \{\oplus_{x\in\Sigma}\, p_x(\mathcal{L}_x^{-1}(s(t(x))), \mathcal{L}_x(s(o(x))))\} - \partial(s)(x) \\
&= \{\oplus_{x\in\Sigma}\, p_x(i_1^{-1} \circ j_1^{-1} \circ (j_2 \circ i_2(s(t(x)))), i_2^{-1} \circ j_2^{-1} \circ (j_2 \circ i_2(s(t(x))))) \\
&\quad + \oplus_{x\in\Sigma}\, p_x(i_1^{-1} \circ j_1^{-1} \circ (j_1 \circ i_1(s(t(x)))), i_2^{-1} \circ j_2^{-1} \circ (j_1 \circ i_1(s(t(x)))))\} \\
&\quad - \{\oplus_{x\in\Sigma}\, p_x(s(o(x)), s(t(x)))\} - \partial(s)(x) \\
&= -2\partial(s)(x).
\end{aligned}$$

The last step above is due to the definition of p_x in the sequence (3). □

Proposition 1.11 *There is an isomorphism* $\hat{X}_\ell(\mathcal{C}) \cong H^1(G,\mathcal{L})$. *The isomorphism is unique up to an orientation of* Ed$\,(G)$.

Proof. Since $(2,\ell)=1$, the above lemma shows that the group $\hat{X}_\ell(\mathcal{C})$ is isomorphic to the cokernel of the coboundary operator d which, by definition, is the group $H^1(G,\, \mathcal{L})$. The uniqueness is trivial. □

Remark 1.12 The above proposition can be viewed as a mild generalization of Grothendieck's result (12.3.8) in [13], where he treats semistable curves with constant sheaves.

Since $\mathrm{Hom}_{S(Y)}(\vartheta_\ell, \pi_*\pi^*\vartheta_\ell) \cong \mathrm{Hom}_{S(\tilde{Y})}(\pi^*\vartheta_\ell, \pi^*\vartheta_\ell)$, there is an étale morphism $\vartheta_\ell \to \pi_*\pi^*\vartheta_\ell$ corresponding to the identity morphism in the latter group. Further consideration at the stalks finds that this morphism is injective. Form the exact sequence of étale sheaves on Y

$$0 \to \vartheta_\ell \to \pi_*\pi^*\vartheta_\ell \to \zeta \to 0 \tag{5}$$

Proposition 1.13 *The dual character group $\hat{X}_\ell(C)$ is embedded in the following exact sequence*

$$0 \longrightarrow \hat{X}_\ell(C) \xrightarrow{\gamma} H^1(Y, \vartheta_\ell) \xrightarrow{\varpi} H^1(Y, \pi_*\pi^*\vartheta_\ell) \longrightarrow 0 \tag{6}$$

Proof. Form the long exact sequence associated to the exact sequence (5)

$$0 \to \Gamma(Y, \vartheta_\ell) \to \Gamma(Y, \pi_*\pi^*\vartheta_\ell) \to \Gamma(Y, \zeta) \to H^1(Y, \vartheta_\ell) \to H^1(Y, \pi_*\pi^*\vartheta_\ell) \to H^1(Y, \zeta).$$

It is easy to see that ζ is such that for $U \xrightarrow{f} Y$ in $((\text{ét})/Y)_{\text{ét}}$, $\zeta(U) = \bigoplus_{x \in f(U) \cap \Sigma} C^1(x, \mathcal{L})$. Hence $\Gamma(Y, \zeta) \cong C^1(G, \mathcal{L})$. Furthermore, We have $\Gamma(Y, \pi_*\pi^*\vartheta_\ell) = \bigoplus_{\amalg Y_i = \tilde{Y}} \Gamma(Y_i, \pi^*\vartheta_\ell) = C^0(G, \mathcal{L})$. Since $(\pi_*\pi^*\vartheta_\ell)_x = \prod_{\pi(y)=x} (\pi^*\vartheta_\ell)_y, \zeta$ is then supported on Σ, i. e., ζ_x is trivial if $x \notin \Sigma$. Therefore $H^1(Y, \zeta) \cong H^1(\Sigma, \zeta|_\Sigma) \cong \bigoplus_{x \in \Sigma} H^1(\{x\}, \zeta_x) = 0$. Now to complete the proof, just apply Proposition 1.11. □

2 Vanishing cycles and sheaf cohomology

Let C be a modular or Shimura curve over $\mathbf{Q}$. We prove in this chapter an extension of Grothendieck's Orthogonality Theorem (cf. [13], Théorème 2.4) using a method modeled on [14]. It requires Deligne's theory of vanishing cycles. Also we prove several important formulas concerning the structures of the cohomology groups of the generic and special fibers of the curve C. Let N be a positive integer, and $\ell \nmid N$ a prime number. For a Dirichlet character $\varepsilon : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, define a locally constant sheaf of $\mathbf{Z}_\ell[\varepsilon]$-modules $\vartheta_\ell = \vartheta_\ell(\varepsilon)$ on C as follows: if C is the modular curve $Y_0(N)$, and $\phi : E \to Y_0(N)$ is the universal elliptic curve, let Θ_ℓ be the sheaf $\mathrm{Symm}^{k-2}R^1\phi_*\mathbf{Z}_\ell$; if C is $V_B(M)$, a Shimura curve associated to an Eichler order of conductor M in an indefinite quaternion algebra $B/\mathbf{Q}$ with $N = M\,\mathrm{Disc}\,B$, let $\phi' : A \to V_B(M)$ be the universal abelian surface; let the sheaf $\mathcal{F}$ be defined by an idempotent splitting $\mathcal{F} \oplus \mathcal{F} = R^1\phi'_*\mathbf{Z}_\ell$, and we put $\Theta_\ell = \mathrm{Symm}^{k-2}\mathcal{F}$. Now we define the sheaf ϑ_ℓ, both on a modular curve and a Shimura curve, to be the largest subsheaf of the sheaf $\Theta_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell[\varepsilon]$ such that $\varepsilon(d)$ acts as the scalar d^{k-2} for all $d \in (\mathbf{Z}/N\mathbf{Z})^\times$. We deal with Shimura curves in the first two sections, and treat the case when C is a modular curve in the last section. There is a word of caution before we move on: In order to apply Deligne's theory, the curve C should be proper. So whenever we have the open modular curve $Y_0(N)$, we will be using on it the parabolic cohomology $H^1_{\mathrm{par}} := \mathrm{im}(H^1_c \to H^1)$ with the sheaf ϑ_ℓ or equivalently, using on the complete modular curve $X_0(N)$ the usual étale cohomology H^1 with the sheaf $j_!\vartheta_\ell$.

2.1 Orthogonality Theorem

Let $C = V_B(M)$ be a Shimura curve associated to an Eichler order of level M of an indefinite quaternion algebra $B/\mathbf{Q}$ with discriminant Disc B. Let p be a prime dividing Disc B. We will continue to use the notation defined in the beginning of Section 1.2. So C_{gen} and Y are the generic and special fibers of the curve $V_B(M)_{/\mathbf{F}_p}$, respectively. For each singular point $x \in \Sigma$, fix a geometric point $\bar{x}$ of x and let $Y_{(x)}$ be the strict localization of Y at $\bar{x}$. Recall that $X_\ell = X_p(V_B(M))_\ell$ is the weight k character group attached to Y. The following exact sequence

$$0 \longrightarrow C_1(x, \mathcal{L}) \xrightarrow{i_x} \hat{\mathcal{L}}(o(x)) \times \hat{\mathcal{L}}(t(x)) \longrightarrow \widehat{(\vartheta_\ell)_x} \longrightarrow 0 \tag{7}$$

is the $\mathbf{Z}_\ell[\varepsilon]$-dual sequence of (3) in Section 1.2. The weight k character group X_ℓ is embedded in the $\mathbf{Z}_\ell[\varepsilon]$-dual sequence of (4)

$$0 \rightarrow X_\ell \rightarrow \oplus_{x\in\Sigma} C_1(x, \mathcal{L}) \rightarrow C_0(G, \mathcal{L})$$

Choose a $\widehat{(\vartheta_\ell)}_x$-basis δ'_x for the group $C_1(x, \mathcal{L})$, and let δ_x be its dual basis in the group $C^1(x, \mathcal{L})$.

Let us recall Deligne's vanishing cycle theory in [6] and [7]. There is an exact sequence of sheaves defined over the special fiber Y

$$0 \rightarrow \vartheta_\ell \rightarrow R\Psi\vartheta_\ell \rightarrow R\Phi\vartheta_\ell \rightarrow 0 \tag{8}$$

The following theorem cites some basic properties of the sheaves $R\Psi\vartheta_\ell$ and $R\Phi\vartheta_\ell$ that we will need in later proofs.

Theorem 2.1 (i)*There is an isomorphism* $H^i(Y, R\Psi\vartheta_\ell) \cong H^i(C_{\text{gen}}, \vartheta_\ell)$ *for any* $i \geq 0$*;*

(ii)*The sheaf* $R\Phi\vartheta_\ell$ *is supported on* Σ*, i.e.,* $(R\Phi\vartheta_\ell)\,|_{Y-\Sigma} = 0$*;*

(iii)*Specialization: There is an exact sequence*

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(Y, \vartheta_\ell)(k-1) & \xrightarrow{\text{sp}} & H^1(Y, R\Psi\vartheta_\ell)(k-1) & \rightarrow & \oplus_{x\in\Sigma}(R\Phi\vartheta_\ell)_x(k-1) \\ & \rightarrow & H^2(Y, \vartheta_\ell)(k-1) & \rightarrow & H^2(Y, R\Psi\vartheta_\ell)(k-1) & \rightarrow & 0 \end{array} \tag{9}$$

(iv)*Cospecialization: The dual of the above exact sequence is the following*

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\tilde{Y}, R\Psi\vartheta_\ell) & \rightarrow & H^0(\tilde{Y}, \vartheta_\ell) & \xrightarrow{\alpha} & \oplus_{x\in\Sigma} H^1_x(Y, R\Psi\vartheta_\ell) \\ & \xrightarrow{\beta} & H^1(Y, R\Psi\vartheta_\ell) & \rightarrow & H^1(Y, \vartheta_\ell) & \rightarrow & 0 \end{array} \tag{10}$$

where α *is the composition of the map* $H^0(\tilde{Y}, \vartheta_\ell) \rightarrow \oplus_{x\in\Sigma} H^0(U_x, \vartheta_\ell)$(*where* $U_x := Y_{(x)} - \{x\}$) *and the sum of the boundary maps* $H^0(U_x, \vartheta_\ell) = H^0(U_x, R\Psi\vartheta_\ell) \rightarrow H^1_x(Y, R\Psi\vartheta_\ell)$.

Proof. The basic theory of vanishing cycles was established in [6]. One can also look into [7] , Chapters 2 and 3 for more direct explanation of these above statements. ☐

Proposition 2.2 *The map* β *in the cospecialization exact sequence* (10) *above induces an embedding*

$$\hat{X}_\ell \overset{\beta}{\hookrightarrow} H^1(Y, \vartheta_\ell).$$

Proof. First observe that the group $H^0(\tilde{Y}, \vartheta_\ell)$ is just another expression for the 0-cochain $C^0(G, \mathcal{L})$. Deligne proved in [7], in the case $\vartheta_\ell = \mathbf{Z}_\ell$, there is an isomorphism

$$H^1_x(Y, R\Psi\vartheta_\ell) \cong C^1(x, \mathcal{L}).$$

It certainly extends to the case of ϑ_ℓ being any constant sheaf, and it is also valid in our case since ϑ_ℓ here is locally constant. Under this identification, the map α in (10) is then the map ∂ in Lemma 1.10. To complete the proof, we need to show that the image of β is contained in $H^1(Y, \vartheta_\ell)$. Consider the long exact sequence of cohomology supported at x associated to (8)

$$H^0_x(Y, R\Phi\vartheta_\ell) \to H^1_x(Y, \vartheta_\ell) \to H^1_x(Y, R\Psi\vartheta_\ell) \to H^1_x(Y, R\Phi\vartheta_\ell) \xrightarrow{\delta} H^2_x(Y, \vartheta_\ell) \tag{11}$$

The group $H^0_x(Y, R\Phi\vartheta_\ell)$ is trivial. Also the composition

$$\eta : \ (R\Phi\vartheta_\ell)_x \xrightarrow{\cong} H^1_x(Y, R\Phi\vartheta_\ell) \xrightarrow{\delta} H^2_x(Y, \vartheta_\ell)$$

gives a commutative diagram

$$\begin{tikzcd}
(R\Phi\vartheta_\ell)_x(k-1) \arrow[r, "\eta(k-1)"] \arrow[d, "\cong"'] & H^2_x(Y, \vartheta_\ell)(k-1) \arrow[d] \\
C_1(x, \mathcal{L}) \arrow[r, "i_x"] & \hat{\mathcal{L}}(o(x)) \times \hat{\mathcal{L}}(t(x))
\end{tikzcd}$$

where the map i_x is the same map in (7). Hence η is injective since i_x is. Knowing this, plus the fact that the map $(R\Phi\vartheta_\ell)_x \to H^1_x(Y, R\Phi\vartheta_\ell)$ is an isomorphism (easily deduced from Theorem 2.1, (ii)), the map δ then has to be an injection. This shows that the map $H^1_x(Y, \vartheta_\ell) \to H^1_x(Y, R\Psi\vartheta_\ell)$ in (11) is an isomorphism. Denote its inverse by β_x. It fits in the following commutative diagram

$$\begin{tikzcd}
\oplus_{x\in\Sigma} H^1_x(Y, \vartheta_\ell) \arrow[r, "\oplus\beta_x^{-1}"] \arrow[d, "\sigma"'] & \oplus_{x\in\Sigma} H^1_x(Y, R\Psi\vartheta_\ell) \arrow[d, "\beta"] \\
H^1(Y, \vartheta_\ell) \arrow[r, "\mathrm{sp}(1-k)"] & H^1(Y, R\Psi\vartheta_\ell).
\end{tikzcd} \tag{12}$$

By Theorem 2.1), (iii), the specialization map sp is injective, hence so is its $(1-k)$-Tate-twist $\mathrm{sp}(1-k)$. As a result, the image of β is inside $H^1(Y, \vartheta_\ell)$. □

We now compare the images of two embeddings β and γ (cf. (6) in §1.2) of the weight k dual character group $\hat{X}_\ell$ in $H^1(Y, \vartheta_\ell)$.

Lemma 2.3 $\mathrm{im}\,\beta = \mathrm{im}\,\gamma$.

Proof. The question is essentially of local nature. Recall that $Y_{(x)}$ is the strict Henselization of Y at the chosen geometric point $\bar{x}$ of $x \in \Sigma$. Set $U_x = Y_{(x)} - \{x\}$. Let σ be the canonical map $\oplus_{x\in\Sigma} H^1_x(Y, \vartheta_\ell) \to H^1(Y, \vartheta_\ell)$ in the commutative diagram (12). Let us rearrange the diagram into the following commutative diagram

$$\begin{tikzcd}
\oplus_{x\in\Sigma} H^1_x(Y, R\Psi\vartheta_\ell) \arrow[r, "\mathrm{id}"] \arrow[d, "\oplus\beta_x"'] & \oplus_{x\in\Sigma} H^1_x(Y, R\Psi\vartheta_\ell) \arrow[d, "\beta"] \\
\oplus_{x\in\Sigma} H^1_x(Y, \vartheta_\ell) \arrow[r, "\sigma"] & H^1(Y, \vartheta_\ell)
\end{tikzcd} \tag{13}$$

Also let γ_x be in the following commutative diagram

$$\begin{tikzcd}
\oplus_{x\in\Sigma} H^0_x(Y, \zeta) \arrow[r, "\cong"] \arrow[d, "\oplus\gamma_x"'] & H^0(Y, \zeta) \arrow[d, "\gamma"] \\
\oplus_{x\in\Sigma} H^1_x(Y, \vartheta_\ell) \arrow[r, "\sigma"] & H^1(Y, \vartheta_\ell)
\end{tikzcd} . \tag{14}$$

We are going to show that for each $x \in \Sigma$, the images of β_x and γ_x in $H^1_x(Y, \vartheta_\ell)$ are the same. Our lemma then clearly follows because of the diagrams (13) and (14). By functoriality, we have a commutative diagram of exact sequences

$$\begin{tikzcd}
H^0_x(Y,\vartheta_\ell) \arrow[r] \arrow[d] & H^0(Y_{(x)},\vartheta_\ell) \arrow[r] \arrow[d] & H^0(U_x,\vartheta_\ell) \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"] & H^1_x(Y,\vartheta_\ell) \arrow[d] \\
H^0_x(Y,\pi_*\pi^*\vartheta_\ell) \arrow[r] \arrow[d] & H^0(Y_{(x)},\pi_*\pi^*\vartheta_\ell) \arrow[r, "p_x"] \arrow[d, "p_x"'] & H^0(U_x,\pi_*\pi^*\vartheta_\ell) \arrow[r] \arrow[d] & H^1_x(Y,\pi_*\pi^*\vartheta_\ell) \arrow[d] \\
H^0_x(Y,\zeta) \arrow[r, "\mathrm{id}"] \arrow[d, "\gamma_x"'] & H^0(Y_{(x)},\zeta) \arrow[r] \arrow[d] & H^0(U_x,\zeta) = 0 \arrow[r] \arrow[d] & H^1_x(Y,\zeta) \arrow[d] \\
H^1_x(Y,\vartheta_\ell) \arrow[r] & H^1(Y_{(x)},\vartheta_\ell) \arrow[r] & H^1(U_x,\vartheta_\ell) \arrow[r] & H^2_x(Y,\vartheta_\ell)
\end{tikzcd}$$

where the horizontal sequences are "long exact sequences with support at x," and the vertical ones are long exact sequences associated with the exact sequence (8). The group $H^0(U_x,\zeta)$ is trivial because the sheaf ζ is supported on $\{x\}$. We may further identify both $H^0(Y_{(x)},\pi_*\pi^*\vartheta_\ell)$ and $H^0(U_x,\pi_*\pi^*\vartheta_\ell)$ with $\mathcal{L}(o(x))\times\mathcal{L}(t(x))$, and $H^0(Y_{(x)},\zeta)$ with $C^1(x,\mathcal{L})$. One checks that the horizontal p_x in the above diagram is an isomorphism, while the vertical one is the natural projection. According to a standard result in homological algebra, there is an identity

$$\gamma_x \circ \mathrm{id}^{-1} \circ p_x = -\partial \circ \mathrm{id}^{-1} \circ p_x.$$

As a consequence, $\mathrm{im}\,\gamma_x = \mathrm{im}\,\partial$. Now observe the commutative diagram

$$\begin{tikzcd}
H^0(U_x,R\Psi\vartheta_\ell) \arrow[r, "\partial'"] & H^1_x(Y,R\Psi\vartheta_\ell) \arrow[d, "\beta_x"] \\
H^0(U_x,\vartheta_\ell) \arrow[r, "\partial"] \arrow[u, "\cong"] & H^1_x(Y,\vartheta_\ell)
\end{tikzcd}$$

where the map ∂' is in fact the projection $p_x : \mathcal{L}(o(x)) \times \mathcal{L}(t(x)) \to C^1(x, \mathcal{L})$. Therefore $\operatorname{im} \beta_x = \operatorname{im} \partial = \operatorname{im} \gamma_x$. $\square$

Let $N_x : R\Phi(\vartheta_\ell)_x(k-1) \to H^1_x(Y, R\Psi\vartheta_\ell)$ be the *variation morphism* Deligne defined in [6], Section 2.1. So for $a \in R\Phi(\vartheta_\ell)_x$ and $\sigma \in I$, N_x is the map

$$N_x(t_\ell^{k-1}(\sigma)a) = -e_x t_\ell^{k-1}(\sigma) \cdot (a\delta_x) \cdot \delta_x,$$

where the symbols I, t_ℓ and e_x are defined in the beginning of Section 1.2, and $(a\delta_x)$ is the coefficient of a in $C_1(x, \mathcal{L})(1-k)$ with δ'_x as the basis. Let $\lambda : X_\ell \to \hat{X}_\ell$ be defined by the commutative diagram

$$\begin{tikzcd}
X_\ell \arrow[r] \arrow[d, "\lambda"'] & \oplus C_1(x, \mathcal{L}) \arrow[d] \\
\hat{X}_\ell & \oplus C^1(x, \mathcal{L}) \arrow[l]
\end{tikzcd}, \tag{15}$$

where the vertical map on the right sends δ'_x to $-e_x\delta_x$. It is easy to see that λ embeds into the following commutative diagram

$$\begin{tikzcd}
H^1(C_{\text{gen}}, \vartheta_\ell)(k-1) \arrow[r, "\beta'"] \arrow[d, "N"'] & X_\ell \arrow[r] \arrow[d, "\lambda"] & \oplus_{x\in\Sigma} R\Phi(\vartheta_\ell)_x(k-1) \arrow[d, "\oplus N_x"] \\
H^1(C_{\text{gen}}, \vartheta_\ell) & \hat{X}_\ell \arrow[l, "\beta"'] & \oplus_{x\in\Sigma} H^1_x(Y, R\Psi\vartheta_\ell) \arrow[l]
\end{tikzcd} \tag{16}$$

where N is the *monodromy logarithm* sending $t_\ell^{k-1}(\sigma) \cdot a$ to $(\sigma - 1)(a)$. And the composition of the maps in the top row is the map in the specialization sequence (9) between the same two groups, while the maps in the bottom row gives β in the cospecialization sequence (10).

Definition 2.4 *Let* $U = H^1(\mathcal{C}_{\mathrm{gen}}, \vartheta_\ell)(k-1)$. *Define the fixed part of* U *to be* $V = U^I$, *and the toric part of* U *to be* $W = (\operatorname{im}\gamma)(k-1)$.

Recall that the Poincaré duality theorem furnishes a perfect paring

$$\varphi : U \times U\,(1-k) \to \mathbf{Z}_\ell[\varepsilon] \tag{17}$$

This pairing is equivariant with respect to the action of Galois. It is Hecke-equivariant only in the sense that

$$\varphi(T_r a, b) = \varphi(a, T_r^* b)$$

where T_r^* is the adjoint operator of T_r. The two actions on the group U make it imperative to use the jargon "Hecke-equivariance" with extra care. So we often refer the first action (when T_r acts as T_r) as the natural Hecke action on $H^1(\mathcal{C}, \vartheta_\ell)$ and call the second action the dual action. For the time being, we will make sure that the relevant maps in our future discussion are Galois-equivariant (seen from the appearances of Tate-twists), and point out the Hecke action whenever necessary. Now we can prove the following

Theorem 2.5 (Orthogonality Theorem) *Let* $V^\perp$ *be the orthogonal complement of* V *with respect to the paring* φ. *Then there is a natural isomorphism* $V^\perp = W(1-k)$.

Proof. By definition, we have $V = \ker N$. Since the map λ is the symmetric bilinear form induced by the quadratic form $-\sum e_x \delta_x^2$ over $\oplus_{x\in\Sigma} C_1(x, \mathcal{L})$, it is then injective. Since β is also injective, we have $\ker N = \ker \beta'$. Since β and β' are duals, we have $\operatorname{im}\beta = (\ker\beta')^\perp$. In view of Lemma 2.3, this completes the proof of the theorem. □

2.2 Reduction formulas of Shimura curves

Let $V_B(M)$ and ϑ_ℓ be defined as in the previous section. Let p be a prime such that $p|\mathrm{Disc}B$, and $\ell > 3$ is a prime such that $(\ell, p) = 1$. Set $N = M \cdot \mathrm{Disc}\ V_B(M)$, and $\bar{\vartheta}_{\ell^i} = \vartheta_\ell/\ell^i\vartheta_\ell$ for any positive integer i. Let $\mathbf{T}_N$ the classical full Hecke algebra associated to the space $S_k(\Gamma_1(N))$. It acts on the group $H^1(V_B(M)\times\bar{\mathbf{Q}}, \vartheta_\ell)$ because of the Shimura isomorphism (cf. [5]). The compatibility of the two Hecke structures on the subgroup $\hat{X}_p(V_B(M))_\ell$, $\mathbf{T}_N$ and the one introduced in Section 1.1, is established in [17], Section 4.

Lemma 2.6 *For each positive integer i, there are a natural isomorphisms*

(i) $H^1(V_B(M)\times\bar{\mathbf{Q}}_p, \bar{\vartheta}_{\ell^i}) \cong H^1(V_B(M)\times\bar{\mathbf{Q}}_p, \vartheta_\ell)\otimes(\mathbf{Z}/\ell^i\mathbf{Z})$;

(ii) $H^1(V_B(M)\times\bar{\mathbf{F}}_p, \bar{\vartheta}_{\ell^i}) \cong H^1(V_B(M)\times\bar{\mathbf{F}}_p, \vartheta_\ell)\otimes(\mathbf{Z}/\ell^i\mathbf{Z})$;

(iii) $H^1(V_B(M)\times\bar{\mathbf{F}}_p, \pi_*\pi^*(\bar{\vartheta}_{\ell^i})) \cong H^1(V_B(M)\times\bar{\mathbf{F}}_p, \pi_*\pi^*\vartheta_\ell)\otimes(\mathbf{Z}/\ell^i\mathbf{Z})$

Proof. For (i), apply cohomology to the exact sequence

$$0 \longrightarrow \vartheta_\ell \xrightarrow{\times\ell^i} \vartheta_\ell \longrightarrow \bar{\vartheta}_{\ell^i} \longrightarrow 0,$$

we get an exact sequence

$$\begin{aligned} H^1(V_B(M)\times\bar{\mathbf{Q}}_p, \vartheta_\ell) &\xrightarrow{\times\ell^i} H^1(V_B(M)\times\bar{\mathbf{Q}}_p, \vartheta_\ell) \longrightarrow H^1(V_B(M)\times\bar{\mathbf{Q}}_p, \bar{\vartheta}_{\ell^i}) \\ &\longrightarrow H^2(V_B(M)\times\bar{\mathbf{Q}}_p, \vartheta_\ell) \xrightarrow{\times\ell^i} H^2(V_B(M)\times\bar{\mathbf{Q}}_p, \vartheta_\ell). \end{aligned}$$

Since $\ell \neq p$, Poincaré duality theorem furnishes an isomorphism

$$H^2(V_B(M)\times\bar{\mathbf{Q}}_p, \vartheta_\ell) \cong \mathrm{Hom}(\Gamma(V_B(M)\times\bar{\mathbf{Q}}_p, \vartheta_\ell), \mathbf{Z}_\ell(1-k)[\varepsilon]).$$

Hence multiplication by ℓ^i on $H^2(V_B(M) \times \bar{\mathbf{Q}}_p, \vartheta_\ell)$ is injective. This proves (i). The last claim can be proved similarly. Since $H^1(V_B(M) \times \bar{\mathbf{F}}_p, \vartheta_\ell)$ is embedded by the specialization map sp in $H^1(C_{\text{gen}}, \vartheta_\ell)$, (ii) is then obvious. □

Definition 2.7 *The weight k group of connected components attached to the curve $V_B(M)$ at the prime p is the group $\Phi_p(V_B(M))_\ell = \operatorname{coker}\,(\lambda : X_p(V_B(M))_\ell(1-k) \to \hat{X}_p(V_B(M))_\ell)$.*

The Tate-twist in the definition is to let λ become a Galois-equivariant embedding.

Theorem 2.8 *There is a natural $\mathbf{T}_N$-equivariant isomorphism of* Gal $(\bar{\mathbf{F}}_p/\mathbf{F}_p)$*-modules*

$$\hat{X}_p(V_B(M))_\ell \otimes (\mathbf{Z}/\ell^i\mathbf{Z}) \cong H^1(V_B(M) \times \bar{\mathbf{F}}_p, \bar{\vartheta}_{\ell^i}). \tag{18}$$

where the Hecke algebra acts naturally on both groups.

Proof. Apply Proposition 1.13 and Lemma 2.6, there is an exact sequence

$$0 \to \hat{X}_p(V_B(M))_\ell \otimes (\mathbf{Z}/\ell^i\mathbf{Z}) \to H^1(V_B(M) \times \bar{\mathbf{F}}_p, \bar{\vartheta}_{\ell^i}) \to H^1(V_B(M) \times \bar{\mathbf{F}}_p, \pi_*\pi^*\bar{\vartheta}_{\ell^i}).$$

To see that the last group above is trivial, observe first that the map

$$\pi^* : \quad ((\text{ét})/V_B(M)_{/\bar{\mathbf{F}}_p})_{\text{ét}} \to ((\text{ét})/\widetilde{V_B(M)}_{/\bar{\mathbf{F}}_p})_{\text{ét}}$$

is exact. Hence its right adjoint π_* preserves injectives ([22], Lemma 1.2). Therefore, there is a natural isomorphism

$$H^1(V_B(M) \times \bar{\mathbf{F}}_p, \pi_*\pi^*\vartheta_{\ell^i}) \cong H^1(\widetilde{V_B(M)} \times \bar{\mathbf{F}}_p, \pi^*\vartheta_{\ell^i}).$$

Since $V_B(M)$ is admissible, our claim reduces to the statement that

$$H^1(\mathbf{P}^1_{\bar{\mathbf{F}}_p}, F) = 0,$$

where F is a constant sheaf. This is trivial since the arithmetic genus of $\mathbf{P}^1$ is 0. □

Theorem 2.9 *There is an exact sequence of* Gal $(\bar{\mathbf{F}}_p/\mathbf{F}_p)$*-modules*

$$\begin{aligned} 0 \longrightarrow H^1(V_B(M) \times \bar{\mathbf{F}}_p, \bar{\vartheta}_{\ell^i}) &\longrightarrow H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_{\ell^i})^I \\ \xrightarrow{\phi} \Phi_p(V_B(M))_\ell[\ell^i] &\to 0. \end{aligned}$$

It is also Hecke-equivariant with $\mathbf{T}_N$ *acts naturally on all groups.*

Proof. The following commutative diagram is derived from a part of the diagram (16) after tensoring each term with $(\mathbf{Z}/\ell^i\mathbf{Z})(1-k)$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_{\ell^i})^I \arrow[r, "\phi"] \arrow[d] & \Phi_p(C)[\ell^i] \arrow[d] \\
H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_{\ell^i}) \arrow[r, "\bar{\beta}'"] \arrow[d, "\bar{N}"'] & {[X_p(V_B(M))_\ell \otimes (\mathbf{Z}/\ell^i\mathbf{Z})](1-k)} \arrow[d, "\bar{\lambda}"'] \\
H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_{\ell^i})(1-k) & \hat{X}_p(V_B(M))_\ell \otimes (\mathbf{Z}/\ell^i\mathbf{Z}) \arrow[l, "\bar{\beta}"'] \arrow[d] \\
 & \Phi_p(C) \otimes (\mathbf{Z}/\ell^i\mathbf{Z}) \arrow[d] \\
 & 0
\end{tikzcd}$$

Notice that we have adjusted the group $\hat{X}_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell^i\mathbf{Z}$ with a Tate-twist in order to make the vertical exact sequence on the right Galois-equivariant. The map ϕ is induced by the map $\bar{\beta}'$. It is surjective since $\bar{\beta}'$ is surjective. Also it is clear that $\ker \phi = \ker \bar{\beta}' = H^1(V_B(M) \times \bar{\mathbf{F}}_p, \vartheta_{\ell^i})(k-1)$. The Hecke-equivariance is due to functoriality. $\square$

Remark 2.10 When $i = 1$, Theorems 2.8 and 2.9are the Shimura curve part of Theorem 5 and Theorem 6, respectively in [16]. The modular curve part of the two theorems will be given as a part of Theorems 2.13 and 2.15 in the next section.

We give one more formula below linking the generic fiber, the special fiber and the character group.

Theorem 2.11 *There is a* $\mathbf{T}_N$*-equivariant exact sequence of* Gal $(\bar{\mathbf{F}}_p/\mathbf{F}_p)$*-modules*

$$\begin{aligned} 0 \;\rightarrow\; & [\hat{X}_p(V_B(M))_\ell(k-1)] \otimes (\mathbf{Z}/\ell^i\mathbf{Z}) \;\rightarrow\; H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \vartheta_{\ell^i})(k-1) \\ \rightarrow\; & X_p(V_B(M))_\ell \otimes (\mathbf{Z}/\ell^i\mathbf{Z}) \;\rightarrow\; 0. \end{aligned}$$

The Hecke algebra's actions on the first two groups are the dual actions, and its action on the last group is induced by its natural action on $\hat{X}_p(V_B(M))_\ell$.

Proof. The pairing φ in (17) induces a natural isomorphism

$$U/V \cong \mathrm{Hom}(V^{\perp}, \mathbf{Z}_\ell[\varepsilon]). \tag{19}$$

By Theorem 2.5, we get

$$\mathrm{Hom}(V^{\perp}, \mathbf{Z}_\ell[\varepsilon]) \cong \mathrm{Hom}(W(1-k), \mathbf{Z}_\ell[\varepsilon]) = \mathrm{Hom}(\hat{X}_p(V_B(M))_\ell, \mathbf{Z}_\ell[\varepsilon]) = X_p(V_B(M))_\ell.$$

So there is an exact sequence of Gal($\bar{\mathbf{F}}_p/\mathbf{F}_p$)-modules

$$0 \rightarrow H^1(V_B(M) \times \bar{\mathbf{F}}_p, \vartheta_\ell)(k-1) \rightarrow H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \vartheta_\ell)(k-1) \rightarrow X_p(V_B(M))_\ell \rightarrow 0. \tag{20}$$

Now tensor the above sequence with $\mathbf{Z}/\ell^i\mathbf{Z}$, and apply Theorem 2.8, we then have the desired exact sequence. The Hecke-equivariance is obtained by tracing the exact sequence back to the Poincaré pairing φ. □

2.3 Reduction formulas of modular curves

In this section we treat the subjects discussed in the previous section with the curve C being a modular curve. It is necessary to separate this case with admissible curves because the reduction of a modular curve at a bad prime yields two components that are also modular curves (hence not the rational curve $\mathbf{P}^1$). There is then no canonical way to define the transition maps if we are to attach a local system to the dual graph of the special fiber Y. But the notion of a local system is only useful to relate our construction of the group $\hat{X}$ in Section 1.2 to Grothendieck's in [13], i. e., that the character group is the first homology group of the dual graph (see [13], formula (12.3.8) and compare to our Proposition 1.11). In what follows all we need is the modular curves analogues of the two reduction formulas, Theorems 2.8 and 2.9.

Let $k \geq 2$ and N be two positive integers, and p a square-free prime factor of N, and ℓ is a prime such that $(\ell, N) = 1$. Let Y_0 be the open modular curve of level N, and $X_0(N) = Y_0(N) \cup \{\text{cusps}\}$ the complete modular curve. Let ϑ_ℓ be the locally constant ℓ-adic sheaf on $Y_0(N)$ we defined in the beginning of this chapter. Recall that $\widetilde{X_0}(N)_{/\bar{\mathbf{F}}_p}$ is the normalization of $X_0(N)_{/\bar{\mathbf{F}}_p}$, and

$$\pi : \ \widetilde{X_0}(N)_{/\bar{\mathbf{F}}_p} \longrightarrow X_0(N)_{/\bar{\mathbf{F}}_p}$$

is the canonical map, and the sheaf ζ is defined by the exact sequence (5).

Definition 2.12 *The weight k dual character group $\hat{X}_p(X_0(N))_\ell$ is the cokernel of the map*

$$\Gamma(Y_0(N)_{/\bar{\mathbf{F}}_p}, \pi_*\pi^*\vartheta_\ell) \to \Gamma(Y_0(N)_{/\bar{\mathbf{F}}_p}, \zeta)$$

induced by the construction map

$$\pi_*\pi^*\vartheta_\ell \to \zeta$$

in the exact sequence (5).

Thanks to Deligne-Rapoport's work, we know completely the structure of the special fiber $X_0(N)_{/\bar{\mathbf{F}}_p}$ — It is two copies of $X_0(N/p)$ intersected at the supersingular points via Frobenius morphisms (cf. [8], Théorème 6.9). Hence by considering the exact sequence (5) locally,

$$0 \longrightarrow (\vartheta_\ell)_x \longrightarrow (\pi_*\pi^*\vartheta_\ell)_x \xrightarrow{p_x} \zeta_x \longrightarrow 0,$$

we find that the sheaf ζ is supported on the set Σ, i. e., $\zeta_x = 0$ if $x \notin \Sigma$. Moreover, for each $x \in \Sigma$, ζ_x is a free $(\vartheta_\ell)_x$-module of rank 1. For a supersingular point $x \in \Sigma$, the group ζ_x will then play the role of the group $C^1(x, \mathcal{L})$ in the previous two sections, while its $\mathbf{Z}_\ell[\varepsilon]$-dual $\zeta_x^\vee$ will substitute for the group $C_1(x, \mathcal{L})$. Fix a ϑ_ℓ-basis δ'_x for ζ_x and write δ_x for its dual in $\zeta_x^\vee$.

Theorem 2.13 (i) *There is an exact sequence*

$$0 \longrightarrow \hat{X}_p(X_0(N))_\ell \xrightarrow{\gamma} H^1_{\mathrm{par}}(Y_0(N) \times \bar{\mathbf{F}}_p, \vartheta_\ell) \xrightarrow{\varpi} H^1_{\mathrm{par}}(Y_0(N) \times \bar{\mathbf{F}}_p, \pi_*\pi^*\vartheta_\ell) \longrightarrow 0. \tag{21}$$

(ii) *There is an exact sequence of* $\mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$*-modules*

$$\begin{array}{ccccc} 0 & \longrightarrow & \hat{X}_p(X_0(N))_\ell \otimes (\mathbf{Z}/\ell^i\mathbf{Z}) & \longrightarrow & H^1_{\mathrm{par}}(Y_0(N) \times \bar{\mathbf{F}}_p, \bar{\vartheta}_{\ell^i}) \\ & \xrightarrow{\varpi} & [H^1_{\mathrm{par}}(Y_0(N/p), \bar{\vartheta}_{\ell^i})]^2 & \longrightarrow & 0. \end{array} \tag{22}$$

Both exact sequences are equivariant for the natural actions of the standard generators $T_r \neq T_p$.

Proof. (i) The sheaf ζ is supported on the set of supersingular points Σ of $Y_0(N)_{/\bar{\mathbf{F}}_p}$. Hence $H^1_{\mathrm{par}}(Y_0(N) \times \bar{\mathbf{F}}_p, \zeta) = 0$. The exactness is now clear by the definition of the group $\hat{X}_p(X_0(N))_\ell$.

(ii) The construction of the sheaf ϑ_ℓ allows us to apply the same argument used in the proof of Lemma 2.6 when we tensor each term in (21) by the group $\mathbf{Z}/\ell^i\mathbf{Z}$. The fact that

$$H^1_{\mathrm{par}}(Y_0(N) \times \bar{\mathbf{F}}_p, \pi_*\pi^*\vartheta_\ell) \cong [H^1_{\mathrm{par}}(Y_0(N/p), \vartheta_\ell)]^2$$

is due Deligne-Rapoport's results mentioned above. □

Definition 2.14 *The weight* k *group of connected components attached to* $X_0(N)$ *at the prime* p *is the group*

$$\Phi_p(X_0(N))_\ell = \mathrm{coker}(\lambda : \ X_p(X_0(N))_\ell(1-k) \to \hat{X}_p(X_0(N))_\ell).$$

Theorem 2.15 *The group $\Phi_p(X_0(N))_\ell[\ell]$ of ℓ-torsion points on $\Phi_p(X_0(N))_\ell$ is trivial. There is a Hecke-equivariant isomorphism of* $\mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$*-modules*

$$H^1_{\mathrm{par}}(Y_0(N) \times \bar{\mathbf{F}}_p, \bar{\vartheta}_\ell) \cong H^1_{\mathrm{par}}(Y_0(N) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)^I$$

where Hecke acts via natural action on both groups.

Proof. We only need to show the first statement. The second statement then easily follows since we have an analogous exact sequence to the one in Theorem 2.9 for modular curves. For $X_0(N)$, the scalars e_x only have 2 or 3 as possible divisors (cf. [25], Proposition 3.2). In particular, we have $(e_x, \ell) = 1$ for all $x \in \Sigma$. Therefore the map $\bar{\lambda}$ (induced by λ mod ℓ) is still injective. Since

$$\mathrm{Card}(\hat{X} \otimes \mathbf{Z}/\ell\mathbf{Z}) = \mathrm{Card}(\mathrm{Hom}(X \otimes \mathbf{Z}/\ell\mathbf{Z}, \mathbf{Z}/\ell\mathbf{Z})) = \mathrm{Card}(X \otimes \mathbf{Z}/\ell\mathbf{Z}),$$

we find that $\bar{\lambda}$ is an isomorphism. Now in the four-term exact sequence

$$0 \longrightarrow \Phi_p(C)[\ell] \longrightarrow \Phi_p(C) \xrightarrow{\times\ell} \Phi_p(C) \longrightarrow \Phi_p(C)/\ell\Phi_p(C) \longrightarrow 0,$$

the last group is trivial. Since the group $\Phi_p(C)$ is finite, the first group above must also be trivial. □

Remark 2.16 The results of this section can be generalized to what Deligne and Rapoport in [8] called "fausses courbes elliptiques." So consider the curve $V_B(N)_{\bar{\mathbf{F}}_p}$ with $p \mid N$. The group of connected components is defined the same as we did in Definition 2.14. Theorem

2.13 extends (with $X_0(N)$ and $Y_0(N)$ replaced by $V_B(N)$ and parabolic cohomology by étale cohomology) because of the work of Carayol [3] and Buzzard [2]. To see that Theorem 2.15 extends, recall Ribet's description of the integer e_x in [25], Section 4 in terms of the group automorphisms of an abelian variety. Namely, for a maximal ideal L in the quaternion algebra B, fix over $\bar{\mathbf{F}}_p$ an abelian surface A and an embedding $L \hookrightarrow A$. The abelian surface A is also furnished with an L-stable cyclic subgroup D of $A[M]$ of order M^2 and cyclic over L. If we let $\mathrm{End}_L(A, D)$ be the ring of L-invariant D-stable endomorphisms of A, we then have the identity

$$e_x = \frac{1}{2} \cdot \#\mathrm{Aut}(A, D).$$

Since there exist two lattices L_1 and L_2 in $\mathbf{C}$, such that

$$\mathrm{Aut}_L(A, D) \subset \mathrm{Aut}(\mathbf{C}/L_1) \times \mathrm{Aut}(\mathbf{C}/L_2),$$

our claim is now clear since it is well-known that the order of any element in the automorphism group of an elliptic curve divides 24.

3 Lowering the level

3.1 Action of Hecke algebra

Let M be a positive integer, and p, q be two distinct primes not dividing M. Throughout this section, we fix a Dirichlet character $\varepsilon : (\mathbf{Z}/M\mathbf{Z})^\times \to \mathbf{C}^\times$. For any N such that $M|N$, ε can be extended naturally to a character of $(\mathbf{Z}/N\mathbf{Z})^\times$ by the map $(\mathbf{Z}/N\mathbf{Z})^\times \to (\mathbf{Z}/M\mathbf{Z})^\times$. We will again denote this extension by ε. Also we write $S_k(\Gamma_0(pqM))$ for $S_k(\Gamma_0(pqM), \varepsilon)$. The weight k Hecke algebra attached to $S_k(\Gamma_0(pqM))$ is as usual $\mathbf{T}$=$\mathbf{T}_{pqM}$. Let $\mathbf{m}$ be a maximal ideal of $\mathbf{T}$ with residue characteristic ℓ. We assume $\ell > k$ and $(\ell, pqM) = 1$. Let $\rho_\mathbf{m}$ be the unique modular Galois representation attached to $\mathbf{m}$. This is a semisimple representation:

$$\rho_\mathbf{m} : \ \mathrm{Gal}\,(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}(2, \mathbf{T}/\mathbf{m}),$$

satisfying

$$\mathrm{trace}\,(\rho_\mathbf{m}(\mathrm{Frob}_r)) = T_r (\mathrm{mod}\ \mathbf{m}), \quad \det\,(\rho_\mathbf{m}(\mathrm{Frob}_r)) = \varepsilon(r)\, r^{k-1} (\mathrm{mod}\ \mathbf{m})$$

for almost all primes r. Furthermore, the representation $\rho_\mathbf{m}$ is unramified at all primes r not dividing ℓpqM, and the above two identities hold for all such primes. For $\mathcal{C}$ a modular curve with level N structure, or a Shimura curve with discriminant D and level M structure satisfying $DM = N$, recall that we have defined on $\mathcal{C}$ the sheaves Θ_ℓ and ϑ_ℓ in Chapter 2. The main purpose of this paper is to compute the $\mathbf{T}/\mathbf{m}$-dimension of the group $H^1[\mathbf{m}] = H^1(\mathcal{C} \times \bar{\mathbf{Q}}, \vartheta_\ell)[\mathbf{m}]$. It is worth noting that we have two actions of $\mathbf{T}$ on the group H^1.

One doesn't know *a priori* that the two actions yield the same kernel on H^1. When $\rho_{\mathfrak{m}}$ is irreducible, since the Eichler-Shimura relation

$$T_r = \mathrm{Frob}_r + < r > \mathrm{Frob}_r^t$$

holds for all primes r not dividing ℓN, the Boston-Lenstra-Ribet theorem shows that the $\mathbf{T}/\mathfrak{m}[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$-module $H^1(k-1)[\mathfrak{m}]$ is semisimple. The Čebotarev and Brauer-Nesbitt theorems then show that $H^1(k-1)[\mathfrak{m}]$ is a direct sum of copies of $\rho_{\mathfrak{m}}$ with $\mathbf{T}$ taking the dual action. Hence $H^1[\mathfrak{m}]$ consists of $\rho_{\mathfrak{m}}^{\vee}$, the dual representation, as direct summands. According to Shimura [30], the Hecke operator T_r acts as $w_N T_r w_N^{-1}$ on $H^1(k-1)$ with the classical w-operator w_N. One thus defines a $\mathbf{T}$-homomorphism

$$\tau : \quad H^1 \longrightarrow H^1(k-1)$$

by sending a to $w_N{\cdot}a{\otimes}\zeta^{\otimes(k-1)}$ with ζ some choice of generator of $\mathbf{Z}_\ell(1)$. This homomorphism is actually an isomorphism since the operator w_N is invertible. Thus the groups $H^1[\mathfrak{m}]$ and $H^1(k-1)[\mathfrak{m}]$ are isomorphic as $\mathbf{T}$-modules after all. Their dimensions as $\mathbf{T}/\mathfrak{m}$-spaces therefore coincide. In our ensuing discussion, we often consider the ramification condition of $\rho_{\mathfrak{m}}$ at a prime r and the action of a Frobenius element in $\rho_{\mathfrak{m}}$. The following two statements give us the convenience to ignore the difference between the two Hecke actions when these factors are our main concerns: (1) $\rho_{\mathfrak{m}}$ is unramified at r if and only if $\rho_{\mathfrak{m}}^{\vee}$ is unramified at r; (2) If $\rho_{\mathfrak{m}}$ is unramified at r, then Frob_r acts in $\rho_{\mathfrak{m}}$ as a scalar if and only if it acts in $\rho_{\mathfrak{m}}^{\vee}$ as the inverse scalar. The verification of these statements is an easy exercise of the definition of the dual representation.

The point of departure of our study of the multiplicity problem is the following Multiplicity One Principle for higher weight modular curves proved by Faltings and Jordan ([12], Theorem 2.1).

Theorem 3.1 *Suppose the representation* $\rho_{\mathfrak{m}}$ *is irreducible. Then* $H^1_{\mathrm{par}}(Y_1(N) \times \bar{\mathbf{Q}}, \bar{\Theta}_\ell)[\mathfrak{m}]$ *is isomorphic to the* $\mathbf{T}/\mathfrak{m}\,[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$*-module corresponding to* $\rho_{\mathfrak{m}}^{\vee}$*. In particular, we have*

$$\dim_{\mathbf{T}/\mathfrak{m}} H^1_{\mathrm{par}}(Y_1(N) \times \bar{\mathbf{Q}}, \bar{\Theta}_\ell)[\mathfrak{m}] = 2.$$

Let $B(pq)$ be the indefinite quaternion algebra over $\mathbf{Q}$ of discriminant pq. The Shimura curve arising from an Eichler order of conductor M in $B(pq)$ is denoted by $V_{B(pq)}(M)$. We begin our approach by defining certain quotients of the Hecke algebra $\mathbf{T}$. Then we study the actions of $\mathbf{T}$ on various groups through these quotients.

There are two natural degeneracy maps

$$\delta_1, \delta_q : \ X_0(pqM) \rightrightarrows X_0(pM) \tag{23}$$

between the two modular curves. From a modular point of view, the curve $X_0(pqM)$ is the moduli space classifying triples (E, C_{pM}, C_q) of an elliptic curve E with two cyclic subgroups of orders pM and p, respectively. The curve $X_0(pM)$ is the space classifying pairs (E, C_{pM}). The map δ_1 then sends the triple (E, C_{pM}, C_q) to the pair (E, C_{pM}), and

δ_q sends (E, C_{pM}, C_q) to the pair $(E/C_q, (C_{pM} \oplus C_q)/C_q)$. They both induce embeddings of $S_k(\Gamma_0(pM))$ into $S_k(\Gamma_0(pqM))$. Specifically, the embedding induced by δ_1 sends a cusp form $f(x) \in S_k(\Gamma_0(pM)$ to the cusp form $f(x) \in S_k(\Gamma_0(pqM)$, while the embedding induced by δ_q maps $f(x)$ to $q^{k-1}f(qx)$. The q-old subspace $S_k(\Gamma_0(pqM))_{q\text{-old}}$ of $S_k(\Gamma_0(pqM))$ is defined as the direct sum of these two embeddings. The corresponding q-new subspace $S_k(\Gamma_0(pqM))_{q\text{-new}}$ is the orthogonal complement to $S_k(\Gamma_0(pqM))_{q\text{-old}}$ with respect to the Petersson inner product on the space $S_k(\Gamma_0(pqM))$. Interchanging the primes q and p, we can symmetrically define the p-old and p-new subspaces. We also define the following composite subspaces:

$$\begin{aligned}
S_k(\Gamma_0(pqM))_{pq\text{-old}} &= S_k(\Gamma_0(pqM))_{p\text{-old}} + S_k(\Gamma_0(pqM))_{q\text{-old}} \\
S_k(\Gamma_0(pqM))_{pq\text{-new}} &= [S_k(\Gamma_0(pqM))_{pq\text{-old}}]^{\perp} \\
S_k(\Gamma_0(pqM))_{p\text{-old}/q\text{-old}} &= S_k(\Gamma_0(pqM))_{p\text{-old}} \cap S_k(\Gamma_0(pqM))_{q\text{-old}} \\
S_k(\Gamma_0(pqM))_{p\text{-old}/q\text{-new}} &= S_k(\Gamma_0(pqM))_{p\text{-old}} \cap S_k(\Gamma_0(pqM))_{q\text{-new}} \\
S_k(\Gamma_0(pqM))_{p\text{-new}/q\text{-old}} &= S_k(\Gamma_0(pqM))_{p\text{-new}} \cap S_k(\Gamma_0(pqM))_{q\text{-old}}
\end{aligned}$$

All these subspaces are stable under the action of $\mathbf{T}$.

Definition 3.2 *Let Ω be the set of symbols*

$$\{p\text{-old}, q\text{-old}, p\text{-new}, q\text{-new}, pq\text{-old}, pq\text{-new}, p\text{-old}/q\text{-old}, p\text{-old}/q\text{-new}, p\text{-new}/q\text{-old}\}.$$

For $* \in \Omega$, the quotient $\mathbf{T}_*$ is the image of $\mathbf{T}$ in the endomorphism ring $\mathrm{End}(S_k(\Gamma_0(pqM))_*)$ *of the space $S_k(\Gamma_0(pqM)_*$.*

As usual, we let $\hat{X}_p(V_{B(pq)}(M))_\ell$ be the dual character group attached to the mod p reduction of the Shimura curve $V_{B(pq)}(M)$. We also let $\hat{X}_q(X_0(qM))_\ell$ (resp. $\hat{X}_q(X_0(pqM))_\ell$) be the dual character group attached to the mod q reduction of the modular curve $X_0(qM)$ (resp. $X_0(pqM)$). Let us reconsider the exact sequence in Theorem 1.6 (ii) in more depth. In there, we used a *definite* quaternion algebra B to construct a finite graph $\Gamma\backslash\Delta$ with an attached local system $\mathcal{L}$. In our present context, the quaternion algebra B is related to the indefinite quaternion algebra $B(pq)$ here by the discriminant identity

$$\text{Disc } B = \frac{\text{Disc } B(pq)}{p} \cdot \infty = q\infty$$

Indeed, here lies deeply Drinfeld's geometric description ([11]) of bad reductions of Shimura curves at primes dividing the discriminants ("Interchanging local invariants at p and ∞"). His theory allows us to identify the dual graph of the special fiber of $V_{B(pq)}$ with $\Gamma\backslash\Delta$ with the prescribed local system $\mathcal{L}$. We refer the reader to [15], Section 4 for more detail on this. By Proposition 1.11 we then find that the group $H^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)_\ell)$ is isomorphic to the dual character group $\hat{X}_p(V_{B_{(pq)}}(M))_\ell$. We may further identify the groups $\hat{X}_q(X_0(qM))^2_\ell$ and $\hat{X}_q(X_0(pqM))_\ell$ with the subgroups of "degree-0 divisors" of the cochain groups in that sequence, i.e.,

$$\hat{X}_q(X_0(qM))^2_\ell \cong \{f \in C^0_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)_\ell) \mid \Sigma_{v \in \text{Ver}(\Delta)} f(v) = 0\},$$

and with a fixed orientation $\text{Ed}(\Delta) = \text{Ed}^+(\Delta) \amalg \text{Ed}^-(\Delta)$ of $\Gamma\backslash\Delta$,

$$\hat{X}_q(X_0(pqM))_\ell \cong \{f \in C^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon)_\ell) \mid \Sigma_{e \in \text{Ed}^+(\Delta)} f(e) = 0\}.$$

Moreover, there is a Hecke-equivariant exact sequence, called the Ribet Exact Sequence,

$$0 \to \hat{X}_q(X_0(qM))_\ell^2 \to \hat{X}_q(X_0(pqM))_\ell \to \hat{X}_p(V_{B(pq)}(M))_\ell \to 0. \tag{24}$$

This important formula was proved by Ribet in [25], Section 4 in the case $(k, \varepsilon) = (2, \mathrm{id})$. The most general version of the Ribet Exact Sequence is the sequence (32) in Section 4.2. Its proof will be explained in a forthcoming joint work by Jordan and Livné [19]. Now recall the exact sequence (21) in Theorem 2.13 (i), if we take the mod q reduction of the modular curve $C = Y_0(pqM)$, the group $H^1_{\mathrm{par}}(Y_0(pM), \vartheta_\ell)^2$ may be regarded as a $\mathbf{T}$-module via the projection ϖ. If we apply Theorem 2.13 again to the $\mathbf{T}$-module $H^1_{\mathrm{par}}(Y_0(pM), \vartheta_\ell)^2$ (for mod p reduction), the group $H^1_{\mathrm{par}}(Y_0(M), \vartheta_\ell)^4$ may be viewed as a $\mathbf{T}$-module as well. The dual character group $\hat{X}_p(V_{B_{(pq)}}(M))_\ell$ is naturally a sub-$\mathbf{T}$-module of the group $H^1(V_{B(pq)}(M), \vartheta_\ell)$. These are the $\mathbf{T}$-module structures we are going to take for these groups in the following theorem.

Theorem 3.3 (i)*The action of the Hecke algebra* $\mathbf{T}$ *on the group* $H^1_{\mathrm{par}}(Y_0(pM), \vartheta_\ell)^2$ *cuts out its q-old quotient* $\mathbf{T}_{q\text{-old}}$, *i.e., an element* $t \in \mathbf{T}$ *acts as* 0 *on* $H^1_{\mathrm{par}}(Y_0(pM), \vartheta_\ell)^2$ *if and only if its image in* $\mathbf{T}_{q\text{-old}}$ *is* 0*;*

(ii)*The action of* $\mathbf{T}$ *on* $H^1_{\mathrm{par}}(Y_0(M), \vartheta_\ell)^4$ *cuts out its p-old/q-old quotient* $\mathbf{T}_{p\text{-old}/q\text{-old}}$*;*

(iii)*The action of* $\mathbf{T}$ *on* $\hat{X}_p(V_{B(pq)}(M))_\ell$ *its pq-new quotient* $\mathbf{T}_{pq\text{-new}}$.

Proof. (i) By functoriality, the two degeneracy maps $\delta_1, \delta_q : X_0(pqM) \to X_0(pM)$ in (23) induce two degeneracy maps

$$\delta_1^*, \delta_q^* : \ H^1_{\text{par}}(Y_0(pM), \vartheta_\ell) \to H^1_{\text{par}}(Y_0(pqM), \vartheta_\ell).$$

These two maps together furnish a map

$$\alpha : \ H^1_{\text{par}}(Y_0(pM), \vartheta_\ell)^2 \to H^1_{\text{par}}(Y_0(pqM), \vartheta_\ell) \tag{25}$$

such that

$$(x, y) \longmapsto -\delta_1^*(x) + \delta_q^*(y).$$

Set $U = H^1_{\text{par}}(Y_0(pqM), \vartheta_\ell)$ and let $U_{q\text{-old}} = \operatorname{im} \alpha$. Define $U^{q\text{-new}}$ by the following exact sequence

$$0 \to U_{q\text{-old}} \to U \to U^{q\text{-new}} \to 0.$$

Dualizing the above sequence and adjusting with a $(1-k)$-Tate-twist, we have

$$0 \to U_{q\text{-new}} \to U \to U^{q\text{-old}} \to 0,$$

where $U_{q\text{-new}} = (U^{q\text{-new}})^\vee(1-k)$, and $U_{q\text{-old}} = (U^{q\text{-old}})^\vee(1-k)$. We claim the map

$$U \to U^{q\text{-old}} \times U^{q\text{-new}}$$

induced by natural projections is an injection. To see this, let $f \in U$ be mapped to (0,0) in the product. Then f is in the group $U_{q\text{-old}} \cap U_{q\text{-new}}$. Since

$$U_{q\text{-new}} = (U^{q\text{-new}})^\vee(1-k) = (U_{q\text{-old}})^\perp(1-k),$$

we have $\psi(f,f) = 0$, where ψ is the Poincaré pairing

$$U \times U \longrightarrow \mathbf{Z}_\ell[\varepsilon](1-k).$$

Since ψ is positive definite, f must be 0 and the claim is proved. Therefore we can represent each element f of U in a unique way by its image $(f^{q\text{-old}}, f^{q\text{-new}})$ in the product $U^{q\text{-old}} \times U^{q\text{-new}}$. To prove (i) of the Proposition, let us reproduce the exact sequence (21) for the curve $Y_0(pqM)$ below

$$0 \to \hat{X}_q(X_0(pqM))_\ell \to H^1_{\mathrm{par}}(Y_0(pqM) \times \bar{\mathbf{F}}_q, \vartheta_\ell) \xrightarrow{\varpi} (H^1_{\mathrm{par}}(Y_0(pM), \vartheta_\ell)^2 \to 0.$$

A result of Jordan and Livné ([17], Theorem 4.8) identifies over $\mathbf{C}$ the group of 1-cochains $C^1_\Gamma(\Delta, \mathcal{L}_{k-2}(\varepsilon))_\ell$ with the classical $S_k(\Gamma_0(pqM))_{q\text{-new}}$. So the first group on the left in the above sequence consists of those elements of the form $f = (0, f^{q\text{-new}})$. Hence an element t acts as 0 on $H^1_{\mathrm{par}}(Y_0(pM), \vartheta_\ell)^2$ if and only if it acts as 0 on the group

$$U(\bar{\mathbf{F}}_q)^{q\text{-old}} = \{f^{q\text{-old}} \mid f \in H^1_{\mathrm{par}}(Y_0(pqM) \times \bar{\mathbf{F}}_q, \vartheta_\ell)\}.$$

Since the map $U_{q\text{-old}} \hookrightarrow U$ factors through the map $U_{q\text{-old}} \hookrightarrow U(\bar{\mathbf{F}}_q)$ induced by α via the mod q reduction of $Y_0(pqM)$, we find that there is an identity

$$U(\bar{\mathbf{F}}_q)^{q\text{-old}} = U^{q\text{-old}}.$$

Therefore t acts as 0 on $H^1_{\mathrm{par}}(Y_0(pM), \vartheta_\ell)^2$ if and only if it acts as 0 on $U^{q\text{-old}}$, which is equivalent to t acting as 0 on $U_{q\text{-old}}$. And part (i) of our proposition is now obvious since the last statement is equivalent to $t = 0$ in $\mathbf{T}_{q\text{-old}}$.

(ii) It is clear now that in the following exact sequence derived from (21)

$$0 \to \hat{X}_q(X_0(pM))_\ell^2 \to H^1_{\mathrm{par}}(Y_0(pM) \times \bar{\mathbf{F}}_p, \vartheta_\ell)^2 \to H^1_{\mathrm{par}}(Y_0(M), \vartheta_\ell)^4 \to 0,$$

the middle group is isomorphic to $U^{q\text{-old}}$. Split the space $S_{q\text{-old}}$ further by its orthogonal subspaces $S_{p\text{-new}/q\text{-old}}$ and $S_{p\text{-old}/q\text{-old}}$, we then represent each $f^{q\text{-old}} \in H^1_{\mathrm{par}}(Y_0(pM), \vartheta_\ell)^2$ by a pair $(f^{p\text{-old}/q\text{-old}}, f^{p\text{-new}/q\text{-old}})$. And we find all elements in the first group on the left in the above sequence have the form $(0, f^{p\text{-new}/q\text{-old}})$. The proof of (ii) is now straightforward.

(iii) This can be obtained by using the sequence (24) and imitating the proof above. □

Remark 3.4 It is not difficult to see that if the $\mathbf{T}$-actions on the three groups above are (induced by) the dual actions, the statements of the theorem remain valid. Also, another proof of the this theorem can be easily obtained if one is willing to apply the Shimura isomorphism ([5], Théorème 2.10).

Proposition 3.5 *Assume the representation ρ_{m} is ramified at p. Let $\mathbf{T}$ acts on the groups below naturally, then*

(i) *The localization of $\hat{X}_q(X_0(qM))_\ell^2$ at $\mathbf{m}$ is trivial.*

(ii) *There is an isomorphism of $\mathbf{T}/\mathbf{m}$-vector spaces*

$$(\hat{X}_q(X_0(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}] \cong (\hat{X}_p(V_{B(pq)}(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}].$$

(iii) *The localization of $\Phi_p(V_{B(pq)}(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z}$ at $\mathbf{m}$ is trivial.*

Again, if $\mathbf{T}$ takes dual action on all the groups above, the statements remain valid.

Proof. We will only consider the case when $\mathbf{T}$ acts naturally.

(i) Let us assume the contrary. Then $[H^1_{\mathrm{par}}(Y_0(qM),\vartheta_\ell)^2]_{\mathbf{m}}$ is nontrivial. By Theorem 3.3 (i), we know $\mathbf{m}$ must be p-old, i. e., a pullback of a maximal ideal in $\mathbf{T}_{p\text{-old}}$. Hence the representation $\rho_{\mathbf{m}}$ is modular of level qM. In particular, $\rho_{\mathbf{m}}$ is unramified at p which contradicts our choice on p made above.

(ii) Tensor each term with $\mathbf{Z}/\ell\mathbf{Z}$ in the exact sequence (24), then localize it at $\mathbf{m}$. Applying (i), we get an isomorphism of $\mathbf{T}/\mathbf{m}$-modules

$$(\hat{X}_q(X_0(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})_{\mathbf{m}} \cong (\hat{X}_p(V_{B(pq)}(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})_{\mathbf{m}}.$$

Now taking kernels of the actions of $\mathbf{m}$ on the two groups above gives the relation we want.

(iii) Take the $\mathbf{Z}_\ell[\varepsilon](k-1)$-dual of the exact sequence (24) and then localize each term in the dual sequence at $\mathbf{m}$. By (i), there is an isomorphism of $\mathbf{T}$-modules

$$[(\hat{X}_q(X_0(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})(1-k)]_{\mathbf{m}} \cong [(\hat{X}_p(V_{B(pq)}(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})(1-k)]_{\mathbf{m}}.$$

By the definition of the group of connected components, we have

$$\begin{aligned}\Phi_p(V_{B(pq)}(M))_\ell)_{\mathbf{m}} &\cong (\hat{X}_p(V_{B(pq)}(M))_\ell)_{\mathbf{m}}/[(X_p(V_{B(pq)}(M))_\ell)(1-k)]_{\mathbf{m}}\\ &\cong (\hat{X}_q(X_0(pqM))_\ell)_{\mathbf{m}}/[(X_q(X_0(pqM))_\ell)(1-k)]_{\mathbf{m}}.\end{aligned}$$

Now tensor $\mathbf{Z}/\ell\mathbf{Z}$ above and the last group becomes zero because of the argument we used in the proof of Theorem 2.15. ☐

Recall we have defined in §1.1 the operator w_r for any prime r dividing the level pqM.

It serves as a link between the Frobenius element Frob_r and the irregular Hecke operator T_r when studying their actions on the character groups. Here are three properties of the operator w_q.

Lemma 3.6 *On the group $\hat{X}_q(X_0(pqM))_\ell$, we have*

(i) $w_q = -T_q$;

(ii) $w_q^2 = \varepsilon(q)q^{k-2}$;

(iii) $w_q = -\mathrm{Frob}_q^{-1}$.

Proof. The first two parts of this lemma are proved in [17] as part of Proposition 3.13 and part of Theorem 3.19, respectively. As for (iii), Faltings and Jordan have shown in [12] that the relation holds on the group $H^1_\Sigma(X_0(pqM)\times\bar{\mathbf{Q}}_q,\vartheta_\ell)$. In the following commutative diagram derived from long exact sequences

$$\begin{tikzcd}
H^1_\Sigma(X_0(pqM)\times\bar{\mathbf{Q}}_q,\vartheta_\ell) \arrow[r, "\mu"] & H^1(X_0(pqM)\times\bar{\mathbf{Q}}_q,\vartheta_\ell) \arrow[r] \arrow[d, "\varpi"'] & H^1(X_0(pqM)-\Sigma,\vartheta_\ell) \arrow[d, "\xi"] \\
 & H^1(X_0(pqM),\pi_*\pi^*\vartheta_\ell) \arrow[r] & H^1(X_0(pqM)-\Sigma,\pi_*\pi^*\vartheta_\ell)
\end{tikzcd}$$

The map ξ is an isomorphism since the sheaf ζ defined in (5) is supported on Σ. As a consequence $\mathrm{im}\,\mu \supseteq \ker\varpi = \hat{X}_q(X_0(pqM))_\ell$. And the claim is now clear. □

Proposition 3.7 *Assume the representation $\rho_{\mathfrak{m}}$ is irreducible. If we also have*

$$\dim_{\mathbf{T}/\mathfrak{m}}(\hat{X}_q(X_0(pqM))_\ell\otimes\mathbf{Z}/\ell\mathbf{Z})[\mathfrak{m}] > 1,$$

then $\rho_{\mathfrak{m}}$ is unramified at q and Frob_q *acts in $\rho_{\mathfrak{m}}$ as a scalar whose square is $\varepsilon(q)$.*

Proof. Due to the definition of the sheaves Θ_ℓ and ϑ_ℓ, the group $H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{Q}}_q,\bar{\vartheta}_\ell)$ is a subgroup of $H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{Q}}_q,\bar{\Theta}_\ell)$. By Theorem 3.1, the group $H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{Q}}_q,\bar{\vartheta}_\ell)[\mathbf{m}]$ is a $\mathbf{T}/\mathbf{m}$-vector space of dimension at most two. Consider the following $\mathbf{T}$-equivariant embeddings

$$(\hat{X}_q(X_0(pqM))_\ell\otimes\mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}]\hookrightarrow H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{Q}}_q,\bar{\vartheta}_\ell)^I[\mathbf{m}]\hookrightarrow H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{Q}}_q,\bar{\vartheta}_\ell)[\mathbf{m}].$$

Our assumption in the proposition then forces all three groups above to be identical. The equality between the second and the third groups above shows that $\rho^\vee_{\mathbf{m}}$ is unramified at q. Hence $\rho_{\mathbf{m}}$ is also unramified at q. By Lemma 3.6, the equality between the first group and the last group shows that Frob_q acts in $\rho^\vee_{\mathbf{m}} = H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{Q}}_q,\bar{\vartheta}_\ell)[\mathbf{m}]$ as the scalar T_q^{-1}. Therefore, the action of Frob_q in $\rho_{\mathbf{m}} = H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{Q}}_q,\bar{\vartheta}_\ell)(k-1)[\mathbf{m}]$ is given by $T_q \bmod \mathbf{m}$. Since $\det(\rho_{\mathbf{m}})$ is the mod ℓ cyclotomic character, we find that

$$\det(\rho_{\mathbf{m}}(\mathrm{Frob}_q)) = \varepsilon(q)q^{k-1} \equiv \varepsilon(q)q^{k-2} \pmod{\mathbf{m}}.$$

From this we easily find

$$q \equiv 1 \pmod{\ell}.$$

This congruence relation indicates that $\mathrm{Frob}_q^2 \equiv \varepsilon(q) \bmod \mathbf{m}$ and the proof is complete. □

3.2 Conjecture "epsilon"

Let N be a positive integer and q is a prime exactly dividing N. Let ℓ and k be two positive integers with k even and $\ell > k$. Let $\mathbf{T} = \mathbf{T}_N$ be the Hecke algebra acting on the space of weight k cusp forms with a Nebentypus ε of conductor dividing N.

Theorem 3.8 *Let* $\rho : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}(2, \bar{\mathbf{F}}_\ell)$ *be an irreducible representation modular of type* (N, k, ε). *Assume that* $(\ell, N) = 1$ *and* ρ *is unramified at* q. *Then* ρ *is modular of level* N/q.

Proof. We first prove the theorem for the following special case:

Case I (ε **is unramified at** q) Since ρ is modular of level N, there is a homomorphism $\omega : \mathbf{T} \to \bar{\mathbf{F}}_\ell$ such that

$$\operatorname{trace}(\rho(\mathrm{Frob}_r)) = \omega(T_r)$$

for almost all prime numbers r. Set $\mathbf{m} = \ker \omega$ and let $\rho_{\mathbf{m}}$ be the modular Galois representation attached to the maximal ideal $\mathbf{m}$. It is easy to see that

$$\rho \cong \rho_{\mathbf{m}}$$

as representations of the Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ over $\bar{\mathbf{F}}_\ell$. So we will replace ρ by $\rho_{\mathbf{m}}$ for the rest of this proof. Set $n = N/q$. It suffices to show that $\mathbf{m}$ is q-old. We prove this in several steps.

(i) The maximal ideal $\mathbf{m}$ is either q-new or q-old.

Suppose this is false. Then we are able to find an element $a = (a_0, a_1)$ in $\mathbf{m}$, where a_0 and a_1 are the images of a in $\mathbf{T}_{q\text{-old}}$ and $\mathbf{T}_{q\text{-new}}$, respectively, such that a_0 is a unit in $\mathbf{T}_{q\text{-old}}$. Multiply by a suitable element in $\mathbf{T}$ if necessary, we may assume that a_0 is 1_{old}, the identity element in $\mathbf{T}_{q\text{-old}}$. By the same argument, we may find an element $b = (b_0, 1_{\text{new}})$ in $\mathbf{m}$. Let $\text{id}_{\mathbf{T}}$ be the identity element of $\mathbf{T}$. Then we have

$$(\text{id}_{\mathbf{T}} - a) \cdot (\text{id}_{\mathbf{T}} - b) = 0.$$

This shows at least one of the two factors on the left-hand side above must be in $\mathbf{m}$. In either case, we conclude that $\text{id}_{\mathbf{T}}$ is an element of $\mathbf{m}$, which is absurd. From now on for the rest of the proof of case I, we assume that $\mathbf{m}$ is q-new. By the Čebotarev Density theorem, there is a prime Q such that $(Q, 2N\ell) = 1$, and Frob_Q acts in $\rho_{\mathbf{m}}$ as a complex conjugation. Let us regard both of the $\mathbf{T}$ and $\mathbf{T}_{QN,Q\text{-old}}$ as subrings of $\text{End}(S_k(\Gamma_0(QN), \varepsilon)^2)$. They share a common subring R generated by the Hecke operators $\{T_r\}_{r \neq Q}$. To avoid possible confusion of the names of the Hecke operator T_Q in different Hecke rings, let us reserve the usual T_Q for the Q^{th} Hecke operator in $\mathbf{T}_{QN,Q\text{-old}}$, and let τ be the Q^{th} Hecke operator in $\mathbf{T}$. Also put $\mathbf{T}' = \mathbf{T}_{QN}$. Therefore, $\mathbf{T}'_{Q\text{-old}} = R[T_Q]$ and $\mathbf{T} = R[\tau]$. Set $\mathcal{R} = R[T_Q, \tau]$. Following [26] we say a maximal ideal $\mathcal{M}$ of $\mathbf{T}'_{Q\text{-old}}$ is *compatible* with $\mathbf{m}$ if the image of $\mathcal{M}$ in $\mathbf{T}'_{Q\text{-old}}$ and $\mathbf{m}$ are contained in a same maximal ideal $\mathcal{I}$ of $\mathcal{R}$.

(ii) There exists a maximal ideal $\mathcal{M}$ of $\mathbf{T}'$ compatible with $\mathbf{m}$.

Apply the second formula of Theorem 3.16 of [17], the action of $\mathbf{T}_Q$ is given by

$$T_Q = \begin{bmatrix} 0 & \varepsilon(Q) \cdot Q^{k-2} \\ Q & \tau \end{bmatrix},$$

a simple calculation then produces the following identity

$$T_Q^2 - \tau \cdot T_Q + \varepsilon(Q) \cdot Q^{k-1} = 0 \tag{26}$$

This shows that the ring $\mathcal{R}$ is integral over $\mathbf{T}$. By Cohen-Seidenberg's Going-up Theorem there is a maximal ideal $\mathcal{I}$ of $\mathcal{R}$, such that $\mathcal{I} \cap \mathbf{T} = \mathbf{m}$. Let $\mathcal{M}'$ be the intersection of $\mathcal{I}$ and $\mathbf{T}'_{Q\text{-old}}$. Then $\mathcal{M}'$ is maximal in $\mathbf{T}'_{Q\text{-old}}$ and its pullback $\mathcal{M}$ in $\mathbf{T}'$ is a maximal ideal compatible with $\mathbf{m}$. Note that $\rho_{\mathbf{m}} \cong \rho_{\mathcal{M}}$ over $\bar{\mathbf{F}}_\ell$.

(iii) The maximal ideal $\mathcal{M}$ is q-new.

We would like to show that $\mathcal{M}$ is the pullback of a maximal ideal in $\mathbf{T}'_{q\text{-new}}$. Consider the following diagram, where arrows are natural projections and R' is the image of R in the ring $\mathrm{End}(S_k(\Gamma_0(N),\varepsilon)_{q\text{-new}})$.

$$\begin{tikzcd}
\mathcal{R} & = & R[T_Q,\tau] & = & \mathcal{R} & & \\
\cup & & \cup & & \cup & & \\
\mathbf{T} \arrow[d] & \supset & R \arrow[d] & \subset & \mathbf{T}'_{Q\text{-old}} \arrow[d] & & \mathbf{T}' \arrow[ll] \arrow[d] \\
\mathbf{T}_{q\text{-new}} & \supset & R' & \subset & \mathbf{T}'_{q\text{-new}/Q\text{-old}} & & \mathbf{T}'_{q\text{-new}} \arrow[ll]
\end{tikzcd}$$

The maximal ideal $\mathcal{I}$ of $\mathcal{R}$, traveling through the route

$$\mathcal{R} \supset \mathbf{T} \to \mathbf{T}_{q\text{-new}} \supset R'$$

with two intersections and a projection, comes down in R' as a nonunit ideal of R' since we assume that $\mathbf{m}$ is q-new. Therefore, following the other route

$$\mathcal{R} \supset \mathbf{T}'_{Q\text{-old}} \to \mathbf{T}'_{q\text{-new}/Q\text{-old}} \supset R',$$

the ideal $\mathcal{I}$ will also become the same nonunit ideal in R'. In particular, the image of $\mathcal{M}'$ in $\mathbf{T}'_{q\text{-new}/Q\text{-old}}$ is not the unit ideal. Pull back this image to $\mathbf{T}'_{q\text{-new}}$, we get an ideal $\neq (1)$.

Since this ideal also coincides the image of $\mathcal{M}$ in $\mathbf{T}'_{q\text{-new}}$, $\mathcal{M}$ is then q-new.

(iv) $\mathcal{M}$ is q-old

Since the q-old quotient $\mathbf{T}'_{q\text{-old}}$ of $\mathbf{T}'$ acts faithfully on the group $H^1_{\text{par}}(Y_0(nQ), \vartheta_\ell)^2$ (this is Theorem 3.3 (i)), it suffices to prove that

$$[H^1_{\text{par}}(Y_0(nQ), \vartheta_\ell)^2]_{\mathcal{M}} \neq 0.$$

Let us assume the contrary. Then Theorem 2.13 furnishes an identity

$$(\hat{X}_q(X_0(nqQ))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathcal{M}] = H^1_{\text{par}}(Y_0(nqQ) \times \bar{\mathbf{F}}_q, \vartheta_\ell)[\mathcal{M}].$$

The representations $\rho_{\mathfrak{m}}$ and $\rho_{\mathcal{M}}$ are isomorphic over $\bar{\mathbf{F}}_\ell$, so $\rho_{\mathcal{M}}$ is unramified at q as well. Therefore, according to Theorem 2.15

$$H^1_{\text{par}}(Y_0(nqQ) \times \bar{\mathbf{F}}_q, \vartheta_\ell)[\mathcal{M}] = H^1_{\text{par}}(Y_0(nqQ) \times \bar{\mathbf{Q}}_q, \vartheta_\ell)[\mathcal{M}].$$

The above two identities plus Theorem 3.1 show that

$$\dim_{\mathbf{T}'/\mathcal{M}}(\hat{X}_q(X_0(nqQ))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathcal{M}] = 2$$

Applying Proposition 3.1 (with the letters n, q and Q here being the letters p, q and M there, respectively), we see that Frob_Q is a scalar in $\rho_{\mathfrak{m}}$. This is a contradiction, since we have chosen Frob_Q such that it acts in $\rho_{\mathfrak{m}}$ as a complex conjugation. In particular, Frob_Q can not act as a scalar.

(v) $\mathcal{M}$ is q-old/Q-old. By Theorem 3.3 (ii), it will be enough to prove that the localization

$[H^1_{\mathrm{par}}(Y_0(n),\vartheta_\ell)^4]_{\mathcal{M}}$ is not trivial. If this is not true, Theorem 2.13 will provide an equality

$$(\hat{X}_Q(X_0(nQ))_\ell^2 \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathcal{M}] = H^1_{\mathrm{par}}(Y_0(nQ)\times \bar{\mathbf{F}}_Q,\vartheta_\ell)^2[\mathcal{M}].$$

The representation $\rho_{\mathcal{M}}$ is unramified at Q. Therefore the group on the right-hand side above is isomorphic to $H^1_{\mathrm{par}}(Y_0(nQ)\times \bar{\mathbf{Q}}_Q,\vartheta_\ell)^2[\mathcal{M}]$. Consider it as a $\mathbf{T}'/\mathcal{M}[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$-module. Since $\mathcal{M}$ is q-old, this group is nontrivial. Its semisimplification is a direct sum of copies of the unique $\mathbf{T}'/\mathcal{M}[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$-module $V_{\mathcal{M}}$ that gives the representation $\rho^\vee_{\mathcal{M}}$. Therefore $V_{\mathcal{M}}$ can be viewed as a subgroup of $(\hat{X}_Q(X_0(nQ))_\ell^2 \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathcal{M}]$. Then $\mathrm{Frob}_Q^{-1}, -w_Q$ and T_Q all agree in $\rho^\vee_{\mathcal{M}}$. This shows Frob_Q is a scalar in $\rho_{\mathcal{M}}$, again contradicting our choice of Frob_Q.

(vi) The ideal $\mathbf{m}_R = \mathbf{m}\cap R$ is q-old.

Let R_0 be the image of R in the ring $\mathrm{End}(S_k(\Gamma_0(N),\varepsilon)^4)$. The meaning of our statement above is to say that $\mathbf{m}_R$ is the pullback of a maximal ideal of R_0. Observe the following diagram where arrows again indicate projection maps.

$$\begin{tikzcd}
\mathcal{R} & = & R[T_Q,\tau] & = & \mathcal{R} \\
\cup & & \cup & & \cup \\
\mathbf{T} \arrow[d] & \supset & R \arrow[d] & \subset & \mathbf{T}'_{Q\text{-old}} \arrow[d] \\
\mathbf{T}_{q\text{-old}} & \supset & R_0 & \subset & \mathbf{T}'_{q\text{-old}/Q\text{-old}}
\end{tikzcd}$$

As we have shown in (v) above, $\mathcal{M}$ is q-old/Q-old. Therefore $\mathrm{im}\,\mathcal{M}'$, the image of $\mathcal{M}'$ in $\mathbf{T}'_{q\text{-old}/Q\text{-old}}$ does not generate the unit ideal. Hence $\mathrm{im}\,\mathcal{M}'\cap R_0 \neq R_0$. Pull back $\mathrm{im}\,\mathcal{M}'\cap R_0$ into R, we get $\mathbf{m}_R$, which is then not the unit ideal in R.

(vii) $\mathbf{m}$ is q-old.

Since $\mathbf{T}$ is a free $\mathbf{Z}$-module of finite rank, τ is therefore integral over R. Hence $\mathbf{T}_{q\text{-old}}$ is integral over R_0. Again by the Going-up Theorem, there is a maximal ideal $\mathbf{m}_{\text{old}}$ of $\mathbf{T}_{q\text{-old}}$ such that $\mathbf{m}_{\text{old}} \cap R_0 = \operatorname{im} \mathcal{M}' \cap R_0$. Let $\mathbf{m}'$ be the pullback of $\mathbf{m}_{\text{old}}$ in $\mathbf{T}$. Clearly $\mathbf{m}' \cap R = \mathbf{m}_R$. We are going to show that $\mathbf{m} = \mathbf{m}'$. Since $\mathbf{m}'$ is q-old, this will complete our proof of the theorem. Consider the image of $\mathbf{T}$ in $\mathbf{T}/\mathbf{m} \times \mathbf{T}/\mathbf{m}'$. First for any element of R, its image can not be (0,1). As for the image of τ, let us apply the Čebotarev Density theorem to the representation $\rho_{\mathbf{m}} \times \rho_{\mathbf{m}'}$. According to it, there is a $T_i \in \mathbf{T}$ with $i \neq Q$, such that T_i and τ have the same image. Therefore τ is also not (0,1) in $\mathbf{T}/\mathbf{m} \times \mathbf{T}/\mathbf{m}'$. By the Chinese Remainder Theorem, this is impossible if $\mathbf{m} \neq \mathbf{m}'$. Now we extend our theorem to the following general case:

Case II (ε is arbitrary mod N) Choose a prime r such that $r \equiv 2 \pmod 3$ and $r \nmid N\ell$. From step (ii) of case I, we find that ρ arises from a form on $\Gamma_1(N) \cap \Gamma_0(r)$, where the Nebentypus associated with this form is the natural mod N extension of ε. By abuse of notation, we denote this character by ε. Since our condition about the representation ρ at q, and because of the fact that $\det(\rho) = \varepsilon\chi^{q-1}$, there exists a Dirichlet character ε' unramified at q and congruent to ε mod $\mathbf{m}$. We now are able to apply Carayol's lemma ([4], Lemme 1) to conclude that ρ arises from a form on $\Gamma_1(N/q) \cap \Gamma_0(qr)$ with ε' as the associated character, provided that we are not in some exceptional case. This exceptional case occurs only if ℓ is 3 and when ρ is induced by a character of $\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\sqrt{-3}))$. Futher, it occurs only if there are supersingular elliptic curves in characteristic 3 with $\Gamma_1(N) \cap \Gamma_0(r)$ structure which have automorphism of order 3. This then requires, in particular, that r is congruent to 1 mod 3. By our choice of r, however, this exceptional case does not occur here. Now we apply what has been proved in case I to get rid of q and r successively and conclude that ρ arises from a form on $\Gamma_1(N/q)$. $\square$

Remark 3.9 The theorem is one of several closely related results with the "lowering the level" theme. Ribet has treated some cases in his paper [28]. Most recently in [9], Diamond has proved a result (Theorem 1.1) that deals this topic in the braodest sense. Another proof of this theorem, by Jordan and Livné, is forthcoming in [20], where they will complete their program announced in [16]. Thanks are due to Ken Ribet, who has suggested to me the method used here in case II.

4 Multiplicity in higher weight

4.1 Ribet's theorem in higher weight

Let p and q be two distinct primes. Let V_B be a Shimura curve arising from a maximal order of the indefinite quaternion algebra $B = B(pq)$ of discriminant pq over $\mathbf{Q}$. For a maximal ideal $\mathbf{m}$ of the Hecke algebra $\mathbf{T}_{pq}$ of weight two cusp forms of level pq with trivial Nebentypus, let $\rho_{\mathbf{m}}$ be the usual modular Galois representation attached to $\mathbf{m}$. In this section we generalize the following Ribet's theorem on multiplicities of Galois representations to higher weight. As Ribet has explained in the introduction of [26], $\rho_{\mathbf{m}}$ can not be unramified at both of the primes p and q if the weight is two. Let us assume it is ramified at p.

Theorem 4.1 *Let J be the jacobian of the Shimura curve V_B. Assume the maximal ideal $\mathbf{m}$ is pq-new and the representation $\rho_{\mathbf{m}}$ is irreducible. Then the multiplicity of $\rho_{\mathbf{m}}$ in $J[\mathbf{m}]$ is one unless $\rho_{\mathbf{m}}$ is unramified at q and the Frobenius element Frob_q at q acts as ± 1 in $\rho_{\mathbf{m}}$. In this latter case, the multiplicity is two.*

Proof. This is [26], Theorem 3. $\square$

The pq-new condition above is to guarantee positive multiplicity, due to the fact proved in Theorem 2.11 that the pq-new quotient of $\mathbf{T}_{pq}$ acts faithfully on J. Let us explain this in more detail in our generalized case. The weight k is any even integer greater than 2. The Shimura curve $V_B(M)$ comes from an Eichler order (the intersection of two maximal orders) of conductor M in the quaternion algebra B. The Hecke algebra $\mathbf{T} = \mathbf{T}_{pqM}$ is the one attached to $S_k(\Gamma_0(pqM), \varepsilon)$, where ε is a character of $(\mathbf{Z}/pqM\mathbf{Z})^\times$. As before, ℓ is the characteristic of the field $\mathbf{k} = \mathbf{T}/\mathbf{m}$ satisfying $\ell > k$ and $(\ell, pqM) = 1$. Let ϑ_ℓ be the sheaf defined on $V_B(M)$ as in Chapter 2. The combination of Theorem 2.8 and Theorem 2.11 provides an exact sequence

$$\begin{aligned} 0 \longrightarrow \quad & \hat{X}_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z} \quad \longrightarrow H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell) \\ \longrightarrow \quad & [X_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z}](1-k) \quad \longrightarrow 0. \end{aligned} \tag{27}$$

According to Theorem 3.3 (iii), if $\mathbf{m}$ is not pq-new, the localization of $\hat{X}_p(V_B(M))_\ell$ at $\mathbf{m}$ is zero. Hence the multiplicity of $\rho_\mathbf{m}$ in $H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)[\mathbf{m}]$ is zero. In order to consider more interesting cases, we thus assume that $\rho_\mathbf{m}$ is pq-new. Let $\bar{\mathbf{m}}$ be the image of $\mathbf{m}$ in $\mathbf{T}_{pq\text{-new}}$, and for $\mathfrak{A} \subseteq \mathbf{T}$, let $V(\mathfrak{A})$ be the set of maximal ideals of $\mathbf{T}$ containing $\mathfrak{A}$. Under the assumption that $\mathbf{m}$ is pq-new, since the character group $X_p(V_B(M))_\ell$ is a finitely generated $\mathbf{T}$-module on which $\mathbf{T}_{pq\text{-new}}$ acts faithfully, we have

$$\begin{aligned} & \operatorname{Supp}_{\mathbf{T}_{pq\text{-new}}}(X_p(V_B(M))_\ell/\bar{\mathbf{m}}X_p(V_B(M))_\ell) \\ & = V(\bar{\mathbf{m}} + \operatorname{Ann}_{\mathbf{T}_{pq\text{-new}}}(X_p(V_B(M))_\ell)) \\ & = V(\bar{\mathbf{m}}) \\ & \neq \varnothing, \end{aligned}$$

So the quotient $X_p(V_B(M))_\ell/\mathbf{m}X_p(V_B(M))_\ell$ is nontrivial. Hence $H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)[\mathbf{m}]$ contains a nonzero subgroup

$$(\hat{X}_p(V_B(M))_\ell/\ell\hat{X}_p(V_B(M))_\ell)[\mathbf{m}] = \operatorname{Hom}(X_p(V_B(M))_\ell/\mathbf{m}X_p(V_B(M))_\ell, \mathbf{Z}_\ell[\varepsilon]),$$

and this will ensure that the multiplicity of $\rho_\mathbf{m}$ in $H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)[\mathbf{m}]$ is positive.

Proposition 4.2 *Let* $\mathbf{m}$ *be* pq*-new. We further assume* $\rho_{\mathbf{m}}$ *is irreducible and unramified at* q*, and* Frob_q *acts in* $\rho_{\mathbf{m}}$ *as a scalar with* $\mathrm{Frob}_q^2 = \varepsilon(q)$*. Then*

$$\dim_{\mathbf{k}}(\hat{X}_q(X_0(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}] = 2.$$

Proof. Let $U = H^1_{\mathrm{par}}(Y_0(pqM), \bar{\vartheta}_\ell)$. Recall $U^{q\text{-new}}$ is defined as the cokernel of the degeneracy map α in (25), and $U_{q\text{-new}}$ is its $\mathbf{Z}_\ell[\varepsilon](1-k)$-dual. The Shimura isomorphism furnishes a Hecke-equivariant isomorphism

$$U_{q\text{-new}} \otimes \mathbf{C} \cong \hat{X}_q(X_0(pqM))_\ell \otimes \mathbf{C}.$$

Since $T_q = -w_q$ on $\hat{X}_q(X_0(pqM))_\ell$, we find it holds on $U_{q\text{-new}}$ as well. By assumption $\mathbf{m}$ is pq-new, it is *a fortiori* q-new. Since $\mathbf{T}_{q\text{-new}}$ acts faithfully on $U_{q\text{-new}}$, this shows $U_{q\text{-new}}[\mathbf{m}]$ is nontrivial and is a direct sum of copies of $\rho_{\mathbf{m}}^\vee$. In particular, $\rho_{\mathbf{m}}^\vee$ is inside $U_{q\text{-new}}[\mathbf{m}]$. Hence $T_q = -w_q$ in $\rho_{\mathbf{m}}^\vee$. Since $\mathrm{Frob}_q = \pm 1$, we see that $q \equiv 1 \pmod{\ell}$. By comparing their actions on the subgroup $\hat{X}_q(X_0(pqM))_\ell[\mathbf{m}]$, we find that $w_q = -T_q = -\mathrm{Frob}^{-1} q = \pm 1$ in $\rho_{\mathbf{m}}^\vee$. Next let us consider the exact sequence induced by the degeneracy map α

$$0 \longrightarrow \ker \alpha[\mathbf{m}] \longrightarrow H^1_{\mathrm{par}}(Y_0(pM), \bar{\vartheta}_\ell)^2[\mathbf{m}] \xrightarrow{\alpha} H^1_{\mathrm{par}}(Y_0(pqM), \bar{\vartheta}_\ell)[\mathbf{m}]. \qquad (28)$$

Combining Theorem 4.3 of [23] and Theorem 1 of [24], Ribet finds the kernel of α is Eisenstein. His methods can extends to higher weight context without difficulty. Hence we find $\ker \alpha$ has trivial extension with the kernel of $\mathbf{m}$. This shows that α becomes an embedding when restricted on $H^1_{\mathrm{par}}(Y_0(pM), \bar{\vartheta}_\ell)^2[\mathbf{m}]$. Since $\rho_{\mathbf{m}}$ is unramified at q, $\mathbf{m}$ is then q-old by

Theorem 3.8. Hence the group $H^1_{\mathrm{par}}(Y_0(pM),\bar{\vartheta}_\ell)^2[\mathfrak{m}]$ is nontrivial. By Theorem 3.1, there is an isomorphism

$$\alpha:\ H^1_{\mathrm{par}}(Y_0(pM),\bar{\vartheta}_\ell)^2[\mathfrak{m}] \xrightarrow{\cong} H^1_{\mathrm{par}}(Y_0(pqM),\bar{\vartheta}_\ell)[\mathfrak{m}].$$

Knowing this, plus the fact (easily deduced from the fourth formula of Theorem 3.16 of [17]) that w_q acts on the group $H^1_{\mathrm{par}}(Y_0(pM),\bar{\vartheta}_\ell)^2$ as the matrix

$$\begin{pmatrix} 0 & -\varepsilon(q)q^{k-2} \\ -1 & 0 \end{pmatrix},$$

we find that

$$H^1_{\mathrm{par}}(Y_0(pqM),\bar{\vartheta}_\ell)[\mathfrak{m}] \subseteq \{(x,y)\in H^1_{\mathrm{par}}(Y_0(pM),\bar{\vartheta}_\ell)^2 \mid x=-w_q y\}.$$

Recall we have used the letter ϖ for the surjection

$$H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{F}}_q,\bar{\vartheta}_\ell)\to H^1_{\mathrm{par}}(Y_0(pM),\bar{\vartheta}_\ell)^2$$

in the exact sequence (22). Set

$$\varpi\,(a)=(\varpi_1(a),\varpi_2(a)).$$

The map α induces via mod q reduction a map

$$\alpha_{/\bar{\mathbf{F}}_q}:\ H^1_{\mathrm{par}}(Y_0(pM)\times\bar{\mathbf{F}}_q,\bar{\vartheta}_\ell)^2\to H^1_{\mathrm{par}}(Y_0(pqM)\times\bar{\mathbf{F}}_q,\bar{\vartheta}_\ell).$$

Sice $\delta_q^*=-w_q\delta_1^*$, the action of $\varpi\circ\alpha_{/\bar{\mathbf{F}}_q}$ on $H^1_{\mathrm{par}}(Y_0(pM)\times\bar{\mathbf{F}}_q,\bar{\vartheta}_\ell)^2[\mathfrak{m}]$ is then given by the matrix

$$\begin{pmatrix} -\varpi_1\circ\delta_1^* & -w_q\cdot\varpi_1\circ\delta_1^* \\ -\varpi_2\circ\delta_1^* & -w_q\cdot\varpi_2\circ\delta_1^* \end{pmatrix}.$$

Therefore,

$$\begin{aligned} &H^1_{\mathrm{par}}(Y_0(pqM), \bar{\vartheta}_\ell)[\mathbf{m}] \subseteq \ker \varpi \circ (\alpha_{/\bar{\mathbf{F}}_q}|_{H^1_{\mathrm{par}}(Y_0(pM)\times\bar{\mathbf{F}}_q, \bar{\vartheta}_\ell)^2[\mathbf{m}]}) \\ &= \ker \varpi = \hat{X}_q(X_0(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z}. \end{aligned}$$

This shows that $\hat{X}_q(X_0(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z}[\mathbf{m}]$ is of k-dimension at least 2. This completes our proof since Theorem 3.1 asserts that this dimension can not exceed 2. ☐

Now we can state and prove the generalization of Theorem 4.1.

Theorem 4.3 *Let* $\mathbf{m}$ *be a maximal ideal of* $\mathbf{T} = \mathbf{T}_{pqM}$ *such that* $\rho_{\mathbf{m}}$ *is irreducible. Assume also that* $\rho_{\mathbf{m}}$ *is ramified at* p*, and the characteristic* ℓ *of the field* $\mathbf{k}=\mathbf{T}/\mathbf{m}$ *is prime to* pqM*. Set* $d = \frac{1}{2}\dim_{\mathbf{k}} H^1_{\mathrm{par}}(V_B(M) \times \bar{\mathbf{Q}}, \bar{\vartheta}_\ell)[\mathbf{m}]$. *Then*

(i) $d = 0$ *if and only if* $\mathbf{m}$ *is not pq-new;*

(ii) $d = 2$ *if and only if* $\mathbf{m}$ *is pq-new, the representation* $\rho_{\mathbf{m}}$*is unramified at* q *and* Frob_q *acts in* $\rho_{\mathbf{m}}$ *as* ± 1*;*

(iii) $d = 1$ *in all other cases.*

Proof. We only need to show (ii). Consider "multiplication by ℓ" on each term of the exact sequence (24)

$$\begin{tikzcd} 0 \arrow[r] & X_p(V_B(M))_\ell \arrow[r] \arrow[d, "\times\ell"] & \hat{X}_p(V_B(M))_\ell \arrow[r] \arrow[d, "\times\ell"] & \Phi_p(V_B(M))_\ell \arrow[r] \arrow[d, "\times\ell"] & 0 \\ 0 \arrow[r] & X_p(V_B(M))_\ell \arrow[r] & \hat{X}_p(V_B(M))_\ell \arrow[r] & \Phi_p(V_B(M))_\ell \arrow[r] & 0 \end{tikzcd} \qquad (29)$$

Applying the Snake Lemma, we get a four-term exact sequence

$$\begin{aligned} 0 \to \quad & \Phi_p(V_B(M))_\ell[\ell] \to X_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z} \\ \to \quad & \hat{X}_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z} \to \Phi_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z} \to 0. \end{aligned}$$

After localizing each term at $\mathbf{m}$, the groups at the two ends disappear (due to Proposition 3.5 (iii)). Hence in the following exact sequence derived from (27)

$$\begin{aligned} 0 \;\to\; (\hat{X}_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}] \quad & \to \quad H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)[\mathbf{m}] \\ \to\; (X_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})(1-k)[\mathbf{m}]. & \end{aligned}$$

the first and last groups have the same dimension over $\mathbf{k}$. According to Proposition 3.5 (ii), this common dimension is the dimension d' of the space $(\hat{X}_q(X_0(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}]$. By Theorem 3.1, $d' \leq 2$. Hence $d \leq 2$ by the above exact sequence. We claim $d = d'$. If $d = 2$, this is clear; if $d = 1$, it is impossible that $d' = 2$ — otherwise, we would have

$$(\hat{X}_p(V_B(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}] = H^1(V_B(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)[\mathbf{m}].$$

This is a contradiction since the action of Gal$(\bar{\mathbf{Q}}/\mathbf{Q})$ on the left group above is unramified, and we have already assumed that $\rho_{\mathbf{m}}$ is ramified at p. This proves our claim. Apply the results in Propositions 3.1 and 4.2 and the proof is now complete.

4.2 The case $\rho_{\mathbf{m}}$ is dominantly ramified at D

Let $B(D)$ be the indefinite quaternion algebra over $\mathbf{Q}$ with discriminant $D = p_1 \cdot p_2 \cdots p_{2n}$, where the p_i's are $2n$ distinct primes. For a positive integer M prime to all the p_i's, let $V_{B(D)}(M)$ be a Shimura curve associated with an Eichler order of conductor M in the quaternion algebra $B(D)$. Let $\mathbf{m}$ be a maximal ideal of residue characteristic ℓ of the Hecke algebra $\mathbf{T} = \mathbf{T}_{DM}$. Set $\mathbf{k}=\mathbf{T}/\mathbf{m}$. As usual, we assume $(\ell, DM) = 1$, $\ell > k$, and we also assume that the modular Galois representation $\rho_{\mathbf{m}}$ is irreducible.

Definition 4.4 *Let r be the number of the primes in the set $\{p_1, p_2, \cdots, p_{2n}\}$ at which ρ_{m} is unramified. Then ρ_{m} is dominantly ramified at the discriminant D if $r \leq n$.*

In general, if ρ_{m} is unramified at r primes of the $2n$ primes in the discriminant D of the curve $V_{B(D)}(M)$, we define $d(n,r)$ to be the maximal dimension of the group $H^1(V_{B(D)}(M), \bar{\vartheta}_\ell)[\mathbf{m}]$ with d and r fixed. We can define, as we did in § 3.1, various subspaces of $S_k(\Gamma_0(DM), \varepsilon)$. For instance, the D-old subspace is the space generated by the images of the degeneracy maps coming from levels D/p, with p ranging through the prime divisors of the discriminant D. The D-new subspace is then the orthogonal complement to the D-old subspace with respect to the Petersson inner product. The D-new quotient $\mathbf{T}_{D\text{-new}}$ of $\mathbf{T}$ is the image of $\mathbf{T}$ in $S_k(\Gamma_0(DM), \varepsilon)_{D\text{-new}}$. The maximal ideal $\mathbf{m}$ is said to be D-new if it is the pullback of a maximal ideal in $\mathbf{T}_{D\text{-new}}$. The purpose of this section is to prove the following

Theorem 4.5 *Assume that* $\mathbf{m}$ *is D-new. If ρ_{m} is dominantly ramified at D, then*

$$d\,(n,r) \leq 2^{r+1}.$$

Proof. By assumption, ρ_{m} is totally ramified at D. Denote $p = p_{2n}$, $q = p_{2n-1}$ and $P = D/pq$. The exact sequence (27) for the curve $V_{B(D)}(M)$ is the following

$$\begin{aligned} 0 \quad \rightarrow \quad & \hat{X}_p(V_{B(D)}(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z} && \rightarrow \quad H^1(V_{B(D)}(M) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell) \\ \rightarrow \quad & [X_p(V_{B(D)}(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z}](1-k) && \rightarrow \quad 0. \end{aligned} \tag{30}$$

Since $\mathbf{m}$ is D-new and the $\mathbf{T}$-action on $\hat{X}_p(V_{B(D)}(M))_\ell$ cuts out $\mathbf{T}_{D\text{-new}}$, this formula shows that $d\,(n,0) \geq 2$. Take kernels of $\mathbf{m}$ in the above sequence. We claim the $\mathbf{k}$-dimensions of the first and the last groups are the same and will be denoted by d. To see this, it suffices to show the identity

$$(\Phi_p(V_{B(D)}(M))_\ell) \otimes \mathbf{Z}/\ell\mathbf{Z})_{\mathbf{m}} = 0 \tag{31}$$

holds. This can be done by an argument similar to the one we made in the proof of Proposition 3.5 (iii). Let us give this argument here again. First observe the fact that $\mathbf{T}$ acts on the group $(\hat{X}_q(V_{B(D/pq)}(qM))^2_\ell$ through its p-old quotient. Since $\rho_{\mathbf{m}}$ is ramified at p, we have an identity

$$(\hat{X}_q(V_{B(D/pq)}(qM))^2_\ell)_{\mathbf{m}} = 0.$$

The generalized Ribet Exact Sequence in this context is the following

$$0 \to \hat{X}_q(V_{B(D/pq)}(qM))^2_\ell \to \hat{X}_q(V_{B(D/pq)}(pqM))_\ell \to \hat{X}_p(V_{B(D)}(M))_\ell \to 0. \tag{32}$$

This then shows that the last two groups are isomorphic after they are localized at $\mathbf{m}$. Therefore,

$$\begin{aligned}
&[\Phi_p(V_{B(D)}(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z}]_{\mathbf{m}} \\
&\cong\ [\hat{X}_p(V_{B(D)}(M))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})_{\mathbf{m}}/(X_p(V_{B(D)}(M))_\ell \otimes \mathbf{Z}(1-k)/\ell\mathbf{Z}]_{\mathbf{m}} \\
&\cong\ [\hat{X}_q(V_{B(D/pq)}(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})_{\mathbf{m}}/(X_q(V_{B(D/pq)}(pqM))_\ell(1-k) \otimes \mathbf{Z}/\ell\mathbf{Z}]_{\mathbf{m}} \\
&= 0
\end{aligned}$$

where the last step is due to Remark 2.16. The sequence (30) then gives an inequality

$$d\,(n,r) \leq 2d. \tag{33}$$

Consider the natural injection

$$(\hat{X}_q(V_{B(P)}(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}] \hookrightarrow H^1(V_{B(P)}(pqM) \times \bar{\mathbf{Q}}_q, \bar{\vartheta}_\ell)[\mathbf{m}].$$

From the argument above we find that d is also the dimension of the $\mathbf{T}/\mathbf{m}$-vector space $(\hat{X}_q(V_{B(P)}(pqM))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathbf{m}]$. If $\rho_{\mathbf{m}}$ is also ramified at q, the injection above can not be an identity since the action of Galois on the group $(\hat{X}_q(V_{B(P)}(pqM))_\ell$ is unramified. Therefore in this case the dimension d satisfies the following inequality

$$d + 1 \leq d\,(n-1, r). \tag{34}$$

If $\rho_{\mathbf{m}}$ is unramified at q, we then simply have

$$d \leq d\,(n-1, r-1). \tag{34'}$$

Combining (33) and (34), we find that under the condition $r < n$, there is an inequality

$$d\,(n, r) \leq 2[d\,(n-1, r) - 1]. \tag{35}$$

The combination of (33) and (34′) yields the relation

$$d\,(n, r) \leq 2d\,(n-1, r-1). \tag{35'}$$

Now apply (35′) repeatedly, we have the following inequality

$$d\,(n, r) \leq 2^r \cdot d\,(n-r, 0). \tag{36}$$

To obtain the value of $d(n-r,0)$, apply (35) successively, we have

$$d(n-r,0) \leq 2^{n-r-1} \cdot d(1,0) - \sum_{i=1}^{n-r-1} 2^i = 2^{n-r} - (2^{n-r}-2) = 2$$

where $d(1,0) = 2$ is due to Theorem 4.3 (iii). Take into account what we have shown in the beginning of the proof, the dimension $d(n-r,0)$ is exactly 2. Now go back to (36), we find that

$$d(n,r) \leq 2^{r+1}. \quad \square$$

Corollary 4.6 *If the representation ρ_{m} is ramified at all the primes in the discriminant, the multiplicity is then equal to one.*

Proof. This is contained in the proof above. ☐

Remark 4.7 Our proposition above shows that the Multiplicity One Principle is preserved by a D-new form f in $S_k(\Gamma_0(DM),\varepsilon)$ if it is not congruent (modulo ℓ) to any other form of lower level dividing DM or, in Mazur's language, if *fusion* does not occur between f and any form coming from any lower level modulo ℓ.

4.3 Existence of higher multiplicity

For the remainder of this chapter, we would like to discuss the existence of higher multiplicities. Let us first consider the case $D = pq$. For simplicity we let $M = 1$ and $\varepsilon = 1$. We shall begin with a new form f in $S_k(\Gamma_0(p))$. Choose a proper prime ℓ (i. e., $(\ell, p) = 1$, and $\ell > k+1$ just to be safe). The cusp form $f \pmod{\ell}$ is then associated to a maximal ideal $\mathbf{m}$ in the Hecke algebra $\mathbf{T}_p$ such that the modular Galois representations attached to f and $\mathbf{m}$, ρ_f and $\rho_{\mathbf{m}}$, respectively, are isomorphic. In light of our remark above, in order to obtain higher multiplicity, we need to create a fusion between f and some form from level pq. This is done by the usual trick of "raising the level," i. e., we use the Čebotarev Density theorem to find a prime q away from $p\ell$ such that $\rho_{\mathbf{m}}(\mathrm{Frob}_q)$ is $+1$ or -1. As is explained in Section 3.2, the two Hecke algebras $\mathbf{T}_p$ and $\mathbf{T}_{pq,q\text{-old}}$ can be regarded as $R[\tau]$ and $R[T_q]$, respectively, where R is the largest common subring they share while τ and T_q are their respective q^{th} Hecke operators. As before, we set $\mathcal{R} = R[T_q, \tau]$. The two operators T_q and τ are related, they satisfy the identity (26)

$$T_q^2 - \tau T_q + q^{k-1} = 0.$$

Since $\rho_{\mathbf{m}}(\mathrm{Frob}_q) = \pm 1$, we have $q \equiv 1 \pmod{\ell}$. Hence $\tau \equiv \pm 2 \pmod{\mathbf{m}}$. Let us insert a word here about a convention: with $\rho_{\mathbf{m}}(\mathrm{Frob}_q)$ set as ± 1, the symbol ± in front of a term means that we take the same parity of signs between that term and $\rho_{\mathbf{m}}(\mathrm{Frob}_q)$; and with ∓ we mean by taking the opposite parity between their signs. As we have shown in the proof of Theorem 3.8, there is a maximal ideal $\mathcal{M}'$ of $R[T_q]$ compatible with $\mathbf{m}$, i. e., both $\mathbf{m}$ and $\mathcal{M}'$ are contained in a same maximal ideal $\mathcal{I}$ of $\mathcal{R}$. Let $\mathcal{M}$ be the preimage of $\mathcal{M}'$ in $\mathbf{T}_{pq}$. There is an obvious embedding

$$\mathbf{T}_p/\mathbf{m} \hookrightarrow \mathcal{R}/\mathcal{I}.$$

View (26) as over $\mathcal{R}/\mathcal{I}$, we find that

$$(T_q \mp 1)^2 = 0.$$

This shows that $T_q \equiv \pm 1 \pmod{\mathcal{I}}$. Thus the embedding above is in fact an isomorphism. Similarly, since τ is ± 2 in the field $\mathcal{R}/\mathcal{I}$, the embedding

$$\mathbf{T}_{pq,q\text{-old}}/\mathcal{M}' \hookrightarrow \mathcal{R}/\mathcal{I}$$

is also an isomorphism. The Hecke-equivariant isomorphism between the residue fields of $\mathbf{m}$ and $\mathcal{M}$ then necessarily induces an isomorphism between the Galois representations $\rho_{\mathbf{m}}$ and $\rho_{\mathcal{M}}$. According to Theorem 4.3, in order to find a four-dimensional kernel $H^1(V_{B(pq)}(1), \vartheta_\ell)[\mathcal{M}]$, we must check that the following four conditions are satisfied:

- (C1) $\rho_{\mathcal{M}}$ is irreducible;
- (C2) $\mathcal{M}$ is pq-new;
- (C3) $\rho_{\mathcal{M}}$ is unramified at q and Frob_q acts in $\rho_{\mathcal{M}}$ as ± 1;
- (C4) $\rho_{\mathcal{M}}$ is ramified at p.

The condition (C1) is easily seen to be satisfied since f is non-Eisenstein (see, for instance, [12], Theorem 3.38). As for (C2), there are several theorems, beginning with Ribet's [27], Theorem 1, on how to add new primes to the level without losing the previous primes in the level. To add more primes to the level we apply Diamond-Taylor's criterion on lifting modular representations ([10], Theorem 3). By the choice of q, we find

$$\operatorname{trace}(\rho_{\mathcal{M}}(\mathrm{Frob}_q)) = \pm 2, \quad \det(\rho_{\mathcal{M}}(\mathrm{Frob}_q)) = 1$$

and

$$q \equiv 1 \pmod{\ell}.$$

The congruence identity in that theorem then holds for the prime q. Hence $\mathcal{M}$ is pq-new. We have selected q to satisfy (C3). For (C4), by conjecture ε we need to select f such that it does not come from a lower level, in this case the only choice of a lower level is one. In general, to determine if f comes from a lower level form or not requires case by case study. But for our purpose, to avoid further calculation, we can simply choose k to be one of the following five integers: $4, 6, 8, 10, 14$. Now f can not possibly come from $S_k(\Gamma_0(1))$ since the space is trivial for such k (cf. [30], Proposition 2.26). We summarize our discussion above as the following

Proposition 4.8 *Let $k = 4, 6, 8, 10, 14$. Fix a prime $\ell > k + 1$. There exist infinitely many distinct primes p and q such that $(pq, \ell) = 1$, and a maximal ideal $\mathcal{M}$ of residue characteristic ℓ in $\mathbf{T}_{pq}$, such that*

$$\dim_{\mathbf{T}/\mathcal{M}} H^1(V_{B(pq)}(1), \vartheta_\ell)[\mathcal{M}] = 4.$$

In weight two, Ribet gives a specific numerical example for the exceptional case in [26]. The only thing needed to achieve this in our case is an auxiliary prime whose existence is guaranteed by the Čebotarev Density theorem. It will be interesting to see such an example.

Finally let us examine the existence of higher multiplicities when D is arbitrary. Our plan is to show a procedure of producing a higher multiplicity kernel for a Shimura curve with four primes in its discriminant from the known existence of higher multiplicities for Shimura curves with two primes in their discriminants. Then induction will provide the construction for arbitrary D. We start with a fixed prime ℓ. Choose two distinct primes r and s away from ℓ. Take a form f in $S_k(\Gamma_0(rs))$ such that the modular Galois representation ρ_f associated to f modulo ℓ is ramified at both r and s. To see a numerical example of such f, we can take $\ell = 7$, $r = 3$, $s = 5$ and f to be the form $f_1 \in S_4(\Gamma_0(15))$ listed in [31]. Hence $f = -q - 4q^2 + 2q^3 + 7q^4 - 2q^5 + \cdots$. Since both $S_4(\Gamma_0(1))$ and $S_4(\Gamma_0(3))$ are trivial (*loc. cit.*), f modulo ℓ can not come from level 1 or 3. The space $S_4(\Gamma_0(5))$ is one dimensional with $g = -q + 4q^2 - 2q^3 - 8q^4 + 5q^5 + \cdots$ as a basis. Since $f \not\equiv m \cdot g \pmod{\ell}$ for any $m \in \mathbf{Z}$, f does not come from level 5 either. Now we use the Čebotarev Density theorem to find two distinct primes p and q not equal to r, s or ℓ such that

$$\{\rho_f(\mathrm{Frob}_p),\ \rho_f(\mathrm{Frob}_q)\} \subseteq \{-1,\ +1\}.$$

By Diamond-Taylor's theorem cited above, there is a maximal ideal $\mathcal{M}$ of $\mathbf{T} = \mathbf{T}_{pqrs}$ such that $\rho_{\mathcal{M}} \cong \rho_f$ and $\mathcal{M}$ is $pqrs$-new. Consider the following Ribet Exact Sequence

$$0 \to \hat{X}_p(V_{B(qs)}(p))_\ell^2 \to \hat{X}_p(V_{B(qs)}(pr))_\ell \to \hat{X}_r(V_{B(pqrs)})_\ell \to 0.$$

The localization of $\hat{X}_p(V_{B(qs)}(p))_\ell^2$ at $\mathcal{M}$ is trivial since $\mathbf{T}$ acts on it through its r-old quotient and $\rho_{\mathcal{M}}$ is ramified at r. Therefore, we have an isomorphism of $\mathbf{T}/\mathcal{M}$-vector spaces

$$(\hat{X}_p(V_{B(qs)}(pr))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathcal{M}] \cong (\hat{X}_r(V_{B(pqrs)})_\ell \otimes \mathbf{Z}/\ell\mathbf{Z})[\mathcal{M}].$$

Let m_1 be the common $\mathbf{T}/\mathcal{M}$-dimension of the above two vector spaces. According to Remark 2.16, we have an exact sequence

$$0 \rightarrow \hat{X}_p(V_{B(qs)}(pr))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z} \rightarrow H^1(V_{B(qs)}(pr) \times \bar{\mathbf{F}}_p, \bar{\vartheta}_\ell) \xrightarrow{\varpi} H^1(V_{B(qs)}(r), \bar{\vartheta}_\ell)^2 \rightarrow 0.$$

Also we have

$$\begin{aligned} H^1(V_{B(qs)}(pr) \times \bar{\mathbf{F}}_p, \bar{\vartheta}_\ell)[\mathcal{M}] &\cong H^1(V_{B(qs)}(pr) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)^I[\mathcal{M}] \\ &\cong H^1(V_{B(qs)}(pr) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)[\mathcal{M}] \end{aligned}$$

since $\rho_{\mathcal{M}}$ is unramified at p. By Theorem 4.3,

$$\dim_{\mathbf{T}/\mathcal{M}} H^1(V_{B(qs)}(pr) \times \bar{\mathbf{Q}}_p, \bar{\vartheta}_\ell)[\mathcal{M}] = 4.$$

So if we set $m_2 = \dim_{\mathbf{T}/\mathcal{M}} H^1(V_{B(qs)}(r), \bar{\vartheta}_\ell)^2[\mathcal{M}]$, we get an inequality

$$m_1 + m_2 \geq 4.$$

We claim $m_1 \geq 2$. Assume the contrary. Then we have $m_2 > 2$. Similarly to (28), the two degeneracy maps

$$\delta_1, \delta_p : \ V_{B(qs)}(pr) \rightrightarrows V_{B(qs)}(r)$$

induce, by functoriality, two maps δ_1^* and δ_p^* between the cohomology groups of these curves (in the opposite direction) and give rise to an exact sequence

$$0 \to \ker\alpha \to H^1(V_{B(qs)}(r), \bar{\vartheta}_\ell)^2 \xrightarrow{\alpha} H^1(V_{B(qs)}(pr), \bar{\vartheta}_\ell)$$

where α sends (x, y) to $-\delta_1^*(x) + \delta_p^*(y)$. One finds $\ker\alpha[\mathcal{M}] = 0$. The calculation is verbatim to that in the proof of Proposition 4.2. Hence we find an isomorphism

$$H^1(V_{B(qs)}(r), \bar{\vartheta})_\ell)^2[\mathcal{M}] \cong H^1(V_{B(qs)}(pr), \bar{\vartheta}_\ell)[\mathcal{M}].$$

Further calculation shows that there is a subgroup $W \subseteq H^1(V_{B(qs)}(r), \bar{\vartheta}_\ell)$ such that

$$H^1(V_{B(qs)}(pr), \bar{\vartheta}_\ell)[\mathcal{M}] = \{(x, -x) | x \in W\}$$

if $\rho_{\mathcal{M}}(\mathrm{Frob}_p) = 1$ and

$$H^1(V_{B(qs)}(pr), \bar{\vartheta}_\ell)[\mathcal{M}] = \{(x, x) | x \in W\}$$

if $\rho_{\mathcal{M}}(\mathrm{Frob}_p) = -1$. Then one finds in either case

$$\begin{aligned} H^1(V_{B(qs)}(pr), \bar{\vartheta}_\ell)[\mathcal{M}] \quad & \subseteq \quad \ker\varpi \circ \alpha_{/\bar{\mathbf{F}}_p} \quad = \quad \ker\varpi \\ = \quad & \hat{X}_p(V_{B(qs)}(pr))_\ell \otimes \mathbf{Z}/\ell\mathbf{Z}. \end{aligned}$$

This shows $m_1 \geq 4$ which contradicts our assumption. So we have proved our claim that $m_1 \geq 2$. Now by Theorem 2.11, we have an exact sequence

$$0 \to \hat{X}_r(V_{B(pqrs)})_\ell \otimes \mathbf{Z}/\ell\mathbf{Z} \to H^1(V_{B(pqrs)} \times \bar{\mathbf{Q}}_r, \bar{\vartheta}_\ell) \to [X_r(V_{B(pqrs)})_\ell \otimes \mathbf{Z}/\ell\mathbf{Z}](1-k) \to 0.$$

Hence we have

$$m_1 \leq m = \dim_{\mathbf{T}/\mathcal{M}} H^1(V_{B(pqrs)} \times \bar{\mathbf{Q}}_r, \bar{\vartheta}_\ell)[\mathcal{M}].$$

Since $\rho_{\mathcal{M}}$ is ramified at r, we must have $m > m_1 \geq 2$. Using induction on the number of prime divisors of D, we then obtain the following

Proposition 4.9 *Let $\mathcal{M}$ be a new maximal ideal of the Hecke algebra $\mathbf{T}_{p_1p_2\cdots p_n\cdot q_1q_2\cdots q_n}$ such that the Galois representation $\rho_{\mathcal{M}}$ is unramified at the p_i's and ramified at the q_i's $(1 \leq i \leq n)$. Assume further that the Frobenius elements at the p_i's act in $\rho_{\mathcal{M}}$ either as $+1$ or -1. Then*

$$\dim_{\mathbf{T}/\mathcal{M}} H^1(V_{B(p_1p_2\cdots p_n\cdot q_1q_2\cdots q_n)}, \bar{\vartheta}_\ell)[\mathcal{M}] \geq 4.$$

It is worth noting that there are two interesting problems still remaining: (1) Find an effective upper bound for the multiplicity in the context of Theorem 4.5. The only occurrence of higher multiplicity we are able to find so far is the case in the above proposition. Even finding a kernel with multiplicity at least three seems to be quite difficult. (2) Describe the multiplicity when ρ_f is dominantly unramified at the discriminant. This is the question raised by Diamond we mentioned in the Introduction.

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