Fuzzy Field Theory as a Random Matrix Model

by

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Abstract

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This dissertation considers the theory of scalar fields on fuzzy spaces from the point of view of random matrices.

First we define random matrix ensembles, which are natural description of such theory. These ensembles are new and the novel feature is a presence of kinetic term in the probability measure, which couples the random matrix to a set of external matrices and thus breaks the original symmetry. Considering the case of a free field ensemble, which is generalization of a Gaussian matrix ensemble, we develop a technique to compute expectation values of the observables of the theory based on explicit Wick contractions and we write down recursion rules for these. We show that the eigenvalue distribution of the random matrix follows the Wigner semicircle distribution with a rescaled radius. We also compute distributions of the matrix Laplacian of the random matrix given by the new term and demonstrate that the eigenvalues of these two matrices are correlated. We demonstrate the robustness of the method by computing expectation values and distributions

for more complicated observables.

We then consider the ensemble corresponding to an interacting field theory, with a quartic interaction. We use the same method to compute the distribution of the eigenvalues and show that the presence of the kinetic terms rescales the distribution given by the original theory, which is a polynomially deformed Wigner semicircle. We compute the eigenvalue distribution of the matrix Laplacian and the joint distribution up to second order in the correlation and we show that the correlation between the two changes from the free field case.

Finally, as an application of these results, we compute the phase diagram of the fuzzy scalar field theory, we find multiscaling which stabilizes this diagram in the limit of large matrices and compare it with the results obtained numerically and by considering the kinetic part as a perturbation.

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Contents

Chapter 1

Introduction

1.1 Fuzzy field theory and random matrices

Both fuzzy spaces and random matrices have a firm place in modern theoretical physics. They arise as object of interest in many areas or are often used as a very useful computational tool. Since fields on fuzzy spaces have a finite number of modes, observables are naturally defined as matrices and there is a straightforward connection to the random matrix theory. Computing expectation values of observables in the fuzzy field theory and in the random matrix theory with certain probability measure is essentially the same calculation. This was first observed by Steinacker in [1, 2].

Indeed, the fuzzy fields very naturally define a wide class of new random matrix ensembles, as was pointed out in [3]. The new feature is the presence of a kinetic term in the measure. This term couples the random matrix to a set of fixed external matrices, which are related to the underlying fuzzy space of the corresponding field theory. The new term also introduces new observables of interest, given by various commutators of the external matrices and the random matrix, and their products.

The computational problem is clear. The new term in the measure does not reduce upon diagonalization to an expression involving only the eigenvalues of the random matrix and the integral over the angular degrees of freedom, which has to be done in order to compute the averages, is not trivial anymore. Therefore the standard approach of saddle point approximation which is used to obtain the results in the limit of large matrices is not usable. In this work, we will present a different line of attack, based on the explicit Wick contractions. And the results will turn out to be surprisingly simple. For a class of ensembles, we will prove that the kinetic term changes the two point function of the theory, but apart from this does not alter the eigenvalue distribution qualitatively. Then, we will give a general argument why this should be true for any result of the theory and also for any theory were the kinetic term is the only symmetry breaking part of the measure.

We consider two kinds of ensembles. A modified Gaussian ensemble, corresponding to a free field and an ensemble with a quartic interaction term, corresponding to a ϕ^4 scalar field theory. For both of these we compute also the correlation between the eigenvalues of the random matrix and its matrix Laplacian, which is given by the kinetic term. From an interacting field theory point of view [4], the whole procedure serves as a non-perturbative treatment of all the terms in the action, which has not been available yet.

As an application of the results in the fuzzy physics, we investigate the phase diagram of the field theory. There has been an extensive numerical and perturbative work on this subject, mostly for the theory on the fuzzy sphere. The phase diagram of the ϕ^4 scalar theory has been obtained numerically [5, 6] and has been partially explained by perturbative treatment of the kinetic term [8], however no non-perturbative treatment has been developed yet. The main motivation behid the study of this problem has been the interplay of commutative and non-commutative theories. Since the planar limit of the theory on the sphere includes a lot of peculiarities of the fuzzy physics, the thorough understanding of the properties of the phase diagram could help us to understand the concept of fuzzy physics better.

In the rest of this introductory section we give a short overview of application of the concepts of fuzzy spaces and random matrix theories in physics. In Chapter 2 we describe the fuzzy spaces and the field theories on them in more detail, concentrating on the aspects that are going to be important for this work. Similarly, in Chapter 3 we give an overview of techniques and results of matrix models. Chapter 4 presents the computation of main results. Here, we compute eigenvalue distribution for the random matrix in the ensemble defined by the free scalar field theory and an interacting scalar field theory on fuzzy sphere and more general fuzzy spaces. We also compute the distribution for the matrix Laplacian and the correlation between the eigenvalues of the two. Then in Chapter 5, we give an overview of the phase structure of the fuzzy field theories and apply our results to the description of their phase diagram. In the concluding Chapter 6, we present several possible paths of further research, that can based on the presented result.

1.2 Non-commutative spaces in physics

Idea of non-commutative geometry arises in the correspondence of commutative C^* algebras and differentiable manifolds. Every manifold comes with a naturally defined algebra of functions and every C^* algebra is an algebra of functions on some manifold. One then defines non-commutative spaces as spaces that correspond to a non-commutative algebra [9, 10]. One introduces a spectral triple of an C^* algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ on which this algebra can be realizes as bounded operators and a special operator $\mathcal{D}$, which will characterize the geometry. Using these three ingredients, it is possible to define differential calculus on a manifold.

More specifically, in a finite dimensional case one arrives at the notion of fuzzy space. Here, the Hilbert space $\mathcal{H}$ is finite dimensional and algebra $\mathcal{A}$ can be realized as an algebra of $N \times N$ matrices. This can be though of as a generalization of a well known notion from quantum mechanics. With large number of quanta the quantum theory is well approximated by the classical theory, with the functions on the classical phase space representing the linear Hermitian operators. The fuzzy space defined by $(\mathcal{A}, \mathcal{H}, \Delta_N)$, with Δ_N the matrix Laplacian, are finite state-approximation to the classical phase space manifold. The finite dimensionality of $\mathcal{H}$ amounts for compactness of the classical version of the manifold.

The original motivation for considering non-commutative manifolds in physics dates back to the early days of quantum field theories. It was suggested by Heisenberg and later formalized by Snyder [11] that the divergences which plague the qft's can be regularized by the space-time non-commutativity. Opposing to other methods, non-commutative space-time keeps its Lorentz invariance. However, since renormalization proved to be effective in providing accurate numerical results, this idea was abandoned.

In the spirit of the original motivation for non-commutative spaces, there has been work by Connes and others to formulate the standard model of particle physics solely in the terms of the spectral triple and the framework of the non-commutative geometry [12, 13, 16]. There have also been attempts to formulate the theory of gravity purely in the terms of the spectral triple [14, 15].

In the description of the Quantum Hall Effect, i.e. dynamics of charged particles in the presence of magnetic field. Originally, the problem was considered in the plane, but fuzzy spaces allow for a natural generalization to higher dimensional curved spaces. When the field is strong enough, particles are confined to the lowest Landau level and the projections of the Hermitian operators on this one level no longer commute, with the magnitude of the magnetic field being related to the non-commutativity parameter [17]. The lowest Landau level states form the Hilbert space $\mathcal{H}$ and considering particle dynamics on a general manifold reproduces the fuzzy version, with the corresponding Hilbert space. We can explore the geometry of the fuzzy space considering the behavior of the electron liquid and even study the case of non-Abelian background field [18, 19, 20].

Something similar happens when considering D-branes in string theory and the effective dynamics of opened strings is described by the non-commutative gauge theory and the D-brane worldvolume becomes a non-commutative space, see [21, 22] for a review. Finally, fuzzy spaces arise in the M-theory as brane solutions [23, 24], see [25] for a review of M-theory and [26] for a review of fuzzy spaces as brane solutions.

1.3 Random matrices in physics and matrix models

Random matrices, as the name suggests, are matrices with random entries given by some overall probability measure, usually with some symmetry. Computing the averages in this theory then amounts to computing matrix integrals with this probability weight.

For a very thorough overview of properties and applications of random matrices, both in physics and beyond, see [27].

Historically, the random matrices arose in the work of Wigner [28, 29], in an attempt to describe the energy levels of heavy nuclei. These are too dense to be described individually and Wigner suggested an statistical approach, where he showed that the level distribution is in a good agreement with the the eigenvalue distribution of a random matrix. This success lies in the universality of the statistical properties of the random matrices. See [30] for overview.

On a very different front, t'Hooft suggested an $1/N_c$ expansion of QCD [31]. The gluons of the theory are $N_c \times N_c$ matrices, here N_c is the number of colors, and since the fluctuations of these need to be integrated out, this is a theory of random matrices. t'Hooft showed, that the $N_c \to \infty$ limit simplifies the theory greatly, since only planar diagrams survive, and corrections of higher orders in $1/N_c$ correspond to topologically more complicated contributions.

This is related to the possibility of using the random matrix theory to discretize two dimensional surfaces. Diagrams are generated by the matrix integrals and when computing large matrix limit of these integrals, we can count the possible discretizations, incorporate more complicated structures or obtain the continuum limit by a suitable rescaling of the parameters of the theory. This is then important in the considerations of conformal field theories and extraction of critical exponents. Similarly in string theory, the world sheet of the string is two dimensional and considering a rescaling of parameters of the theory, one can generate two dimensional surfaces of all genera, i.e. string interactions. See [32, 33, 34, 35].

Random matrix theory has a wide application in the condensed matter physics, again mostly due to the universality properties. The large number of electrons or other constituents that come into play corresponds to the limit of a large dimension of the matrix. One can either study thermodynamical properties of some closed system, then the random matrix is the Hamiltonian H. Or one can study transport properties, where the matrix of interest is the scattering matrix S. See [36, 37] for review. Since level repulsion is a characteristic for both random matrices and chaotic systems, it is believed that random matrix theory will lead to the theory of quantum chaos [37, 38, 39].

From a pure mathematical point of view random matrices and their statistical properties have applications in fields as combinatorics, graph theory, theory of knots and many more. Let us mention one quite remarkable and unexpected appearance. Random matrices have a clear connection to Riemann conjecture. Namely, the two-point correlation function of the zeros of the zeta function $\zeta(z)$ on the critical line seems to be the same as the two-point correlation function of the eigenvalues of a random Hermitian matrix from a simple Gaussian ensemble [40] and also share other statistical properties with random matrix ensembles [41]. However, there is no true understanding of the reason behind these observations.

Chapter 2

Fuzzy spaces and fuzzy field theory

In this section, we introduce the concept of the fuzzy space and the field theory on such spaces. The first section introduces the basic ideas and illustrates them on the simplest case, the fuzzy sphere. Then, a more detailed discussion of the construction of fuzzy $\mathbb{C}P^n$ follows. Finally, we describe the field theory on these spaces and mention some basic features of these theories.

Excellent reviews on the fuzzy spaces and fuzzy field theories are [42, 43, 44, 45] and on more general non-commutative spaces [21, 22]. More mathematical details can be found in reviews [9, 46].

2.1 An appetizer - The Fuzzy Sphere

First, let us give a flavor of what the idea behind the non-commutative spaces is and what the main features are, with emphasis on the non-commutative version of the two sphere S_F^2. More details will follow in later sections.

Every manifold comes with a naturally defined associative algebra of functions with point-wise multiplication. This algebra is generated by the coordinate functions of the manifold and is from the definition commutative. As it turns out, this algebra contains all the information about the original manifold and we can describe geometry of the manifold purely in terms of the algebra. Also, every commutative algebra is an algebra of functions on some manifold. Therefore, what we get is

$$\text{commutative algebras} \longleftrightarrow \text{differentiable manifolds.}$$

See [46] for details. A natural question to ask is whether there is a similar expression for non-commutative algebras, or

$$\text{non-commutative algebras} \longleftrightarrow \text{???}$$

Quite obvious answer is no, there is no space to put on the other side of the expression. Coordinates on all the manifolds commute and that is the end of the story. So, as is often the case, we define new objects, called non-commutative manifolds, that are going to fit on the right hand side. Namely we look how aspects of the regular commutative manifolds are encoded into their corresponding algebras and we call the non-commutative manifold object, that would be encoded in the same way in a non-commutative algebra.

This is going to introduce non-commutativity among the coordinates. This notion is not completely new, as one recalls the commutation relations of the quantum mechanics $[x^i, p_j] = i\hbar\delta^i_j$. In classical physics, the phase space of the theory was a regular manifold. However in quantum theory we introduce non-commutativity between (some) of the coordinates and therefore the phase space of the theory becomes non-commutative. One of the most fundamental consequences of the commutation relations is the uncertainty principle. The exact position and momentum of the particle can not be measured and therefore we can not specify a single particular point of the phase space. Similarly, if there is non-commutativity between the coordinates, there is a corresponding uncertainty principle in measurement of coordinates. The notion of a space-time point stops to make sense, since we can not exactly say, where we are. This is the motivation behind the name of the fuzzy space.

In practice, we often 'deform' a commutative space into its non-commutative analogue. In this way we get non-commutative spaces that give a desired commutative limit and also writing such a space from scratch is very difficult. An example of such deformation is already mentioned phase space of quantum mechanics or the more general case of non-commutative flat space $\mathbb{R}^2_\theta$, given by

$$[x_i, x_j] \equiv x_i x_j - x_j x_i = i\theta_{ij}, \tag{2.1}$$

for some constant, anti-symmetric tensor θ^{ij}. To illustrate the procedure of deformation better let us consider a different example and show how this works for a two-sphere [47, 48, 49].

The regular two sphere is defined as the set of points with a given distance from the origin, i.e. $\sum_{i=1}^3 x_i^2 = R^2$. This comes with an understood condition on commutativity of the coordinates $x_i x_j - x_j x_i = 0$. Coordinate functions constrained in this way generate the algebra of all the functions on the sphere. [1]

Now we define the fuzzy two sphere by the coordinates $\hat{x}_i$, which obey the following conditions

$$\sum_{i=1}^{3} \hat{x}_i^2 = \rho^2 \quad , \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta\varepsilon_{ijk}\hat{x}_k, \tag{2.2}$$

where ρ, θ are parameters describing the fuzzy sphere, in a similar way as R did describe the regular sphere. The radius of the original sphere was encoded in the sum of the squares of the coordinates, so we will call ρ the 'radius' of the non-commutative sphere. We see, that such $\hat{x}$'s are achieved

[1] Note that this is technically not the easiest way to do so. It is easier to introduce only two coordinates θ, φ on the sphere and define the algebra of functions not by the generators, but by the basis, e.g. the spherical harmonics. However the two sphere defined in our way is easier deformed into the non-commutative analogue.

by a spin-j representation of the $SU(2)$. If we chose

$$\hat{x}_i = \frac{2r}{\sqrt{(2j+1)^2-1}} L_i \quad , \quad [L_i, L_j] = i\varepsilon_{ijk} L_k \quad , \quad \sum_{i=1}^{3} L_i^2 = j(j+1), \tag{2.3}$$

where L_i's are the generators of $SU(2)$, we get

$$\sum_{i=1}^{3} \hat{x}_i^2 = \frac{4\rho^2}{N^2-1}\left(\frac{N-1}{2}\right)\left(\frac{N-1}{2}+1\right) = \rho^2 \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}}, \tag{2.4}$$

with $N = 2j+1$ the dimension of the representation. Matrices $\hat{x}_i$ become coordinates on the non-commutative sphere. Note, that the limit $N \to \infty$ removes the non-commutativity, since $\theta \to 0$, and we recover a regular sphere with radius r. This explains a rather strange choice of parametrization in (2.3). Also note, that this way we got a series of spaces, one for each j (or N). The important fact is that the coordinates still do have the $SU(2)$ symmetry and therefore it makes sense to talk about this object as spherically symmetric. This explains the particular choice of deformation in (2.2).

The non-commutative analogue of the derivative is the L-commutator, since it captures the change under a small translation, which is rotation in the case of the sphere. The integral of a function becomes a trace, since it is a scalar product on the space of matrices. As we sill see in the next section, both of these have the correct commutative limits.

Spherical harmonic functions Y_l^m form a basis of the algebra of functions on the regular sphere. These are labeled by $l = 0, 1, 2, \ldots$ and by $m = -l, -l+1, \ldots, l-1, l$ and this basis is infinite. If we truncate this algebra, namely take the following set of functions

$$Y_l^m \quad , \quad m = -l, -l+1, \ldots, l-1, l \quad \& \quad l = 0, 1, \ldots, N-1 \ , \tag{2.5}$$

we recover a different algebra. This is obviously not the algebra of functions on the regular sphere and also to make this algebra closed, it can be seen that we need to introduce some non-trivial commutation rules [48]. And one can check, that the N^2 independent matrices generated by (2.3) are in one-to-one correnspondence with this truncated set of spherical harmonics. This means, that in the limit of large N the algebra we recover is truly the algebra of functions on regular commutative sphere S^2.

Here we can see in a different way why the fuzzy-ness introduces short distance structure. l measures the momentum of the mode and cutting off the modes we have introduced the highest possible momentum. This in turn introduces the shortest possible distance to measure.

2.2 Construction of fuzzy spaces

Manifolds with a symplectic structure admit construction of a fuzzy analog. In general, co-adjoint orbit of a compact semisimple Lie group enjoys being a symplectic manifold and such spaces can be quantized in a very natural way, described towards at the end of this section. The Hilbert space which arises in this process is a unitary irreducible representation of this group and this is in a very intimate relationship to quantization of the phase space of classical mechanics to a quantum Hilbert space mentioned in the previous section. We will now describe this procedure in some detail for $\mathbb{C}P^n$.

Construction of fuzzy spaces which are not derived from a co-adjoint orbit of a semisimple Lie group, for example higher dimensional spheres, is more involved. One starts from a larger space which is a co-adjoint orbit and the irrelevant factors are projected out [50, 51]

2.2.1 Construction of fuzzy $\mathbb{C}P^n$

In this section, we will present the construction of the fuzzy version of the complex projective spaces, following [52, 53], some of the original references being [54, 55]. To do this, we will consider $\mathbb{C}P^n$ as $SU(n+1)/U(n)$. This follows from the standard definition of $\mathbb{C}P^n$ as lines in $\mathbb{C}^{n+1}$ which go through the origin, i.e. the identification $z \sim \lambda z, z \in \mathbb{C}^{n+1}$ for some $\lambda \in \mathbb{C}$. Since there is a natural action of $g \in SU(n+1)$ on z by $z' = gz$, we get the advertised relation for $SU(n+1)/U(n)$. This means, that the functions on $\mathbb{C}P^n$ are those functions on $SU(n+1)$, which are $U(n)$ invariant. This allows for construction of the basis of functions on $\mathbb{C}P^n$ in terms of the Wigner $\mathcal{D}$-functions of $SU(n+1)$. We will do this and then show how these are in one-to-one correspondence with $N \times N$ matrices, which are functions on the fuzzy $\mathbb{C}P^n$.

Functions on $\mathbb{C}P^n$

Consider the totally symmetric representation of $SU(n+1)$ of rank k. The dimension of this representation is

$$N_k = \frac{(n+k)!}{n!k!}. \tag{2.6}$$

Now consider the following functions on $\mathbb{C}P^n$

$$\Psi^k_m(g) = \sqrt{N_k}\mathcal{D}^n_{m,-k}(g), \tag{2.7}$$
$$\mathcal{D}^k_{m,-k}(g) = \langle k, m| \, \hat{g} \, |k, -k\rangle \,, \tag{2.8}$$

where g is an element of $SU(n+1)$, $\hat{g}$ is its corresponding operator in this representation, $|k, -k\rangle$ denotes the lowest weight state and $|k, m\rangle \,, m = 1, 2, \dots, N_k$ are the states in the representation. The fact that we consider only the lowest weight state $|k, -k\rangle$ ensures, that these functions on $SU(n+1)$ are correctly invariant under $U(n)$. The $\mathcal{D}$-functions can be constructed explicitly using the coherent state representation and the local complex coordinates for $\mathbb{C}P^n$ [56]. They are completely symmetric holomorphic functions of order k, but this will not be needed for what follows.

The basis for functions on regular $\mathbb{C}P^n$ is then given by the union of all the functions (2.8) for $k = 0, 1, 2, \dots$.

Now consider space of $N_L \times N_L$ matrices, where N_L is given by (2.6) for

$k = L$, where L is the cut-off on the number of the modes. These are going to form a basis of functions on the fuzzy $\mathbb{C}P^n_F$. We need to show, that as $L \to \infty$ these are in one to one correspondence with the functions on regular $\mathbb{C}P^n$, that the matrix product becomes the usual commutative product in this limit and we need to define object like derivatives and integrals.

Symbols and large N-limit of matrices

We associate an $N_L \times N_L$ matrix A with a function $A(g)$ on the classical $\mathbb{C}P^n$ by

$$A(g) = \sum_{mm'} \mathcal{D}^n_{m,-n}(g) A_{mm'} \mathcal{D}^{*n}_{m',-n}(g). \tag{2.9}$$

We call this function a symbol corresponding to A and this object is going to be essential and in the large N_L, or large L, limit, the matrix A will tend to it's symbol $A(g)$. We can also define a product between the functions on $\mathbb{C}P^n$ (symbols), which we will call the star product and denote it $\star$, by

$$A(g) \star B(g) \equiv (AB)(g). \tag{2.10}$$

This means that the star product of the two symbols is given by a symbol of the product of the two matrices. This product is not commutative and deforms the original product on $\mathbb{C}P^n$ by contributions that are suppressed by powers of $1/N_L$, see [57, 58]. The symbol corresponding to the commutator of two matrices becomes the Poisson bracket on $\mathbb{C}P^n$ [59]

$$([A,B])(g) = \frac{i}{L}\{A,B\} + \mathcal{O}\left(1/L^2\right). \tag{2.11}$$

This reflects the standard procedure of quantum mechanics, where the Poisson brackets of the observables get replaced by commutators of the corresponding operators. The trace of the matrix becomes an integral

$$\mathrm{Tr}\,(A) = \sum_i A_{ii} = N \int dg\, D^k_{m,-n} A_{mm'} \mathcal{D}^{*n}_{m',-n}(g) = N \int dg\, A(g), \tag{2.12}$$

where dg is the Haar measure on $SU(n+1)$ and we have used the orthogonality property of the Wigner $\mathcal{D}$ functions.

Matrix-function correspondence

The last step is to show how the matrices correspond to the functions on commutative $\mathbb{C}P^n$. Functions on fuzzy $\mathbb{C}P^n_F$ are $N_L \times N_L$ matrices and we can identify the coordinate matrices. These we define as

$$X_A = -\frac{R}{\sqrt{C_2(L)}} T_A, \tag{2.13}$$

where T_A are the generators of $SU(n+1)$ in the symmetric rank k representation and $C_2(k)$ is the value of quadratic Casimir in this representation and R is the radius of the $\mathbb{C}P^n$. In the large L limit, the symbol of X_A is $S_{A\,n^2+2n}(g) \equiv 2\mathrm{Tr}\left(g^T t_A g^* t_{n^2+2n}\right)$. Here, t_A's are the generators of the $SU(n+1)$ that was used to define the $\mathbb{C}P^n$ and t_{n^2+2n} is the generator of the $U(1)$ direction in the $U(n)$ subgroup of $SU(n+1)$. Moreover, for any matrix function of the coordinate matrices $F(X_A)$, the corresponding symbol is given given as a function of the coordinate functions $F(S_{A\,n^2+2n}(g))$. This shows, that in the large L limit, the matrices and the functions coincide.

Finally, considering a general matrix M generated by the coordinate matrices (2.13), defining the derivative operator as

$$-iD_A M \equiv [T_A, M] \approx -\frac{i}{L}\frac{nL}{\sqrt{2n(n+1)}}\{S_{A\,n^2+2n}, M\}, \tag{2.14}$$

one can check that D_A's follow the appropriate $SU(n+1)$ algebra conditions. Also, in explicit realization in terms of complex coordinates on $\mathbb{C}P^n$ these reduce to known expressions for derivatives.

The fuzzy sphere as a special case $\mathbb{C}P^1_F$

The sphere is $\mathbb{C}P^1$ and the explicit formulae should also illustrate rather abstract treatment of the previous section.

We will deal with representations of $SU(2)$, which are given by the standard angular momentum theory. The spin-j representation is given by the maximal angular momentum $L = j/2$ and $N = 2j + 1 = L + 1$. Generators of the group are angular momentum matrices L_i and relation (2.13) becomes (2.3). The number of spherical harmonics, which form a basis on regular sphere, up to a certain angular momentum is $\sum_{j=0}^{L}(2j+1) = (L+1)^2 = N^2$, which is the number of independent $N \times N$ matrices.

The basis of matrices is given by the symmetrized products of the coordinate matrices X_i, i.e. $id, X_i, X_{(i}X_{j)}, \ldots$. The same is true for the spherical harmonics, which are formed by the products of S_{i3} with contractions removed, functions with up to k factors of S_{i3} correspond to spherical harmonics of angular momentum up to k. There is thus one-to-one correspondence between the matrices and spherical harmonics and as $L \to \infty$, the matrix algebra corresponding to fuzzy sphere becomes the algebra of functions on regular sphere.

2.2.2 Limits of $\mathbb{C}P^n_F \to \mathbb{R}^{2n}$

After the construction of the fuzzy $\mathbb{C}P^n_F$, one is left with several possibilities of taking the large N limit. Let us discuss those in some more detail.

Commutative $\mathbb{C}P^n$

Most of the previous section was devoted to proving that when one takes the $N_L \to \infty$ limit with a fixed radius R, we recover the commutative version of $\mathbb{C}P^n$. Matrices become functions, etc. However, let us stress here that this correspondence is geometrical. If we define some structure on top of the geometry, like field theory for example, we are not guaranteed to obtain the commutative counterpart. And as we will see, this is, as a rule, not the case.

Non-commutative $\mathbb{R}^{2n}_\theta$

Here, it is useful to consider the explicit realization of the fuzzy $\mathbb{C}P^n_F$ coordinates as

$$x_i = \frac{R}{\sqrt{\frac{n}{2(n+1)}L^2 + \frac{n}{2}L}} L_i \quad , \quad [x_i, x_j] = i\frac{R}{\sqrt{\frac{n}{2(n+1)}L^2 + \frac{n}{2}L}} f_{ijk}x_k \quad , \tag{2.15}$$

where $R = \sqrt{g^{ij}x_i x_j}$ is the radius of the $\mathbb{C}P^n$ and L_i are the generators of $SU(n+1)$ in the corresponding representation. If we now scale the radius as $R^2 = L\theta\frac{n}{n+1}$, we blow up the $\mathbb{C}P^n$ around one point. What is left is a non-commutative $\mathbb{R}^{2n}_\theta$. We will show this for the case of fuzzy sphere, more general details can be found in [54]. For $n = 1$ we find

$$[x_1, x_2] = i\frac{2\sqrt{\frac{N\theta}{2}}}{\sqrt{N^2-1}}\sqrt{\frac{N\theta}{2} - x_1 - x_2} \to i\theta, \tag{2.16}$$

which means that x_1, x_2 are coordinates on the non-commutative $\mathbb{R}^2_\theta$.

Commutative $\mathbb{R}^{2n}$

If we scale radius with a power of N smaller than $\frac{1}{2}$, i.e. $R^2 = N^{1-\varepsilon}, 0 > \varepsilon > 1$, the commutators of the left-over coordinates vanish and we obtain the commutative $\mathbb{R}^{2n}$.

2.2.3 Quantization of Poisson manifolds

To conclude, let us note that the presented approach was an explicit realization of a more general concept of quantizing a Poisson manifold [60, 61, 62]. We will not worry too much about the proper definitions and existence of what we are about to work with and will simply assume that the objects can be defined and behave nicely.

We start with a manifold equipped with an anti-symmetric bracket $\{.,.\}$ satisfying the Jacobi identity. Quantization map is then defined as a map between the algebra of functions on the manifold and a matrix algebra $\mathcal{A}(\mathbb{C})$,

$$\mathcal{I} \; : \; f(x) \to F \in \mathcal{A}. \tag{2.17}$$

Algebra $\mathcal{A}$ is generated by the images of the coordinate functions $X^\mu = \mathcal{I}(x^\mu)$. To be a good quantization, $\mathcal{I}$ has to satisfy

$$\begin{aligned} \mathcal{I}(fg) - \mathcal{I}(f)\mathcal{I}(g) &\to 0, \\ \frac{1}{\theta}\big[\mathcal{I}(i\{f,g\}) - [\mathcal{I}(f),\mathcal{I}(g)]\big] &\to 0, \\ \text{as } \theta &\to 0, \end{aligned} \tag{2.18}$$

where we have defined a parameter θ by $\{x_\mu, x_\nu\} = \theta\,\theta_0^{\mu\nu}(x)$. The star product is then defined as a pullback of the matrix product using $\mathcal{I}$,

$$f \star g \equiv \mathcal{I}^{-1}\big(\mathcal{I}(f)\mathcal{I}(g)\big). \tag{2.19}$$

The integral of a function over the manifold then tends to the trace of the corresponding matrix

$$\int d\Omega\, f \to \mathrm{Tr}\left(\mathcal{I}(f)\right), \tag{2.20}$$

where $d\Omega$ is the volume defined by the Poisson structure $\theta\,\theta_0^{\mu\nu}$. Finally, when we consider the manifold to be embedded in some $\mathbb{R}^d$ by coordinate functions x^a, images of these define the coordinate matrices $X^a \equiv \mathcal{I}(x^a)$. These matrices then follow the same embedding conditions as the original coordinate functions did.

When the algebra $\mathcal{A}$ is finite dimensional, we refer to the resulting non-commutative space as a fuzzy space.

Finally, note, that (2.9) is a particular choice of the quantization map $\mathcal{I}$.

2.3 Scalar field theories on fuzzy spaces

As in the case of the regular space, the scalar field on the fuzzy space is a power series in the coordinate functions and thus an element of the algebra discussed in the previous section. It is itself an $N \times N$ matrix, which can be expressed as

$$M = \sum_{l,A} c_A^l T_A^l \quad , \quad l = 0, 1, \ldots, L \quad , \quad A = 1, 2, \ldots, \dim(n, l) \tag{2.21}$$

where T_m^l are polarization tensors, analogue of the spherical harmonic functions

$$\begin{aligned} \sum_{i=1}^{(n+1)^2-1} [L_i, [L_i, T_A^l]] &= l(l+n) T_A^l, \\ \mathrm{Tr}\left(T_A^l T_B^{l'}\right) &= \delta^{ll'} \delta_{AB}, \\ \sum_A \left(T_A^l T_A^{l'}\right)_{ij} &= \frac{\dim(n,l)}{N} \delta^{ll'} \delta_{ij}, \end{aligned} \tag{2.22}$$

where L_α the $SU(n+1)$ generators in the N dimensional representation. The field theory is defined by the action. In what follows we work with the Euclidean signature.

The field theory on the fuzzy space is then defined by "starring" all the products in the commutative action. The free field action is then

$$S_0[\phi] = \int dx \left(\frac{1}{2} (\partial_i \phi) \star (\partial_i \phi) + \frac{1}{2} \mu^2 \phi \star \phi \right). \tag{2.23}$$

By construction, this action has the desired commutative limit. As we have seen in the previous section, when we represent the field ϕ with a $N \times N$ matrix M, the integral becomes a trace and ∂_i becomes a commutator with L_i and the star product is just the regular matrix product. Therefore, this action becomes

$$\begin{aligned} S_0 &= \frac{V_n}{N}\mathrm{Tr}\left[-\frac{1}{2}\frac{1}{R^2}[L_\alpha, M][L_\alpha, M] + \frac{1}{2}\mu^2 M^2\right] =, \\ &= \frac{V_n}{N}\left[\frac{1}{2}\frac{1}{R^2}\mathrm{Tr}\left(M[L_\alpha,[L_\alpha,M]]\right) + \frac{1}{2}\mu^2\mathrm{Tr}\left(M^2\right)\right], \end{aligned} \tag{2.24}$$

with V_n the volume of $\mathbb{C}P^n$.

We will rescale the fields to absorb the $1/N$ and the volume factors. Using the expansion (2.21) the free field action becomes

$$S_0 = \sum_{A,l}\frac{1}{2}\left(l(l+n)+\mu^2\right)(c_A^l)^2 \tag{2.25}$$

The correlator of the two components of the field is

$$\langle c_A^l c_B^{l'} \rangle = \frac{1}{l(l+n)+\mu^2}\delta^{ll'}\delta_{AB} \tag{2.26}$$

which is expression analogous to the usual propagator $(p^2 - m^2)^{-1}$. Using this, one can compute the free field correlation functions. The full interacting field action is then given as $S = S_0 + S_I$, with

$$S_I[M] = \sum g_k\left(\frac{N}{V_n}\right)^{1-\frac{k}{2}}\mathrm{Tr}\left(M^k\right) = \sum \tilde{g}_k\mathrm{Tr}\left(M^k\right). \tag{2.27}$$

The field theory is defined by the functional correlations

$$\langle F[M]\rangle = \frac{\int dM\ e^{-S}F[M]}{\int dM\ e^{-S}}, \tag{2.28}$$

or by the generating function for the correlators

$$Z(J) = \frac{1}{\int dM\ e^{-S}}\int dM\ e^{-S+Tr(JM)}, \tag{2.29}$$

or in any other usual way. One then derives the Feynman rules and Feynman diagrams [43].

We will mention one of the features of the fuzzy field theories which has no commutative analogue and which will be important for what we need in the later parts of this dissertation, when we consider the phase structure of the fuzzy field theory in the section 5. This is the UV/IR mixing and is the result of the non-locality of the theory, which the non-commutativity necessarily introduces [63]. At the level of diagrams, this yields a difference between the planar and non-planar diagrams of the theory. This is because the vertex is no longer invariant under any permutation of the legs. Factors

$$\mathrm{Tr}\left(T^{l_1}_{m_1}T^{l_2}_{m_2}T^{l_3}_{m_3}T^{l_4}_{m_4}\right)c^{l_1}_{m_1}c^{l_2}_{m_2}c^{l_3}_{m_3}c^{l_4}_{m_4} \tag{2.30}$$

posses only a cyclic symmetry and diagrams with a loop between matrices $1-2$ and matrices $1-3$ will give a different contribution.

In the case of the fuzzy sphere, this translates into a the finite, non-vanishing difference between the planar and the non-planar contributions to the two point functions at one loop, which is equal to

$$\frac{2}{N^2-1}\sum_{l=0}^{N-1}\frac{l(l+1)(2l+1)}{l(l+1)+\mu^2}. \tag{2.31}$$

At the level of large N effective action, this introduces a non-local, momentum dependent contribution, which can not be canceled by a counter term and thus is referred to as a non-commutative anomaly [64]. When treating the sphere as an approximation of the flat space and taking the large R limit as in the section 2.2.2, this results into an infrared divergence in the non-planar contribution to the one loop two point function, which is the UV/IR mixing [65].

The high energy and low energy regimes of the theory are not independent. When one computes the one loop propagator of a non-commutative field theory on $\mathbb{R}^d$, the non-planar diagrams become regulated by the non-commutativity due to some oscillating factors [22], which act as an effective regulator proportional to $1/\sqrt{\theta}$. However in the limit of very small momentum of the external particle, this regulator vanishes and low momentum process gets divergent contributions from high energy virtual particles. The short distance structure of the theory modifies long distance physics.

In order to remove this mixing and reproduce the expected commutative limit, we need to redefine the naive action (2.24). One possible procedure is to define a normal ordering of the interaction term, which results into the cancellation of the extra non-commutative contribution [66]. This leads into the following modification of the action for M^4 theory

$$S[M] = \frac{V_n}{N}\mathrm{Tr}\left[\frac{1}{2}\frac{1}{R^2}MC_2\mathcal{Z}(C_2)M + \frac{1}{2}M\left(t - \frac{g}{2}\mathcal{R}(t)\right)M + gM^4\right], \quad (2.32)$$

where $C_2M = [L_\alpha, [L_\alpha, M]]$ is the quadratic Casimir of $SU(2)$. Function $\mathcal{Z}(C_2)$ is a power series in C_2 starting with $1 + \kappa Q(C_2)$ and κ is related to μ^2 and g. Both $\mathcal{Z}$ and $\mathcal{R}$ are related to the renormalization of the wave function and the mass of the theory.

A different approach was suggested in [67]. Here, the UV/IR mixing is treated as an effect of the asymmetry between the high energy and the low energy regimes of the non-commutative theory and the models is modified to restore this symmetry. This is achieved by introduction of a harmonic oscillator-like term into the action, which modifies the dispersion relation.

Chapter 3

Random matrix models

In this chapter, we will describe some standard results of the theory of random matrices. We will introduce basic concepts and then concentrate mostly on the computation of the eigenvalue distributions for the random matrix, as this is of our primary interest.

In this section, we follow mostly the discussion of [33, 34, 68]. A more thorough treatment can be found in [30], or a more mathematically minded in [69].

We should also mention that there is a very powerful method to analyze the random matrix models that is not going to be mentioned here, since we will not use it later. This method uses certain set of orthogonal polynomials, which are orthogonal with respect to the probability measure. All the quantities of interest can then be expressed in terms of quantitis relate to these polynomials and also a systematic expansion in the size of the matrix can be performed. We refer to the references above for details about this method and also for different methods, which are used to analyze random matrices but will not be discussed here.

3.1 The basics and the planar limit

3.1.1 The ensembles

As mentioned in the introduction, random matrix theory is given by the ensemble of matrices M with the integration measure dM and the probability measure $\exp(-S(M))$. The expected value of a function $f(M)$ of the random matrix is then computed as

$$\langle f \rangle = \frac{1}{Z} \int dM e^{-S(M)} f(M). \tag{3.1}$$

Here, the normalization $1/Z$ is such that $\langle 1 \rangle = 1$.

The ensemble usually has some symmetry and it is assumed that the integration measure dM, as well as the weight $S(M)$ and any reasonable function $f(M)$ are symmetric also.

The choice of the matrix ensemble is then dictated by the physical or mathematical setup. Most usually, matrix M is hermitian, real symmetric or quaternionic self-dual, with $SU(N)$, $SO(N)$ and $Sp(2N)$, or a sub-group, being the symmetry group. Matrix M can be also directly an unitary, orthogonal or symplectic matrix. In some mathematical applications, such as statistics or number theory, more complicated matrices, which need not to be square, arise, see e.g. [70].

The most general choice of the probability measure is then dictated by the choice of the ensemble and the symmetry group. To be more precise, let M be a square $N \times N$ matrix with symmetry group G, with the action $M \to gMg^{-1}$ with $g \in G$. We then write the measure as $e^{-N^2 S(M)}$, the factor of N^2 makes it explicit that the measure is of the same order as dM and thus contributes. The action $S(M)$ is then finite in the limit of large N. Then, the most general invariant measure is given by

$$S(M) = \frac{1}{N} \sum_{n=0}^{N} g_n \mathrm{Tr}\left(M^n\right), \tag{3.2}$$

where the factor of $1/N$ ensures proper large N limit, as the sum of N eigenvalues is of the order N. Any other invariant function can be re-expressed in terms of the first N traces. The constant and the linear terms are usually not considered, since they can be absorbed into redefinition of M. Also, the M^2 term is usually considered separately and we write

$$S(M) = \frac{1}{2}\mu^2 \frac{1}{N}\mathrm{Tr}\left(M^2\right) + \sum_{n=3}^{N} g_n \frac{1}{N}\mathrm{Tr}\left(M^n\right). \tag{3.3}$$

This is because of the fact that the $n \geq 3$ introduce "interactions" into the theory, which means that this renders also some of the higher order correlation functions nontrivial. Theory with only $\mathrm{Tr}\left(M^2\right)$ term can be solved exactly and the extra terms can be considered as a perturbation.

We will treat the measure dM explicitly only for the case of $N \times N$ Hermitian matrices. In general case, upon the diagonalization of the matrix, the measure will become

$$dM = J(\lambda) d\Lambda dU, \tag{3.4}$$

where $d\Lambda$ is the measure on the space of eigenvalues λ of M, dU is the measure of the symmetry group G and $J(\lambda)$ is Jacobian corresponding to the change of variables from M_{ij} to λ and U. In the case of Hermitian matrices, $G = SU(N)$ and the integration measure is given by

$$dM = \prod_{i=1}^{n} M_{ii} \prod_{i<j} d\mathrm{Re}M_{ij}\, d\mathrm{Im}M_{ij}. \tag{3.5}$$

We can diagonalize the matrix

$$M = U^{\dagger} \Lambda U, \tag{3.6}$$

where $\Lambda = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_N\right)$ is the diagonal matrix of the eigenvalues of M.

The integration measure then becomes

$$dM = \left(\prod_{i<j}(\lambda_i - \lambda_j)^2\right)\left(\prod_{i=1}^{N} d\lambda_i\right) dU. \tag{3.7}$$

There are number of ways how to compute the Jacobian, which turned out to be the square of the Vandermonde determinant in this case. Thanks to the invariance of the measure, J depends only on λ's, and we can compute it in the vicinity of $U = 1$. Here

$$dM = dU^\dagger \Lambda + d\Lambda + \Lambda dU = d\Lambda + \Lambda dU - dU\Lambda, \tag{3.8}$$

where we have used $dU^\dagger = -dU$. This means that

$$dM_{ij} = d\lambda_i \delta_{ij} + (\lambda_i - \lambda_j) dU_{ij}, \tag{3.9}$$

the change of variables is diagonal and the Jacobian is just the product of the factors $(\lambda_i - \lambda_j)$.

3.1.2 Planar limit as the leading order in the limit of large matrices

One is usually interested in the behavior of the results in the limit of a very large matrix. For the case of $N \times N$ Hermitian matrices this means $N \to \infty$ limit. There are many reasons for this, being mathematical, physical and also practical.

From the mathematical point of view, the quantities like eigenvalue distribution become continuous in the large matrix limit. Also in this limit, certain properties of eigenvalue distribution become independent of the exact probability distribution and depend only on the symmetry group of the ensemble. This notion is called universality and allows to study certain properties on the simplest ensembles [30].

When random matrices describe a physical system, large matrix limit is the limit of large number of constituents. When studying the spectra of large nuclei, this means large number of levels. In condensed matter this means large number of electrons or other particles of interest. When one uses the random matrix to describe a lattice or to discretize some surface, large N limit is the continuum limit. In all these cases the limit is well justified by the physics of the problem.

When one considers field theory on a fuzzy space, large N limit is the limit of commutative theory and if the fuzzy structure was introduced to regulate the theory, this removes the regulator. Also, if there is non-commutativity present in the nature, it is very small and this justifies the limit of large N. The subleading contributions then represent the non-commutative correction.

And lastly, the large N limit provides a considerable simplification to the calculations. As we will see, the diagrams involved in computation of the averages give contributions with different powers of N and this power is related to the topology of the diagram. The leading order is given by the planar diagrams and this simplifies greatly the resulting combinatorial problem. The subleading corrections can then be computed systematically.

To compute expectation values of invariant functions $f(M)$, one has to compute correlators of the form $\left\langle \mathrm{Tr}\left(M^k\right)[W(M)]^l\right\rangle$. To compute these, we use the Wick's theorem. One has to sum over all possible pairing of M's in the expectation value, weighted by a propagator $\overbrace{M_{ij}M_{kl}}$ for each pairing. Since the contractions are done with the Gaussian, or free, measure, the propagator is given by the inverse of the quadratic part of (3.3) and is equal to

$$\overbrace{M_{ij}M_{kl}} = \frac{1}{N\mu^2}\delta_{il}\delta_{jk}. \tag{3.10}$$

To compute the higher correlators, we use method of fat graphs due to 't Hooft [31]. We will indicate the matrix M in the diagrams by a double line, which represent the double index structure of M_{ij}. Since the matrix is in the adjoint representation of $SU(N)$, indexes i and j can be viewed as in

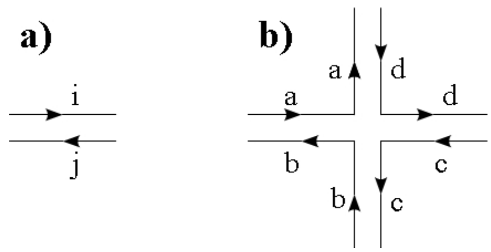

Figure 3.1: a) shows the double-line notation of a matrix element M_{ij}, b) shows the double line notation of a quartic vertex $M_{ab}M_{bc}M_{cd}M_{da}$.

the fundamental and anti-fundamental representation and the lines and the arrows reflect this structure. This is illustrated in the Figure 3.1. Vertices are then given by a star of n double lined legs. Let us illustrate this at the computation of the first order correction to the two point function $\langle M_{ij}M_{kl}\rangle$ of the theory with quartic interaction $g\mathrm{Tr}\left(M^4\right)$.

The relevant diagrams are shown in the Figure 3.2a. There are two possible kinds of contractions, one planar 3.2b and one non-planar 3.2c. They give respective contributions

$$\text{planar} = Ng_4\frac{1}{N\mu^2}\delta_{ia}\delta_{jb}\frac{1}{N\mu^2}\delta_{bl}\delta_{ik}\frac{1}{N\mu^2}\delta_{dd}\delta_{ic} = \frac{1}{N}g\left(\frac{1}{\mu^2}\right)^3\delta_{ik}\delta_{jl} \quad (3.11)$$

and

$$\begin{aligned}\text{non-planar} &= Ng_4\frac{1}{N\mu^2}\delta_{ia}\delta_{jb}\frac{1}{N\mu^2}\delta_{ab}\delta_{cd}\frac{1}{N\mu^2}\delta_{dk}\delta_{cl} \\ &= \frac{1}{N^2}g_4\left(\frac{1}{\mu^2}\right)^3\delta_{ij}\delta_{kl}. \end{aligned} \quad (3.12)$$

Following the summation of indexes on delta functions we observe that the overall factor of N depends on the number of closed lines in our fat graph. Each line produces, after the summation of all but one of the indexes, factor of δ_{ii}, which then gives a factor of N when summed over the last index.

Now comes a crucial observation for a general diagram. Let it have E

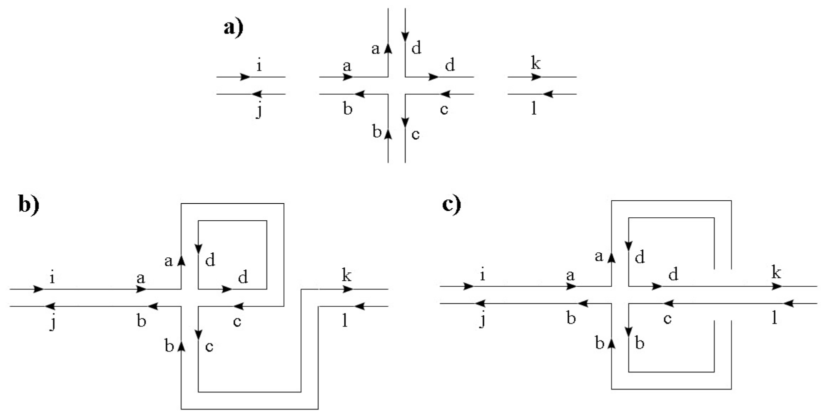

Figure 3.2: a) shows the general form of the diagram contribution to the g^1 order of the two point function, b) shows the planar contribution, c) shows the non-planar contribution.

propagators, or edges, V_n vertices with n legs and h closed loops. The factor for this diagram is then

$$\left(\frac{1}{N\mu^2}\right)^E N^h \prod_{n=3}^{N}(-Ng_n)^{V_n}, \tag{3.13}$$

where $V = \sum V_n$ is the total number of vertices. If we now consider the diagram as a Riemann surface of genus g, we have the topological relation

$$2 - 2g = h - E + V. \tag{3.14}$$

So (3.13) can be rewritten as

$$N^2\left[\left(\frac{1}{\mu^2}\right)^E \prod(-g_n)^{V_n}\right]\left[\frac{1}{N^2}\right]^g. \tag{3.15}$$

We see that the first factor is given by the type of the diagram. The second factor is given by the topology of the diagram and diagrams of the same type, with higher genus g are all suppressed by a factor of $1/N^{2g}$. This also means that the leading order contribution is given by the $g = 0$, i.e. planar diagrams.

To connect the two-point functions (3.11) and (3.12) to this result, we need to realize that the two diagrams can not be realized as Riemann surfaces, since they are not an average of an invariant function of the form $\mathrm{Tr}\left(M^k\right)$. In order to be able to do so, we need to contract the two external matrices with $\delta_{ik}\delta_{jl}$, i.e. to close the loops, and consider them as a new, 2-point vertex, which brings an extra factor of N. Then, we see that the planar diagram really has N^2 dependence, and the non-planar N^0.

3.2 The saddle point approximation and eigenvalue distributions

In this section, we will show how to obtain the eigenvalue distribution of the random matrix without explicit computation of any expectation values. We will later show that doing that and counting the diagrams leads to the same result as computed here.

We absorb the Jacobian in (3.7) into the action and obtain a theory governed by the effective measure

$$N^2 S_{eff}(\lambda) = N^2 \left[\frac{1}{2}\mu^2 \frac{1}{N} \sum_{i=1}^{N} \lambda_i^2 + \frac{1}{N} W(\lambda) - 2\frac{1}{N^2} \sum_{i<j} \log|\lambda_i - \lambda_j| \right] \tag{3.16}$$

where we have denoted the $n \geq 3$ part of the measure as

$$W(\lambda) = \sum_k g_k \sum_{i=1}^{N} \lambda_i^k. \tag{3.17}$$

As mentioned before, the effective action is now finite in the limit of large N, since the sum over N eigenvalues is of the order N^1. Therefore as $N \to \infty$, the integral in (3.1) will be dominated by the saddle-point configuration of the eigenvalues λ_i^E, which extremizes the effective action (3.16). The average of an invariant function $f(M)$ is then given, in the large N limit, by

$$\langle f \rangle = \sum_{i=1}^{N} f\left(\lambda_i^E\right). \tag{3.18}$$

We vary the action with respect to λ_i to obtain

$$\mu^2 \lambda_i^E + W'(\lambda_i^E) = \frac{1}{N} \sum_{i \neq j} \frac{1}{\lambda_i^E - \lambda_j^E}, \tag{3.19}$$

for $i = 1, 2, \ldots, N$. For solutions of this equation λ_i^E, we formally define the eigenvalue distribution

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i^E). \tag{3.20}$$

This function becomes continuous in the large N limit and for the moment we will assume that it vanishes outside a finite interval $\mathcal{C}$. For a function of the eigenvalues $f(\lambda)$ we then have in the large N limit

$$\frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \to \int_{\mathcal{C}} d\lambda \, \rho(\lambda) f(\lambda), \tag{3.21}$$

which is used to change (3.19) to

$$\mu^2 \lambda + W'(\lambda) = 2P \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}. \tag{3.22}$$

Here, $P \int$ denotes the principal value of the integral. We introduce a function, called the planar resolvent,

$$\omega_0(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda}. \tag{3.23}$$

In terms of this function the eigenvalue distribution can be computed using the discontinuity equation

$$\rho(\lambda) = -\frac{1}{2\pi i}\left[\omega_0(\lambda + i\varepsilon) - \omega_0(\lambda - i\varepsilon)\right], \tag{3.24}$$

and equation (3.22) becomes an equation for resolvent

$$\omega_0(z + i\varepsilon) - \omega_0(z - i\varepsilon) = -\mu^2\lambda - W'(\lambda). \tag{3.25}$$

3.2.1 One-cut assumption

If $\mathcal{C}$ is given by the interval $[b, a]$, the equation (3.25) is solved by

$$\omega_0(z) = \frac{1}{2}\oint_{\mathcal{C}} \frac{dz'}{2\pi i} \frac{\mu^2 z' + W'(z')}{z - z'} \sqrt{\frac{(z-a)(z-b)}{(z'-a)(z'-b)}}. \tag{3.26}$$

We have used the asymptotics $\omega_0(z) \to 1/z$ as $z \to \infty$, which follows from (3.23) and normalization of $\rho(\lambda)$. This asymptotic behavior also yields

$$\begin{aligned} \oint_{\mathcal{C}} \frac{dz'}{2\pi i} \frac{\mu^2 z' + W'(z')}{\sqrt{(z'-a)(z'-b)}} &= 0, \\ \oint_{\mathcal{C}} \frac{dz'}{2\pi i} \frac{z'\left[\mu^2 z' + W'(z')\right]}{\sqrt{(z'-a)(z'-b)}} &= 2. \end{aligned} \tag{3.27}$$

Finally, for a polynomial potential (3.26) can be solved as

$$\omega_0(z) = \frac{1}{2}\left[\mu^2 z + W'(z) - M(z)\sqrt{(z-a)(z-b)}\right], \tag{3.28}$$

where

$$M(z) = \oint_0 \frac{dz'}{2\pi i} \frac{\mu^2/z' + W'(1/z')}{1 - zz'} \frac{1}{\sqrt{(1-az')(1-bz')}}, \tag{3.29}$$

with contour around $z = 0$.

Knowing the distribution function, one can now compute various expectation values of the theory. For example the normalized traces of powers of

M are given by

$$\frac{1}{N}\left\langle \mathrm{Tr}\left(M^k\right)\right\rangle = \int_{\mathcal{C}} d\lambda\, \lambda^k \rho(\lambda). \tag{3.30}$$

Similarly, the planar free energy is given by

$$F_0 = -\log\int dM\, e^{-N^2 S(M)} = -\log e^{-N^2 S_{eff}[\rho(\lambda)]} = N^2 S_{eff}[\rho(\lambda)]. \tag{3.31}$$

Note that

$$\begin{aligned} S_{eff}[\rho(\lambda)] &= \frac{1}{2}\mu^2\int d\lambda\ \lambda^2\rho(\lambda) + \sum_k g_k\int d\lambda\ \lambda^k\rho(\lambda) \\ &- 2\int d\lambda\, d\lambda'\ \rho(\lambda)\rho(\lambda')\log|\lambda-\lambda'| \end{aligned} \tag{3.32}$$

We should also mention that the resolvent is connected to the moments of the distribution by

$$z\omega(z) = \frac{1}{N}\sum_{k=0}^{\infty}\frac{\left\langle \mathrm{Tr}\left(M^k\right)\right\rangle}{z^k}. \tag{3.33}$$

This is indeed the defining relation for the full resolvent with contributions from diagrams of any topology. The planar part is then given by the planar diagrams of $\left\langle \mathrm{Tr}\left(M^k\right)\right\rangle$, which together with (3.30) gives our original definition of planar resolvent (3.23).

Let us illustrate these rather formal developments on two simple examples, which will be also needed in the later chapters.

Wigner semicircle distribution

The simplest example is clearly the no potential case $W(\lambda) = 0$. The first of the conditions (3.27) yields $a + b = 0$, which is expected for the symmetric potential. The second condition then gives $a = 2/\mu$. Equation (3.29) then yields $M(z) = \mu^2$ and we finally obtain

$$\omega_0(z) = \frac{1}{2}\left(\mu^2 z - \mu^2\sqrt{z^2 - \frac{4}{\mu^2}}\right), \tag{3.34}$$

and using the discontinuity equation (3.24), we obtain the celebrated Wigner semicircle law

$$\rho(\lambda) = \frac{\mu^2}{2\pi}\sqrt{\frac{4}{\mu^2} - \lambda^2}\ ,\ \lambda^2 < \frac{4}{\mu^2}. \tag{3.35}$$

Note that this is normalized to 1. If we used the original normalization of $\rho(\lambda)$, which is the number of eigenvalues N, this would become

$$\rho(\lambda) = \frac{\mu^2}{2\pi}\sqrt{\frac{4N}{\mu^2} - \lambda^2}\ ,\ \lambda^2 < \frac{4N}{\mu^2}. \tag{3.36}$$

Deformation due to quartic potential

We now introduce a simple interaction potential $W(\lambda) = g\lambda^4$, which corresponds to the term $g\mathrm{Tr}\left(M^4\right)$ in the measure. Again, the first of the conditions (3.27) yields $b = -a$ and the second then gives

$$ga^4 + (2ga^2 + \mu^2)a^2 = 4. \tag{3.37}$$

The radius of the distribution is then given by

$$a^2 = \frac{1}{6g}\left(\sqrt{\mu^4 + 48g} - \mu^2\right). \tag{3.38}$$

Equation (3.29) then yields $M(z) = 4gz^2 + 2ga^2 + \mu^2$ and we finally obtain

$$\omega_0(z) = \frac{1}{2}\left[4gz^3 + \mu^2 z - (4gz^2 + 2ga^2 + \mu^2)\sqrt{z^2 - a^2}\right], \tag{3.39}$$

and

$$\rho(\lambda) = \frac{1}{\pi}\left(\frac{\mu^2}{2} + ga^2 + 2g\lambda^2\right)\sqrt{a^2 - \lambda^2}\ ,\ \lambda^2 < a^2. \tag{3.40}$$

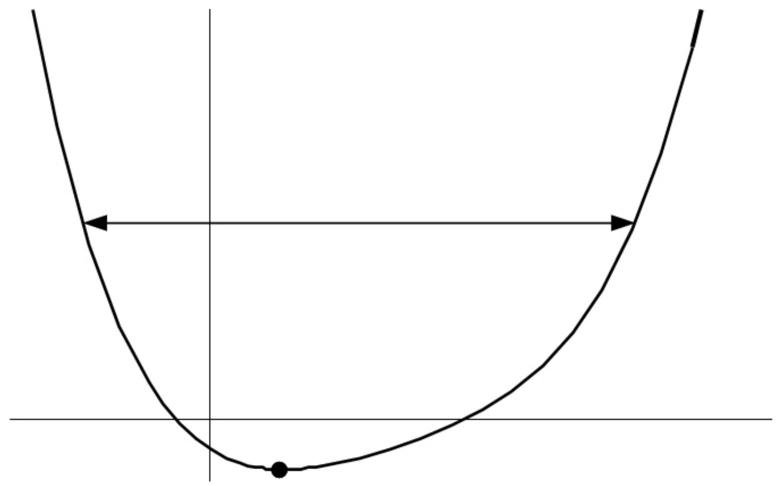

Figure 3.3: Eigenvalues do not collapse into the minimum of the potential, but are spread out over a finite interval due to the repulsion in the effective action.

These results were first obtained in [71]. Again, this distribution is normalizes to 1. If we used the normalization to N, the result for the square of the radius would be

$$a^2 = \frac{1}{6g}\left(\sqrt{\mu^4 + 48gN} - \mu^2\right). \tag{3.41}$$

3.2.2 Multi-cut solutions

There is a very natural analogue between the eigenvalue dynamics and the N particle gas. Looking at the effective action (3.16) we see that the Vandermond determinant introduces a repulsive effective potential between the eigenvalues and competes with the original potential term. The eigenvalues want to collapse into the minimum of the potential, but due to the repulsion they are going to occupy some finite interval around the minimum, as in the Figure 3.3.

The situation gets more complicated when the potential has more than one minimum. If the minima are deep enough, the gas of particles can split into two or more disjoint parts, each located at one minimum. In the language of the eigenvalue density, the one-cut assumption is no longer valid and we need to assume a more complicated support of the distribution.

This is very clearly illustrated in the solution (3.40) for the quartic potential. Allowing for negative μ^2, the eigenvalue distribution becomes negative if $\mu^4 > 16g$. This indicates that the solution can not be interpreted as a probability distribution and one has to start from scratch.

We illustrate the approach on the case of two minima of the potential. We will therefore assume that the support $\mathcal{C}$ is given by the union of two intervals $[a, b]$ and $[c, d]$. Equation (3.28) the becomes

$$\omega_0(z) = \frac{1}{2}\left[\mu^2 z + W'(z) - M(z)\sqrt{(z-a)(z-b)(z-c)(z-d)}\right] \quad (3.42)$$

and the endpoints of the intervals are given by

$$\begin{aligned} \oint_{\mathcal{C}} \frac{dz'}{2\pi i}\frac{\mu^2 z' + W'(z')}{\sqrt{(z'-a)(z'-b)}} &= 0, \\ \oint_{\mathcal{C}} \frac{dz'}{2\pi i}\frac{z'\left[\mu^2 z' + W'(z')\right]}{\sqrt{(z'-a)(z'-b)}} &= 0, \\ \oint_{\mathcal{C}} \frac{dz'}{2\pi i}\frac{(z')^2\left[\mu^2 z' + W'(z')\right]}{\sqrt{(z'-a)(z'-b)}} &= 2. \end{aligned} \quad (3.43)$$

Recall that these are given by the $1/z$ asymptotics of $\omega_0(z)$. There are too few conditions to determine the endpoints! In general for s-cut case, there are $s+1$ conditions but $2s$ unknowns to be determined. Here, we have to make an extra assumption.

One can show that assuming the free energy of the system to be minimal gives exactly the extra $s-1$ conditions that are needed. One introduces the filling fraction x_s of the eigenvalues in the s-th cut as

$$x_s = \int_{\mathcal{C}_s} d\lambda\, \rho(\lambda)\ ,\ \sum_s x_s = 1, \quad (3.44)$$

and the variation of the free energy as a function of x_s should vanish. This condition has also a very intuitive physical meaning. Recall the problem as a N particle gas of eigenvalues and lets get back to the two cut case. These now sit in two wells. If we now try to move one eigenvalue from one well to the other, this should cost us no energy in the equilibrium case. Since if the energy difference was negative, the eigenvalues would tunnel through the potential barrier until the situation would become balanced. For a positive energy difference, the eigenvalues would tunnel the other way. This condition gives

$$\int_b^c M(z)\sqrt{(z-a)(z-b)(z-c)(z-d)} = 0. \tag{3.45}$$

Two-cut solution to quartic potential

This procedure then yields the solution for the two-cut solution in the case of $\mu^2 < 0$ and potential $g\mathrm{Tr}\left(M^4\right)$ to be

$$\rho(\lambda) = \frac{2g\lambda}{\pi}\sqrt{(-a-\lambda)(-b-\lambda)(b-\lambda)(a-\lambda)}, \tag{3.46}$$

where the support is $\mathcal{C} = [-a,-b] \cup [b,a]$ and

$$a^2 = \frac{|\mu^2|}{4g} + \frac{1}{\sqrt{g}}\ ,\ b^2 = \frac{|\mu^2|}{4g} - \frac{1}{\sqrt{g}}. \tag{3.47}$$

It is rather reassuring to note that for $\mu^4 = 16g$, the two intervals connect and two-cut assumption is no longer valid, which is the same condition we obtained using the analysis of the one-cut solution.

The analysis of the eigenfunction distribution can be summarized in the phase-diagram, which is shown in the Figure 3.4. The diagram indicates the phase of the theory, i.e. either the one-cut or two-cut solution, as a function of the parameters μ^2 and g. If the parameters lie in the region I, the eigenvalue distribution has one cut. If they are in region II, the solution is a two-cut distribution. The critical line is given by $\mu^4 = 16g$. For $\mu^2 > 0$

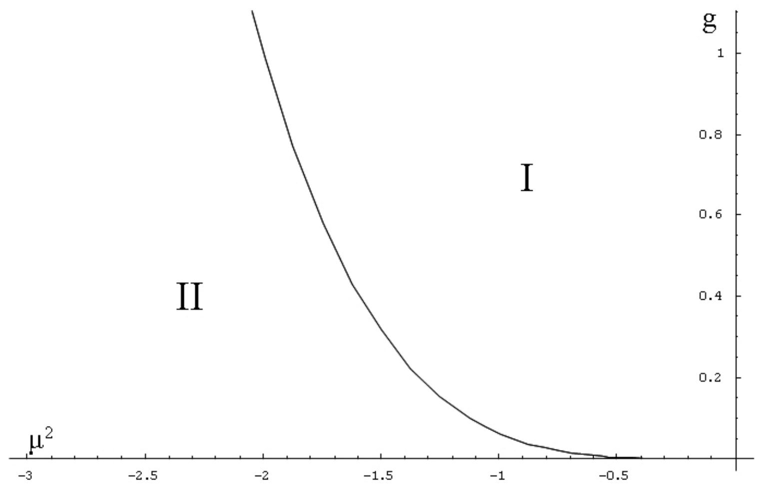

Figure 3.4: Phase diagram of the $W(M) = g\mathrm{Tr}\left(M^4\right)$ theory. See text for details.

the theory is always in the one-cut phase.

Chapter 4

Random matrix approach to scalar fuzzy fields

This chapter includes the core findings of the presented dissertation. We show how the scalar field theory on a fuzzy space can be described in the language of random matrices and that there is a natural class of random matrix ensembles which are defined by the scalar fields.

These new matrix ensembles are characterized by a new term in the measure, which corresponds to the kinetic term of the field theory action. As mentioned in the introduction, this extra term involves external matrices, which breaks the invariance of the measure and thus the diagonalization and reduction to the eigenvalue problem is no longer available. However, the results we find are remarkably simple. For a class of ensembles we will prove that the features of the distribution of the eigenvalues do not change. And the only change that occurs is in the relevant two point function of the theory, which enters the distribution as a parameter.

These new ensembles have new observables, defined by the commutators of M and the external matrices. We will compute the expectation values of some of the new observables including matrix Lapalcian and we will show that these are correlated with the eigenvalues of the random matrix M.

In the first part, we develop the approach on the case of free theory. In the second part, we generalize the approach to the case of an interacting theory, namely the quartic interaction. Finally, we present an argument, why one expects these simple results to be a generic feature of the fuzzy field theories.

This chapter is based on [3] and [4].

4.1 Results for the free theory

Motivated by the action for the free scalar field theory on fuzzy $\mathbb{C}P^n_F$ (2.24), we will a new random matrix ensemble, given by probability measure

$$S[M] = \frac{1}{2}\mathrm{Tr}\left(M[L_\alpha,[L_\alpha,M]]\right) + \frac{1}{2}\mu^2\mathrm{Tr}\left(M^2\right). \tag{4.1}$$

The external matrices L_α are the generators of $SU(n+1)$ in the N dimensional representation. We absorb the factor of N^2 in (3.16) into the definition of the matrix and couplings. We also absorb the volume factor, which arose in the field theory. This is important in the section 5.5, where we consider multiscaling of the theory. Our approach will be suitable for a more general measure with a general kinetic term of the form

$$S[M] = \frac{1}{2}\mathrm{Tr}\left(M\mathcal{K}M\right) + \frac{1}{2}\mu^2\mathrm{Tr}\left(M^2\right), \tag{4.2}$$

where we assume that the action of $\mathcal{K}$ on T^l_A (2.21) depends on l and m only, i.e. that T^l_A are eigenmatrices of $\mathcal{K}$. Also, to be able to make the connection to the underlying fuzzy space and field theory, we will expect any reasonable $\mathcal{K}$ to respect the $SU(n+1)$ symmetry of this space. Finally, we define

$$B = \mathcal{K}M. \tag{4.3}$$

4.1.1 Eigenvalue distribution of the random matrix

We will be interested in the distributions of eigenvalues in the random matrix ensemble given by the probability distribution (4.2). As mentioned before, the usual approach of diagonalization and reduction to the eigenvalue problem is not possible in this case and we will derive recursion rules for the moments of the distribution using explicit Wick contractions.

The following is done for the case of the fuzzy sphere S_F^2. As we will see, the treatment is general and the difference is going to be only in the expression for the two point functions f, g, h, (4.6), which will be considered in section 5.3.

Looking at the distribution (4.2) and the expansion of the random matrix M in terms of the basis T_A^l (2.21) we see that the correlation of two components of the field is

$$\overbrace{c^l_m c^{l'}_{m'}} = \delta^{ll'}\delta_{mm'}G(l), \tag{4.4}$$

where $G(l)$ is the propagator given by the kinetic term. For example for the standard Laplacian kinetic term, we get $G(l) = 1/[(l(l+1)+\mu^2]$. This then yields the two point function of the form

$$\langle (MM)_{ij}\rangle = \frac{1}{N}\sum_{l=0}^{N-1}(2l+1)G(l)\delta_{ij} \equiv f\delta_{ij}, \tag{4.5}$$

where f is defined by this equation. Similarly, we can define the propagators

$$\langle (B\,B)_{ij}\rangle = g\,\delta_{ij}\ ,\ \langle (M\,B)_{ij}\rangle = \langle (B\,M)_{ij}\rangle = h\,\delta_{ij}. \tag{4.6}$$

Before we proceed with the actual calculation, we need to derive the following important result valid in the large N limit. For any matrix product, the expectation value is proportional to the identity matrix. To see this, consider $\langle Z_{ij} \rangle$ where Z is a matrix of the form

$$Z = M^{m_1}\, B^{b_1}\, M^{m_2}\, B^{b_2} \cdots \tag{4.7}$$

In evaluating the expectation value of Z_{ij} by Wick contractions, the leading term in the large N limit will involve only the planar contractions. As a result, in the leading term, there will be at least one case where two adjacent matrices are contracted. Denoting $Q_1 Q_2$ the two adjacent matrices, and $\mathcal{L}$ and $\mathcal{R}$ the matrices to the left and right of $Q_1 Q_2$, and using (4.6), this contraction will give us a term of the form

$$\langle Z_{ij} \rangle \approx \langle \mathcal{L}_{ia}\, \mathcal{R}_{bj} \rangle \, \langle (Q_1 Q_2)_{ab} \rangle \sim \langle (\mathcal{L}\mathcal{R})_{ij} \rangle \, q \tag{4.8}$$

where Q_1, Q_2 stand for either M or B and q is the appropriate function among f, g, h. This step reduces the correlator to one with lower number of matrices M or B, with the two nearby matrices that were Wick contracted deleted. Iterating, we see that

$$\langle Z_{ij} \rangle = z\, \delta_{ij} \tag{4.9}$$

where $z = \langle \mathrm{Tr}\,(Z) \rangle / N$ is a scalar quantity. We thus see that the correlators f, g, h are the only information we need when computing expectation values of this form.

We are now ready to investigate the moments of eigenvalue distribution of M. According to (3.30), these are given by $\left\langle \mathrm{Tr}\left(M^{2m}\right)\right\rangle$. We expect only the even moments to be non-vanishing due to the symmetry of the measure. We now single out one of the matrices in this trace and consider the contraction of this matrix with other matrix in the product, i.e.

$$\left\langle (M^{2m-2-p})_{il} \overbrace{M_{lr} (M^p)_{rj} M_{ji}} \right\rangle = c\ \delta_{li}\delta_{rj} \left\langle (M^{2m-2-p})_{il} (M^p)_{rj} \right\rangle , \tag{4.10}$$

where the two matrices that are being contracted are connected by the usual clip. Due to the planarity of the diagram, there are no contractions between the two groups of matrices. We can therefore write

$$\left\langle (M^{2m-2-p})_{il}\overbrace{M_{lr}(M^p)_{rj}M_{ji}}\right\rangle = \frac{f}{N}\left\langle \mathrm{Tr}\left(M^p\right)\right\rangle \left\langle \mathrm{Tr}\left(M^{2m-2-p}\right)\right\rangle . \quad (4.11)$$

Summing over all possible contractions, i.e. over all possible values of p yields

$$\left\langle \mathrm{Tr}\left(M^{2m}\right)\right\rangle = \frac{f}{N}\sum_{p=0}^{m-1}\left\langle \mathrm{Tr}\left(M^{2p}\right)\right\rangle \left\langle \mathrm{Tr}\left(M^{2(m-1-p)}\right)\right\rangle \;,\; m \geq 1. \quad (4.12)$$

The condition on m arises from the fact that we need at least one matrix M to be able to consider any contractions. We now define the rescaled moments

$$F_{2m} = \frac{1}{N}\left\langle \mathrm{Tr}\left[\left(\frac{M}{2\sqrt{f}}\right)^{2m}\right]\right\rangle , \quad (4.13)$$

which are going to be finite and in the terms of which the recursion rule becomes

$$4F_{2m} = \sum_{p=0}^{m-1} F_{2p}F_{2(m-1-p)}. \quad (4.14)$$

Note that F_0 enters here as an initial condition and is not given by this formula. But clearly $F_0 = \langle \mathrm{Tr}\ id\rangle /N = 1$. One could immediately recognize equation (4.14) as a property of the Catalan numbers and identify F_{2m} with a rescaled version of these. However, to illustrate better the method we will be using later, we will do something different. We will define the function

$$\phi(t) = \sum_{m=0}^{\infty} t^{2m}F_{2m}, \quad (4.15)$$

which is the moment generating function for the distribution $\rho(x)$, the distribution of the eigenvalues of the matrix M. If we now multiply (4.14) by

t^{2m} and sum over all m, we obtain

$$4\sum_{m=1}^{\infty} t^{2m}F_{2m} = t^2\sum_{m=1}^{\infty}\sum_{p=0}^{m-1} t^{2p}F_{2p}t^{2(m-1-p)}F_{2(m-1-p)}, \tag{4.16}$$
$$4\big(\phi(t)-1\big) = t^2\phi^2(t). \tag{4.17}$$

This is an easy quadratic equation for $\phi(t)$. We chose the solution which has the correct $t\to 0$ limit and obtain

$$\phi(t) = 2\frac{1-\sqrt{1-t^2}}{t^2}. \tag{4.18}$$

Looking back at the definition of the resolvent (3.33) we find

$$\omega(\lambda) = \frac{1}{\lambda}\phi(1/\lambda) \tag{4.19}$$

and with the discontinuity equation (3.24) we see that such generating function corresponds to the distribution

$$\rho(x) = \frac{2}{\pi}\sqrt{1-x^2}. \tag{4.20}$$

As expected, this is nothing else than the normalized Wigner distribution, which has Catalan numbers as the moments.[1] From the definition of the moments (4.13) we see that the radius of the distribution is $2\sqrt{f}$. We will get to the explicit computation of f in the section 4.1.5. Here, let us only note that the radius in the original Gaussian ensemble was $2\sqrt{N}/\mu$. The two point correlator in that case is N/μ^2, so we see that the kinetic term rescaled the radius of the distribution, by changing it's two point function, but nothing else has changed apart from that.

If we now define the rescaled moments of the distribution of the matrix

[1]More precisely the $2n$-th moment of the semicircle distribution of radius R is $c_n(R/2)^{2n}$, with c_n the n-th Catalan number.

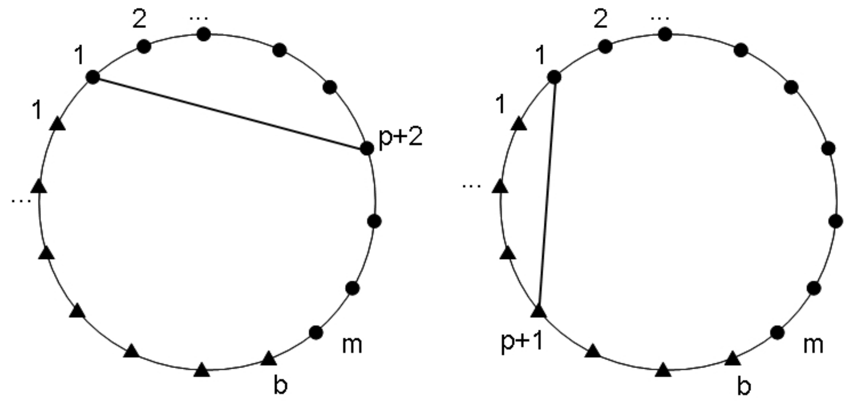

Figure 4.1: Two types of contractions contributing to (4.29). Circles represent expectation values of matrices. The dots M's, the triangles B's and the line represents the contraction. Planarity ensures that there are going to be no contractions between the two parts divided by the line.

B

$$G_{2m} = \frac{1}{N} \left\langle \mathrm{Tr} \left[\left(\frac{B}{2\sqrt{g}} \right)^{2m} \right] \right\rangle , \tag{4.21}$$

we find out that the whole process leads to the same result and these generate the semicircle distribution with radius $2\sqrt{g}$.

4.1.2 The joint eigenvalue distribution for M and B

We will now derive the recurrence relations for the moments of the joint distribution of the eigenvalues of the matrix M and the matrix B. This means we need to compute $\langle \mathrm{Tr} \left(M^m B^b \right) \rangle$. In evaluating this by Wick contractions, consider the contractions of an M matrix next to the series of B's. There are two types of contractions, as indicated in the Figure 4.1. First, considering the contraction of this matrix M with another M matrix, we get

$$\langle (M^{m-2-p})_{ij} \overbrace{M_{jk}(M^p)_{kr} M_{rs}}(B^b)_{si}\rangle = f\,\frac{1}{N}\langle \mathrm{Tr}M^p\rangle\,\langle \mathrm{Tr}(M^{m-2-p}\,B^b)\rangle \tag{4.22}$$

As before, the planarity property has again split the expectation value into two. Similarly considering the contraction of the chosen M matrix with one of the B's results into

$$\langle (M^{m-1})_{ij}\,\overbrace{M_{jk}\,(B^p)_{kr} B_{rs}}(B^{b-p-1})_{si}\rangle = h\,\frac{1}{N}\langle \mathrm{Tr}B^p\rangle\ \langle \mathrm{Tr}(M^{m-1}B^{b-p-1})\rangle. \tag{4.23}$$

Combining these two and summing over all possible contractions, i.e. allowing for all possible values of p, we get the recursion rule

$$\begin{aligned}\langle \mathrm{Tr}(M^m\,B^b)\rangle &= f\,\frac{1}{N}\sum_{p=0}^{m-2}\langle \mathrm{Tr}M^p\rangle\,\langle \mathrm{Tr}(M^{m-2-p}B^b)\rangle \\ &+ h\,\frac{1}{N}\sum_{p=0}^{b-1}\langle \mathrm{Tr}(M^{m-1}\,B^{b-p-1})\rangle\,\langle \mathrm{Tr}B^p\rangle. \end{aligned}\tag{4.24}$$

Since one M is chosen to be contracted with all other matrices, this relation applies when $m \geq 1$.

In a similar way, we can single out a B matrix adjacent to the series of M matrices and consider its contractions. This leads to the recursion rule, for $b \geq 1$,

$$\begin{aligned}\langle \mathrm{Tr}(M^m\,B^b)\rangle &= g\,\frac{1}{N}\sum_{p=0}^{b-2}\langle \mathrm{Tr}B^p\rangle\,\langle \mathrm{Tr}(M^m\,B^{b-2-p})\rangle \\ &+ h\,\frac{1}{N}\sum_{\alpha=0}^{m-1}\langle \mathrm{Tr}(M^{m-p-1}\,B^{b-1})\rangle\,\langle \mathrm{Tr}M^p\rangle. \end{aligned}\tag{4.25}$$

These recursion rules may also be viewed as the Schwinger-Dyson equations for expectation values calculated via the functional integral

$$\langle \mathcal{O}\rangle = \int [dM]\,e^{-S(M)}\ \mathcal{O}. \tag{4.26}$$

This equation can be obtained, for example, by considering the identity

$$\int [dM]\, \frac{\partial}{\partial M_{ji}} \left[(M^{m-1}\, B^{b-1})_{ji}\, e^{-S} \right] = 0. \tag{4.27}$$

We will not consider the simplifications of such matrix integrals, but rather proceed to the direct solution of the recursion rules, as in the previous section.

To solve the recursion rules, we define the normalized correlator

$$W_{m,b} = \frac{1}{N} \left\langle \operatorname{Tr} \left[\left(\frac{M}{2\sqrt{f}} \right)^m \left(\frac{B}{2\sqrt{g}} \right)^b \right] \right\rangle \tag{4.28}$$

The two recursion rules become

$$\begin{aligned} 4W_{m,b} &= \sum_{p=0}^{m-2} W_{p,0} W_{m-2-p,b} + \gamma \sum_{p=0}^{b-1} W_{m-1,b-1-p} W_{0,p} \quad m \geq 1, && (4.29) \\ 4W_{m,b} &= \sum_{p=0}^{b-2} W_{0,p} W_{m,b-p-2} + \gamma \sum_{p=0}^{m-1} W_{m-1-p,b-1} W_{p,0} \quad b \geq 1, && (4.30) \end{aligned}$$

where

$$\gamma = \frac{h}{\sqrt{f\, g}} \tag{4.31}$$

As we did before, we define the generating function

$$\phi(t,s) = \sum_{m,b=0}^{\infty} W_{m,b}\, t^m\, s^b \tag{4.32}$$

We set $W_{0,0} = 1$, which follows from the expectation value of the identity and is the normalization condition for the distribution. The recursion rules now become

$$\begin{aligned} 4\Big(\phi(t,s) - \phi(s) \Big) &= t^2 \phi(t,s)\phi(t) + \gamma t s \phi(t,s)\phi(s), && (4.33) \\ 4\Big(\phi(t,s) - \phi(t) \Big) &= s^2 \phi(t,s)\phi(s) + \gamma t s \phi(t,s)\phi(t), && (4.34) \end{aligned}$$

where $\phi(t)$ and $\phi(s)$ are the generating functions corresponding to the eigenvalue distributions of M and B and are given by (4.17). Solving one of these equations for $\phi(t,s)$ we find

$$\phi(t,s) = \frac{4\phi(s)}{4 - t^2\phi(t) - \gamma ts\phi(s)} = \frac{4\phi(s)}{\frac{4}{\phi(t)} - \gamma ts\phi(s)} = \frac{\phi(t)\phi(s)}{1 - \frac{1}{4}\gamma ts\phi(s)\phi(t)}. \tag{4.35}$$

Clearly both the equations give the same result for $\phi(t,s)$, which serves as a consistency check. Using the explicit formula for $\phi(t)$ and $\phi(s)$ (4.18) we get

$$\phi(t,s) = \frac{4}{\left(1+\sqrt{1-t^2}\right)\left(1+\sqrt{1-s^2}\right) - \gamma ts}. \tag{4.36}$$

Also, as a consequence of (4.6,4.31), we have for the constant γ

$$-1 \le \gamma \le 1 \tag{4.37}$$

The value of γ is crucial. For $\gamma = 0$, in particular, the above generating function is the product of the generating functions of two independent Wigner distributions, while for nonzero γ we have correlations.

We now need to find the distribution corresponding to the generating function (4.35). We will present a general method to invert generating functions of this form to obtain the distribution. This method is a generalization of approach used in [3] and is presented in [4].

If the generating function is expressed as

$$\phi(t_1,\dots,t_n) = f\Big(t_1\phi(t_1),\dots,t_n\phi(t_n)\Big)\phi(t_1)\dots\phi(t_n), \tag{4.38}$$

then the corresponding distribution is given by

$$\rho(x_1,\dots,x_n) = \sum_{\varepsilon_i=\pm1} \varepsilon_1\dots\varepsilon_n F(e^{i\varepsilon_1\theta_1},\dots,e^{i\varepsilon_n\theta_n}), \tag{4.39}$$

where

$$F(z_1,\ldots,z_n)=\left(\prod_{j=1}^{n}\frac{z_j}{i\pi}\right)f(2z_1,\ldots,2z_n) \tag{4.40}$$

and $x_n=\cos\theta_n$.

We will prove the one-dimensional case. The generalization to more dimensions is then straightforward. Let us have a generating function of the form,

$$\Phi(t)=f\big(t\phi(t)\big)\phi(t), \tag{4.41}$$

where $\phi(t)=2/(1+\sqrt{1-t^2})$ is the generating function for the Wigner semicircle distribution. We expand this as

$$\Phi(t)=\sum_{n=0}^{\infty}a_n t^n\phi^{n+1}(t). \tag{4.42}$$

Using the explicit formula for $\phi(t)$, we obtain

$$\begin{aligned} t^n\phi^{n+1}(t) &= \sum_{k=0}^{[n/2]}\binom{n+1}{2k+1}\left(\frac{2}{t}\right)^n(1-t^2)^k\phi(t) \\ &- 2\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}\left(\frac{2}{t}\right)^n(1-t^2)^k. \end{aligned} \tag{4.43}$$

Now follows a crucial observation. If a generating function $g(t)$ corresponds to the distribution $\rho(x)$, then the nonsingular part of $g(t)/t^n$ generates $x^n\rho(x)$ for any $n\geq 0$. In previous expression, the second term contains only singular terms. However, the left hand side is clearly nonsingular and therefore the second term cancels the singular part of the first term but does not contribute otherwise. After resumming the first term of (4.43), we see that $t^n\phi^{n+1}(t)$ generates the following distribution

$$\begin{aligned}\rho_n(x) &= \frac{\rho_0(x)2^n x^n}{2\sqrt{1-\frac{1}{x^2}}}\left[\left(1+\sqrt{1-\frac{1}{x^2}}\right)^{n+1}-\left(1-\sqrt{1-\frac{1}{x^2}}\right)^{n+1}\right]= \\ &= \frac{2^n}{i\pi}\left[\left(x+i\sqrt{1-x^2}\right)^{n+1}-\left(x-i\sqrt{1-x^2}\right)^{n+1}\right]. \qquad (4.44)\end{aligned}$$

Since $|x|\leq 1$, we can write $x=\cos\theta$ for some $\theta\in[0,\pi]$ and $x\pm i\sqrt{1-x^2}=e^{\pm i\theta}$. This yields

$$\rho_n(x)=\frac{2^n e^{i(n+1)\theta}-2^n e^{-i(n+1)\theta}}{i\pi}, \qquad (4.45)$$

the final distribution is then

$$\rho(x)=\sum_{n=0}^{\infty}a_n\rho_n(x)=\frac{1}{i\pi}\left(e^{i\theta}f(2e^{i\theta})-e^{-i\theta}f(2e^{-i\theta})\right), \qquad (4.46)$$

which is the desired formula (4.40) for the one-dimensional case.

Looking at (4.35), we see that $\phi(t,s)$ is of the form (4.38). Using this method, we compute the joint probability distribution of the eigenvalues of M and B to be

$$\rho(x,y)=\rho(x)\rho(y)\frac{1-\gamma^2}{(1-\gamma^2)^2-4\gamma(1+\gamma^2)xy+4\gamma^2(x^2+y^2)}. \qquad (4.47)$$

This is symmetric in x and y. The integral of $\rho(x,y)$ over y is nontrivial, it can be calculated and gives $\rho(x)$ as it should, giving a check on the expression for $\rho(x,y)$.

Although the quadratic form in x,y in the denominator is not positive definite, the above distribution is positive for $\gamma^2\leq 1$. Indeed, for $\gamma>0$ the negative eigenvalue of the quadratic form corresponds to the eigenvector $x=y$. Putting $x=y=1$, their maximal value given the Wigner distribution prefactor $\rho(x)\rho(y)$, the denominator becomes $(1-\gamma)^4$. Similarly, for $\gamma<0$ the negative eigenvalue corresponds to $x=-y$ and choosing $x=-y=1$

the denominator becomes $(1+\gamma)^4$. For $\gamma^2 = 1$ the distribution appears to be singular, but taking the limit we see that it becomes $\rho(x)\delta(x-y)$ for positive γ and $\rho(x)\delta(x+y)$ for negative γ.

To see this, we write $\gamma = 1 - \varepsilon$ and $y = x + u\varepsilon$ and to the leading order in ε, we obtain

$$\begin{aligned} \int dy\; f(y)\rho(x,y) &= f(x)\rho(x)\frac{2}{\pi}\sqrt{1-x^2}\frac{1}{2}\underbrace{\int du\,\frac{1}{1+u^2-x^2}}_{\frac{\pi}{\sqrt{1-x^2}}} + \dots \\ &= f(x)\rho(x), \end{aligned} \tag{4.48}$$

where the ellipsis stands for terms which vanish in the $\varepsilon \to 0$ limit. This then justifies writing $\rho(x,y) = \rho(x)\delta(x-y)$.

For $\gamma^2 > 1$, however, the distribution becomes negative, signaling the nonexistence of a probability interpretation in that case. We can understand this by noticing that the correlation of the variables x and y is calculated as

$$\frac{\langle xy\rangle}{\sqrt{\langle x^2\rangle\,\langle y^2\rangle}} = \left.\frac{2\partial_t\partial_s\phi}{\sqrt{\partial_t^2\phi\,\partial_s^2\phi}}\right|_{t=s=0} = \gamma \tag{4.49}$$

Since correlations of stochastic variables have to be between -1 and 1, this is also the allowed range of γ. For $\gamma = \pm 1$ the two variables are fully (anti)correlated and thus equal to (minus) each other, justifying the delta-functions.

In conclusion, we obtain a correlated distribution for the eigenvalues of M and B, while their marginal distributions remain Wigner semicircles of radii $2\sqrt{f}$ for M and $2\sqrt{g}$ for B, as seen from the scaling factors in (4.28).

Since the two point functions of the matrices M and B were the only relevant quantities for computation of the joint distribution of the previous section, we can define "linear" functions of matrix M as

$$\begin{aligned} A_1 &= \sum_{m,l} c^l_m a_1(l,m) T^l_m, \\ A_2 &= \sum_{m,l} c^l_m a_2(l,m) T^l_m \end{aligned} \tag{4.50}$$

where $a_{1,2}$ are some functions of l and m. When we compute the distributions for these matrices the whole procedure will go through basically intact and we recover the same results. Namely the joint distribution of the eigenvalues of A_1 and A_2 is going to be given by

$$\rho(x,y) = \rho(x)\rho(y)\frac{1-\gamma_{12}^2}{(1-\gamma_{12}^2)^2 - 4\gamma_{12}(1+\gamma_{12}^2)xy - 4\gamma_{12}^2(x^2+y^2)} \tag{4.51}$$

and

$$\gamma_{12} = \frac{\langle \mathrm{Tr}\,(A_1 A_2)\rangle}{\sqrt{\alpha_1\alpha_2}}, \tag{4.52}$$

where $\alpha_i = \left\langle \mathrm{Tr}\left(A_i^2\right)\right\rangle$. The distributions of eigenvalues of matrices A_1, A_2 have radius $2\sqrt{\alpha_i}$. In terms of the original definition we have

$$\langle \mathrm{Tr}\,(A_1 A_2)\rangle = \frac{1}{N}\sum_{l=0}^{N-1} G(l) \sum_{m=-l}^{l} a_1(l,m) a_2(l,m), \tag{4.53}$$

$$\alpha_i = \frac{1}{N}\sum_{l=0}^{N-1} G(l) \sum_{m=-l}^{l} a_i^2(l,m). \tag{4.54}$$

4.1.3 Joint distribution of eigenvalues of three matrices

We now turn to computation of the joint distribution of three matrices of the form (4.50), i.e. we look for $\rho(x, y, z)$ such that

$$\begin{aligned} W_{a,b,c} &= \frac{1}{N}\left\langle \mathrm{Tr}\left[\left(\frac{A_1}{2\sqrt{\alpha_1}}\right)^a\left(\frac{A_2}{2\sqrt{\alpha_2}}\right)^b\left(\frac{A_3}{2\sqrt{\alpha_3}}\right)^c\right]\right\rangle \\ &= \int dxdydz\ x^a y^b z^c \rho(x,y,z). \end{aligned} \tag{4.55}$$

Our approach will be again to write down recursion rules for the moments $W_{a,b,c}$ coming from the Wick contractions. Focusing on one of the matrices A_1, there are three different types of contractions

$$\begin{aligned} \left\langle \mathrm{Tr}\left(A_1^a A_2^b A_3^c\right)\right\rangle &\to \frac{\left\langle \mathrm{Tr}\left(A_1^2\right)\right\rangle}{N}\left\langle \mathrm{Tr}\left(A_1^{p-2}\right)\right\rangle\left\langle \mathrm{Tr}\left(A_2^b A_3^c\right)\right\rangle \\ &\to \frac{\sqrt{\left\langle \mathrm{Tr}\left(A_1 A_2\right)\right\rangle}}{N}\left\langle \mathrm{Tr}\left(A_1^{a-1} A_2^p\right)\right\rangle\left\langle \mathrm{Tr}\left(A_2^{b-p-1} A_3^c\right)\right\rangle \\ &\to \frac{\sqrt{\left\langle \mathrm{Tr}\left(A_1 A_3\right)\right\rangle}}{N}\left\langle \mathrm{Tr}\left(A_1^{a-1} A_2^b A_3^p\right)\right\rangle\left\langle \mathrm{Tr}\left(A_3^p\right)\right\rangle. \end{aligned} \tag{4.56}$$

Summing over all possible contractions we obtain

$$\begin{aligned} 4W_{a,b,c} &= \sum_{p=0}^{a-2} W_{p,0,0} W_{a-p-2,b,c} + \gamma_{12}\sum_{p=0}^{b-1} W_{a-1,p,0} W_{0,b-p-1,c} + \\ &+ \gamma_{13}\sum_{p=0}^{c-1} W_{0,0,p} W_{a-1,b,c-p-1} \qquad a \geq 1. \end{aligned} \tag{4.57}$$

This rule holds only for $a \geq 1$. The quantities of the form $W_{0,0,p}$ and $W_{0,b,c}$ are moments of one- and two- matrix distributions and are given by (4.29,4.30). In this recursion rule they enter as the initial condition.

As before in the case of two matrices, there are more recursion formulas to write. Considering contractions of one of the A_2 matrices and one of the

A_3 matrices we obtain two more equations

$$\begin{aligned} 4W_{a,b,c} &= \sum_{p=0}^{b-2} W_{0,p,0}W_{a,b-p-2,c} + \gamma_{12}\sum_{p=0}^{a-1} W_{p,0,0}W_{a-p-1,b-1,c} \\ &+ \gamma_{23}\sum_{p=0}^{c-1} W_{a,0,p}W_{0,b-1,c-p-1} \qquad b \geq 1, \end{aligned} \tag{4.58}$$

$$\begin{aligned} 4W_{a,b,c} &= \sum_{p=0}^{b-2} W_{0,0,p}W_{a,b,c-p-2} + \gamma_{13}\sum_{p=0}^{a-1} W_{0,p,0}W_{a-p-1,b,c-1} \\ &+ \gamma_{23}\sum_{p=0}^{b-1} W_{a,p,0}W_{0,b-1-p,c-1} \qquad c \geq 1. \end{aligned} \tag{4.59}$$

We solve these by defining the generating function

$$\phi(t,s,u) = \sum_{a,b,c} W_{a,b,c}t^a s^b u^c \tag{4.60}$$

and by rewriting the recursion relations as equations for $\phi(t,s,u)$,

$$\begin{aligned} 4\Big(\phi(t,s,u)-\phi(0,s,u)\Big) &= t^2\phi(t,0,0)\phi(t,s,u) + \gamma_{12}ts\phi(t,s,0)\phi(0,s,u) \\ &+ \gamma_{13}tu\phi(0,0,u)\phi(t,s,u), \end{aligned} \tag{4.61}$$

$$\begin{aligned} 4\Big(\phi(t,s,u)-\phi(t,0,u)\Big) &= s^2\phi(0,s,0)\phi(t,s,u) + \gamma_{12}ts\phi(t,0,0)\phi(t,s,u) \\ &+ \gamma_{23}su\phi(t,0,u)\phi(0,t,u), \end{aligned} \tag{4.62}$$

$$\begin{aligned} 4\Big(\phi(t,s,u)-\phi(0,s,u)\Big) &= u^2\phi(0,0,u)\phi(t,s,u) + \gamma_{13}tu\phi(0,s,0)\phi(t,s,u) \\ &+ \gamma_{23}su\phi(t,s,0)\phi(0,t,u). \end{aligned} \tag{4.63}$$

We require $\phi(0,0,0) = W_{0,0,0} = 1$ as a normalization condition. Now, choosing appropriate variables to be zero, we can solve these equations. For example in the first relation, setting $s = u = 0$ we get

$$4\phi(t,0,0) - 4 = t^2\phi^2(t,0,0), \tag{4.64}$$

which is the same equation we arrived at in the two-matrix case, with solution

$$\phi(t,0,0) \equiv \phi(t) = \frac{2}{1+\sqrt{1-t^2}}, \tag{4.65}$$

and similarly for the case of $\phi(0,s,0)$ and $\phi(0,0,u)$.

Setting $u = 0$ in (4.61) we get an equation for $\phi(t,s,0)$ that solves again for the formula obtained in the previous section

$$\phi(t,s,0) = \frac{\phi(t)\phi(s)}{1-\frac{1}{4}\gamma_{12}ts\phi(t)\phi(s)}. \tag{4.66}$$

The same quantity could be obtained from (4.62) by setting $u = 0$. Doing this, we would get the same result, which serves as a consistency check. Next, we set $s = 0$ in (4.61) and obtain

$$\phi(t,0,u) = \frac{\phi(t)\phi(u)}{1-\frac{1}{4}\gamma_{13}tu\phi(t)\phi(u)}. \tag{4.67}$$

Again, setting $s = 0$ in (4.63) gives the same formula as a check. Finally $t = 0$ in (4.62) or (4.63) yields

$$\phi(0,s,u) = \frac{\phi(s)\phi(u)}{1-\frac{1}{4}\gamma_{23}su\phi(s)\phi(u)}. \tag{4.68}$$

These three results follow directly from the initial condition we use for W's.

Plugging these into one of the original equations we get the final formula for the generating function

$$\phi(t,s,u) = \frac{\phi(t)\phi(s)\phi(u)}{\Big(1-\frac{1}{4}st\gamma_{12}\phi(s)\phi(t)\Big)\Big(1-\frac{1}{4}tu\gamma_{13}\phi(t)\phi(u)\Big)\Big(1-\frac{1}{4}su\gamma_{23}\phi(s)\phi(u)\Big)} \tag{4.69}$$

With no surprise, this is the formula we obtain from either of the three equations. Applying the inversion procedure from the previous section and summarized in equation (4.39) yields a very complicated formula of the form

$$\rho(x,y,z) = \rho(x)\rho(y)\rho(z) \times \tilde{\rho}_3(x,y,z). \tag{4.70}$$

An explicit formula for $\tilde{\rho}_3(x,y,z)$ is given in the Appendix A.

A couple of important observations can be made. In the case $\gamma_{12,23,13} = 0$, the factor $\tilde{\rho}_3(x,y,z)$ becomes 1 and the matrices are correctly uncorrelated. In the case of two of the three γ's vanishing the factor becomes (if x and y are the two variables for which the correlation does not vanish)

$$\tilde{\rho}_3(x,y,z) = \frac{1-\gamma^2}{(1-\gamma^2)^2 - 4\gamma(1+\gamma^2)xy + 4\gamma^2(x^2+y^2)} \tag{4.71}$$

and we obtain $\rho(x,y,z) = \rho(x,y)\times\rho(z)$. If $\gamma_{12,23,31} \to 1$, we get $\rho(x,y,z) \to \rho(x)\delta(x-y)\delta(x-z)$ and if $\gamma_{12,23} \to -1, \gamma_{23} \to 1$, we get$\rho(x,y,z) \to \rho(x)\delta(x+y)\delta(x+z)$, i.e. fully correlated or anticorrelated distributions, as expected. Finally in the case of $\gamma_{12} = \gamma_{13} = \gamma, \gamma_{23} \to 1$, we obtain $\rho(x,y,z) \to \rho(x,y)\delta(y-z)$.

4.1.4 $MBMB$ joint distribution

To get the four-point joint distribution of matrices $MBMB$, we need to compute the following quantities

$$W_{a,b,c,d} = \frac{1}{N}\left\langle \mathrm{Tr}\left[\left(\frac{M}{2\sqrt{f}}\right)^a \left(\frac{B}{2\sqrt{g}}\right)^b \left(\frac{M}{2\sqrt{f}}\right)^c \left(\frac{B}{2\sqrt{g}}\right)^d\right]\right\rangle. \tag{4.72}$$

Using the same procedure with the explicit Wick contractions of one of the matrices in the M^a group and planarity of the diagrams we find the following large N recursion rule

$$\begin{aligned} 4W_{a,b,c,d} &= \sum_{p=0}^{a-2} W_{p,0,0,0}W_{a-p-2,b,c,d} + \gamma\sum_{p=0}^{b-1} W_{0,p,0,0}W_{a-1,b-p-1,c,d} + \\ &+ \sum_{p=0}^{c-1} W_{0,b,p,0}W_{a-1,0,c-p-1,d} + \gamma\sum_{p=0}^{b-1} W_{a-1,0,0,p}W_{0,b,c,d-p-1} \end{aligned} \tag{4.73}$$

for $a \geq 1$. Introducing the generating function

$$\phi(t,s,u,v) = \sum_{a,b,c,d} W_{a,b,c,d}\; t^a s^b u^c v^d \tag{4.74}$$

this becomes

$$\begin{aligned}
&4\Big(\phi(t,s,u,v) - \phi(0,s,u,v)\Big) = \\
&\quad = t^2\phi(t,0,0,0)\phi(t,s,u,v) + \gamma ts\phi(0,s,0,0)\phi(t,s,u,v) + \\
&\quad +tu\phi(0,s,u,0)\phi(t,0,u,v) + \gamma tv\phi(t,0,0,v)\phi(0,s,u,v).
\end{aligned} \tag{4.75}$$

As before, we can consider contractions of a matrix from other groups of matrices and we get three more equations for the generating function. These are

$$\begin{aligned}
&4\Big(\phi(t,s,u,v) - \phi(t,0,u,v)\Big) = \\
&\quad = s^2\phi(0,s,0,0)\phi(t,s,u,v) + \gamma su\phi(0,0,u,0)\phi(t,s,u,v) + \\
&\quad +sv\phi(0,0,u,v)\phi(t,s,0,v) + \gamma st\phi(t,s,0,0)\phi(t,0,u,v), \qquad (4.76)\\
&4\Big(\phi(t,s,u,v) - \phi(t,s,0,v)\Big) = \\
&\quad = u^2\phi(0,0,u,0)\phi(t,s,u,v) + \gamma uv\phi(0,0,0,v)\phi(t,s,u,v) + \\
&\quad +ut\phi(t,0,0,v)\phi(t,s,u,0) + \gamma us\phi(0,s,u,0)\phi(t,s,0,v), \qquad (4.77)\\
&4\Big(\phi(t,s,u,v) - \phi(t,s,u,0)\Big) = \\
&\quad = v^2\phi(0,0,0,v)\phi(t,s,u,v) + \gamma vt\phi(t,0,0,0)\phi(t,s,u,v) + \\
&\quad +vs\phi(t,s,0,0)\phi(0,s,u,v) + \gamma vu\phi(0,0,u,v)\phi(t,s,u,0). \qquad (4.78)
\end{aligned}$$

We again have $\phi(0,0,0,0) = W_{0,0,0,0} = 1$. Then, choosing appropriate variables to be zero, we can solve these for the generating function pretty much the same way we did in the case of three matrices. The equations for $\phi(t,0,0,0), \phi(t,s,0,0), \phi(t,s,u,0)$ are going to be the very same ones as for $\phi(t,0,0), \phi(t,s,0), \phi(t,s,u)$ and will give the same results. We can make the notation more compact by defining

$$\psi(t,s) = 1 - \frac{1}{4}\gamma_{ts} ts\phi(t)\phi(s), \tag{4.79}$$

where γ_{ts} is the correlation parameter between the matrices corresponding to the variables t and s. This way

$$\phi(\tau,\sigma) = \frac{\phi(\tau)\phi(\sigma)}{\psi(\tau,\sigma)}, \tag{4.80}$$

$$\phi(\tau,\sigma,\rho) = \frac{\phi(\tau)\phi(\sigma)\phi(\rho)}{\psi(\tau,\sigma)\psi(\tau,\rho)\psi(\sigma,\rho)}, \tag{4.81}$$

where τ, σ, ρ are any of t, s, u, v. Finally (4.75) becomes

$$\begin{aligned} \phi(t,s,u,v) &= \frac{\phi(t)\phi(s)\phi(u)\phi(v)}{\psi(t,s)\psi(t,v)\psi(u,v)\psi(s,u)\psi(s,v)\psi(t,u)} \\ &\quad \times \left[1 - \frac{tsuv}{16}\phi(t)\phi(s)\phi(u)\phi(v)\right]. \end{aligned} \tag{4.82}$$

Using the method described in detail in the previous section and summarized in equation (4.39) yields the four-point distribution of the form

$$\rho(x,y,z,w) = \rho(x)\rho(y)\rho(z)\rho(w) \times \tilde{\rho}_4(x,y,z,w), \tag{4.83}$$

with explicit formula for $\tilde{\rho}_4(x,y,z,w)$ given in the Appendix A. Here we just observe that the factor $\tilde{\rho}_4(x,y,z,w)$ is 1 when $\gamma = 0$, which gives two independent distributions. In the case of $\gamma \to 1$, we get $\rho(x,y,z,w) \to \rho(x)\delta(x-y)\delta(z-w)\delta(x-w)$, and in the case of $\gamma \to -1$ we get $\rho(x,y,z,w) \to \rho(x)\delta(x+y)\delta(z+w)\delta(x+w)$, i.e. the variables correctly correlate or anti-correlate.

4.1.5 Explicit formulas for f, g, h, γ and large N scalings

Scaling and special cases

As we mentioned earlier, our approach can be used in a more general case than just the Laplacian kinetic term. Therefore, let us consider the case of a kinetic term proportional to l^α for large values of l, i.e. $\mathcal{K}T_A^l \stackrel{l\to\infty}{\to} l^\alpha T_A^l$. Also, let us assume that $\alpha \geq 0$. Going back to the definition of the two point functions f, h, g, it is straightforward to see that in the large N limit

$$\begin{aligned} h &\sim \frac{1}{N}\sum_{l=0}^{N-1}\frac{l\,l^\alpha}{l^\alpha} \sim N, \\ g &\sim \frac{1}{N}\sum_{l=0}^{N-1}\frac{l\,l^{2\alpha}}{l^\alpha} \sim N^{1+\alpha}. \end{aligned} \tag{4.84}$$

Both of these sums diverge in the limit of large N. However, depending on the value of α, the sum in f can be convergent, when dominated by the contribution of small values of l. We thus find out

$$\begin{aligned} \alpha < 2 \quad f &\sim \frac{1}{N}\sum_{l=0}^{N-1} l^{1-\alpha} = N^{1-\alpha}, \\ \alpha > 2 \quad f &\sim N^{-1}. \end{aligned} \tag{4.85}$$

From the definition of γ, we see that for $\alpha < 2$ the correlation is finite. For $\alpha > 2$, the correlation is proportional to $N^{1-\alpha/2}$ and thus vanishes.

The same analysis holds also for the case of the $\mathbb{C}P_F^n$. In this case, the value of α for which the behavior of f changes becomes $\alpha = 2n$.

The case of $\alpha = 2$, which includes the usual Laplacian kinetic term requires special treatment. But before that, let us repeat this procedure for correlations of the matrix M with its matrix Lapalcian $[L_\alpha, [L_\alpha, M]]$ in a theory with a general kinetic term.

We will denote the corresponding quantities relating to the matrix Laplacian and not to the matrix B by a prime. We will take $G(l)$ to go like l^β for large values of l, we see that

$$\begin{aligned} f &\sim \frac{1}{N}\sum(2l+1)\,G(l) \sim \frac{1}{N}\sum l^{\beta+1} \\ g' &\sim \frac{1}{N}\sum l^{\beta+5} \\ h' &\sim \frac{1}{N}\sum l^{\beta+3} \end{aligned} \tag{4.86}$$

There are four possible cases.

- $\beta > -2$:
  In this case $f \sim N^{\beta+1}$, $g' \sim N^{\beta+5}$, $h' \sim N^{\beta+3}$ and so, $\gamma \sim N^0 \sim 1$.
- $-4 < \beta < -2$:
  Here $f \sim N^{-1}$, $g' \sim N^{\beta+5}$, $h' \sim N^{\beta+3}$ and so, $\gamma' \sim N^{\frac{1}{2}(\beta+2)} \to 0$.
- $-6 < \beta < -4$:
  $f \sim N^{-1}$, $g' \sim N^{\beta+5}$, $h' \sim N^{-1}, \gamma' \sim N^{-\frac{1}{2}(\beta+6)} \to 0$.
- $\beta < -6$:
  In this case, $f \sim N^{-1}$, $g' \sim N^{-1}$, $h' \sim N^{-1}$ and hence, $\gamma' \sim 1$.

For $\beta > -2$ we get explicitly

$$\gamma' = \frac{\sqrt{(\beta+2)(\beta+6)}}{\beta+4}. \tag{4.87}$$

The case of no kinetic term, $\beta = 0$ gives $\gamma = \sqrt{3}/2$. The case of $\beta = -2$ is the case of the Laplacian kinetic term, which we will treat more closely.

The Laplacian kinetic term

We will explicitly compute the two-point functions in the case of the Laplacian kinetic term, i.e. $G(l) = 1/(l(l+1)+\mu^2)$. The MM correlator is given by

$$f = \frac{1}{N}\langle \mathrm{Tr}\,(MM)\rangle = \frac{1}{N}\sum_{l=0}^{N-1}\frac{2l+1}{l(l+1)+\mu^2} \approx \int_0^1 dx \frac{2Nx+1}{Nx(Nx+1)+\mu^2} \tag{4.88}$$

where we have introduced $x = l/N$ and changed the summation to integration $\frac{1}{N}\sum_l \to \int dx$. This integral is now easy to compute and we obtain

$$f = \frac{2}{N}\log\left(\frac{N}{\mu}\right), \tag{4.89}$$

where we have included only the leading large N term and the first term to include μ. Similarly for the other two correlators we obtain

$$g = \frac{N^3}{2} - N\mu^2, \tag{4.90}$$
$$h = N - \frac{2\mu^2}{N}\log\left(\frac{N}{\mu}\right). \tag{4.91}$$

This yields

$$\gamma = \frac{1 - \frac{2\mu^2}{N^2}\log\left(\frac{N}{\mu}\right)}{\sqrt{\log\left(\frac{N}{\mu}\right)}} \approx \frac{1}{\sqrt{\log N}}. \tag{4.92}$$

The same is true for γ' in the cae of $\beta = -6$.

The massless case

The action describing a massless scalar on the fuzzy $\mathbb{C}P^n_F$, $\mu = 0$ in the action (4.2), merits special attention. This is the critical case $\beta = -2$ in the scaling of $G(l)$ at large l, corresponding to a very weak vanishing of $\gamma \sim (\log N)^{-1/2}$ at the large N limit computed above. Further, for $\mu = 0$ the constant mode of the field (the trace of M) drops out of the action and must be eliminated from the calculation.

In fact, taking the limit $\mu \to 0$ in this case is somewhat nontrivial, as we can see from (4.89) and for $\mu \sim 1/\log N$ or smaller we obtain in the large N limit

$$f = \frac{1}{N}(\mu^{-1} + 2\log N) \ , \quad g = \frac{1}{2}N^3 \ , \quad h = N \tag{4.93}$$

which leads to a Wigner radius of $2\sqrt{f} = 2\sqrt{(\mu^{-1} + 2\log N)/N}$ for the eigenvalue distribution of M. This not only diverges as μ goes to zero, but is also misleading: for such low values of μ the planarity property of matrix expectation values fails, since the trace part of M contributes to the same order (or higher) than the traceless part and arises in all diagrams (planar and nonplanar). Such contributions give rise to a Gaussian, rather than Wigner, distribution. To understand this better, we decompose M in its trace part c_0 and its traceless ($l \neq 0$) part $\tilde{M}$

$$M = \frac{c_0}{\sqrt{N}} + \tilde{M} = \frac{c_0}{\sqrt{N}} + \sum_{l>0,\, A} c_A^l \, T_A^{(l)} \tag{4.94}$$

The trace and traceless parts decouple. The eigenvalues of $\tilde{M}$ have a Wigner distribution at large N with radius $2\sqrt{2\log N/N}$, arising from f with $l = 0$ dropped, while the single mode of the trace part contributes a shift distributed as a Gaussian with spread $1/\sqrt{N\mu}$. Since the two distributions are independent, the total eigenvalue distribution will be given by their convolution. For $\mu \gg 1/\log N$ the spread of the Gaussian is much smaller than the Wigner radius and the convolution essentially gives back the Wigner. For $\mu \ll 1/\log N$, on the other hand, the Gaussian dominates. For $\mu \sim 1/\log N$ we get an intermediate distribution.

In order to obtain the massless result, therefore, we have to omit the trace part of the matrix (which has vanishing action) and substitute $\tilde{M}$ for M. The result is again a Wigner semicircle of radius $\sqrt{8\log N/N}$ for $\tilde{M}$, correlated weakly ($\gamma \sim (\log N)^{-1/2}$) with a Wigner semicircle for B of radius $2\sqrt{g} = \sqrt{2N^3}$.

Mass rescaling and the correlation γ

In the original scaling, the mass term did not play any role in the value of γ, since the large N limit in the sums (4.84) washed away the effect of μ^2 in the denominator. The same happened for the case $\alpha \leq 2$ also for the two point function f, which was independent of μ^2 in the large N limit.

However, there is a different scaling of the mass terms of the action and upon a rescaling of μ^2, the correlation γ is always finite and depends also on μ^2.

First, let us see how this works for a Laplacian kinetic term, i.e. $\alpha = 2$. If we rescale the mass $\mu^2 \to N^2\tilde{\mu}^2$, both terms are going to contribute in the large N limit. Namely, if we repeat the calculation (4.88) we obtain

$$f = \frac{1}{N} \log\left(1 + \frac{1}{\tilde{\mu}^2}\right) , \tag{4.95}$$

where we have kept only the leading-order contribution. Similar calculation then yields also

$$h = N\left[1 - \tilde{\mu}^2 \log\left(1 + \frac{1}{\tilde{\mu}^2}\right)\right] , \tag{4.96}$$

$$g = N^3\left[\frac{1}{2} - \tilde{\mu}^2 + \tilde{\mu}^4 \log\left(1 + \frac{1}{\tilde{\mu}^2}\right)\right] \tag{4.97}$$

and finally finite

$$\gamma = \frac{\left[1 - \tilde{\mu}^2 \log\left(1 + \frac{1}{\tilde{\mu}^2}\right)\right]}{\sqrt{\log\left(1 + \frac{1}{\tilde{\mu}^2}\right)\left[\frac{1}{2} - \tilde{\mu}^2 + \tilde{\mu}^4 \log\left(1 + \frac{1}{\tilde{\mu}^2}\right)\right]}} . \tag{4.98}$$

Note that this is finite in the limit of very large $\tilde{\mu}$ and tends to $\sqrt{3}/2$. This is the value for correlation between the eigenvalues of matrix Laplacian of M and M itself in the case of no kinetic term and is the same as we have obtained before from formula (4.87).

From the Euler-Maclaurin formula

$$\begin{aligned}\sum_{l=0}^{N-1} f(l) &= \int_0^{N-1} f(x)dx + \frac{1}{2}\Big(f(N-1) - f(0)\Big) \\ &+ \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!}\left[f^{(2p-1)}(N-1) - f^{(2p-1)}(0)\right], \end{aligned} \tag{4.99}$$

where B_{2p} are the Bernoulli numbers, we see that all the subleading terms come with negative power of μ^2 and thus the scaling $\mu^2 \to N^2\tilde{\mu}^2$ does not make any of them relevant.

This procedure is going to work for any $\alpha > 2$. We rescale $\mu^2 \to \tilde{\mu}^2 N^\alpha$ and obtain

$$\begin{aligned} f &= \frac{N}{N^{1+\alpha}} N \int_0^1 dx\, \frac{2x}{x^\alpha + \tilde{\mu}^2} = N^{1-\alpha}\frac{1}{\tilde{\mu}^2}\ {}_2F_1\left(\frac{2}{\alpha}, 1, \frac{2}{\alpha}+1, -\frac{1}{\mu^2}\right), \\ g &= \frac{N^{1+\alpha}}{N^{1+\alpha}} N \int_0^1 dx\, \frac{2x^{1+\alpha}}{x^\alpha + \tilde{\mu}^2} = N\left[1 - {}_2F_1\left(\frac{2}{\alpha}, 1, \frac{2}{\alpha}+1, -\frac{1}{\mu^2}\right)\right], \\ h &= \frac{N^{1+2\alpha}}{N^{1+\alpha}} N \int_0^1 dx\, \frac{2x^{1+2\alpha}}{x^\alpha + \tilde{\mu}^2} \\ &= N^{1+\alpha}\left[\frac{2}{2+\alpha} - \tilde{\mu}^2 + \tilde{\mu}^2\ {}_2F_1\left(\frac{2}{\alpha}, 1, \frac{2}{\alpha}+1, -\frac{1}{\mu^2}\right)\right]. \end{aligned} \tag{4.100}$$

These clearly give a finite γ, but now with a contribution form the mass term also. We compute f, g, h for the $\mathbb{C}P_F^n$ theory in section 5.3.

One could now consider also the correlations of the matrix Laplacian of M and M under this rescaling. As for the case of the Laplacian kinetic term, also for $\beta = -6$ a logarithmic vanishing of γ' changes to a finite value in the large N limit. For the intermediate cases of $\beta \in (-6, -2)$ no rescaling is possible to keep the correlation finite.

4.2 Interacting theory

We are now ready to introduce the interaction to the free action (4.2). We will consider a quartic interaction potential

$$S_{int} = \tilde{g}\ \mathrm{Tr}\left(M^4\right) \quad , \quad \tilde{g} = g/N. \tag{4.101}$$

The factor of N in the definition of the coupling constant will become clear shortly, but the motivation is evident from the matrix measure (3.3) and occurs after rescaling M to absorb the overall N^2 factor. As mentioned in the introduction, the case without the kinetic term is well known [68, 71] and the result is a polynomial correction to the Wigner semicircle distribution.

Expanding the interaction part in power series in $\tilde{g}$ yields for an average of an observable $\mathcal{O}(M)$,

$$\begin{aligned} \langle \mathcal{O} \rangle &= \frac{1}{Z}\sum_{a=0}^{\infty}\frac{(-\tilde{g})^a}{a!}\int dM e^{-S_0(M)}\mathcal{O}(M)\left[\mathrm{Tr}\left(M^4\right)\right]^a = \\ &= \frac{1}{Z}\sum_{a=0}^{\infty}\frac{(-\tilde{g})^a}{a!}\left\langle \mathcal{O}(M)\left[\mathrm{Tr}\left(M^4\right)\right]^a\right\rangle, \end{aligned} \tag{4.102}$$

where

$$\begin{aligned} Z &= \sum_{a=0}^{\infty}\frac{(-\tilde{g})^a}{a!}\int dM e^{-S_0(M)}\left[\mathrm{Tr}\left(M^4\right)\right]^a \\ &= \sum_{a=0}^{\infty}\frac{(-\tilde{g})^a}{a!}\left\langle\left[\mathrm{Tr}\left(M^4\right)\right]^a\right\rangle. \end{aligned} \tag{4.103}$$

We see that evaluating the average in the interacting theory can be done using averages of the free theory. We just need to pick correct diagrams that contribute to the expectation value on the rhs of (4.102). The diagrams containing vacuum bubbles, i.e. parts, where some of the vertexes from $\left[\mathrm{Tr}\left(M^4\right)\right]^a$ contract only among themselves, will be canceled by the $1/Z$ factor. Therefore we can write

$$\langle \mathcal{O} \rangle = \sum_{a=0}^{\infty}\frac{(-\tilde{g})^a}{a!}\left\langle \mathcal{O}(M)\left[\mathrm{Tr}\left(M^4\right)\right]^a\right\rangle_{0,con}, \tag{4.104}$$

where the subscript will indicate that we consider only diagrams that do not contain disconnected vacuum bubbles and that the contractions are to be taken using the free theory measure.

Now we also see the motivation for the $1/N$ factor in the definition of the coupling constant $\tilde{g}$. Since each trace in $\left\langle \mathcal{O}(M) \left[\mathrm{Tr}\left(M^4\right)\right]^a \right\rangle$ raises large N dependence of this expression by one power of N, the different expectation values in the sum (4.104) are going to be all of the same order, given by $\mathcal{O}(M)$. This means that the terms of the sum will contribute in the large N limit.

Before we proceed with computation of the eigenvalue distribution of M, let us stress one point. The contractions in (4.104) are done using the free measure. But we have already seen that in the free theory, the kinetic term only rescaled the radius of the original distribution. And therefore we expect the same in the interacting case, namely that all the distributions of the M^4-theory with no kinetic term will survive also in the full theory, only with a rescaled variable and possibly rescaled coupling g.

4.2.1 Eigenvalue distribution of the matrix M

As in the previous section, to obtain the eigenvalue distribution of M, we will first compute the moment generating function

$$\phi_1(t) = \sum_{m=0}^{\infty} t^m F_m = \sum_{m=0}^{\infty} t^{2m} F_{2m}. \tag{4.105}$$

where F_m are properly rescaled moments $\left\langle \mathrm{Tr}\left(M^{2m}\right)\right\rangle$. The potential is even and therefore odd moments vanish.

From (4.104) we see that we need to investigate quantities of the form

$$F_a^{2m} = \frac{1}{N^{1+a}} \left\langle \mathrm{Tr}\left[\left(\frac{M}{\sqrt{f}}\right)^{2m}\right] \mathrm{Tr}\left[\left(\frac{M}{\sqrt{f}}\right)^{4}\right] \overset{a \text{ times}}{\cdots\cdots\cdots} \mathrm{Tr}\left[\left(\frac{M}{\sqrt{f}}\right)^{4}\right] \right\rangle_{0,con} . \tag{4.106}$$

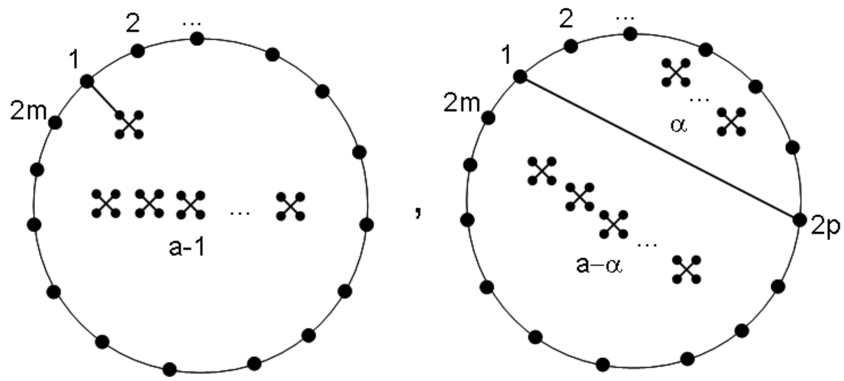

Figure 4.2: Two types of terms contributing to (4.108).

Expressions

$$F_{2m} = \sum_{a=0}^{\infty} \frac{(-g)^a}{a!} F_a^{2m} \tag{4.107}$$

are then going to be finite and will give $2m$-point correlators of M's in the interacting theory. Let us stress again that in this section, recursion rules (4.14,4.29,4.30) for the free correlators change a little, due to a different normalization of the distribution of M.

We will now write down the recursion rules for F_a^{2m} in a very similar way we did in previous cases. The contributing planar diagrams are going to be of the form of $2m$ points on a circle with four-point vertexes inside this circle. The number of vertexes is a, and the points connect with the vertexes with none of the lines intersecting. If we look at one point on the circle, there are two different types of contractions we can make.

- First is with a vertex, then the three legs of the vertex become effectively three new points on the circle, which now has $2m + 2$ points, but there are only $a - 1$ vertexes left and there are $4a$ different legs that we can connect the first point to. This contraction gives the first line of (4.108).

- The second type of contraction is with a different point on the circle, then the circle splits into two circles, each new circle can have different number of vertexes in it - one circle α and the other $a-\alpha$, with a proper combinatorial factor. These contractions give the second line of (4.108).

The two types of contractions are illustrated in the Figure 4.2. We need to be a little careful. If there are no points on any circle to connect with, we can not have any vertexes in this circle, as this would produce a disconnected diagram. Therefore we need to set $F_a^0 = 0$ for any $a \neq 0$ and $F_0^0 = 1$.

Summing over all possible contractions, we obtain

$$\begin{aligned} \left\langle \mathrm{Tr}\left(M^{2m}\right)\left[\mathrm{Tr}\left(M^4\right)\right]^a\right\rangle = 4afN\left\langle \mathrm{Tr}\left(M^{2m+2}\right)\left[\mathrm{Tr}\left(M^4\right)\right]^{a-1}\right\rangle + \\ +\frac{f}{N}\sum_{p=1}^{m}\sum_{\alpha=0}^{a}\binom{a}{\alpha}\left\langle \mathrm{Tr}\left(M^{2(p-1)}\right)\left[\mathrm{Tr}\left(M^4\right)\right]^{\alpha}\right\rangle\left\langle \mathrm{Tr}\left(M^{2(m-p)}\right)\left[\mathrm{Tr}\left(M^4\right)\right]^{a-\alpha}\right\rangle \end{aligned} \tag{4.108}$$

and using the definition (4.106) this becomes

$$F_a^{2m} = 4aF_{a-1}^{2m+2} + \sum_{p=1}^{m}\sum_{\alpha=0}^{a}\frac{a!}{\alpha!(a-\alpha)!}F_\alpha^{2(p-1)}F_{a-\alpha}^{2(m-p)}. \tag{4.109}$$

Immediately we note that if we define $\tilde{F}_a^{2m} = F_a^{2m}/a!$, this simplifies into

$$\tilde{F}_a^{2m} = 4\tilde{F}_{a-1}^{2(m+1)} + \sum_{p=0}^{m-1}\sum_{\alpha=0}^{a}\tilde{F}_\alpha^{2p}\tilde{F}_{a-\alpha}^{2(m-1-p)}. \tag{4.110}$$

Multiply the recurrence relation by $(-g)^a$ and sum over $a = 1$ to ∞ to obtain

$$F_{2m} - F_0^{2m} = -4gF_{2(m+1)} + \sum_{p=0}^{m-1}\left[F_{2p}F_{2(m-1-p)} - F_0^{2p}F_0^{2(m-1-p)}\right]. \tag{4.111}$$

As a consequence of the free case recursion rules (4.14), the terms involving

F_0's will cancel. This is a result of the initial condition and similar terms will also cancel for this reason later. We, therefore, get

$$F_{2(m+1)} = \frac{1}{4g}\left[\sum_{p=0}^{m-1} F_{2p}F_{2(m-1-p)} - F_{2m}\right]. \tag{4.112}$$

This expression holds however only for $m \geq 1$, so we need to specify F_0 and F_2, which enter here as an initial condition and need to be extracted directly from (4.110). From the definition of F_a^0, it is clear that $F_0 = 1$, consistent with the normalization of the distribution. F_2 is essentially the planar dressed propagator of the interacting theory and for a while let us go further without specifying it.

We multiply the previous formula by $t^{2(m+1)}$ and sum over $m = 1$ to ∞ to obtain

$$\phi_1(t) - 1 - t^2 F_2 = \frac{1}{4g}\left[t^4\phi_1^2(t) - t^2\big(\phi_1(t) - 1\big)\right] \tag{4.113}$$

or

$$t^4\phi_1^2 - \left(4g + t^2\right)\phi_1 + \left(4g + t^2 + 4gt^2F_2\right) = 0, \tag{4.114}$$

which gives

$$\phi_1(t) = \frac{4g + t^2 - \sqrt{(4g+t^2)^2 - 4t^4(4g+t^2) - 16F_2gt^6}}{2t^4}. \tag{4.115}$$

So we are left to specify the two-point function F_2.

Going back to (4.110), we can use the recursion rule to generate a number of $\tilde{F}_a^{2m}$. After some trial and error using the integer factorization of the terms we can guess the formula for $\tilde{F}_a^{2m}$ in the following form

$$\tilde{F}_a^{2m} = 12^a\left(\frac{(2m)!}{m!(m-1)!}\right)\left(\frac{(m+2a-1)!}{(m+a+1)!a!}\right). \tag{4.116}$$

Not surprisingly, this is indeed the formula of g expansion of the $2m$-point correlator given in the appendix of [71] and this will be taken as a motivation for using the following expression for F_2

$$F_2 = \sum_{a=0}^{\infty} 12^a 2 \frac{(2a)!}{(a+2)!a!} = \frac{(1+48g)^{3/2} - 1 - 72g}{864g} = \frac{2}{3} a^2 (4 - a^2), \quad (4.117)$$

where $a^2 = (\sqrt{1+48g} - 1)/24g$. It would be interesting to see whether one can extract F_2 in a closed form directly from the recursion rule (4.110) without solving for $\tilde{F}_a^{2m}$. Or one could try to prove that (4.116) solves the recursion rule (4.110) by explicit computation or some inductive method. We have attempted this, but the problem is more complicated and we will proceed with assumption (4.117).

We are taking F_2 to be the expression obtained by the standard methods [71]. It might seem that we are assuming something we are trying to prove, but this is not the case. Motivated by the behavior of the first terms of the recurrence, we assume that the initial condition we are about to use in (4.112) is the same as for the case of no kinetic term. We then compute the generating function $\phi_1(t)$ that is given by the moments (4.112), and if it turns out to be the generating function of [71], we conclude that this assumption leads to all the correlators being of the form as in the theory with no kinetic term.

We plug the formula (4.117) for F_2 into the the generating function (4.115), which after some algebra can be brought into the form

$$\phi_1(t) = \frac{t^2 + 4g}{2t^4} - \frac{1}{t^2} \left(\frac{1}{2} + 4ga^2 + \frac{2g}{t^2} \right) \sqrt{1 - 4a^2 t^2}. \quad (4.118)$$

This is the standard generating function for the distribution of M with no kinetic term and $\mathrm{Tr}\left(M^4\right)$ interaction obtained in section 3.2. The discussion goes along those lines, defining the resolvent and using the discontinuity equation. In the end, we obtain the distribution

$$\rho_1(x) = \frac{1}{\pi} \left(\frac{1}{2} + 4ga^2 + 2gx^2 \right) \sqrt{4a^2 - x^2}. \quad (4.119)$$

This is the polynomial deformation to the Wigner semicircle distribution, with the radius $2a$. Looking back at (4.106), the variable in the case of the unscaled matrix is $x \to x/\sqrt{f}$ and we need to replace $g \to f^2 g$. To see this better, we should carry the explicit factors of f in the calculation. Such calculation would yield an extra factor of f^{m+2a} in (4.116). From the definition of $\phi_1(t)$, we can see that f^m rescales t by $\sqrt{f}$, which in turn scales x by $1/\sqrt{f}$ and from (4.107) we can see that f^{2a} rescales g by f^2. The final formula is

$$\rho_1(x) = \frac{1}{\pi}\left(\frac{1}{2f} + 4gfa^2 + 2gx^2\right)\sqrt{4a^2 f - x^2}. \tag{4.120}$$

Therefore starting from the recurrence relation (4.109) we have been able to recover the distribution of eigenvalues of the random matrix ensemble with weight (4.101). In the next section, we will generalize this approach to different observables of the interacting theory.

As mentioned in the introduction, the authors of [8] have computed the same distribution treating the kinetic part of the action as a perturbation. After using the explicit formula (4.95) for f, the expression (4.120) reduces to their result. To see this, one has to introduce a parameter ε in front of the kinetic term, do the expansion of f in powers of ε and take $\varepsilon = 1$ at the end of the calculation. We will see this explicitly in section 5.4.

4.2.2 Eigenvalue distribution of the matrix B and the joint MB distribution

As in the free case, the theory now includes new observables involving the matrix $B = \mathcal{K}M$. In the free case, this matrix followed the Wigner semicircle distribution, as did the underlying matrix M. Now, the situation is going to be different, since contractions of B with matrix M in the interaction vertex are going to turn even pure B correlators into mixed MB ones.

We will discuss the distribution of eigenvalues of B and the joint distribution for M and B and as in section 4.1.2, we will derive the recursion rules for the two generating functions

$$\phi_2(s) = \sum s^{2n} G_{2n}, \tag{4.121}$$

$$\phi(t,s) = \sum_{m,b} t^m s^b W_{m,b} \tag{4.122}$$

and

$$G_{2b} = \sum_{a=0}^{\infty} \frac{(-g)^a}{a!} G_a^{2b} = \sum_{a=0}^{\infty} (-g)^a \tilde{G}_a^{2b} \tag{4.123}$$

$$W_{m,b} = \sum_{a=0}^{\infty} \frac{(-g)^a}{a!} W_a^{m,b} = \sum_{a=0}^{\infty} (-g)^a \tilde{W}_a^{m,b} \tag{4.124}$$

are the moments of the distributions, with

$$G_a^{2b} = \frac{1}{N^{1+a}} \left\langle \mathrm{Tr}\left[\left(\frac{B}{\sqrt{g}}\right)^{2b}\right] \mathrm{Tr}\left[\left(\frac{M}{\sqrt{f}}\right)^{4}\right] \overset{a \text{ times}}{\cdots\cdots} \mathrm{Tr}\left[\left(\frac{M}{\sqrt{f}}\right)^{4}\right] \right\rangle_{0,con}, \tag{4.125}$$

$$W_a^{m,b} = \frac{1}{N^{1+a}} \left\langle \mathrm{Tr}\left[\left(\frac{M}{\sqrt{f}}\right)^{m} \left(\frac{B}{\sqrt{g}}\right)^{b}\right] \mathrm{Tr}\left[\left(\frac{M}{\sqrt{f}}\right)^{4}\right] \overset{a \text{ times}}{\cdots\cdots} \mathrm{Tr}\left[\left(\frac{M}{\sqrt{f}}\right)^{4}\right] \right\rangle_{0,con}. \tag{4.126}$$

Using the same approach as before, it is now quite easy to write down the recurrence rule for $\tilde{G}_a^{2m}$

$$\tilde{G}_a^{2b} = 4\gamma \tilde{W}_{a-1}^{3,2b-1} + \sum_{p=0}^{b-1} \sum_{\alpha=0}^{a} \tilde{G}_\alpha^{2p} \tilde{G}_{a-\alpha}^{2(b-1-p)}, \tag{4.127}$$

where the first term comes from the contraction of B with a vertex and the second term from the contraction of B with a different matrix B. This holds only for $b \geq 1$. It is also not too difficult to write down the recursion rules for $\tilde{W}$'s. Here, we can again obtain two different recursions, considering contraction of the first M matrix or the first B matrix. These two are

$$\tilde{W}_a^{m,b} = \sum_{p=0}^{m-2} \sum_{\alpha=0}^{a} \tilde{F}_\alpha^p \tilde{W}_{a-\alpha}^{m-p-2,b} + \gamma \sum_{p=0}^{b-1} \sum_{\alpha=0}^{a} \tilde{G}_\alpha^p \tilde{W}_{a-\alpha}^{m-1,b-1-p} + 4\tilde{W}_{a-1}^{m+2,b} \quad (4.128)$$

holding for $m \geq 1, a \geq 1$ and

$$\tilde{W}_a^{m,b} = \sum_{p=0}^{b-2} \sum_{\alpha=0}^{a} \tilde{G}_\alpha^p \tilde{W}_{a-\alpha}^{m,b-p-2} + \gamma \sum_{p=0}^{m-1} \sum_{\alpha=0}^{a} \tilde{F}_\alpha^p \tilde{W}_{a-\alpha}^{m-1-p,b-1} + 4\gamma \tilde{W}_{a-1}^{m+3,b-1} \quad (4.129)$$

for $b \geq 1, a \geq 1$. Note that for $b = 0$ in the first and $m = 0$ in the second equation we recover the recursion rules for $\tilde{F}$'s and $\tilde{G}$'s respectively. Also in these expressions, $\tilde{W}_0^{m,b}$ are treated as initial values, given by the recursion rules of the free case (4.294.30).

Setting $m = 1$ in the first of the recursion, we get

$$4\gamma \tilde{W}_{a-1}^{3,b} = \gamma \tilde{W}_a^{1,2b-1} - \gamma^2 \sum_{k=0}^{b-1} \sum_{\alpha=0}^{a} \tilde{G}_\alpha^{2k} \tilde{G}_{a-\alpha}^{2(b-1-k)}, \quad (4.130)$$

where we have used the fact that $W_a^{0,b} \equiv G_a^b$ is nonzero only for even b. Using this in the the recursion rule for $\tilde{G}_a^{2b}$ (4.127) we find

$$\tilde{G}_a^{2b} = \gamma \tilde{W}_a^{1,2b-1} + (1-\gamma^2) \sum_{p=0}^{b-1} \sum_{\alpha=0}^{a} \tilde{G}_\alpha^{2p} \tilde{G}_{a-\alpha}^{2(b-1-p)}. \quad (4.131)$$

Multiplying this equation by $(-g)^a$ and summing over all a yields

$$G_{2b} = \gamma W_{1,2b-1} + (1-\gamma^2) \sum_{p=0}^{b-1} G_{2p} G_{2(b-1-p)}. \quad (4.132)$$

Note that for the cancellation of the "0" terms, we had to use $W_0^{1,2b-1} =$

γG_0^{2b}, which is going to be discussed shortly.

$$\begin{aligned} \phi_2(s) - 1 &= (1-\gamma^2)s^2\phi_2^2(s) + \gamma s \left[\sum_{b=1}^{\infty} W_{1,2b-1}s^{2b-1}\right] = \\ &= (1-\gamma^2)s^2\phi_2^2(s) + \gamma s \underbrace{\left[\sum_{b=0}^{\infty} W_{1,b}s^{b}\right]}_{W_1(s)}, \end{aligned} \tag{4.133}$$

where we have used the fact that $W_{1,b}$ vanishes for even b. We see that this is very different from the equation for $\phi_1(t)$, as expected due to the different role of M and B in the interaction. The solution is given by

$$\phi_2(s) = \frac{1 - \sqrt{1 - 4s^2(1-\gamma^2)(1+\gamma s W_1)}}{2(1-\gamma^2)s^2}. \tag{4.134}$$

From the condition $\phi_2 \to \phi_0$ in the limit of $g \to 0$, we see that in this limit one has to have $W_1 \to \gamma s \phi_0^2$. We will prove that this is indeed the case in the next section. Note that $W_0^{1,2b-1} = \gamma G_0^{2b}$, used to get (4.130) is a consequence of this, see the text after (4.141).

In the following, we will drop the argument of ϕ_1 and ϕ_2. It will be understood that the former is always function of t and the latter a function of s.

From the recursion rules for $\tilde{W}$'s, we can derive an equation for the generating function of the two-point distribution $\phi(t,s)$. To do this, we express (4.128) as

$$\begin{aligned} W_{m,b} - W_0^{m,b} &= \sum_{p=0}^{m-2}\left[F_p W_{m-p-2,b} - F_0^p W_0^{m-p-2,b}\right] \\ &+ \gamma \sum_{p=0}^{b-1}\left[G_p W_{m-1,b-1-p} - G_0^p W_0^{m-1,b-p-1}\right] \\ &- 4g W_{m+2,b}. \end{aligned} \tag{4.135}$$

The terms with a subscript 0 will cancel, since they follow the relations for the free quantities. Continuing the procedure we arrive at

$$\begin{aligned}\phi(t,s)-\phi_2 &= t^2\phi_1\phi(t,s)+\gamma ts\phi_2\phi(t,s)\\ &- \frac{4g}{t^2}\left[\phi(t,s)-\phi_2-tW_1(s)-t^2W_2(s)\right], \quad (4.136)\end{aligned}$$

where we have denoted $W_2(s)=\sum_{b=0}^{\infty}W_{2,b}s^b$. A similar equation can be derived from the second recurrence rule

$$\begin{aligned}\phi(t,s)-\phi_1 &= s^2\phi_2\phi(t,s)+\gamma ts\phi_1\phi(t,s)\\ &- \frac{4g\gamma s}{t^3}\left[\phi(t,s)-\phi_2-tW_1(s)-t^2W_2(s)\right]. \quad (4.137)\end{aligned}$$

Before, we have used multiple equations for the generating functions as a consistency check. At this point, we trust the procedure enough to use these as independent equations and to eliminate one of the unknown functions. Doing this and using the equation (4.133), we get the final formula for $\phi(t,s)$,

$$\phi(t,s)=\frac{\phi_2\Big(t\phi_1-\gamma s\phi_2\Big)}{t-\gamma s\phi_2+\gamma stW_1}. \quad (4.138)$$

So to compute $\phi_2(s)$ and $\phi(t,s)$ we need to specify the function $W_1(s)$.

Let us stress here that the coefficient γ in these expressions is the very same correlation as in expressions of section 4.1. This shows that even though we consider different matrix ensemble, it still knows about the underlying fuzzy sphere it was built on, which is therefore encoded solely in the kinetic term of (4.2).

4.2.3 Leading order $W_1(s)$ in γ

In the case of $\tilde{F}_a^{2m}$, we have been able to guess the solution of the recursion rule. However for the case of $\tilde{W}_a^{1,2b+1}$ the situation is more involved and no easy guess is possible. Moreover, simple analysis of the integer factorization of the first couple of terms shows that such a guess might be very difficult and the explicit formula more complicated than a simple product of factorials.

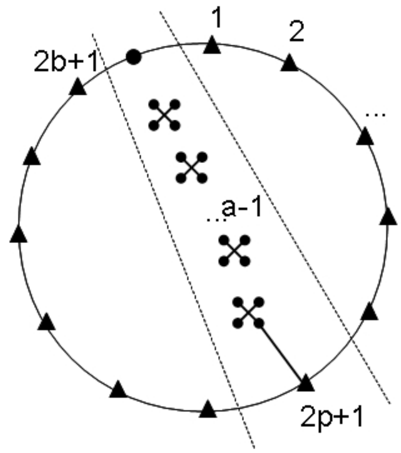

Figure 4.3: Triangles represent matrices B. With only one MB contraction, the diagram splits into two parts of matrices B and contractions among M's. There are no contractions of B's with the lone matrix M on the circle, since then we would have a vacuum bubble or another factor of γ.

Therefore, we will compute $\tilde{W}_a^{1,2b+1}$ only in the leading order in γ to get the first nontrivial contribution due to the MB contraction. In such case, the contributing diagrams have only one MB contraction, leading to one factor of γ. Let us stress that the results we will obtain are still exact in g, and we make no assumption about the magnitude of the coupling.

As the Figure 4.3 illustrates, we get the following expression for the diagrams with one MB contraction

$$\tilde{W}_a^{1,2b+1} = 4\gamma \tilde{W}_{a-1}^{4,0} \sum_{p=0}^{b} c_p c_{b-p} = \gamma \tilde{F}_a^2 c_{b+1}, \tag{4.139}$$

where we have used the recursion rule (4.110). This holds for $a \geq 1$. For $a = 0$ we simply have $W_0^{1,2b+1} = \gamma \sum c_p c_{b-p} = \gamma c_{b+1} = \gamma F_0^2 c_{b+1}$. This then yields

$$W_{1,2b+1} = \sum_{a=0}^{\infty} (-g)^a W_a^{1,2b+1} = \gamma c_{b+1} F_2 \tag{4.140}$$

and

$$W_1 = \sum_{b=0}^{\infty} s^b W_{1,b} = \sum_{b=0}^{\infty} s^{2b+1} W_{1,2b+1} = \gamma F_2 \frac{\phi_0(s) - 1}{s} = \gamma s \phi_0^2(s) F_2. \quad (4.141)$$

In the last step, we have used the equation for the generating function of the Wigner distribution $\phi_0(s) - s^2\phi_0^2(s) = 1$. Note that in the free theory case, there are no vertexes to contract with and the first order in γ is the full formula. In this limit $F_2 \to 1$, and we get the advertised behavior of W_1 in the free field case used in the previous section. We use this result in the equation (4.134) to obtain the leading contribution to the distribution ϕ_2,

$$\begin{aligned} \phi_2(s) &= \phi_0(s) + \gamma^2 \phi_0(s) \frac{1 - \phi_0(s) + s^2\phi_0^2(s) F_2}{2 - \phi_0(s)} \\ &= \phi_0(s) + \gamma^2 (F_2 - 1)\phi_0(s) \frac{s^2\phi_0^2(s)}{1 - s^2\phi_0^2(s)}. \end{aligned} \quad (4.142)$$

Expanding (4.138) up to second order in γ yields

$$\begin{aligned} \phi(t,s) &= \phi_0(s)\phi_1(t) + \gamma s \phi_0^2(s) \frac{\phi_1(t) - 1}{t} + \\ &+ \gamma^2 \left[\phi_1(t)(F_2 - 1)\phi_0(s) \frac{s^2\phi_0^2(s)}{1 - s^2\phi_0^2(s)} + \frac{(1 - F_2 t^2)\phi_1(t) - 1}{t^2} s^2 \phi_0^3(s) \right]. \end{aligned} \quad (4.143)$$

We now need to invert these to obtain the distributions. The γ^0 order is clearly $\rho_0(y)\rho_1(x)$, as these are generated by $\phi_0(s)$ and $\phi_1(t)$. For γ^1 order, we use a fact mentioned in the section 4.1.2 in the proof of formula (4.39). We argued that the non-singular part of $\phi(t)/t^n$ generates the distribution $x^n \rho(x)$, where $\rho(x)$ is generated by $\phi(t)$. This way, we see that

$$\begin{aligned} \frac{\phi_1(t) - 1}{t} &\to x\rho_1(x), \\ s\phi_0^2(s) = \frac{\phi_0(s) - 1}{s} &\to y\rho_0(y). \end{aligned} \quad (4.144)$$

The second line could be obtained also by a direct use of the formula (4.39) and this formula is used to invert the γ^2 part of $\phi_2(s)$ also. The first part of γ^2 term in $\phi(t,s)$ is this second order contribution from $\rho_2(y)$. The inversion of the second part is again done by using the previous trick and we obtain

$$\begin{aligned} \frac{(1-F_2t^2)\phi_1(t)-1}{t^2} &\quad\to\quad x^2(1-F_2x^2)\rho_1(x), \\ s^2\phi_0^3(s) = \frac{(1-s^2)\phi_0(s)-1}{s^2} &\quad\to\quad y^2(1-y^2)\rho_1(x), \end{aligned} \tag{4.145}$$

After some algebra, we find the final result for the distribution of the eigenvalues of matrix B and the joint distribution of M and B, valid up to second order in the correlation γ, to be

$$\begin{aligned} \rho_2(y) &= \rho_0(y)\left(1+\gamma^2(1-F_2)\frac{y^2-2}{y^2-4}\right), \\ \rho(x,y) &= \rho_1(x)\rho_2(y)\Big(1+\gamma xy+\gamma^2x^2(1-F_2x^2)y^2(1-y^2)\Big). \end{aligned} \tag{4.146}$$

As in the case of the distribution of M, we have to change $g\to f^2g$, $x\to x/\sqrt{f}$, $y\to y\sqrt{g}$ to get the distributions of the unscaled matrices. Note that both of these become the appropriate free expressions but there is something new in the second formula. One could guess that the interacting result for $\rho(x,y)$ would be just the free case with the one matrix marginals replaced by the interacting expressions. The extra factor of F_2 shows that this is not the case.

4.3 A general argument

To conclude this section, let us comment on this rather surprising finding. We have seen that the presence of the kinetic terms only changes the two point function of the theory, but apart from that does not change anything. As we did in the Chapter 3, let us separate the quadratic part of the action from the rest and write (being rather schematic in the notation)

$$\begin{aligned} &\int dM\, e^{-\mathrm{Tr}(M\mathcal{K}M)/2-\mu^2\mathrm{Tr}\left(M^2\right)/2-W(M)} = \\ = \quad &\int d\Lambda\, J(\Lambda) e^{-W(\Lambda)} \int dU\, e^{-\mathrm{Tr}(M\mathcal{K}M)/2-\mu^2\mathrm{Tr}\left(\Lambda^2\right)/2}. \end{aligned} \tag{4.147}$$

where we have diagonalized the matrix M. The integral over the angular degrees of freedom is now rather complicated. Previous analytical work was based on perturbative expansion of the kinetic term and explicit integration. As higher order terms are included, the expressions become very involved. However, we have done the integral! In the section 4.1.1 we have basically shown that the integral is equal to

$$\int dU\, e^{-\mathrm{Tr}(M\mathcal{K}M)/2-\mu^2\mathrm{Tr}\left(\Lambda^2\right)/2} = e^{-\frac{1}{2f}\mathrm{Tr}\left(\Lambda^2\right)}, \tag{4.148}$$

with f given by (4.5). At this point this seems a little like a miracle, but we will happily take it. Since $M\mathcal{K}M$ is quadratic in M, this integration is universal for any potential $W(M)$, which includes only powers of M^3 and greater. And therefore, if we are computing an expected value of a function that does not involve the angular degrees of freedom, the result is going to be the same as with the theory of no kinetic term, but a changed two point function f. Traces including powers of B or more general matrix (4.50) do involve angular degrees of freedom and this argument can not be used.

For Laplacian kinetic term, this integral was approached in [7, 8]. The perturbative expansion of the kinetic term was set up and then the integral was calculated using group theoretical methods. The results included multitraces and some of them are presented in the Section 5.1. It would be interesting to compare these results and see, how such a complicated looking perturbative process results in this simple formula.

Chapter 5

Phase diagram of scalar field theories on the fuzzy spaces

As was mentioned in the section 2.3, commutative limit of the fuzzy scalar field theory is much more complicated than one would expect. The UV/IR mixing of the non-commutative theory on the plane is related to the non-commutative anomaly on the fuzzy sphere [64]. This, on the other hand was argued to be the source of a distinct phase in the phase diagram of the theory, which is classically not present and which survives the commutative limit [72, 73].

In this chapter, we will review this concept and some past work and then we will apply the results of the previous section, which will provide a non-perturbative analytic description of the phase diagram. Most of the work so far has been concentrated on the case of the fuzzy sphere, however our results will be applicable also in the case of more complicated fuzzy spaces, for which there is no numerical data available at the moment.

Unfortunately, as we will shortly see, our findings are rather anti-climatic. Results of [8] suggested that the kinetic term plays a role in the change of the matrix model phase diagram from Figure 3.4 into the full fuzzy diagram with the new phase emerging. However, in the following we will show that this is not true and once we consider the kinetic term non-perturbatively, we get the pure matrix model phase diagram again. Thus, to explain the fuzzy features, one has to go further.

We first summarize the previous work, then show how the results of the section 4.2 become relevant. After that, we present the phase diagram for the fuzzy sphere and also for a general $\mathbb{C}P^n_F$ and show that we can consistently rescale the parameters of the theory to obtain a diagram, which is independent of N.

5.1 Phases of the theory, numerical simulations and perturbative treatment

Description of the phases of the theory in flat space

Scalar ϕ^4 theory on commutative $\mathbb{R}^2$ allows for two phases [74]. In the symmetric phase, field configurations oscillate around the symmetric vacuum $\phi = 0$ and a phase which spontaneously breaks the $\phi \to -\phi$ symmetry of the theory and the phase transition between these two is of second order. In this phase, the field oscillates around one of the minima of the potential. Later, using the lattice techniques, the critical line of the theory has been identified [75].

The phase structure of the non-commutative theories [73, 76, 77] predicted existence of a third, striped phase. In this phase, the field is non-vanishing in both of the minima of the potential, if those are sufficiently deep. Existence of this third phase is a non-commutative effect and the phase does not disappear in the commutative limit of the theory. This is the result of the UV/IR mixing and reflects the fact that the naive non-commutative theory does not reproduce the expected commutative counterpart. Modification of the action (2.32) removes the non-commutative anomaly and reproduces the expected commutative limit, it is natural to expect that such modification would remove this phase.

Since non-commutative theory on the plane can be obtained from the theory on the fuzzy sphere, a third phase of the theory is expected also there.

Numerical simulations for the fuzzy sphere

In [78], the three phases of the scalar field theory on the fuzzy sphere, disordered, uniform ordered and non-uniform ordered, were identified. The explicit formula for the critical lines between the uniform/disordered phase and the non-uniform/disordered phase was obtained, but the non-uniform/uniform phase transition was not accessible.

The phase transition between the one-cut and the two-cut phase has been identified and carefully studied in [79]. The split of the eigenvalue density was observed and it was shown that a gap around the zero eigenvalue develops for a critical value of the coupling. The coordinates of this critical point have been found for a range of values of g and μ^2.

In [6] the whole phase diagram of the theory was obtained. The result is shown in the Figure 5.1. All three phases of the theory are identified, with the corresponding boundaries of co-existence. This work was also able to pin down the triple point, with coordinates, in our notation,

$$(\mu^2, g) = (-0.8 \pm 0.08, 0.15 \pm 0.05). \tag{5.1}$$

Let us note that to obtain this result, results of [78] and some extrapolation were necessary.

The results of [79] and [6] agree on the location of the one-cut and two-cut solution phase transition boundary. For summary of some earlier numerical results see [80] and references therein.

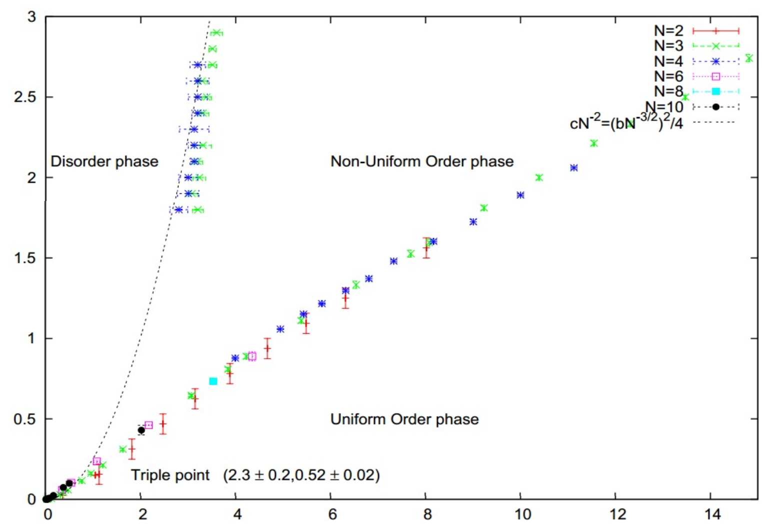

Figure 5.1: The phase diagram for the fuzzy sphere obtained numerically in [6]. The horizontal axis is related to $-\mu^2$ and the vertical axis is related to the coupling g.

Perturbative treatment for the fuzzy sphere

The previous analytic attempts to describe the phase diagram of the scalar field theory on the fuzzy sphere [7, 8] rely on perturbative expansion of the kinetic part of the action. One expands

$$\begin{aligned} e^{-\varepsilon\beta \mathrm{Tr}(M[L_\alpha,[L_\alpha,M]])} &= 1-\varepsilon\beta \mathrm{Tr}\left(M[L_\alpha,[L_\alpha,M]]\right) \\ &+ \frac{\varepsilon^2\beta^2}{2}\left[\mathrm{Tr}\left(M[L_\alpha,[L_\alpha,M]]\right)\right]^2+\dots . \end{aligned} \tag{5.2}$$

When we want to diagonalize the matrix M in the expressions here, the angular degrees of freedom U in $M = U^\dagger \Lambda U$ do still enter the calculation. The terms $[\mathrm{Tr}\left(M[L_\alpha,[L_\alpha,M]]\right)]^n$ are expressed as characters in a $2n$-fold product of the fundamental representation of $SU(N)$, which then become sums of characters in irreducible representations. The integration over the Haar measure dU is then done using an orthogonality relation for these irreducible representations. Eigenvalues enter through the characters of Λ, accompanied by numerical factors coming from the traces of the products (characters) of the generators of $SU(N)$. The results are then reexponentiated to obtain a new effective action.

For example for the fuzzy sphere, the following effective action was obtained (in our notation)

$$\begin{aligned}\beta S_{eff}[\rho(\lambda)] &= \beta\left(\varepsilon - \frac{\varepsilon^2\beta}{3}\int d\lambda\,\lambda^2\rho(\lambda) + \frac{1}{2}\tilde{\mu}^2\right)\int d\lambda\,\lambda^2\rho(\lambda) \\ &+ \beta\tilde{g}\int d\lambda\,\lambda^4\rho(\lambda) - \int d\lambda d\lambda'\,\log|\lambda-\lambda'|\rho(\lambda'). \qquad (5.3)\end{aligned}$$

Note the appearance of a multitrace term in the effective action, which is a general feature of this approach. Theory with this effective action is then solved using the usual saddle point methods mentioned in the section 3.2 and the following eigenvalue distribution is obtained

$$\rho(\lambda) = \frac{1}{2\pi}\left(\frac{4}{R^2} - \tilde{g}\beta R^2 + 4\tilde{g}\beta\lambda^2\right)\sqrt{R^2-\lambda^2}, \qquad (5.4)$$

with the radius R of the distribution given by

$$12\varepsilon + R^2\beta\frac{\varepsilon^2}{16}\left[R^3\tilde{g}\beta^2\frac{\varepsilon^2}{16} + 4R^2\left(\beta\frac{\varepsilon}{4} - 9\tilde{g}\right) - 24\left(\frac{\varepsilon}{4} + \frac{1}{2}\tilde{\mu}^2\right)\right] = 0. \qquad (5.5)$$

Note that the equation for the distribution is the same as in the pure potential case (3.40) and only the expression for R changes. Also note that even though this expression contains terms of the second order in ε, we expect this to hold only up to order ε^1. This is because the ε^2 part of the previous equation gets contribution from ε^3 part of the kinetic term contribution, which is omited. The critical line described by this distribution is then given by

$$\tilde{\mu}^2 = -4\sqrt{\frac{\tilde{g}}{\beta}} - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{12}\sqrt{\frac{\beta}{\tilde{g}}}. \qquad (5.6)$$

In [8], also the other two phases of the theory were identified and the triple

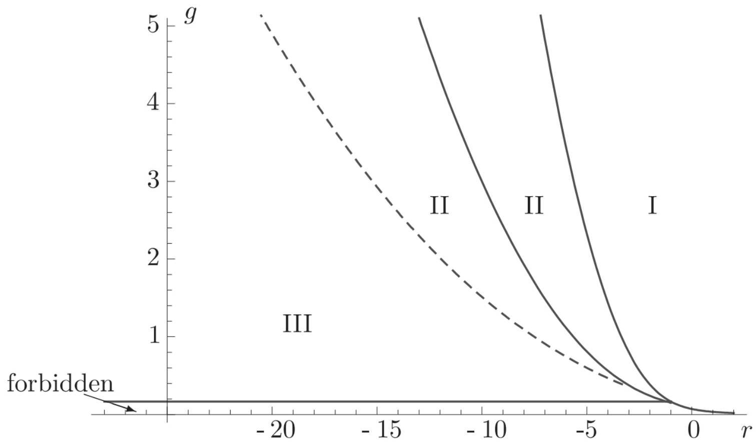

Figure 5.2: Phase diagram for the fuzzy sphere S_F^2 presented in [8]. The parameter r is related to μ. See the text for description of the diagram.

point was localized at $(\tilde{\mu}^2, g) = (-\frac{1}{2}, \beta/3)$. The phase diagram of this paper (for $\beta = \frac{1}{2}$) is depicted in the Figure 5.2. Region I is the one-cut solution corresponding to the disordered phase, region II is the double-cut, or the non-uniform ordered phase and region III is the asymmetric single-cut phase, or uniform ordered phase. There is a forbidden parameter space region, for which the distribution is not well defined.

In [8], the same analytic perturbative procedure was used to carry out the calculation of the distribution for the theory on fuzzy $\mathbb{C}P^2$ and Figure 5.3 shows the computed phase diagram presented there. The solid line gives lower boundary of the one-cut phase, the dashed line is the boundary of the forbidden phase region. The two-cut phase is allowed in between these two lines. The dotted line is the lower left boundary of the possible existence of the asymmetric phase. It is however not known, where this phase is going to be realized.

The boundary between the one-cut and two-cut domains is given by

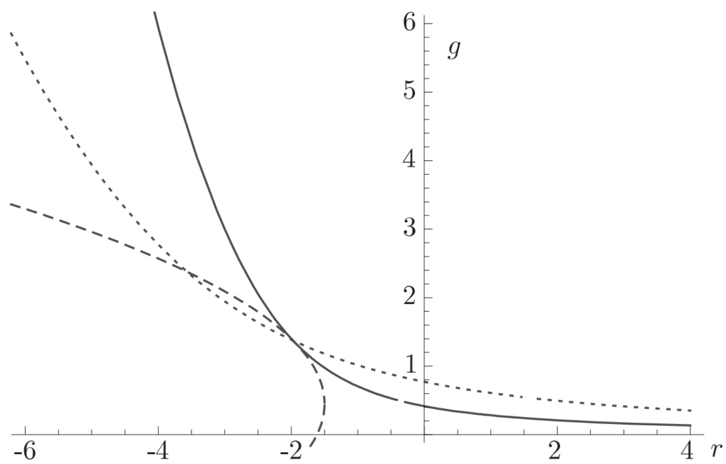

Figure 5.3: Phase diagram for the fuzzy $\mathbb{C}P_F^2$ presented in [8]. The parameter r is related to μ. See the text for description of the diagram.

(again in our notation and normalization)

$$\tilde{\mu}^2 = -\frac{4}{3}\varepsilon - 4\sqrt{\frac{g}{\beta}} + \varepsilon^2 \frac{16\beta\varepsilon + 60\sqrt{g\beta}}{540g}. \tag{5.7}$$

5.2 One-cut/two-cut phase transition

In this section, we will consider the scalar field on fuzzy space at finite inverse temperature β. This means we introduce a factor of β into the measure (4.2) or the path integral weight (2.28).

As discussed in the section 3.2.2, if we allow for negative values of μ^2, the eigenvalue distribution can become negative and can not be interpreted as a probability distribution. This suggests a failure of the one-cut assumption and the topology of the support of the distribution changes. In the same chapter, we have seen that the condition for this phase transition was

$$\mu^4 = 16g, \tag{5.8}$$

with $\mu^2 < 0$. Since the distribution for the ensemble with the kinetic term in the measure (4.2) is the original distribution with two point function f, we expect this to become

$$f^2 = \frac{1}{16g} \tag{5.9}$$

and $f < 0$, which means

$$f = -\frac{1}{\sqrt{16g}}. \tag{5.10}$$

Without any surprise, solving

$$\frac{1}{2f} + 4gfa^2 = 0 \quad , \quad a^2 = \frac{\sqrt{1+48gf^2}-1}{24gf^2} \tag{5.11}$$

yields $f^2 = 1/16g$, which is (5.9). Introduction of β can be simply taken into account by $g \to \beta g$ and $f \to f/\beta$, which yields

$$f = -\sqrt{\frac{\beta}{16g}}. \tag{5.12}$$

We should keep in mind that the two point function f in this expression is understood with respect to the original measure (4.2), so one needs to make the change $g \to \beta g$ and $f \to f/\beta$ also to the eigenvalue distribution (4.120) to obtain

$$\begin{aligned} \rho(x) &= \frac{1}{\pi}\left(\frac{\beta}{2f} + 4a^2 fg + 2g\beta x^2\right)\sqrt{4a^2 f/\beta - x^2}, \\ a^2 &= \beta\frac{\sqrt{1+48gf^2/\beta}-1}{24gf^2}. \end{aligned} \tag{5.13}$$

This equation defines the critical line in the phase diagram of the theory and can be used to obtain phase diagrams similar to the phase diagram of the original matrix model show in the Figure 3.4. In principle, there is another critical line in this diagram.

As mentioned in the section 3.2.2, the two-cut solution is not entirely determined by the normalization condition. There is one free parameter left, the filling fraction x, i.e. the fraction of eigenvalues in one of the cuts. In some part of the parameter spaces, the theory admits a solution with $x = 1$, i.e. with all the eigenvalues in one of the minima of the potential. The critical line for this is given by[1]

$$f = -\sqrt{\frac{\beta}{60g}}. \tag{5.14}$$

For the original matrix model, this solution is however never assumed, since the lowest free energy is in the case of $x = 1/2$, in compliance with the symmetry of the problem. However, if one introduces symmetry breaking in a way that does not change the potential, but changes the free energy, there might be a region in the parameter space, where different values of x are energetically preferable. We will therefore show this line to denote the possible location of the values of the parameters, where an asymmetric single cut solution is possible.

The expression for the eigenvalue distribution (5.13) as well as the conditions (5.12) and (5.14) involve explicit powers of N. To make sure that our results still make sense in the large N limit, we will have to rescale the coupling constant g and possibly the matrix M. To see, what kind of scaling is required, we will need the explicit formulas for f.

5.3 Two point functions and scaling for different spaces

In this section, we will give explicit formulas for the required two point functions for number of different spaces and we will discuss the scaling of these in the limit of large matrices.

[1]This can be derived by plugging the ansatz for an asymmetric single cut solution into the equation for $\omega_0(z)$ (3.42) and see, when this gives a consistent solution.

5.3.1 Fuzzy sphere

This case was discussed in detail in the section 4.1.5. There, we have computed

$$f = \frac{1}{N} \log\left(1 + \frac{1}{\tilde{\mu}^2}\right). \tag{5.15}$$

Looking at (5.13), we see that we need to scale $g \to N^2\tilde{g}$, so that gf^2 does no scale. This scaling also leaves a^2 constant in the large N limit and thus scaling $M \to M/\sqrt{N}$ will keep the radius $4a^2 f$ finite and the distribution (5.13) intact. We will discuss the general scaling of the theory later in the section 5.5.

There is however something strange going on in the previous formula and we need to address this issue before we proceed.

Namely, if we allow for negative values of $\tilde{\mu}^2$, this expression becomes ill defined for $-1 \leq \mu^2 \leq 0$. To understand this problem better, we need to go to the original, unscaled formula for f (4.5), i.e.

$$f = \frac{1}{N} \sum_{l=0}^{N-1} \frac{2l+1}{l(l+1) - |\mu^2|}. \tag{5.16}$$

Without any scaling, this expression diverges in the limit $N \to \infty$ for any value of $|\mu^2|$ that is not an integer equal to $l(l+1)$ for some l, since the sum is dominated by the large l terms. If this is the case, there is a mode in the spectrum that has a vanishing propagator and we encounter the same problems as with the massless case in the section 4.1.5. And the solution to the problem is also the same, we simply remove this problematic mode from the spectrum and then recover the divergent formula (5.16) for f.

However if we try to rescale μ with N, we might run into trouble. For $N^2|\tilde{\mu}^2| > N^2$, there is no problem. All the terms in the denominator of (5.16) are finite and negative and the sum is finite (and negative). However if $N^2|\tilde{\mu}^2| < N^2$ we need to be careful. If $|\tilde{\mu}^2|$ is irrational, there are no zero denominators in the sum. However the actual value of the sum is a function that is not well behaved. Moreover, if $|\mu^2|$ is rational, there is a number of modes with zero propagator, and this number increases with N. Therefore we can not simply remove them from the spectrum and the theory is not well defined.

Therefore we conclude that for $-1 < \tilde{\mu}^2 < 0$ rescaling $\mu \to N\tilde{\mu}$ is not well defined and can not be made. Without the rescaling, as discussed above, the theory is effectively massless. This means that $f \sim \log N/N$ in the large N limit.

As mentioned before, our approach is well suited for theories with a more general kinetic term. Also mentioned before, such theory was proposed to cure the problem of the problem of the planar limit of the theory on the sphere. It is straightforward to write down and evaluate the propagator for the kinetic term (2.32)

$$
\begin{aligned}
f &= \frac{1}{N}\sum_{l=0}^{N-1}\frac{2l+1}{l(l+1)+\frac{\tilde{\kappa}}{N^2}l^2(l+1)^2+N^2\tilde{\mu}^2} \\
&\approx \frac{1}{N}\frac{2}{\sqrt{-1+4\tilde{\mu}^2\tilde{\kappa}}}\arctan\left(\frac{\sqrt{-1+4\tilde{\mu}^2\tilde{\kappa}}}{1+2\tilde{\mu}^2}\right). \qquad (5.17)
\end{aligned}
$$

For negative values of $\tilde{\mu}^2$, this becomes

$$
f(\tilde{\mu}^2<0) = \frac{1}{N}\frac{2}{\sqrt{1+4|\tilde{\mu}^2|\tilde{\kappa}}}\operatorname{arctanh}\left(\frac{\sqrt{1+4|\tilde{\mu}^2|\tilde{\kappa}}}{1-2|\tilde{\mu}^2|}\right). \qquad (5.18)
$$

Since $\tilde{\kappa}$ is function of $\tilde{\mu}$ and $\tilde{g}$, equation (5.12) with this f becomes a very complicated and needs to be approached numerically. We will not elaborate on this, mainly because our results will turn out to be incomplete. We will just conclude that once the complete results are considered, this f should change the phase diagram and remove the effect of the non-commutative anomaly.

5.3.2 Fuzzy $\mathbb{C}P^n_F$

The fuzzy versions of $\mathbb{C}P^n_F$ were discussed in the Chapter 2. The scalar field theories on these were also mentioned there. Looking at the formula for the kinetic term of the action (4.1), it is straightforward to generalize the expression (4.88) for the case of propagator on $\mathbb{C}P^n_F$ and we get

$$f = \frac{1}{N}\sum_{l=0}^{L}\frac{\dim(n,l)}{l(l+n)+\mu^2}. \tag{5.19}$$

As in the fuzzy sphere case, N is the dimension of the representation and L is the angular momentum cut-off. $\dim(n,l)$ is the dimension of the rank-l representation of $SU(n+1)$. We again convert the sum into an integral with introducing $x = l/L$ and compute

$$\begin{aligned} f &= \frac{L^n}{n\big((n-1)!\big)^2}\frac{1}{\mu^2}\int_0^1 dx \frac{x^{2n-1}}{\frac{L^2x^2}{\mu^2}+1} \\ &= \frac{L^n}{n\big((n-1)!\big)^2}\frac{1}{\mu^2}\,{}_2F_1\left(n,1,n+1,-\frac{L^2}{\mu^2}\right), \end{aligned} \tag{5.20}$$

where ${}_2F_1$ is the hyper-geometric function. As we can see, to keep μ in the expression we need to rescale $\mu \to N^{2/n}\tilde{\mu}$, which together with $L = (n!N)^{1/n}$ yields

$$f = N^{1-\frac{2}{n}}\frac{(n-1)!}{\tilde{\mu}^2}\,{}_2F_1\left(n,1,n+1,-\frac{(n!)^{2/n}}{\tilde{\mu}^2}\right). \tag{5.21}$$

For $n = 1$, we recover the expression (4.95) for the fuzzy sphere. For next couple of values of n we get the following

$$\begin{aligned} n=2 \qquad f &= 1-\frac{1}{2}\tilde{\mu}^2\log\left(1+\frac{2}{\tilde{\mu}^2}\right), \\ n=3 \qquad f &= N^{1/3}\left[\left(\frac{3}{4}\right)^{1/3}-\frac{\tilde{\mu}^2}{6^{1/3}}+\frac{1}{6}\tilde{\mu}^4\log\left(1+\frac{6^{2/3}}{\tilde{\mu}^2}\right)\right], \\ n=4 \qquad f &= 2\sqrt{N}\left[\sqrt{\frac{2}{3}}-\frac{1}{4}\tilde{\mu}^2+\frac{1}{4\sqrt{6}}\tilde{\mu}^4-\frac{1}{48}\tilde{\mu}^6\log\left(1+\frac{2\sqrt{6}}{\tilde{\mu}^2}\right)\right]. \end{aligned} \tag{5.22}$$

For completeness, let us give expressions for h and g in the case of general $\mathbb{C}P_F^n$. Generalization of formula (4.6) is straightforward and we obtain

$$\begin{aligned} h &= N\frac{(n!)^{1+2/n}n!}{(n+1)}\frac{1}{\tilde{\mu}^2}\,{}_2F_1\left(n+1,1,n+2,-\frac{1}{\tilde{\mu}^2}\right), \\ g &= N^{1+\frac{2}{n}}\frac{(n!)^{1+4/n}}{(n+2)}\frac{1}{\tilde{\mu}^2}\,{}_2F_1\left(n+2,1,n+3,-\frac{1}{\tilde{\mu}^2}\right). \end{aligned} \tag{5.23}$$

This gives finite γ in the large N limit also in this case, thanks to the scaling of μ

As in the case of the sphere, also for the $\mathbb{C}P_F^n$, we can rescale the coupling constant $g \to N^{4/n-2}\tilde{g}$. Rescaling the matrix $M \to N^{(n-2)/(2n)}M$ then keeps the radius finite and preserves the form of the distribution.

This procedure works for any kinetic term that has divergent sum in (5.19) and mass, coupling and the matrix can be consistently rescaled to obtain (5.12,5.13), which are now scale free. If the sum in the expression for f does not diverge, there is no trouble at all, we do not need to rescale the mass and coupling and the matrix can be rescaled to achieve the same. We will not give the explicit formula for a general kinetic term here.

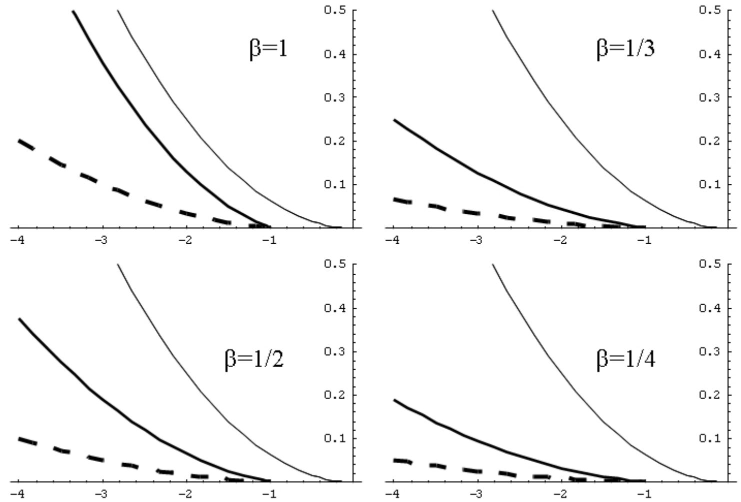

Figure 5.4: Phase diagrams for the fuzzy sphere at different values of β. Solid line denotes the boundary of one-cut/two-cut transition, dashed line shows the border of possible existence of asymmetric one-cut solution and the thin line is the one-two cut critical line for a model with no kinetic term.

5.4 Phase diagrams

5.4.1 Fuzzy sphere

Using (5.12) and (5.15) together yields for the one-cut critical line

$$\tilde{g} = \frac{\beta}{16\left[\log\left(1+\frac{1}{\tilde{\mu}^2}\right)\right]^2}, \tag{5.24}$$

$$\tilde{\mu}^2 = -\frac{1}{1-e^{-\sqrt{\frac{\beta}{16\tilde{g}}}}}. \tag{5.25}$$

Lines for several values of β are plotted in the Figure 5.4.

To compare with the perturbative results of [8], we need to redo the calculation with a factor of ε in front of the kinetic term and expand (5.25) in powers of ε. This then yields

$$\tilde{\mu}^2 = -\frac{\varepsilon}{1 - e^{-\varepsilon\sqrt{\frac{\beta}{16\tilde{g}}}}} = -4\sqrt{\frac{\tilde{g}}{\beta}} - \frac{\varepsilon}{2} - \varepsilon^2 \frac{1}{48}\sqrt{\frac{\beta}{\tilde{g}}} + \dots \ . \tag{5.26}$$

Note that the first two terms of this expansion exactly match the first two terms of (5.6). It is also clear how the higher order contributions change the phase diagram computed using the perturbative methods, Figure 5.2. The critical line, together with the nontrivial triple point, gets pushed in the negative direction of $\tilde{\mu}^2$, with the tail of the curve approaching the $g = 0$ axis. Also note that the tail changes direction from positive $\tilde{\mu}^2$ to negative $\tilde{\mu}^2$ with each higher order correction.

5.4.2 Fuzzy $\mathbb{C}P^n_F$

As in the case of the fuzzy sphere, corresponding equations for $\mathbb{C}P^n_F$, (5.12) and (5.21) yield the critical line

$$\tilde{g} = \frac{\beta (n!)^{4/n}\big((n-1)!\big)^2}{16\left[\tilde{\mu}^2\ {}_2F_1\ (n, 1, n+1, -\frac{1}{\tilde{\mu}^2})\right]^2}. \tag{5.27}$$

Several lines for different values of β and n are plotted in the Figure 5.5.

The above formula can not be inverted to give an expression for critical $\tilde{\mu}$. However, to compare with the previous results, we can solve the equation for a given n up to a given order in ε. For $n = 2$ we then obtain

$$\tilde{\mu}^2 = -4\sqrt{\frac{\tilde{g}}{\beta}} - \frac{4}{3}\varepsilon - \frac{1}{18}\sqrt{\frac{\beta}{g}}\varepsilon^2 + \dots \ . \tag{5.28}$$

We again get agreement with (5.7). The same comparison with the perturbative phase diagram as before holds here. The critical line gets pushed in the negative direction of the $\tilde{\mu}^2$, with the non-trivial triple point approaching the $\tilde{g} = 0$ axis.

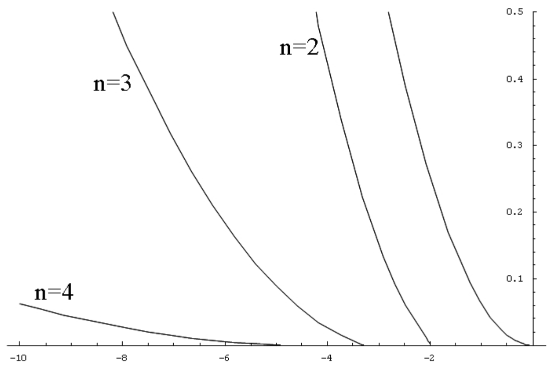

Figure 5.5: Phase diagrams for the fuzzy $\mathbb{C}P^n_F$ for several values of n. The lines denote the one-two cut critical line and the fourth line denotes the critical line for model with no kinetic term.

5.5 Large N multiscaling

In this last section, we would like to connect these results to the field theories on fuzzy and commutative versions of $\mathbb{R}^{2n}$. As discussed in the section 2.2.2, these can be obtained as an appropriate limit of $\mathbb{C}P^n_F$, with the radius R being scaled as $N \to \infty$. To do this, we need to change two things in our previous calculations. First of all the action (2.24) includes a volume factor that has been previously absorbed into the definition of the field and coupling constant. And second, there is a factor of R^2 in the kinetic term. The complete action then reads

$$S[M] = \frac{V_n}{N}\left[\frac{\varepsilon}{R^2}\mathrm{Tr}\left(M[L_\alpha,[L_\alpha,M]]\right) + \frac{1}{2}\mu^2\mathrm{Tr}\left(M^2\right) + \frac{g}{N}\mathrm{Tr}\left(M^4\right)\right], \tag{5.29}$$

where V_n is the volume of the $\mathbb{C}P^n$ given by $V_n = \left(2+\frac{1}{n}\right)^n \frac{\pi^n}{n!}R^{2n} = C_n R^{2n}$ and L_α are the generators of $SU(n+1)$ in the N dimensional representation. It is now straightforward to redo the calculation in this general case and one obtains the following formulas for f

$$f = N\frac{(n-1)!}{\mu^2}\,{}_2F_1\left(n,1,n+1,-\frac{\varepsilon N^{2/n}(n!)^{2/n}}{\mu^2 R^2}\right), \tag{5.30}$$

the eigenvalue distribution

$$\rho(x) = \frac{1}{\pi}\left(\frac{V_n\beta}{2fN} + 4a^2 f\frac{N}{V_n\beta}g\frac{V_n\beta}{N} + 2g\frac{V_n\beta}{N}x^2\right)\sqrt{\frac{4a^2fN}{V_n\beta} - x^2}, \quad (5.31)$$

where the square of the radius is given by

$$\frac{4a^2fN}{V_n\beta} = \frac{\sqrt{1+48g\frac{V_n\beta}{N}\left(\frac{N}{V_n\beta}\right)^2 f^2} - 1}{6g\frac{V_n\beta}{N}f\frac{N}{V_n\beta}}. \quad (5.32)$$

Now, we perform a general multiscaling of the parameters. We replace

$$R \to RN^{\theta_R}\ ,\ \mu \to \mu N^{\theta_\mu}\ ,\ M \to MN^{\theta_x}\ ,\ g \to gN^{\theta_g}\ ,\ \beta \to \beta N^{\theta_\beta}. \quad (5.33)$$

We will omit the tildes over the scale-free variables from now on. This means that

$$\begin{aligned} f &= N^{1-2\theta_\mu}\frac{(n-1)!}{V_n\mu^2}\,{}_2F_1\left(n, 1, n+1, -\frac{\varepsilon N^{2/n-2\theta_R-2\theta_\mu}(n!)^{2/n}}{\mu^2R^2}\right), \\ 4a^2f/\beta &= N^{-1-\theta_\gamma+2\theta_\mu}\frac{\mu^2}{6g(n-1)!} \\ &\times \frac{\sqrt{1+\frac{48g(n-1)!^2}{V_n\mu^4}N^{3-\theta_\beta+\theta_\gamma-4\theta_\mu-2n\theta_R}\,{}_2F_1(\ldots)} - 1}{{}_2F_1(\ldots)}. \end{aligned} \quad (5.34)$$

We now need to chose the conditions for the scaling. We will do this to preserve as much information and features of the original theory as possible. This means that the argument of the hypergeometric function can not scale with N, i.e.

$$\frac{2}{n} - 2\theta_R - 2\theta_\mu = 0. \quad (5.35)$$

We will also impose the condition

$$3 - \theta_\beta + \theta_\gamma - 4\theta_\mu - 2n\theta_R = 0, \quad (5.36)$$

which ensures that the form of the radius of the distribution does not change. Finally, we require the radius to scale the same way as x^2 does, i.e.

$$-1 - \theta_\gamma + 2\theta_\mu = 2\theta_x. \tag{5.37}$$

There are going to be two scalings left undetermined by these conditions. We chose these to be θ_R, as we will want to scale R freely when making connection to $\mathbb{R}^{2n}$, and θ_β, which can be chosen to fix the overall scaling of the action. Solving the three conditions, we find

$$\theta_\mu = \frac{1}{n} - \theta_R \ , \ \theta_g = 2n\theta_R - 4\theta_R + \theta_\beta - 3 + \frac{4}{n} \ , \ \theta_x = -n\theta_R + \theta_R - \frac{\theta_\beta}{2} + 1 - \frac{1}{n}. \tag{5.38}$$

We now plug these back into the equation for the eigenvalue density and obtain

$$\rho(x) = \frac{V_n\beta}{\pi}\left(\frac{1}{2u} + \rho^2 g + 2gx^2\right)\sqrt{\rho^2 - x^2} \tag{5.39}$$

where the scale free radius is given by

$$\rho^2 = \frac{\sqrt{1 + \frac{48gu^2}{V_n\beta}} - 1}{6gu} \tag{5.40}$$

where u is the scale free part of the two point function f. This clearly shows that we can rescale the parameters of the theory, including the radius R in such a way that preserves the form of the results we have obtained. Therefore the distribution (5.39), with the corresponding phase structure discussed in previous section, is going to be relevant also for the case of $\mathbb{R}^{2n}$, both commutative and non-commutative. The corresponding scaling of radius R is given in section 2.2.2 and results into

- $\theta_R = 0$ is the limit of the commutative $\mathbb{C}P^n$ of radius R. In this case

$$\theta_\mu = \frac{1}{n} \ , \ \theta_g = -3 + \frac{4}{n} + \theta_\beta \ , \ \theta_x = -\frac{\theta_\beta}{2} + 1 - \frac{1}{n}. \tag{5.41}$$

- $\theta_R = \frac{1}{2n}$ is the limit of the non-commutative $\mathbb{R}^{2n}_\theta$ with the non-commutativity parameter θ given by $\tilde{R} = \frac{1}{2}\theta$. In this case, we obtain

$$\theta_\mu = \frac{1}{2n} \quad , \quad \theta_g = -2 + \frac{2}{n} + \theta_\beta \quad , \quad \theta_x = -\frac{\theta_\beta}{2} + \frac{1}{2} - \frac{1}{2n}. \tag{5.42}$$

- $\theta_R = \frac{1-\varepsilon}{2n}$ for some value of $0 < \varepsilon < 1$ is the limit of the commutative $\mathbb{R}^{2n}$.

To see what this scaling does at the level of the action, we plug the rescaled parameters (5.38) back into the action (5.29). Traces in the the mass and interaction terms behave roughly as N. The trace in the kinetic term behaves roughly as

$$\sum_{l=0}^{L} \frac{\dim(n,l)}{N} l(l+n) = \frac{1}{N} L^{2n+2} = N^{1+\frac{2}{n}}. \tag{5.43}$$

We also need to keep in mind that the factor of $1/N$ in the kinetic term was there due to normalization of the distribution. To remove it, we take $\gamma \to \gamma + 1$ and we find out that the mass and interaction terms both scale with N^1 and the kinetic term scales with $N^{1-\frac{2}{n}}$. Including the behavior of the traces, the action scales homogeneously and is of order N^2, as desired from the matrix model considerations of section 3.2.

5.6 Comparison with perturbative and numerical results

As mentioned before, the analysis in the past concentrated mostly on the fuzzy sphere.

Comparison of these results with the numerical work of [6] and the perturbative work of [8] starts with the scaling. The scaling found in both of these papers is

$$\theta_\beta = -\frac{1}{2} \quad , \quad \theta_g = \frac{5}{2} \quad , \quad \theta_x = -\frac{1}{4}. \tag{5.44}$$

When one takes the difference in the scaling in the overall volume factor into account, we see that the corresponding choice of θ_β in our case is $+\frac{1}{2}$, since then the action scales with an overall factor of $1/\sqrt{N}$. This then gives the same θ_g and θ_x in the $n = 1$ and $\theta_R = 0$ case of (5.38). This was expected, since the overall scaling of the action using either (5.38) or (5.44) is the same. This means that we are on the correct track, the scale free parameters are the same and we can compare our phase diagram, Figure 5.4 with the phase diagram obtained in [6].

However, the phase structure of the theory computed numerically is not the same. Our phase diagrams do not include the non-uniform, or the asymmetric one-cut, phase. This is due to the fact that the kinetic term produced the effective action, which was a pure matrix model again, only with a rescaled mass term. As this matrix model does not break the reflection symmetry of the original action, we did not expect this symmetry breaking phase to appear. However, the story is different once $1/N$ corrections are considered. In [81], it was shown that there are solutions, which produce the same planar eigenvalue density, but are inequivalent at the next to leading order and in this order also break the $M \to -M$ symmetry. Also, one needs to be careful when rescaling the parameters of the theory, as it is not clear how this affects the $1/N$ contributions.

In general, what needs to be done is the following. One needs to introduce a symmetry breaking term $\sigma \mathrm{Tr}\left(M^3\right)$ into the action.[2] Do the calculation to lower orders in N, do the multiscaling and restore the symmetry of the potential in some fashion. After doing this, we expect the phase transition to appear, the asymmetric phase to develop and the triple point to shift. The proximity of the trivial triple point $(-1, 0)$ to the numerically obtained one (5.1) makes such conjecture plausible. [2] suggests that considering a matrix model corresponding to the renormalized scalar field theory might also give a non-trivial triple point.

[2] This is not the easiest way to do it. A term $\bar{\sigma}\mathrm{Tr}\,(M)$ is going to be easier. One just needs to be careful not to spoil the validity of (4.148) by terms which are linear or quadratic in M. The cubic addition and the linear addition can be connected by a shift in the matrix M.

The perturbative treatment of the kinetic term however found the asymmetric phase in the phase diagram. This is identifiable as a relict of perturbative approach. When we consider successive ε contributions to the critical line (5.25), we can observe the triple point of the phase diagram in Figure 5.2 to approach the trivial triple point $(-1, 0)$. Also, the forbidden region of the parameters space backs away and the boundary between the symmetric two-cut and asymmetric one-cut solutions goes down to the μ^2 axis. This is due to the fact that the terms neglected in (5.3), which are lower order in ε are leading order in N and do contribute to the final result. The same is true for the phase diagram of the $\mathbb{C}P^2$ phase diagram.

Chapter 6

Conclusions and outlook

The main results and conclusions

In this work, we have described and investigated a class of new random matrix ensembles. The novel element is the presence of a new term in the probability measure, which couples the random matrix to a set of external matrices and thus breaks the invariance of the measure. This prevents the diagonalization of the matrix and one can not use the standard reduction to the eigenvalue problem. We have developed a method to compute the large N limit of the expectation values in this ensemble using the explicit Wick contractions and relating different expectation values via recurrence relations.

First, we have shown that the eigenvalue distribution of the modified Gaussian ensemble remains the Wigner semicircle and the new term in the action rescales its radius. This happened as a result of the change of the underlying two point function of the ensemble. We have then shown that the same is the case for a more complicated ensemble with a quartic self-interaction in the measure. The mass and the coupling of the theory got rescaled, but the qualitative features of the distribution did not change. We have conjectured this to be generic, since the non-trivial angular integration was indirectly done in the Gaussian case.

The external matrices present in the measure give new observables into the ensemble. We have computed expectation values and distributions for a class of these. Namely, for the matrix representing action of the new term on the random matrix $B = \mathcal{K}M$, and products of B with the random matrix M itself. We have also considered a more general functions of the random matrix, for example the matrix Laplacian. This generality shows that our method can be used in different settings and is general and robust.

We have found that the eigenvalues of M and B are correlated and we have given the joint probability distribution for these. We have shown that in the Gaussian ensemble and in the ensemble with quartic interaction, this distribution is different beyond the change of one matrix marginals and the interaction changes how these two matrices are correlated.

The new ensembles had a very clear physical motivation. The new term in the measure corresponds to a kinetic term of a scalar field theory on some fuzzy space. This is because such fields are described by finite dimensional matrices and the kinetic term is commutators with generators of the symmetry group of the underlying fuzzy space. These generators are the external matrices of the matrix model and therefore the choice of the extra term in the measure for the ensemble is dictated by the choice of the fuzzy space. The standard kinetic term of the field theory is the quadratic Casimir operator, which is the double commutator in the cases discussed here. But a more complicated and more general term can be considered and thus our approach is applicable for more complicated field theories. This term is only constrained to be quadratic in the field/random matrix and to respect the symmetry of the fuzzy space, both of which are naturally met.

Importantly, the results for the matrix ensemble are non-perturbative. This means the contribution of all orders in the coupling is considered and thus the results we obtain on the field theory side are non-perturbative also.

This was important for the consideration of the phase structure of different fuzzy field theories. We have shown how the kinetic term changes the boundary of existence of the one-cut solution of the matrix eigenvalue distribution, which is the disordered phase of the field theory. We have found scaling of the parameters of the theory, which stabilizes the phase diagram and which agrees with scaling obtained by numerical simulations. This complements, and goes beyond, the previous work, where the extra term was treated as a perturbation. However the corrected triple point of the theory is placed back on the axis of no coupling and the phase diagram contains only two phases, the one-cut/disordered phase and the two-cut/non-uniform ordered phase. The third phase of the fuzzy theory found in the numerical analysis, the asymmetric one-cut, or the uniform ordered phase, is not present. We have concluded that this phase should appear once the multi-scaling is investigated more carefully and also subleading contributions to the distribution are considered.

Outlook and further prospects

To conclude this dissertation, we will present several questions and possibly interesting topics, which arise.

There were two questions in section 4.2 that do deserve further interest. First is the proof that expression (4.116) for the coefficients $\tilde{F}_a^{2m}$ does indeed solve the the recursion rules (4.110) derived from the explicit Wick contractions. To do this, perhaps an iterative approach with use of some generalization of the Vandermonde identity [82]

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} \tag{6.1}$$

could work. Alternately, one could try to extract the formula (4.117) for F_2 directly from the recursion rules. Second point that would be interesting to elaborate more on is the full solution to the eigenvalue distribution for the matrix B and the joint distribution in the interacting case (4.146). We have presented the solution up to the second order in γ. One could try to extract the expression for $W_1(s)$, which is the only unknown parameter, directly from recursion rules (4.128), or alternatively use the three equations for the generating functions $\phi_2(s), \phi(t,s)$ we have obtained as follows. Express these as power series in γ,

$$\phi_2(s) = \sum_{n=0}^{\infty} \gamma^{2n} \phi_2^{(2n)}(s) \quad , \quad \phi(t,s) = \sum_{n=0}^{\infty} \gamma^{n} \phi^{(n)}(t,s) \quad ,$$
$$W_1(s) = \gamma \sum_{n=0}^{\infty} \gamma^{2n} W_1^{(2n+1)}(s) \quad , \quad W_2(s) = \sum_{n=0}^{\infty} \gamma^{2n} W_2^{(2n)}(s) \quad , \tag{6.2}$$

plug (4.134) in (4.137,4.136) and collect the terms that are the same order in γ. By the standard argument that these two need to hold independently of value of γ, one gets conditions on the $W_1^{(2n+1)}(s), W_2^{(2n)}(s), \phi^{(n)}(t,s)$. Since W_1 contains only odd and W_2 only even powers of γ, starting with the γ^0 order in (4.136) we can iterate, plug the results of one calculation into the next equation and work our way up order by order and in principle find expression for $\phi(t,s), \phi_2(s)$ up to any given order. Doing this, one can hope to get a feeling what is going on, guess a formula for the γ^n contribution and prove it iteratively.[1] Once the full solution is found, it would be interesting to investigate the phase structure of the matrix B.

When studying the one-cut/two-cut phase transition of the theory in the Chapter 5, we have first computed the eigenvalue distribution, which was consequence of the recursion rules (4.110) and looked for the values of the parameters, for which this distribution became negative. It would be both interesting and illuminating to try to look for the failure of the one-cut assumption, and emergence of the two-cut phase, directly in the terms of the moments F_{2m} or in the governing recursion rule.

When generalizing the results, one should start with considering a general ϕ^{2k} theory and repeat the procedure of the section 4.2.1. One could then look for the aspects of non-renormalizability of certain theories, like

[1] It is straightforward to check that this procedure gives the same γ^2 contribution (4.142,4.143) we have computed by explicit calculation of $W_1^{(1)}$.

ϕ^6 on $\mathbb{R}^4$ or ϕ^4 on $\mathbb{R}^6$, that are reflected in the regularized matrix versions. This was suggested in [7] and [8], but was not elaborated on neither there, nor here. Some very interesting structure of M^6 matrix model is revealed in [83] and the straightforward connection of this model to the ϕ^6 theory conjectured here makes those results relevant.

Further, more complicated fuzzy spaces and the corresponding matrix ensembles can be considered. Scalar field theory on the fuzzy 4-sphere has been constructed by a modification of the kinetic term of the theory on $\mathbb{C}P^3$ [51]. This will allow for computation of the two point function f and analysis of the theory. Similarly, one can consider matrix quantum mechanics as a fuzzy field theory on $\mathbb{R} \times S^2_F$, which is again well suited for our approach. The perturbative treatment of the kinetic term mentioned in the section 5.1 has been used for such theory also [84]. The phase structure of the theory was investigated there and there are also numerical results available [85]. Finally, one can consider a more general field theory, like a fuzzy version of gauge theories [20, 86], and theories on more general spaces, like the fuzzy torus [60, 87].

The matrix models with certain type of $SU(N)$ symmetry breaking in the action were considered before [88]. Integrals of the form

$$\int dU e^{\beta \operatorname{Tr}\left(M_1 U M_2 U^\dagger\right)}, \tag{6.3}$$

were computed and it was later realized that these are a special case of a more general result of Harish-Chandra [89]. It would be interesting to see, how do our results, especially (4.148), connect to this.

This discussion arose in the context of two matrix models, governed by an action of the form

$$\begin{aligned} S &= \frac{1}{2}\operatorname{Tr}\left(M_1 \mathcal{K} M_1\right) + \frac{\mu_1^2}{2}\operatorname{Tr}\left(M_1^2\right) + \frac{1}{2}\operatorname{Tr}\left(M_2 \mathcal{K} M_2\right) + \frac{\mu_2^2}{2}\operatorname{Tr}\left(M_2^2\right) \\ &+ S_{int}(M_1, M_2). \end{aligned} \tag{6.4}$$

Our method and results are very much usable in this setting. For example if the interaction term is given by $S_{int} = c\mathrm{Tr}\,(M_1 M_2)$, we can immediately use the results of the section 4.1.2 and conclude that the distributions of eigenvalues of M_1 and M_2 are going to be Wigner semicircles and that their joint eigenvalue distribution is given by (4.47). The correlation γ is then connected to c and $\mu_{1,2}^2$. For more complicated interactions of the matrices, it is possible to write down recursion rules for $\mathrm{Tr}\left(M_1^a M_2^b\right)$ in a similar way as we did in the section 4.2.2 and try to solve them.

We have shown how the random matrix approach finds use in the phase diagram considerations of the fuzzy field theory. Obviously, one should look for the application of the new matrix ensembles in other areas, where matrix models are used, most notably the condensed matter physics and string theories.

Finally, there remains an issue already addressed in the previous section. The lack of the asymmetric phase in the phase diagram calls for an explanation. To do this, a more general matrix model needs to be considered, namely

$$S(M) = \mathrm{Tr}\left[\frac{1}{2}\mu^2 M^2 + \sigma M^3 + g M^4\right]. \tag{6.5}$$

This choice of symmetry breaking allows us to use the conjecture of section 4.3. We should compute the phase structure of such theory and remove the symmetry breaking term, keeping in mind that a subleading term, or even terms of all genera, might be of importance after the multiscaling of the parameters.

Appendices

Appendix A

Explicit formulas for three and four matrix distributions

The factor $\tilde{\rho}_3(x, y, z)$ multiplying $\rho(x)\rho(y)\rho(z)$ in formula (4.70) is given by fraction with the following numerator

$$
\begin{aligned}
1 - g_{12}^2 - g_{13}^2 + g_{12}^2 g_{13}^2 - g_{12} g_{13} g_{23} + g_{12}^3 g_{13} g_{23} + g_{12} g_{13}^3 g_{23} - g_{12}^3 g_{13}^3 g_{23} \\
- g_{23}^2 + g_{12}^2 g_{23}^2 + g_{13}^2 g_{23}^2 - g_{12}^2 g_{13}^2 g_{23}^2 + g_{12} g_{13} g_{23}^3 - g_{12}^3 g_{13} g_{23}^3 - g_{12} g_{13}^3 g_{23}^3 \\
+ g_{12}^3 g_{13}^3 g_{23}^3 - 4 g_{12}^2 g_{13}^2 x^2 + 4 g_{12} g_{13} g_{23} x^2 + 4 g_{12}^2 g_{13}^2 g_{23}^2 x^2 - 4 g_{12} g_{13} g_{23}^3 x^2 \\
+ 4 g_{12} g_{13}^2 xy - 4 g_{13} g_{23} xy - 4 g_{12}^2 g_{13} g_{23} xy + 4 g_{12}^2 g_{13}^3 g_{23} xy + 4 g_{12} g_{23}^2 xy \\
- 4 g_{12} g_{13}^2 g_{23}^2 xy - 4 g_{12}^3 g_{13}^2 g_{23}^2 xy + 4 g_{12}^2 g_{13} g_{23}^3 xy + 4 g_{12} g_{13} g_{23} y^2 - 4 g_{12} g_{13}^3 g_{23} y^2 \\
- 4 g_{12}^2 g_{23}^2 y^2 + 4 g_{12}^2 g_{13}^2 g_{23}^2 y^2 + 4 g_{12}^2 g_{13} xz - 4 g_{12} g_{23} xz - 4 g_{12} g_{13}^2 g_{23} xz \\
+ 4 g_{12}^3 g_{13}^2 g_{23} xz + 4 g_{13} g_{23}^2 xz - 4 g_{12}^2 g_{13} g_{23}^2 xz - 4 g_{12}^2 g_{13}^3 g_{23}^2 xz \\
+ 4 g_{12} g_{13}^2 g_{23}^3 xz - 4 g_{12} g_{13} yz + 4 g_{12}^2 g_{23} yz + 4 g_{13}^2 g_{23} yz - 4 g_{12}^2 g_{13}^2 g_{23} yz \\
- 4 g_{12} g_{13} g_{23}^2 yz + 4 g_{12}^3 g_{13} g_{23}^2 yz + 4 g_{12} g_{13}^3 g_{23}^2 yz - 4 g_{12}^2 g_{13}^2 g_{23}^3 yz \\
+ 4 g_{12} g_{13} g_{23} z^2 - 4 g_{12}^3 g_{13} g_{23} z^2 - 4 g_{13}^2 g_{23}^2 z^2 + 4 g_{12}^2 g_{13}^2 g_{23}^2 z^2
\end{aligned}
$$

and denominator

$$1-2\gamma_{12}^2+\gamma_{12}^4-2\gamma_{13}^2+4\gamma_{12}^2\gamma_{13}^2-2\gamma_{12}^4\gamma_{13}^2+\gamma_{13}^4-2\gamma_{12}^2\gamma_{13}^4+\gamma_{12}^4\gamma_{13}^4-2\gamma_{23}^2+4\gamma_{12}^2\gamma_{23}^2$$

$$-2\gamma_{12}^4\gamma_{23}^2+4\gamma_{13}^2\gamma_{23}^2 8\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2+4\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2-2\gamma_{13}^4\gamma_{23}^2+4\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2$$

$$-2\gamma_{12}^4\gamma_{13}^4\gamma_{23}^2+\gamma_{23}^4-2\gamma_{12}^2\gamma_{23}^4+\gamma_{12}^4\gamma_{23}^4-2\gamma_{13}^2\gamma_{23}^4+4\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4-2\gamma_{12}^4\gamma_{13}^2\gamma_{23}^4+\gamma_{13}^4\gamma_{23}^4$$

$$2\gamma_{12}^2\gamma_{13}^4\gamma_{23}^4+\gamma_{12}^4\gamma_{13}^4\gamma_{23}^4+4\gamma_{12}^2x^2+4\gamma_{13}^2x^2-16\gamma_{12}^2\gamma_{13}^2x^2$$

$$+4\gamma_{12}^4\gamma_{13}^2x^2+4\gamma_{12}^2\gamma_{13}^4x^2-8\gamma_{12}^2\gamma_{23}^2x^2-8\gamma_{13}^2\gamma_{23}^2x^2+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2-8\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2x^2$$

$$-8\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2x^2+4\gamma_{12}^2\gamma_{23}^4x^2+4\gamma_{13}^2\gamma_{23}^4x^2-16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4x^2+4\gamma_{12}^4\gamma_{13}^2\gamma_{23}^4x^2+4\gamma_{12}^2\gamma_{13}^4\gamma_{23}^4x^2$$

$$+16\gamma_{12}^2\gamma_{13}^2x^4-32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^4+16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4x^4-4\gamma_{12}xy-4\gamma_{12}^3xy$$

$$+8\gamma_{12}\gamma_{13}^2xy+8\gamma_{12}^3\gamma_{13}^2xy-4\gamma_{12}\gamma_{13}^4xy-4\gamma_{12}^3\gamma_{13}^4xy+8\gamma_{12}\gamma_{23}^2xy$$

$$+8\gamma_{12}^3\gamma_{23}^2xy-16\gamma_{12}\gamma_{13}^2\gamma_{23}^2xy-16\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2xy+8\gamma_{12}\gamma_{13}^4\gamma_{23}^2xy-+8\gamma_{12}^3\gamma_{13}^4\gamma_{23}^2xy$$

$$-4\gamma_{12}\gamma_{23}^4xy4\gamma_{12}^3\gamma_{23}^4xy+8\gamma_{12}\gamma_{13}^2\gamma_{23}^4xy+8\gamma_{12}^3\gamma_{13}^2\gamma_{23}^4xy-4\gamma_{12}\gamma_{13}^4\gamma_{23}^4xy$$

$$-4\gamma_{12}^3\gamma_{13}^4\gamma_{23}^4xy-16\gamma_{12}\gamma_{13}^2x^3y-16\gamma_{12}^3\gamma_{13}^2x^3y+32\gamma_{12}\gamma_{13}^2\gamma_{23}^2x^3y+32\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2x^3y$$

$$-16\gamma_{12}\gamma_{13}^2\gamma_{23}^4x^3y-16\gamma_{12}^3\gamma_{13}^2\gamma_{23}^4x^3y+4\gamma_{12}^2y^2-8\gamma_{12}^2\gamma_{13}^2y^2+4\gamma_{12}^2\gamma_{13}^4y^2$$

$$+4\gamma_{23}^2y^2-16\gamma_{12}^2\gamma_{23}^2y^2+4\gamma_{12}^4\gamma_{23}^2y^2-8\gamma_{13}^2\gamma_{23}^2y^2+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^2-8\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2y^2+4\gamma_{13}^4\gamma_{23}^2y^2$$

$$-16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2y^2+4\gamma_{12}^4\gamma_{13}^4\gamma_{23}^2y^2+4\gamma_{12}^2\gamma_{23}^4y^2-8\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4y^2$$

$$+4\gamma_{12}^2\gamma_{13}^4\gamma_{23}^4y^2+16\gamma_{12}^2\gamma_{13}^2x^2y^2+16\gamma_{12}^2\gamma_{23}^2x^2y^2+16\gamma_{13}^2\gamma_{23}^2x^2y^2-96\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2y^2$$

$$+16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2x^2y^2+16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2x^2y^2+16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4x^2y^2+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^4y^2-16\gamma_{12}\gamma_{23}^2xy^3$$

$$-16\gamma_{12}^3\gamma_{23}^2xy^3+32\gamma_{12}\gamma_{13}^2\gamma_{23}^2xy^3+32\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2xy^3-16\gamma_{12}\gamma_{13}^4\gamma_{23}^2xy^3-16\gamma_{12}^3\gamma_{13}^4\gamma_{23}^2xy^3$$

$$-64\gamma_{12}\gamma_{13}^2\gamma_{23}^2x^3y^3-64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2x^3y^3+16\gamma_{12}^2\gamma_{23}^2y^4-32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^4+16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2y^4$$

$$+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2y^4-4\gamma_{13}xz+8\gamma_{12}^2\gamma_{13}xz-4\gamma_{12}^4\gamma_{13}xz-4\gamma_{13}^3xz+8\gamma_{12}^2\gamma_{13}^3xz-4\gamma_{12}^4\gamma_{13}^3xz$$

$$+8\gamma_{13}\gamma_{23}^2xz-16\gamma_{12}^2\gamma_{13}\gamma_{23}^2xz+8\gamma_{12}^4\gamma_{13}\gamma_{23}^2xz+8\gamma_{13}^3\gamma_{23}^2xz-16\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xz+8\gamma_{12}^4\gamma_{13}^3\gamma_{23}^2xz$$

$$-4\gamma_{13}\gamma_{23}^4xz+8\gamma_{12}^2\gamma_{13}\gamma_{23}^4xz-4\gamma_{12}^4\gamma_{13}\gamma_{23}^4xz-4\gamma_{13}^3\gamma_{23}^4xz+8\gamma_{12}^2\gamma_{13}^3\gamma_{23}^4xz$$

$$-4\gamma_{12}^4\gamma_{13}^3\gamma_{23}^4xz-16\gamma_{12}^2\gamma_{13}x^3z-16\gamma_{12}^2\gamma_{13}^3x^3z+32\gamma_{12}^2\gamma_{13}\gamma_{23}^2x^3z+32\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2x^3z$$

$$-16\gamma_{12}^2\gamma_{13}\gamma_{23}^4x^3z-16\gamma_{12}^2\gamma_{13}^3\gamma_{23}^4x^3z-4\gamma_{23}yz+8\gamma_{12}^2\gamma_{23}yz-4\gamma_{12}^4\gamma_{23}yz+8\gamma_{13}^2\gamma_{23}yz$$

$$-16\gamma_{12}^2\gamma_{13}^2\gamma_{23}yz+8\gamma_{12}^4\gamma_{13}^2\gamma_{23}yz-4\gamma_{13}^4\gamma_{23}yz+8\gamma_{12}^2\gamma_{13}^4\gamma_{23}yz-4\gamma_{12}^4\gamma_{13}^4\gamma_{23}yz$$

$$-4\gamma_{23}^3yz+8\gamma_{12}^2\gamma_{23}^3yz-4\gamma_{12}^4\gamma_{23}^3yz+8\gamma_{13}^2\gamma_{23}^3yz-16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3yz$$

$$+8\gamma_{12}^4\gamma_{13}^2\gamma_{23}^3yz-4\gamma_{13}^4\gamma_{23}^3yz+8\gamma_{12}^2\gamma_{13}^4\gamma_{23}^3yz-4\gamma_{12}^4\gamma_{13}^4\gamma_{23}^3yz+16\gamma_{12}\gamma_{13}x^2yz$$

$$+16\gamma_{12}^3\gamma_{13}x^2yz+16\gamma_{12}\gamma_{13}^3x^2yz+16\gamma_{12}^3\gamma_{13}^3x^2yz-16\gamma_{12}^2\gamma_{23}x^2yz$$

$$-16\gamma_{13}^2\gamma_{23}x^2yz+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}x^2yz-16\gamma_{12}^4\gamma_{13}^2\gamma_{23}x^2yz-16\gamma_{12}^2\gamma_{13}^4\gamma_{23}x^2yz$$

$$-32\gamma_{12}\gamma_{13}\gamma_{23}^2x^2yz-32\gamma_{12}^3\gamma_{13}\gamma_{23}^2x^2yz-32\gamma_{12}\gamma_{13}^3\gamma_{23}^2x^2yz-32\gamma_{12}^3\gamma_{13}^3\gamma_{23}^2x^2yz$$

$$-16\gamma_{12}^2\gamma_{23}^3x^2yz-16\gamma_{13}^2\gamma_{23}^3x^2yz+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3x^2yz-16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^3x^2yz$$

$$-16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^3x^2yz+16\gamma_{12}\gamma_{13}\gamma_{23}^4x^2yz+16\gamma_{12}^3\gamma_{13}\gamma_{23}^4x^2yz+16\gamma_{12}\gamma_{13}^3\gamma_{23}^4x^2yz$$

$$+16\gamma_{12}^3\gamma_{13}^3\gamma_{23}^4x^2yz-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}x^4yz-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3x^4yz-16\gamma_{12}^2\gamma_{13}xy^2z$$

$$-16\gamma_{12}^2\gamma_{13}^3xy^2z+16\gamma_{12}\gamma_{23}xy^2z+16\gamma_{12}^3\gamma_{23}xy^2z-32\gamma_{12}\gamma_{13}^2\gamma_{23}xy^2z$$

$$-32\gamma_{12}^3\gamma_{13}^2\gamma_{23}xy^2z+16\gamma_{12}\gamma_{13}^4\gamma_{23}xy^2z+16\gamma_{12}^3\gamma_{13}^4\gamma_{23}xy^2z-16\gamma_{13}\gamma_{23}^2xy^2z$$

$$+64\gamma_{12}^2\gamma_{13}\gamma_{23}^2xy^2z-16\gamma_{12}^4\gamma_{13}\gamma_{23}^2xy^2z-16\gamma_{13}^3\gamma_{23}^2xy^2z+64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xy^2z$$

$$-16\gamma_{12}^4\gamma_{13}^3\gamma_{23}^2xy^2z+16\gamma_{12}\gamma_{23}^3xy^2z+16\gamma_{12}^3\gamma_{23}^3xy^2z$$

$$-32\gamma_{12}\gamma_{13}^2\gamma_{23}^3xy^2z-32\gamma_{12}^3\gamma_{13}^2\gamma_{23}^3xy^2z+16\gamma_{12}\gamma_{13}^4\gamma_{23}^3xy^2z+16\gamma_{12}^3\gamma_{13}^4\gamma_{23}^3xy^2z$$

$$-16\gamma_{12}^2\gamma_{13}\gamma_{23}^4xy^2z-16\gamma_{12}^2\gamma_{13}^3\gamma_{23}^4xy^2z+64\gamma_{12}\gamma_{13}^2\gamma_{23}x^3y^2z+64\gamma_{12}^3\gamma_{13}^2\gamma_{23}x^3y^2z$$

$$-64\gamma_{12}^2\gamma_{13}\gamma_{23}^2x^3y^2z-64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2x^3y^2z+64\gamma_{12}\gamma_{13}^2\gamma_{23}^3x^3y^2z+64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^3x^3y^2z$$

$$-16\gamma_{12}^2\gamma_{23}y^3z+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}y^3z-16\gamma_{12}^2\gamma_{13}^4\gamma_{23}y^3z-16\gamma_{12}^2\gamma_{23}^3y^3z$$

$$+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3y^3z-16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^3y^3z-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}x^2y^3z+64\gamma_{12}\gamma_{13}\gamma_{23}^2x^2y^3z$$

$$+64\gamma_{12}^3\gamma_{13}\gamma_{23}^2x^2y^3z+64\gamma_{12}\gamma_{13}^3\gamma_{23}^2x^2y^3z+64\gamma_{12}^3\gamma_{13}^3\gamma_{23}^2x^2y^3z$$

$$-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3x^2y^3z-64\gamma_{12}^2\gamma_{13}\gamma_{23}^2xy^4z-64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xy^4z+4\gamma_{13}^2z^2-8\gamma_{12}^2\gamma_{13}^2z^2$$

$$+4\gamma_{12}^4\gamma_{13}^2z^2+4\gamma_{23}^2z^2-8\gamma_{12}^2\gamma_{23}^2z^2+4\gamma_{12}^4\gamma_{23}^2z^2-16\gamma_{13}^2\gamma_{23}^2z^2$$
$$+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2z^2-16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2z^2+4\gamma_{13}^4\gamma_{23}^2z^2-8\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2z^2$$
$$+4\gamma_{12}^4\gamma_{13}^4\gamma_{23}^2z^2+4\gamma_{13}^2\gamma_{23}^4z^2-8\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4z^2+4\gamma_{12}^4\gamma_{13}^2\gamma_{23}^4z^2$$
$$+16\gamma_{12}^2\gamma_{13}^2x^2z^2+16\gamma_{12}^2\gamma_{23}^2x^2z^2+16\gamma_{13}^2\gamma_{23}^2x^2z^2-96\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2z^2$$
$$+16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2x^2z^2+16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2x^2z^2+16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4x^2z^2+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^4z^2$$
$$-16\gamma_{12}\gamma_{13}^2xyz^2-16\gamma_{12}^3\gamma_{13}^2xyz^2+16\gamma_{13}\gamma_{23}xyz^2-32\gamma_{12}^2\gamma_{13}\gamma_{23}xyz^2$$
$$+16\gamma_{12}^4\gamma_{13}\gamma_{23}xyz^2+16\gamma_{13}^3\gamma_{23}xyz^2-32\gamma_{12}^2\gamma_{13}^3\gamma_{23}xyz^2+16\gamma_{12}^4\gamma_{13}^3\gamma_{23}xyz^2$$
$$-16\gamma_{12}\gamma_{23}^2xyz^2-16\gamma_{12}^3\gamma_{23}^2xyz^2+64\gamma_{12}\gamma_{13}^2\gamma_{23}^2xyz^2+64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2xyz^2$$
$$-16\gamma_{12}\gamma_{13}^4\gamma_{23}^2xyz^2-16\gamma_{12}^3\gamma_{13}^4\gamma_{23}^2xyz^2+16\gamma_{13}\gamma_{23}^3xyz^2-32\gamma_{12}^2\gamma_{13}\gamma_{23}^3xyz^2$$
$$+16\gamma_{12}^4\gamma_{13}\gamma_{23}^3xyz^2+16\gamma_{13}^3\gamma_{23}^3xyz^2-32\gamma_{12}^2\gamma_{13}^3\gamma_{23}^3xyz^2+16\gamma_{12}^4\gamma_{13}^3\gamma_{23}^3xyz^2$$
$$-16\gamma_{12}\gamma_{13}^2\gamma_{23}^4xyz^2-16\gamma_{12}^3\gamma_{13}^2\gamma_{23}^4xyz^2+64\gamma_{12}^2\gamma_{13}\gamma_{23}x^3yz^2+64\gamma_{12}^2\gamma_{13}^3\gamma_{23}x^3yz^2$$
$$-64\gamma_{12}\gamma_{13}^2\gamma_{23}^2x^3yz^2-64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2x^3yz^2+64\gamma_{12}^2\gamma_{13}\gamma_{23}^3x^3yz^2+64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^3x^3yz^2$$
$$+16\gamma_{12}^2\gamma_{13}^2y^2z^2+16\gamma_{12}^2\gamma_{23}^2y^2z^2+16\gamma_{13}^2\gamma_{23}^2y^2z^2-96\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^2z^2+$$
$$16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2y^2z^2+16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2y^2z^2+16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4y^2z^2-64\gamma_{12}\gamma_{13}\gamma_{23}x^2y^2z^2$$
$$-64\gamma_{12}^3\gamma_{13}\gamma_{23}x^2y^2z^2-64\gamma_{12}\gamma_{13}^3\gamma_{23}x^2y^2z^2-64\gamma_{12}^3\gamma_{13}^3\gamma_{23}x^2y^2z^2+128\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2y^2z^2$$
$$-64\gamma_{12}\gamma_{13}\gamma_{23}^3x^2y^2z^2-64\gamma_{12}^3\gamma_{13}\gamma_{23}^3x^2y^2z^2-64\gamma_{12}\gamma_{13}^3\gamma_{23}^3x^2y^2z^2-64\gamma_{12}^3\gamma_{13}^3\gamma_{23}^3x^2y^2z^2$$
$$+64\gamma_{12}^2\gamma_{13}\gamma_{23}xy^3z^2+64\gamma_{12}^2\gamma_{13}^3\gamma_{23}xy^3z^2-64\gamma_{12}\gamma_{13}^2\gamma_{23}^2xy^3z^2-64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2xy^3z^2$$
$$+64\gamma_{12}^2\gamma_{13}\gamma_{23}^3xy^3z^2+64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^3xy^3z^2+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^4z^2-16\gamma_{13}\gamma_{23}^2xz^3$$
$$+32\gamma_{12}^2\gamma_{13}\gamma_{23}^2xz^3-16\gamma_{12}^4\gamma_{13}\gamma_{23}^2xz^3-16\gamma_{13}^3\gamma_{23}^2xz^3+32\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xz^3$$
$$-16\gamma_{12}^4\gamma_{13}^3\gamma_{23}^2xz^3-64\gamma_{12}^2\gamma_{13}\gamma_{23}^2x^3z^3-64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2x^3z^3-16\gamma_{13}^2\gamma_{23}yz^3$$
$$+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}yz^3-16\gamma_{12}^4\gamma_{13}^2\gamma_{23}yz^3-16\gamma_{13}^2\gamma_{23}^3yz^3+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3yz^3$$
$$-16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^3yz^3-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}x^2yz^3+64\gamma_{12}\gamma_{13}\gamma_{23}^2x^2yz^3+64\gamma_{12}^3\gamma_{13}\gamma_{23}^2x^2yz^3$$

$$+64\gamma_{12}\gamma_{13}^3\gamma_{23}^2x^2yz^3+64\gamma_{12}^3\gamma_{13}^3\gamma_{23}^2x^2yz^3-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3x^2yz^3$$

$$+64\gamma_{12}\gamma_{13}^2\gamma_{23}xy^2z^3+64\gamma_{12}^3\gamma_{13}^2\gamma_{23}xy^2z^3-64\gamma_{12}^2\gamma_{13}\gamma_{23}^2xy^2z^3-$$

$$64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xy^2z^3+64\gamma_{12}\gamma_{13}^2\gamma_{23}^3xy^2z^3+64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^3xy^2z^3-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}y^3z^3$$

$$-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3y^3z^3+16\gamma_{13}^2\gamma_{23}^2z^4-32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2z^4+16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2z^4$$

$$+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2z^4-64\gamma_{12}\gamma_{13}^2\gamma_{23}^2xyz^4-64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2xyz^4+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^2z^4$$

The factor $\tilde{\rho}_4(x,y,z,w)$ multiplying $\rho(x)\rho(y)\rho(z)\rho(w)$ in formula (4.83) is given by

$$\frac{(1-\gamma^2)^3\left[1+\gamma^6-\gamma^2(1+\gamma^2)(1+4xz+4yw)+4\gamma^3(x+z)(w+y)\right]}{denominator}$$

with the denominator

$$(1+\gamma^2)^8-4\gamma(1+\gamma^{14})(x+z)(w+y)8\gamma^2(1+\gamma^{12})\times$$

$$\Big[x^2+y^2+z^2+w^2+2(xy+zw)(xw+yz)+4xyzw\Big]$$

$$-4\gamma^3(1+\gamma^{10})(x+z)(w+y)\Big[-5+4(x^2+y^2+z^2+w^2)+4(xz+wy)+16xyzw\Big]$$

$$+16\gamma^4(1+\gamma^8)\Big[-3(x^2+y^2+z^2+w^2)+(x^4+y^4+z^4+w^4)+3(x^2+z^2)(y^2+w^2)+4(x^2z^2+y^2w^2)$$

$$-4xyzw-2(xy+zw)(xw+yz)+8xyzw(x^2+y^2+z^2+w^2)+4(x^2+z^2)(y^2+w^2)(xz+wy)$$

$$+16xyzw(xz+wy)+16x^2y^2z^2w^2\Big]$$

$$-4\gamma^5(1+\gamma^6)(x+z)(w+y)\Big[9-12(x^2+y^2+z^2+w^2+xz+wy)+16(x^2+z^2)(y^2+w^2)$$

$$+16(x^2z^2+y^2w^2)+16xz(x^2+z^2)+16yw(y^2+w^2)+48xyzw+64xyzw(xz+wy)\Big]$$

$$8\gamma^6(1+\gamma^4)\Big[15(x^2+y^2+z^2+w^2)-8(x^4+y^4+z^4+w^4)-2(xy+zw)(xw+yz)$$

$$-24(x^2+z^2)(y^2+w^2)-32(x^2z^2+y^2w^2)-4xyzw+32(x^2y^2z^2+\text{perm})+16x^2z^2(x^2+z^2)$$

$$+16y^2w^2(y^2+w^2)+8(x^2+z^2)(y^4+w^4)+8(x^4+z^4)(y^2+w^2)$$

$$+32xyzw(xz(x^2+z^2)+yw(y^2+w^2))+32xyzw(x^2+z^2)(y^2+w^2)$$

$$+64xyzw(x^2z^2+y^2w^2)+64xyzw(xy+zw)(xw+yz)+32(x^3y^2z^3+\text{cycl})+128x^2y^2z^2w^2\Big]$$

$$-4\gamma^7(1+\gamma^2)(x+z)(w+y)\Big[-5+8(x^2+y^2+z^2+w^2)+8(xz+yw)$$

$$-16(x^2+z^2)(y^2+w^2)-16xz(x^2+z^2+xz)-16yw(y^2+w^2+yw)-64xyzw$$

$$+64xyzw(x^2+y^2+z^2+w^2)+128xyzw(xz+yw)+64(x^3z^3+y^3w^3)+64(x^2y^2z^2+\text{perm})\Big]$$

$$32\gamma^8\Big[-5(x^2+y^2+z^2+w^2)+3(x^4+y^4+z^4+w^4)+9(x^2+z^2)(y^2+w^2)$$

$$+2(xy+zw)(xw+yz)+12(x^2z^2+y^2w^2)+4xyzw-8x^2z^2(x^2+z^2)-8y^2w^2(y^2+z^2)$$

$$-4(x^2+z^2)(y^4+w^4)-4(x^4+z^4)(y^2+w^2)-8xyzw(x^2+y^2+z^2+w^2)-16xyzw(xz+yw)$$

$$-16(x^2y^2z^2+\text{perm})-4(x^2+z^2)(y^2+w^2)(xz+wy)+8(x^4z^4+y^4w^4)+64x^2y^2z^2w^2$$

$$+32xyzw(x^2z^2+y^2w^2)+16xyzw(x^2+z^2)(y^2+w^2)+16xyzw(xz(x^2+z^2)+yw(y^2+w^2))$$

$$+32xyzw(xy+zw)(xw+yz)+16(x^3y^2z^3+\text{cycl})+8(x^2y^2z^4+x^2y^4z^2+x^4y^2z^2+\text{perm})\Big]$$

Note that all the polynomials have the desired symmetry $x \leftrightarrow z, y \leftrightarrow w, (x,y) \leftrightarrow (z,w)$.

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