Uniqueness Theorems for Some Nonlinear Parabolic Equations

by

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Abstract

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We study the uniqueness of solutions of the Cauchy problem of two nonlinear parabolic equations in this thesis.

We first study the uniqueness of the solutions of the initial value problem associated with the ∞-Laplacian operators:

$$\begin{cases} u_t = \Delta_\infty u \\ u|_{t=0} = u_0 \end{cases} \tag{1}$$

where Δ_∞ denotes the ∞-Laplacian. We prove the uniqueness of solutions of (1) in a class of functions with exponential growth.

We also study the uniqueness of the solutions of the evolution associated

with the minimal surface equation:

$$\begin{cases} u_t = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \text{ in } S_T \\ u|_{t=0} = u_0 \end{cases} \tag{2}$$

where $S_T = \mathbb{R}^N \times (0, T), T > 0$.

We obtain a new uniqueness class of solutions for (2), which is reminiscent of the classical results for the heat equation.

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Contents

Chapter 1

Introduction

1.1 Motivation

There is a vast literature (cf. [1],[5],[7],[8]-[11],[16] [20]-[22] and others) devoted to non-linear versions of the Cauchy problem associated with the heat equation. For example, Bènilan, Crandall, and Pierre [11] studied the uniqueness of the initial value problem associated with the porous medium equation, and DiBenedetto and Herrero [9] studied the uniqueness of the initial value problem associated with the p-Laplacian. All those previous results show that the uniqueness of solutions can only hold if we impose some conditions on the growth of the solution $u(x,t)$ as $|x| \to \infty$. In the second chapter, we consider the uniqueness of the solutions of the following initial value problem:

$$\begin{cases} u_t = \Delta_\infty u \\ u|_{t=0} = u_0 \end{cases} \tag{1.1}$$

where Δ_∞ denotes the infinity-Laplacian given by

$$\Delta_\infty u = \frac{\langle D^2 u Du, Du\rangle}{|Du|^2}$$

Here and below the solution u is a real-valued function, Du and D^2u denote respectively its gradient and Hessian matrix. We also denote $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the classical Euclidean norm and inner product in $\mathbb{R}^N$.

For $n = 1$ the equation $u_t = \Delta_\infty u$ is nothing but the classical heat equation, and the well-known counterexample of Tihonov shows that there exists a nonvanishing solution to the classical heat equation with $u_0 \equiv 0$. By adding dummy variables, we obtain a counterexample to the uniqueness also in higher dimensions. It would be interesting to know if the optimal growth rate that guarantees uniqueness for the limit case, i.e. $p = \infty$ is $O(e^{a|x|^2})$ as in the case of the heat equation. We give a partial answer to the above conjecture in this thesis.

For quasilinear parabolic equations, Barles et al.[1] obtained a general comparison result for solutions with polynomial growth but with a restriction on the rate of polynomial growth of the initial data. Many authors studied the corresponding Cauchy problem associated with the infinity-Laplacian operator(see Juutinen and Kawohl[17] etc.). Juutinen and Kawohl obtained a comparison principle of solutions and the uniqueness assuming a linear growth of the solution as $|x| \to \infty$. The aim of the second chapter is to revisit the subject and obtain a new growth condition that guarantees the uniqueness. We prove a new comparison principle and obtain a uniqueness result in a class of solutions with exponential growth at infinity. It is also worth noting that many solutions grow exponentially. For instance, it was shown in [17] that $Ce^{\mu t}\cosh(\sqrt{\mu}|x|), \mu > 0$ is twice differentiable everywhere and satisfies the equation in the viscosity sense also at the points where the spatial gradient vanishes. This particular solution grows exponentially at infinity.

The object of the third chapter is to distinguish a new uniqueness class for solutions of the following Cauchy problem:

$$\begin{cases} u_t = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \text{ in } S_T \\ u|_{t=0} = u_0 \end{cases} \tag{1.2}$$

where $S_T = \mathbb{R}^N \times (0,T)$, $T > 0$.

In the case of Cauchy problem for the classical heat equation,

$$\begin{cases} u_t = \Delta u, \quad \text{on} \quad S_T = \mathbb{R}^N \times (0,T), \\ u|_{t=0} = u_0 \end{cases} \tag{1.3}$$

such classes are well-known. A classical theorem of Tikhonov says that if $u(x,t)$ solves the Cauchy problem for the heat equation with initial data $u_0(x) = 0$ and

$$|u(x,t)| \le ae^{b|x|^2},$$

for some positive constants a and b for all $t > 0, x \in \mathbb{R}^N$, then $u \equiv 0$(see [27]). Without additional conditions on the behavior of the solution for large values of $|x|$, the solution generally cannot be expected to be unique if the domain is unbounded. The broadest class is the Tacklind class(see [26]). Tacklind's theorem asserts that if a solution of the Cauchy problem for the heat equation on the strip $\mathbb{R}^N \times (0, T)$ with a zero initial function satisfies the inequality

$$u(x,t) \leq Ce^{|x|h(|x|)}, \quad \forall (x,t) \in \mathbb{R}^N \times (0,T)$$

with a monotonically nondecreasing function h such the integral

$$\int_0^T \frac{dr}{h(r)}$$

diverges, then $u \equiv 0$ in $\mathbb{R}^N \times (0, T)$.

Gushchin (see[16]) distinguished a uniqueness class which is very close to the Tacklind class.

Theorem 1.1.1. *(Proposition 1, [16]) Let $u = u(x,t)$ be a solution of $u_t = \Delta u$ with $u_0 = 0$. Assume that, for some $x_0 \in \mathbb{R}^N$ and for all $R > 1$,*

$$\int_0^T \int_{B(x_0,R)} u^2(x,t)dxdt \leq \exp(g(R)) \tag{1.4}$$

where $g(r)$ is a positive increasing function on $[1, \infty)$ such that

$$\int_1^\infty \frac{r}{g(r)}dr = \infty \tag{1.5}$$

then $u \equiv 0$.

The above growth conditions (1.4) and (1.5) are optimal for the uniqueness of solutions of the Cauchy problem for the heat equation.

For complete manifolds, similar results have been obtained by Grigor'yan [13] and Karp and Li [19].

Starting from the 1980's, many authors (see [11], [9], [1] etc.) studied uniqueness and existence of solutions of nonlinear versions of the Cauchy problem (1.3). In the second part of this thesis we also study the uniqueness of solutions of the nonlinear evolution associated with the minimal surface equation (1.2).

We obtain a new condition on the growth of solutions that guarantees the uniqueness of solutions as $|x| \to \infty$. Some weaker results were obtained earlier by Barles et al. in [1] as a by-product of a more general investigation.

1.2 Main results

The main result of the first part of this thesis is the following comparison principle:

Theorem 1.2.1. *Let* $u(x,t)$ *and* $v(y,t)$ *be, respectively, a viscosity subsolution and a viscosity supersolution of (1.1). If there exists a constant* $C > 0$

such that

$$u(x,t) - u(y,t) \leq C \exp(\beta(|x|^2 + |y|^2))|x-y| \tag{1.6}$$

$$v(x,t) - v(y,t) \geq -C \exp(\beta(|x|^2 + |y|^2))|x-y| \tag{1.7}$$

for some $\beta > 0$, *for all* $(x,t) \in \mathbb{R}^N \times [0,T]$ *and*

$$u^*(x,0) \leq u_0(x) \leq v_*(x,0)$$

then

$$u(x,t) \leq v(x,t), \quad (x,t) \in \mathbb{R}^N \times [0,T].$$

Remark 1.2.2. From the assumptions (1.5) and (1.6), we can deduce that

$$u(x,t) \leq C_0 e^{\beta'|x|^2} \quad \text{and} \quad v(x,t) \geq -C_0 e^{\beta'|x|^2},$$

for some $\beta' > 0$. For instance, we can take $y = 0$ in (1.5), and since $u(x,t)$ is upper semicontinous function, $u(0,t)$ is bounded on the closed interval $[0,T]$. Similarly, we can get $v(x,t) \geq -C_0 e^{\beta'|x|^2}$ from (1.6).

Using this comparison principle, we can prove a new uniqueness result:

Theorem 1.2.3. *Let* $u(x,t)$ *and* $v(y,t)$ *be two viscosity solutions of (1.1). If there exists a constant* $C > 0$ *such that*

$$u(x,t) - u(y,t) \leq C \exp(\beta(|x|^2 + |y|^2))|x-y| \tag{1.8}$$

$$v(x,t) - v(y,t) \geq -C \exp(\beta(|x|^2 + |y|^2))|x-y| \tag{1.9}$$

for some $\beta > 0$*, for all* $(x,t) \in \mathbb{R}^N \times [0,T]$ *and*

$$u(x,0) = v(x,0)$$

then we have $u \equiv v$*.*

Our comparison result should be compared with the following result of Juutinen and Kawohl assuming a linear growth of the solutions as $|x| \to \infty$:

Theorem 1.2.4. *(see [17],Theorem 4.9)Let* u *and* v *be a viscosity subsolution and a viscosity supersolution, respectively, of (1.1) in* $R^N \times (0,T)$ *such that there exists* $K > 0$ *and a modulus of continuity* ω *so that*

(A1) $u(x,t) \leq K(|x|+1)$ *and* $v(x,t) \geq -K(|x|+1)$ *for all* $(x,t) \in \mathbb{R}^N \times (0,T)$*;*

(A2) $u(x,0) - v(y,0) \leq \omega(|x-y|)$ *for all* $x, y \in \mathbb{R}^N$*;*

(A3) $u(x,0) - v(y,0) \leq K(|x-y|+1)$ *for all* $x, y \in \mathbb{R}^N$*.*

Then $u \leq v$ *in* $\mathbb{R}^N \times (0,T)$*.*

The condition (1.6) could be replaced by the upper bound of the gradient of u and v, hence we state the following:

Theorem 1.2.5. *Let* $u(x,t)$ *and* $v(y,t)$ *be two absolutely continuous viscosity solutions of (1.1). If*

$$|Du(x,t)| \leq C\exp(\beta|x|^2), \quad \forall (x,t) \in \mathbb{R}^N \times [0,T] \tag{1.10}$$

$$|Dv(x,t)| \leq C\exp(\beta|x|^2), \quad \forall (x,t) \in \mathbb{R}^N \times [0,T] \tag{1.11}$$

for some $\beta > 0$*,* $C > 0$ *and*

$$u(x,0) = v(x,0)$$

for every $(x,t) \in \mathbb{R}^N \times [0,T]$*, then* $u \equiv v$*.*

The third chapter of this thesis is dedicated to the study of the uniqueness of solutions of the Cauchy problem associated with the minimal surface equation:

$$\begin{cases} u_t = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \text{ in } S_T \\ u|_{t=0} = u_0 \end{cases} \tag{1.12}$$

where $S_T = \mathbb{R}^N \times (0,T), T > 0$.

We first obtain a uniqueness result, which is reminiscent of the classical result, Theorem 1.1.1:

Proposition 1.2.6. *Let* $u = u(x,t)$ *be a solution to Cauchy problem (1.12) with* $u_0 = 0$*. Assume that, for some* $x_0 \in \mathbb{R}^N$ *and for all* $R > 0$*,*

$$\int_0^T \int_{B(x_0,R)} \frac{u^2(x,t)}{\sqrt{1+|\nabla u|^2}} dxdt \le \exp(f(R)) \tag{1.13}$$

where $f(R)$ *is a positive increasing function on* $[1,\infty)$ *such that*

$$\int_1^\infty \frac{r}{f(r)} dr = \infty \tag{1.14}$$

then $u \equiv 0$*.*

Our main uniqueness result for the Cauchy problem associated with the minimal surface operators is as follows:

Theorem 1.2.7. *Let $u(x,t)$ and $v(x,t)$ be two solutions to the Cauchy problem associated with the minimal surface operators (1.12). Assume that, for some $x_0 \in \mathbb{R}^N$ and for all $R > 0$,*

$$\int_0^T \int_{B(x_0,R)} u^2(x,t)dxdt \leq \exp(f(R)) \tag{1.15}$$

$$\int_0^T \int_{B(x_0,R)} v^2(x,t)dxdt \leq \exp(f(R)) \tag{1.16}$$

where $f(R)$ is a positive increasing function on $[1,\infty)$ such that

$$\int_1^\infty \frac{r}{f(r)}dr = \infty$$

If, furthermore, $u(x,t) \to v(x,t)$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ as $t \to 0$, then $u \equiv v$.

In [1], Barles et al. obtained a uniqueness result for the Cauchy problem (1.12) for viscosity solutions that satisfy a polynomial growth condition with a certain range of powers.

Chapter 2

The ∞-Laplacian heat equation

2.1 Notations and definitions

The appropriate notion of solution when dealing with (1.1) is that of viscosity solution. As there is more than one way of introducing the concept, we fix ideas in the next definition (cf. Juutinen and Kawohl[17]). For more background concerning viscosity solutions, see Giga et al.[12] and Crandall et al.[2] (and cf. Dodziuk [5]).

Definition 2.1.1. Let S_T denote the strip $\mathbb{R}^N \times (0, T)$. An upper semi-continuous function $u(x, t)$ is a viscosity subsolution of $u_t = \Delta_\infty u$ in S_T if, whenever $(x_0, t_0) \in S_T$ and $\varphi(x, t) \in C^2(S_T)$ such that

$$\varphi(x_0, t_0) = u(x_0, t_0)$$

and

$$\varphi(x, t) > u(x, t), \quad \forall (x, t) \in S_T, \quad (x, t) \neq (x_0, t_0)$$

then

$$\begin{cases} \varphi_t(x_0,t_0) \le \Delta_\infty \varphi(x_0,t_0), & \text{if} \quad D\varphi(x_0,t_0) \neq 0 \\ \varphi_t(x_0,t_0) \le \Lambda(D^2\varphi(x_0,t_0)), & \text{if} \quad D\varphi(x_0,t_0) = 0 \end{cases}$$

where $\Lambda(D^2\varphi(x_0,t_0))$ denotes the largest eigenvalue of the Hessian matrix of φ at the point (x_0,t_0).

Analogously, a lower semicontinuous function $v(x,t)$ is a viscosity supersolution of $v_t = \Delta_\infty v$ in S_T if, whenever $(x_0,t_0) \in S_T$ and $\varphi(x,t) \in C^2(S_T)$ such that

$$\varphi(x_0,t_0) = v(x_0,t_0)$$

and

$$\varphi(x,t) < v(x,t), \quad \forall (x,t) \in S_T, \quad (x,t) \neq (x_0,t_0)$$

then

$$\begin{cases} \varphi_t(x_0,t_0) \ge \Delta_\infty \varphi(x_0,t_0), & \text{if} \quad D\varphi(x_0,t_0) \neq 0 \\ \varphi_t(x_0,t_0) \ge \lambda(D^2\varphi(x_0,t_0)), & \text{if} \quad D\varphi(x_0,t_0) = 0 \end{cases}$$

where $\lambda(D^2\varphi(x_0,t_0))$ denotes the smallest eigenvalue of the Hessian matrix of φ at the point (x_0,t_0).

Finally, a continuous function $u(x,t)$ is a viscosity solution of (1) in S_T if it is both a viscosity subsolution and a viscosity supersolution.

Remark 2.1.2. The following type of function is a very useful barrier function for viscosity solutions:

$$\phi(x,t) = \exp\left(2\beta(|x|^2+1)e^{\gamma t}\right)$$

The derivatives of the function are computed as follows:

$$\phi_i = 4\beta x_i e^{\gamma t} \exp\left(2\beta(|x|^2+1)e^{\gamma t}\right)$$

$$\phi_{ij} = 16\beta^2 x_i x_j e^{2\gamma t} \exp\left(2\beta(|x|^2+1)e^{\gamma t}\right)$$

$$\phi_{ii} = 16\beta^2 x_i^2 e^{2\gamma t} \exp\left(2\beta(|x|^2+1)e^{\gamma t}\right) + 4\beta e^{\gamma t} \exp\left(2\beta(|x|^2+1)e^{\gamma t}\right)$$

$$\phi_t = 2\gamma\beta(|x|^2+1)e^{\gamma t} \exp\left(2\beta(|x|^2+1)e^{\gamma t}\right)$$

Hence

$$\begin{aligned}\Delta_\infty\phi - \phi_t &= \left(16\beta^2|x|^2e^{2\gamma t} + 4\beta n e^{\gamma t} - 2\gamma\beta(|x|^2+1)e^{\gamma t}\right)\exp\left(2\beta(|x|^2+1)e^{\gamma t}\right)\\ &= \left((16\beta^2e^{\gamma t} - 2\gamma\beta)|x|^2e^{\gamma t} + (4n\beta - 2\gamma\beta)e^{\gamma t}\right)\exp\left(2\beta(|x|^2+1)e^{\gamma t}\right)\end{aligned}$$

if we choose positive numbers $\gamma > 2n$ and β such that $e^{\gamma T} \leq \frac{\gamma}{8\beta}$, then $\Delta_\infty\phi - \phi_t < 0$. If $\gamma = 8\beta e$, then $e^{\gamma t} \leq e$ when $t < \frac{1}{\gamma}$.

We denote by

$$z^*(x,t) = \limsup_{s\searrow 0}\{z(y,\tau) : |x-y| \leq s, |t-\tau| \leq s\}$$

the upper envelope of a given funtion $z(x,t)$. The definition of lower envelope $z_*(x,t)$ is analogous, with liminf replacing limsup.

We recall the notion of the second-order semi-jets of a function which plays a role of derivatives up to the second-order in usual calculus. Semi-jets are infinitesimal quantities. We shall give an equivalent definition of viscosity solutions by using semi-jets. Such an infinitesimal interpretation of viscosity solutions is useful in proving comparison principles.

Definition 2.1.3. The parabolic super 2-jet of a continuous function z at a point $(\omega, r) \in \mathbb{R}^N \times (0, T)$, denoted by $\mathcal{P}^{2,+}(z(\omega, r))$, is the set of all $(\tau, q, Z) \in \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$ such that

$$\begin{aligned} z(x,t) \leq & z(\omega, r) + \tau(t-r) + q \cdot (x-\omega) + \frac{1}{2} Z(x-\omega) \cdot (x-\omega) \\ & + o(|t-r| + |x-\omega|^2), \forall (x,t) \in R^N \times (0,T) \end{aligned}$$

Analogously, $(\hat{\tau}, \hat{q}, \hat{Z}) \in \mathcal{P}^{2,-}(z(\omega, r))$ if $(\hat{\tau}, \hat{q}, \hat{Z}) \in -\mathcal{P}^{2,+}(-z(\omega, r))$.

2.2 A Comparison Principle

We first introduce a family of auxiliary functions. For $\alpha, \varepsilon, \eta > 0$ and $\beta_0 > \max\{\frac{\beta}{2}, \frac{\beta'}{2}, 1\}$, define

$$\Psi(x,y,t) = \exp((\beta_0|x+y|^2+1)e^{\eta t})(\frac{e^{\beta_0|x-y|^2}-1}{\varepsilon^2} + \alpha) \tag{2.1}$$

$$= K(x+y)P(x-y) \tag{2.2}$$

where β is the same as in Remark 1.2.2.

Here we have set

$$K(w) = \exp((\beta_0|w|^2 + 1)e^{\eta t}), P(z) = \frac{e^{\beta_0|z|^2} - 1}{\varepsilon^2} + \alpha$$

The function $\Psi(x, y, t)$ serves as a suitable barrier for the function

$$\omega(x, y, t) = u(x, t) - v(y, t). \tag{2.3}$$

We also prepare the following fundamental result of User's Guide to viscosity solutions([2], Theorem 8.3).

Lemma 2.2.1. *Let $u(x, t)$ be upper semicontinuous and let $v(x, t)$ be lower semicontinuous. Let $\chi(x, y, t)$ be continuously differentiable in $t \in (0, T)$ and twice continuously differentiable in $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$). Suppose that there exists $(\hat{x}, \hat{y}, \hat{t}) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, T)$ such that*

$$u(x, t) - v(x, t) - \chi(x, y, t) \leq u(\hat{x}, \hat{t}) - v(\hat{x}, \hat{t}) - \chi(\hat{x}, \hat{y}, \hat{t})$$

for all $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, T)$. Assume further that there exists $\omega > 0$ such that, for every $M > 0$, there is a $C_M > 0$ such that $\tau_1, \tau_2 \leq C_M$ whenever

$$(\tau_1, q_1, X) \in \mathcal{P}^{2,+}(u(x, t)); (\tau_2, q_2, Y) \in \mathcal{P}^{2,+}(-v(y, t));$$

$$|x - \hat{x}| + |y - \hat{y}| + |t - \hat{t}| \leq \omega$$

and

$$|u(x, t)| + |q_1| + ||X|| \leq M; |v(x, t)| + |q_2| + ||Y|| \leq M;$$

then for each $\theta > 0$, there exists $X, Y \in S^N$ such that

(a) $(\tau_1, D_x\chi(\hat{x}, \hat{y}, \hat{t}), X) \in \overline{P}^{2,+}(u(\hat{x}, \hat{t}))$;

$(\tau_2, D_y\chi(\hat{x}, \hat{y}, \hat{t}), Y) \in \overline{P}^{2,+}(-v(\hat{x}, \hat{t}))$;

(b) $\tau_1 + \tau_2 = \chi_t(\hat{x}, \hat{y}, \hat{t})$;

(c)

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A + \theta A^2$$

where $A = D^2_{(x,y)}\chi(\hat{x}, \hat{y}, \hat{t})$.

The main tool to prove the uniqueness part of our main theorem is the following comparison principle for the ∞-Laplacian heat equation.

Theorem 2.2.2. *Let $u(x,t)$(respectively, $v(x,t)$) be a viscosity subsolution (respectively, viscosity supersolution) of (1.6). If*

$$u(x,t) - u(y,t) \leq C \exp(\beta(|x|^2 + |y|^2))|x - y|, \tag{2.4}$$

$$v(x,t) - v(y,t) \geq -C \exp(\beta(|x|^2 + |y|^2))|x - y|, \tag{2.5}$$

for some $\beta > 0$ and $C > 0$, and if

$$u^*(x, 0) \leq u_0(x) \leq v_*(x, 0) \tag{2.6}$$

then

$$u(x,t) \leq v(x,t), \quad (x,t) \in \mathbb{R}^N \times [0, T]. \tag{2.7}$$

The following lemma is used in the proof of the comparison principle.

Lemma 2.2.3. *Let* $0 \leq T' < T'' < T$*. Under the assumptions of Theorem 2.2.2, the supremum of* $\omega - \Psi$ *is finite and is attained at a point* $(\hat{x}, \hat{y}, \hat{t}) \in \mathbb{R}^N \times \mathbb{R}^N \times [T', T'']$*. If, in addition, there exists* $(\tilde{x}, \tilde{t}) \in \mathbb{R}^N \times [T', T'']$ *such that*

$$u(\tilde{x}, \tilde{t}) > v(\tilde{x}, \tilde{t})$$

then we can choose parameters α *and* ε *small enough in order to have a positive supremum and* $|\hat{x} - \hat{y}| < 1$*.*

Proof. We first prove that $\omega(x, y, t) - \Psi(x, y, t) \to -\infty$ as $|x|, |y| \to \infty$ (uniformly with respect to $t \in [0, T]$). Since

$$\begin{aligned}
\Psi(x, y, t) &= \exp((\beta_0|x+y|^2 + 1)e^{\eta t})(\frac{e^{\beta_0|x-y|^2} - 1}{\varepsilon^2} + \alpha) \\
&\geq \exp((\beta_0|x+y|^2 + 1)e^{\eta t}) \min\{\alpha, \frac{1}{\varepsilon^2}\}(\exp(\beta_0|x-y|^2) - 1 + 1) \\
&\geq \min\{\alpha, \frac{1}{\varepsilon^2}\}(\exp(\beta_0|x-y|^2 + \beta_0|x+y|^2))
\end{aligned}$$

From the above inequality and Remark 1.2.2, it follows that

$$\begin{aligned}\omega(x,y,t)-\Psi(x,y,t) &\leq \omega(x,y,t)\\ &\qquad -\min\{\alpha,\frac{1}{\varepsilon^2}\}\left(\exp(\beta_0|x-y|^2+\beta_0|x+y|^2)\right)\\ &\leq 2C_0\exp(\beta'(|x|^2+|y|^2))\\ &\qquad -\min\{\alpha,\frac{1}{\varepsilon^2}\}\left(\exp(\beta_0|x-y|^2+\beta_0|x+y|^2)\right)\\ &= 2C_0\exp(\frac{\beta'}{2}(|x-y|^2+|x+y|^2))\\ &\qquad -\min\{\alpha,\frac{1}{\varepsilon^2}\}\left(\exp(\beta_0|x-y|^2+\beta_0|x+y|^2)\right)\end{aligned}$$

which proves the claim since $\beta_0 > \frac{\beta'}{2}$. Since $\Psi(x,y,t)$ is smooth and $\omega(x,y,t)$ is upper semicontinuous, we deduce that there exists $(\hat{x},\hat{y},\hat{t}) \in \mathbb{R}^N \times \mathbb{R}^N \times [T',T'']$ where

$$\max_{\mathbb{R}^N\times\mathbb{R}^N\times[T',T'']}\left(\omega(x,y,t)-\Psi(x,y,t-T')\right)$$

is achieved.

If there exists $(\tilde{x},\tilde{t}) \in \mathbb{R}^N \times [T',T'']$ such that

$$u(\tilde{x},\tilde{t}) > v(\tilde{x},\tilde{t})$$

we have

$$\begin{aligned}u(\hat{x},\hat{t})-v(\hat{y},\hat{t})-\Psi(\hat{x},\hat{y},\hat{t}-T') &\geq u(\tilde{x},\tilde{t})-v(\tilde{x},\tilde{t})\\ &\qquad -\alpha\exp((\beta_0|2\tilde{x}|^2+1)e^{\eta(\tilde{t}-T')})\end{aligned}$$

The right side is positive if α is chosen to be sufficiently small. Hence

$$u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) - \Psi(\hat{x},\hat{y},\hat{t}-T') > 0$$

We see from remark 1.2.2 that

$$\begin{aligned}
&\exp((\beta_0|\hat{x}+\hat{y}|^2+1)e^{\eta(\hat{t}-T')})(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha) \\
&\le u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) \\
&\le C_0(e^{\beta'|\hat{x}|^2}+e^{\beta'|\hat{y}|^2}) \\
&\le 2C_0 e^{\beta'|\hat{x}|^2+\beta'|\hat{y}|^2}
\end{aligned}$$

Using the fact that $\beta'|x|^2+\beta'|y|^2 = \dfrac{\beta'}{2}(|x+y|^2+|x-y|^2)$, it follows that

$$\begin{aligned}
&\exp(\beta_0|\hat{x}+\hat{y}|^2+1)e^{\eta(\hat{t}-T')})(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha) \\
&\le 2C_0\exp(\frac{\beta'}{2}|\hat{x}+\hat{y}|^2+\frac{\beta'}{2}|\hat{x}-\hat{y}|^2)
\end{aligned}$$

Since $\beta_0 > \dfrac{\beta'}{2}$ and $e^{\eta(\hat{t}-T')} \ge 1$, we obtain $\beta_0 e^{\eta(\hat{t}-T')} > \dfrac{\beta'}{2}$. Thus

$$\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2} \le 2C_0\exp(\frac{\beta'}{2}(|\hat{x}-\hat{y}|^2))$$

therefore

$$e^{\beta_0|\hat{x}-\hat{y}|^2} \le 2C_0\varepsilon^2\exp(\frac{\beta'}{2}|\hat{x}-\hat{y}|^2)+1$$

and we obtain

$$\begin{aligned}
e^{(\beta_0-\frac{\beta'}{2})|\hat{x}-\hat{y}|^2} &\le 2C_0\varepsilon^2 + \frac{1}{\exp(\frac{\beta'}{2}|\hat{x}-\hat{y}|^2)} \\
&\le 2C_0\varepsilon^2+1
\end{aligned}$$

Using power series expansion of the left side, we obtain

$$|\hat{x}-\hat{y}|^2 \le \frac{2C_0\varepsilon^2}{\beta_0 - \frac{\beta'}{2}}$$

hence if we choose ε so small such that $\dfrac{2C_0\varepsilon^2}{\beta_0 - \frac{\beta'}{2}} \le 1$, then we conclude that $|\hat{x}-\hat{y}| \le 1$. □

Proof of Theorem 2.2.2. We will show that there exists no point $(\tilde{x},\tilde{t}) \in \mathbb{R}^N \times [0,T]$ with $u(\tilde{x},\tilde{t}) > v(\tilde{x},\tilde{t})$. Any such point would have to lie in one of the strips $\mathbb{R}^N \times [T_p, T_{p+1}]$ with, $T_p = pT_1, p = 0,1,2,3,...$ and we will show inductively, that $(\tilde{x},\tilde{t})$ can not lie in any of these strips, where T_1 will be fixed later. We start with the first strip $\mathbb{R}^N \times [0,T_1]$ and assume, by contradiction, that (2.7) is violated in a strip $\mathbb{R}^N \times [0,T_1]$. Due to Lemma 2.2.3(with $T' = 0, T'' = T_1$), the conclusions of Lemma 2.2.1 with $T' = 0, T'' = T_1$ are valid in this strip. Recalling that Ψ is defined by (2.1) and ω by (2.3), we can then assume that there exists $(\hat{x},\hat{y},\hat{t})$ in this strip such that

$$\sup_{\mathbb{R}^N\times\mathbb{R}^N\times[0,T_1]} \omega(x,y,t) - \Psi(x,y,t) = \omega(\hat{x},\hat{y},\hat{t}) - \Psi(\hat{x},\hat{y},\hat{t})$$

We first consider the case when the supremum is achieved at a point such that $\hat{t} > 0$. By Lemma 2.2.1, applied with $\chi(x,y,t) = \Psi(x,y,t)$, for each

$\theta > 0$ there exist $\tau_1, \tau_2 \in \mathbb{R}$ and $X, Y \in S^N$ such that

$$(\tau_1, D_x\Psi(\hat{x}, \hat{y}, \hat{t}), X) \in \overline{P}^{2,+}(u(\hat{x}, \hat{t}))$$

$$(\tau_2, -D_y\Psi(\hat{x}, \hat{y}, \hat{t}), Y) \in \overline{P}^{2,-}(v(\hat{x}, \hat{t}))$$

$$\tau_1 - \tau_2 = \Psi_t(\hat{x}, \hat{y}, \hat{t})$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \theta A^2 \tag{2.8}$$

and

$$A = D^2_{(x,y)}\Psi(\hat{x}, \hat{y}, \hat{t})$$

On the other hand, we can write (2.8) as, for all $\zeta, \xi \in \mathbb{R}^N$

$$\begin{aligned} \langle X\zeta, \zeta\rangle - \langle Y\xi, \xi\rangle \leq & \langle D^2_{ww}\Psi(\zeta + \xi), \zeta + \xi\rangle + 2\langle D^2_{wz}\Psi(\zeta + \xi), \zeta - \xi\rangle \\ & + \langle D^2_{zz}\Psi(\zeta - \xi), \zeta - \xi\rangle + \theta\langle A^2(\zeta, \xi), (\zeta, \xi)\rangle \end{aligned} \tag{2.9}$$

Since $u(x, t)$(respectively, $v(x, t)$) is a viscosity subsolution (respectively, viscosity supersolution) of (1.6),

$$\tau_1 \leq \frac{\langle XD_x\hat{\Psi}, D_x\hat{\Psi}\rangle}{|D_x\hat{\Psi}|^2} \quad \text{and} \quad \tau_2 \geq \frac{\langle YD_y\hat{\Psi}, D_y\hat{\Psi}\rangle}{|D_y\hat{\Psi}|^2}$$

Subtracting the previous two inequalities, we obtain

$$\tau_1 - \tau_2 = \frac{\partial\Psi}{\partial t} \leq \frac{\langle XD_x\hat{\Psi}, D_x\hat{\Psi}\rangle}{|D_x\hat{\Psi}|^2} - \frac{\langle YD_y\hat{\Psi}, D_y\hat{\Psi}\rangle}{|D_y\hat{\Psi}|^2}$$

In (2.9), we can choose $\zeta = \dfrac{D_x\hat{\Psi}}{|D_x\hat{\Psi}|}$ and $\xi = \dfrac{D_y\hat{\Psi}}{|D_y\hat{\Psi}|}$, noticing that $|\zeta| = |\xi| = 1$. Using these two vectors and letting θ go to 0, it follows from (2.9) that

$$\eta(\beta_0|\hat{x}+\hat{y}|^2+1)e^{\eta\hat{t}}\Psi \le \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3 \tag{2.10}$$

where

$$\mathcal{A}_1 = \langle D^2_{ww}\Psi(\zeta+\xi), \zeta+\xi\rangle$$

$$\mathcal{A}_2 = 2\langle D^2_{wz}\Psi(\zeta+\xi), \zeta-\xi\rangle$$

$$\mathcal{A}_3 = \langle D^2_{zz}\Psi(\zeta-\xi), \zeta-\xi\rangle$$

Next we estimate $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$.

(a). Estimate of $\mathcal{A}_1$.

We have

$$\begin{aligned}\mathcal{A}_1 = ||D^2_{ww}\Psi|||\zeta+\xi|^2 &\le (4\beta_0^2|\hat{x}+\hat{y}|^2e^{2\eta\hat{t}}\Psi + 2\beta_0Ne^{\eta\hat{t}}\Psi)\cdot 4\\ &\le (16\beta_0^2|\hat{x}+\hat{y}|^2e^{\eta\hat{t}} + 8\beta_0N)e^{\eta\hat{t}}\Psi\\ &\le c_1e^{\eta\hat{t}}(\beta_0|\hat{x}+\hat{y}|^2+1)\Psi\end{aligned}$$

where $c_1 = 8N\beta_0$. Here we have used $e^{\eta\hat{t}} < e$, i.e. $\eta\hat{t} < 1$, and η will be fixed later.

(b). Estimate of $\mathcal{A}_2$.

An explicit computation gives $D^2_{wz}\Psi = DK \otimes DP$, recall that

$$K = \exp((\beta_0|\hat{x}+\hat{y}|^2+1)e^{\eta\hat{t}}), P = \frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2} + \alpha$$

We assume without loss of generality that $\hat{x} \neq \hat{y}$ or equivalently $DP \neq 0$, since otherwise $\mathcal{A}_2$ would be 0 and causes no problem. Then we obtain

$$\begin{aligned}
\mathcal{A}_2 &\leq 2|DK||DP||\zeta+\xi||\zeta-\xi| \\
&\leq 16\beta_0|\hat{x}+\hat{y}|e^{\eta\hat{t}}\exp((\beta_0|\hat{x}+\hat{y}|^2+1)e^{\eta\hat{t}})2\beta_0|\hat{x}-\hat{y}|\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}}{\varepsilon^2} \\
&\leq 32\beta_0^2|\hat{x}+\hat{y}|e^{\eta\hat{t}}K(1+\frac{1}{\alpha\varepsilon^2})P \\
&\leq 16\beta_0^2(|\hat{x}+\hat{y}|^2+1)e^{\eta\hat{t}}K(1+\frac{1}{\alpha\varepsilon^2})P \\
&\leq c_2 e^{\eta\hat{t}}(\beta_0|\hat{x}+\hat{y}|^2+1)\Psi
\end{aligned}$$

where $c_2 = 16\beta_0^2(1+\frac{1}{\alpha\varepsilon^2})$ and we have used $|\hat{x}-\hat{y}| < 1$ and $\beta_0 > 1$.

(c). Estimate of $\mathcal{A}_3$

The computation of $D^2_{zz}\Psi$ gives

$$\begin{aligned}
\mathcal{A}_3 &\leq K|D^2P||\zeta-\xi||\zeta-\xi| \\
&\leq (16\beta_0^2|\hat{x}-\hat{y}|^2 + 8\beta_0 N|\hat{x}-\hat{y}|)K\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}}{\varepsilon^2} \\
&\leq (16\beta_0^2+8\beta_0 N)(1+\frac{1}{\alpha\varepsilon^2})\Psi \\
&\leq c_3 e^{\eta\hat{t}}(\beta_0|\hat{x}+\hat{y}|^2+1)\Psi
\end{aligned}$$

where $c_3 = (16\beta_0^2+8\beta_0 N)(1+\frac{1}{\alpha\varepsilon^2})$ and we have used $|\hat{x}-\hat{y}| < 1$.

Plugging the above estimates in (2.10), we obtain

$$\eta(\beta_0|\hat{x}+\hat{y}|^2+1)e^{\eta\hat{t}}\Psi \le \sum_{i=1}^{3}\mathcal{A}_i \le (c_1+c_2+c_3)e^{\eta\hat{t}}(\beta_0|\hat{x}+\hat{y}|^2+1)\Psi \quad (2.11)$$

Here c_1, c_2 and c_3 depend on β_0, α and ε. For any specific choice of the parameters β_0, α and ε, we can choose $\eta = \eta_0 > c_1 + c_2 + c_3$ and set $T_1 = \min\{\frac{1}{\eta_0}, T\}$, we get a contradiction. Thus, for all η sufficiently large, we must have $\hat{t} = 0$.

From the previous step, we know that the maximum of the function

$$u(x,t) - v(y,t) - \Psi(x,y,t)$$

is achieved at a point $(\hat{x}, \hat{y}, 0)$. *Since $\hat{t} = 0$, we won't have the factor $e^{\eta\hat{t}}$ in the following calculations.* It follows from the assumption $u(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) > 0$ that there exists $\delta > 0$ such that

$$0 < \delta \le u(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) - \alpha\exp(\beta_0 e^{\eta\tilde{t}}|2\tilde{x}|^2+1) \quad (2.12)$$

$$\le u(\hat{x},0) - v(\hat{y},0) - \exp(\beta_0|\hat{x}+\hat{y}|^2+1)\left(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha\right) \quad (2.13)$$

for α sufficiently small. Then the above inequality, (2.5) and (2.6) imply

$$\begin{aligned}
\delta \leq & u(\hat{x},0) - u(\hat{y},0) + u(\hat{y},0) - v(\hat{y},0) \\
& - \exp(\beta_0|\hat{x}+\hat{y}|^2+1)(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha) \\
\leq & C\exp(\beta(|\hat{x}|^2+|\hat{y}|^2))|\hat{x}-\hat{y}| \\
& - \exp(\beta_0|\hat{x}+\hat{y}|^2+1)(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha) \\
\leq & C\exp(\frac{\beta}{2}(|\hat{x}+\hat{y}|^2+|\hat{x}-\hat{y}|^2))|\hat{x}-\hat{y}| \\
& - \exp(\beta_0|\hat{x}+\hat{y}|^2+1)(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha)
\end{aligned}$$

Using the fact that $|\hat{x}-\hat{y}| < 1$, we obtain

$$\begin{aligned}
\delta \leq & C'\exp(\frac{\beta}{2}(|\hat{x}+\hat{y}|^2))|\hat{x}-\hat{y}| \\
& - \exp(\beta_0(|\hat{x}+\hat{y}|^2+1))(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha)
\end{aligned}$$

with $C' = Ce^{\beta/2}$.

Notice that

$$\begin{aligned}
C'|\hat{x}-\hat{y}| &= \frac{C'\varepsilon}{\sqrt{\beta_0}\sqrt{2}} \cdot \frac{\sqrt{2}\sqrt{\beta_0}|\hat{x}-\hat{y}|}{\varepsilon} \\
&\leq \frac{1}{2}(\frac{C'^2\varepsilon^2}{2\beta_0} + \frac{2\beta_0|\hat{x}-\hat{y}|^2}{\varepsilon^2}) \\
&= \frac{C'^2\varepsilon^2}{4\beta_0} + \frac{\beta_0|\hat{x}-\hat{y}|^2}{\varepsilon^2}
\end{aligned}$$

Setting $\frac{C'^2}{4\beta_0} = C_1$ and observing that

$$\beta_0|\hat{x}-\hat{y}|^2 \leq e^{\beta_0|\hat{x}-\hat{y}|^2} - 1$$

we obtain

$$
\begin{aligned}
\delta &\le \exp(\frac{\beta}{2}(|\hat{x}+\hat{y}|^2))(C_1\varepsilon^2 + \frac{\beta_0|\hat{x}-\hat{y}|^2}{\varepsilon^2}) \\
&\quad - \exp(\beta_0(|\hat{x}+\hat{y}|^2+1))(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha) \\
&\le C_1\varepsilon^2\exp(\frac{\beta}{2}(|\hat{x}+\hat{y}|^2)) - \alpha\exp(\beta_0(|\hat{x}+\hat{y}|^2+1)) \\
&\le 0
\end{aligned}
$$

If we set $\alpha e^{\beta_0} = C_1\varepsilon^2$ and note that $\beta_0 > \frac{\beta}{2}$, we get a contradiction also for the case $\hat{t} = 0$. Finally, we see that there can exist no point $(\tilde{x}, \tilde{t}) \in \mathbb{R}^N \times [0, T_1]$ where $u(\tilde{x}, \tilde{t}) > v(\tilde{x}, \tilde{t})$. In particular, we have $u(\hat{x}, T_1) \le v(\hat{x}, T_1)$. If $T_1 = T$, we are done with the proof.

If the point $(\tilde{x}, \tilde{t}) \in \mathbb{R}^N \times [T_1, 2T_1]$, then we use $e^{\eta(\hat{t}-T_1)}$ instead of $e^{\eta\hat{t}}$. The above argument applies, we conclude that the supremum is actually achieved at a point $(\hat{x}, \hat{y}, T_1)$. In this case, as in the derivation of (2.13), we obtain

$$
\begin{aligned}
0 < \delta &\le u(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) - \alpha\exp(\beta_0 e^{\eta(\tilde{t}-T_1)}|2\tilde{x}|^2+1) \\
&\le u(\hat{x},T_1) - v(\hat{y},T_1) - \exp(\beta_0|\hat{x}+\hat{y}|^2+1)(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha)
\end{aligned}
$$

Hence

$$\begin{aligned}\delta &\le u(\hat{x},T_1)-u(\hat{y},T_1)+u(\hat{y},T_1)-v(\hat{y},T_1)\\&-\exp(\beta_0|\hat{x}+\hat{y}|^2+1)(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha)\\&\le u(\hat{x},T_1)-u(\hat{y},T_1)-\exp(\beta_0|\hat{x}+\hat{y}|^2+1)(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha)\\&\le C\exp(\beta|\hat{x}|^2+\beta|\hat{y}|^2))|\hat{x}-\hat{y}|\\&-\exp(\beta_0|\hat{x}+\hat{y}|^2+1)(\frac{e^{\beta_0|\hat{x}-\hat{y}|^2}-1}{\varepsilon^2}+\alpha)\end{aligned}$$

where we have used the fact that $u(\hat{y},T_1)-v(\hat{y},T_1)\le 0$ from the first strip. The same argument for the first strip applies in this case, and hence for all the points $(x,t)\in\mathbb{R}^N\times[T_1,2T_1]$ in the second strip we have

$$u(x,t)\le v(x,t)$$

Since such $\tilde{x},\tilde{t}$ must lie in some strip $\mathbb{R}^N\times[T_p,T_{p+1}]$ with $T_p=pT_1, p=0,1,2,3,\cdots$, and the above argument leads to a contradiction in any of such strip, iteratively. Thus the proof of Theorem 2.2.2 is complete.

Remark 2.2.4. If $T_1=T$, then condition (2.4) could be replaced with

$$u_0(x)-u_0(y)\le C\exp(\beta(|x|^2+|y|^2))|x-y|.$$

2.3 Uniqueness

In this final section, we prove the uniqueness of solutions of the Cauchy problem (1.1), in the class of functions with exponential growth.

Theorem 2.3.1. *Let $u(x,t)$ and $v(y,t)$ are two viscosity solutions of (1.1). If*

$$u(x,t) - u(y,t) \leq C\exp(\beta(|x|^2 + |y|^2))|x-y|, \tag{2.14}$$

$$v(x,t) - v(y,t) \geq -C\exp(\beta(|x|^2 + |y|^2))|x-y|, \tag{2.15}$$

for some $\beta > 0$ and $C > 0$, and if

$$u(x,0) = v(x,0) \tag{2.16}$$

then $u \equiv v$.

Proof. Since $u(x,t)$ and $v(y,t)$ are two viscosity solutions of (1.6), $u(x,t)$ (respectively, $v(x,t)$) is a viscosity subsolution (respectively, viscosity supersolution) of (1.6). Therefore, by Theorem 2.2.2, we have

$$u(x,t) \leq v(x,t), \quad (x,t) \in \mathbb{R}^N \times [0,T]$$

Similarly, $v(x,t)$(respectively, $u(x,t)$) is a viscosity subsolution (respectively, viscosity supersolution) of (1.6). Notice that by symmetry in x, y, u actually satisfies (2.15) and v also satisfies (2.14). Therefore, by Theorem 2.2.2, we have also

$$v(x,t) \leq u(x,t), \quad (x,t) \in \mathbb{R}^N \times [0,T]$$

Therefore

$$u(x,t) \equiv v(x,t), \quad (x,t) \in \mathbb{R}^N \times [0,T].$$

The proof is thus complete. □

Chapter 3

The parabolic minimal surface equation

3.1 Formulation of the problem

This chapter is devoted to studying the uniqueness property of solutions to the time-dependent minimal surface equation. We consider the following parabolic equation:

$$u_t = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \tag{3.1}$$

The Dirichlet problem associated with this equation had been considered by Lichnewsky and Temam(cf. [21]), they proved the existence and uniqueness results of the so-called pseudosolution for the evolution problem associated with the minimal surface problem on $\Omega \times (0, T)$ for some open bounded domain $\Omega \subset \mathbb{R}^N$. Lieberman (cf. [22], [23]) also studied the Cauchy-Dirichlet problem and obtained the following existence result for cylindrical domains:

Theorem 3.1.1. *(Theorem 12.18 [23]) Let* $\Omega = \omega \times (0,T)$ *with* $\partial\omega \in C^2$ *and denote by* H' *the mean curvature of* $\partial\omega$*. Then the Cauchy Dirichlet problem* $u_t = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$ *in* Ω*,* $u = \varphi$ *on* $\mathcal{P}\Omega$ *is solvable for arbitrary* $\varphi \in H_{1+\beta}$ *with* $\varphi_t \in L^\infty$ *if and only if* $\sup|\varphi_t| \leq (n-1)\inf H'$.

In this chapter, we obtain a new uniqueness class of solutions of Cauchy problem associated with the parabolic minimal surface equation:

$$\begin{cases} u_t = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \text{ in } S_T \\ u|_{t=0} = u_0 \end{cases} \tag{3.2}$$

where $S_T = \mathbb{R}^N \times (0,T), T > 0$.

In [1], Barles et al. proved a uniqueness result for viscosity solutions of (3.2) under a certain polynomial growth condition on the solutions.

3.2 Preliminaries

We first define two functions which will be used extensively in the proof of the theorems in the sequel. Set $\rho(x)$ to be the distance function from the ball B_R, that is,

$$\rho(x) = (d(x,x_0) - R)_+$$

If we set $s = 2b - a$ so that, for all $t \in [a,b]$,

$$b - a \leq s - t \leq 2(b-a)$$

then

$$\xi(x,t) = -\frac{\rho^2(x)}{4(s-t)} \leq -\frac{\rho^2(x)}{8(b-a)} \leq 0$$

The distributional gradient $\nabla\rho$ is in $L^\infty(\mathbb{R}^N)$ and satisfies the inequality $|\nabla\rho| \leq 1$, which implies, for any $s \neq t$,

$$|\nabla\xi(x,t)| \leq \frac{\rho(x)}{2(t-s)}$$

Since

$$\frac{\partial\xi}{\partial t} = -\frac{\rho^2(x)}{4(t-s)^2}$$

we obtain

$$\frac{\partial\xi}{\partial t} + |\nabla\xi|^2 \leq 0$$

For a given $R > 0$, we define a function $\varphi(x)$ by

$$\varphi(x) = \min\left(\left(3 - \frac{d(x,x_0)}{R}\right)_+, 1\right)$$

From the definition, we see that φ is a Lipschitz function with Lipschitz constant $1/R$ and $|\nabla\varphi| \leq 1/R$.

Consider the function $u\varphi^2 e^\xi$ as a function of x for any fixed $t \in [a,b]$. Since it is obtained from locally Lipschitz functions by taking product and composition, this function is also locally Lipschitz on $\mathbb{R}^N$. Thus, $u\varphi^2 e^\xi \in W^1_c(\mathbb{R}^N)$.

We now present a result used in the proof of our theorems, which uses the ideas of Grigor'yan[13] and Gushchin [15].

Lemma 3.2.1. *Suppose $u(x,t) \in L^2_{\text{loc}}(R^N)$ for all $t \in (0,T)$ and*

$$\int_K u^2(x,t)dx$$

is continuous in t for every compact set K. Assume that for $0 < a < b \leq T$, u satisfies

$$\int_{B_R} u^2(x,b)dx - \int_{B_{4R}} u^2(x,a)dx \leq \frac{4}{R^2}$$

where R satisfies

$$b - a \leq \frac{R^2}{8f(4R)}. \tag{3.3}$$

If $u(x,t) \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ as $t \to 0$, and if $f(r)$ is a positive increasing function on $[1,\infty)$ such that

$$\int_1^\infty \frac{r}{f(r)}dr = \infty \tag{3.4}$$

then $u \equiv 0$.

Proof. Fix $R > 0$ and $t \in [0,T)$. For any non-negative integer k, set

$$R_k = 4^k R$$

and for any $k \geq 1$, choose a (so far arbitrary) number τ_k to satisfy the condition

$$0 < \tau_k \leq c\frac{R_k^2}{f(R_k)} \tag{3.5}$$

where $c = 1/128$, since if $f(r) = r^2$, then $b - a \leq \dfrac{R^2}{8f(4R)} = \dfrac{1}{128}$ and we always want to choose $\tau_k = c\dfrac{R_k^2}{f(R_k)} = c \leq b - a$.

We next define a decreasing sequence of times $\{t_k\}$ inductively by $t_0 = t$ and $t_k = t_{k-1} - \tau_k$. If $t_k > 0$, then the function u satisfies all the conditions of this lemma with $a = t_k$ and $b = t_{k-1}$. Hence

$$\int_{B_{R_{k-1}}} u^2(\cdot, t_{k-1})dx \leq \int_{B_{R_k}} u^2(\cdot, t_k)dx + \frac{4}{R_{k-1}^2} \tag{3.6}$$

If it happens that $t_k = 0$ for some k, then by the initial condition we obtain

$$\int_{B_{R_k}} u^2(\cdot, t_k)dx = 0.$$

In this case, it follows from (3.6) that

$$\int_{B_R} u^2(\cdot, t)dx \leq \sum_{i=1}^{\infty} \frac{4}{R_{i-1}^2} = \frac{C}{R^2}$$

which implies by letting $R \to \infty$ that $u \equiv 0$.

Hence to finish the proof, it suffices to construct, for any $R > 0$ and $t \in (0, T)$, a sequence $\{t_k\}$ as above that vanishes at a finite k. The condition

$t_k = 0$ is equivalent to $t = \tau_1 + \tau_2 + \cdots + \tau_k$. The only restriction is (3.5). The hypothesis that $f(r)$ is an increasing function implies that

$$\int_R^\infty \frac{rdr}{f(r)} \leq \sum_{k=0}^\infty \int_{R_k}^{R_{k+1}} \frac{rdr}{f(r)} \leq \sum_{k=0}^\infty \frac{R_{k+1}^2}{f(R_k)}$$

Therefore, the sequence $\{\tau_k\}$ can be chosen to satisfy simultaneously (3.5) and

$$\sum_{k=1}^\infty \tau_k = \infty$$

By diminishing some of τ_k, we can achieve $t = \tau_1 + \tau_2 + \cdots + \tau_k$ for any finite t, which finishes the proof. □

Remark 3.2.2. It is essential that for every R,

$$\sum_{k=0}^\infty \frac{R_{k+1}^2}{f(R_k)} = \infty.$$

Otherwise, we may find some convergent sequence, for instance if $f(r) = r^3$, then

$$\sum_{k=0}^\infty \frac{R_{k+1}^2}{f(R_k)} = \sum_{k=0}^\infty \frac{(4^{k+1}R)^2}{(4^k R)^3} = \frac{C}{R}$$

If R is sufficient large (as was taken in the proof of the Lemma 3.2.1), then

$$\sum_{k=0}^\infty \frac{R_{k+1}^2}{f(R_k)}$$

would be sufficient small, and therefore we may not be able to get $b - a = \tau_1 + \tau_2 + \cdots + \tau_k$ for time interval (a, b) and finite k.

3.3 Uniqueness results

We have the following uniqueness result for the Cauchy problem for the parabolic minimal surface equation:

Proposition 3.3.1. *Let $u = u(x,t)$ be a solution to Cauchy problem (3.2) with $u_0 = 0$. Assume that, for some $x_0 \in \mathbb{R}^N$ and for all $R > 0$,*

$$\int_0^T \int_{B(x_0,R)} \frac{u^2(x,t)}{\sqrt{1+|\nabla u|^2}} dxdt \leq \exp(f(R)) \tag{3.7}$$

where $f(R)$ is a positive increasing function on $[1,\infty)$ such that

$$\int_1^\infty \frac{r}{f(r)} dr = \infty \tag{3.8}$$

then $u \equiv 0$.

Proof. Multiplying equation (3.1) by $u\varphi^2 e^\xi$(φ and ξ were defined at the beginning of section 3.2), and integrating over $[a,b] \times \mathbb{R}^N$, we obtain

$$\int_a^b \int_{\mathbb{R}^N} \frac{\partial u}{\partial t} u\varphi^2 e^\xi \, dxdt = \int_a^b \int_{\mathbb{R}^N} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) u\varphi^2 e^\xi \, dxdt$$

Since both functions u and ξ are smooth in $t \in [a,b]$, the time integral on the left hand side can be computed as follows:

$$\int_a^b \int_{\mathbb{R}^N} \frac{\partial u}{\partial t} u\varphi^2 e^\xi \, dxdt = \frac{1}{2} u^2\varphi^2 e^\xi |_a^b - \frac{1}{2} \int_a^b \frac{\partial \xi}{\partial t} u^2 \varphi^2 e^\xi dt$$

Applying the product rule and the chain rule to compute $\nabla(u\varphi^2 e^\xi)$, we obtain

$$\begin{aligned}
&-\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot\nabla(u\varphi^2 e^\xi)\\
&=\frac{1}{\sqrt{1+|\nabla u|^2}}\left(-|\nabla u|^2\varphi^2 e^\xi-\langle\nabla u,\nabla\xi\rangle u\varphi^2 e^\xi-2\langle\nabla u,\nabla\varphi\rangle u\varphi e^\xi\right)
\end{aligned}$$

and we also have

$$\begin{aligned}
&-|\nabla u|^2\varphi^2 e^\xi-\langle\nabla u,\nabla\xi\rangle u\varphi^2 e^\xi-2\langle\nabla u,\nabla\varphi\rangle u\varphi e^\xi\\
&\leq-|\nabla u|^2\varphi^2 e^\xi+|\nabla u||\nabla\xi||u|\varphi^2 e^\xi+\frac{1}{2}|\nabla u|^2\varphi^2 e^\xi+2|\nabla\varphi|^2u^2 e^\xi
\end{aligned}$$

Thus

$$\begin{aligned}
&-\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot\nabla(u\varphi^2 e^\xi)\\
&\leq\frac{1}{\sqrt{1+|\nabla u|^2}}\left(\left(-\frac{1}{2}|\nabla u|^2+|\nabla u||\nabla\xi||u|\right)\varphi^2 e^\xi+2|\nabla\varphi|^2u^2 e^\xi\right)
\end{aligned}$$

Combining the above calculations and using the properties of ξ, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} u^2\varphi^2 e^{\xi} dx|_a^b &= \int_a^b \int_{\mathbb{R}^N} \frac{\partial \xi}{\partial t} u^2\varphi^2 e^{\xi} dxdt \\
&+ 2\int_a^b \int_{\mathbb{R}^N} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) u\varphi^2 e^{\xi}\, dxdt \\
&\leq 4\int_a^b \int_{\mathbb{R}^N} \frac{1}{(1+|\nabla u|^2)^{1/2}} |\nabla \varphi|^2 u^2 e^{\xi} dxdt \\
&+ \int_a^b \int_{\mathbb{R}^N} \frac{1}{\sqrt{1+|\nabla u|^2}} \left(\left(-|\nabla \xi|^2 u^2 - |\nabla u|^2 + 2|\nabla u||\nabla \xi||u|\right) \varphi^2 e^{\xi}\right) \\
&= 4\int_a^b \int_{\mathbb{R}^N} \frac{1}{(1+|\nabla u|^2)^{1/2}} |\nabla \varphi|^2 u^2 e^{\xi} dxdt \\
&- \int_a^b \int_{\mathbb{R}^N} \frac{1}{\sqrt{1+|\nabla u|^2}} \left(\left(|\nabla \xi||u| - |\nabla u|\right)^2 \varphi^2 e^{\xi}\right) \\
&\leq 4\int_a^b \int_{\mathbb{R}^N} \frac{1}{(1+|\nabla u|^2)^{1/2}} |\nabla \varphi|^2 u^2 e^{\xi} dxdt
\end{aligned}$$

This immediately implies

$$\begin{aligned}
\int_{B_R} u^2(x,b) e^{\xi(x,b)} dx \leq &\int_{B_{4R}} u^2(x,a) e^{\xi(x,a)} dx \\
&+ \frac{4}{R^2} \int_a^b \int_{B_{4R}\setminus B_{2R}} \frac{1}{(1+|\nabla u|)^{1/2}} u^2 e^{\xi} dxdt
\end{aligned} \tag{3.9}$$

Consequently, we can drop the factor e^{ξ} on the left hand side because $\xi = 0$ in B_R and drop the factor of e^{ξ} in the first integral on the right hand side because $\xi \leq 0$. Clearly, if $x \in B_{4R} \setminus B_{2R}$, then $\rho(x) > R$. Since

$$\xi(x,t) = -\frac{\rho^2(x)}{4(s-t)} \leq -\frac{\rho^2(x)}{4(b-a)} \leq 0$$

it follows from the above inequalities that

$$\xi(x,t) \leq -\frac{R^2}{8(b-a)} \quad \text{in} \quad [a,b] \times B_{4R} \setminus B_{2R}$$

We obtain from (3.9)

$$\int_{B_R} u^2(x,b)dx \leq \int_{B_{4R}} u^2(x,a)dx + \frac{4}{R^2}\exp(-\frac{R^2}{8(b-a)}) \int_a^b \int_{B_{4R}} \frac{1}{(1+|\nabla u|^2)^{1/2}} u^2 dxdt$$

Therefore using

$$\int_a^b \int_{B_{4R}} \frac{1}{(1+|\nabla u|^2)^{1/2}} u^2 dxdt \leq e^{f(4R)}$$

we obtain:

$$\int_{B_R} u^2(x,b)dx \leq \int_{B_{4R}} u^2(x,a)dx + \frac{4}{R^2}\exp(-\frac{R^2}{8(b-a)} + f(4R))$$

Hence

$$\int_{B_R} u^2(x,b)dx - \int_{B_{4R}} u^2(x,a)dx \leq \frac{4}{R^2}$$

Therefore, by Lemma 3.2.1, $u \equiv 0$. □

It is worth noting that the above result is reminiscent of the classical result (Theorem 1.1.1) for the Cauchy problem for the heat equation.

Corollary 3.3.2. *Let $u(x,t)$ be a solution to (3.2) satisfying*

$$|u(x,t)| \leq Ce^{a|x|^2}, \forall t \in (0,T],\ and\, x \in \mathbb{R}^N \tag{3.10}$$

and $u_0 = 0$*, then* $u(x,t) \equiv 0$*. Furthermore, the same is true if* u *satisfies instead of (3.10) the condition*

$$|u(x,t)| \leq Ce^{f(|x|)}, \forall t \in (0,T], \text{ and} x \in \mathbb{R}^N \tag{3.11}$$

where $f(r)$ *is a convex increasing function satisfying (3.8).*

Proof. Since (3.10) is a particular case of (3.11), when $f(x) = c|x|^2$. Hence we need only consider (3.11), which implies

$$\int_0^T \int_{B(x_0,R)} \frac{u^2(x,t)}{\sqrt{1+|\nabla u|^2}} dxdt \leq CR^n e^{f(R)} = Ce^{g(R)} \tag{3.12}$$

where $g(r) := f(r) + n \log r$. The convexity of f implies that $\log r \leq Cf(r)$ for large r. Hence $g(r) \leq Cf(r)$ and function $g(r)$ also satisfies (3.8). Therefore, by Proposition 3.3.1, the proof is complete. □

Before generalizing Proposition 3.3.1, we first derive some elementary facts. By the product rule for divergence, we have

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= \operatorname{div}(\nabla u)\frac{1}{\sqrt{1+|\nabla u|^2}} + \nabla u \cdot \nabla\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right) \\
&= \frac{\Delta u}{\sqrt{1+|\nabla u|^2}} + \nabla u \cdot \nabla\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right)
\end{aligned}$$

Then we have

$$\begin{aligned}
(u-v)_t &= \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\\
&= \operatorname{div}\int_0^1 \left[\frac{d}{ds}\left(\frac{\nabla(su+(1-s)v)}{\sqrt{1+|\nabla(su+(1-s)v)|^2}}\right)\right] ds\\
&= \operatorname{div}\int_0^1 \frac{\nabla(u-v)}{\sqrt{1+|\nabla(su+(1-s)v)|^2}} ds\\
&- \operatorname{div}\int_0^1 \frac{\langle\nabla(u-v), \nabla(su+(1-s)v)\rangle\nabla(su+(1-s)v)}{(1+|\nabla(su+(1-s)v)|^2)^{3/2}} ds
\end{aligned}$$

Therefore $\omega = u - v$ satisfies

$$\omega_t = (a^{i,j}(x,t)\omega_i)_j \tag{3.13}$$

where

$$\begin{aligned}
a^{i,j}(x,\tau) &= \left(\int_0^1 \frac{1}{\sqrt{1+|\nabla(su+(1-s)v)|^2}} ds\right)\delta_{ij}\\
&- \int_0^1 \left(\frac{1}{\sqrt{1+|\nabla(su+(1-s)v)|^2}}\right)^3 (su+(1-s)v)_i(su+(1-s)v)_j ds
\end{aligned}$$

where $A(x,\tau) = (a^{i,j}(x,\tau))_{N\times N}$ is a positive semidefinite matrix satisfying

$$0 \le a^{i,j}(x,\tau)\xi_i\xi_j \le \alpha_0(x,\tau)|\xi|^2, \quad \forall \xi \in \mathbb{R}^N$$

where we take $\alpha_0(x,\tau) = \displaystyle\int_0^1 \frac{1}{\sqrt{1+|\nabla(su+(1-s)v)|^2}} ds.$

Here we obtain a generalization of Proposition 3.3.1:

Theorem 1.2.7 *Let $u(x,t)$ and $v(x,t)$ be two distinct solutions to the*

Cauchy problem (3.2). Assume that, for some $x_0 \in \mathbb{R}^N$ *and for all* $R > 0$,

$$\int_0^T \int_{B(x_0,R)} u^2(x,t)dxdt \leq \exp(f(R))$$
$$\int_0^T \int_{B(x_0,R)} v^2(x,t)dxdt \leq \exp(f(R))$$

where $f(R)$ *is a positive increasing function on* $[1, \infty)$ *such that*

$$\int_1^\infty \frac{r}{f(r)} dr = \infty \tag{3.14}$$

If, furthermore, $u(x,t) \to v(x,t)$ *in* $L^2_{\text{loc}}(\mathbb{R}^N)$ *as* $t \to 0$, *then* $u \equiv v$.

Proof. Multiplying equation (3.13) by $\omega\varphi^2 e^\xi$ (φ and ξ were defined at the beginning of this section), and integrating over $[t_1, t_2] \times \mathbb{R}^N$, we obtain

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \frac{\partial \omega}{\partial t} \omega \varphi^2 e^\xi \, dxdt = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (a^{i,j}(x,t)\omega_i)_j \omega \varphi^2 e^\xi \, dxdt \tag{3.15}$$

Since both functions ω and ξ are smooth in $t \in [a, b]$, the time integral on the left hand side can be computed as follows:

$$\frac{1}{2} \int_{t_1}^{t_2} \frac{\partial(\omega^2)}{\partial t} \varphi^2 e^\xi \, dt = \frac{1}{2} \omega^2 \varphi^2 e^\xi |_{t_1}^{t_2} - \frac{1}{2} \int_{t_1}^{t_2} \frac{\partial \xi}{\partial t} \omega^2 \varphi^2 e^\xi dxdt$$

If we choose the test function $\omega\varphi^2 e^\xi$, then

$$\begin{aligned}\nabla(\omega\varphi e^{\xi/2})A(x,t)\nabla(\omega\varphi e^{\xi/2}) &= \omega^2(x,t)\nabla(\varphi e^{\xi/2})A(x,t)\nabla(\varphi e^{\xi/2}) \\ &+ 2\omega(x,t)\varphi e^{\xi/2}\nabla(\varphi e^{\xi/2})A(x,t)\nabla\omega(x,t) \\ &+ \varphi^2 e^\xi \nabla\omega(x,t)A(x,t)\nabla\omega(x,t)\end{aligned}$$

We combine the above results and note that

$$\nabla \omega A(x,t)\nabla(\omega\varphi^2 e^{\xi}) = a^{i,j}(x,t)\omega_i(\omega\varphi^2 e^{\xi})_j$$

to obtain

$$\begin{aligned}
&\nabla(\omega\varphi e^{\xi/2})A(x,t)\nabla(\omega\varphi e^{\xi/2}) - \omega^2(x,t)\nabla(\varphi e^{\xi/2})A(x,t)\nabla(\varphi e^{\xi/2})\\
&\quad = 2\omega(x,t)\varphi e^{\xi/2}\nabla(\varphi e^{\xi/2})A(x,t)\nabla\omega(x,t)\\
&\qquad + \varphi^2 e^{\xi}\nabla\omega(x,t)A(x,t)\nabla\omega(x,t)\\
&\quad = 2\omega(x,t)\varphi e^{\xi}\nabla\varphi A(x,t)\nabla\omega(x,t) + \omega(x,t)\varphi^2 e^{\xi}\nabla\xi A(x,t)\nabla\omega(x,t)\\
&\qquad + \varphi^2 e^{\xi}\nabla\omega(x,t)A(x,t)\nabla\omega(x,t)\\
&\quad = \nabla\omega A(x,t)\nabla(\omega\varphi^2 e^{\xi})\\
&\quad = a^{i,j}(x,t)\omega_i(\omega\varphi^2 e^{\xi})_j
\end{aligned}$$

In conclusion,

$$\begin{aligned}
&\int_{B_{\sigma+R}} \omega^2(x,t_2)\varphi^2 e^{\xi}dx - \int_{B_{\sigma+R}} \omega^2(x,t_1)\varphi^2 e^{\xi}dx\\
&\quad + 2\int_{t_1}^{t_2}\int_{B_{\sigma+R}} \nabla(\omega\varphi e^{\xi/2})A(x,t)\nabla(\omega\varphi e^{\xi/2})dxdt\\
&= 2\int_{t_1}^{t_2}\int_{B_{\sigma+R}} \omega^2(x,t)\nabla(\varphi e^{\xi/2})A(x,t)\nabla(\varphi e^{\xi/2})dxdt\\
&\quad + \int_{B_{\sigma+R}}\int_{t_1}^{t_2} \frac{\partial\xi}{\partial t}\omega^2\varphi^2 e^{\xi}dxdt
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\int_{B_R} \omega^2(x,t_2)\varphi^2 e^{\xi} dx - \int_{B_R} \omega^2(x,t_1)\varphi^2 e^{\xi} dx \\
&\quad + 2\int_{t_1}^{t_2}\int_{B_R} \nabla(\omega\varphi e^{\xi/2}) A(x,t) \nabla(\omega\varphi e^{\xi/2}) dxdt \\
&\le \int_{B_{\sigma+R}} \omega^2(x,t_2)\varphi^2 e^{\xi} dx - \int_{B_{\sigma+R}} \omega^2(x,t_1)\varphi^2 e^{\xi} dx \\
&\quad + 2\int_{t_1}^{t_2}\int_{B_{\sigma+R}} \nabla(\omega\varphi e^{\xi/2}) A(x,t) \nabla(\omega\varphi e^{\xi/2}) dxdt \\
&\quad + \int_{B_{\sigma+R}\setminus B_R} \omega^2(x,t_1)\varphi^2 e^{\xi} dx \\
&= 2\int_{t_1}^{t_2}\int_{B_{\sigma+R}} \omega^2(x,t) \nabla(\varphi e^{\xi/2}) A(x,t) \nabla(\varphi e^{\xi/2}) dxdt \\
&\quad + \int_{B_{\sigma+R}}\int_{t_1}^{t_2} \frac{\partial \xi}{\partial t}\omega^2\varphi^2 e^{\xi} dxdt + \int_{B_{\sigma+R}\setminus B_R} \omega^2(x,t_1)\varphi^2 e^{\xi} dx
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_{B_R} \omega^2(x,t_2)\varphi^2 e^{\xi} dx \\
&\le 2\int_{t_1}^{t_2}\int_{B_{\sigma+R}} \omega^2(x,\tau)\alpha_0(x,\tau)(|\nabla(\varphi e^{\xi/2})|^2) dxd\tau \\
&+ \int_{B_{\sigma+R}}\int_{t_1}^{t_2} \frac{\partial \xi}{\partial t}\omega^2\varphi^2 e^{\xi} dxdt + \int_{B_{\sigma+R}} \omega^2(x,t_1)\varphi^2 e^{\xi} dx
\end{aligned}
$$

Now we estimate $|\nabla(\varphi e^{\xi/2})|^2$ as follows

$$
\begin{aligned}
|\nabla(\varphi e^{\xi/2})|^2 &= |e^{\xi/2}\nabla\varphi + \varphi\nabla e^{\xi/2}|^2 \\
&\le (e^{\xi/2}|\nabla\varphi| + \varphi|\nabla e^{\xi/2}|)^2 \\
&= e^{\xi}|\nabla\varphi|^2 + \varphi^2|\nabla e^{\xi/2}|^2 + 2\varphi e^{\xi/2}|\nabla\varphi||\nabla e^{\xi/2}| \\
&= e^{\xi}|\nabla\varphi|^2 + \frac{1}{4}\varphi^2 e^{\xi}|\nabla\xi|^2 + \varphi e^{\xi}|\nabla\varphi||\nabla\xi| \\
&= e^{\xi}|\nabla\varphi|^2 + \frac{1}{4}\varphi^2 e^{\xi}|\nabla\xi|^2 + e^{\xi}\frac{1}{\sqrt{2}}\varphi|\nabla\xi|\frac{\sqrt{2}}{1}|\nabla\varphi| \\
&\le e^{\xi}|\nabla\varphi|^2 + \frac{1}{4}\varphi^2 e^{\xi}|\nabla\xi|^2 + \frac{1}{2}e^{\xi}\left((\frac{1}{\sqrt{2}}\varphi|\nabla\xi|)^2 + (\sqrt{2}|\nabla\varphi|)^2\right) \\
&= 2e^{\xi}|\nabla\varphi|^2 + \frac{1}{2}\varphi^2 e^{\xi}|\nabla\xi|^2
\end{aligned}
$$

Combining the above estimates

$$
\begin{aligned}
&\int_{B_R} \omega^2(x,t_2)\varphi^2 e^{\xi}dx \\
&\le 2\int_{t_1}^{t_2}\int_{B_{\sigma+R}} \omega^2(x,\tau)\alpha_0(x,\tau)(2e^{\xi}|\nabla\varphi|^2 + \frac{1}{2}\varphi^2 e^{\xi}|\nabla\xi|^2)dxd\tau \\
&+ \int_{B_{\sigma+R}}\int_{t_1}^{t_2}\frac{\partial\xi}{\partial t}\omega^2\varphi^2 e^{\xi}dxdt + \int_{B_{\sigma+R}} \omega^2(x,t_1)\varphi^2 e^{\xi}dx
\end{aligned}
$$

This immediately implies

$$
\begin{aligned}
\int_{B_R} \omega^2(x,t_2)e^{\xi(x,t_2)}dx \le &\int_{B_{4R}} \omega^2(x,t_1)e^{\xi(x,t_1)}dx \\
&+ \frac{4}{R^2}\int_{t_1}^{t_2}\int_{B_{4R}\setminus B_{2R}} \alpha_0(x,t)\omega^2 e^{\xi}dxdt
\end{aligned}
\tag{3.16}
$$

Consequently, we can drop the factor e^{ξ} on the left hand side because $\xi = 0$ in B_R and drop the factor of e^{ξ} in the first integral on the right hand side because $\xi \leq 0$. Clearly, if $x \in B_{4R} \setminus B_{2R}$, then $\rho(x) > R$. Since

$$\xi(x,t) = -\frac{\rho^2(x)}{4(s-t)} \leq -\frac{\rho^2(x)}{4(t_2-t_1)} \leq 0,$$

it follows from the above inequalities that

$$\xi(x,t) \leq -\frac{R^2}{8(t_2-t_1)} \quad \text{in} \quad [t_1,t_2] \times B_{4R} \setminus B_{2R}$$

We obtain from (3.16)

$$\begin{aligned}\int_{B_R} \omega^2(\cdot,t_2)dx \leq & \int_{B_{4R}} \omega^2(x,t_1)dx \\ & + \frac{4}{R^2}\exp(-\frac{R^2}{8(t_2-t_1)}) \int_{t_1}^{t_2}\int_{B_{4R}} \alpha_0\omega^2 dxdt\end{aligned}$$

Therefore using

$$\begin{aligned}\int_0^T\int_{B_{4R}} u^2(x,t)dxdt \leq e^{f(4R)} \\ \int_0^T\int_{B_{4R}} v^2(x,t)dxdt \leq e^{f(4R)}\end{aligned}$$

and

$$\alpha_0(x,t) = \int_0^1 \frac{1}{\sqrt{1+|\nabla(su+(1-s)v)|^2}}ds \leq 1,$$

we obtain:

$$\begin{aligned}\int_{B_R} \omega^2(x,t_2)dx \leq & \int_{B_{4R}} \omega^2(x,t_1)dx \\ & + \frac{16}{R^2}\exp(-\frac{R^2}{8(t_2-t_1)} + f(4R))\end{aligned}$$

Hence

$$\int_{B_R} \omega^2(x,t_2)dx \leq \int_{B_{4R}} \omega^2(x,t_1)dx + \frac{16}{R^2}$$

The uniqueness is thus proved as in Lemma 3.2.1, that is, $u \equiv v$. □

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