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ON BOUNDED ELEMENTS AND CENTRALIZERS OF GENERALIZED UNIFORM SUBGROUPS OF LOCALLY COMPACT GROUPS.

The City University of New York, Ph.D., 1973
Mathematics

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ON BOUNDED ELEMENTS AND CENTRALIZERS OF GENERALIZED UNIFORM SUBGROUPS OF LOCALLY COMPACT GROUPS

by

Kwan-Yuk Claire Law Sit

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1973

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

April 27, 1973

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§ 0 Introduction.

Let G be a locally compact group and H a closed subgroup of G such that G/H admits a finite invariant measure (we call such a subgroup a generalized uniform subgroup of G). In [1], Borel showed that the centralizer $Z_G(H)$ of H in G is equal to the center Z(G) of G when G is a semi-simple connected Lie group without compact factors. Recently, Greenleaf-Moskowitz-Rothschild [6] extended Borel's result to those connected Lie groups G with the property that Z(G) = B(G) where B(G) is the set of bounded elements of G (an element x of G is bounded if the conjugacy class of x has compact closure). In § 4 we prove a result analogous to that of Greenleaf-Moskowitz-Rothschild for linear algebraic groups (see Corollary 4.5).

In § 1 we inquire, more generally, for a generalized uniform subgroup H of a locally compact group G when $Z_G(H)$ is an $[FD]^-$ group (see § 1 for all compactness conditions on locally compact groups). S. P. Wang [22] showed that $Z_G(H)$ is a [Z] group and therefore an $[FD]^-$ group (see [8]) when G is a connected Lie group and H is discrete. In [14], using Borel's density theorem above D. H. Lee showed that $Z_G(H)$ is an $[FD]^-$ group for connected Lie groups without assuming that H is discrete. Greenleaf-Moskowitz-Rothschild [6] also obtained the same result as D. H. Lee by other means (see below). Again, utiliz-ing Borel's density theorem and a modification of a lemma of Lee we prove the following theorem in § 1.

1.2. <u>Theorem</u>. Let G be a locally connected loacally compact group and H a generalised uniform subgroup of G. Then $Z_G(H)$ is an $[FD]^-$ group.

From this we are able to deduce that $Z_G(H)$ is an $[FD]^-$ group for a large class of locally compact groups G (see 1.10, 1.11 and 1.16). We also prove that under certain additional conditions on the group G, $Z_G(H)$ is a $[Z]$ group and also abelian in certain cases (see 1.13, 1.14, 1.15 and 1.17). However, there is an example of a totally disconnected group G such that $Z_G(H)$ is not an $[FD]^-$ group (see § 1).

In order to investigate the situation for totally disconnected groups G we are led to study the structure of $B(G)$ for linear algebraic groups G over a locally compact field.

Here we recall some definitions and facts about topological and algebraic groups. Let $\mathfrak{A}(G)$ (resp. $\mathfrak{I}(G)$) be the group of topological (resp. inner) automorphisms of the locally compact group G. Following Tits we call an automorphism α of G an automorphism of bounded displacement (a. b. d.) if the set $\{ \alpha(g)\, g^{-1} \mid g \in G \}$ has compact closure. For $x \in G$, x is said to be bounded if the conjugacy class (the $\mathfrak{I}(G)$-orbit) Γ_x of x has compact closure. The set $B(G)$ of all bounded elements of G is a normal subgroup of G and it is easy to see that $B(G)$ is exactly the set of those x in G such that the inner automorphism α_x is an a. b. d. .

Let F be a locally compact non discrete field of characteristic zero. Hence F is either the field $\underset{\sim}{R}$ of real numbers, the field $\underset{\sim}{C}$ of complex numbers or a finite extension of $\underset{\sim}{Q}_p$. By an F-group we mean an algebraic subgroup $\underset{\sim}{G}$ of $GL(n)$ defined over F where $GL(n)$ denotes the group of $n{\times}n$ invertible matrices over an algebraically closed extension of the field F. By the unipotent radical of $\underset{\sim}{G}$ we mean the maximal (Zariski) connected normal unipotent subgroup $\underset{\sim}{N}$ of $\underset{\sim}{G}$. It is well known that $\underset{\sim}{N}$ is defined and split (triangularizable) over F. By an F-split torus in $\underset{\sim}{G}$ we mean an algebraic subgroup $\underset{\sim}{T}$ of $\underset{\sim}{G}$ defined and diagonalizable over F. An F-group $\underset{\sim}{G}$ is said to be F-split or split over F if $\underset{\sim}{G}$ has a maximal torus which is F-split. We denote by G (resp. N, T) the subgroup of all F-rational points in $\underset{\sim}{G}$ (resp. $\underset{\sim}{N}$, $\underset{\sim}{T}$); in other words $G = \underset{\sim}{G} \cap GL(n, F)$. It is known that G is Zariski-dense in $\underset{\sim}{G}$. For the general theory of algebraic groups, the reader is referred to [2].

For the group G we shall mainly consider the topology induced by that of the field F and unless otherwise stated whenever we mention any topological propertity in connection with G, we shall always refer to this topology. On the other hand, we only consider the Zariski topology in $\underset{\sim}{G}$. An automorphism α of $\underset{\sim}{G}$ is F-rational (or an F-automorphism) if α is defined over F. For such F-automorphism α, $\alpha(G) \subseteq G$ and therefore the retriction $\alpha_{|G}$ of α to G is in $\mathfrak{A}(G)$ and is in fact analytic. For an automorphism α of $\underset{\sim}{G}$, we say that α is an a. b. d. if the restriction $\alpha_{|G}$ is an a. b. d. of G, i.e. $\{ \alpha(g)g^{-1} \mid g \in G \}$ has compact closure.

In [20], Tits studied the automorphisms of bounded displacement of connected Lie groups and obtained a structure theorem for $B(G)$. In § 2 of this paper, we study the F-automorphisms of bounded displacement of F-groups $\underset{\sim}{G}$ in the spirit of Tits and obtained a structure theorem for $B(G)$. The main results there are the following three theorems :

2.1 <u>Theorem</u>. Let $\underset{\sim}{G}$ be a connected F-group and α an F-automorphism of $\underset{\sim}{G}$. Then α is an a. b. d. if and only if $\alpha(n) = n$ for all $n \in \underset{\sim}{N}$ and for any F-split torus $\underset{\sim}{T}$ of $\underset{\sim}{G}$ $\alpha(t) = t$ for all $t \in \underset{\sim}{T}$. Moreover when α is an a. b. d., $\alpha(n) = n$ for any unipotent element n in G.

2.2 <u>Theorem</u>. Let $\underset{\sim}{G}$ be a connected F-group. Then

$$B(G) = \bigcap \{ Z_G(T) \mid \underset{\sim}{T} \text{ is an F-split torus in } \underset{\sim}{G} \} \bigcap Z_G(N) ;$$

in particular $B(G)$ is closed.

2.3 <u>Theorem</u>. Let $\underset{\sim}{G}$ be a connected F-group and α be an F-rational a. b. d. . If $\underset{\sim}{G}$ is F-split, then $\alpha = \mathrm{id}$. In particular, $B(G) = Z(G)$.

Let G be a locally compact group, Greenleaf-Moskowitz-Rothschild [5] defined a layering of G as follows : let D be closed $\mathcal{J}(G)$-invariant set in G. A layering of G terminating at D is collection of closed $\mathcal{J}(G)$-invariant subsets $G = X_0 \supset X_1 \supset \cdots \supset X_m = D$ such that, for each $x \in X_j - X_{j+1}$, $j = 0, 1, \cdots, m-1$, there exists a relative neighbourhood V of x in X_j with infinitely many pairwise disjoint conjugates. Since X_{j+1} is closed, we may assume that $V \subset X_j - X_{j+1}$; then the conjugates $\alpha_g(V)$ also lie in this "layer". They [5] applied

Tits' result on B(G) to prove that if G is a connected locally compact group then there exists a layering of G terminating at B(G) and any finite central measure of G is supported in B(G). In a later paper [6], they proved further that B(G) is an $[FD]^-$ group and for a generalized uniform subgroup H of G, $Z_G(H) \subset B(G)$ and therefore $Z_G(H)$ is an $[FD]^-$ group. (The last result being that of D. H. Lee mentioned in the second paragraph.)

In § 3, we use techniques suggested in [5] and [18] and prove the following theorem :

3.1 <u>Theorem</u>. Let $\underset{\sim}{G}$ be a connected F-group. Then there exists a layering of G terminating at B(G).

In § 4, we apply the results obtained in §§ 2,3 to give a supplementary answer for the incomplete investigation in § 2. Specifically we prove the following theorem :

4.3 <u>Theorem</u>. Let $\underset{\sim}{G}$ be a connected F-group and H a generalized uniform subgroup of G. Then $Z_G(H)$ is an $[FD]^-$ group. We note that when the field F is a p-adic field G is totally disconnected.

In § 4, we also prove that any finite central measure on the algebraic group G is supported on B(G) (see 4.1) and that if the generalized uniform subgroup H of an algebraic group G is a centralizer of a point in G, then H is a uniform subgroup, i. e., G/H is compact (see 4.2).

It is my pleasure to thank my thesis advisor Professor Martin Moskowitz who suggested the problem and gave generous advice through the writing of this paper. I would also like to thank Professor Herve Jacquet who has kindly let me consult him on all matters connected with algebraic groups.

§ 1 Centralizer of a generalized uniform subgroup.

Throughout this section G denotes a locally compact group and H a closed subgroup of G. A measure μ on the left coset space G/H is invariant if $\mu(xE) = \mu(E)$ for all $x \in G$ and all Borel sets E in G/H. We say that H is a generalized uniform subgroup if the space G/H admits a finite invariant measure. Our main concern in this section is to prove theorems 1.1 and 1.2. In the course of proving these two theorems we also obtain some results telling when an $[FC]^-$ group is an $[FD]^-$ group (see 1.5 and 1.8).

Recall that an element $g \in G$ is said to be periodic if g is contained in a compact subgroup of G. We denote by $P(G)$ the set of all periodic elements of G ; $P(G)$ needs not be closed nor be a subgroup of G. But in the case where G is an $[FC]^-$ group $P(G)$ is a closed characteristic subgroup of G (see Lemma E below).

1.1 Theorem. Suppose that G is connected and that H is a generalized uniform subgroup of G. Then $P(Z_G(H))$ is compact.

1.2 Theorem. Suppose that G is locally connected and that H is a generalized uniform subgroup of G. Then the commutator subgroup of $Z_G(H)$ has compact closure.

Before proceeding to details of the proof of the main theorems we introduce some notations and list certain classes of locally compact groups as well as several results from various places for the convenience of future reference :

For subsets D, E of G, we denote by $\{D, E\}$ the set

$$\{ [d,e] = ded^{-1}e^{-1} \mid d \in D,\ e \in E \},$$

by $\langle E \rangle$ the subgroup generated by E and by E^{-} the closure of E. The commutator subgroup of any subgroup A of G is denoted by A'.

(1) [IN] = class of locally compact groups G such that G possesses a compact $\mathcal{I}(G)$-invariant neighbourhood of the identity, 1 .

(2) [SIN] = class of locally compact groups G such that every neighbourhood of 1 in G contains an $\mathcal{I}(G)$-invariant neighbourhood of 1.

(3) $[\mathrm{FC}]^{-}$ = class of locally compact groups G such that every element of G is bounded.

(4) $[\mathrm{FD}]^{-}$ = class of locally compact groups G such that G'^{-} is compact.

(5) $[\mathrm{FIA}]^{-}$ = class of locally compact groups G such that $\mathcal{I}(G)$ has compact closure in $\mathfrak{A}(G)$ where $\mathfrak{A}(G)$ is made into a topological group with the compact-open topology (for example see [9]).

(6) [Z] = class of locally compact groups G such that G/ Z(G) is compact.

<u>Lemma</u> A (Dietzmann Lemma, see [13, P.154]. This result was first proved in [9].) If E is a periodic subset of G such that E consists of bounded elements and that E^{-} is compact, then $\langle E \rangle^{-}$ is a compact subgroup of G.

Lemma B [15, p.185]. If G is discrete and is a finitely generated periodic $[FC]^-$ group, then G is finite.

Lemma C [9, p.21]. A compactly generated $[FC]^-$ group G which contains a dense subset of periodic elements is compact.

Lemma D [19, Theorem 3D, p.596]. (This result is stated in [19] without proof. A proof can be found in [15].) Let G be an $[FC]^-$ group, then there exists a compact normal subgroup K of G such that G is an extension $1 \longrightarrow K \longrightarrow G \longrightarrow V \times D \longrightarrow 1$, where V is a vector group and $D \in [FC]^-$ is discrete.

Lemma E (A special case of [9, Theorem 3.16, p.21]). Let G be an $[FC]^-$ group, then $P(G)$ is a closed characteristic $[FC]^-$ subgroup of G and G is an extension

$$1 \longrightarrow P(G) \longrightarrow G \longrightarrow V \times D \longrightarrow 1,$$

where V is a vector group, D a discrete torsionfree abelian group, and $V \times D \in [FC]^-$.

Lemma F [16, p.22]. Let $H \subset L$ be closed subgroups of G. If G/H has finite invariant measure μ, then G/L and L/H both admit finite invariant measures of which μ is a product.

We prove theorems 1.1 and 1.2 in a series of lemmas :

1.3 Lemma. Let H be a generalized uniform subgroup of G, then $Z_G(H)$ is an $[FC]^-$ group.

Proof. See [14, p.197]. There he only dealt with analytic groups ; but the proof is valid for any locally compact group.

1.4 Corollary. Let H be a generalized uniform subgroup of G.

If G is such that $P(G)$ is compact, then $Z_G(H) \in [FD]^-$. In particular if G is aperiodic (i.e. $P(G) = \{1\}$), then $Z_G(H)$ is abelian.

Proof. Since $Z_G(H)$ is an $[FC]^-$ group, it follows from Lemma E that $Z_G(H)' \subset P(Z_G(H))$. As $P(Z_G(H))$ is contained in $P(G)$, the corollary follows easily.

1.5 Lemma. Let G be an $[FC]^-$ group and E be a subset of G such that for all $e \in E$, the inner automorphism α_e of G induced by e leaves E stable ($\alpha_e(E) \subset E$). Suppose that the set $\{E,E\}^-$ is compact and that there exists a finite set $\{x_1 = 1, x_2, \cdots, x_n\}$ of distinct elements of G such that $G = E \bigcup x_2E \bigcup \cdots \bigcup x_nE$. Then G is an $[FD]^-$ group.

Proof. Set $C = \{x_i x_j \mid i, j = 1, \cdots, n\}$,

$$D_1 = \{C,C\},\ D_2 = \{E,E\},\ D_3 = \{C,E\},\ D_4 = \{E,C\}$$

and $D = D_1 \bigcup D_2 \bigcup D_3 \bigcup D_4$.

Then D_1 is finite and D_2 has compact closure. We write D_3 as

$$\bigcup_{c\in C} c\{e c^{-1} e^{-1} \mid e \in E\}.$$

Since every element in G has bounded $\mathcal{J}(G)$-orbit and C is a finite set, it follows that D_3 has compact closure. Similarly D_4 has compact closure and hence D^- is compact. Noticing that $D \subset G' \subset P(G)$, we may apply Lemma A to conclude that $\langle D\rangle^-$ is a compact subgroup of G. Thus 1.5 will be proved if we show that $\langle D\rangle = G'$. For this, it suffices to show that $\{G,G\} \subset \langle D\rangle$. Indeed, take any $g, g' \in G$, we have $g = x_i e$ and $g' = x_j e'$ for some $1 \leq i, j \leq n$ and $e, e' \in E$. Hence

$$\begin{aligned}&[g,g'] \\ &= [x_i,e][e,e'][e'ee'^{-1},x_ix_j][x_i,x_j][x_jx_i,e'][e',x_j],\end{aligned}$$

where each factor in the product is an element of D. So $[g,g']$ is in $\langle D\rangle$ and the proof of 1.5 is complete.

1.6 <u>Corollary</u>. Suppose that $G \varepsilon [FC]^-$ is discrete and that H is a subgroup of G such that G/H is finite. Then $H \varepsilon [FD]^-$ implies $G \varepsilon [FD]^-$.

<u>Proof</u>. It is obvious that H is a subset of G which satifies all the hypothesis of 1.5. Hence G' is compact (finite).

1.7 <u>Lemma</u>. Suppose there exists a compact normal subgroup K of G such that G/K is an $[FD]^-$ group, then G is an $[FD]^-$ group.

<u>Proof</u>. Since $G'^-K/K = (G/K)'^-$ is compact, therefore G'^-K and hence G'^- are compact.

1.8 <u>Lemma</u>. Suppose that G is an $[FC]^-$ group and that H is an $[FD]^-$ subgroup of G such that G/H is compact. Then G is an $[FD]^-$ group.

<u>Proof</u>. By Lemma D, there exists a compact normal subgroup K of G such that G is an extension $1 \longrightarrow K \longrightarrow G \longrightarrow V \times D \longrightarrow 1$, where V is a vector group and $D \varepsilon [FC]^-$ is discrete. Observing that HK/K is a closed subgroup of G/K with $(G/K)/(HK/K)$ compact, in view of Lemma 1.7, we may assume that $G = V \times D$.

Since V is an open normal subgroup of G, the canonical projection $\eta : G \longrightarrow G/V$ is a both open and closed continuous homomorphism. Hence both $\eta(G)/\eta(H)$ and $\eta(H)'^{-}$ ($= \eta(H'^{-})$) are compact (finite). Therefore by (1.6), we have $\eta(G)'^{-} = D'^{-}$ finite. Since $G'^{-} = D'^{-}$, the proof of (1.8) is thus complete.

1.9 *Lemma* (Zassenhaus, Auslander). Let G be a connected Lie group with its radical R, $\pi : G \longrightarrow G/R$ the projection and L a closed subgroup of G. If the 1-component L_o of L is solvable, then the 1-component $(\pi(L)^{-})_o$ of $\pi(L)^{-}$ is solvable.

1.10 *Corollary*. Let G be a connected Lie group whose semi-simple part is compact and L be a closed subgroup of G such that L_o is solvable. Then L is compactly generated.

Remark. (1.10) was obtained by D. H. Lee using Lemma 1.9 (see [14, Lemma C]) which in turn was proved by H. C. Wang [22, Theorem A] under the further assumption that R is simply connected. In order to apply Wang's proof to the present case, we first generalize two technical lemmas in [22].

Let E be a subset of G, by $\mathcal{L}_n(E)$ we mean the set of commutators

$$[g_1,[g_2,[\cdots[g_{n-1},g_n]\cdots]]], \quad g_i \in E$$

of length n, and by $\lim\limits_{n} \mathcal{L}_n(E) = 1$, we mean that given any neighborhood U of 1, there exists an integer n_o such that $\mathcal{L}_n(E) \subset U$ for all $n \geq n_o$.

1.11 *Lemma* (Auslander). Let S be a normal subgroup of a connected Lie group G and $\pi : G \longrightarrow G/S$ be the projection. If S is isomorphic to $T \times V$, where T is a real torus and V a real vector space, then there exists a neighborhood W of 1 in G/S such that for any compact subset K of G with $\pi(K) \subset W$, $\lim_n \mathcal{L}_n(E) = 1$.

In the case where S is isomorphic to a real vector space, (1.11) is [22, Lemma 1, p.210].

Proof of (1.11). Let $\phi : G^\sim \longrightarrow G$ be a universial covering of G, $V^\sim = \phi^{-1}(T \times V)$, then $\phi_{|V^\sim} : V^\sim \longrightarrow T \times V$ is a universial covering of $T \times V$. Let $\pi^\sim : G^\sim \longrightarrow G^\sim/V^\sim$ be the projection, then we have the induced universal covering $\phi^\sim : G^\sim/V^\sim \longrightarrow G/T \times V$ which satisfies the following commutative diagram :

$$\begin{tikzcd}
G^\sim \arrow[r, "\pi^\sim"] \arrow[d, "\phi"] & G^\sim/V^\sim \arrow[d, "\phi^\sim"] \\
G \arrow[r, "\pi"] & G/T \times V
\end{tikzcd}$$

Since $V^\sim$ is a vector space, we have by [22, Lemma 1] a neighborhood $W^\sim$ of $1^\sim$ in $G^\sim/V^\sim$ such that for any subset $K^\sim$ of $G^\sim$ with $\pi^\sim(K^\sim) \subset W^\sim$, $\lim_n \mathcal{L}_n(K^\sim) = 1^\sim$.

Suppose we have the following :

(1) for any compact subset K of G with $\pi(K) \subset \phi^\sim(W^\sim)$, there exists a compact subset $K^\sim$ of $G^\sim$ such that $\phi(K^\sim) = K$ and $\pi(K^\sim) \subset W^\sim$.

Then $W = \phi^\sim(W^\sim)$ is a neighborhood of 1 in $G/T \times V$ such that for any compact subset K of G with $\pi(K) \subset W$, $\lim_n \mathcal{L}_n(K) = 1$. Indeed, let U be any neighborhood of 1 in G, then $\phi^{-1}(U)$ is a neighborhood of $1^\sim$ in $G^\sim$. So there exists an n_o such that $\mathcal{L}_n(K^\sim) \subset \phi^{-1}(U)$ for all $n \geq n_o$. Hence

$$\mathcal{L}_n(K) = \mathcal{L}_n(\phi(K^\sim)) = \phi(\mathcal{L}_n(K^\sim)) \subset U, \quad n \geq n_o.$$

Thus the proof of (1.11) is complete, if we establish (1).

In order to prove (1), we show first that for any $k \in K$, there exists an element $k^{\sim} \in G^{\sim}$ such that $\phi(k^{\sim}) = k$ and $\pi(k^{\sim}) \in W^{\sim}$. Indeed, let $k \in K$, then there exists an $k^* \in W^{\sim}$ such that $\pi(k) = \phi^{\sim}(k^*)$ and hence a $k_1^{\sim} \in G^{\sim}$ such that $\pi^{\sim}(k_1^{\sim}) = k^*$. Therefore

$$\pi(k) = \phi^{\sim}(\pi^{\sim}(k_1^{\sim})) = \pi(\phi(k_1^{\sim})) \quad \text{or} \quad \phi(k_1^{\sim})k^{-1} \in T \times V.$$

Thus there exists an $h^{\sim} \in V^{\sim}$ such that $\phi(h^{\sim}) = \phi(k_1^{\sim})k^{-1}$. Let $k^{\sim} = (h^{\sim})^{-1}k_1^{\sim}$, then $\phi(k^{\sim}) = k$ and $\pi^{\sim}(k^{\sim}) \in W^{\sim}$.

Now let U_k be an open neighborhood of $k^{\sim}$ in $G^{\sim}$ with U_k^- compact and

$$k^{\sim} \in U_k \subset U_k^- \subset (\pi^{\sim})^{-1}(W^{\sim}).$$

Then $\{ \phi(U_k) \mid k \in K \}$ is an open covering of K, and hence there is a finite subcovering $\phi(U_{k_1}) \cup \cdots \cup \phi(U_{k_n}) \supset K$. So the set

$$K^{\sim} = \phi^{-1}(K) \cap (U_{k_1}^- \cup \cdots \cup U_{k_n}^-)$$

has the properties specified in (1). The proof of (1.11) is thus complete.

1.12 Lemma. Let G, S, π, W be as in Lemma 1.11, L be a closed subgroup of G such that L_o is solvable. Then $L\,\pi^{-1}(W)$ generates a solvable group.

In the case where S is isomorphic to a real vector space, (1.12) is [22, Lemma 2, p.210]. That (1.12) follows from (1.11) is exactly the same as [22, Lemma 2] from [22, Lemma 1] ; so we omit the proof here.

The proof of (1.9) is similar to that of [22, Theorem A] with one modification. In the induction step, we replace $G/[R,R]$ by $G/[R,R]^-$ and the replacement is justified since R is topological solvable.

We are now in the position to prove Theorems 1.1 and 1.2.

Proof of (1.1). We show first that it suffices to prove (1.1) in the case where G is an analytic group. Let K be a compact normal subgroup of G with G/K an analytic group and let $\eta : G \longrightarrow G/K$ be the natural projection which is continuous, open and closed. Then $\eta(H)$ is a generalized uniform subgroup of $\eta(G)$. Hence by assumption $P(Z_{\eta(G)}(\eta(H)))$ is compact. It follows from Lemma 1.3 and Lemma E that $P(Z_G(H))$ is a closed subgroup of $Z_G(H)$. And it is easy to see that $\eta(P(Z_G(H)))$ is a closed subgroup contained in $P(Z_{\eta(G)}(\eta(H)))$. Thus $\eta(P(Z_G(H)))$ and hence $P(Z_G(H))$ is compact. This completes the reduction to the analytic case.

Now we assume that G is an analytic group. Let R be the radical of G and C be the maximal connected normal compact subgroup of a semi-simple part of G. As RC/C is a normal subgroup of G/R, RC is a normal subgroup of G and G/RC is a semi-simple connected Lie group without compact factors. Let $\pi : G \longrightarrow G/RC$ be the natural projection. As $\pi(H)^-$ is a closed subgroup of $\pi(G)$ such that $\pi(G)/\pi(H)^- (\cong G/(HRC)^-)$ admits a finite invariant measure, so by Borel's density theorem, we have

$$Z_{\pi(G)}(\pi(H)^-) = Z(\pi(G)).$$

Since

$$\pi(P(Z_G(H))) \subset \pi(Z_G(H)) \subset Z_{\pi(G)}(\pi(H)^-),$$

it follows that $\pi(P(Z_G(H)))$ is an abelian discrete normal subgroup of $\pi(G)$ and hence it is finitely generated. (We note here that $(Z_G(H)\,RC)/RC = \pi(Z_G(H))$ is abelian ; this fact will be used in the proof of Corollary 1.18.) . Hence by Lemma **B**, we have $\pi(P(Z_G(H)))$ finite. Now

$$P(Z_G(H))/(P(Z_G(H)) \bigcap RC) \cong \pi(P(Z_G(H)))$$

and

$$P(Z_G(H)) \bigcap RC \subset P(Z_{RC}(H)),$$

in order to prove $P(Z_G(H))$ to be compact, it suffices to show that $P(Z_{RC}(H))$ is compact.

Set $P = P(Z_{RC}(H))$ and P_o the connected component of 1 of P. Then P_o which is a compactly generated periodic $[FC]^-$ group is compact (see Lemma C). Let S be the semi-simple part of P_o and $\mathfrak{A}(S)$ (resp. $\mathcal{J}(S)$) be the topological (resp. inner) automorphism group of S. As S is normal in P, we can define a homomorphism $\psi : P \longrightarrow \mathfrak{A}(S)$ by $\psi(x) = \alpha_{x|S}$, $x \in P$. Since $\mathfrak{A}(S)/\mathcal{J}(S)$ is finite and $P/\psi^{-1}(\mathcal{J}(S)) \cong \psi(P)/\mathcal{J}(S)$, therefore $\psi^{-1}(\mathcal{J}(S)) = Z_P(S)\,S$ has finite index in P. Thus P will be compact if we show that $Z_P(S)$ is compactly generated.

Since $P_o = Z(P_o)_o \cdot S$, it is easy to see that $Z_P(S)_o = Z(P_o)_o$. Now $Z_P(S)_o \subset Z_P(S) \subset RC$, therefore it follows from (1.10) that $Z_P(S)$ is compactly generated. The proof of (1.1) is complete.

Proof of (1.2). We may assume that H is normal in G. Since the normalizer $N_G(H)$ of H in G is a closed subgroup of G containing H, it follows from Lemma F that $N_G(H)/H$ admits a finite (positive) invariant measure. Also we have $Z_G(H) \subset N_G(H)$ and $Z_{N_G(H)}(H) = Z_G(H)$.

Assuming that H is normal in G, we have G/H compact. Let $\pi : G \longrightarrow G/H$ be the projection, then $\{ \pi(x G_o) \mid x \in G \}$, where G_o denotes the 1-component of G, is an open covering of G/H. Let $x_1 = 1, x_2, \cdots, x_m$ be distinct elements of G such that

$$G/H = \bigcup_{i=1}^{m} \pi(x_i G_o) \qquad \text{and} \qquad G = \bigcup_{i=1}^{m} x_i G_o H.$$

Hence

$$Z_G(H) = \bigcup_{i=1}^{m} Z_{x_i G_o H}(H),$$

and we may arrange the x_i's such that

$$Z_{x_i G_o H}(H) \neq \emptyset \quad \text{for} \quad 1 \leq i \leq n \leq m$$

and

$$Z_{x_i G_o H}(H) = \emptyset \quad \text{for} \quad n < i \leq m.$$

For $2 \leq i \leq n$, pick $a_i \in Z_{x_i G_o H}(H)$, it is easy to see that

$$Z_{x_i G_o H}(H) = a_i Z_{G_o H}(H).$$

Thus $Z_{G_o H}(H)$ is a subgroup of the $[FC]^-$ group $Z_G(H)$ with finite index. Therefore in view of (1.8), it suffices to show that $Z_{G_o H}(H)$ is an $[FD]^-$ group.

Since $G_o / G_o \cap H (\cong G_o H/H)$ is compact, it follows from (1.1) that $P(Z_{G_o}(G_o \cap H))$ is compact. Thus it remains to show that

$$Z_{G_o H}(H)' \subset P(Z_{G_o}(G_o \cap H)).$$

Indeed let $c, c' \in Z_{G_oH}(H)$ with $c = gh$, $g \in G_o$ and $h \in H$. Then

$$[c,c'] = g h c' h^{-1} g^{-1} c'^{-1} = g c' g^{-1} c'^{-1} \in G_o.$$

Hence

$$Z_{G_oH}(H)' \subset G_o \cap Z_{G_oH}(H) = Z_{G_o}(H) \subset Z_{G_o}(G_o \cap H).$$

From Lemma E, we see that all elements of $Z_{G_oH}(H)'$ are periodic. Hence we have $Z_{G_oH}(H)'$ contained in $P(Z_{G_o}(G_o \cap H))$ and this completes the proof of (1.2).

From (1.2), we have, in particular the following theorem :

1.13 <u>Theorem</u>. Let G be a Lie group and H a generalized uniform subgroup of G. Then $Z_G(H) \in [FD]^-$.

Recall that a weak pro-Lie group is a locally compact group G which contains a compact normal subgroup K of G such that G/K is a Lie group. We obtain the following corollary of (1.13) :

1.14 <u>Corollary</u>. Let G be a weak pro-Lie group and H be a generalized uniform subgroup of G. Then $Z_G(H) \in [FD]^-$.

<u>Proof.</u> Let K be a compact normal subgroup of G such that G/K is a Lie group and set $Z = Z_G(H)$. By Lemma F, we see that HK/K is a generalized uniform subgroup of G/K. Hence it follows from (1.13) that $Z_{G/K}(HK/K) \in [FD]^-$. As $ZK/K \subset Z_{G/K}(HK/K)$, therefore we have $ZK/K \in [FD]^-$. Thus by (1.7), we have ZK in $[FD]^-$ and hence $Z \in [FD]^-$.

1.15 <u>Remark</u>. Suppose K is a compact normal subgroup of G such that $Z_{G/K}(HK/K)$ is an $[FD]^-$ group, then as shown above, we have $Z_G(H)$ an $[FD]^-$ group.

1.16 Corollary. Suppose that G is solvable and connected and that H is a generalized uniform subgroup of G. Then $Z_G(H)'^-$ is compact and central.

In the special case where G is analytic and H is discrete this corollary was proved by S. P. Wang [24].

Proof of (1.16). Since G is connected, there exists a compact normal subgroup K of G such that G/K is an analytic group. Therefore by (1.14), we have $Z_G(H)'^-$ compact. Let $\pi : G \longrightarrow G/K$ be the natural projection. Then $\pi(G)'^-$ is the nilradical of $\pi(G)$. Let L be the unique compact subgroup of $\pi(G)'^-$. Then L is central in $\pi(G)'^-$ and hence normal in $\pi(G)$. Therefore $\pi^{-1}(L)$ is normal in G and by [12, Iwasawa], $\pi^{-1}(L)$ is central in G. Now $\pi(Z_G(H)'^-)$ is a compact subgroup contained in L, thus $Z_G(H)'^- \subset \pi^{-1}(L)$ and the corollary follows.

1.17 Corollary. Let G be a connected simply connected Lie group such that $G/\mathrm{rad}(G)$ has no compact factors and H be a generalized uniform subgroup of G. Then $Z_G(H)$ is abelian.

This corollary has been proved by D. H. Lee [14] and in the special case where G is solvable and H discrete, it was proved by R. Tolimieri [21]. In fact, we can obtain the same result on $Z_G(H)$ by slightly weakening the conditions on G (see Corollary 1.18). Corollary 1.18 was also proved by Greenleaf-Moskowitz-Rothschild by different methods [6].

1.18 Corollary. Let G be a connected Lie group such that $G/\mathrm{rad}(G)$ has no compact factors and let H be a generalized uniform subgroup of G. If the center $Z(N)$ of the nilradical N of G is simply connected, then $Z_G(H)$ is abelian.

Proof. As mentioned in the proof of (1.1), we have $(Z_G(H)\,R\,C)/R\,C$ abelian where $R = \mathrm{rad}(G)$ and C is the maximal connected compact normal subgroup of a semi-simple part of G which is trivial in this case. Therefore we have $Z_G(H)'^{-} \subset R$. Let K be a maximal compact subgroup of R that contains $Z_G(H)'^{-}$; then K is connected and hence abelian (see [12]). Let L be the identity component of the normalizer of K in R. Since K is a compact normal abelian subgroup of L, it follows from [12] that $K \subset Z(L)$ and in fact, $K \subset Z(L)_o$. Now

$$Z(L)_o \subset \text{nilradical of } L = L'^{-} \subset R'^{-} = N,$$

and so K is the unique maximal compact subgroup of N and hence $K \subset Z(N)$. Since $Z(N)$ contains no compact subgroups, $K = \{1\}$ and $Z_G(H)$ is abelian.

1.19 Corollary. Let G be an [IN] group and H be a generalized uniform subgroup of G. Then $Z_G(H) \in [FD]^{-}$.

Proof. As any [IN] group is a weak pro-Lie group [9, Theorem 2.11], the corollary follows from (1.14) immediately. However, we shall give a different proof which is elementary.

As in the proof of (1.2), we may assume that H is normal in G and that G/H is compact. Let V be a compact $\mathscr{I}(G)$-invariant neighborhood of 1 in G and U be an open symmetric neighborhood of 1 in G such that $U^2 \subset V$. It can be seen easily that we can select m distinct points $x_1 = 1, x_2, \cdots, x_m$ in G such that

$$G = \bigcup_{i=1}^{m} x_i U H \qquad \text{and} \qquad Z_G(H) = \bigcup_{i=1}^{n} Z_{x_i U H}(H)$$

where $1 \leq n \leq m$ and

$$Z_{x_i U H}(H) \neq \emptyset \quad \text{for} \quad 1 \leq i \leq n,$$

$$Z_{x_i U H}(H) \neq \emptyset \quad \text{for} \quad n < i \leq m.$$

For each $1 \leq i \leq n$, pick an $a_i \in Z_{x_i U H}(H)$ with $a_1 = 1$, then it is easy to see that $x_i U H \subset a_i V H$. Thus

$$Z_G(H) = \bigcup_{i=1}^{n} Z_{a_i V H}(H) = \bigcup_{i=1}^{n} a_i Z_{V H}(H) .$$

If we show that $Z_{V H}(H)$ is $\mathcal{J}(G)$- invariant and that $\{ Z_{V H}(H), Z_{V H}(H) \}^-$ is compact, then we shall have $Z_G(H) \in [FD]^-$ by (1.5). It is obvious that $Z_{V H}(H)$ is $\mathcal{J}(G)$-invariant since both $V H$ and $Z_G(H)$ are $\mathcal{J}(G)$-invariant and $Z_{V H}(H) = V H \bigcap Z_G(H)$. Furthermore, we have $\{ Z_{V H}(H), Z_{V H}(H) \}$ contained in $V V^{-1}$. Indeed, for any $x, x' \in Z_{V H}(H)$, let $x' = v h$ with $v \in V$ and $h \in H$, then

$$[x, x'] = x v h x^{-1} h^{-1} v^{-1} = x v x^{-1} v^{-1} \in V V^{-1}.$$

Hence $\{ Z_{V H}(H), Z_{V H}(H) \}^-$ is compact and this completes the proof of (1.19).

If we impose some conditions on the generalized uniform subgroup H in stead of on G, then there are many cases where $Z_G(H)$ becomes an $[FD]^-$ group or even a $[Z]$ group (since $[FD]^- \subset [Z]$, see [8, p.331]), as show in the proposition below.

1.20 <u>Proposition</u>. Let G be a locally compact group and H be a generalized uniform subgroup of G.

(i) If H is such that H/H_o is compact, then $Z_G(H) \in [FD]^-$.

(ii) If (a) H is a Lie group such that $\mathrm{rad}(H_o)$ is compact or (b) H is compactly generated such that H_o is open and compact, then $Z_G(H) \in [FD]^-$.

(iii) If H is an $[FD]^-$ group, then so is $Z_G(H)$.

(iv) If H is abelian, then $Z_G(H) \in [Z]$.

(v) If $\mathcal{I}(H)$ is closed in $\mathcal{A}(H)$, then $Z_G(H) \in [Z]$.

In particular, if one of the following conditions is satisfied, then $\mathcal{I}(H)$ is closed in $\mathcal{A}(H)$ and (v) follows.

(a) H is a semi-simple Lie group [7, p.337].

(b) H is a connected Lie group and has a lattice [4]. (In particular, H is a $\underset{\sim}{Q}$-rational simply connected nilpotent Lie group — classical result of Malcev).

(c) H is a $[Z]$ group.

(d) H is discrete and finitely generated. (In the particular case where G is an analytic group and H discrete (hence finitely), it was proved by S. P. Wang [23] that $Z_G(H)$ is an $[Z]$ group.)

<u>Proof of (1.20)</u>. As mentioned in the proof of (1.2), we may assume that H is normal in G and that G/H is compact.

(v) Let $\eta : G \longrightarrow \mathcal{A}(H)$ be the continuous homomorphism defined by $\eta(g) = \alpha_{g|H}$. Then $\eta^{-1}(\mathcal{I}(H)) = Z_G(H)\,H$ is a closed subgroup of G and hence $Z_G(H)\,H/H$ is compact. Since the map $Z_G(H) \longrightarrow Z_G(H)\,H/H$ is continuous and the compactness of $Z_G(H)\,H/H$ implies that $Z_G(H)$ is σ-compact, so the map is open and hence $Z_G(H)/Z_G(H) \bigcap H$ is topological isomorphic to $Z_G(H)\,H/H$. Now $Z_G(H) \bigcap H \subset$ center$(Z_G(H))$, therefore $Z_G(H)/$center$(Z_G(H))$ is compact and (v) is proved.

(iv) It is easy to see that $H \subset$ center$(Z_G(H)) \subset Z_G(H)$. Since, by assumption, G/H is compact, it follows that $Z_G(H) \in [Z]$.

(iii) Assuming that H is a normal $[FD]^-$ subgroup of G, we have H'^- a compact normal subgroup of G. Therefore H/H'^- is an abelian normal subgroup of G/H'^- with compact quotient and it follows from (iv) that $Z_{G/H'^-}(H/H'^-) \in [Z]$ and in particular in $[FD]^-$. Thus by Remark 1.15, we have $Z_G(H) \in [FD]^-$.

(ii) (a) Here $\mathrm{rad}(H_o)$ is a compact normal subgroup of G and $H/\mathrm{rad}(H_o)$ is a semi-simple Lie group with compact quotient in $G/\mathrm{rad}(H_o)$. Since $\mathcal{I}(H/\mathrm{rad}(H_o))$ is closed in $\mathfrak{A}(H/\mathrm{rad}(H_o))$ [see 7, p.337], it follows from (v) that $Z_{G/\mathrm{rad}(H_o)}(H/\mathrm{rad}(H_o))$ is an $[FD]^-$ group and hence $Z_G(H) \in [FD]^-$ by (1.15).

(b) Since H/H_o is discrete and finitely generated, it follows that $\mathcal{I}(H/H_o)$ is closed in $\mathfrak{A}(H/H_o)$ and as in the proof of (a) above, we have $Z_G(H) \in [FD]^-$.

(i) Since both H/H_o and $(G/H_o)/(H/H_o)$ are compact, we have G/H_o and hence G/G_o compact. Therefore G is a weak pro-Lie group and (i) follows from (1.14).

We now give an example of a totally disconnected group G with a discrete $[FC]^-$ generalized subgroup H such that $Z_G(H)$ is **not** an $[FD]^-$ group. (See § 4 for a class of totally disconnected groups such that $Z_G(H) \in [FD]^-$.)

Let H be a weak direct sum of countably many copies of a finite simple group, discretely topologized. It is obvious that $H \in [FC]^- \bigcap [SIN] = [FIA]^-$ [see 8, Theorem 4.1, p. 325], that is, $\mathcal{I}(H)^-$ is compact. Let G be the semi-direct product $H \times_\eta \mathcal{I}(H)^-$, where $\eta(\alpha)(h) = \alpha(h)$, $h \in H$ and $\alpha \in \mathcal{I}(H)^-$, with the product topology [see 11]. Then H is a closed normal subgroup of G with G/H compact. An element $(h, \alpha) \in Z_G(H)$ if and only if for all $h' \in H$,

$$(h', id) = (h, \alpha)(h', id)(h, \alpha)^{-1} = (h\alpha(h')h^{-1}, id),$$

in other words, if and only if $\alpha = \alpha_{h^{-1}}$.

Now consider the map $\psi : H \longrightarrow Z_G(H)$ defined by $\psi(h) = (h, \alpha_{h^{-1}})$. It is obvious that ψ is an one-one onto continuous map such that $\psi(hk) = \psi(h)\psi(k)$, $h,k \in H$ (i.e. ψ is an anti-homomorphism). Since H is a countable infinite discrete group, $Z_G(H) = \psi(H)$ is a countable union of closed subsets, namely $\bigcup_{h \in H} \{(h, \alpha_{h^{-1}})\}$. Hence one of the sets $\psi(\{h\}) = \{(h, \alpha_{h^{-1}})\}$ has a non-void interior and it follows that $Z_G(H)$ is discrete and ψ is a homeomorphism. We claim that $Z_G(H)'^-$ is not compact. For if $Z_G(H)$ were compact, then using the fact that ψ is an anti-homorphism and a homeomorphism, we have $\psi(H'^-) = \psi(H)'^- = Z_G(H)'^-$ and hence H'^- would be compact. But $H = H'^-$ is infinite (not compact). So the claim is proved.

§ 2 The structure of B(G).

Throughout this section F is a locally compact non discrete field of characteristic zero (i.e. F is either $\underset{\sim}{R}$, $\underset{\sim}{C}$ or a finite extension of $\underset{\sim}{Q}_p$), $\underset{\sim}{G}$ a connected F-group and G its F-rational points. By a Borel subgroup of $\underset{\sim}{G}$ we mean a maximal solvable connected subgroup $\underset{\sim}{L}$ of $\underset{\sim}{G}$. Any closed subgroup of $\underset{\sim}{G}$ which contains a Borel subgroup is called a parabolic subgroup. Recall that an automorphism α of $\underset{\sim}{G}$ is an a. b. d. of $\underset{\sim}{G}$ if the rstriction $\alpha_{|G}$ of G is an a. b. d. of G, i.e. $\alpha(G) \subset G$ and $\{ \alpha(g) g^{-1} \mid g \in G \}$ has compact closure. An element $x \in G$ is bounded if the inner automorphism α_x of $\underset{\sim}{G}$ is an a. b. d. and $B(G)$ is the set of bounded elements . We denote by $\underset{\sim}{N}$ the unipotent radical of $\underset{\sim}{G}$ and $\mathfrak{J}$ the collection of all F-split tori $\underset{\sim}{T}$ in $\underset{\sim}{G}$. Section 2 concerns itself with the proof of the following three theorems.

2.1 Theorem. Let $\underset{\sim}{G}$ be a connected F-group and α an F-automorphism of $\underset{\sim}{G}$. Then α is an a. b. d. if and only if $\alpha(n) = n$ for all $n \in \underset{\sim}{N}$ and for all $\underset{\sim}{T} \in \mathfrak{J}$, $\alpha(t) = t$, for all $t \in \underset{\sim}{T}$. Moreover if α is an a. b. d., then $\alpha(n) = n$ for any unipotent element n in G.

2.2 Theorem. Let $\underset{\sim}{G}$ be a connected F-group. Then

$$B(G) = \bigcap \{ Z_G(T) \mid T \in \mathfrak{J} \} \bigcap Z_G(N) ;$$

in particular $B(G)$ is closed.

2.3 Theorem. Let $\underset{\sim}{G}$ be a connected F-group and α be an F-rational a. b. d. of $\underset{\sim}{G}$. If $\underset{\sim}{G}$ is F-split, then $\alpha = \mathrm{id}$ and in particular $B(G) = Z(G)$.

Theorems 2.1 and 2.2 prove the analogue of Tits' results for real analytic groups while (2.3) is analogus to various results of Tits and Greenleaf-Moskowitz-Rothschild.

We begin our proof with the following lemma :

2.4 Lemma. Suppose $X, Y \in M(n, F)$ are nilpotent matrices such that the set $\{(\exp tX)(\exp -tY) \mid t \in F\}$ has compact closure. Then $X = Y$.

Proof. Since X and Y are nilpotent, $P(t) = (\exp tX)(\exp -tY)$ is a polynomial in t with coefficients in $M(n, F)$. In other words each entry $P_{ij}(t)$ of $P(t)$ is a polynomial in t with coefficients in F. That the set $\{ P(t) \mid t \in F \}$ has compact closure is equivalent to that each set $\{ |P_{ij}(t)| \mid t \in F \}$ is bounded. Hence each $P_{ij}(t)$ is a constant and $P(t)$ is a constant matrix. Thus $\frac{d}{dt} P = 0$ and in particular

$$0 = \frac{d}{dt} P(t) \Big|_{t=0} = -Y + X .$$

This proves the lemma.

Remark. Suppose that G is a complex analytic group and $\mathfrak{g}$ its Lie algebra. If $X, Y \in \mathfrak{g}$ are such that $\{(\exp tX)(\exp -tY) \mid t \in \underset{\sim}{C}\}$ has compact closure in G, then $X = Y$ by Liouville's theorem. Hence by a similar argument as in the proof of (2.1) below we have that any complex analytic automorphism of bounded displacement of G is trivial on the closure of the range $\exp \mathfrak{g}$. In particular any complex analytic group of type (E) has no non trivial complex analytic a. b. d. (cf. [5, Theorem 10.5]).

Before proving (2.1), we recall some facts about a torus. The group of all characters of a torus $\underset{\sim}{T}$ defined and split over F is the set of all group homomorphisms $\psi : \underset{\sim}{T} \longrightarrow GL(1)$. Since $\underset{\sim}{T}$ is F-split, all the characters of $\underset{\sim}{T}$ are defined over F. Now $\underset{\sim}{T}$ is isomorphic to $GL(1)^r$ for some natural number r, so we may write for each $t \in \underset{\sim}{T}$, $t = (a_1, \cdots, a_r)$, $a_i \in GL(1)$ and take the F-rational points T of $\underset{\sim}{T}$ to be $GL(1, F)^r$. The mappings $\sigma_i : \underset{\sim}{T} \longrightarrow GL(1)$ defined by $\sigma_i(t) = a_i$, $t \in \underset{\sim}{T}$, $i = 1, \cdots, r$ form a basis for the group of characters of $\underset{\sim}{T}$. Therefore for each character ψ of $\underset{\sim}{T}$,

$$\psi(t) = \prod_{i=1}^{r} \sigma_i(t)^{m(i)} = \prod_{i=1}^{r} a_i^{m(i)}$$

for some integers $m(i)$.

We claim that if the set $\{ \psi(t) \mid t \in T \}$ is bounded (with respect to the topology of F), then $\psi(t) = 1$ for all $t \in \underset{\sim}{T}$. Indeed, for each $i = 1, \cdots, r$, let $t = (1, \cdots, 1, a_i, 1, \cdots, 1)$, $a_i \in F^{\times}$, then we have $\psi(t) = a_i^{m(i)}$ and $\{ a_i^{m(i)} \mid a_i \in F^{\times} \}$ bounded. Therefore for each $i = 1, \cdots, r$, $m(i) = 1$ or equivalently $\psi = 1$.

<u>Proof of (2.1)</u>. First we show the only if part. Let α be an F-rational a. b. d. of $\underset{\sim}{G}$ and $\underset{\sim}{T} \in \mathcal{T}$, then $\alpha(\underset{\sim}{T})$ is again an F-split torus. Let $\{ v_1, \cdots, v_n \}$ be a basis of F^n consisting of simultaneous eigen vectors of $\alpha(\underset{\sim}{T})$ and $w = \sum_{i=1}^{n} b_i v_i$ be a simultaneous eigen vector of $\underset{\sim}{T}$. We have for each $s \in \alpha(\underset{\sim}{T})$, $s(v_i) = \psi_i(s) v_i$, $i = 1, \cdots, n$ and for each $t \in \underset{\sim}{T}$, $t(w) = \phi(t) w$ where each ψ_i, $i = 1, \cdots, n$ (resp. ϕ) is a character of $\alpha(\underset{\sim}{T})$ (resp. $\underset{\sim}{T}$).

Since α is an a. b. d. , the set

$$(1) \qquad \{ \alpha(t)\, t^{-1} \mid t \in T \}$$

has compact closure and hence so does the set

$$\{ \alpha(t)\, t^{-1}(w) \mid t \in T \}$$

of transforms of w by elements of (1). Now for all $t \in \underset{\sim}{T}$,

$$\alpha(t)\, t^{-1}(w) = \phi(t^{-1})\, \alpha(t)(\textstyle\sum b_i v_i) = \sum b_i\, \phi(t^{-1})\, \psi_i(\alpha(t)) v_i .$$

Denote by ϕ^{-1} the map defined by $\phi^{-1}(t) = \phi(t^{-1})$. It is easy to see that for each $i = 1, \cdots, n$, $\phi^{-1} \cdot \psi_i \circ \alpha$ is a character of $\underset{\sim}{T}$ and that for $b_i \neq 0$, $\phi^{-1} \cdot \psi_i \circ \alpha$ is bounded on T. Hence for $b_i \neq 0$, we have $\phi^{-1} \cdot \psi_i \circ \alpha = 1$ or $\phi(t) = \psi_i(\alpha(t))$ for all $t \in \underset{\sim}{T}$. Hence for all $t \in \underset{\sim}{T}$, we have

$$t(w) = \phi(t) w = \textstyle\sum b_i\, \psi_i(\alpha(t)) v_i = \sum b_i\, \alpha(t)(v_i) = \alpha(t)(w).$$

Since w is an arbitrary eigen vector for t and both t and $\alpha(t)$ are semi-simple, we have $\alpha(t) = t$ for all $t \in \underset{\sim}{T}$.

Now we show that for any unipotent element $g \in G$, $\alpha(g) = g$. For then it follows from this that α is trivial on $\underset{\sim}{N}$ since α is defined over F and N is Zariski-dense in $\underset{\sim}{N}$. Thus let $d\alpha$ be the differential of α and X be the nilptent matrix in $M(n, F)$ such that $\exp X = g$. Then $d\alpha(X)$ is again nilpotent. Since α is an a. b. d., the set

$$\{ \alpha(\exp tX)(\exp tX)^{-1} \mid t \in F \} = \{ (\exp t\, d\alpha(X))(\exp -tX) \mid t \in F \}$$

has compact closure. So by (2.4), we have $d\alpha(X) = X$. Hence $\alpha(\exp X) = \exp d\alpha(X) = \exp X$ or $\alpha(g) = g$. Thus the only if part and the last statement of (2.1) are proved.

To prove the if part, we show first that for any $\underset{\sim}{T} \in \mathcal{T}$,

$g \in G$, $[\alpha(g)\, g^{-1}, t] = 1$ for all $t \in \mathbf{T}$. Since $g^{-1}\mathbf{T}g$ is again a torus defined and split over F, we have, for all $t \in \mathbf{T}$,

$$g^{-1}t\,g = \alpha(\,g^{-1}t\,g\,) = \alpha(g)^{-1}t\,\alpha(g)\,.$$

And so $[\,\alpha(g)g^{-1}, t\,] = 1$.

Now let $\mathbf{G} = \mathbf{D}\cdot\mathbf{N}$ and $G = D \cdot N$ where $\mathbf{D}$ is a Levi F-subgroup of $\mathbf{G}$. Let $\mathbf{P}$ be a minimal parabolic subgroup of $\mathbf{D}$ and $\mathbf{M}$ be a Levi F-subgroup of $\mathbf{P}$. Then there exists a maximal F-split torus $\mathbf{S}$ in $\mathbf{M}$ such that $\mathbf{M} = Z_{\mathbf{D}}(\mathbf{S})$ and M/S is compact [see 3, Corollary 4.16 and Proposition 9.3]. Hence there is a compact set C in M such that $M = C\,S$. By a theorem of Bruhat and Tits [see for example 8], we have $H = KMK$ where K is a compact subgroup of H. Hence $G = KCSKN$ and for any g in G, we have $g = k\,c\,s\,k'\,n$ where $k, k' \in K$, $c \in C$, $s \in S$ and $n \in N$. So

$$\begin{aligned}
\alpha(g)\, g^{-1} &= \alpha(k)\alpha(c)\alpha(s)\alpha(k')\alpha(n)\, n^{-1}\, k'^{-1}\, s^{-1}\, c^{-1}\, k^{-1} \\
&= \alpha(k)\alpha(c)\, s\, \alpha(k')\, k'^{-1}\, s^{-1}\, c^{-1}\, k^{-1} \\
&= \alpha(k\,c\,k')(k\,c\,k')^{-1},
\end{aligned}$$

because $\alpha(n) = n$ and $[\,\alpha(k')k'^{-1}, s\,] = 1$. Hence the set $\{\,\alpha(g)\, g^{-1} \mid g \in G\,\}$ has compact closure and the proof of (2.1) is complete.

Remark. In the above proof, we note that the compactness of K and C in fact, implies that the set $\{\,\alpha(g)\, g^{-1} \mid g \in G\,\}$ is compact and hence closed. In particular, if $x \in B(G)$, then the conjugacy class Γ_x of x in G is closed. (In general, for any $x \in G$, Γ_x is locally closed but needs not to be closed.)

We are now in the position to prove (2.2) :

Recall that $x \in B(G) \Longleftrightarrow \alpha_x$ is an a. b. d. of $\underset{\sim}{G}$. It follows from (2.1) that

$$B(G) = \bigcap_{\underset{\sim}{T} \in \mathfrak{J}} Z_G(T) \bigcap Z_G(N).$$

As $Z_{\underset{\sim}{G}}(\underset{\sim}{N})$ and $Z_{\underset{\sim}{G}}(\underset{\sim}{T})$, $\underset{\sim}{T} \in \mathfrak{J}$, are defined over F, so is the subgroup

$$\underset{\sim}{B} = \bigcap_{\underset{\sim}{T} \in \mathfrak{J}} Z_{\underset{\sim}{G}}(\underset{\sim}{T}) \bigcap Z_{\underset{\sim}{G}}(\underset{\sim}{N})$$

and $B(G)$ is exactly the F-rational points of $\underset{\sim}{B}$; in particular $B(G)$ is closed.

2.5 Corollary. Let $\underset{\sim}{G}$ be a connected F-group and $\underset{\sim}{B}$ be as above. Then $\underset{\sim}{B}$ is a normal F-subgroup of $\underset{\sim}{G}$ with its unipotent radical central and every F-split torus of $\underset{\sim}{B}$ is central in $\underset{\sim}{B}$. If $\underset{\sim}{G}$ is reductive, then $\underset{\sim}{B}$ is also reductive and F-anisotropic (any F-split torus of $\underset{\sim}{B}$ is central in $\underset{\sim}{B}$. See [3].).

Proof. Since $B(G)$ is normal in G and since G is Zariski-dense in $\underset{\sim}{G}$, $\underset{\sim}{B}$ is normal in $\underset{\sim}{G}$. The unipotent radical $\underset{\sim}{U}$ of $\underset{\sim}{B}$ is a normal subgroup of $\underset{\sim}{G}$ contained in $\underset{\sim}{N}$ and hence $[\underset{\sim}{U}, \underset{\sim}{B}] = 1$. The rest of the corollary is obvious.

Remark. When $\underset{\sim}{B}$ is reductive, $B(G)$ is compactly generated [3, Theorem 13.4]. When $\underset{\sim}{B}$ is not reductive, then in the case where F is an p-adic field, $B(G)$ is not compactly generated since F and hence the rational points U of $\underset{\sim}{U}$ is not compactly generated while as in the case where $F = \underset{\sim}{R}$ or $\underset{\sim}{C}$, then $B(G)$ is compactly generated since in this case both the rational points of a Levi subgroup of $B(G)$ and U are compactly generated. In fact, $B(G)$

is compactly generated for any connected locally compact group G [6, Proposition 2]. Though $B(G)$ is not necessary compactly generated for algebraic groups G, it is always an $[FD]^{-}$ group as shown in the following corollary. (For definition of $[FD]^{-}$ groups, see § 1.)

2.6 <u>Corollary</u>. Let $\underset{\sim}{G}$ be a connected F-group. Then $B(G)$ is an $[FD]^{-}$ group.

<u>Proof</u>. Let $\underset{\sim}{B}$ be as above and $\underset{\sim}{B}^{o}$ the connected component of $\underset{\sim}{B}$, then $\underset{\sim}{B}^{o} \cap GL(n, F)$ has finite index in $B(G)$. Hence in view of Lemma 1.8, we may suppose that $\underset{\sim}{B}$ is connected.

Let $\underset{\sim}{B} = \underset{\sim}{M} \cdot \underset{\sim}{U}$ and $B(G) = M \cdot U$ where $\underset{\sim}{M}$ is a Levi F-subgroup of $\underset{\sim}{B}$ and $\underset{\sim}{U}$ its unipotent radical. By (2.5), U is central in M and hence $B(G)' = M'$. Thus it suffices to show $M \in [FD]^{-}$. To see this, we observe first that M is compactly generated [3, Theorem 13.4]. Now the G-orbits of each point of $B(G)$ has compact closure, in particular so does the $B(G)$-orbit. Hence $B(G)$ is an $[FC]^{-}$ group and again the closed subgroup M of $B(G)$ is an $[FC]^{-}$ group. Therefore by a result of Grosser and Moskowitz [9, Theorem 3.20] that the class of compactly generated $[FC]^{-}$ groups coincides with the class of $[FD]^{-}$ groups, we have M an $[FD]^{-}$ group and this completes the proof of (2.6).

<u>Proof of (2.3)</u>. Let $\underset{\sim}{T}$ be an F-split maximal torus of $\underset{\sim}{G}$ and $\underset{\sim}{L}$ be a Borel subgroup of $\underset{\sim}{G}$ that contains $\underset{\sim}{T}$. If we prove that $\alpha(g) = g$ for all $g \in \underset{\sim}{L}$, then by a result of Borel [2, Corollary 11.4, p.263], we shall have $\alpha(g) = g$ for all $g \in \underset{\sim}{G}$.

Now $\underset{\sim}{L} = \underset{\sim}{T} \cdot \underset{\sim}{L}_u$ where $\underset{\sim}{L}_u$ is the subgroup of unipotent elements of $\mathbf{L}$ [2, p.244]. By (2.1), we have $\alpha(g) = g$ for all $g \in \mathbf{T}$ and for all unipotent elements $g \in G$ and in particular for all $g \in L_u$. Since α is defined over F and L_u is Zariski-dense in $\mathbf{L}_u$, it follows that $\alpha(g) = g$ for all $g \in \mathbf{L}_u$. Thus $\alpha(g) = g$ for all $g \in \mathbf{L}$ and α is the identity mapping of $\mathbf{G}$.

It is obvious that $Z(G) \subset B(G)$. To see the converse inclusion, we note that for any $x \in B(G)$, the inner automorphism α_x of $\mathbf{G}$ is an F-rational a. b. d.. So $\alpha_x = \mathrm{id}$ and $x \in Z(G)$. The proof of (2.3) is complete.

Examples. For a connected F-split group $\mathbf{G}$, we know by (2.3) that there are no non-trivial F-rational a. b. d. of $\mathbf{G}$, but $\mathbf{G}$ may admit non-trivial non F-rational a. b. d.. For example, take $G = \mathbf{G} = \mathbf{C}^{\times}$, then $\mathbf{G}$ is a (Zariski) connected algebraic group defined and split over $\mathbf{C}$. The map $\alpha : \mathbf{G} \longrightarrow \mathbf{C}^{\times}$ defined by $\alpha(z) = \bar{z}$ for all $z \in \mathbf{G}$ is obviously an a. b. d.; but α is not defined over $\mathbf{C}$. However if we regard $\mathbf{C}^{\times}$ as an algebraic over $\mathbf{R}$, i.e., as the $\mathbf{R}$-rational points of the algebraic group

$$\mathbf{G} = \left\{ \begin{bmatrix} z & w \\ -w & z \end{bmatrix} \;\middle|\; z, w \in C , z^2 + w^2 \neq 0 \right\} .$$

Then α is $\mathbf{R}$-rational and is an a. b. d. of $\mathbf{C}^{\times}$. For here $\mathbf{G}$ is not $\mathbf{R}$-split.

§ 3 Layerings of G.

We shall use same notations as developed in § 2. For the definition of layerings, the reader is referred to the introduction. The main object of this section is to prove Theorem 3.1. We show that there exist layerings of G terminating at $Z_G(N)$ and at $\bigcap Z_G(T)$ $(\underset{\sim}{T} \in \mathcal{T})$ respectively. Then by means of the following Lemma 3.2 adapted from [5] and the structure of B(G), we obtain a layering of G terminating at B(G).

3.1 Theorem. Let $\underset{\sim}{G}$ be a connected F-group. Then there exists a layering of G terminating at B(G).

We prove (3.1) in a series of lemmas and propositions :

3.2 Lemma [5, Lemma 2.2]. Let C, D be two closed $\mathcal{J}(G)$-invariant subsets of a locally compact group G such that there are layerings $G = X_0 \supset \cdots \supset X_m = C$ and $G = Y_0 \supset \cdots \supset Y_k = D$. Then there exists a layering of G that terminates with $C \bigcap D$.

3.3 Lemma. Let F^r be equipped with the topology induced by the norm of F. Then for each $x \neq 0 \in F^r$, there exists an open neighborhood E of x in F^r and a sequence of natural numbers $k_1 < k_2 < \cdots$ such that $\delta(i)E$ are pairwise disjoint where we set $\delta(i) = p^{k_i}$ with $p > 1$ any natural number if $F = \underset{\sim}{R}$ or $\underset{\sim}{C}$, and $\delta(i) = 1/p^{k_i}$ with p equal to the characteristic of the residue field if F is non-euclidean.

Proof. We denote by $|x|$ the norm of $x \in F^r$. Let $x \neq 0 \in F^r$, then $|x| \neq 0$. Let d be a real number such that $d < |x|$. The set $E = \{ x + y \mid y \in F^r, |y| < d \}$ is obviously an open neighborhood of x. Now for any $x + y \in E$, we have

$$0 < |x| - d < |x| - |y| \leq |x + y| \leq |x| + |y| < |x| + d.$$

Let k_1 be the smallest natural number such that

$$|x| + d < p^{k_1}(|x| - d)$$

where p is the characteristic of the residue field if F is non-euclidean and $p > 1$ is any natural number if $F = \underset{\sim}{R}$ or $\underset{\sim}{C}$.

For $i \geq 2$, we define k_i to be the smallest natural number such that

$$p^{k_{i-1}}(|x| + d) < p^{k_i}(|x| - d).$$

Then the $\delta(i)E$'s are pairwise disjoint since

$$\sup_{z \in E} |\delta(i-1)z| = |\delta(i-1)| \sup_{z \in E} |z| \leq p^{k_{i-1}}(|x| + d) <$$

$$< p^{k_i}(|x| - d) \leq |\delta(i)| \inf_{z \in E} |z| = \inf_{z \in E} |\delta(i)z|.$$

3.4 <u>Proposition</u>. Let $\underset{\sim}{G}$ be a connected F-group and $\underset{\sim}{N}$ be its unipotent radical. Then there exists a layering $G = X_0 \supset X_1 \supset \cdots \supset X_m = Z_G(N)$ of closed normal subgroups of G such that for each $x \in X_j - X_{j+1}$, there exists a relative neighborhood V of x in X_j such that V has countably many disjoint conjugates.

<u>Proof.</u> Let $N = N^0 \supset N^1 \supset \cdots \supset N^m = \{1\}$ be the central descending sequence of N. Then for each j, N^j is a closed normal subgroup of G and N^j/N^{j+1} is topological isomorphic to F^r for some natural number r. For each $j = 0, \cdots, m$, define

$$X_j = \{ x \in G \mid [N, x] \subset N^j \}.$$

Then we have that $G = X_0 \supset X_1 \supset \cdots \supset X_m = Z_G(N)$ and that

$X_j \subset N^j$, $j = 0, \cdots, m$. As each X_j/N^j is the centralizer of N/N^j in G/N^j, so each X_j/N^j is a closed normal subgroup of G/N^j and hence each X_j is a closed normal subgroup of G.

We show that the subgroups X_j provide the desired layering of G. For $j = 0, \cdots, m-1$, let $x \in X_j - X_{j+1}$, since $[N, x]$ is contained in N^j but not in N^{j+1}, there is an element $u \in N$ such that $[u, x] \in N^j - N^{j+1}$. As the mapping $y \longmapsto [u, y]$ from X_j to N^j is continuous, it follows that there exists a compact relative neighborhood V of x in X_j such that $[u, y] \in N^j - N^{j+1}$ for all $y \in V$; since X_{j+1} is closed, we may suppose further that $V \subset X_j - X_{j+1}$.

Let $\sim : G \longrightarrow G/N^{j+1}$ be the natural projection and $\eta : X_j^{\sim} \longrightarrow (N^j)^{\sim}$ be the continuous mapping defined by $\eta(y^{\sim}) = [u^{\sim}, y^{\sim}]$, $y \in X_j$. Since $(N^j)^{\sim}$ is topological isomorphic to F^r for some natural number r and $\eta(x^{\sim}) \neq 1^{\sim}$, we have, by (3.3), an open set E containing $\eta(x^{\sim})$ in $(N^j)^{\sim}$ and a sequence of natural numbers $k_1 < k_2 < \cdots$ such that $E^{\delta(i)}$ are pairwise disjoint, where $\delta(i) = p^{k_i}$ if $F = \underset{\sim}{R}$ or $\underset{\sim}{C}$ and $\delta(i) = 1/p^{k_i}$ otherwise. Shrink V if necessary, so that $\eta(V^{\sim}) \subset E$.

Let $W = (N^j)^{\sim} \bigcap V^{\sim} V^{\sim -1}$ and $W^{1/2} = \{ w^{1/2} \mid w \in W \}$ (note here that W is a symmetric neighborhood of $1^{\sim}$ in $(N^j)^{\sim}$). Then we have :

(1) for large i ($i \geq i_0$ say), $\eta(V^{\sim})^{\delta(i)} \cdot W^{1/2}$ pairwise disjoint.

Indeed, since $|1/2\delta(i)| \longrightarrow 0$ as $i \longrightarrow \infty$, W is compact and $\eta(\tilde{V})$ is a compact set contained in the open set E, there exists an i_o such that for all $i \geq i_o$, $\eta(\tilde{V}) \cdot W^{1/2\delta(i)} \subset E$. Hence for all $i \geq i_o$,

$$\eta(\tilde{V})^{\delta(i)} \cdot W^{1/2} = (\, \eta(\tilde{V}) \cdot W^{1/2\delta(i)}\,)^{\delta(i)} \subset E^{\delta(i)}$$

are pairwise disjoint.

We claim that for $i \geq i_o$, $u^{\delta(i)}\, V u^{-\delta(i)}$ are pairwise disjoint. We observe first that every element in N has a unique i-th root since $\exp_{|N}$ is a bijection. Now for any $y \in V$, since $[u, y] \notin N^{j+1}$, it is obvious that for any natural number i, both $[u, y]^i$ and $[u, y]^{1/i}$ are not in N^{j+1}; in other words, both $\eta(\tilde{y})^i$ and $\eta(\tilde{y})^{1/i}$ are distinct from $\tilde{1}$.

Next we prove that for all $y \in V$, $i = 1, 2, \cdots$, we have

$$(2) \qquad \tilde{u}^{1/i}\tilde{y}\,\tilde{u}^{-1/i} = \eta(\tilde{y})^{1/i}\,\tilde{y}$$

$$(2') \qquad \tilde{u}^{i}\,\tilde{y}\,\tilde{u}^{-i} = \eta(\tilde{y})^{i}\,\tilde{y}.$$

Consider any $u_1, u_2 \in N$ and $y \in V$, $[u_1, y] \in N^j$ implies that $[u_2, [u_1, y]] \in N^{j+1}$, hence $[\tilde{u}_1, \tilde{y}]\,\tilde{u}_2 = \tilde{u}_2[\tilde{u}_1, \tilde{y}]$. Therefore for any $y \in V$ and $i, k = 1, 2, \cdots$, we have

$$(3) \qquad [\tilde{u}^{k/i}, \tilde{y}]\,[\tilde{u}^{1/i}, \tilde{y}]$$

$$= \tilde{u}^{1/i}(\tilde{u}^{k/i}\,\tilde{y}\,\tilde{u}^{-k/i}\,\tilde{y}^{-1})(\tilde{y}\,\tilde{u}^{-1/i}\,\tilde{y}^{-1})$$

$$= [\tilde{u}^{(k+1)/i}, \tilde{y}].$$

Applying (3) $i - 1$ times to the product $[\tilde{u}^{1/i}, \tilde{y}]^i$, we have $[\tilde{u}^{1/i}, \tilde{y}] = [\tilde{u}, \tilde{y}] = \eta(\tilde{y})$, or equivalently, $[\tilde{u}^{1/i}, \tilde{y}] = \eta(\tilde{y})^{1/i}$. Hence (2) follows. Similarly applying (3) $i - 1$

times to the product $[u^\sim, y^\sim]^i$, we obtain (2').

Now suppose that there were $k > i \geq i_o$ such that $u^{\delta(i)} V u^{-\delta(i)} \bigcap u^{\delta(k)} V u^{-\delta(k)} \neq \emptyset$. Then there would be y_1, y_2 in V such that $u^{\delta(i)} y_1 u^{-\delta(i)} = u^{\delta(k)} y_2 u^{-\delta(k)}$. Hence by (2) and (2'), we would have

$$\eta(y_1^\sim)^{\delta(i)} y_1^\sim = \eta(y_2^\sim)^{\delta(k)} y_2^\sim$$

or

$$\eta(y_1^\sim)^{\delta(i)} \eta(y_2^\sim)^{-\delta(k)} = y_2^\sim y_1^{\sim -1} \varepsilon (N^j)^\sim \bigcap V^\sim V^{\sim -1} = W$$

Thus there would be w_1, $w_2 \varepsilon W^{1/2}$ such that

$$\eta(y_1^\sim)^{\delta(i)} w_1 = \eta(y_2^\sim)^{\delta(k)} w_2 .$$

This contradicts to (1). Hence the claim is proved and the proof of (3.4) is complete.

3.5 <u>Lemma</u>. Let $\underset{\sim}{T}$ be an F-split torus and $\psi_1 \neq \psi_2$ be two characters of $\underset{\sim}{T}$. Then there exists a $t_o \varepsilon T$ such that the norm $|\psi_1(t_o)\psi_2(t_o)^{-1}| \neq 1$.

<u>Proof.</u> As mentioned in section 2, we may assume that $\underset{\sim}{T} = GL(1)^r$, $T = GL(1, F)^r$ for some natural number r and $\psi_i(t) = \prod a_j^{m(i,\, j)}$ where $t = (a_1, \cdots, a_r) \varepsilon \underset{\sim}{T}$, $i = 1, 2$ and $m(i, j)$ are some suitable integers. Since $\psi_1 \neq \psi_2$, there exists a j, $1 \leq j \leq r$ such that $m(1, j) \neq m(2, j)$.

Let $t_o = (1, \cdots, 1, a_j, 1, \cdots, 1)$ with $a_j \varepsilon F^\times$ such that $|a_j| \neq 1$. Then obviously $\psi_1(t_o)\psi_2(t_o)^{-1} = a_j^{m(1,j)-m(2,j)}$ has norm different from 1.

3.6 <u>Proposition</u>. Let $\underset{\sim}{T}$ be an F-split torus of $\underset{\sim}{G}$. Then for any $x \in G - Z_G(T)$, there exists a compact neighborhood V of x such that $V \subset G - Z_G(T)$ and V has countably many disjoint conjugates.

Proof. Since $\underset{\sim}{T}$ is a torus defined and split over F, there exists a basis $v_1, \cdots, v_n$ of F^n such that $t(v_i) = \psi_i(t)\, v_i$, $t \in \underset{\sim}{T}$, $i = 1, \cdots, n$ where each ψ_i is a character of $\underset{\sim}{T}$. For each $x \in G$, we have $x(v_i) = \sum_{j=1}^{n} x_{i\,j}\, v_j$, $i = 1, \cdots, n$, where each $x_{i\,j}$ is a function defined over F and $x_{i\,j}|_G$ is continuous.

Now

$$x \in Z_G(T) = Z_G(\,T\,) \Longleftrightarrow (x\,t)_{i\,j} = (t\,x)_{i\,j},\quad t \in T,\quad 1 \leq i, j \leq n$$

$$\Longleftrightarrow x_{i\,j}\,\psi_i(t) = \psi_j(t)\, x_{i\,j}$$

$$\Longleftrightarrow x_{i\,j} = 0 \quad \text{or} \quad x_{i\,j} \neq 0 \quad \text{and} \quad \psi_i = \psi_j.$$

Therefore

$$x \in G - Z_G(T) \Longleftrightarrow \exists\ i, j \quad \text{such that} \quad x_{i\,j} \neq 0 \quad \text{and} \quad \psi_i \neq \psi_j$$

So if we set $J = \{\, (i\,,j) \mid 1 \leq i,\ j \leq n,\ \psi_i \neq \psi_j \,\}$ and $U_{i\,j} = \{\, x \in G \mid x_{i\,j} \neq 0 \,\}$, then it is obvious that each $U_{i\,j}$ is an open set in G and that

$$G - Z_G(T) = \bigcup \{\, U_{i\,j} \mid (i\,,j) \in J \,\}.$$

Now let $x \in G - Z_G(T)$, then there exists $(i\,,j) \in J$ such that $x \in U_{i\,j}$. For this fixed $(i\,,j)$, let V be a compact neighblrhood of x in G such that $V \subset U_{i\,j}$. Let c, c' be such that

$$0 < c = \min_{x \in V} |x_{i\,j}| \leq \max_{x \in V} |x_{i\,j}| = c' .$$

Since $\psi_i \neq \psi_j$, it follows from (3.5) that there exists a $t_o \in T$ such that $|\psi_i(t_o)\,\psi_j(t_o)^{-1}| = a \neq 1$. Also for any $t \in \underset{\sim}{T}$ and any integer k, we have

$$(t^k x\, t^{-k})_{i\,j} = \psi_i(t)^k x_{i\,j}\, \psi_j(t)^{-k} = (\psi_i(t)\,\psi_j(t)^{-1})^k x_{i\,j}$$

and in particular, we have

$$|t_o^{\,k} x\, t_o^{\,-k}| = a^k\, |x_{i\,j}| .$$

So if we pick an integer k such that $a^k c > c'$, then the sequence of conjugates $V_m = t_o^{\,mk} V t_o^{\,-mk}$ are pairwise disjoint since

$$\min_{x \in V_{m+1}} |x_{i\,j}| = a^{(m+1)k} c > a^{mk} c' = \max_{x \in V_m} |x_{i\,j}| .$$

3.7 <u>Proposition.</u> Let $\underset{\sim}{G}$ be a connected F-group. Then $G \subset \bigcap Z_G(T)$ $(\underset{\sim}{T} \in \mathfrak{T})$ is a 1-layering of G.

<u>Proof.</u> We recall that $\bigcap Z_G(T)$ $(\underset{\sim}{T} \in \mathfrak{T})$ is a closed normal subgroup of G. Now for any $x \in G - \bigcap Z_G(T)$, there exists a $\underset{\sim}{T} \in \mathfrak{T}$ such that $x \in G - Z_G(T)$. Hence it is easy to see that the proposition follows from (3.6).

It is now clear that (3.1) follows from (3.2), (3.4) and (3.7).

<u>Remark.</u> In the proof of (3.4), if we define

$$X_j = \{\, x \in G \mid [N, \alpha(x)] \subset N^j,\ \alpha \in \mathcal{B} \,\}$$

where $\mathcal{B}$ denotes the group of all F-rational a. b. d. of $\underset{\sim}{G}$, then it can be shown that $G = X_0 \supset \cdots \supset X_m = Z_G(N)$ is a layering of G with the additional propertity that the X_j's are

$\mathfrak{B}$-invariant. Also the 1-layering $G \supset \bigcap Z_G(T)$ $(\underset{\sim}{T} \in \mathfrak{T})$ is $\mathfrak{B}$-invariant. Hence we can obtain a $\mathfrak{B}$-invariant layering of G terminating at $B(G)$.

§ 4 Applications.

We use same notations as developed in the previous sections. Recall that in § 1 we investigated when $Z_G(H)$ is an $[FD]^-$ group if H is a generalized uniform subgroup of a locally compact group G. We know that in many cases $Z_G(H)$ is an $[FD]^-$ group (for instance when G is locally connected), but $Z_G(H)$ fails to be an $[FD]^-$ group for a certain totally disconnected group G. We are now in a position to give a class of totally disconnected groups G, namely the class of p-adic algebraic groups, for which $Z_G(H)$ is an $[FD]^-$ group (see (4.4) below).

Recall that a measure μ on a locally compact group G is said to be central if μ is $\mathcal{J}(G)$-invariant, i.e., if $\mu(\alpha_x A) = \mu(A)$ for all $\alpha_x \in \mathcal{J}(G)$ and all Borel sets A in G. We have the following results :

4.1 Theorem. Let $\underset{\sim}{G}$ be a connected F-group. Then any finite central measure on G is supported on $B(G)$.

4.2 Theorem. Let $\underset{\sim}{G}$ be a connected F-group and H a generalized uniform subgroup of G such that H is a centralizer C_x of some $x \in G$. Then G/H is compact.

4.3 Theorem. Let $\underset{\sim}{G}$ be a connected F-group and H a generalized uniform subgroup of G. Then $Z_G(H)$ is an $[FD]^-$ group.

All these theorems prove analogous results of Greenleaf-Moskowitz-Rothschild for connected locally compact groups and real analytic groups. When $F = \underset{\sim}{R}$, (4.2) is also a (slightly generalized) result of a particular case of a theorem of Mostow [16, Theorem 7.1] which states that if G is a connected Lie group and H a generalized uniform subgroup of G with finitely many connected components, then G/H is compact.

Making use of (3.1) and a simple induction argument would yield (4.1) and so we omit the proof here (the reader may also refer to [5]).

Before proving (4.2), we recall some facts in harmonic analysis. Let $\underset{\sim}{G}$ be a connected F-group and G its F-rational points. Since F is σ-compact, it follows that G and the conjugate class Γ_x of $x \in G$ are σ-compact. As G/C_x and Γ_x are standard Borel spaces, the canonical bijection $\eta : G/C_x \longrightarrow \Gamma_x$ is a continuous Borel isomorphism. Furthermore, η is equivariant with respect to the left translation of G on G/C_x and the action of $\mathcal{J}(G)$ on Γ_x. Thus (finite) left invariant measure on G/C_x can be transferred to (finite) $\mathcal{J}(G)$-invariant measure on Γ_x and vise versa. The latter measure can be regarded as central measure on G supported on Γ_x.

Proof. of (4.2). Since G/C_x admits a finite invariant measure μ, we may regard μ as a finite central measure on G supported on

Γ_x. Hence $x \in \Gamma_x \subset B(G)$ and therefore Γ_x is compact (see remark on p.29). Applying the following lemma from [6], we have G/C_x compact.

Lemma [6, Lemma 1]. Let G be a locally compact group, then G/C_x is compact if and only if Γ_x is compact.

Proof of (4.3). Since $B(G)$ is an $[FD]^-$ group (Corollary 2.6) and since any closed subgroup of an $[FD]^-$ group is again an $[FD]^-$ group, the proof of (4.3) will be complete if we establish the following proposition.

4.4 Proposition. Let $\underset{\sim}{G}$ be a connected F-group and H a generalized uniform subgroup of G, then $Z_G(H) \subset B(G)$.

Proof. Let $x \in Z_G(H)$. Then $H \subset C_x$ and it follows from Lemma F in § 1 that G/C_x admits a finite invariant measure. Hence, as in the proof of (4.2), we have $x \in \Gamma_x \subset B(G)$ and so the proposition is established.

As an immediate consequence of (4.4), we have the following :

4.5 Corollary. Suppose that $\underset{\sim}{G}$ is a connected F-group such that $B(G) = Z(G)$ and H a generalized uniform subgroup of G. Then $Z_G(H) = Z(G)$.

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Autobiography

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