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Analysis of the Spectrum of SL(3,Z)\SL(3,R)/SO(3,R)

by

Nam-Jong Moh

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2001

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Abstract

Analysis of the Spectrum on SL(3,Z)\SL(3,R)/SO(3,R)

by

Nam-Jong Moh

Advisor: Jonathan Huntley

In this thesis we make a systematic study of the spectral properties of the space **SL(3,Z)\SL(3,R)/SO(3,R).** In particular we use Rayleigh quotient method to give a lower bound for the first non-trivial eigenvalue of the Laplace-Beltrami operator, we use the Selberg trace formula to obtain Weyl's law with an error term and we make a detailed study of Whittaker functions to obtain a comparison test that allows one to determine when two cusp forms are the same if sufficiently many Fourier coefficients agree. We conclude by showing what would be true if conjectures related to the Riemann hypothesis were true.

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어머니 고맙습니다

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CHAPTER 1

Introduction

Since the early work of Jacobi on θ functions to study the representation of integers as sums of squares, automorphic forms in various disguises have played an important role in number theory. Notable among these is Ramanujan's discriminant function whose reciprocal is intimately connected with the study of the partition function in number theory. The automorphic properties of this function, which includes periodicity with period one, give rise to a Fourier expansions whose study has been a central theme in number theory for over a century. The estimate of the size of the Fourier coefficients is a key element in these applications. Hecke explicitly introduced the concept of automorphic forms in the 1920's. He assumes one has an analytic function on the upper half plane, that has only polynomial growth at infinity, and with respect to some discrete subgroup Γ of $SL(2, \mathbb{R})$, the group of symmetries of the upper half plane, satisfies

$$f(\gamma z) = (cz + d)^k f(z)$$

where γ, an element of Γ equals $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In particular, we have that f is periodic with respect to $\mathbb{X}$, so we have a Fourier expansion.

In the 1940's, Maass generalized this notion by considering functions that are not necessarily holomorphic but are eigenfunctions of the Laplace-Beltrami operators. These functions have been of tremendous usefulness in the study of

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the Riemann zeta function. and in various aspects of the study of Kloosterman sums. One of particular interest here is the eigenvalue λ of the Laplace-Beltrami operator. How large the minimal eigenvalue denoted as λ_1 is is of a great interest. When Γ is non compact. we have continuous spectrum created by the Eisenstein series. and it is interesting to find out if

1) there are eigenvalues below the continuous spectrum

2) there exist actual L^2 eigenfunctions.

3) how many Fourier coefficients are needed to determine a cusp form

We will study these questions in a different setting. It has become evident that the study of automorphic forms on more general groups is of importance in number theory. One sees this early on with Hilbert modular forms. but in the 1950's a far greater generalization began. Selberg's seminal investigations in the 1950's are a good example. and there are many others. One difficulty of the generalization is that the analysis needed to study the forms becomes increasingly complicated. Often this can be done. at least in principle. by general techniques: however. one often needs results that can be used explicitly for other purposes.

In this thesis we study spectral properties of the rank 2 homogeneous space

$$\mathbb{X} = SL\,(3\mathbb{Z}) \backslash SL\,(3.\mathbb{R})\,/SO\,(3.\mathbb{R})\,.$$

We study several problems. and use a variety of techniques. After some more detailed discussion of notations and definitions, we will proceed as follows. We first. in chapter 2. will consider the question whether the first eigenvalue of the Laplacian

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is in the continuous spectrum. This is related to the infinite place Ramanujan conjecture which asserts that all non-trivial eigenfunctions have eigenvalues whose representation parameters would be tempered, meaning $\mathrm{Re}(\nu_1) = \mathrm{Re}(\nu_2) = \frac{1}{3}$, in the notation we will introduce later. This is a reprise of older work and uses variational techniques. We next use the Selberg trace formula to establish Weyl's law with an error term, in chapter 3. In chapter 4, we change directions. Here we examine the Fourier expansion of cusp forms, and study certain special functions, Whitaker functions and K-Bessel functions, and apply our knowledge to study the spectrum, in particular, we study spectral multiplicity. Finally we show, using a variant of the Riemann hypothesis, that we can conjecturally obtain a much sharper estimate.

We now make explicit our notations and definitions, and state in more detail what we will show. Let G be a reductive group, which in our case will always be $GL(3,\mathbb{R})$, the group of invertible 3×3 matrices with real coefficients, $PGL(3,\mathbb{R})$, the quotient of $GL(3,\mathbb{R})$ by the center, which is isomorphic to $\mathbb{R}$, or $SL(3,\mathbb{R})$, subgroup of $GL(3,\mathbb{R})$ consisting of matrices of determinant 1. Let K denote the maximal compact subgroup, and let $H = G/K$. This is an analogue of the classical upper half plane. It is a symmetric space.

For the case $G = PGL(3,\mathbb{R})$, we can put coordinates on the space H in the following manner. From the Iwasawa decomposition, we can write an element $g \in G$ as a product $g = nak$, with n upper triangular a diagonal and k compact. The collection of n forms a group N of upper triangular matrices. The collection of a forms an abelian group A. The collection of k forms a compact group. As H

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consist of equivalence classes of elements mod K we need no coordinates for K.

Following Bump, we write an element of N by $\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$, and an element of A as $\begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The fact that we are in $PGL(3, \mathbb{R})$ is why the 3,3 entry can be assumed to be identically 1, and if we work on $SL(3, \mathbb{R})$ we can assume the 3,3 entry is the inverse of the product of the other two diagonal entries. We write this point as τ.

$PGL(3, \mathbb{R})$ acts on H by matrix multiplication, and there is a natural metric on H such that the action is by isometries. It is given by

$$g_{ij} = \begin{array}{c|ccccc|c} & y_1 & y_2 & x_1 & x_3 & x_2 & \\ \hline y_1 & y_2^2 & \frac{1}{2} y_1 y_2 & 0 & 0 & 0 & \\ y_2 & \frac{1}{2} y_1 y_2 & y_1^2 & 0 & 0 & 0 & \\ x_1 & 0 & 0 & y_2^2 + x_2^2 & -x_2 & 0 & \frac{1}{y_1^2 y_2^2} \\ x_3 & 0 & 0 & x_2 & 1 & 0 & \\ x_2 & 0 & 0 & 0 & 0 & y_1^2 & \end{array}$$

From this we get a natural invariant volume element

$$dV = \frac{dx_1 dx_2 dx_3 dy_1 dy_2}{y_1^3 y_2^3}.$$

H also has a ring of invariant differential operators. By this we mean that the

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operators D are such that for ϕ smooth on H, and

$$D\varphi\left(g\circ\tau\right)=D\phi\left(\tau\right)|_{g\circ\tau}$$

for any $g\in G$. In essence the fudge factor term in the chain rule disappears. We can of course extend the class of allowable functions.

In our case, a study of the center of the universal enveloping algebra of G shows that the ring has two non-trivial generators, that in our notation can be taken to be

$$\begin{aligned}\Delta_1 &= y_1^2\frac{\partial^2}{\partial y_1^2}+y_2^2\frac{\partial^2}{\partial y_2^2}-y_1y_2\frac{\partial^2}{\partial y_1\partial y_2}+y_1^2\left(x_2^2+y_2^2\right)\frac{\partial^2}{\partial x_3^2}\\ &\quad +y_1^2\frac{\partial^2}{\partial x_1^2}+y_2^2\frac{\partial^2}{\partial x_2^2}+2y_1^2x_2\frac{\partial^2}{\partial x_1\partial x_3}\end{aligned}$$

and

$$\begin{aligned}\Delta_2 &= -y_1^2y_2\frac{\partial^3}{\partial y_1^2\partial y_2}+y_1y_2^2\frac{\partial^3}{\partial y_1\partial y_2^2}-y_1^3y_2^2\frac{\partial^3}{\partial x_3^2\partial y_1}+y_1y_2^2\frac{\partial^3}{\partial x_2^2\partial y_1}\\ &\quad -2y_1^2y_2x_2\frac{\partial^3}{\partial x_1\partial x_3\partial y_2}+\left(-x_2^2+y_2^2\right)y_1^2y_2\frac{\partial^3}{\partial x_3^2\partial y_2}-y_1^2y_2\frac{\partial^3}{\partial x_1^2\partial y_2}\\ &\quad +2y_1^2y_2^2\frac{\partial^3}{\partial x_1\partial x_2\partial x_3}+2y_1^2y_2x_2\frac{\partial^3}{\partial x_2\partial x_3^2}+y_1^2\frac{\partial^2}{\partial y_1^2}-y_2^2\frac{\partial^2}{\partial y_2^2}\\ &\quad +2y_1^2x_2\frac{\partial^2}{\partial x_1\partial x_3}+y_1^2\left(x_2^2+y_2^2\right)\frac{\partial^2}{\partial x_3^2}+y_1^2\frac{\partial^2}{\partial x_1^2}-y_2^2\frac{\partial^2}{\partial x_2^2}.\end{aligned}$$

The first of these is the Laplace-Beltrami operator, and could have been calculated from the metric. The ring of invariant operators is commutative.

If H is as above, we let Γ be a discrete subgroup of G. This acts on H and we can consider the quotient $\mathbb{X}=\Gamma\backslash H$. We will take $SL(3,\mathbb{Z})$ for Γ although the construction can be made in considerably greater generality. A key fact in this case is that $\mathbb{X}$ is not compact, but still has finite volume. $\mathbb{X}$ has one cusp,

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when y_1 and y_2 go to infinity. The cusp geometrically looks like two copies of $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$ that are connected at a point as both y_1 and y_2 tend to infinity. Depending on what one wishes to do, it can be advantageous to see $\mathbb{X}$ as a non compact manifold, or as a fundamental domain, by which we mean an open subset $\mathcal{D}$ of H such that $\mathcal{D} \cap g\mathcal{D} = \emptyset$ for $g \neq 1$ but $\bigcup_{g\in G} g\overline{\mathcal{D}} = H$.

Our goal is to describe, as well as we can, the spectrum of the ring of invariant differential operators. We first make the observation that for $\mathbb{X}$ there is continuous spectrum spanned by two types of Eisenstein series. The first is found by summing

$$I_{(\nu_1,\nu_2)}(\tau) = y_1^{2\nu_1+\nu_2} y_2^{2\nu_2+\nu_1},$$

we get

$$E_{(\nu_1,\nu_2)}(\tau) = \sum_{g\in\Gamma_\infty\backslash\Gamma} I_{(\nu_1,\nu_2)}(g\circ\tau).$$

This expression is convergent in the plane $\operatorname{Re}(\nu_1)$, $\operatorname{Re}(\nu_2) > 1$, and it has a meromorphic continuation to all values of $\nu_1$and ν_2. In particular it is analytic when $\operatorname{Re}(\nu_1) = \operatorname{Re}(\nu_2) = \frac{1}{3}$. We also have an infinite collection of families of Eisenstein series, associated to elements of the discrete spectrum of $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$. These objects constitute the entire continuous spectrum. This spectrum is absolutely continuous. It spans $[1,\infty)$ if we parametrize by the Laplacian eigenvalue, and thus continuous spectrum exists. Moreover, any point spectrum will be of finite multiplicity. The functions contributing to the point spectrum are in $L^2(\mathbb{X})$. It will be shown in the next chapter that with the exception of the constant functions, all L^2 eigenfunctions will be embedded in the continuous spectrum. It is known that any such functions must be cuspidal, meaning

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$$\int_0^1\int_0^1 \phi\left(\begin{pmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{pmatrix}\tau\right) d\xi_1 d\xi_3 = 0. \text{ and}$$

$$\int_0^1\int_0^1 \phi\left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\tau\right) d\xi_2 d\xi_3 = 0 \text{ for all } \tau \in \mathbb{X}.$$

These functions enjoy the property that when either y_1 or $y_2 \to \infty$ the functions vanish. Much of our work in chapter 4 makes this more precise.

The Eisenstein series and the cusp forms (and constants) are called automorphic forms for Γ: in particular they are periodic in the y variables. and hence enjoy a Fourier expansion. From elliptic regularity (they are eigenfunctions of the Laplacian) the expression will converge. all summations can be interchanged. and we may for example differentiate the expression. etc. If $\phi(\tau)$ is a cusp form. its expansion is

$$\phi(\tau) = \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} n_1^{-1}n_2^{-1}a_{n_1,n_2}W_{1,1}^{(\nu_1,\nu_2)}\left(\begin{pmatrix} n_1n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g\tau\right).$$

where

$$\Gamma^2 = \left\{\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z},\ ad-bc = \pm 1\right\}.$$

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$\Gamma^2_\infty = \Gamma^2 \cap \Gamma_\infty$, and Γ_∞ is the group of 3×3 upper triangular unipotent matrices with integer coefficients. Here $W_{1,1}^{(\nu_1,\nu_2)}$ is a Whittaker function. In Bump's notation

$$W_{1,1}^{(\nu_1,\nu_2)}(\tau, w_1) = W_{\nu_1\nu_2}(y_1, y_2)e^{2\pi i(x_1+x_2)} .$$

Vinogradov and Takhtadzhyan established an integral equality for $W_{\nu_1\nu_2}$, and the equality was generalized by Stade into $GL(n)$ cases [St]. The result is

$$\begin{aligned} W_{\nu_1,\nu_2}(y_1, y_2) &= \pi^{-2}(y_1)^{1+(\nu_1-\nu_2)/2}\,(y_2)^{1+(\nu_2-\nu_1)/2} \\ &\quad \times \int_0^\infty K_\nu\left(y_1\sqrt{1+x}\right) K_\nu\left(y_2\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}dx \end{aligned}$$

where $\nu = (3\nu_1 + 3\nu_2 - 2)/2$. This equality is very convenient and we are using it many times in the chapter 4 where we study the Whittaker function to estimate the number of linearly independent cusp forms with eigenvalue λ.

In chapter 3, we use the Selberg trace formula for $SL(3,\mathbb{Z})$ to establish the existence of cusp forms. Weyl's law was established by Stade and Wallace [S-W] but we obtain error estimates. Finally, in chapter 5, we obtain, using standard conjectures about the location of zeros of L-functions of automorphic forms, a logarithmic bound in λ for the multiplicity of an eigenvalue rather than polynomial multiplicity in λ.

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CHAPTER 2

Minimum Bound for the First Non-trivial Eigenvalue

In this chapter we show that the first cuspidal eigenvalue λ_1 is a number that is embedded in the continuous spectrum. The continuous spectrum is $[1, \infty)$, and we will show $\lambda_1 > \frac{3\pi^2}{10} > 2.96088$. From the theory of Eisenstein series, we know that there are no L^2 eigenvalues other than the constant, as they must be residues of poles of Eisenstein series. Hence we conclude that with the exception of $\lambda_0 = 0$, the eigenvalue for the constant function, all L^2 eigenvalues are embedded in the continuous spectrum. This result can be considered as an analogue of Selberg's eigenvalue conjecture. The result is also implied by the Ramanujan conjecture, but our result does not imply this as we do not consider the third order operator.

The proof is based on the Rayleigh-Ritz method and we shall now give it. This proof is a reprise of work in [C-H-J-T]. We thank the authors for giving permission to reproduce it here.

If $\Delta\phi = \lambda\phi$,

$$\frac{\int_D \phi \Delta \bar{\phi} dV}{\int_D |\phi|^2 dV} = \lambda.$$

Integrating by parts, we get

$$\frac{\int_D |\nabla\phi|^2 dV}{\int_D |\phi|^2 dV} = \lambda.$$

where the gradient ∇ is

$$|\nabla f|^2 = y_1^2 \left|\frac{\partial f}{\partial y_1}\right|^2 + y_2^2 \left|\frac{\partial f}{\partial y_2}\right|^2 - \mathrm{Re}\, y_1 y_2 \frac{\partial f}{\partial y_1}\frac{\partial f}{\partial y_2} + y_1^2\left(x_2^2 + y_2^2\right)\left|\frac{\partial f}{\partial x_3}\right|^2$$

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$$+y_1^2\left|\frac{\partial f}{\partial x_1}\right|^2+y_2^2\left|\frac{\partial f}{\partial x_2}\right|^2+2\,\mathrm{Re}\,y_1^2x_2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_3}.$$

A theorem of Grenier [G] gives an explicit description of this fundamental domain which is as following. Let $w^{-1}=y_1$, $v^{-3/2}=y_2^2y_1$, then a fundamental domain $\mathcal{D}$ for $SL(3,\mathbb{Z})$ is given by:

(i) $v^{3/2}\le v^{3/2}(1-x_2+x_3)^2+w(1-x_1)^2+w^{-1}$

(ii) $v^{3/2}\le v^{3/2}(x_2-x_3)^2+w(1-x_1)^2+w^{-1}$

(iii) $v^{3/2}\le v^{3/2}x_2^2+w$

(iv) $v^{3/2}\le v^{3/2}x_3^2+wx_1^2+w^{-1}$

(v) $1\le w^{-2}+x_1^2$

(vi) $0\le x_2\le\frac{1}{2}$

(vii) $0\le x_1\le\frac{1}{2}$

(viii) $-\frac{1}{2}\le x_3\le\frac{1}{2}$.

But for our purposes we only need the following lemma.

Lemma 2.1. *Siegel set S, described as $|x_3|<\frac{1}{2}$, $0<x_1<\frac{1}{2}$, $0<x_2<\frac{1}{2}$, $y_1>\frac{\sqrt{3}}{2}$, $y_2>\frac{\sqrt{3}}{2}$ contains the domain, and ten copies of the fundamental domain contains the Siegel set.*

This allow us to replace the rather complicated fundamental domain with a set that looks like an infinite rectangle in five dimensions.

From the above, we obtain that

$$\frac{\int_S|\nabla\phi|^2\,dV}{\int_S|\phi|^2\,dV}<10\lambda$$

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as

$$\int_{\mathcal{D}} |\nabla\phi|^2\, dV < 10 \int_{\mathcal{D}} |\nabla\phi|^2\, dV,$$

and

$$\int_{\mathcal{D}} |\nabla\phi|^2\, dV < \int_{S} |\phi|^2\, dV.$$

We now proceed to estimate $\frac{\int_S |\nabla\phi|^2 dV}{\int_S |\phi|^2 dV}$. Unfortunately, the Fourier expansion given in the introduction cannot be used due to its non-orthogonality so that when we square ϕ we will have more coefficients than we can control. However, we can work as follows. In the proof of the Fourier expansion, given in [B] page 66, we first consider the expansion with respect to the abelian group of elements $g = \begin{pmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{pmatrix}$, and we have $\phi(\tau) = \sum \phi_{n_1}^{n_3}(\tau)$ where

$$\phi_{n_1}^{n_3}(\tau) = \int_0^1 \int_0^1 \phi(g\tau)\, e^{-2\pi i (n_1\xi_1 + n_3\xi_3)} d\xi_1 d\xi_3.$$

As we have a cusp form, $\phi_0^0 = 0$.

Let

$$\Gamma_1^2 = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \right\}.$$

and Γ_∞^2 be its stabilizer at infinity. We can write

$$\phi(\tau) = \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \sum_{n_1=1}^{\infty} \phi_{n_1}^0 (\gamma \circ \tau)$$

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We know ϕ is C^∞ so we may interchange integration and summation. so Parseval's theorem implies. as we have abelian Fourier expansions,

$$\frac{\int_S \sum_{n_1=1}^{\infty} \left| \sum_{\gamma \in \Gamma_\infty^2 \setminus \Gamma_1^2} \nabla \phi_{n_1}^0 (\gamma \circ \tau) \right|^2 dV}{\int_S \sum_{n_1=1}^{\infty} \left| \sum_{\gamma \in \Gamma_\infty^2 \setminus \Gamma_1^2} \phi_{n_1}^0 (\gamma \circ \tau) \right|^2 dV} < 10\lambda.$$

The invariance of ∇ allow us to rewrite this as

$$\frac{\int_S \sum_{n_1=1}^{\infty} \left| \left[\sum_{\gamma \in \Gamma_\infty^2 \setminus \Gamma_1^2} \nabla \phi_{n_1}^0 (\tau) \right]_{\gamma \circ \tau} \right|^2 dV}{\int_S \sum_{n_1=1}^{\infty} \left| \left[\sum_{\gamma \in \Gamma_\infty^2 \setminus \Gamma_1^2} \phi_{n_1}^0 (\tau) \right]_{\gamma \circ \tau} \right|^2 dV} < 10\lambda.$$

where the notation means the evaluation at $\gamma \circ \tau$. Integration by parts of the form $\phi_{n_1}^0$, we get the above inequality in terms of the Laplacian. and now we consider the action of Laplacian Δ on these functions. As $\phi_{n_1}^0$ is independent of x_3. the terms involving x_3 annihilate $\phi_{n_1}^0$. Also all terms involving y_1. y_2. and x_2 are positive operators (compare with (2.31) on page 32 of Bump.) So

$$\Delta \phi_{n_1}^0 \geq -y_1^2 \frac{\partial^2}{\partial x_1^2} \phi_{n_1}^0 = y_1^2 4\pi^2 n_1^2 \phi_{n_1}^0.$$

but $y_1^2 > \frac{3}{4}$ in the Siegel set. So

$$\Delta \phi_{n_1}^0 \geq \frac{3}{4} 4\pi^2 n_1^2 \phi_{n_1}^0 = 3\pi^2 n_1^2 \phi_{n_1}^0.$$

So

$$10\lambda > \frac{\int_S \sum_{n_1=1}^{\infty} \sum_{\gamma \in \Gamma_\infty^2 \setminus \Gamma_1^2} \Delta \left[\phi_{n_1}^0 (\tau) \right]_{\gamma \circ \tau} \left[\phi_{n_1}^0 (\tau) \right]_{\gamma \circ \tau} dV}{\int_S \sum_{n_1=1}^{\infty} \left| \left[\sum_{\gamma \in \Gamma_\infty^2 \setminus \Gamma_1^2} \phi_{n_1}^0 (\tau) \right]_{\gamma \circ \tau} \right|^2 dV} \geq 3\pi^2.$$

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and so

$$\lambda > \frac{3\pi^2}{10},$$

as was asserted.□

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CHAPTER 3

Weyl's Law with Error Estimate

3.1 History and Statement of Results

The study of the spectrum of the Laplacian on spaces has been a subject of great interest in this century. The behavior of this spectrum has applications to physics. geometry and number theory. It was H. Weyl who first systematically studied the topic and obtained an important result. He showed that for a bounded domain in $\mathbb{R}^n$ with reasonable boundary conditions. the following asymptotic relation holds

$$N(\lambda) \sim C\lambda^{\frac{n}{2}}, \quad \text{as } \lambda \to \infty,$$

where $N(\lambda)$ denotes the dimension of the space of eigenfunctions of the Laplacian with eigenvalue less than λ, and C is a constant depends only on n, the dimension of the domain. and the volume of the domain [We]. Similar asymptotic results of this kind are called Weyl's law. Since then. many authors have put efforts on generalizations of Weyl's law to other Riemannian spaces. In particular the spaces $\Gamma\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$, and $\Gamma\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R})$, where Γ is a co-finite discrete but not necessarily co-compact subgroup of $SL(2,\mathbb{R})$, are considered very interesting because of their relevance to automorphic forms. One of those results which we are going to use in this thesis is the Weyl's law on $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R})$ by Stade and Wallace [S-W].

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It is, for many purposes, useful to have an error term for the asymptotic result. Courant [C] obtained

$$N(\lambda) = C\lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}} \log \lambda)$$

for the bounded domains. The proofs of these results use the technique of Dirichlet-Neumann bracketing, and the fact that the eigenvalues can be explicitly computed for cubes.

For compact manifolds, Minakshisundaram and Pleijel [M-P] use estimates on the heat kernel to derive Weyl's result. Using the wave equation, Hormander [H] obtains a Weyl's law with an error term. That is

$$N(\lambda) = C\lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}}).$$

For non-compact manifolds Weyl's law is in general false. The difficulty is continuous spectrum that does not allow the use of the above techniques in their original form. However, if the space is a finite volume symmetric space, then other techniques are available. For symmetric spaces associated to $SL(2,\mathbb{R})$ and a discrete group of isometries Γ, Selberg's trace formula [S] can be used to obtain that Weyl's law is true if one adds an extra term that accounts for the continuous spectrum. Also, a reasonable error term can be obtained. Moreover, if Γ is a congruence group of $SL(2,\mathbb{R})$ the additional term is smaller than the term accounting for the square integrable eigenfunctions, so Weyl's law holds. Actually, slightly more is true. We are now considering non-compact spaces, and so the spaces have cusps. With a finite number of exceptions all of the square integrable eigenforms are cusp forms.

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In the space on which we are going to work

$$\mathbb{X} = SL(3, \mathbb{Z})\backslash SL(3, \mathbb{R})/SO(3, \mathbb{R}),$$

one would expect for this space that similar results hold. That is $N(\lambda) = C\lambda^{\frac{5}{2}} + O(\lambda^{\frac{5-1}{2}})$. As we mentioned earlier. Stade and Wallace [S-W], using the version of the trace formula for $\mathbb{X}$ derived by Wallace through a series of articles and summarized in [W], established Weyl's law for this space. The theorem 1 of [S-W] says as $\lambda \to \infty$

$$N(\lambda) \sim \frac{vol\mathbb{X}}{(4\pi)^{\frac{5}{2}}\Gamma(\frac{7}{2})}\lambda^{\frac{5}{2}}.$$

They used Gaussians as test functions to obtain the asymptotic behavior at small time for the trace of the heat kernel. The result then follows from a standard Tauberian Argument.

In this thesis we show how this result can be combined with a different set of test functions to obtain an error estimate for Weyl's law. We state this result explicitly.

Theorem 3.1. *Let* $\mathbb{X} = SL(3, \mathbb{Z})\backslash SL(3, \mathbb{R})/SO(3, \mathbb{R})$ *,and* $N(\lambda)$ *denote the dimension of the space of cusp forms on* $\mathbb{X}$ *with Laplace eigenvalue less than* λ*. Then*

$$N(\lambda) = C\lambda^{\frac{5}{2}} + O(\lambda^{2})$$

with

$$C = \frac{Vol(\mathbb{X})}{(4\pi)^{(\frac{5}{2})}\Gamma(\frac{7}{2})}.$$

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Here C is the constant that one expects for Weyl's law. It depends only on the volume and the dimension of the space . The proof will be obtained by estimating the number of cusp forms with Laplace eigenvalue between λ, and $\lambda - 2\sqrt{\lambda} + 1$. We should note that cusp forms are joint eigenfunctions of the Laplace-Beltrami operator and an independent third order differential operator as $SL(3, \mathbb{R})$ has rank 2. We will not use this operator explicitly. Our test functions are the translations of functions that approximate the characteristic function of a disc instead of Gaussians as in [S-W].

In the course of the proof, we will give an estimate on the number of cusp forms that come from representations that violate the Ramanujan conjecture, which in representation theory language states that all cusp forms arise from principal series representations. This set is conjectured to be empty, but very little is known about it. In chapter 2, it was shown that no "small" eigenvalues exist: however, we do not know about the existence of "large" non-Ramanujan eigenvalue. Because of this reason, we state the result as a theorem.

Theorem 3.2. *The dimension of the space of cusp forms with Laplace eigenvalue less than λ that violate the Ramanujan conjecture is $O(\lambda^{\frac{3}{2}})$.*

3.2 Notations and the Selberg Trace Formula

We will let λ denote the eigenvalue corresponding to an L^2 eigenfunction of the Laplace-Beltrami operator on $\mathbb{X}$. Furthermore, the eigenfunction is actually a joint eigenfunction of the entire ring of invariant differential operators, as that is a commutative ring. Although we don't use the third order independent operator

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explicitly, by using the Selberg trace formula we are implicitly using the fact we are working with automorphic forms, which are joint eigenfunctions of the entire ring.

For the purpose of using the trace formula, it is necessary to write λ in terms of two parameters. In [S-W] two parameters s and t are used to describe the eigenvalues. In this notation $\lambda = s(1-s) + \frac{1}{3}t(1-t)$. This notation works well for them as they write the trace formula in terms of Helgason transforms found in Teras [T] and harmonic analysis on groups. However, we find it more convenient to work with a formula that relates a function in two variables to its Euclidean Fourier transform. We then have all of the tools of classical harmonic analysis at our disposal, and this will make it easier to perform our analysis with a greater value of test functions as we will not, as in [S-W], have to explicitly evaluate Fourier transforms. We use parameters α and β which are related to the earlier parameters by $\alpha = -i(s - \frac{1}{2}), \beta = -i(t - \frac{1}{2})$. We also change the metric so that our λ corresponds to 3λ in the notation of [S-W]. This makes no real difference, as this change in the number of cusp forms will be accounted for by the change of volume that it creates. In this notation we have $\lambda = 3\alpha^2 + \beta^2 + 1$. Also, from representation theory it is known that either both α and β are real or $-\operatorname{Im}\sqrt{3}\alpha = \operatorname{Im}\beta, \frac{1}{6} < \operatorname{Re}\sqrt{3}\alpha = \operatorname{Re}\beta < \frac{1}{2}$.

It is also convenient to make the following change of notation from that in [S-W]. On page 243, at the beginning of the second section, diagonal matrices a are introduced. They are elements of the group A of diagonal matrices in $SL(3,\mathbb{R})$. A

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given a is described as $a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & (a_1a_2)^{-1} \end{pmatrix}$. Here we have $a_i > 0$. Replacing a_i by e^{b_i} we obtain a pair of real numbers. We use this to replace the Helgason transforms, that are essentially Mellin transforms in [S-W], by Fourier transforms.

We now state the trace formula in the form that we will use.

Proposition 3.1. *Let g be a smooth, compactly supported function on $\mathbb{R}^2$ such that its Fourier transform $\hat{g}$ is Weyl invariant. Let (s_n, t_n) denote the (α, β) that correspond to an eigenvalue λ corresponding to a square-integrable automorphic form. Then*

$$\sum_{n=0}^{\infty} \hat{g}\,(s_n, t_n) = (ID) + (HYP) + (LOX) + (PAR)$$

The term on the left is the spectral term, and the 4 terms on the right of the equation are as follows.

$$\begin{aligned}(ID) &= \frac{vol\mathbb{X}}{(4\pi)^{\frac{5}{2}}\Gamma(\frac{7}{2})}\int_{\mathbb{R}^2} \hat{g}\,(s, t)t(t-3s)(t+3s) \\ &\quad \tanh \pi t \tanh \frac{\pi(t+3s)}{2} \tanh \frac{\pi(t-3s)}{2} dsdt,\end{aligned}$$

$$\begin{aligned}(HYP) &= \sum_{\epsilon_1\epsilon_2\epsilon_3=1} \mathrm{Reg}[Z(\epsilon_1)]Cl[Z(\epsilon_1)]\left|\epsilon_1^2\epsilon_2\right| \cdot \\ &\quad [(\epsilon_1-\epsilon_3)(\epsilon_1-\epsilon_2)(\epsilon_2-\epsilon_3)]^{-1}g(\log\epsilon_1, \log\epsilon_2).\end{aligned}$$

$$(LOX) = \sum_{(r,\theta)} \frac{|\ln r_0|\, Cl[Z(r)]}{1-2r^{-3}\cos\theta + r^{-6}} \int_{\mathbb{R}^2} \hat{g}\,(s, t)\frac{r^{1-s}e^{-2\theta\,\mathrm{Im}(t)}}{1+e^{-2\pi\,\mathrm{Im}(t)}}dsdt,$$

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$$(PAR) = A_1 \hat{g}(0,0) + A_2 \hat{g}(-\frac{1}{6}, \frac{1}{2}) + A_3 \hat{g}(\frac{1}{6}, \frac{1}{2}).$$

We must explain the meaning of the terms. The spectral term and the term (ID) are self explanatory, and in (PAR) the A_i are constants that need not be computed for our purposes. For the term (HYP), the sum is over triples $\epsilon_1, \epsilon_2, \epsilon_3$ where the ϵ_i are real such that $|\epsilon_1| > |\epsilon_2| > |\epsilon_3|$ and there is, by definition, a hyperbolic matrix in $SL(3,\mathbb{Z})$ with ϵ_i as its eigenvalues. The ϵ_i give rise to a number field. Reg denotes the regulator, and Cl the narrow class number. We also note that the logarithms occur in the function g because of our earlier change of variable $a = e^b$. Also the lack of a dependence on ϵ_3 is illusory, as the product of the ϵ_i is 1. In the term (LOX), the sum is over pairs (r,θ) with $r > 0, r \neq 1, 0 < \theta \leq \pi$ such that some matrix in $SL(3,\mathbb{Z})$ has eigenvalues $r^{-2}, re^{i\theta}, re^{-i\theta}$. If we denote the matrix by q, then r_0 is such that the centralizer of q has generator conjugate to $\begin{pmatrix} r_0^{-2} & 0 & 0 \\ 0 & r_0 e^{i\theta} & 0 \\ 0 & 0 & r_0 e^{-i\theta} \end{pmatrix}$. Finally, we noted that the test functions must be Weyl invariant. This is a generalization of the fact that for the classical trace formula, the test functions must be even. In our notation if we let $\sqrt{3}s$ denote the x-axis and t the y-axis, then two points are related by the Weyl group if

$$(\sqrt{3}s_1, t_1) = (e^{\frac{n\pi i}{3}}\sqrt{3}s_2, e^{\frac{n\pi i}{3}}t_2)$$

where n is a integer. This is simply a translation of representation theory facts into our notation.

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3.3 Proof of the Theorems

Weyl's law has already been established; thus to prove theorem 3.1 it is sufficient to estimate the number of cusp forms with Laplace eigenvalue between $\lambda - 2\sqrt{\lambda} + 1$ and λ. We shall now proceed to do this. In the course of the proof we will also prove Theorem 3.2.

We note that there are two ways that such an eigenvalue can exist. We can, in our parameters, assume S and T are real and have

$$\lambda - 2\sqrt{\lambda} + 1 \leq 3S^2 + T^2 \leq \lambda.$$

(Recall $\lambda = 3\alpha^2 + \beta^2 + 1$) We may also have S and T complex, and satisfying the restrictions given in Section 2, namely

$$-\operatorname{Im}\sqrt{3}S = \operatorname{Im}T$$

$$\frac{1}{6} < \operatorname{Re}\sqrt{3}S = \operatorname{Re}T < \frac{1}{2}$$

Actually, the complex parameters may be related to the above restriction by a Weyl transformation as described in section 2.

We will handle the two situations separately, in both cases the trace formula will be the main tool.

We first consider the case of real parameters. As stated above, we will actually estimate the number of eigenvalues with $\lambda - 2\sqrt{\lambda} + 1 \leq 3S^2 + T^2 \leq \lambda$. This is obviously sufficient. The point is that these inequalities describe an annulus of inner radius $\sqrt{\lambda} - 1$ and outer radius $\sqrt{\lambda}$ if along the horizontal axis we put $\sqrt{3}S$ and along the vertical axis we put T.

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We would like to put into the trace formula $\hat{g}$ equal to the characteristic function of the annulus. This is clearly not a legitimate test function so we must approximate it. We start with a function $\hat{g}_0$ such that g_0, its inverse Fourier transform, is compactly supported, smooth, radially symmetric and nonnegative. This will mean that $\hat{g}_0$ is Schwartz class and extends to an analytic function of two complex variables with exponential growth. We may also choose our functions so that

$$\hat{g}_0 > 0.$$

$$1 < \hat{g}_0 < 2.$$

if

$$3S^2 + T^2 < 1.$$

and

$$\hat{g}_0 < \frac{2}{(3S^2 + T^2)^k}.$$

for

$$3S^2 + T^2 > 1.$$

Here k may be taken to be a large positive constant. We may also assume that $\hat{g}_0 > 0$ for purely imaginary values of S and T. Standard arguments in Fourier analysis guarantee that such a function exists.

For our test function. we choose translates of $\hat{g}_0$ such that the point $(0,0)$ is translated to a point (S_0, T_0) that is in the annulus. We then sum over enough translates to cover the entire annulus. This will take no more than the smallest

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integer greater than $4\pi\sqrt{\lambda}$ translates. Here by cover, we mean that we want the test function to be at least one in the annulus. By the positivity of $\hat{g}_0$ we see that if we estimate the contribution to the spectral side of the equation for any given translate, we obtain an estimate for the test function by summing the estimates of the translates. We now let (S_0, T_0) be a point in the annulus and we let

$$\hat{g}_{(S_0,T_0)}$$

be the translate of $\hat{g}_0$ that is centered at (S_0, T_0), combined with its five Weyl transform related functions, so that it approximates six characteristic functions.

Lemma 3.1.

$$\sum_{n=0}^{\infty} \hat{g}_{(S_0,T_0)}\,(s_n, t_n) = O(\lambda^{\frac{3}{2}})$$

when $s, t \in \mathbb{R}$.

Remark: By summing over the $O(\sqrt{\lambda})$ such functions in the lemma we will have that real (S, T) contribute $O(\lambda^2)$ to the spectral side of the equation, proving part of the theorem.

Proof of Lemma 3.1: We must obtain estimates for the terms on the right hand side of the trace formula.

We first analyze the identity term. The contribution of this term with the original function $\hat{g}_0$ is clearly $O(1)$. The translation to the point $(\sqrt{3}S_0, T_0)$ will be $O(\lambda^{\frac{3}{2}})$, as the integrand is up to very small error, approximating by overestimating a cubic polynomial multiplied by twice the characteristic function of a circle of

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radius one. Of course, we are using the rapid decay of our test functions outside of a circle of radius one.

The hyperbolic term involves the inverse Fourier transform of

$$\hat{g}_{(\sqrt{3}S_0, T_0)} \; .$$

As

$$\hat{g}_{(\sqrt{3}S_0, T_0)}$$

is just a translation of $\hat{g}_0$, this function will simply be g_0 multiplied by a complex exponential. It is clear that from the compact support of g_0 it gives a $O(1)$ contribution, as we only have a finite sum. The same will be true for such a function multiplied by a complex exponential. We thus have an $O(1)$ contribution.

The loxodromic term is dominated in the annulus, up to some constant, by the test function given in [S-W], when the Gaussian chosen is still greater than $\frac{1}{e}$ in the annulus. The decaying part of the test functions can be easily estimated as we may choose k in our original assumptions arbitrarily large. We thus get a $O(\lambda)$ contribution for each

$$\hat{g}_{(\sqrt{3}S_0, T_0)} \; .$$

The parabolic terms give us $O(1)$ for each $\hat{g}_{(\sqrt{3}S_0, T_0)}$ as we are evaluating at individual points, and as we translate the function the result gets even smaller due to the rapid decay of the original test function. We thus have proved Lemma 3.1.

To complete the proof of Theorem 3.1 we will need to prove Theorem 3.2. We now assume that S and T are not real, but satisfy the other possible conditions, described above. We will once again use translates of $\hat{g}_0$. This time we will translate

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so that we cover the lines $T = 0, T = 3S, T = -3S$ up to the point $3\alpha^2+\beta^2+1 < \lambda$. We will need $O(\lambda^{\frac{1}{2}})$ such translates. We once again use the trace formula. For all terms but the identity, the arguments are identical to those previously used. The identity term now gives a better estimate. We now are only working with what is approximately a quadratic polynomial, as one of $t, (t-3s), (t+3s)$ does not go to infinity as the parameters go to infinity. We thus get an $O(\lambda)$ contribution from each $\hat{g}_{(\sqrt{3}S_0, T_0)}$ yielding an over all contribution of $O(\lambda^{\frac{3}{2}})$. This proves Theorem 3.2 and thus we have proved Theorem 3.1. Once again, it should be emphasized that we actually expect that the contribution from the non-real S and T to only consist of the constant function.

We conclude this chapter by remarking that a more refined analysis of the terms is likely to be possible. This would allow us to use the techniques in [H-T] to prove Weyl's law directly and to perform the refined local analysis found there, leading to "local" Weyl's laws and more precise estimates. In particular, our results show that "on average" every ball centered at $(\sqrt{3}S_0, T_0)$, with radius less than $\sqrt{\lambda}$ has approximately $C\lambda^{\frac{3}{2}}$ where C is a constant depending on the ball. Also we expect to improve the estimates slightly. These issues will be discussed elsewhere.

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CHAPTER 4

A Comparison of Cusp Forms on $GL(3,\mathbb{Z})\backslash PGL(3,\mathbb{R})/O(3)$.

4.1 Preliminary Discussion and Results

The structure of the argument we use to prove the theorem 4.1 is a mild variation of the Siegel-Maass method. This method for $SL(2,\mathbb{Z})$ cusp forms can be found in Terras [T]. The use of this method is possible because of the Fourier expansion of a cusp form with Whittaker functions as kernels as in Bump [B]. The main idea of the method is that we derive an inequality which has the maximum of a cusp form on smaller side of the inequality and a factor times the maximum of a cusp form on the larger side of the inequality. Then draw a contradiction. That is the factor of the larger side of the inequality gets smaller than 1, and this contradiction forces the cusp form to be identically zero.

In doing so, we make a use of the integral expression of Whitaker functions which contains two K-Bessel functions as in Stade [St] p 318. Bump and Huntley use this expression to get an asymptotic expansion of Whittaker functions with a certain condition on arguments. They substitute both K-Bessel functions with an asymptotic expansion of K-Bessel function around infinity.

As we will discuss in detail in section 4.2, Bump and Huntley's asymptotic expansion can be used to prove that a finite number of Fourier coefficients determine a cusp form. However, in the proof of the theorem 4.1 we only use Bump and

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Huntley's asymptotic expansion of Whitaker function to get a minimum bound of $|W_{\nu_1,\nu_2}(1,1)|$, and we use the asymptotic expansions of K-Bessel functions to overestimate absolute values of Whittaker functions.

Actually what we are using is an error estimate of K-Bessel functions. This overestimation for a large argument is constructed by Olver [O2]. The use of this overestimate is needed to get our result mainly because of the Δ factors, whose definition will be on p 28, in the arguments of Whittaker functions.

As we will see in section 4.2, the arguments of Whittaker functions are $n_1\Delta y_1$, and $\frac{n_2}{\Delta^2}y_2$. This asymmetry of the arguments (appearances of Δ and Δ^{-2}) is the source of two difficulties of using Bump and Huntley's asymptotic expansion of Whittaker functions. First, we have some cases which do not satisfy the condition of validity of Bump and Huntley's expansion (for example when $n_1 \geq k\lambda$, n_2 is small, and $\Delta \geq n_1^{\frac{1}{3}}$, then we have $(n_1\Delta)^{-(-\frac{1}{2}+\varepsilon)} \times \frac{n_2}{\Delta^2}$ small for any $\varepsilon > 0$.) Second, we have cases that none of $n_1\Delta y_1$, and $\frac{n_2}{\Delta^2}y_2$ is large enough (for example when $n_2 = k\lambda$ and n_1 is small but Δ is say $(k\lambda)^{\frac{1}{2}}$.) As we will see in the later sections, we need at least one of these two arguments to be large enough (to use [B-H] expansion, one of them must be larger than $k\lambda$) so $e^{-n_1\Delta y_1}$, or $e^{-\frac{n_2}{\Delta^2}y_2}$ be small enough to create a contradiction.

To overcome these difficulties, we are going to work on K-Bessel functions, and make the contribution of n_1 to be $O(\lambda^{\frac{1}{2}})$ and the contribution of n_2 to be $O(\lambda^{\frac{3}{2}})$ rather than more natural $O(\lambda)$ contributions from both n_1 and n_2. We are exploiting the fact that when argument of a K-Bessel function is larger in the

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order than the parameter, we can use Olver's error estimate of K-Bessel functions a numerical tool for our purpose. That is we overestimate K-Bessel functions by a positive exponential of $e^{-n_1 \Delta y_1}$, or $e^{-\frac{n_2}{\Delta^2} y_2}$ as at least one of the arguments is larger than $k\lambda^{\frac{1}{2}}$.

We will work in terms of ν rather than λ. This can be justified by the relationship between λ and ν_1, ν_2 in [B] p 33. That is

$$\lambda = 3\left(\nu_1^2 + \nu_1\nu_2 + \nu_2^2 - \nu_1 - \nu_2\right).$$

But our

$$\nu = \frac{3\nu_1 + 3\nu_2 - 2}{2}.$$

So the order of ν is $O(\lambda^{\frac{1}{2}})$.

We now state the main results of this chapter.

Theorem 4.1. *If Fourier coefficients a_{n_1,n_2} of a cusp form ϕ on*

$$\mathbb{X} = GL(3,\mathbb{Z})\backslash PGL(3,\mathbb{R})/O(3)$$

are zero when max $\left(n_1, n_2^{\frac{1}{3}}\right) \leq O\left(\lambda^{\frac{1}{2}}\right)$, then ϕ is identically zero.

Corollary 4.2. *Let ϕ and ψ be cusp forms on $\mathbb{X}$, and a_{n_1,n_2} and b_{n_1,n_2} be their Fourier coefficients respectively.. If $a_{n_1,n_2} = b_{n_1,n_2}$ when max $\left(n_1, n_2^{\frac{1}{3}}\right) \leq O\left(\lambda^{\frac{1}{2}}\right)$, then $\phi = \psi$.*

Notice that the theorem 4.1 implies that the multiplicity of an eigenvalue λ is $O\left(\lambda^2\right)$, which is an obvious implication of theorem 3.1. However, theorem 3.1 says nothing about Fourier coefficients of the cusp form, and theorem 4.1 doesn't imply actual existence of cusp forms for a certain interval of λ.

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Before we continue, we quote the theorem of [B] page 65 precisely, and describe a action of Γ_1^2 on y_1 and y_2. Let Γ_∞ be the group of 3×3 upper triangular unipotent matrices with integer coefficients. Also let:

$$\Gamma^2 = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} | a, b, c, d \in \mathbb{Z},\ ad - bc = \pm 1 \right\}$$

and

$$\Gamma^2_\infty = \Gamma^2 \cap \Gamma_\infty.$$

and, denote the subgroup of index 2 in Γ^2, as Γ_1^2, which is the group of those elements of determinant one.

The theorem says:

If ϕ is a cusp form of type ν_1, and ν_2 there exist coefficients a_{n_1,n_2} for positive integers n_1, and n_2 such that

$$\phi(\tau) = \sum_{g \in \Gamma^2_\infty \backslash \Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} n_1^{-1} n_2^{-1} a_{n_1,n_2} W_{1,1}^{(\nu_1,\nu_2)} \left(\begin{pmatrix} n_1 n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g\tau \right).$$

Here

$$\tau = \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Now

$$\begin{pmatrix} n_1 n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} y_1' y_2' & y_1' x_2' & x_3' \\ 0 & y_1' & x_1' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (cx_2 + d)\,\Delta^{-1} & -cy_2\Delta^{-1} & 0 \\ cy_2\Delta^{-1} & (cx_2 + d)\,\Delta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\Delta = |c\,(x_2 + iy_2) + d|$. For our purpose we may ignore x_j, because they do not contribute to the magnitude of the Whitaker function. Solving equations for y_1', and y_2', we find the action of Γ_1^2 on y_1 and y_2. That is $y_1' = n_1 \Delta y_1$, and $y_2' = \frac{n_2 y_2}{\Delta^2}$. This theorem and the action will be used in sections 4.2 and 4.3.

4.2 Motivations of Theorem 4.1.

By theorem 3.1, we know the bound of the dimension of a cusp form with the eigenvalue λ of the Laplacian on $\mathbb{X}$ is $O(\lambda^2)$ because

$$N(\lambda + 1) - N(\lambda) = C\left((\lambda + 1)^{\frac{5}{2}} - \lambda^{\frac{5}{2}}\right) + O(\lambda^2) = O(\lambda^2).$$

However we do not know which ones of coefficients a_{n_1,n_2} are actually non zero on the Fourier expansion of a cusp form. That is we might have a non zero coefficients a_{n_1,n_2} when n_1, or n_2 are extremely large. However, theorem 4.1 shows otherwise.

In [B-H], they constructed an asymptotic expansion of Whitaker functions as y_1, or $y_2 \to \infty$ using a variation of Watson's lemma.

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The theorem 1 in [B-H] says:

$$\sqrt{\frac{3\pi}{2}}W_{\nu_1,\nu_2}(y_1,y_2)\sim\sum_{n=0}^{\infty}W^n_{\nu_1,\nu_2}(y_1,y_2),$$

where for certain constants $c(j,\nu_1,\nu_2)$, we have

$$W^n_{\nu_1,\nu_2}(y_1,y_2) = y_1^{\frac{1}{3}}y_2^{\frac{1}{3}}\left[\sum_{-n}^{n}c(j,\nu_1,\nu_2)\,y_1^{\frac{2j}{3}}y_2^{\frac{-2j}{3}}\right] \times\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{-(6n+1)/4}e^{-\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

This asymptotic expansion is valid when $y_1\to\infty$, and $y_1^{-\alpha}y_2$ is kept equal to a positive constant, or when $y_2\to\infty$, and $y_1y_2^{-\alpha}$ is kept equal to a positive constant, for any $-\frac{1}{2}<\alpha<\infty$. In particular, it is valid if y_1, and y_2 both are large.

Furthermore, they computed $c(j,\nu_1,\nu_2)$, by a recursive method which is described in the last part of the article, as the following.

$$W^0_{\nu_1,\nu_2}(y_1,y_2)=y_1^{\frac{1}{3}}y_2^{\frac{1}{3}}\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{-\frac{1}{4}}e^{-\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{\frac{3}{2}}}.$$

$$W^1_{\nu_1,\nu_2}(y_1,y_2) = \left[\left(\frac{5}{18}+\frac{\lambda}{2}\right)\left(y_1y_2^{-\frac{1}{3}}+y_1^{-\frac{1}{3}}y_2\right)+\left(\frac{35}{72}+\lambda\right)y_1^{\frac{1}{3}}y_2^{\frac{1}{3}}\right] \times\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{-\frac{7}{4}}e^{-\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

$$W^2_{\nu_1,\nu_2}(y_1,y_2)=$$

$$[\left(-\frac{35}{648}-\frac{\lambda}{36}-\frac{\mu}{4}+\frac{\lambda^2}{8}\right)y_1^{\frac{5}{3}}y_2^{-1}+\left(-\frac{245}{1296}-\frac{\lambda}{16}-\frac{\mu}{2}+\frac{\lambda^2}{2}\right)y_1y_2^{-\frac{1}{3}}$$
$$+\left(-\frac{805}{3456}-\frac{5\lambda}{72}+\frac{3\lambda^2}{4}\right)y_1^{\frac{1}{3}}y_2^{\frac{1}{3}}+\left(-\frac{245}{1296}-\frac{\lambda}{16}-\frac{\mu}{2}+\frac{\lambda^2}{2}\right)y_1^{-\frac{1}{3}}y_2$$
$$+\left(-\frac{35}{648}-\frac{\lambda}{36}-\frac{\mu}{4}+\frac{\lambda^2}{8}\right)y_1^{-1}y_2^{\frac{5}{3}}]\times\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{-\frac{13}{4}}e^{-\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

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and

$$W^3_{\nu_1,\nu_2}(y_1, y_2) =$$

$$\begin{aligned}
&[\left(\frac{665}{34992}+\frac{\lambda}{1296}+\frac{19\mu}{72}-\frac{\lambda\mu}{8}-\frac{7\lambda^2}{144}+\frac{\lambda^3}{48}\right) y_1^{\frac{7}{3}} y_2^{-\frac{5}{3}} \\
&+\left(\frac{5495}{46656}+\frac{17\lambda}{2592}+\frac{103\mu}{96}-\frac{\lambda\mu}{2}-\frac{173\lambda^2}{576}+\frac{\lambda^3}{8}\right) y_1^{\frac{5}{3}} y_2^{-1} \\
&+\left(\frac{53725}{186624}-\frac{55\lambda}{20736}+\frac{65\mu}{48}-\frac{5\lambda\mu}{8}-\frac{55\lambda^2}{72}+\frac{5\lambda^3}{16}\right) y_1 y_2^{-\frac{1}{3}} \\
&+\left(\frac{761915}{2239488}-\frac{175\lambda}{10368}-\frac{295\lambda^2}{288}+\frac{5\lambda^3}{12}\right) y_1^{\frac{1}{3}} y_2^{\frac{1}{3}} \\
&+\left(\frac{53725}{186624}-\frac{55\lambda}{20736}-\frac{65\mu}{48}+\frac{5\lambda\mu}{8}-\frac{55\lambda^2}{72}+\frac{5\lambda^3}{16}\right) y_1^{-\frac{1}{3}} y_2 \\
&+\left(\frac{5495}{46656}+\frac{17\lambda}{2592}-\frac{103\mu}{96}+\frac{\lambda\mu}{2}-\frac{173\lambda^2}{576}+\frac{\lambda^3}{8}\right) y_1^{-1} y_2^{\frac{5}{3}} \\
&+\left(\frac{665}{34992}+\frac{\lambda}{1296}-\frac{19\mu}{72}+\frac{\lambda\mu}{8}-\frac{7\lambda^2}{144}+\frac{\lambda^3}{48}\right) y_1^{-\frac{5}{3}} y_2^{\frac{7}{3}}] \\
&\times \left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{-\frac{19}{4}} e^{-\left(y_1^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{\frac{3}{2}}}
\end{aligned}$$

Notice that when we divide the zeroth term by the first term of Bump Huntley asymptotic expansion when both $|y_1|$. and $|y_2|$ are lager than $|\lambda|$.

$$\begin{aligned}
&\frac{W^0_{\nu_1,\nu_2}(y_1, y_2)}{W^1_{\nu_1,\nu_2}(y_1, y_2)} \\
&= \frac{(y_1y_2)^{1/3}\left(y_1^{2/3}+y_2^{2/3}\right)^{-1/4}}{\left[\left(\frac{5}{18}+\frac{\lambda}{2}\right)\left(y_1y_2^{-1/3}+y_1^{-1/3}y_2\right)+\left(\frac{35}{72}+\lambda\right)(y_1y_2)^{1/3}\right]\left(y_1^{2/3}+y_2^{2/3}\right)^{-7/4}} \\
&= \frac{(y_1y_2)^{1/3}\left(y_1^{2/3}+y_2^{2/3}\right)^{3/2}}{k\lambda\cdot\max\left(y_1y_2^{-1/3}, y_1^{1/3}y_2^{1/3}, y_1^{-1/3}y_2\right)}.
\end{aligned}$$

where k is a constant. and if this quotient is larger than 1, we can use the Bump and Huntley's asymptotic expansion to overestimate the Whitaker functions. That is the zeroth term is larger than the first term. More precisely, from the recursive

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method used in [B-H] to get the $c(j, \nu_1, \nu_2)$, we know that the highest power of λ is increased by 1 as n increases for each term in the asymptotic expansion of a Whittaker function, and the exponents of $\left(y_1^{2/3} + y_2^{2/3}\right)$ is decreased by $\frac{3}{2}$. This implies that the asymptotic expansion of a Whittaker function can be overestimated by a convergent geometric series times $exp\left(-\left(y_1^{2/3} + y_2^{2/3}\right)^{3/2}\right)$ when $|y_1|$, and $|y_2|$ are lager than λ.

In order to show the bound of the dimension of a cusp form, we use the Fourier expansion of a cusp form in the theorem of Bump. Since

$$W_{1,1}^{(\nu_1,\nu_2)}(\tau, w_1) = W_{\nu_1\nu_2}(y_1, y_2)e^{2\pi i(x_1+x_2)} ,$$

we want to have an overestimate of

$$\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right|$$

using Bump and Huntley's asymptotic expansion where y_1° and y_2° are such that the absolute value of a cusp form $\phi(y_1^\circ, y_2^\circ)$ is the maximum. Furthermore, in order to use Siegel and Maass method, we want these overestimates to be in the form of some positive power of

$$e^{-\left(n_1\Delta y_1^\circ + \frac{n_2 y_2^\circ}{\Delta^2}\right)}$$

multiplied by a constant where at least one of $n_1\Delta y_1^\circ$, and $\frac{n_2 y_2^\circ}{\Delta^2}$ is larger than λ.

First we consider, when $\Delta = 1$. Then, by the above argument, when we have both n_1, and $n_2 \geq \lambda$, we have

$$|W_{\nu_1,\nu_2}(n_1 y_1^\circ, n_2 y_2^\circ)| \leq k\left(e^{-\left(n_1\Delta y_1^\circ + \frac{n_2 y_2^\circ}{\Delta^2}\right)}\right)^\varepsilon .$$

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where $\varepsilon > 0$. Now let the coefficients a_{n_1,n_2} of the Fourier expansion of the cusp form equal zero when $\max(n_1, n_2) \le 4\lambda$. Then the number of a_{n_1,n_2} that are equal to zero is $16\lambda^2 = O(\lambda^2)$ which is within the range of our aim.

Suppose $n_1 > n_2$. Then we may assume that we have $n_1 > 4\lambda > \lambda > n_2$. We can apply Bump and Huntley's asymptotic expansion because $n_2 y_2^\circ$ does not approach to zero. Then we have .

$$\sqrt{\tfrac{3\pi}{2}} W_{\nu_1,\nu_2}(n_1 y_1^\circ, n_2 y_2^\circ) \sim$$

$$(n_1y_1^\circ)^{\frac{1}{3}} (n_2y_2^\circ)^{\frac{1}{3}} \left((n_1y_1^\circ)^{\frac{2}{3}} + (n_2y_2^\circ)^{\frac{2}{3}}\right)^{-\frac{1}{4}} e^{-\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

$$+[\left(\tfrac{5}{18}+\tfrac{\lambda}{2}\right)\left((n_1y_1^\circ)(n_2y_2^\circ)^{-\frac{1}{3}} + (n_1y_1^\circ)^{-\frac{1}{3}}(n_2y_2^\circ)\right)$$

$$+\left(\tfrac{35}{72}+\lambda\right)(n_1y_1^\circ)^{\frac{1}{3}}(n_2y_2^\circ)^{\frac{1}{3}}]$$

$$\times\left((n_1y_1^\circ)^{\frac{2}{3}} + (n_2y_2^\circ)^{\frac{2}{3}}\right)^{-\frac{7}{4}} e^{-\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

$$+[\left(-\tfrac{35}{648}-\tfrac{\lambda}{36}-\tfrac{\mu}{4}+\tfrac{\lambda^2}{8}\right)(n_1y_1^\circ)^{\frac{5}{3}}(n_2y_2^\circ)^{-1}$$

$$+\left(-\tfrac{245}{1296}-\tfrac{\lambda}{16}-\tfrac{\mu}{2}+\tfrac{\lambda^2}{2}\right)(n_1y_1^\circ)(n_2y_2^\circ)^{-\frac{1}{3}}$$

$$+\left(-\tfrac{805}{3456}-\tfrac{5\lambda}{72}+\tfrac{3\lambda^2}{4}\right)(n_1y_1^\circ)^{\frac{1}{3}}(n_2y_2^\circ)^{\frac{1}{3}}$$

$$+\left(-\tfrac{245}{1296}-\tfrac{\lambda}{16}-\tfrac{\mu}{2}+\tfrac{\lambda^2}{2}\right)(n_1y_1^\circ)^{-\frac{1}{3}}(n_2y_2^\circ)$$

$$+\left(-\tfrac{35}{648}-\tfrac{\lambda}{36}-\tfrac{\mu}{4}+\tfrac{\lambda^2}{8}\right)(n_1y_1^\circ)^{-1}(n_2y_2^\circ)^{\frac{5}{3}}]$$

$$\times\left((n_1y_1^\circ)^{\frac{2}{3}} + (n_2y_2^\circ)^{\frac{2}{3}}\right)^{-\frac{13}{4}} e^{-\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

$$+[\left(\tfrac{665}{34992}+\tfrac{\lambda}{1296}+\tfrac{19\mu}{72}-\tfrac{\lambda\mu}{8}-\tfrac{7\lambda^2}{144}+\tfrac{\lambda^3}{48}\right)(n_1y_1^\circ)^{\frac{7}{3}}(n_2y_2^\circ)^{-\frac{5}{3}}$$

$$+\left(\tfrac{5495}{46656}+\tfrac{17\lambda}{2592}+\tfrac{103\mu}{96}-\tfrac{\lambda\mu}{2}-\tfrac{173\lambda^2}{576}+\tfrac{\lambda^3}{8}\right)(n_1y_1^\circ)^{\frac{5}{3}}(n_2y_2^\circ)^{-1}$$

$$+\left(\tfrac{53725}{186624}-\tfrac{55\lambda}{20736}+\tfrac{65\mu}{48}-\tfrac{5\lambda\mu}{8}-\tfrac{55\lambda^2}{72}+\tfrac{5\lambda^3}{16}\right)(n_1y_1^\circ)(n_2y_2^\circ)^{-\frac{1}{3}}$$

$$+\left(\tfrac{761915}{2239488}-\tfrac{175\lambda}{10368}-\tfrac{295\lambda^2}{288}+\tfrac{5\lambda^3}{12}\right)(n_1y_1^\circ)^{\frac{1}{3}}(n_2y_2^\circ)^{\frac{1}{3}}$$

$$+\left(\tfrac{53725}{186624}-\tfrac{55\lambda}{20736}-\tfrac{65\mu}{48}+\tfrac{5\lambda\mu}{8}-\tfrac{55\lambda^2}{72}+\tfrac{5\lambda^3}{16}\right)(n_1y_1^\circ)^{-\frac{1}{3}}(n_2y_2^\circ)$$

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$$+\left(\frac{5495}{46656}+\frac{17\lambda}{2592}-\frac{103\mu}{96}+\frac{\lambda\mu}{2}-\frac{173\lambda^2}{576}+\frac{\lambda^3}{8}\right)(n_1y_1^\circ)^{-1}(n_2y_2^\circ)^{\frac{5}{3}}$$

$$+\left(\frac{665}{34992}+\frac{\lambda}{1296}-\frac{19\mu}{72}+\frac{\lambda\mu}{8}-\frac{7\lambda^2}{144}+\frac{\lambda^3}{48}\right)(n_1y_1^\circ)^{-\frac{5}{3}}(n_2y_2^\circ)^{\frac{7}{3}}]$$

$$\times\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{-\frac{19}{4}}e^{-\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{\frac{3}{2}}}.$$

Now compare this expansion with the expansion of $\sqrt{\frac{3\pi}{2}}W_{\nu_1,\nu_2}(\lambda,\lambda)$ term by term. First the zeroth term of $\sqrt{\frac{3\pi}{2}}W_{\nu_1,\nu_2}(n_1y_1^\circ,n_2y_2^\circ)$.

$$(n_1y_1^\circ)^{\frac{1}{3}}(n_2y_2^\circ)^{\frac{1}{3}}\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{-\frac{1}{4}}e^{-\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

is smaller than the zeroth term of $\sqrt{\frac{3\pi}{2}}W_{\nu_1,\nu_2}(\lambda,\lambda)$.

$$\lambda^{\frac{2}{3}}\left(2\lambda^{\frac{2}{3}}\right)^{-\frac{1}{4}}e^{-\left(2\lambda^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

as λ gets large because then we have

$$e^{-\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{\frac{3}{2}}}<e^{-\left(2\lambda^{\frac{2}{3}}\right)^{\frac{3}{2}}}.$$

Comparing term by term for arbitrary n, we have the following.

$$\sqrt{\frac{3\pi}{2}}W_{\nu_1,\nu_2}(n_1y_1^\circ,n_2y_2^\circ)\sim$$

$$\sum_{n=0}^{\infty}(n_1y_1^\circ)^{\frac{1}{3}}(n_2y_2^\circ)^{\frac{1}{3}}\left[\sum_{-n}^{n}c(j,\nu_1,\nu_2)(n_1y_1^\circ)^{\frac{2j}{3}}(n_2y_2^\circ)^{\frac{-2j}{3}}\right]$$

$$\times\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{-(6n+1)/4}e^{-\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{\frac{3}{2}}}$$

$$\le\sum_{n=0}^{\infty}(n_1y_1^\circ)^{\frac{1}{3}}(n_2y_2^\circ)^{\frac{1}{3}}\left[\sum_{-n}^{n}c(j,\nu_1,\nu_2)(n_1y_1^\circ)^{\frac{2j}{3}}(n_2y_2^\circ)^{\frac{-2j}{3}}\right]$$

$$\times\left((n_1y_1^\circ)^{\frac{2}{3}}+(n_2y_2^\circ)^{\frac{2}{3}}\right)^{-(6n+1)/4}e^{-\left(\frac{n_1y_1^\circ}{4}+\frac{n_1y_1^\circ}{10}+2\sqrt{2}\lambda+n_2y_2^\circ\right)}$$

$$\le\sum_{n=0}^{\infty}\lambda^{\frac{1}{3}}\lambda^{\frac{1}{3}}\left[\sum_{-n}^{n}c(j,\nu_1,\nu_2)\lambda^{\frac{2j}{3}}\lambda^{\frac{-2j}{3}}\right]$$

$$\times\left(\lambda^{\frac{2}{3}}+\lambda^{\frac{2}{3}}\right)^{-(6n+1)/4}e^{-2\sqrt{2}\lambda}\left(e^{-(n_1y_1^\circ+n_2y_2^\circ)}\right)^{\frac{1}{4}}$$

$$\le\sqrt{\frac{3\pi}{2}}W_{\nu_1,\nu_2}(\lambda,\lambda)\left(e^{-(n_1y_1^\circ+n_2y_2^\circ)}\right)^{\frac{1}{4}}.$$

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This comparison is true even though n is lager than n_1. because if then

$$\left((n_1y_1^\circ)^{\frac{2}{3}} + (n_2y_2^\circ)^{\frac{2}{3}}\right)^{-(6n+1)/4}$$

will dominate the entire polynomial part. So we have shown

$$W_{\nu_1,\nu_2}(n_1y_1^\circ, n_2y_2^\circ) \le \left(e^{-(n_1y_1^\circ+n_2y_2^\circ)}\right)^{\frac{1}{4}}$$

when $n_1 > 4\lambda > \lambda > n_2$. By the symmetry of Bump and Huntley's expansion, when $n_2 > 4\lambda > \lambda > n_1$, we have $W_{\nu_1,\nu_2}(n_1y_1^\circ, n_2y_2^\circ) \le \left(e^{-(n_1y_1^\circ+n_2y_2^\circ)}\right)^{\frac{1}{4}}$ also.

So the remaining cases are when $\Delta > 1$. For the cases of $n_1 > n_2$, we can not always use Bump and Huntley's asymptotic expansion because $(n_1\Delta)^{-(-\frac{1}{2}+\varepsilon)} \times \frac{n_2}{\Delta^2}$ may not be a constant (when Δ is larger than say $(n_1)^{\frac{1}{2}}$) and so do not satisfy the criterion of the theorem. However the integral expression of the Whittaker function

$$W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \tfrac{n_2y_2^\circ}{\Delta^2}) =$$

$$\pi^{-2}(n_1\Delta y_1^\circ)^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2y_2^\circ}{\Delta^2}\right)^{1+(\nu_2-\nu_1)/2}$$

$$\int_0^\infty K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx$$

gives $W_{\nu_1,\nu_2}\left(n_1\Delta y_1^\circ, \frac{n_2y_2^\circ}{\Delta^2}\right) \le \left(e^{-(n_1y_1^\circ+n_2y_2^\circ)}\right)^{\frac{1}{4}}$ as shown in the proof of the theorem 4.1 in the following section. The main argument is the first K-Bessel function decays much faster than the second one grows as Δ gets larger regardless of x.

When $n_2 > n_1$, we have following cases. First when $n_1\Delta \ge 4\lambda$, then we may have $(n_1\Delta)^{\frac{1}{2}} \times \frac{n_2}{\Delta^2} > k$ if n_1 is large where k is a positive constant. Then we are in the similar situation as $n_1 > n_2$, and $\Delta = 1$. However when $(n_1\Delta)^{\frac{1}{2}} \times \frac{n_2}{\Delta^2} < k$ (i.e.

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neither n_1 nor n_2 is large enough) we can't apply Bump and Huntley's expansion, and when $n_1\Delta < 4\lambda$, we may have cases when none of $n_1\Delta$ and $\frac{n_2}{\Delta^2}$ is larger than 4λ. So we can not use the same argument. And this asymmetry of $n_1\Delta$ and $\frac{n_2}{\Delta^2}$ prevents our result in this section becomes just a corollary of that expansion, and motivated us to use different approach.

However, we can still get a significant result from Bump and Huntley's expansion. That is the bound of the dimension of the cusp forms with the eigenvalue λ of Laplacian on $GL(3,\mathbb{R})/Z(GL(3,\mathbb{R}))O(3)$ is $O(\lambda^4)$ and unlike the result from the chapter 3, we know the locations of non zero Fourier coefficients. The argument is as following. Instead of letting the coefficients a_{n_1,n_2} of the Fourier expansion of the cusp form equal zero when $\max(n_1,n_2) \le 4\lambda$ as above, we let $a_{n_1,n_2} = 0$ for all n_1, and n_2 when $\max\left(n_1,n_2^{\frac{1}{3}}\right) \le 4\lambda$. So we only need to care about the cases when $\max\left(n_1,n_2^{\frac{1}{3}}\right) > 4\lambda$.This ensures at least one of $n_1\Delta y_1^\circ$, and $\frac{n_2 y_2^\circ}{\Delta^2}$ larger than 4λ. That is, if $\frac{n_2 y_2^\circ}{\Delta^2} < 4\lambda$, we have $\Delta^2 > 16\lambda^2$ which forces $n_1\Delta > 4\lambda$. And so we may use the above argument.

This result can be improved slightly without a major change in the proof. That is, what we really care is the magnitude of $e^{-\left((n_1\Delta y_1^\circ)^{\frac{2}{3}}+\left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}}$, and if $\left((n_1\Delta y_1^\circ)^{\frac{2}{3}}+\left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} \ge 4\lambda$, the above argument works. Since

$$\frac{d\left((n_1\Delta y_1^\circ)^{\frac{2}{3}}+\left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}}{d\Delta}$$
$$= \frac{3}{2}\left((n_1\Delta y_1^\circ)^{\frac{2}{3}}+\left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(\frac{2}{3}(n_1 y_1^\circ)^{\frac{2}{3}}\Delta^{-\frac{1}{3}}-\frac{2}{3}2n_2 y_2^\circ\Delta^{-\frac{7}{3}}\right),$$

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by letting

$$(n_1 y_1^\circ)^{\frac{2}{3}} - 2n_2 y_2^\circ \Delta^{-2} = 0,$$

we find that $\left((n_1\Delta y_1^\circ)^{\frac{2}{3}} + \left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}$ is minimum at

$$\Delta = \left(\frac{2n_2 y_2^\circ}{n_1 y_1^\circ}\right)^{\frac{1}{3}}.$$

So by replacing Δ in $\left((n_1\Delta y_1^\circ)^{\frac{2}{3}} + \left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}$ by $\left(\frac{2n_2 y_2^\circ}{n_1 y_1^\circ}\right)^{\frac{1}{3}}$, we know that if

$$(n_1 y_1^\circ)^{\frac{2}{3}} (n_2 y_2^\circ)^{\frac{1}{3}} \geq 4\lambda.$$

then we may use the above argument. So we only need to let $a_{n_1,n_2} = 0$ when $n_1 < \lambda$. and $n_2 \leq \frac{(4\lambda)^3}{n_1^2}$. So the number of the Fourier coefficients a_{n_1,n_2} witch are zero can be overestimated by $2\int_1^\lambda \frac{(4\lambda)^3}{n_1^2} dn_1 = 2(4\lambda)^3 [-\frac{1}{n_1}]|_1^\lambda = O(\lambda^3)$. seeing n_1 as real numbers rather than integers.

4.3 Proof of Theorem 4.1

The theorem in Bump page 65 says if ϕ is a cusp form of type ν_1. and ν_2 there exist coefficients a_{n_1,n_2} for positive integers n_1. and n_2 such that

$$|\phi(\tau^\circ)| \leq \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \left| n_1^{-1} n_2^{-1} a_{n_1,n_2} W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2}) \right|. \tag{4.3.1}$$

Since ϕ is a cusp form, there exist τ° such that $\phi(\tau^\circ)$ is the maximum of ϕ, and

$$W_{1,1}^{(\nu_1,\nu_2)}(\tau, w_1) = W_{\nu_1\nu_2}(y_1, y_2) e^{2\pi i(x_1+x_2)}$$

[B] (3.46), and so we have

$$|\phi(\tau^\circ)| \leq \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \left| n_1^{-1} n_2^{-1} a_{n_1,n_2} W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2}) \right|.$$

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But by Bump (4.9) and (4.12), which say

$$\phi_{n_1,n_2}(\tau) = \int_0^1\int_0^1\int_0^1 \phi\left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{pmatrix}\tau\right) e(-n_1\xi_1 - n_2\xi_2).$$

and

$$\phi_{n_1,n_2}(\tau) = a_{n_1,n_2}\,|n_1 n_2|^{-1}\, W_{1,1}^{(\nu_1,\nu_2)}\left(\begin{pmatrix} n_1 n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\tau\right).$$

we have

$$|\phi(\tau^\circ)| \;\geq\; \left|\phi\left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} n_1 n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\tau\right)\right|$$

$$\geq\; \left|\phi_{n_1,n_2}\left(\begin{pmatrix} n_1 n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\tau\right)\right|.$$

and so

$$|\phi(\tau^\circ)| \geq \left|a_{n_1,n_2}\,|n_1 n_2|^{-1}\, W_{\nu_1,\nu_2}\left(\begin{pmatrix} n_1 n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} n_1 n_2 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\tau\right)\right|.$$

So we have

$$|a_{n_1,n_2} n_1^{-1} n_2^{-1}| \leq \frac{|\phi(\tau^\circ)|}{|W_{\nu_1\nu_2}(y_1, y_2)|}$$

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for any fixed non-zero y_1, and y_2. Here we want to show $W_{\nu_1\nu_2}(y_1, y_2)$ does not decay as fast as

$$e^{-\frac{1}{4}k|\nu|}$$

in the exponential order when $|\nu_1|$, and(or) $|\nu_2| \to \infty$. The reason is because in the following lemmas 1, 2, and 3, we will show that

$$\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right| \le e^{-\frac{1}{4}k|\nu}$$

when $\max(n_1, n_2^{\frac{1}{3}}) \ge k|\nu|$, and $\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right|$ is the numerator of the Fourier expansion of a cusp form. But we can choose y_1, and y_2 to be any number we want. So we may choose them as an after fact. That is we choose these y_1, and y_2 after we choose ν_1, and ν_2 whose absolute values are large enough. Then we choose y_1, and y_2 what ever the numbers that make $|W_{\nu_1\nu_2}(y_1, y_2)|$ maximum or at least the numbers that $|W_{\nu_1\nu_2}(y_1, y_2)|$ is large enough for our purpose. A pair of numbers which serves our purpose is $y_1 = y_2 = 1$. We will show

$$|W_{\nu_1\nu_2}(1, 1)| \ge \lambda^{-3}$$

at the end of the section. Assuming this is true, and by replacing $n_1^{-1}n_2^{-1}a_{n_1,n_2}$ in (4.3.1) by $\frac{|\phi(\tau^\circ)|}{|W_{\nu_1\nu_2}(1,1)|}$, we have

$$|\phi(\tau^\circ)| \le \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{|\phi(\tau^\circ)|}{|W_{\nu_1\nu_2}(1,1)|}\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right|. \quad (4.3.2)$$

Assume $a_{n_1,n_2} = 0$ for all n_1, and n_2 such that $\max(n_1, n_2^{\frac{1}{3}}) \le k|\nu|$, where $\nu = (3\nu_1 + 3\nu_2 - 2)/2$ and k is a fixed but large enough constant. Here the number

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of coefficients a_{n_1,n_2} which equal zero is $k^4 |\nu| |\nu|^3 = O(\lambda^2)$. We will show

$$\sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ,\frac{n_2y_2^\circ}{\Delta^2})\right|$$

in (4.3.2) is less than some positive power of $e^{-|\nu|}$. Then since

$$|W_{\nu_1\nu_2}(1,1)| \geq \lambda^{-3},$$

$$\sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{|W_{\nu_1\nu_2}(1,1)|}\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ,\frac{n_2y_2^\circ}{\Delta^2})\right|$$

is less than some positive power of $e^{-|\nu|}$, which is a contradiction unless

$$\phi(\tau^\circ) = 0.$$

Then this proves the dimension of a cusp form ϕ is $O(\lambda^2)$, and the theorem 4.1.

For non zero coefficients, we have the following 3 cases. $n_1 \geq n_2^{\frac{1}{3}}$, or $n_2^{\frac{1}{3}} \geq n_1$ and $\frac{n_2}{\Delta^2} \leq k|\nu|$, or $n_2^{\frac{1}{3}} \geq n_1$ and $\frac{n_2}{\Delta^2} > k|\nu|$. In the second case, we may assume $n_2^{\frac{1}{3}} > k|\nu|$, and $n_2 \leq k|\nu|\Delta^2$ so $\Delta \geq k|\nu|$. In the third case $\Delta < \left(\frac{n_2}{k|\nu|}\right)^{\frac{1}{2}}$. So we have

$$\sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ,\frac{n_2y_2^\circ}{\Delta^2})\right|$$

$$\leq \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1\geq n_2^{\frac{1}{3}}}^{\infty}\sum_{n_2=1}^{n_1^3}\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ,\frac{n_2y_2^\circ}{\Delta^2})\right|$$
$$+\sum\sum_{\Delta\geq k|\nu|}^{\infty}\sum_{n_1=1}^{n_2^{\frac{1}{3}}}\sum_{n_2=(k|\nu|)^3}^{\infty}\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ,\frac{n_2y_2^\circ}{\Delta^2})\right|$$
$$+\sum\sum_{\Delta<(\frac{n_2}{k|\nu|})^{\frac{1}{2}}}\sum_{n_1=1}^{n_2^{\frac{1}{3}}}\sum_{n_2=(k|\nu|)^3}^{\infty}\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ,\frac{n_2y_2^\circ}{\Delta^2})\right|.$$

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Here and the inequality that follows lemma 4.3, we are using the fact that g is a matrix. $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with determinant 1, and a, b, c, and d are integers.

We will show

Lemma 4.1. *If* $\max\left(n_1, n_2^{\frac{1}{3}}\right) \geq k\,|\nu|$, *and* $n_1 \geq n_2^{\frac{1}{3}}$, *then*

$$\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right| \leq e^{-\frac{1}{4}n_1\Delta y_1^\circ} \leq e^{-\frac{1}{4}k|\nu|}.$$

Lemma 4.2. *If* $\max\left(n_1, n_2^{\frac{1}{3}}\right) \geq k\,|\nu|$, $n_2^{\frac{1}{3}} \geq n_1$, *and* $\frac{n_2}{\Delta^2} \leq k\,|\nu|$,

then

$$\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right| \leq e^{-\frac{1}{4}n_1\Delta y_1^\circ} \leq e^{-\frac{1}{4}k|\nu|}.$$

Lemma 4.3. *If* $\max\left(n_1, n_2^{\frac{1}{3}}\right) \geq k\,|\nu|$, $n_2^{\frac{1}{3}} \geq n_1$, *and* $\frac{n_2}{\Delta^2} \geq k\,|\nu|$,

then

$$\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right| \leq e^{-\frac{1}{2}\frac{n_2 y_2^\circ}{\Delta^2}} \leq e^{-\frac{1}{4}k|\nu|}.$$

Then we have

$$\sum_{g\in\Gamma_\infty^2\backslash\Gamma^2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right|$$

$$\leq \sum_{d=0}^{0}\sum_{c=1}^{\infty}\sum_{n_1\geq n_2^{\frac{1}{3}}}^{\infty}\sum_{n_2=1}^{n_1^3}\left|e^{-\frac{1}{4}n_1\Delta y_1^\circ}\right| + \sum_{d=1}^{\infty}\sum_{c=0}^{\infty}\sum_{n_1\geq n_2^{\frac{1}{3}}}^{\infty}\sum_{n_2=1}^{n_1^3}\left|e^{-\frac{1}{4}n_1\Delta y_1^\circ}\right|$$

$$+\sum_{\Delta\geq k|\nu|}\sum_{n_1=1}^{\infty}\sum_{n_2=(k|\nu|)^3}^{n_2^{\frac{1}{3}}}\sum^{\infty}\left|e^{-\frac{1}{4}n_1\Delta y_1^\circ}\right| + \sum_{\Delta<\left(\frac{n_2}{k|\nu|}\right)^{\frac{1}{2}}}\sum\sum_{n_1=1}^{n_2^{\frac{1}{3}}}\sum_{n_2=(k|\nu|)^3}^{\infty}\left|e^{-\frac{1}{2}\frac{n_2 y_2^\circ}{\Delta^2}}\right|$$

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Here the first two summations are actually to be multiplied by 4 because c and d could be negative but for our purpose, a big O estimation a factor of any finite constant is irrelevant. These are quadruple geometric series. By changing the order of summations if necessary, we get the above is less than $\left(e^{-k|\nu|}\right)^{\varepsilon}$. so $|\phi(\tau^{\circ})| \leq |\phi(\tau^{\circ})| \left(e^{-k|\nu|}\right)^{\varepsilon}$ where the actual size of ε being immaterial as long as it is a positive constant. So we only need to prove above 3 lemmas and the claim we made earlier. That is, $|W_{\nu_1\nu_2}(1,1)| \geq \lambda^{-3}$.

Proof of lemma 4.1.

In these cases, we have $n_1 \geq n_2^{\frac{1}{3}}$.

$$W_{\nu_1,\nu_2}(n_1\Delta y_1^{\circ}, \tfrac{n_2 y_2^{\circ}}{\Delta^2})$$

$$= \pi^{-2}(n_1\Delta y_1^{\circ})^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2 y_2^{\circ}}{\Delta^2}\right)^{1+(\nu_2-\nu_1)/2}$$
$$\times \int_0^{\infty} K_{\nu}\left(n_1\Delta y_1^{\circ}\sqrt{1+x}\right) K_{\nu}\left(\frac{n_2 y_2^{\circ}}{\Delta^2}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}dx$$
$$= \pi^{-2}(n_1\Delta y_1^{\circ})^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2 y_2^{\circ}}{\Delta^2}\right)^{1+(\nu_2-\nu_1)/2}$$
$$\times \Big(\int_0^{x^{\circ}} K_{\nu}\left(n_1\Delta y_1^{\circ}\sqrt{1+x}\right) K_{\nu}\left(\frac{n_2 y_2^{\circ}}{\Delta^2}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}dx$$
$$+ \int_{x^{\circ}}^{\infty} K_{\nu}\left(n_1\Delta y_1^{\circ}\sqrt{1+x}\right) K_{\nu}\left(\frac{n_2 y_2^{\circ}}{\Delta^{\circ}}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4}dx\Big)$$

where

$$x^{\circ} = \frac{n_2^2 y_2^{\circ 2}}{\Delta^4 - n_2^2 y_2^{\circ 2}}.$$

This is where

$$\frac{n_2 y_2^{\circ}}{\Delta^2}\sqrt{\frac{x+1}{x}} = 1.$$

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The polynomial part of the outside of the parenthesis is immaterial in terms of magnitude of $W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})$.

When $0 \le x \le x^\circ$,

$$\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}} \ge 1.$$

and when $x > x^\circ$,

$$\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}} < 1.$$

So

$$\left|\int_0^{x^\circ} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right) K_\nu\left(\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4} dx\right|$$

$$< \int_0^{x^\circ} \left|K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)\right| \left|K_\nu(1)\right| \left|x^{(3\nu_1-3\nu_2-4)/4}\right| dx$$

because of the following reason which also explains that $K_\nu(1) = O(e^{\frac{\pi}{2}|\nu|})$.

We know $-\frac{1}{2} < Re(\nu) < \frac{1}{2}$. By [L] (5.10.24), for $Re(z) > 0$ and $Re(\nu) > -\frac{1}{2}$,

$$\begin{aligned} K_\nu(z) &= \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zt}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt \\ &= \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{1}{2}\right)^\nu z\nu \int_1^\infty e^{-(z-1)t}e^{-t}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt \end{aligned}$$

By inspecting this we know $|K_\nu(z)| \le |K_\nu(1)|$ when $z > 1$. Furthermore

$$\begin{aligned} &|K_\nu(1)| \\ &= \left|\frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{1}{2}\right)^\nu \int_1^\infty e^{-t}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt\right| \\ &\le \left|\frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{1}{2}\right)^\nu\right| \\ &\quad \times \left(\left|\int_1^2 e^{-t}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt\right| + \left|\int_2^\infty e^{-t}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt\right|\right). \end{aligned}$$

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Since $Re(\nu - \frac{1}{2}) < 0$, we have

$$\begin{aligned}
&|K_\nu(1)| \\
&\le \left|\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{1}{2}\right)^\nu\right|\left(\left|\int_1^2 e^{-t}(t-1)^{\nu-\frac{1}{2}}\,dt\right| + \left|\int_2^\infty e^{-t}dt\right|\right) \\
&= \left|\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{1}{2}\right)^\nu\right| e^{-1}\left(\left|\int_0^1 e^{-t}t^{\nu-\frac{1}{2}}dt\right| + e^{-1}\right) \\
&\le \left|\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{1}{2}\right)^\nu\right| e^{-1}\left(\left|\frac{e^{-t}t^{\nu+\frac{1}{2}}}{\nu+\frac{1}{2}}\right|_0^1 + \left|\int_0^1 \frac{e^{-t}t^{\nu+\frac{1}{2}}}{\nu+\frac{1}{2}}dt\right| + e^{-1}\right).
\end{aligned}$$

Since $Re(\nu + \frac{1}{2}) > 0$, and so $\left|t^{\nu+\frac{1}{2}}\right| \le 1$, and

$$\begin{aligned}
&|K_\nu(1)| \\
&\le \left|\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{1}{2}\right)^\nu\right| e^{-1}\left(\left|\frac{e^{-1}}{\nu+\frac{1}{2}}\right| + \left|\int_0^1 \frac{e^{-t}}{\nu+\frac{1}{2}}dt\right| + e^{-1}\right) \\
&= \left|\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{1}{2}\right)^\nu\right| e^{-1}\left(\frac{e^{-1}}{\nu+\frac{1}{2}} + e^{-1}\right).
\end{aligned}$$

But $\left|\Gamma\left(\nu+\frac{1}{2}\right)\right| \sim e^{-\frac{\pi}{2}|\nu+\frac{1}{2}|}$ by Sterling's formula. So

$$|K_\nu(1)| = O\left(e^{\frac{\pi}{2}|\nu+\frac{1}{2}|}\right).$$

By Olver [O.II], we have

$$\begin{aligned}
K_\nu(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}(1+\gamma_1), \\
\text{where } |\gamma_1| &\le e^{|(\nu^2-\frac{1}{4})z^{-1}|}\left(\frac{(\nu^2-\frac{1}{4})}{z}\right) \quad \text{for } |ph(z)| \le \frac{\pi}{2}.
\end{aligned}$$

So

$$|K_\nu(z)| \le \left|\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left(1 + e^{|(\nu-\frac{1}{2})(\nu+\frac{1}{2})z^{-1}|}\left(\frac{(\nu-\frac{1}{2})(\nu+\frac{1}{2})}{z}\right)\right)\right|.$$

If $z > 4\pi\left|\nu+\frac{1}{2}\right|$, then

$$|K_\nu(z)| \le \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left|\left(1 + e^{|(\nu-\frac{1}{2})|}\left(\nu - \frac{1}{2}\right)\right)\right|$$

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as $|\nu|$ is large enough.

Then by letting $n_1 > 4\pi\left|\nu + \frac{1}{2}\right|$ (here 4π is not really anything fundamental, it could be any large enough constant for our purpose), and overestimating the first $k-$Bessel function, we obtain

$$\left|\int_0^{x^\circ} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx\right|$$

$$\leq \int_0^{x^\circ} e^{-\frac{3}{4}n_1\Delta y_1^\circ\sqrt{1+x}}\left|K_\nu(1)\right|x^{(3\nu_1-3\nu_2-4)/4}dx.$$

We have already shown $\left|K_\nu(1)\right| = O\left(e^{\frac{\pi}{2}\left|\nu+\frac{1}{2}\right|}\right)$ which can be absorbed by $e^{-\frac{3}{4}n_1\Delta y_1^\circ\sqrt{1+x}}$.

So

$$\left|\int_0^{x^\circ} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx\right| \leq e^{-\frac{3}{4}n_1\Delta y_1^\circ\sqrt{1+x}}.$$

For

$$\int_{x^\circ}^{\infty} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx.$$

we use Lebedev [L] (5.7.1)

$$I_\nu(z) = \sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{\nu+2n}}{\Gamma(n+1)\Gamma(n+\nu+1)}.$$

and [L] (5.7.2)

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu z}$$

where $|z| < \infty$, $\nu \neq 0, \pm1, \pm2, \cdots$, and $\arg(z) < \pi$. Notice that when x is within the limits of the integral,

$$\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}} < \infty.$$

So we can use these equalities for the second K-Bessel function.

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Then

$$\left|\int_{x^\circ}^{\infty} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx\right|$$

$$\leq \int_{x^\circ}^{\infty} e^{-\frac{3}{4}n_1\Delta y_1^\circ\sqrt{1+x}}\left|\frac{\pi}{2\sin\nu\pi}\right|$$

$$\times\left\{\sum_{n=0}^{\infty}\left|\frac{\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{-\nu+2n}}{\Gamma(n+1)\Gamma(n-\nu+1)}\right|-\left|\frac{\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{\nu+2n}}{\Gamma(n+1)\Gamma(n+\nu+1)}\right|\right\}x^{(3\nu_1-3\nu_2-4)/4}dx.$$

$|\sin\nu\pi|$ is not small. In fact, $\sin\nu\pi = \frac{e^{-i\nu\pi}-e^{i\nu\pi}}{2i}$ and the imaginary part of ν has a large absolute value, so one of the exponential expressions is large. All of Γ functions on denominators are not too small for our case. That is

$$\Gamma(n\pm\nu+1)\geq(n\pm\nu)!e^{-\pi|\nu|}$$

which will be proven shortly. $\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^2 \leq \frac{1}{4}$ when x is within the limits of integration. Then

$$\left|\int_{x^\circ}^{\infty} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx\right|$$

$$\leq \int_{x^\circ}^{\infty} e^{-\frac{3}{4}n_1\Delta y_1^\circ\sqrt{1+x}}|\{|\Gamma(-\nu+1)|^{-1}\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{-\nu}\sum_{n=0}^{\infty}\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{2n}$$

$$-|\Gamma(\nu+1)|^{-1}\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{\nu}\sum_{n=0}^{\infty}\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{2n}\}|x^{(3\nu_1-3\nu_2-4)/4}dx$$

$$\leq \int_{x^\circ}^{\infty} e^{-\frac{3}{4}n_1\Delta y_1^\circ\sqrt{1+x}}e^{\pi|\nu+1|}|\{\left(\frac{2\Delta^2}{n_2 y_2^\circ}\sqrt{\frac{x}{x+1}}\right)^{\nu}\sum_{n=0}^{\infty}\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{2n}$$

$$-\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{\nu}\sum_{n=0}^{\infty}\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^{2n}x^{(3\nu_1-3\nu_2-4)/4}dx$$

$$\leq \int_{x^\circ}^{\infty} e^{-\frac{3}{4}n_1\Delta y_1^\circ\sqrt{1+x}}e^{\pi|\nu+1|}\frac{1}{1-\left(\frac{n_2 y_2^\circ}{2\Delta^2}\sqrt{\frac{x+1}{x}}\right)^2}$$

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$$\times \left| \left\{ \left(\frac{2\Delta^2}{n_2 y_2^\circ} \sqrt{\frac{x}{x+1}} \right)^\nu - \left(\frac{n_2 y_2^\circ}{2\Delta^2} \sqrt{\frac{x+1}{x}} \right)^\nu \right\} \right| x^{(3\nu_1 - 3\nu_2 - 4)/4} dx$$
$$\leq e^{-\frac{1}{2} n_1 \Delta y_1^\circ}.$$

So we can conclude

$$\left| W_{\nu_1,\nu_2}(n_1 \Delta y_1^\circ, \frac{n_2}{\Delta^2}) \right| \leq e^{-\frac{\pi}{2} n_1 \Delta y_1^\circ}$$

when $n_1 \geq n_2^{\frac{1}{3}}$ if we show the Γ functions in the denominators are not too small. Since $\Gamma(n \pm \nu + 1) \geq (n \pm \nu)!\ \Gamma(\pm\nu)$ we only need to show $\Gamma(\pm\nu)$ are not too small. Stirling's Formula [L] (1.4.23) is

$$\Gamma(z) = e^{\left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln 2\pi} \left[1 + O\left(|z|^{-1}\right)\right].$$

Let $\nu = x + iy$. Then we know $-\frac{1}{2} < x < \frac{1}{2}$. and $y \to \pm\infty$. So $1 > x + \frac{1}{2} > 0$. If $y > 0$.

$$\begin{aligned}
\Gamma(-\nu) &= \Gamma(-x - iy) \\
&= e^{\left(-x - iy - \frac{1}{2}\right) \ln(-x - iy) - x - iy - \frac{1}{2} \ln 2\pi} \left[1 + O\left(|-x - iy|^{-1}\right)\right] \\
&= e^{\left(-\left(x + \frac{1}{2}\right) - iy\right)\left(\ln(|-x - iy|) + i\left(-\frac{\pi}{2} + 2J\pi\right)\right) + x + iy + \frac{1}{2} \ln 2\pi} \left[1 + O\left(|-x - iy|^{-1}\right)\right] \\
&= e^{-\left(x + \frac{1}{2}\right) \ln(|-x - iy|) - iy \ln(|-x - iy|) - i\left(x + \frac{1}{2}\right)\left(-\frac{\pi}{2} + 2J\pi\right) + y\left(-\frac{\pi}{2} + 2J\pi\right) + x + iy + \frac{1}{2} \ln 2\pi} \\
&\quad \times \left[1 + O\left(|-x - iy|^{-1}\right)\right]
\end{aligned}$$

So

$$|\Gamma(-\nu)| \geq e^{-\ln(x+y) - y\frac{\pi}{2}} \geq e^{-\pi|\nu|}.$$

If $y < 0$.

$$\Gamma(-\nu) = \Gamma(-x - iy)$$

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$$= e^{-\left(x+\frac{1}{2}\right)\ln(|-x-iy|)-iy\ln(|-x-iy|)-i\left(x+\frac{1}{2}\right)\left(-\frac{\pi}{2}+2J\pi\right)+y\left(\frac{\pi}{2}+2J\pi\right)+x+iy+\frac{1}{2}\ln 2\pi}$$

$$\times \left[1+O\left(|-x-iy|^{-1}\right)\right].$$

So

$$|\Gamma(-\nu)| \geq e^{-\ln(x+y)+y\frac{\pi}{2}} \geq e^{-\pi|\nu|}.$$

Like wise when $y \to \pm\infty$.

$$\Gamma(\nu) = e^{\left(x-\frac{1}{2}+iy\right)\left(\ln(|x+iy|)+i\left(\pm\frac{\pi}{2}+2J\pi\right)\right)-x-iy+\frac{1}{2}\ln 2\pi}\left[1+O\left(|-x-iy|^{-1}\right)\right]$$

$$= e^{\left(x-\frac{1}{2}\right)\ln(|x+iy|)+iy\ln(|x+iy|)+i\left(x-\frac{1}{2}\right)\left(\pm\frac{\pi}{2}+2J\pi\right)\mp y\left(\frac{\pi}{2}+2J\pi\right)-x-iy+\frac{1}{2}\ln 2\pi}$$

$$\times \left[1+O\left(|-x-iy|^{-1}\right)\right]$$

Since $-1 < x - \frac{1}{2} < 0$,

$$|\Gamma(\nu)| \geq e^{-\ln(x+y)\mp y\frac{\pi}{2}} \geq e^{-\pi|\nu|} \qquad \square$$

Proof of lemma 4.2.

$$\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})\right| \leq e^{-\frac{1}{4}n_1\Delta y_1^\circ} \leq e^{-\frac{1}{4}k|\nu|} \text{ when } n_2^{\frac{1}{3}} \geq n_1 \text{ and } \frac{n_2}{\Delta^2} \leq k|\nu|.$$

Here we have $\Delta \geq k|\nu|$ because $n_2 \geq (k|\nu|)^3$, and $\Delta^2 \geq \frac{n_2}{k|\nu|}$. Then

$$W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2})$$

$$= \pi^{-2}(n_1\Delta y_1^\circ)^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{1+(\nu_2-\nu_1)/2}$$

$$\int_0^\infty K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx$$

$$= \pi^{-2}(n_1\Delta y_1^\circ)^{1+(\nu_1-\nu_2)/2}\left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{1+(\nu_2-\nu_1)/2}$$

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$$\times(\int_0^{x^\circ} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx$$

$$+\int_{x^\circ}^{\infty} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\Delta^\circ}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx)$$

where $x^\circ = \frac{n_2^2 y_2^{\circ 2}}{\Delta^4 - n_2^2 y_2^2}$ as in the Lemma 1. So when $0 \le x \le x^\circ$, $\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}} \ge 1$, and when $x > x^\circ$, $\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}} < 1$.

Since $\Delta \ge k|\nu|$, we have $n_1\Delta y_1^\circ\sqrt{1+x} \ge k|\nu|$. By the same argument as the proof of the Lemma 1, we have

$$\left|\int_0^{x^\circ} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx\right| \le e^{-\frac{3}{4}n_1\Delta y_1^\circ\sqrt{1+x}}.$$

and

$$\left|\int_{x^\circ}^{\infty} K_\nu\left(n_1\Delta y_1^\circ\sqrt{1+x}\right)K_\nu\left(\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)x^{(3\nu_1-3\nu_2-4)/4}dx\right| \le e^{-\frac{1}{2}n_1\Delta y_1^\circ}.$$

And again we conclude

$$\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2}{\Delta^2})\right| \le e^{-\frac{1}{2}n_1\Delta y_1^\circ} \text{ when } n_2^{\frac{1}{3}} \ge n_1 \text{and } \frac{n_2}{\Delta^2} \le k|\nu|. \quad \square$$

Proof of the Lemma 4.3.

$$\left|W_{\nu_1,\nu_2}(n_1\Delta y_1^\circ, \frac{n_2y_2^\circ}{\Delta^2})\right| \le e^{-\frac{1}{2}\frac{n_2y_2^\circ}{\Delta^2}} \le e^{-\frac{1}{4}k|\nu|} \text{ when } n_2^{\frac{1}{3}} \ge n_1 \text{ and } \frac{n_2}{\Delta^2} \ge k|\nu|.$$

Here we overestimate the second K Bessel function by the similar way as in the proof of Lemma 1.That is

$$K_\nu\left(\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right) \le e^{-\frac{1}{2}\frac{n_2y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}}.$$

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This is justified because of the condition $\frac{n_2}{\Delta^2} \geq k\,|\nu|$ which is the same condition as for the first K Bessel function in the proof of the Lemma 1.

The first K Bessel function here does not give any trouble because $n_1 \Delta y_1^\circ \sqrt{1+x} \geq 1$. And better yet, when $x \geq (k\,|\nu|)^2$, we have

$$\left|K_\nu\left(n_1 \Delta y_1^\circ \sqrt{1+x}\right)\right| \leq e^{-\frac{3}{4} n_1 \Delta y_1^\circ \sqrt{1+x}}.$$

So

$$\begin{aligned}
&\left|W_{\nu_1,\nu_2}\left(n_1 \Delta y_1^\circ, \frac{n_2 y_2^\circ}{\Delta^2}\right)\right| \\
\leq\ & \pi^{-2} \left|(n_1 \Delta y_1^\circ)^{1+(\nu_1-\nu_2)/2} \left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{1+(\nu_2-\nu_1)/2}\right| \\
&\times \int_0^\infty \left|K_\nu\left(n_1 \Delta y_1^\circ \sqrt{1+x}\right)\right| \left|K_\nu\left(\frac{n_2 y_2^\circ}{\Delta^2}\sqrt{\frac{x+1}{x}}\right)\right| \left|x^{(3\nu_1-3\nu_2-4)/4}\right| dx \\
\leq\ & \pi^{-2} \left|(n_1 \Delta y_1^\circ)^{1+(\nu_1-\nu_2)/2} \left(\frac{n_2 y_2^\circ}{\Delta^2}\right)^{1+(\nu_2-\nu_1)/2}\right| \\
&\times \int_0^\infty \left|K_\nu\left(n_1 \Delta y_1^\circ \sqrt{1+x}\right)\right| e^{-\frac{3\pi}{4} n_1 \Delta y_1^\circ \sqrt{1+x}} \left|x^{(3\nu_1-3\nu_2-4)/4}\right| dx \\
\leq\ & e^{-\frac{1}{2}\frac{n_2 y_2^\circ}{\Delta^2}}. \qquad \square
\end{aligned}$$

Finally we have to prove the claim we have made in the beginning of the proof of the theorem. That is

$$|W_{\nu_1,\nu_2}(1,1)| \geq \frac{1}{\lambda^3}.$$

We have

$$\begin{aligned}
&W_{\nu_1,\nu_2}(y_1, y_2) \\
=\ & \pi^{-2} y_1^{1+(\nu_1-\nu_2)/2} y_2^{1+(\nu_2-\nu_1)/2} \int_0^\infty K_\nu\left(y_1\sqrt{1+x}\right) K_\nu\left(y_2\sqrt{\frac{1+x}{x}}\right) x^{(3\nu_1-3\nu_2-4)/4} dx.
\end{aligned}$$

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We also have

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zt}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt$$

for $Re\,(z) > 0$ and $Re\,(\nu) > -\frac{1}{2}$. [L] (5.10.24). So we have

$$\begin{aligned}
& W_{\nu_1,\nu_2}(y_1, y_2) \\
= \;& \pi^{-2} y_1^{1+(\nu_1-\nu_2)/2} y_2^{1+(\nu_2-\nu_1)/2} \\
& \times \int_0^\infty \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{y_1\sqrt{1+x}}{2}\right)^\nu \int_1^\infty e^{-ty_1\sqrt{1+x}}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt \\
& \times \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{y_2\sqrt{x+1}}{2\sqrt{x}}\right)^\nu \int_1^\infty e^{-ty_2\sqrt{\frac{x+1}{x}}}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt \\
& x^{(3\nu_1-3\nu_2-4)/4}dx. \\
= \;& \frac{y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2}}{\pi 4^\nu\left(\Gamma\left(\nu+\frac{1}{2}\right)\right)^2}\int_0^\infty\int_1^\infty e^{-ty_1\sqrt{1+x}}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt \\
& \times \int_1^\infty e^{-ty_2\sqrt{\frac{x+1}{x}}}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt\frac{(1+x)^{(3\nu_1-3\nu_2-2)/2}}{x^{(3\nu_2-1)/2}}dx
\end{aligned}$$

So

$$\begin{aligned}
W_{\nu_1,\nu_2}(\lambda, \lambda) \;=\;& \frac{\lambda^{3\nu_1-3\nu_2}}{\pi 4^\nu\left(\Gamma\left(\nu+\frac{1}{2}\right)\right)^2}\int_0^\infty\int_1^\infty e^{-t\lambda\sqrt{1+x}}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt \\
& \int_1^\infty e^{-t\lambda\sqrt{\frac{x+1}{x}}}(t-1)^{\nu-\frac{1}{2}}(t+1)^{\nu-\frac{1}{2}}\,dt\frac{(1+x)^{(3\nu_1+3\nu_2-2)/2}}{x^{(3\nu_2-1)/2}}dx.
\end{aligned}$$

But by [B-H] theorem 1.Since there is no complex value involved in $e^{t(\lambda-1)\sqrt{1+x}}$ and $e^{t(\lambda-1)\sqrt{\frac{x+1}{x}}}$. and

$$t(\lambda-1)\left(\sqrt{1+x}+\sqrt{\frac{x+1}{x}}\right) \geq 2\sqrt{2}(\lambda-1).$$

we have

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$$
\begin{aligned}
&|W_{\nu_1,\nu_2}(1,1)| \\
&\quad \geq \left| \frac{e^{2\sqrt{2}(\lambda-1)}}{\lambda^{3\nu_1+3\nu_2}} \right| \\
&\quad\quad \times \left| \begin{array}{l} \frac{\lambda^{3\nu_1+3\nu_2}}{\pi 4^{\nu}\left(\Gamma\left(\nu+\frac{1}{2}\right)\right)^2} \int\limits_0^{\infty} \int\limits_1^{\infty} e^{-t\lambda\sqrt{1+x}} (t-1)^{\nu-\frac{1}{2}} (t+1)^{\nu-\frac{1}{2}}\, dt \\ \times \int\limits_1^{\infty} e^{-t\lambda\sqrt{\frac{x+1}{x}}} (t-1)^{\nu-\frac{1}{2}} (t+1)^{\nu-\frac{1}{2}}\, dt \frac{(1+x)^{(3\nu_1+3\nu_2-2)/2}}{x^{(3\nu_2-1)/2}} dx \end{array} \right| \\
&\quad \geq \frac{e^{2\sqrt{2}(\lambda-1)}}{\lambda^3} |W_{\nu_1,\nu_2}(\lambda,\lambda)| \\
&\quad = \frac{e^{2\sqrt{2}(\lambda-1)}}{\lambda^3} k \sqrt{\frac{2}{3\pi}} \frac{\lambda^{1/2}}{2^{1/4}} e^{-2\sqrt{2}\lambda} \\
&\quad \geq \frac{1}{\lambda^3}. \qquad \square
\end{aligned}
$$

This conclude the proof of the theorem 4.1. □

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CHAPTER 5

Further Study

The unconditional results given previously are probably possible for improvements. It is quite possible that the multiplicity of the spectrum is always equal to either zero or one. We do not know a way to show this. however we will show. under a generalization of the Riemann hypothesis. a strong multiplicity theorem.

Attached to a cusp form ϕ we have an $L-$series

$$L(s,\phi)=\sum_{n=1}^{\infty}a_{1,n}n^{-s}$$

which satisfies

$$\Phi(s)L(s,\phi)=\tilde{\Phi}(1-s)L\left(1-s,\tilde{\phi}\right)$$

with

$$L\left(s,\tilde{\phi}\right)=\sum_{n=1}^{\infty}a_{n,1}n^{-s}.$$

$$\Phi(s)=\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1-2\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{s+\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{s-1+\nu_1+2\nu_2}{2}\right)$$

and

$$\tilde{\Phi}(s)=\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1-\nu_1-2\nu_2}{2}\right)\Gamma\left(\frac{s-\nu_1+\nu_2}{2}\right)\Gamma\left(\frac{s-1+2\nu_1+\nu_2}{2}\right).$$

It can be shown that

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}a_{n_1,n_2}n_1^{-s_1}n_2^{-s_2}=\frac{L\left(s,\tilde{\phi}\right)L(s,\phi)}{\zeta(s_1+s_2)}. \qquad \text{Bump (9.2)}$$

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From these facts we conclude that the $a_{n_1,1}$ determine the cusp form. We can also factor

$$L(s,\phi) = \prod_{p=1}^{\infty}\left(1 - a_{1,p}p^{-s} + a_{p,1}p^{-2s} - p^{-3s}\right)^{-1}$$
$$= \prod_{p=1}^{\infty}\left((1-\alpha_1(p))(1-\alpha_2(p))(1-\alpha_3(p))p^{-s}\right)^{-1}.$$

Given two cusp forms ϕ and ψ, with same ν_i with $\alpha_i(p)$ for ϕ and $\beta_i(p)$ for ψ, and Fourier coefficients a_{n_1,n_2} for ϕ, and b_{n_1,n_2} for ψ we can define the Rankin Selberg convolution

$$L(s,\phi) = \prod_p \prod_{i,j}\left(1 - \alpha_i(p)\beta_j(p)p^{-s}\right)^{-1}.$$

This has meromorphic continuation and satisfies the functional equation

$$\Phi(s,\phi\times\psi)\,L(s,\phi\times\psi) = \tilde{\Phi}\left(1-s,\tilde{\phi}\times\tilde{\psi}\right)L\left(1-s,\tilde{\phi}\times\tilde{\psi}\right)$$

with $\tilde{\phi}$, and $\tilde{\psi}$ the cusp forms related to ϕ and ψ by the functional equation earlier

$$\Phi(s,\phi\times\psi) = \pi^{-\frac{3s}{2}}\prod_i\prod_j \Gamma\left(\frac{s+-4\nu_1-2\nu_2}{2}\right)\Gamma\left(\frac{s+2\nu_1-2\nu_2}{2}\right)\Gamma\left(\frac{s-1+2\nu_1+4\nu_2}{2}\right)$$

and as ν_i are the same for ϕ and ψ

$$\Phi\left(s,\tilde{\phi}\times\tilde{\psi}\right) = \Phi(s,\phi\times\psi).$$

Note:The Rankin-Selberg method works when ϕ and ψ do not have the same ν's, but this is the case we need and the formulas are slightly simpler, except for the case $\psi = \tilde{\phi}$. We actually have analytic continuation. When $\psi = \tilde{\phi}$, we have a simple pole at $s = 1$.

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We now assume that all non-trivial zeros are on the line $\mathrm{Re}(s) = \frac{1}{2}$. We let ϕ and ψ be two cusp forms with same ν_1 and ν_2. Denote $L\left(s, \phi \times \bar{\phi}\right) = L_1(s)$ and $L(s, \phi \times \psi) = L_2(s)$. and write $L_1(s) = \sum \frac{c_n}{n^s}$ and $L_2(s) = \sum \frac{d_n}{n^s}$ with the c_i for $i < x$ are determined by $a_{i,j}$. i and $j < x$. and similarly with the d_i. We consider the integrals

$$I_j(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{L_j'(s)}{L_j(s)} \frac{x^s}{s} ds. \qquad j = 1. \text{ or } 2$$

As in the proof of the explicit formula for the Riemann zeta function. this is

$$\sum_{n \le x} c_n \Lambda(n) n^{-s} \text{ for } L_1(s)$$
$$\sum_{n \le x} d_n \Lambda(n) n^{-s} \text{ for } L_2(s).$$

with

$$\Lambda(n) = \left\{ \begin{array}{ll} \log p \text{ if } n = p^a \text{ where } p \text{ is a prime} & \\ 0 & \text{otherwise.} \end{array} \right\}$$

Letting m be the maximum of the absolute values of the sum of ν's in the various Γ terms. and letting $\sigma = s + it$ we get that the sum of the nontrivial zeros satisfy

$$\sum_{p} \frac{1}{1 + (t - \mathrm{Im}(p))^2} = O(\log(|t| + |m| + 2)).$$

Thus the contribution of the nontrivial zeros is

$$O\left(x^{\frac{1}{2}} (\log(x + m))^2\right).$$

This tells us that for $L_1(s)$ as we have a pole at $s = 1$

$$I_1(x) = x + O\left(x^{\frac{1}{2}} (\log(x + m))^2\right)$$

but for the entire $L_2(s)$ we get

$$I_2(x) = O\left(x^{\frac{1}{2}} (\log(x + m))^2\right).$$

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However if $c_n = d_n$ for $n \leq x$, $I_1(s) = I_2(s)$ and c_n and d_n are determined by $a_{1,n}$ and $b_{1,n}$ for $n < x$. In this case we have

$$
\begin{aligned}
x &= O\left(x^{\frac{1}{2}} (\log (x+m))^2\right) \\
x^{\frac{1}{2}} &= O\left((\log (x+m))^2\right)
\end{aligned}
$$

which is a contradiction if all terms are equal for $n < (\log(x+m))^4$. As m is approximately $\sqrt{\lambda}$ we thus conclude that

$$M(\lambda) = O\left((\log(x+m))^4\right).$$

We actually could have simply assumed the ν_i not the same for each cusp form, but simply both were such that m for ϕ, and m for ψ satisfied some bound, calling it $\sqrt{\lambda}$.

We conclude by mentioning several ways for further investigation. We find the gap in the range of validity for the asymptotic expansion to be quite interesting. We suspect that some bizarre behavior may be occurring in this range. It might be interesting to attempt to investigate this region numerically, and see if, for example, we have oscillatory behavior. We might also try different techniques. It is feasible that a differential equations technique might work. We also should try compare our results to those found from representation theory. We also will make efforts to extend the results on Weyl's law to obtain a local Weyl's law, as in [H-T].

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REFERENCES

[B] D. Bump. *Automorphic Forms on* $GL(3.\mathbb{R})$.
Springer Lecture Note in Mathematics #1083. (1984)

[B-H] D. Bump and J. Huntley. *Unramified Whittaker Functions for* $GL(3.\mathbb{R})$. Journal D'Analyse Mathematique. Vol. 65 (1995). 19-44.

[C-H-J-T] S. Catto. J. Huntley. J. Jorgenson. and D. Tepper.
On an Analogue of Selberg's Eigenvalue Conjecture for $SL_3(\mathbb{Z})$.
Proc. Amer. Math. Soc. Vol. 126. #12. Dec (1998). 3455-3459.

[C-H-M-T] S. Catto. J. Huntley. N. Moh. and D. Tepper.
Weyl's Law with Error Estimate.
Contemporary Mathematics Vol. 251. (1999). 107-116.

[D] H. Davenport. *Multiplicative Number Theory.* 2^{nd} ed..
Springer-Verlag. New York. 1980.

[De] N. G.. de Bruijn. *Asymptotic Methods in Analysis.*
Dover. New York. 1981.

[Er] A. Erdélyi. *Asymptotic Expansions.* 1956. Dover. New York.

[G] U. Grenier. *Fundamental Domains for* $P_n/GL_n(\mathbb{Z})$. Ph.D. Thesis. UCSD. 1986.

[H1] J. Huntley. *Multiplicity One Theorem for Automorphic Forms.*
Ph.D. Thesis. Stanford. 1987.

[H2] J. Huntley. *Spectral Multiplicity on Product of Hyperbolic Spaces.*
Proc. Amer. Math. Soc. Vol. 111. #1. Jan (1991). 1-11.

[H3] J. Huntley. *Spectral Multiplicity on* $GL_n(\mathbb{R})$.
Trans. Amer. Math. Soc. Vol. 332. #2. Aug (1992). 875-888.

[H-J-L1] J. Huntley, J. Jorgenson, R. Lundelius, *On the Asymptotic Behavior of Couning Functions Associated to Degenerating Hyperbolic Riemann Surfaces.* Journal of Functional Analysis MSRI Preprint No. 021-95.

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[H-J-L2] J. Huntley. J. Jorgenson. R. Lundelius. *Continuity of Small Eigen Functions on Degenerating Riemann surfaces.* Bol. Soc. Mat. Mex (1995)

[H-T] J. Huntley and D. Tepper. *A Local Weyl's Law. The Angular Distribution and Mutiplicity of Cusp Forms on Product Spaces*, Trans. Amer. Math. Soc. Vol. 330. #1. Mar. (1992). 97-110.

[L] N. Lebedev *Special Functions and Their Applications.* Dover Books. New York. 1972.

[L-P] P. Lax. R. Phillips: *Scattering Theory for Automorphic Forms.* Annals of Math Studies (1978)

[M-P] S. Minakshisundaram. A. Pleijel. *Some Properties of the Eigenfunctions of the Laplace Operator on Riemannian Manifolds.* Can. J. Math.. Vol. 1 (1949) 242-256.

[O1] F. Olver. *Asymptotics and Special Functions*. Academic Press. New York.1974.

[O2] F. Olver. *Error Bounds for Asymptotic Expansions. with an Application to Cylinder Functions of Large Arguement.* In *Asymptotic Solutions of Differential Equations and Their Applications* (C. H. Wilcox ed.) (1964) 167-183. Wiley. New York.

[S] A. Selberg. *Harmonic Anlysis and Discontinuous Groups in Weakly Symmetric Riemann Space with Applications to Dirichlet Series.* J. Indian Math. Soc. Vol. 20 (1956). 47-87.

[St] E. Stade. *On Explicit Formulas for* $GL(n, \mathbb{R})$ – *Whittaker Functions.* Duke Math. J. Vol. 60 (1989) 313-362.

[S-W] E. Stade. D. Wallace: *Weyl's Law for* $SL(3, \mathbb{Z})$. Pacific Journal of Mathematics. Vol. 173 No1 (1996) 241-259.

[T] A. Terras. *Harmonic Analysison Symmetric Spaces and Applications*, Springer-Verlag, New York. 1985.

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[W] D. Wallace, *The Selberg trace formula for* $SL(3,\mathbb{Z})$. Trans. Amer. Math. Soc., 345 (1994) 1-36.

[We] H. Weyl, *Das Asymptoticsche Verteilungsgesetz der Eigenwerte Linearer Partieller Differentialgleichungen.* Math. Ann. 71 (1911) 441-479.

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