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A computational study of the factor groups of the lower central series of a certain free product

Vulis, Marina, Ph.D.

City University of New York, 1994

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300 N. Zeeb Rd.
Ann Arbor, MI 48106

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A Computational Study of the Factor Groups of the Lower Central Series of a Certain Free Product

by

Marina Vulis

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12/23/93 — Michael Anshel

Date — Chair of Examining Committee

12/28/93 — Józef Dodziuk

Date — Executive Officer

Professor Michael Anshel

Professor Józef Dodziuk

Professor Burton Randol

Supervisory Committee

The City University of New York

Abstract

A Computational Study of the Factor Groups

of the Lower Central Series of a Certain Free Product

by

Marina Vulis

Adviser: Professor Michael Anshel

Let G denote the free product of three cyclic groups of order 2. The purpose of this study is the computational investigation of the factor groups of the lower central series of G. The motivation for studying this particular example is twofold. First, we wish to gain better insight into the performance of algorithms that have emerged over the past two decades in computational group theory. Further we would like to contribute to better understanding of free products of cyclic groups and their lower central series by learning more about these specific examples.

We begin by introducing certain fundamental concepts from the commutator calculus and demonstrate how they can be used in computing presentations of the factor groups of the lower central series of G.

Next, we present the results of computational experiments with the finite nilpotent factors of the lower central series of G. We introduce the concept of supergirth. We

examine the Cayley graphs of these factor groups and generate conjectures based on our computations about both their supergirths and diameters. We identify the factor group G/G_3 as a known nilpotent group of order 64.

We continue the examination of the group G by using the classical representation of G as a subgroup of the modular group (the group $PSL(2, Z)$ of linear fractional transformations). This allows us to relate our study of G with the concepts of the trace function, the level function and the parabolic class number, all of which are integral to the study of the modular group Γ. We prove a formula for the level of the terms of the lower central series of G in Γ.

In the final chapter we formulate a hybrid algorithm for computing free bases for the terms of the lower central series of G using two well-known rewriting processes and specify a power commutator presentation for each of the factor groups G/G_n.

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To my mother, Alla Inzel, I owe more than I ever can put in words.

Finally, I must thank my son Daniel, who did not fuss about my studying.

Dedication

To my father, Lazar Inzel, whom I loved and admired so very much.

Contents

Chapter 1

The Lower Central Series of a Free Group $F(r)$ of Finite Rank

In this chapter we introduce certain fundamental concepts from the commutator calculus that will be used in computing presentations of the factor groups of the lower central series of G.

1.1 Basic Commutators

In this section, we introduce basic commutators and give examples of basic commutators as generators of the factor groups of the lower central series of a free group F on r generators:

$$F(r) = \langle x_1, x_2, \ldots, x_r \rangle.$$

Let us recall a few well-known definitions. The *commutator* of two elements a, b of a group H, denoted (a,b), is the element $a^{-1}b^{-1}ab$. We sometimes abbreviate $((a,b),c)$ by (a,b,c). For subgroups H_1, H_2 of H, the notation $[H_1, H_2]$ will mean the subgroup of H generated by all the commutators (g,h), where $g \in H_1$, $h \in H_2$.

The *lower central series* of a group H [7, p. 293] is the descending series:

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$$

which is defined inductively by $H = H_1$, $H_2 = [H,H]$, $H_3 = [H,H_2]$, $\ldots$, $H_n = [H, H_{n-1}], \ldots$

It can be shown that each H_{n+1} is a normal subgroup of H_n, and is in fact a *fully invariant subgroup* of H, i.e., is invariant under all endomorphisms of H.

If H_N is trivial for some N, the group H is called *nilpotent.*

Following R. Prener [10, p. 10], we define the basic commutators c_i, $i = 1, 2, \ldots$, their dimensions $D(c_i)$ and their ordering:

Definition 1.1

i) The *basic commutators of dimension 1* are the generators: $c_i = x_i$, $i = 1, \ldots, r$, ordered by their subscripts. (Note that they cannot in general be expressed as (x,y) commutators.)

ii) Having defined basic commutators of dimensions less than n and their ordering, we recursively define the *basic commutators of dimension* n to be all the $c_k = (c_i, c_j)$ such that:

a) c_i and c_j are basic commutators,

b) $D(c_i) + D(c_j) = n$,

c) $c_i > c_j$,

d) if $c_i = (c_k, c_l)$ (i.e., $D(c_i) > 1$), then $c_j \geq c_l$.

iii) If $D(c_i) > D(c_j)$ then $c_i > c_j$.

iv) If $D(c_i) = D(c_j) > 1$ (i.e., $c_i = (c_{i_1}, c_{j_1})$ and $c_j = (c_{i_2}, c_{j_2})$), then $c_i > c_j$ if and only if either

a) $c_{i_1} > c_{i_2}$, or

b) $c_{i_1} = c_{i_2}$ and $c_{j_1} > c_{j_2}$.

It is clear from the definition that any 2 c_j's are comparable.

We follow Prener in using the term *dimension* $D(c_i)$, rather than Marshall Hall [5, pp. 165–166] in using the term *weight* $\omega(c_i)$. In addition, while Hall defined the basic commutators of equal dimension to be ordered arbitrarily with respect to each other, we follow Waldinger [11] and Prener [10, p. 34], in imposing an order (iv) to make our computations more efficient. This is, in fact, the order in which the basic commutators are listed by Algorithm 1.2.

Basic commutators play a significant role in the lower central series of a free group. They are shown to be the generators of the quotients F_n/F_{n+1}, which in fact are free abelian groups. The number of basic commutators of dimension n is given by the Witt formula [5, p. 169]:

$$M_r(n) = \frac{1}{n}\sum_{d|n}\mu(d)r^{n/d}$$

where $\mu(d)$ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute a few examples for the free group of rank 3, which will be of interest to us later:

$$\begin{aligned}
M_3(1) &= \mu(1) \times 3^1 = 3, \\
M_3(2) &= \frac{1}{2}\mu(1) \times 3^2 + \mu(2)3^1 = \frac{1}{2}(9-3) = 3, \\
M_3(3) &= \frac{1}{3}\mu(1) \times 3^3 + \mu(3)3^1 = \frac{1}{3}(27-3) = 8, \\
M_3(4) &= \frac{1}{4}\mu(1) \times 3^4 + \mu(2)3^2 + \mu(4)3^1 = \frac{1}{4}(81-9) = 18, \\
M_3(5) &= \frac{1}{5}\mu(1) \times 3^5 + \mu(5)3^1 = \frac{1}{5}(243-3) = 48.
\end{aligned}$$

We can write an algorithm for listing the basic commutators of a given dimension $n > 1$, provided the ones of lower dimensions have been already computed. Here and further we will use $x_{n;m}$ to denote the mth basic commutator c_s of dimension n. For $c_i = (c_k, c_l)$ (i.e., of dimension > 1) we define *left parent* $p_l(c_i) = c_k$ and *right parent* $p_r(c_i) = c_l$.

ALGORITHM 1.2 (Compute the basic commutators of dimension n)

Inputs:

n, the dimension of interest > 1,

$x_{k;l}$, $k < n$, known basic commutators.

Output:

$x_{n;l}$, $1 \le l \le M(n)$, basic commutators of dimension n.

Step 1. Let $k = 1$.

Step 2. For o from $\lceil n/2 \rceil$ to $n - 1$:

(Consider all pairs of dimensions o and $n - o$ such that $o \ge n - o$.)

Step 3. For i from 1 to $M(o)$:

Step 4. For j from 1 to $M(n - o)$:

(Consider all basic commutators of these dimensions.)

Step 5. If $x_{o;i} > x_{n-o;j}$ and ($o > 1$ or $x_{n-o;j} > p_r(x_{o;i})$) then

(Select the pairs that satisfy Definition 1.1.)

Step 6. Let $x_{n;k} = (x_{o;i}, x_{n-o;j})$ and increment k.

Step 7. Next j, i, o.

Step 8. Assert $k = M(n) + 1$.

We use this algorithm as a step in another algorithm:

ALGORITHM 1.3 (List basic commutators of dimensions $\leq m$)

Step 1. Let $x_{1;i} = x_i$, the generators.

Step 2. For n from 2 to m do:

Step 3. Invoke Algorithm 1.2.

Step 4. Output $x_{n;i} = x_i$, $i = 1, \ldots, M(n)$.

Step 5. Next n.

Let us list the basic commutators of $F(3)$ of dimensions 1, 2, 3 and 4.

$$
\begin{aligned}
x_{1;1} &= x_1 \\
x_{1;2} &= x_2 \\
x_{1;3} &= x_3 \\
x_{2;1} &= (x_2, x_1) \\
x_{2;2} &= (x_3, x_1) \\
x_{2;3} &= (x_3, x_2) \\
x_{3;1} &= (x_{2;1}, x_1) = ((x_2, x_1), x_1) \\
x_{3;2} &= (x_{2;1}, x_2) = ((x_2, x_1), x_2) \\
x_{3;3} &= (x_{2;1}, x_3) = ((x_2, x_1), x_3)
\end{aligned}
$$

$$
\begin{aligned}
x_{3;4} &= (x_{2;2}, x_1) = ((x_3, x_1), x_1) \\
x_{3;5} &= (x_{2;2}, x_2) = ((x_3, x_1), x_2) \\
x_{3;6} &= (x_{2;2}, x_3) = ((x_3, x_1), x_3) \\
x_{3;7} &= (x_{2;3}, x_2) = ((x_3, x_2), x_2) \\
x_{3;8} &= (x_{2;3}, x_3) = ((x_3, x_2), x_3) \\
x_{4;1} &= (x_{2;2}, x_{2;1}) = ((x_3, x_1), (x_2, x_1)) \\
x_{4;2} &= (x_{2;3}, x_{2;1}) = ((x_3, x_2), (x_2, x_1)) \\
x_{4;3} &= (x_{2;3}, x_{2;2}) = ((x_3, x_2), (x_3, x_1)) \\
x_{4;4} &= (x_{3;1}, x_1) = ((x_{2;1}, x_1), x_1) \\
x_{4;5} &= (x_{3;1}, x_2) = ((x_{2;1}, x_1), x_2) \\
x_{4;6} &= (x_{3;1}, x_3) = ((x_{2;1}, x_1), x_3) \\
x_{4;7} &= (x_{3;2}, x_2) = ((x_{2;1}, x_2), x_2) \\
x_{4;8} &= (x_{3;2}, x_3) = ((x_{2;1}, x_2), x_3) \\
x_{4;9} &= (x_{3;3}, x_3) = ((x_{2;1}, x_3), x_3) \\
x_{4;10} &= (x_{3;4}, x_1) = ((x_{2;2}, x_1), x_1) \\
x_{4;11} &= (x_{3;4}, x_2) = ((x_{2;2}, x_1), x_2) \\
x_{4;12} &= (x_{3;4}, x_3) = ((x_{2;2}, x_1), x_3)
\end{aligned}
$$

$$
\begin{aligned}
x_{4;13} &= (x_{3;5}, x_2) = ((x_{2;2}, x_2), x_2) \\
x_{4;14} &= (x_{3;5}, x_3) = ((x_{2;2}, x_2), x_3) \\
x_{4;15} &= (x_{3;6}, x_3) = ((x_{2;2}, x_3), x_3) \\
x_{4;16} &= (x_{3;7}, x_2) = ((x_{2;3}, x_2), x_2) \\
x_{4;17} &= (x_{3;7}, x_3) = ((x_{2;3}, x_2), x_3) \\
x_{4;18} &= (x_{3;8}, x_3) = ((x_{2;3}, x_3), x_3)
\end{aligned}
$$

1.2 The Collection Process

In his dissertation, R. Prener used the collection process, which allows one to rewrite an element of the group expressed in terms of basic commutators into a product in which the basic commutators are ordered in the sense of Definition 1.1.

We view an arbitrary product of n basic commutators

$$w = c_{i_1} \ldots c_{i_m} c_{i_{m+1}} \cdots c_{i_n}$$

as a product of the *collected part* and the *uncollected part*, either of which may be empty. The *collected part* $c_{i_1} \ldots c_{i_m}$ is the largest initial word having the properties $i_k \leq i_{k+1}$ for $1 \leq k < m$ and $i_m < i_j$ for $m < j < n$. The *uncollected part* $c_{i_{m+1}} c_{i_{m+2}} \cdots c_{i_n}$ is the final remainder where $i_{m+1} > i_j$ for some $j > m+1$. In practical applications, we often start with a word w in basic commutators of dimension 1, i.e., in generators.

If the uncollected part is not empty, we can choose in it some "out of order" j such that $i_j < i_{j-1}$, and apply the following

ALGORITHM 1.4 (Collection step)

Inputs:

$w = c_{i_1} \dots c_{i_m} c_{i_{m+1}} \dots c_{i_n}$, a word in basic commutators with non-empty uncollected part;

j such that $i_j < i_{j-1}$.

Output:

w', equal to w in $F(r)$.

Step 1. Compute w' by replacing $c_{i_{j-1}} c_{i_j}$ in w by $c_{i_j} c_{i_{j-1}} (c_{i_{j-1}}, c_{i_j})$.

For example, in the word $x_3 x_1 x_2$ the collected part is empty and x_1 is the only out of order commutator. Applying the collection step yields $x_1 x_3 (x_3, x_1) x_2$. Note that $(x_3, x_1) = x_{2;2}$.

LEMMA 1.5 *The new term* $c_n = (c_{i_{j-1}}, c_{i_j})$ *is a basic commutator whose dimension satisfies*

$$D(c_n) = D(c_{i_{j-1}}) + D(c_{i_j}).$$

Indeed, Marshall Hall [5, p. 166] showed that the basic commutators are precisely the commutators that may arise during collection. When

$$w = c_{i_1} \dots c_{i_k} \dots c_{i_{j-1}} c_{i_j} \dots c_{i_n}$$

is replaced by

$$w = c_{i_1} \dots c_{i_k} \dots c_{i_j} c_{i_{j-1}} (c_{i_{j-1}}, c_{i_j}) \dots c_{i_n},$$

the out of order j moves one position to the left. The new commutator $(c_{i_{j-1}}, c_{i_j})$ belongs on the right of $c_{i_{j-1}}$ and c_{i_j} in the ordering.

We may try to iterate the collection step:

ALGORITHM 1.6 (Collection process)

Input:

$w = c_{i_1} \dots c_{i_m} c_{i_{m+1}} \dots c_{i_n}$, a word in basic commutators.

Output:

w', a word in basic commutators in collected form.

Step 1. Attempt to choose a j such that $i_j < i_{j-1}$.

Step 2. If the search fails then the algorithm terminates.

Step 3. Apply Algorithm 1.4 to w and j

Step 4. Go to step 1.

Because a new basic commutator is added at every collection step, and these new commutators are not necessarily ordered, the collection process need not always terminate.

In addition, we have not specified which out of order j we choose in Step 1 of Algorithm 1.6 as the input to Algorithm 1.4 when the search turns up several choices. Many search strategies are possible. We say that we *collect on the right (on the left)* if we always choose the rightmost (leftmost) j that is out of order. We may also first search for the out of order commutator of the smallest dimension and, if there are several, choose the rightmost. We may also loosen the criterion for termination by requiring only that the uncollected part not contain any basic commutators of dimension less than some threshold value. This version of the process always terminates.

For example, consider the word $w = x_3x_1x_2$ in $F(3)$. It is not in collected form because x_3 is to the left of x_1 and x_2. A collection process yields:

$$\begin{aligned} w &= x_3x_1x_2 = x_1x_3(x_3, x_1)x_2 = x_1x_3x_{2;2}x_2 = x_1x_3x_2x_{2;2}(x_{2;2}, x_2) = \\ &= x_1x_3x_2x_{2;2}x_{3;5} = x_1x_2x_3(x_3, x_2)x_{2;2}x_{3;5} = x_1x_2x_3x_{2;3}x_{2;2}x_{3;5} = \\ &= x_1x_2x_3x_{2;2}x_{2;3}(x_{2;3}, x_{2;2})x_{3;5} = x_1x_2x_3x_{2;2}x_{2;3}x_{4;3}x_{3;5} = \\ &= x_1x_2x_3x_{2;2}x_{2;3}x_{3;5}x_{4;3}(x_{4;3}, x_{3;5}) = x_1x_2x_3x_{2;2}x_{2;3}x_{3;5}x_{4;3}x_{7;k}, \end{aligned}$$

where $(x_{4;3}, x_{3;5}) = x_{7;k}$ for some k.

Chapter 2

The Lower Central Series of G

Let G be the free product of three groups of order 2 given by the presentation

$$\langle x_1, x_2, x_3; x_1^2, x_2^2, x_3^2 \rangle.$$

We are interested in the quotient groups of the lower central series of G.

2.1 G-Simple Basic Commutators

Following R. Prener [10], we will continue to use the term *basic commutator* to refer to the images in G of the basic commutators of the free group $F(3) = \langle x_1, x_2, x_3 \rangle$ under the natural projection.

We define a special set of basic commutators which R. Prener has proven to be the free generators of G_2, the commutator subgroup of G. R. Prener considered more general factor groups of F_r, with additional relations of the form (x_i, x_j), and imposed additional constraints on the basic commutators containing generators that commute. We simplify his definition [10, p. 10] for our G:

DEFINITION 2.1 The *G-simple basic commutators (GSBC's)* are the basic commutators of the form:

$$g = (\ldots(x_{i_1}, x_{i_2}), \ldots), x_{i_p}) \quad 2 \leq p \leq r,$$

where $r = 3$ and x_{i_j}'s are distinct generators of G.

Definitions 1.1 and 2.1 imply the following

LEMMA 2.2 *A commutator $g = (\ldots(x_{i_1}, x_{i_2}), \ldots), x_{i_p})$ is a GSBC if and only if $i_1 > i_2$ and, for $p = 3$, $i_2 < u_3$.*

Direct computations prove

LEMMA 2.3 *The following 5 basic commutators are G-simple:*

$$\begin{aligned}
e_{2;1} &= (x_2, x_1) = x_2 x_1 x_2 x_1, \\
e_{2;2} &= (x_3, x_1) = x_3 x_1 x_3 x_1, \\
e_{2;3} &= (x_3, x_2) = x_3 x_2 x_3 x_2, \\
e_{3;1} &= (x_2, x_1, x_3) = x_1 x_2 x_1 x_2 x_3 x_2 x_1 x_2 x_1 x_3, \\
e_{3;2} &= (x_3, x_1, x_2) = x_1 x_3 x_1 x_3 x_2 x_3 x_1 x_3 x_1 x_2.
\end{aligned}$$

Therefore G_2 has rank 5. The rank of G_2 was also derived by M. Anshel and R. Prener [2], and is given by:

$$\begin{aligned} \text{Rank}(G_2) &= rn - n\sum_{i=1}^{r}(\frac{1}{q_i}) - (n-1) = \\ &= 3 \times 8 - 8 \times (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) - (8-1) = \\ &= 5, \end{aligned} \tag{2.4}$$

where q_i's are the orders of the generators, $n = 8$ is their product, and $r = \text{Rank}(G) = 3$. Later we will show yet another way to compute the rank of G_2.

2.2 The Essential Commutators, the Fundamental Commutators, and the Quotient Groups of the Lower Central Series of G

In this section, we define the essential commutators of dimension n and the fundamental commutators of weight n, derive the formulas for the number of essential commutators and for the number of fundamental commutators. Then we describe the presentation, compute the order of the quotient group G/G_{n+1}, and give a few examples. Finally, we obtain the order of the quotient group G_n/G_{n+1}.

R. Prener demonstrated that the quotient groups G_n/G_{n+1} are generated by special types of basic commutators called fundamental commutators. We will define fundamental commutators in two steps, defining first the essential commutators.

DEFINITION 2.5 (Prener, [10], p. 34) A basic commutator c is an *essential commutator* if:

i) c is a GSBC; or,

ii) $c = (c_l, c_r)$, where c_l and c_r are essential commutators.

Note that in case (ii), c is not a GSBC.

For example, $e_{5;1} = (e_{3;1}, e_{2;1})$ and $e_{5;2} = (e_{3;1}, e_{2;2})$ are essential commutators, since $e_{3;1}$, $e_{2;1}$, $e_{2;2}$ are GSBC's. Therefore $e_{10;1} = (e_{5;2}, e_{5;1})$ is also essential.

Now we will derive a recursive formula for the number of essential commutators of dimension n, denoted by E_n. First, let us examine some preliminary examples.

The number of basic commutators of dimension 2 is:

$$E_2 = \binom{3}{2} = 3.$$

It follows from Definition 2.5 that the essential commutators of dimension 3 are only the GSBC's of dimension 3. They are

$$e_{3;1} = ((x_2, x_1), x_3) = x_1x_2x_1x_2x_3x_2x_1x_2x_1x_3,$$

$$e_{3;2} = ((x_3, x_1), x_2) = x_1x_3x_1x_3x_2x_3x_1x_3x_1x_2,$$

and hence the number of essential commutators of dimension 3 is seen to be

$$E_3 = 2.$$

Let us consider higher dimensions:

$$E_4 = \binom{E_2}{2} = 3,$$

because $4 = 2 + 2$, and

$$E_5 = E_2 \times E_3 = 3 \times 2 = 6,$$

because $5 = 3 + 2$. Further,

$$E_6 = \binom{E_3}{2} + E_4 \times E_2 = 1 + 3 \times 3 = 10.$$

Analogously,

$$E_7 = E_4 \times E_3 + E_5 \times E_2 = 3 \times 2 + 6 \times 3 = 24.$$

$$E_8 = \binom{E_4}{2} + E_5 \times E_3 + E_6 \times E_2 = 3 + 6 \times 2 + 10 \times 3 = 45.$$

To derive a general formula for E_n, we need to consider the even n and the odd n separately.

When n is even, we partition n into

$$n = \frac{n}{2} + \frac{n}{2} = 2 + n - 2 = 3 + n - 3 = \cdots = \frac{n}{2} - 1 + \frac{n}{2} + 1.$$

For the first partition, the number of essential commutators of dimension n is $\binom{E_{n/2}}{2}$. For the other partitions, it is the sum of the products of the numbers of the essential commutators of lower dimensions adding up to n and greater or equal to 2.

When n is odd, we partition n into

$$n = 2 + n - 2 = 3 + n - 3 = \cdots = \frac{n-1}{2} + n - \frac{n-1}{2}.$$

For this partition, the number of essential commutators is the sum of the products of the numbers of the essential commutators of lower dimensions adding up to n and greater or equal to 2.

With this analysis we conclude that:

$$E_n = \begin{cases} \sum_{k=2}^{(n-1)/2} E_k \times E_{n-k} & n \text{ odd}, \\ \binom{E_{n/2}}{2} + \sum_{k=2}^{n/2-1} E_k \times E_{n-k} & n \text{ even}. \end{cases} \tag{2.6}$$

DEFINITION 2.7 (Prener, [10, p. 31]) For a word R in F, we define the *dimension* of R, denoted $D(R)$, to be m if R lies in F_m, but not in F_{m+1}. For a word R in G, we define the *weight* of R, denoted $W(R)$, to be n if R lies in G_n, but not in G_{n+1}.

LEMMA 2.8 (Prener, [10, p. 32]) *If the word $R = \prod x_j$ is in G then*

$$W(R) \geq D(R),$$

where $D(R)$ is the dimension of R when the x_j's are considered in F.

We have now built the basis for

DEFINITION 2.9 (Prener, [10, p. 35])

i) The *fundamental commutators of weight 2* are the essential commutators of dimension 2.

ii) Having defined the fundamental commutators of weight k, we define the *fundamental commutators of weight* $k+1$ to be the essential commutators of dimension $k+1$ together with the squares of all fundamental commutators of weight k.

R. Prener shows Definition 2.7 of dimension and weight to be consistent with those of Definitions 1.1 and 2.9.

LEMMA 2.10 *For any fundamental commutator f of weight w there exists a unique essential commutator e such that $w = e^{2^v}$ and $D(e) = w - v$.*

We will use $e_{i;j}$ to denote the jth fundamental commutator of dimension i.

2.3 The Images of Basic Commutators in G

We now consider the images of basic commutators under the natural projection $\sigma\colon F(3) \to G$.

In G the words of $F(3)$ are subjected to cancellations, such as

$$x_j^k = \begin{cases} x_j & \text{odd } k, \\ 1 & \text{even } k. \end{cases}$$

For example,

$$\begin{aligned} x_{3;1} &= ((x_2, x_1), x_1) = \\ &= x_1^{-1} x_2^{-1} x_1 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2 x_1 x_1 = \\ &= x_1 x_2 x_1 x_2 x_1 x_2 x_1 x_2 = \\ &= (x_2, x_1)^2 = \\ &= x_{2;1}^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} x_{3;2} &= (x_2, x_1), x_2) = \\ &= x_1^{-1} x_2^{-1} x_1 x_2 x_2^{-1} x_2^{-1} x_1^{-1} x_2 x_1 x_2 = \\ &= x_1 x_2 x_1 x_2^3 x_1 x_2 x_1 x_2 = \\ &= x_1 x_2 x_1 x_2 x_1 x_2 x_1 x_2 = \\ &= (x_1, x_2)^2. \end{aligned}$$

Note that (x_1, x_2) is not a basic commutator.

Cancellations can also occur during the collection process. For example,

$$x_1 x_2 x_1 = x_1 x_1 x_2 (x_2, x_1) = x_2 (x_2, x_1) = x_2 x_{2;1}.$$

Now let us compare the number of generators of the factor groups F_n/F_{n+1} and G_n/G_{n+1}. The rank of G_n/G_{n+1}, which is equal to the number of fundamental commutators of weight n, will be discussed later on.

$$\text{Rank}(F_1/F_2) = \text{Rank}(G_1/G_2) = 3,$$

$$\text{Rank}(F_2/F_3) = \text{Rank}(G_2/G_3) = 3.$$

However

$$\text{Rank}(F_3/F_4) = 8$$

and

$$\text{Rank}(G_3/G_4) = 5.$$

The basis for F_3/F_4 consists of the following basic commutators:

$$x_{3;1} = (x_{2;1}, x_1) = ((x_2, x_1), x_1),$$

$$x_{3;2} = (x_{2;1}, x_2) = ((x_2, x_1), x_2),$$

$$x_{3;3} = (x_{2;1}, x_3) = ((x_2, x_1), x_3),$$

$$x_{3;4} = (x_{2;2}, x_1) = ((x_3, x_1), x_1),$$

$$x_{3;5} = (x_{2;2}, x_2) = ((x_3, x_1), x_2),$$

$$x_{3;6} = (x_{2;2}, x_3) = ((x_3, x_1), x_3),$$

$$x_{3;7} = (x_{2;3}, x_2) = ((x_3, x_2), x_2),$$

$$x_{3;8} = (x_{2;3}, x_3) = ((x_3, x_2), x_3).$$

The basis for G_3/G_4 is the following:

$$e_{2;1}^2 = x_2x_1x_2x_1x_2x_1x_2x_1,$$

$$e_{2;2}^2 = x_3x_1x_3x_1x_3x_1x_3x_1,$$

$$e_{2;3}^2 = x_3x_2x_3x_2x_3x_2x_3x_2,$$

$$e_{3;1} = ((x_2, x_1), x_3) = x_1x_2x_1x_2x_3x_2x_1x_2x_1x_3,$$

$$e_{3;2} = ((x_3, x_1), x_2) = x_1x_3x_1x_3x_2x_3x_1x_3x_1x_2.$$

Thus, the 8 basic commutators of dimension 3 in F_3/F_4 are mapped onto the 5 basic commutators of weight 3 in G_3/G_4, which turn out to be fundamental commutators.

Now we will list the fundamental commutators of weights ≤ 5.

According to Definition 2.9, there are 3 fundamental commutators of weight 2:

$$e_{2;1}, e_{2;2}, e_{2;3}$$

There are $3 + 2 = 5$ fundamental commutators of weight 3:

$$e_{3;1}, e_{3;2}, e_{2;1}^2, e_{2;2}^2, e_{2;3}^2.$$

There are 5 + 3 = 8 fundamental commutators of weight 4:

$$e_{4;1}, e_{4;2}, e_{4;3}, e^2_{3;1}, e^2_{3;2}, e^4_{2;1}, e^4_{2;2}, e^4_{2;3}.$$

There are 8 + 6 = 14 fundamental commutators of weight 5:

$$e_{5;1}, e_{5;2}, e_{5;3}, e_{5;4}, e_{5;5}, e_{5;6}, e^2_{4;1}, e^2_{4;2}, e^2_{4;3}, e^4_{3;1}, e^4_{3;2}, e^8_{2;1} e^8_{2;2}, e^8_{2;3}.$$

We observe that $P_k = P_{k_1} + E_k$. We have proved

LEMMA 2.11 (Prener, [10, p. 45]) *The number of fundamental commutators of weight k is given by*

$$P_k = \sum_{i=2}^{k} E_i.$$

Thus, for example,

$$P_2 = 3,$$

$$P_3 = 5,$$

$$P_4 = 8,$$

$$P_5 = 8 + 6 = 14,$$

$$P_6 = 14 + 10 = 24,$$

$$P_7 = 24 + 24 = 48,$$

$$P_8 = 48 + 45 = 93,$$

$$P_9 = 93 + 110 = 203,$$

$$P_{10} = 203 + 228 = 431.$$

In his dissertation, R. Prener gave a presentation for the quotient groups of the lower central series of G.

LEMMA 2.12 (Prener, [10, p. 46]) *The quotient group G/G_{n+1} is given by the presentation*

$$\langle x_1, x_2, x_3, e_{2;1}, e_{2;2}, e_{2;3}, e_{3;1}, \ldots, e_{r;s};$$

$$x_1^2, x_2^2, x_3^2, \psi_1, \psi_2, \ldots, \psi_{r;s}, e_{2;1}^{2^{u_{2;1}}}, \ldots, e_{r;s}^{2^{u_{r;s}}}, m_1, m_2, \ldots, m_t\rangle, \quad (2.13)$$

where $e_{i;j}$ are the essential commutators of dimension $\leq n$, $\psi_{i;j}$ are the words in x_1, x_2, x_3 which define $e_{i;j}$, $u_{i;j} = n + 1 - D(e_{i;j})$, and m_j are the essential commutators of dimension $n + 1$.

An example of this quotient is

$$G/G_2 = \langle x_1, x_2, x_3; x_1^2, x_2^2, x_3^2, e_{2;1}, e_{2;2}, e_{2;3}\rangle.$$

In fact, G/G_2 is the abelianization of G and is isomorphic to the direct product of three cyclic groups of order 2. The last three relators, (x_2, x_1), (x_3, x_1), and (x_3, x_2), indicate that the group G/G_2 is abelian. In particular, the cosets of G modulo G_2 are just G_2, x_1G_2, x_2G_2, x_3G_2, $x_1x_2G_2$, $x_1x_3G_2$, $x_2x_3G_2$, $x_1x_2x_3G_2$.

The following examples are more involved:

$$G/G_3 = \langle x_1, x_2, x_3, e_{2;1}, e_{2;2}e_{2;3}; x_1^2, x_2^2, x_3^2, e_{2;1}^2, e_{2;2}^2, e_{2;3}^2, e_{3;1}, e_{3;2}\rangle. \qquad (2.14)$$

$$G/G_4 = \langle x_1, x_2, x_3, e_{2;1}, e_{2;2}, e_{2;3}, e_{3;1}, e_{3;2}; x_1^2, x_2^2, x_3^2, e_{2;1}^{2^2}, e_{2;2}^{2^2}, e_{2;3}^{2^2}, e_{3;1}^2, e_{3;2}^2, e_{41}, e_{42}, e_{43}\rangle.$$

R. Prener also proved a formula for the order of G/G_{n+1}:

LEMMA 2.15 (Prener [10, p. 46]) *The order of the quotient G/G_{n+1} is given by*

$$|G/G_{n+1}| = 2^3 2^{E_n}(2^2)^{E_{n-1}}(2^3)^{E_{n-2}}\dots(2^{n-1})^{E_2}. \qquad (2.16)$$

Let us perform a few calculations.

$$|G/G_2| = 2^3 = 8,$$

$$|G/G_3| = 2^3 2^3 = 2^6 = 64,$$

$$|G/G_4| = 2^3 2^2 (2^2)^3 = 2^{11} = 2048,$$

$$|G/G_5| = 2^3 2^3 (2^2)^2 (2^3)^3 = 2^{19},$$

$$|G/G_6| = 2^3 2^6 (2^2)^3 (2^3)^2 (2^4)^3 = 2^{33},$$

$$|G/G_7| = 2^3 2^{10} (2^2)^6 (2^3)^3 (2^4)^2 (2^5)^3 = 2^{57},$$

$$|G/G_8| = 2^{57} \times 2^{48} = 2^{105},$$

$$|G/G_9| = 2^{105} \times 2^{93} = 2^{198},$$

$$|G/G_{10}| = 2^{198} \times 2^{203} = 2^{401}.$$

By (2.16) and the Third Isomorphism Theorem,

$$|G_n/G_{n+1}| = \frac{|G/G_{n+1}|}{|G/G_n|} = \frac{2^3 2^{E_n}(2^2)^{E_{n-1}}(2^3)^{E_{n-2}}\dots(2^{n-1})^{E_2}}{2^3 2^{E_{n-1}}(2^2)^{E_{n-2}}(2^3)^{E_{n-3}}\dots(2^{n-2})^{E_2}} =$$

$$= 2^{E_n}2^{E_{n-1}}\dots 2^{E_2} = 2^{P_n}.$$

Hence,

COROLLARY 2.17 *The order of the quotient group G_n/G_{n+1} is given by*

$$|G_n/G_{n+1}| = 2^{P_n}.$$

We observe that, for example,

$$|G_2/G_3| = 2^3;$$

$$|G_3/G_4| = 2^5;$$

$$|G_4/G_5| = 2^8;$$

$$|G_5/G_6| = 2^{14}.$$

It follows from the definition of the lower central series that the quotient group G_n/G_{n+1} lies in the center of the group G/G_{n+1}. The order of the center of G/G_{n+1} is greater than or equal to the order of G_n/G_{n+1}. On the other hand, G_n/G_{n+1} is an abelian group of exponent 2 [10, p. 32].

For example, the factor group G_2/G_3 is generated by $e_{2;1}$, $e_{2;2}$, $e_{2;3}$. The elements of the group (modulo G_3) are:

$$e_{2;1},\ e_{2;2},\ e_{2;3},\ e_{2;1}e_{2;2},\ e_{2;1}e_{2;3},\ e_{2;2}e_{2;3},\ e_{2;1}e_{2;2}e_{2;3},\ 1.$$

The quotient G_3/G_4 is generated by $e_{2;1}^2$, $e_{2;2}^2$, $e_{2;3}^2$, $e_{3;1}$, $e_{3;2}$; the rank of the group is 5. The order of the group is computed by the following sum:

$$\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} + 1 = 2^n - 1 + 1 = 2^n = 32.$$

2.4 Some Experiments with the Essential Commutators of G

We have seen that the only essential commutators of G of dimensions 2 and 3 are the GSBC's of these dimensions. The algorithm for listing the essential commutators of higher dimensions follows immediately from Definition 2.5.

ALGORITHM 2.18 (Compute the essential commutators of G of dimension n)

Inputs:

n, the dimension of interest > 3;

$e_{k;l}$, $k < n$, $1 \leq l \leq E_{n-1}$, known essential commutators of dimensions $< n$.

Output:

$e_{n;l}$, $1 \le l \le E_n$, essential commutators of dimension n.

Step 1. Let $k = 1$.

Step 2. For o from $\lceil n/2 \rceil$ to $n-2$:

Step 3. For i from 1 to E_o:

Step 4. For j from 1 to E_{n-o}:

(Consider all essential commutators of these dimensions.)

Step 5. If $e_{o;i} > e_{n-o;j}$ then

(Select the pairs that satisfy Definition 1.1.)

Step 6. Let $e_{n;k} = (e_{o;i}, e_{n-o;j})$, and increment k.

Step 7. Next j, i, o.

Note that $k = E_n + 1$ upon termination. Note also that the condition $e_{o;i} > e_{n-o;j}$ in Step 5 amounts to verifying that, for even n and $o = n/2$, $i > j$.

This algorithm was machine-implemented by D. Vulis. Running it for weights $n \le 15$, we have examined all possible partitions $n = a + b$ $(a \ge b)$ and all the essential commutators of dimension n that arise as commutators of essential commutators of dimensions a and b. The following table presents the maximum and the minimum lengths of essential commutators as reduced words in G for these partitions.

Dimension	Partition $a + b$	Minimum	Maximum
4	2+2	14	16
5	3+2	20	28
6	3+3	38	38
	4+2	28	40
	Overall	28	40
7	4+3	34	46
	5+2	44	64
	Overall	34	64
8	4+4	46	58
	5+3	38	76
	6+2	56	88
	Overall	38	88
9	5+4	64	88
	6+3	64	98
	7+2	68	136
	Overall	64	136
10	5+5	72	106

	6+4	50	108
	7+3	76	146
	8+2	76	184
	Overall	50	184
11	6+5	84	134
	7+4	68	156
	8+3	86	196
	9+2	128	280
	Overall	68	280
12	6+6	98	154
	7+5	76	174
	8+4	90	208
	9+3	130	288
	10+2	100	376
	Overall	76	376
13	7+6	104	202
	8+5	78	222
	9+4	140	298

	10+3	106	386
	11+2	140	568
	Overall	78	568
14	7+7	132	238
	8+6	100	252
	9+5	122	328
	10+4	114	392
	11+3	136	578
	12+2	144	760
	Overall	100	760
15	8 + 7	132	292
	9 + 6	126	348
	10 + 5	136	416
	11 + 4	160	582
	12 + 3	154	768
	13 + 2	160	1144
	Overall	126	1144

The relationship between the minimum lengths and the structure of G and its lower central series is explained in the following section. We have been unable to discern an elegant relationship between U_n, the maximum length of the essential commutators of dimension n, and the lower central series of G. However, they too possess a pattern:

$$\begin{aligned}
U_5 - U_4 = U_6 - U_5 &= 12 = 3 \times 2^2, \\
U_8 - U_7 = U_7 - U_6 &= 24 = 3 \times 2^4, \\
U_{10} - U_9 = U_9 - U_8 &= 48 = 3 \times 2^5, \\
U_{12} - U_{11} = U_{11} - U_{10} &= 96 = 3 \times 2^6, \\
U_{14} - U_{13} = U_{13} - U_{12} &= 192 = 3 \times 2^7, \\
U_{15} - U_{14} &= 384 = 3 \times 2^8.
\end{aligned}$$

2.5 The Cayley Graphs of G/G_{n+1}

In this section we study the Cayley graphs of the factor groups G/G_{n+1} and present the results of computational experiments about their supergirths and diameters.

Given a finite group H and a set of generators $Y = \{y_1, y_2, \ldots, y_r\}$, all of which are of order 2, we construct the *Cayley graph of H with respect to Y* by taking its vertices to be the elements of H and including an edge between h_1 and h_2 if and only if $h_2 = h_1 y_i$ (and hence $h_1 = h_2 y_i$) for some $y_i \in Y$.

Clearly, a path along the edges corresponds to a word in the generators. If a word in the generators of the group is equal to the identity, then the path corresponding to this word is a closed path, and if such a word is reduced, then it corresponds to a cycle.

DEFINITION 2.19 [3, p. 56] The *girth* of a graph is the length of the shortest cycle in the graph.

We extend the concept of girth for a Cayley graph.

DEFINITION 2.20 The *supergirth* of a Cayley graph is the length of the shortest cycle in the graph that contains all the generators.

Each group G/G_{n+1} is a finite group with a known presentation (2.13), and we can study the corresponding Cayley graph.

The Cayley graph for G/G_2 with generators x_1, x_2, x_3 is a cube, whose eight vertices are the eight elements of the group. The length of a shortest cycle containing all three generators (e.g., $x_1x_2x_3x_1x_2x_3$) is 6, and hence the supergirth of the graph is 6. However there are shorter relators that do not contain all three generators:

$$e_{2;1} = (x_2, x_1) = (x_2x_1)^2,$$

$$e_{2;2} = (x_3, x_1) = (x_3x_1)^2,$$

$$e_{2;3} = (x_3, x_2) = (x_3x_2)^2,$$

and the girth of this Cayley graph is 4.

Now we consider the group G/G_3. This is a group of order 64. It follows from the presentation (2.13) that the relators are $e_{3;1}, e_{3;2}, e_{2;1}^2 e_{2;2}^2 e_{2;3}^2$. To determine the supergirth, we need to consider the words in the generators x_1, x_2, x_3 which are relators. In particular, $e_{3;1}$ and $e_{3;2}$ are the elements of G_3 and have the shortest length and determine the supergirth of G/G_3:

$$e_{3;1} = x_1 x_2 x_1 x_2 x_3 x_2 x_1 x_2 x_1 x_3,$$

$$e_{3;2} = x_1 x_3 x_1 x_3 x_2 x_3 x_1 x_3 x_1 x_2.$$

Once again, the powers of the essential commutators of dimension 2 produce a girth that is smaller than the supergirth.

$$\begin{aligned} & e_{2;1}^2 = (x_2, x_1)^2 = (x_2 x_1)^4, \\ = \;& e_{2;2}^2 = (x_3, x_1)^2 = (x_3 x_1)^4, \\ = \;& e_{2;3}^2 = (x_3, x_2)^2 = (x_3 x_2)^4, \end{aligned}$$

and the girth of this Cayley graph is 8.

Clearly, the graph has various other cycles corresponding to the words in G_3. It follows from the definition of the lower central series that $e_{4;1}$, for example, is also an element of

G_3, hence it corresponds to a cycle in the Cayley graph of the group G/G_3, but has length 14.

On the other hand, we can characterize G/G_3 as one of the groups of order 64 described by Hall and Senior in [4]. The authors described 267 groups of order 64. The group G/G_3 belongs to the family Γ_9 and is numbered 144 in their list. It has 31 elements of order 2 and 32 elements of order 4 (there are in fact 37 such groups). The order of the group of the automorphisms of G/G_3 is $2^9 \times 6$. The group G/G_3 contains 13 self-conjugate subgroups of order 8.

Next we consider the group G/G_4. From the presentation we see that the relators $e_{4;1}$, $e_{4;2}$, $e_{4;3}$ determine the supergirth. The length of $e_{4;1}$ is 16, but the lengths of $e_{4;2}$ and $e_{4;3}$ is 14 because of cancellations. This is in fact both the girth and the supergirth of the Cayley graph of G/G_4, because the relators

$$\begin{aligned} & e_{2;1}^4 = (x_2, x_1)^4 = (x_2 x_1)^8; \\ = \; & e_{2;2}^4 = (x_3, x_1)^4 = (x_3 x_1)^8; \\ = \; & e_{2;3}^4 = (x_3, x_2)^4 = (x_3 x_2)^8 \end{aligned}$$

have length 16. Since there is no cancellation in the powers of $e_{2;j}$'s, the girth of the Cayley graph of G/G_n is equal to the supergirth in dimensions $n \geq 4$.

The following table summarizes the number of essential commutators (according to (2.6)), the supergirths and their location in G_n's, $n \leq 15$.

n	#	Supergirth	at
3	2	10	$e_{3;1} = (x_2, x_1, x_3)$, $e_{3,2} = (x_3, x_1, x_3)$
4	3	14	$e_{4;2} = (e_{2;3}, e_{2;1})$, $e_{4;3} = (e_{2;3}, e_{2;2})$
5	6	20	$e_{5;1} = (e_{3;1}, e_{3;2})$, $e_{5;5} = (e_{3;2}, e_{2;2})$
6	10	28	$e_{6;7} = (e_{4;2}, e_{2;3})$, $e_{6;10} = (e_{4;3}, e_{2;3})$
7	24	34	$e_{7;6} = (e_{4;3}, e_{3;2})$, $e_{7;8} = (e_{5;1}, e_{2;2})$
8	45	38	$e_{8;4} = (e_{5;1}, e_{3;1})$, $e_{8;13} = (e_{5;5}, e_{3;2})$
9	110	64	$e_{9;13} = (e_{5;5}, e_{4;1})$, $e_{9;38} = (e_{6;10}, e_{3;2})$
10	228	50	$e_{10;35} = (e_{6;7}, e_{4;2})$, $e_{10;45} = (e_{6;10}, e_{4;3})$
11	552	68	$e_{11;78} = (e_{7;6}, e_{4;3})$
12	1228	76	$e_{12;82} = (e_{7;7}, e_{5;1})$, $e_{12;164} = (e_{7;20}, e_{5;5})$
13	2952	78	$e_{13;259} = (e_{8;4}, e_{5;1})$, $e_{13;317} = (e_{8;13}, e_{5;5})$
14	6858	100	$e_{14;633} = (e_{8;36}, e_{6;7})$, $e_{14;726} = (e_{8;45}, e_{6;10})$
15	16516	126	$e_{15;1460} = (e_{9;38}, e_{6;10})$

We observe that the supergirth does not increase monotonically with the dimension. Nor is it true that the essential commutators that determine supergirths are themselves composed of essential commutators that determine supergirths in their lower dimensions. For example, $e_{9;13} = (e_{5;5}, e_{4;1})$ is composed of the essential commutator $e_{4;1}$ of length 16, which is not minimal one, but there are only three $e_{4;j}$'s, and two of them have minimal length. Similarly, neither $e_{8;36}$ nor $e_{8;45}$, used for weight 14, are minimal. On the other hand, our computational experiments support the following:

CONJECTURE 2.21 *The number of essential commutators whose length is minimal (i.e., is the supergirth) does not exceed 2 in each dimension.*

We now consider the diameters of these Cayley graphs.

DEFINITION 2.22 Let G be a connected graph. The *distance* between two vertices x and y is the length of (i.e., the number of edges in) a shortest path from x to y and is denoted $d(x, y)$. The maximum distance between any two vertices of G is called the *diameter* of G.

LEMMA 2.23 (Alon and Milman [1, p. 86]) *Let γ be a generating set of a group H closed under inversion and containing the identity. The diameter of the Cayley graph of H with respect to γ is the smallest number k such that $\gamma^k = \gamma \cdot \gamma \cdot \cdots \cdot \gamma = H$.*

In other words, every element of G is a product of k elements of γ (some possibly trivial), and at least one element of G cannot be written as a product of fewer than k elements of γ.

In our case, $\gamma = \{x_1, x_2, x_3, e\}$. We can compute the diameters of some G/G_{n+1}'s.

First, we consider G/G_2. This is an abelian group of order 8. The set γ^2 does not include the single element $x_1x_2x_3$, while the 64 elements of the set γ^3 are equivalent to the 8 distinct words of G/G_2:

$\{x_1, x_2, x_3, e\}^3 = \{x_1, x_2, x_3, e\} \times \{x_1, x_2, x_3, e\} \times \{x_1, x_2, x_3, e\} = \{x_1x_1x_1, x_1x_1x_2, x_1x_1x_3,$

$x_1x_1e, x_1x_2x_1, x_1x_2x_2, x_1x_2x_3, x_1x_2e, x_1x_3x_1, x_1x_3x_2, x_1x_3x_3, x_1x_3e, x_1ex_1, x_1ex_2, x_1ex_3,$

$x_1ee, x_2x_1x_1, x_2x_1x_2, x_2x_1x_3, x_2x_1e, x_2x_2x_1, x_2x_2x_2, x_2x_2x_3, x_2x_2e, x_2x_3x_1, x_2x_3x_2, x_2x_3x_3,$

$x_2x_3ex_2ex_1, x_2ex_2, x_2ex_3, x_2ee, x_3x_1x_1, x_3x_1x_2, x_3x_1x_3, x_3x_1ex_3x_2x_1, x_3x_2x_2, x_3x_2x_3,$

$x_3x_2ex_3x_3x_1, x_3x_3x_2, x_3x_3x_3, x_3x_3ex_3ex_1, x_3ex_2, x_3ex_3, x_3ee, ex_1x_1, ex_1x_2, ex_1x_3, ex_1e,$

$ex_2x_1, ex_2x_2, ex_2x_3, ex_2e, ex_3x_1, ex_3x_2, ex_3x_3, ex_3eeex_1, eex_2, eex_3, eee\} = \{x_1, x_2, x_3, e,$

$x_1x_2x_1, x_1, x_1x_2x_3, x_1x_2e, x_1x_3x_1, x_1x_3x_2, x_1, x_1x_3, e, x_1x_2, x_1x_3, x_1, x_2, x_2x_1x_2, x_2x_1x_3,$

$x_2x_1e, x_1, x_2, x_3, e, x_2x_3x_1, x_2x_3x_2, x_2, x_2x_3x_2x_1, e, x_2x_3, x_2, x_3, x_3x_1x_2, x_3x_1x_3, x_3x_1, x_3x_2x_1,$

$x_3, x_3x_2x_3, x_3x_2, x_1, x_2, x_3, e, x_3x_1, x_3x_2, e, x_3, e, x_1x_2, x_1x_3, x_1, x_2x_1, e, x_2x_3, x_2, x_3x_1, x_3x_2, e,$

$x_3x_1, x_2, x_3, e\} = \{x_1, x_2, x_3, e, x_1x_2x_3, x_1x_2, x_1x_3, x_2x_3\}.$

Thus, γ^3 contains 8 distinct elements: e, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, $x_1x_2x_3$. Therefore, the diameter of the Cayley graph of G/G_2 is 3.

Now we consider G/G_3. This is a group of order 2^6. Our computations show that 4 of its elements cannot be written as a product of fewer than 6 generators: $x_1x_2x_1x_3x_2x_3$, $x_1x_2x_3x_1x_2x_3$, $x_1x_2x_3x_1x_3x_2$, $x_1x_2x_3x_2x_1x_3$. Hence its diameter is 6.

Alon and Milman [1, Prop. 5.1] prove an upper bound for the diameter of a Cayley graph of a group H with respect to a set of generators of order 2 that involves $\log_2 |H|$. These 2 examples support the following

CONJECTURE 2.24 *The diameter of the Cayley graph of the group* G/G_{n+1} *is* $\log_2 |G/G_{n+1}|$.

Chapter 3

G as a Subgroup of the Modular Group

In this chapter we describe G as a a subgroup of the modular group (the group $PSL(2, Z)$ of linear fractional transformations).

3.1 The Modular Group Γ

In numerous papers, Morris Newman described the free product of cyclic groups of order 2 as a subgroup of the modular group. We begin by describing the modular group Γ.

DEFINITION 3.1 The group of linear fractional transformations which map both the interior of the upper half-plane onto itself and the real axis onto itself given by:

$$z \mapsto \frac{az+b}{cz+d},$$

where $ad - bc = 1$ and $a, b, c, d \in Z$, is called the *modular group* and will be denoted by Γ.

The modular group is isomorphic to the special projective linear group

$$PSL(2, Z) = SL(2, Z)/\{\pm I\}, \tag{3.2}$$

where $SL(2, Z)$ is the group of 2×2 matrices with determinant 1. Thus, Γ is identified with the special linear group modulo its center.

The transformations

$$S(z) = z + 1, \quad T(z) = -\frac{1}{z}$$

generate Γ and correspond to the matrices:

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is well-known that Γ is the free product:

$$\Gamma = \langle x, y; x^2 = 1, y^3 = 1 \rangle,$$

where x corresponds to T and y corresponds to $TS = \dfrac{-1}{z+1}$:

$$x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

and hence:

$$x^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I, \quad y^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

It follows from (3.2) that both I and $-I$ are identified with the identify in the quotient Γ.

The Kurosh subgroup theorem states that every non-trivial subgroup H of a free product $G = \prod A_i$ is itself a free product:

$$H = F * \prod x_j^{-1} U_j x_j,$$

where F is a free group (possibly trivial), and each U_j is a subgroup of some A_i (see, for example, [5, p. 315]).

Thus, any subgroup of Γ is either a free group or a free product of the conjugates of $\{x\}$ and $\{y\}$. Our group G can be viewed in several ways [9, pp. 142–143]. It is the normal closure of x in Γ, denoted by $\mathrm{gp}_\Gamma(x)$. Alternatively, G is generated by the cubes of the elements of Γ, and hence has index 3 in Γ. In this form it is denoted by Γ^3. In fact, M. Newman [8, p. 481] proved

THEOREM 3.3 *The group Γ^3 is the free product of three cyclic groups of order 2, and the following holds:*

$$[\Gamma : \Gamma^3] = 3,$$

$$\Gamma = \Gamma^3 + y\Gamma^3 + y^2\Gamma^3,$$

$$\Gamma^3 = \{x, yxy^2, y^2xy\}.$$

The elements of Γ^3 may be characterized by the requirement that the sum of the exponents of y be divisible by 3.

We deduce that the group G may be described as the free product

$$\{x\} * \{yxy^2\} * \{y^2xy\}.$$

To prove the theorem, Newman considered the group generated by the set $\{x, yxy^2, y^2xy\}$, which is normal in Γ and lies in Γ^3. The sum of the exponents of y in each generator is indeed a multiple of 3. From this he showed that $\Gamma = K + yK + y^2K$, which implies that $\Gamma^3 = K$. He proved that K has defining relators $x^2 = (yxy^2)^2 = (y^2xy)^2 = 1$ by showing that no generator belongs to the group generated by the other two.

We also see that the generators x, yxy^2 and y^2xy can be expressed as 2×2 matrices:

$$x_1 = x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$x_2 = yxy^2 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix},$$

$$x_3 = y^2xy = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}.$$

This allows us to express reduced words in G as products of these matrices.

3.2 The Traces of the Essential Commutators

In this section we discuss the traces of the matrices representing the essential commutators.

Our computational experiments show that the traces of essential commutators exhibit a pattern.

Each essential commutator $e_{2;j}$ of dimension 2 has trace 7.

Each essential commutator $e_{3;j}$ of dimension 3 has trace 83.

Each essential commutator $e_{4;j}$ of dimension 4 has trace 322.

However, the traces of the the essential commutators of dimensions greater than 4 are no longer constant. For dimension 5, for example, there are three values for the trace:

2918 (for $e_{5;1}, e_{5;5}$)

11662 (for $e_{5;2}, e_{5;4}$)

72898 (for $e_{5;3}, e_{5;6}$).

We observe that for dimension 5 the smallest trace's absolute value belongs to the elements $e_{5;1}$ and $e_{5;5}$, which also yields the supergirth of the group G/G_5.

For dimension 6, the values for the traces are the following:

9841502 (for $e_{6;1}$)

119558 (for $e_{6;2}, e_{6;5}, e_{6;7}, e_{6;9}, e_{6;10}$)

28579718 (for $e_{6;4}$)

813562 (for $e_{6;6}$).

Here again the smallest trace corresponds to the two elements $e_{6;7}$ and $e_{6;10}$ having the shortest length, i.e., to the supergirth of the group G/G_6.

We can make the same observation for dimension 7. The essential commutator $e_{7;6}$ has the shortest length, and the trace of this element is 2335718, which is the smallest trace's absolute value for the elements of dimension 7.

Our computational experiments support the following

CONJECTURE 3.4 *In the group G/G_{n+1} the relator with the smallest trace, which is an essential commutator of dimension $n+1$, also has the shortest length, corresponding to the supergirth.*

Indeed, it seems plausible that the product of the fewest matrices will have the smallest trace.

3.3 Relation to the Principal Congruence Subgroup of Γ

In this section we present the results of computational experiments with essential commutators, viewed as matrices.

The commutator subgroup G_2 of G is a free group on five generators:

$$e_{2;1} = (x_2, x_1) = x_2 x_1 x_2 x_1 = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

$$e_{2;2} = (x_3, x_1) = x_3 x_1 x_3 x_1 = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$$

$$e_{2;3} = (x_3, x_2) = x_3 x_2 x_3 x_2 = \begin{pmatrix} 8 & 3 \\ -3 & -1 \end{pmatrix}$$

$$e_{3;1} = (x_2, x_1, x_3) = x_1 x_2 x_1 x_2 x_3 x_2 x_1 x_2 x_1 x_3 = \begin{pmatrix} 46 & 63 \\ 27 & 37 \end{pmatrix}$$

$$e_{3;2} = (x_3, x_1, x_2) = x_1 x_3 x_1 x_3 x_2 x_3 x_1 x_3 x_1 x_2 = \begin{pmatrix} 37 & 27 \\ 63 & 46 \end{pmatrix}.$$

All these GSBC's lie in $\Gamma(3)$, the group of all 2×2 matrices congruent to I modulo 3. Since these are the generators, $G_2 < \Gamma(3)$.

Now, $\Gamma(3)$ has index 12 in Γ. Further, the quotient group $\Gamma/\Gamma(3)$ is the group $PSL(2,3)$, a group of order 12, and is isomorphic to the alternating group A_4. On the other hand, the commutator subgroup G_2 has index 8 in G and index 24 in Γ itself.

Now, the group G_3, the next term of the lower central series of G, is contained in G_2 and hence in $\Gamma(3)$. The question naturally arises whether the group G_3 can be contained in $\Gamma(9)$, the subgroup of Γ of all 2×2 matrices congruent to I modulo 9? Even though

$$e_{3;1} = \begin{pmatrix} 46 & 63 \\ 27 & 37 \end{pmatrix}$$

is congruent to I modulo 9, as are $e_{3;2}$, $e_{4;i}$, in the case the other elements of G_3, the answer is "no", because the index of G_3 in Γ is

$$[\Gamma : G_3] = 2^6 \times 3 = 192,$$

and the index

$$[\Gamma : \Gamma(9)] = 1/2 \times 9^3 \times (1 - 1/3^2) = 324,$$

and hence $\Gamma(9)$ is a smaller group than G_3.

In conclusion, for each dimension n we present the smallest m such that

$$e_{n;j} \equiv I \pmod m \quad 1 \leq j \leq E_n.$$

n	m
3	$9 = 3^2$
4	$18 = 2 \times 3^2$
5	$54 = 2 \times 3^3$
6	$54 = 2 \times 3^3$
7	$162 = 2 \times 3^4$

8 $324 = 2^2 \times 3^4$

9 $486 = 2 \times 3^5$

10 $1458 = 2 \times 3^6$

11 $8748 = 2^2 \times 3^7$

We do not readily discern a pattern in the powers of 2's and 3's. On the other hand, the fundamental commutators of corresponding weights do not seem to possess such properties. For example, for the fundamental commutators of weight 5 we obtain $m = 947$ and 747.

3.4 The Level and the Parabolic Class Number of G and the Terms of the Lower Central Series of G in Γ and Their Free Rank

In this section we prove a formula for calculating the level of $G_1, G_2, \ldots, G_n$ in the modular group Γ.

DEFINITION 3.5 (M. Newman, [9, p. 145]) Let $z = xy$ be the element of Γ corresponding to S:

$$z = xy = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Given a normal subgroup H of Γ of finite index, the *level* of H is defined to be the least positive integer n such that $z^n \in H$.

Knowing the level and the index $[\Gamma : H]$, we can compute another characteristic:

DEFINITION 3.6 (M. Newman, [9, p. 151]) Let H be a subgroup of Γ of finite index μ. The number of distinct conjugates $u^{-1}Pu$, where P is the infinite cyclic group generated by $z = xy$, and u runs over H, is called the *parabolic class number* t of H.

M. Newman showed [9, p. 151]) that $t = \frac{\mu}{L}$, where L is the level of H in Γ. He also showed [9, p. 144] that if H is a normal subgroup of Γ of finite index μ, $H \neq \Gamma, \Gamma^2, \Gamma^3$, then H is a free group whose rank is

$$\text{Rank}(H) = 1 + \frac{\mu}{6}. \tag{3.7}$$

We can use this formula to obtains the ranks of G_n, $n > 1$.

Before proving our theorem about the level, let us explicitly compute the levels for several terms of the lower central series of G, viewing them as subgroups of the modular group Γ. Since all the terms of the lower central series are fully invariant in G [7, p. 74], they are normal in Γ.

Viewing our $G = G_1$ as Γ^3, a normal subgroup of Γ, we see that

$$(xy)^3 = xyxyxy = (x)(yxy^2)(y^2xy) = x_1x_2x_3 \in G,$$

and hence its level in Γ is 3. The index μ of G_n is $3 \times |G/G_n|$, with the order of G/G_n given by (2.16).

In general, the level of a subgroup divides its index in Γ. In our case, this amounts to 3 dividing 3.

Next, we consider the commutator subgroup G_2, the second term of the lower central series of G.

The level of G_2 must be a multiple of 3 and must divide 24 — the index of G_2 in Γ. Let us examine $(xy)^6$:

$$(xy)^6 = xyxyxyxyxyxy = x_1x_2x_3x_1x_2x_3.$$

We will collect this word as follows:

$$x_1x_2x_3x_1x_2x_3 = x_1x_2x_1x_3(x_3,x_1)x_2x_3 = x_1x_2x_1x_3e_{2;2}x_2x_3 =$$

$$= x_1x_1x_2e_{2;1}x_3e_{2;2}x_2x_3 = x_2e_{2;1}x_3e_{2;2}x_2x_3 = x_2x_3e_{2;1}(e_{2;1},x_3)e_{2;2}x_2x_3 =$$

$$= x_2x_3e_{2;1}e_{3;1}e_{2;2}x_2x_3 = x_2x_3x_2x_3e_{2;1}e_{3;1}e_{2;2}(e_{2;1}e_{3;1}e_{2;2},x_2x_3) =$$

$$= e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}(e_{2;1}e_{3;1}e_{2;2},x_2x_3).$$

Clearly, the last term is in G_2.

By (3.7) the rank of G_2 is $1 + \frac{24}{6} = 5$, which coincides with the result in (2.4).

Next, let us experiment with the level of G_3. We will collect $(xy)^{12}$ as follows:

$$(xy)^{12} = (xyxyxy)^4 = (x_1x_2x_3)^4 = ((x_1x_2x_3)^2)^2 =$$

$$= e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}(e_{2;1}e_{3;1}e_{2;2}, x_2x_3)e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}(e_{2;1}e_{3;1}e_{2;2}, x_2x_3).$$

Notice that in the above word the exponent sum for each of $e_{2;1}$, $e_{2;2}$ and $e_{2;3}^{-1}$ is 2, and the remaining parts already lie in G_3. Hence, after collecting elements we can show that the entire word is an element of G_3.

First, denote $(e_{2;1}e_{3;1}e_{2;2}, x_2x_3)$ by m, which is obviously an element of G_3. Thus we have:

$$e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}me_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}m = e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}m(m, e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}).$$

The part $m(m, e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}) \in G_3$, therefore we need only to collect the remaining part.

$$e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2} = e_{2;3}^{-1}e_{2;3}^{-1}e_{2;1}e_{3;1}e_{2;2}(e_{2;1}e_{3;1}e_{2;2}, e_{2;3}^{-1})e_{2;1}e_{3;1}e_{2;2}.$$

Since $(e_{2;1}e_{3;1}e_{2;2}, e_{2;3}^{-1}) \in G_3$ we can push it to the right by collecting, and it follows from (2.13) that $e_{2;3}^{-1}e_{2;3}^{-1}$ is in G_3, we need only to consider:

$$e_{2;1}e_{3;1}e_{2;2}e_{2;1}e_{3;1}e_{2;2} = e_{2;1}e_{2;1}e_{3;1}e_{2;2}(e_{3;1}e_{2;2}, e_{2;1})e_{3;1}e_{2;2}.$$

Since $(e_{3;1}e_{2;2}, e_{2;1})e_{3;1}$ lies in G_3, and it can be pushed to the right, and since $(e_{2;1})^2$ also lies in G_3, we see that $e_{3;1}e_{2;2}e_{2;2}$ is also in G_3.

With this we conclude that the level of G_3 in the modular group Γ is indeed 12. The natural conjecture is that the level of G_4, the next term of the lower central series, is 24. Consider:

$$(xy)^{24} = (xyxyxy)^8 = x_1x_2x_3x_1x_2x_3x_1x_2x_3x_1x_2x_3x_1x_2x_3x_1x_2x_3x_1x_2x_3x_1x_2x_3.$$

Notice that in the above expression, each of the words $x_1x_2x_1x_2$, $x_2x_3x_2x_3$ and $x_2x_3x_2x_3$ has the exponent sum 4. On the other hand, $e_{2;1}^{2^2}$, $e_{2;2}^{2^2}$ and $e_{2;1}^{2^2}$ are relators in the factor group G/G_4 (2.13). Hence, using the collection process, we can show that $(xy)^{24}$ is an element of G_4.

Let us summarize our observations:

Group	Level L	Index μ	Parabolic class number t	Rank
$G_1 = G$	$3 = 3 \times 2^0$	3	1	3
G_2	$6 = 3 \times 2^1$	$24 = 3 \times 2^3$	4	5
G_3	$12 = 3 \times 2^2$	$192 = 3 \times 2^6$	16	33
G_4	$24 = 3 \times 2^3$	$6144 = 3 \times 2^{11}$	256	2049

Now we will show by induction that the level L of G_n in Γ can be computed by the following formula:

$$L_n = 3 \times 2^{n-1} \tag{3.8}$$

We have seen this formula to be true for $n = 1, 2, 3, 4$. Suppose it is true for $n = k$:

$$L_k = 3 \times 2^{k-1}.$$

We must show that holds for $n = k + 1$. Indeed, by the induction hypothesis,

$$(xy)^{3\times 2^{k-1}} \in G_k, \tag{3.9}$$

Notice that after collecting the word $(xy)^{3\times 2^{k-1}} = (x_1x_2x_3)^{2^{k-1}}$, we obtain, for example, $(x_1x_2x_1x_2)^{2^{k-2}}M$, where M is an element of G_k, and $(x_1x_2x_1x_2)^{2^{k-2}} = (e_{2;1}^{-1})^{2^{k-2}}$ is also an element of G_k by presentation (2.13).

Now we will show that the level of G_{k+1} in Γ is 3×2^k, i.e. $(xy)^{3\times 2^k} \in G_{k+1}$. Consider

$$(xy)^{3\times 2^k} = ((xy)^{3\times 2^{k-1}})^2 = ((x_1x_2x_3)^{2^{k-1}})^2 = ((x_1x_2x_3)^{2^k}.$$

The above expression contains twice as many $x_1x_2x_1x_2$ as (3.9), i.e., after the collection process the exponent of $x_1x_2x_1x_2 = e_{2;1}^{-1}$ doubles, becoming 2^{k-1}, while x_3 raised to an even power vanishes. Hence $(x_1x_2x_1x_2)^{2^{k-1}} = (e_{2;1}^{-1})^{2^{k-1}}$ lies in G_{k+1} (2.13), the next term of the lower central series. In addition, the right part of the collected word also lies in G_{k+1}, which proves

THEOREM 3.10 *Let $G = \Gamma^3$. Then the level of the n-th term of the lower central series of G in Γ is:*

$$L_n = 3 \times 2^{n-1}.$$

COROLLARY 3.11 *The parabolic class number of G_n is*

$$t = \frac{2^3 2^{E_{n-1}} (2^2)^{E_{n-2}} (2^3)^{E_{n-3}} \ldots (2^{n-2})^{E_2}}{2^{n-1}}.$$

Chapter 4

Algorithms

In this chapter we apply some algorithms to the group G and its factor groups.

4.1 A Hybrid Algorithm for Deriving Presentations of G and G_n's Using the Reidemeister-Schreier Rewriting Process and the Collection Process

We first outline the Reidemeister-Schreier process [7, pp. 86-98].

Let $E = \langle A; R \rangle$ be a group with a known presentation, and G be a subgroup of E. F denotes the free group on A. We wish to write a presentation for G.

We choose a set T of representatives K, one for each coset of G in E, such that:

i) The representative of the identity coset is 1,

ii) for distinct $K \in T$, the cosets KG are distinct,

iii) $\cup_{K\in T} KG = E$,

iv) every initial segment of an element of T is itself an element of T.

Next, for a coset representative $K \in T$ and an element $a_\nu \in A$, we define the *generating symbol:*

$$S_{K,a_\nu} = Ka_\nu(\overline{Ka_\nu})^{-1},$$

where $\overline{x}$, for $x \in E$, is the representative of the coset xG.

Let

$$U = a_{\nu_1}^{\epsilon_1} \cdots a_{\nu_n}^{\epsilon_n} \quad \epsilon_i = \pm 1$$

be a word in G written in terms of the generators of E. We define the rewriting process

$$\tau(U) = S_{K_1,a_{\nu_1}}^{\epsilon_1} \cdots S_{K_n,a_{\nu_n}}^{\epsilon_n},$$

where

$$K_i = \begin{cases} \overline{a_{\nu_1}^{\epsilon_1} \cdots a_{\nu_{i-1}}^{\epsilon_{i-1}}} & \epsilon_i = 1, \\ \overline{a_{\nu_1}^{\epsilon_1} \cdots a_{\nu_i}^{\epsilon_i}} & \epsilon_i = -1. \end{cases}$$

Then G has the presentation

$$\langle S_{K,a_\nu}, \ldots; S_{K,a_\nu} = \tau(Ka_\nu(\overline{Ka_\nu})^{-1}), \ldots, \tau(Kr_\mu K^{-1})\rangle. \qquad (4.1)$$

We will first use the Reidemeister-Schreier rewriting process to write a presentation for Γ^3 as a subgroup of the modular group Γ. In our case, E is the modular group Γ with the presentation $\langle x, y; x^2, y^3\rangle$. We wish to obtain a presentation for the subgroup Γ^3 of index 3.

We choose the coset representatives of Γ^3 to be the set $\{1, y, y^2\}$. Then: $\overline{x} = 1$, since the representative of an element is 1 if and only if this element is in the subgroup, and indeed, $x = x^3$ is the element of Γ^3. Further, $\overline{xy} = \overline{\overline{x}y} = \overline{y} = y$, [7, Theorem 2.7, iv, p. 89]; and $\overline{y^2x} = \overline{y^2} = y^2$, since $x\Gamma^3 = \Gamma^3$.

Next, we list the generating symbols S_{K,a_ν}:

$$S_{1,x}, S_{1,y}, S_{y,x}, S_{y,y}, S_{y^2,x}, S_{y^2,y},$$

where

$$S_{1,x} = 1x(\overline{1x})^{-1} = x,$$

$$S_{1,y} = 1y(\overline{1y})^{-1} = 1,$$

$$S_{y,x} = yx(\overline{yx})^{-1} = yxy^{-1} = yxy^2,$$

$$S_{y,y} = yy(\overline{yy})^{-1} = y^2y^{-2} = 1,$$

$$S_{y^2,x} = y^2x(\overline{y^2x})^{-1} = y^2xy^{-2} = y^2xy,$$

$$S_{y^2,y} = y^2y(\overline{y^2y})^{-1} = 1.$$

Then, by Schreier's theorem, Γ^3 has a presentation (4.1), in which the relators $\tau(KrK^{-1})$ are:

$$\tau(1x^21^{-1}) = \tau(1xx1) = S_{1,x}S_{\overline{x},x}S_{\overline{x^2},1} = S_{1,x}S_{1,x},$$

$$\tau(1y^31^{-1}) = \tau(1yyy1) = S_{1,y}S_{y,y}S_{y^2,y} = 1,$$

$$\tau(yx^2y^{-1}) = \tau(1yxxy^{-1}) = S_{1,y}S_{y,x}S_{\overline{yx},x}(S_{\overline{yx^2y^{-1}},y})^{-1} = S_{y,x}S_{y,x},$$

$$\tau(yy^3y^{-1}) = \tau(yyyyy^{-1}) = S_{1,y}S_{y,y}S_{y^2,y} = 1,$$

$$\tau(y^2x^2y^{-2}) = \tau(yyxxy) = S_{1,y}S_{y,y}S_{y^2,x}S_{\overline{y^2x},x}S_{\overline{y^2x^2},y} = S_{y^2,x}S_{y^2,x},$$

$$\tau(y^2y^3y^{-2}) = \tau(yyyyyy^{-2}) = 1.$$

These computations show that the generators $S_{1,x}$, $S_{y,x}$, $S_{y^2,x}$ are of order 2. Let us denote

$$x = x_1, \quad yxy^2 = x_2, \quad y^2xy = x_3.$$

Hence the presentation for Γ^3 is:

$$\langle x_1, x_2, x_3; x_1^2, x_2^2, x_3^2 \rangle.$$

Next, we will use the Reidemeister-Schreier process and the collection process to write a presentation for G_3 as a subgroup of G_2, with generators expressed in terms of G-simple basic commutators.

We recall that G_2 is a free group with five generators:

$$G_2 = \langle e_{2;1}, e_{2;2}, e_{2;3}, e_{3;1}, e_{3;2} \rangle.$$

The coset representatives of G_2/G_3 are

$$1, \ e_{2;1}, \ e_{2;2}, \ e_{2;3}, \ e_{2;1}e_{2;2}, \ e_{2;1}e_{2;3}, \ e_{2;2}e_{2;3}, \ e_{2;1}e_{2;2}e_{2;3}.$$

For each coset representative K and for each generator a_ν of G_2, we collect the generating symbol $S_{K,a_\nu} = Ka_\nu(\overline{Ka_\nu})^{-1}$. For example, consider

$$S_{e_{2;1}e_{2;2}e_{2;3},e_{2;1}} = e_{2;1}e_{2;2}e_{2;3}e_{2;1}(\overline{e_{2;1}e_{2;2}e_{2;3}e_{2;1}})^{-1}.$$

Let us collect the term

$$\begin{aligned} t &= e_{2;1}e_{2;2}e_{2;3}e_{2;1} = \\ &= e_{2;1}e_{2;2}e_{2;1}e_{2;3}(e_{2;3}, e_{2;1}) = \\ &= e_{2;1}e_{2;2}e_{2;1}e_{2;3}e_{4;i} = \\ &= e_{2;1}e_{2;1}e_{2;2}(e_{2;2}, e_{2;1})e_{2;3}e_{4;i} = \\ &= e_{2;1}^2e_{2;2}e_{4;j}e_{2;3}e_{4;i} = \\ &= e_{2;1}^2e_{2;2}e_{2;3}e_{4;j}(e_{4;j}, e_{2;3})e_{4;i} = \\ &= e_{2;1}^2e_{2;2}e_{2;3}e_{4;j}e_{6;k}e_{4;i} \end{aligned}$$

The uncollected part $u = e_{4;j}e_{6;k}e_{4;i}$ lies in G_3 and $uG_3 = G_3$. Hence, the coset representative of t just that of the collected part $c = e_{2;1}^2e_{2;2}e_{2;3}$.

Further, using the property $\overline{ab} = \overline{\bar{a}b}$ [7, p. 89], we see that the coset representative $\bar{t} = \bar{c} = \overline{e_{2;1}^2e_{2;2}e_{2;3}} = \overline{\overline{e_{2;1}^2}e_{2;2}e_{2;3}}$. But the coset representative of $e_{2;1}^2$ is 1, since it lies in G_3. Hence $\bar{t} = e_{2;2}e_{2;3}$.

Therefore the generating symbol is $e_{2;1}e_{2;2}e_{2;3}e_{2;1}(e_{2;2}e_{2;3})^{-1} = e_{2;1}e_{2;2}e_{2;3}e_{2;1}e_{2;3}^{-1}e_{2;2}^{-1}$.

We apply similar analysis to all the $5 \times 8 = 40$ generating symbols:

$$S_{1,e_{2;1}} = e_{2;1}(\overline{e_{2;1}})^{-1} = 1$$

$$S_{1,e_{2;2}} = 1$$

$$S_{1,e_{2;3}} = 1$$

$$S_{1,e_{3;1}} = e_{3;1}(\overline{e_{3;1}})^{-1} = e_{3;1}$$

$$S_{1,e_{3;2}} = e_{3;2}(\overline{e_{3;2}})^{-1} = e_{3;2}$$

$$S_{e_{2;1},e_{2;1}} = e_{2;1}e_{2;1}(\overline{e_{2;1}e_{2;1}})^{-1} = e_{2;1}^2$$

$$S_{e_{2;1},e_{2;2}} = e_{2;1}e_{2;2}(\overline{e_{2;1}e_{2;2}})^{-1} = e_{2;1}e_{2;2}e_{2;2}^{-1}e_{2;1}^{-1} = 1$$

$$S_{e_{2;1},e_{2;3}} = e_{2;1}e_{2;3}(\overline{e_{2;1}e_{2;3}})^{-1} = e_{2;1}e_{2;3}e_{2;3}^{-1}e_{2;1}^{-1} = 1$$

$$S_{e_{2;1},e_{3;1}} = e_{2;1}e_{3;1}(\overline{e_{2;1}e_{3;1}})^{-1} = e_{2;1}e_{3;1}e_{2;1}^{-1}$$

$$S_{e_{2;1},e_{3;2}} = e_{2;1}e_{3;2}(\overline{e_{2;1}e_{3;2}})^{-1} = e_{2;1}e_{3;2}e_{2;1}^{-1}$$

$$S_{e_{2;2},e_{2;1}} = e_{2;2}e_{2;1}(\overline{e_{2;2}e_{2;1}})^{-1} = e_{2;2}e_{2;1}(\overline{e_{2;1}e_{2;2}(e_{2;2}, e_{2;1})})^{-1} = e_{2;2}e_{2;1}e_{2;2}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;2},e_{2;2}} = e_{2;2}e_{2;2}(\overline{e_{2;2}e_{2;2}})^{-1} = e_{2;2}^2$$

$$S_{e_{2;2},e_{2;3}} = e_{2;2}e_{2;3}(\overline{e_{2;2}e_{2;3}})^{-1} = e_{2;2}e_{2;3}e_{2;3}^{-1}e_{2;2}^{-1} = 1$$

$$S_{e_{2;2},e_{3;1}} = e_{2;2}e_{3;1}(\overline{e_{2;2}e_{3;1}})^{-1} = e_{2;2}e_{3;1}e_{2;2}^{-1}$$

$$S_{e_{2;2},e_{3;2}} = e_{2;2}e_{3;2}(\overline{e_{2;2}e_{3;2}})^{-1} = e_{2;2}e_{3;2}e_{2;2}^{-1}$$

$$S_{e_{2;3},e_{2;1}} = e_{2;3}e_{2;1}(\overline{e_{2;3}e_{2;1}})^{-1} = e_{2;3}e_{2;1}e_{2;3}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;3},e_{2;2}} = e_{2;3}e_{2;2}(\overline{e_{2;3}e_{2;2}})^{-1} = e_{2;3}e_{2;2}e_{2;3}^{-1}e_{2;2}^{-1}$$

$$S_{e_{2;3},e_{2;3}} = e_{2;3}e_{2;3}(\overline{e_{2;3}e_{2;3}})^{-1} = e_{2;3}^{2}$$

$$S_{e_{2;3},e_{3;1}} = e_{2;3}e_{3;1}(\overline{e_{2;3}e_{3;1}})^{-1} = e_{2;3}e_{3;1}e_{2;3}^{-1}$$

$$S_{e_{2;3},e_{3;2}} = e_{2;3}e_{3;2}(\overline{e_{2;3}e_{3;2}})^{-1} = e_{2;3}e_{3;2}e_{2;3}^{-1}$$

$$S_{e_{2;1}e_{2;2},e_{2;1}} = e_{2;1}e_{2;2}e_{2;1}(\overline{e_{2;1}e_{2;2}e_{2;1}})^{-1} = e_{2;1}e_{2;2}e_{2;1}(\overline{e_{2;1}e_{2;1}e_{2;2}(e_{2;2},e_{2;1})})^{-1} =$$

$$= e_{2;1}e_{2;2}e_{2;1}e_{2;2}^{-1}$$

$$S_{e_{2;1}e_{2;2},e_{2;2}} = e_{2;1}e_{2;2}e_{2;2}(\overline{e_{2;1}e_{2;2}e_{2;2}})^{-1}e_{2;1}e_{2;2}e_{2;2}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;2},e_{2;3}} = e_{2;1}e_{2;2}e_{2;3}(\overline{e_{2;1}e_{2;2}e_{2;3}})^{-1} = e_{2;1}e_{2;2}e_{2;3}(e_{2;1}e_{2;2}e_{2;3})^{-1} = 1$$

$$S_{e_{2;1}e_{2;2},e_{3;1}} = e_{2;1}e_{2;2}e_{3;1}(\overline{e_{2;1}e_{2;2}e_{3;1}})^{-1} = e_{2;1}e_{2;2}e_{3;1}e_{2;2}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;2},e_{3;2}} = e_{2;1}e_{2;2}e_{3;2}(\overline{e_{2;1}e_{2;2}e_{3;2}})^{-1} = e_{2;1}e_{2;2}e_{3;2}e_{2;2}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;3},e_{2;1}} = e_{2;1}e_{2;3}e_{2;1}(\overline{e_{2;1}e_{2;3}e_{2;1}})^{-1} = e_{2;1}e_{2;3}e_{2;1}(\overline{e_{2;1}e_{2;1}e_{2;3}(e_{2;3},e_{2;1})})^{-1} =$$

$$= e_{2;1}e_{2;3}e_{2;1}e_{2;3}^{-1}$$

$$S_{e_{2;1}e_{2;3},e_{2;2}} = e_{2;1}e_{2;3}e_{2;2}(\overline{e_{2;1}e_{2;3}e_{2;2}})^{-1} = e_{2;1}e_{2;3}e_{2;2}(\overline{e_{2;1}e_{2;2}e_{2;3}(e_{2;3},e_{2;2})})^{-1} =$$

$$= e_{2;1}e_{2;3}e_{2;2}e_{2;3}^{-1}e_{2;2}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;3},e_{2;3}} = e_{2;1}e_{2;3}e_{2;3}(\overline{e_{2;1}e_{2;3}e_{2;3}})^{-1} = e_{2;1}e_{2;3}e_{2;3}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;3},e_{3;1}} = e_{2;1}e_{2;3}e_{3;1}(\overline{e_{2;1}e_{2;3}e_{3;1}})^{-1} = e_{2;1}e_{2;3}e_{3;1}e_{2;3}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;3},e_{3;2}} = e_{2;1}e_{2;3}e_{3;2}(\overline{e_{2;1}e_{2;3}e_{3;2}})^{-1} = e_{2;1}e_{2;3}e_{3;2}e_{2;3}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;2}e_{2;3},e_{2;1}} = e_{2;2}e_{2;3}e_{2;1}(\overline{e_{2;2}e_{2;3}e_{2;1}})^{-1} = e_{2;2}e_{2;3}e_{2;1}(\overline{e_{2;2}e_{2;1}e_{2;3}(e_{2;3}, e_{2;1})})^{-1} =$$

$$= e_{2;2}e_{2;3}e_{2;1}(\overline{e_{2;1}e_{2;2}e_{2;3}})^{-1} = e_{2;2}e_{2;3}e_{2;1}e_{2;3}^{-1}e_{2;2}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;2}e_{2;3},e_{2;2}} = e_{2;2}e_{2;3}e_{2;2}(\overline{e_{2;2}e_{2;3}e_{2;2}})^{-1} = e_{2;2}e_{2;3}e_{2;2}(\overline{e_{2;2}e_{2;2}e_{2;3}(e_{2;3}, e_{2;2})})^{-1} =$$

$$= e_{2;2}e_{2;3}e_{2;2}e_{2;3}^{-1}$$

$$S_{e_{2;2}e_{2;3},e_{2;3}} = e_{2;2}e_{2;3}e_{2;3}(\overline{e_{2;2}e_{2;3}e_{2;3}})^{-1} = e_{2;2}e_{2;3}e_{2;3}e_{2;2}^{-1}$$

$$S_{e_{2;2}e_{2;3},e_{3;1}} = e_{2;2}e_{2;3}e_{3;1}(\overline{e_{2;2}e_{2;3}e_{3;1}})^{-1} = e_{2;2}e_{2;3}e_{3;1}e_{2;3}^{-1}e_{2;2}^{-1}$$

$$S_{e_{2;2}e_{2;3},e_{3;2}} = e_{2;2}e_{2;3}e_{3;2}(\overline{e_{2;2}e_{2;3}e_{3;2}})^{-1} = e_{2;2}e_{2;3}e_{3;2}e_{2;3}^{-1}e_{2;2}^{-1}$$

$$S_{e_{2;1}e_{2;2}e_{2;3},e_{2;1}} = e_{2;1}e_{2;2}e_{2;3}e_{2;1}(\overline{e_{2;1}e_{2;2}e_{2;3}e_{2;1}})^{-1} = e_{2;1}e_{2;2}e_{2;3}e_{2;1}(e_{2;2}e_{2;3})^{-1} =$$

$$= e_{2;1}e_{2;2}e_{2;3}e_{2;1}e_{2;3}^{-1}e_{2;2}^{-1}$$

$$S_{e_{2;1}e_{2;2}e_{2;3},e_{2;2}} = e_{2;1}e_{2;2}e_{2;3}e_{2;2}(\overline{e_{2;1}e_{2;2}e_{2;3}e_{2;2}})^{-1} = e_{2;1}e_{2;2}e_{2;3}e_{2;2}(e_{2;1}e_{2;3})^{-1} =$$

$$= e_{2;1}e_{2;2}e_{2;3}e_{2;2}e_{2;3}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;2}e_{2;3},e_{2;3}} = e_{2;1}e_{2;2}e_{2;3}e_{2;3}(\overline{e_{2;1}e_{2;2}e_{2;3}e_{2;3}})^{-1} = e_{2;1}e_{2;2}e_{2;3}e_{2;3}(e_{2;1}e_{2;2})^{-1} =$$

$$= e_{2;1}e_{2;2}e_{2;3}e_{2;3}e_{2;2}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;2}e_{2;3},e_{3;1}} = e_{2;1}e_{2;2}e_{2;3}e_{3;1}(\overline{e_{2;1}e_{2;2}e_{2;3}e_{3;1}})^{-1} = e_{2;1}e_{2;2}e_{2;3}e_{3;1}e_{2;3}^{-1}e_{2;2}^{-1}e_{2;1}^{-1}$$

$$S_{e_{2;1}e_{2;2}e_{2;3},e_{3;2}} = e_{2;1}e_{2;2}e_{2;3}e_{3;2}(\overline{e_{2;1}e_{2;2}e_{2;3}e_{3;2}})^{-1} = e_{2;1}e_{2;2}e_{2;3}e_{3;2}e_{2;3}^{-1}e_{2;2}^{-1}e_{2;1}^{-1}$$

Thus 33 symbols are not equal to the identity. Since G_2 is a free group, the relations $\tau(KrK^{-1})$ are empty.

Hence, the subgroup G_3 has no relators, and it is is a free group on 33 free generators obtained above.

4.2 Power Commutator Presentations

In this section, we introduce a power-commutator presentation of a p-group and show that this presentation is the same as the presentation obtained by R. Prener for the group G/G_{n+1} in [10, p. 46; (2.13) above].

DEFINITION 4.2 [6, p. 143] A *p-group* is one whose every element has order of a power of p, where p is a fixed prime.

Clearly, the order of a finite p-group is p^n, $n \in N$. In particular, the factor group G/G_{n+1} is a 2-group.

Now consider a central series (not necessarily the lower central series) of a finite p-group A of order p^n:

$$A = A_0 > A_1 > \ldots > A_{n-1} > A_n = E,$$

in which A_{i-1}/A_i is cyclic of order p, $1 \leq i \leq n$. Take $a_i \in A_{i-1}/A_i$, $1 \leq i \leq n$, so that every element $a \in A_{i-1}$ has a unique normal form

$$a = a_{i+1}^{\alpha_{i+1}} \ldots a_n^{\alpha_n},$$

where $0 \leq \alpha_j < p$, for $i < j \leq n$.

LEMMA 4.3 (D.L. Johnson, [6, p. 171]) *The group G has the presentation*

$$\langle a_1, \ldots, a_n; P, C \rangle,$$

where P is the set of powers given by

$$a_j^p = \prod_{k=j+1}^{n} a_k^{\alpha(j,k)}, \quad 1 \leq i \leq n,$$

and C is the set of commutators given by

$$(a_j, a_k) = \prod_{k=j+1}^{n} a_k^{\alpha(i,j,k)}, \quad 1 \leq i < j \leq n.$$

DEFINITION 4.4 (D.L. Johnson, [6, p. 171]) The above presentation for A is called a *power-commutator presentation* of A.

Notice that this presentation for a p-group is the same as the presentation obtained by R. Prener for the group G/G_{n+1} (2.13).

For example, consider again the presentation (2.14) for the group G/G_3:

$$\langle x_1, x_2, x_3, e_{2;1}, e_{2;2}, e_{2;3};$$

$$x_1^2, x_2^2, x_3^2, e_{2;1} = (x_2, x_1), e_{2;2} = (x_3, x_1), e_{2;3} = (x_3, x_2), e_{2;1}^2, e_{2;2}^2, e_{2;3}^2, (e_{2;1}, x_3), (e_{2;2}, x_2)\rangle.$$

If we denote the generators in the above presentation by $a_1, \ldots, a_6$, we will obtain

$$\langle a_1, a_2, a_3, a_4, a_5, a_6;$$

$$a_1^2, a_2^2, a_3^2, a_4^2, a_5^2, a_6^2, a_4 = (a_2, a_1), a_5 = (a_3, a_1), a_6 = (a_3, a_2), (a_4, a_3), (a_5, a_2)\rangle,$$

which is exactly the power-commutator presentation for a p-group.

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