Drawdowns, Drawups and Their Applications

by

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To Yiming,
and my parents.

Abstract

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Advisor: Olympia Hadjiliadis

This thesis studies the probability characteristics of drawdown and drawup processes of general linear diffusions. The drawdown process is defined as the current drop from the running maximum, while the drawup process is defined as the current rise over the running minimum. Attention is drawn to the first hitting times of the drawdown and the drawup processes, also known as the drawdown and the drawup respectively, and their applications in managing financial risks and detecting abrupt changes in random processes. The probabilities that the drawdown of a units precedes the drawup of equal size are derived in a biased simple random walk model and a drifted Brownian motion model. It is then shown that there exists an analytical formula for the Laplace transform of the drawdown of a units when it precedes the drawup of b units. The above problem can be related to the arbitrage-free pricing of a digital option related to the drawdowns and the drawups. Several static and semi-static replications are developed to hedge the risk exposure of these options. Finally, we study the properties of the drawups as a means of detecting abrupt changes in random processes with multi-source observations. In particular, we study extensions of the cumulative sum (CUSUM) stopping rule, which is the drawup of the log-likelihood ratio process. It is shown that the N-CUSUM stopping rule is at least second-order asymptotically optimal as the meantime to the first false alarm tends to infinity.

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Chapter 1

Introduction

This thesis is a collection of three related works on drawdowns and drawups. The first part establishes the main probabilistic result, i.e., the derivation of the joint distribution of drawdowns and drawups in various models. The second part is a study of replication strategies of two new exotic digital options based on drawdowns and drawups. The third part considers an application of drawups in the problem of quickest detection of abrupt changes in random processes with multi-source observations.

Drawdown processes, and their counterparts, drawup processes, have been extensively studied in the financial risk management literature. The drawdown of a given process is defined as the drop of the present value from the running maximum. The drawdown and the maximum drawdown have been customarily used as risk measures in finance in that they measure the current drop of a stock price, index or value of a portfolio from its running maximum. Similarly, the drawup of a given process is defined as the current rise of the present value over the running minimum. It can be perceived as a performance measure of the return. Over the last few decades, risk management of drawdowns and portfolio optimization with drawdown constraints has become increasingly important among the practitioners. Grossman & Zhou [32], Cvitanic & Karatzas [23], Chekhlov, Uryasev & Zabarankin [21] studied portfolio optimization under constraints on the drawdown process. Douady, Shiryaev & Yor [26] studied the expectation of the maximum drawdown for standard Brownian motion. Magdon-Ismail et. al. [53] determined the distribution of the maximum drawdown of drifted Brownian motion, based on which they described another time-adjusted measure of performance known as the Calmar ratio (see Magdon & Atiya [54]). Other works which describe drawdown processes as dynamic measures of risk include Vecer [88, 89], Pospisil & Vecer [63], Pospisil, Vecer & Xu [66]. For an overview of the existing techniques for analysis of market crashes as well as a collection of empirical studies of the drawdown process and the maximum drawdown process, please refer to Sornette [76].

The drawdowns and the drawups are the first hitting times of the drawdown and the drawup processes to levels a and b, respectively. They are closely related to the maximum drawdown and the maximum drawup over a time-horizon T. Taylor in 1975 (see [85]) derived the exact formula of the joint Laplace transform of the drawdown stopping time and the maximum stopped at that moment in a drifted Brownian motion model. Later, Lehoczky in 1977 (see [49]) extended the above result in a general diffusion model. Recently, Meilijson [55] proved that the drawdown can be viewed as the optimal exercise time of a certain type of look-back American put option. The joint distribution of drawdown stopping times and drawup stopping times are first considered in Hadjiliadis & Vecer [38], and Pospisil, Vecer & Hadjiliadis [65]. They derived the probability that a drawdown stopping time precedes a drawup stopping time in an infinite time-horizon. An application of drawdown stopping times and drawup stopping times in trading with constant-rate transaction cost can be found in Lochowski [50].

In the work that appears in Chapter 2, the joint distribution of drawdown and drawup stopping times is studied. In particular, we characterize the event that the drawdown stopping time precedes the minimum of the drawup stopping time and a pre-specified time-horizon T. The derivation is first accomplished in the case that $a = b$, by drawing the connection of the relevant event to the range process. We then consider the case $a \neq b$ and derive the Laplace transform of the drawdown stopping time when it precedes the drawup stopping time through path decomposition. These results extend the work of Taylor [85] and Lehoczky [49] by relating the drawdown stopping times to the drawup stopping times. In a recent paper of Salminen & Vallois [71], the joint distribution of the maximum drawdown and the maximum drawup over $[0, t]$ is studied in a drifted Brownian motion model; yet it is not possible to extract information on the joint distribution of the drawdown and the drawup stopping times from their paper. On the other hand, the results in Hadjiliadis & Vecer [38], and Pospisil, Vecer & Hadjiliadis [65] can be regarded as special cases of the results in Chapter 2, when the time-horizon is infinite.

Drawdown processes do not only provide dynamic measures of risk, but can also be viewed as measures of "relative regret". Similarly drawup process can be viewed as measures of "relative satisfaction". Thus, a drawdown or a drawup of a certain number of units may signal the time in which an investor may choose to change his/her investment position depending on his/her perception of future moves of the market and his/her risk aversion. Using the results in our paper we are able to calculate the probability that a relative drawdown of $(100\times\alpha)\%$ occurs before a relative drawup of $(100\times\beta)\%$ in a finite time-horizon. On the other hand, a digital option on the event that the relative drawdown occurs before the relative drawup could also be seen as a means of protection. Chapter 2 provides a closed-form formula for the risk-neutral price of this digital option at time 0 both in the case of an infinite maturity and in the case of a finite maturity.

Drawdown and drawup processes also arise in the problem of quickest detection of abrupt changes in a stochastic process. In particular, consider the situation in which a diffusion process is sequentially observed. At some unknown point in time, possibly as a result of the onset of a signal, the dynamics of the process change abruptly in one of two possible opposite directions in the drift. Drawdowns and drawups then provide a detection mechanism of the change-point for each of the possible changes. More specifically, the drawup of the log-likelihood ratio process is known as the cumulative sum (CUSUM) stopping rule, which was first introduced by Page [60] in 1954, and whose optimality was proven in discrete-time models by Moustakides [56], in the continuous-time Brownian motion model by Beibel [9] and Shiryaev [73], and in a continuous-time Itô process model by Moustakides [57].

On the other hand the two-sided CUSUM stopping rule used to detect two-side changes in random processes was introduced by Barnard [5] in 1959. Distributional properties of the two-sided CUSUM (2-CUSUM) stopping rule were subsequently studied by Van Dobben de Bruyn [25], Bissell [10], Woodall [91], and Khan [45, 46, 48]. Its optimality properties were studied and established by Lorden [51], Dragalin [27], Hadjiliadis [33, 34], Hadjiliadis, Hernandez-Del-Valle & Stamos [35], Hadjiliadis & Moustakides [36] and Hadjiliadis & Poor [37]. For an overview of these results please refer to Poor & Hadjiliadis [62]. A challenging problem in engineering is the detection and identification of such signals when they are only present for a finite period of time. These signals are known as transient signals. Using the results in this work, it is possible to derive closed-form formulas for the probability of misidentification of the direction of the change in the drift when the signal has exponential life. Moreover, using the results in Chapter 2 for drifted Brownian motion, we derive this probability when the transient signal is present for a finite period of time T.

Chapter 2 is mainly concerned with probabilistic results related to the drawdowns and the drawups. The rest of the thesis focuses on two applications of drawdowns and drawups in finance and engineering. In particular, in Chapter 3 we focus on replication strategies of the digital option introduced in Chapter 2. We consider two special cases: $a = K, b = \infty$ and $a = b = K$. The first claim pays \$1 at expiry T if and only if the spot has drawn down by at least \$$K$ over $[0, T]$, while the second claim pays \$1 at T if and only if the time at which the drawdown first reaches K precedes the earlier of T and the time at which the drawup first reaches K. Both of these instruments clearly provide protection against adverse movements in the market. In this work we present model-free static hedges of the second claim using one-touch knockouts and their spreads. Then under symmetry and continuity assumptions, we also derive semi-static hedges of both claim using one-touch knockouts, single barrier one-touches and vanilla options.

As pointed out earlier, the maximum drawdown of an asset or portfolio is commonly used as a measure of the risk of holding that asset over $[0, T]$. Consequently, a risk averse investor who is concerned that this risk measure realizes to a value larger than expected would presumably be interested in being compensated for large realizations of maximum drawdown. A digital call written on the maximum drawdown pays a fixed amount of money, say one dollar, if the maximum drawdown over $[0, T]$ is excessively large at T. Hence, the payoff at T is $\mathbb{1}_{\{\tau_K^D \leq T\}}$ for some strike $K > 0$. The premium for this digital call is analogous to an insurance premium.

Maximum drawdown is commonly used to evaluate the risk of a hedge fund over a specific time period. An asset manager who knows in advance that his portfolio risk is being evaluated wholly or in part by the portfolio's maximum drawdown is exposed to large positive realizations of maximum drawdown. In particular, it is not uncommon for managers who experience large maximum drawdowns to see their funds under management rapidly diminish. Since performance fees are typically proportional to funds under management, these fees would similarly diminish. By purchasing a digital call before any such maximum drawdown is realized, a portfolio manager can insure against the loss of income.

The premium for this digital call can be cheapened if the payoff is lessened. One way to do this is to further introduce dependence of the terminal payoff on the time it takes for a realization of a drawup of a pre-specified level. If the investor holding the digital call is also long the underlying asset, then it seems reasonable the investor would be willing to give up some of the payoff if a drawup occurs first, in return for reduced premium. We have

$$\mathbb{1}_{\{\tau_K^D \le \tau_K^U \wedge T\}} = \mathbb{1}_{\{\tau_K^D \le T\}} - \mathbb{1}_{\{\tau_K^U \le \tau_K^D \le T\}}.$$

Consider a claim that pays $\mathbb{1}_{\{\tau_K^D \le \tau_K^U \wedge T\}}$ dollars at T. In words, the claim pays one dollar at its expiry date T if and only if a drawdown of size K precedes the earlier of a drawup of the same size and expiry. For brevity, we refer to this claim as a digital call on a K-drawdown preceding a K-drawup. Such a payoff would be of interest to anyone who is more concerned about the downside than the upside, or at least more so than the market is. The payoff from the digital call on the K-drawdown preceding a K-drawup will be smaller than the payoff from a co-terminal digital call on maximum drawdown with strike K because of the possibility that a K-drawup precedes a K-drawdown.

A financial intermediary who provides a digital call on maximum drawdown or K-drawdown preceding a K-drawup to clients is typically faced with the problem of hedging the exposure and marking the position after the sale. If there exists a hedging strategy which perfectly replicates the payoff of such a digital call under a set of reasonable assumptions, then the mark-to-market value of this replicating portfolio can be used to mark the position of this digital call. Under the continuity and martingale assumption, Cheridito, Nikeghbali & Platen [22] consider a dynamic hedging of options with payoff triggered by the maximum drawdown, as do Pospisil & Vecer [63, 64], which involves continuous trading. In this work, we look for a hedging strategy which achieves a perfect replication with the least possible time instances in which trading is involved. This kind of strategy is undoubtedly more robust than a dynamic hedging strategy. Such a replication is also known as static[1] and was introduced in Breeden and Litzenberger [14]. It was further studied in Bowie and Carr [13], Carr and Chou [17], Carr, Ellies & Gupta [18], Carr & Madan [19], Derman, Ergener & Kani [24], and Sbuelz [72].

In the work that appears in Chapter 3, we show that there exists a robust

[1]Some authors consider robust (model-free) replicating portfolio which superhedges or subhedges the target claim. For example, Hao [40] developed static super- and sub-replication strategies of double touch barrier options.

static hedge of the digital call on the K-drawdown preceding a K-drawup. This hedge uses positions in one-touch knockouts. We then develop simple sufficient conditions on the underlying asset price dynamics which allow for semi-robust replicating strategies to hedge the digital call on maximum drawdown with one-touch knockouts. One-touch knockouts do trade liquidly in the over-the-counter (OTC) currency options market. Our strategy replicates perfectly under a symmetry condition and provided that the running maximum increases only continuously. One-touch knockouts are not necessarily available for all currency pairs, hedging and marking requires the development of additional simple sufficient conditions on the underlying asset price dynamics which allow for alternative replicating strategies using more liquid instruments. In particular, if we enforce symmetry condition and additionally assume that the running minimum decreases continuously, then we can develop replicating strategies that use only single barrier one-touches, or even path-independent options. Note that for all of the above strategies, hedging requires only occasional trading, typically only when maxima or minima change. As vanilla options are not necessarily available for all currency pairs, one can always impose further dynamical restrictions and resort to classical dynamic hedging. Whenever a model allows the payoff of vanilla options to be dynamically replicated with the underlying asset, it can be used in conjunction with our results to replicate the payoff of calls on maximum drawdown with the same instruments.

As indicated above, the hedging strategies have been ordered by strengthening the sufficient conditions on price dynamics under which hedging can occur. However, as the hedging strategies decrease in robustness, they increase in terms of the liquidity of the assets used in the hedge. Thus, the choice of hedging strategy depends on the user's tolerance for model risk and on the nature of the market.

In the last Chapter, the problem of quickest detection of abrupt changes is revisited. We consider the situation in which we receive observations from N sources, and the onset of a signal can occur at different times across observations from different sources. In our formulation, we consider the case of equal-strength and unequal-strength signals across these sources, which in discrete-time models corresponds to the case of the same and different out-of-control distributions. We also assume that the N observed processes are independent, which constitutes an assumption consistent with the fact that the N change-points can be different. The goal is to detect the moment of the first change-point as soon as possible, while controlling the false alarms. In this formalism, we seek a stopping rule T that detects a change-point τ while at the same time controls the meantime to the first false alarm. In other words, at each decision time point, t, we want to discriminate between the two states of the process: the state $\{t < \tau\}$ and the state $\{t \geq \tau\}$. More specifically, the stopping rule T minimize the detection delay of the change under the constraint on the meantime to the first false alarm.

To address this problem we propose the N-CUSUM stopping rule (see for example, [58, 81, 82, 83, 84]). The N-CUSUM stopping rule consists of running N one dimensional CUSUM schemes in parallel, each designed to detect the respective changes. Optimality properties of the N-CUSUM stopping rule in the case that the N change-points are identical have been studied by Tartakovsky [78] and Moustakides [56]. More recently, the case of different change-points was considered by Raghavan & Veeravalli [67]. However, in their configuration it is assumed that the change-points propagate according to a specific distribution, and this propagation depends on the unknown identity of the first source affected. In our setup, we do not impose any assumption on the distribution of change-points.

In the work that appears in Chapter 4, an extended Lorden's performance index (see [51]) is proposed as a performance measure for the detection delay of a stopping rule T. In other words, the worst detection delay over all paths and over all change-points, is considered. The goal is to minimize the worst case detection delay, subject to a constraint in the meantime to the first false alarm. We first consider the continuous Brownian motion model, where the incoming signal results a drift change that is related to the signal strength. The Brownian motion model is a good approximation when observations are taken at a high frequency and when the magnitudes of the changes are small. To investigate the problem when this is not the case, we also consider general discrete-time models, where the incoming signal results in a shift in the distribution of observations.

In the Brownian motion model, the derivation is achieved by bounding above the detection delay of the unknown optimal stopping rule by the detection delay of the proposed N-CUSUM stopping rule and below by the detection delay of a one-dimensional CUSUM stopping rule. Using results of Magdon-Ismail et. al. [53], we get the exact formula for the expected detection delay of the N-CUSUM stopping rule. We analyze the asymptotic expansion of this formula, and compare it with the results in Shiryaev [73]. It is shown that, the N-CUSUM stopping rule is at least second-order asymptotically optimal as the meantime to the first false alarm tends to infinity. Moreover, it is interesting that, in the case in which one of the signals is the weakest, the N-CUSUM enjoys third-order asymptotic optimality.

In the discrete-time models, we derive similar bounds for the detection delay of the unknown optimal stopping rule and derive asymptotic expansions for the expected detection delays using results in Dragalin, Tartakovsky & Veeravalli [28], Khan [47], Moustakides[56], Tartakovsky [79]. Based on overshoot characteristics of the models, we prove that the N-CUSUM stopping rule is at least second-order asymptotically optimal as the meantime to the first alarm tends to infinity. Moreover, when exactly one of the distribution shifts achieves the smallest Kullback-Leibler distance from the initial regime (before the change), the N-CUSUM is third-order asymptotically optimal.

Chapter 2

Joint Distribution of Drawdowns and Drawups

In this work we study drawdowns and drawups of general diffusion processes. The drawdown process is defined as the current drop of the process from its running maximum, while the drawup process is defined as the current increase over its running minimum. The drawdown and the drawup are the first hitting times of the drawdown and the drawup processes respectively. We characterize the joint distribution of drawdowns and drawups, and apply the results to a problem of interest in financial risk-management and to the problem of transient signal detection and identification of two-sided changes in the drift of general diffusion processes.

The remaining of the chapter is structured in the following way: definitions are introduced in Section 2.1. In Section 2.2, we derive the probability that the drawdown precedes the drawup of equal size in a random walk model and in a Brownian motion model. In Section 2.3, we proceed to consider general linear diffusion dynamics, and derive the Laplace transform of the drawdown of a units when it precedes the drawup of b units, in the cases $a = b$ (Theorem 2.3), $a > b$ (Theorem 2.4) and $a < b$ (Theorem 2.5). The special case of a drifted Brownian motion model is revisited in Section 2.4, where we also derive the analytical density $p^{(\mu)}(t; a, b)$ by analytical inversion of the Laplace transform. We then present applications of our results in a problem of risk-management and the problem of transient signal detection and identification of two-sided alternatives in Section 2.5. Finally, we conclude with some closing remarks in Section 2.6.

2.1 Drawdown and Drawup Processes

We begin with mathematical definitions of the first hitting time, drawdown, drawup and range processes in the most general setting.

Definition 2.1. *Let $X_\cdot = \{X_t; t \geq 0\}$ be a real-valued stochastic process, u be a real number. The first hitting time of $X_\cdot$ to u, which is denoted by τ_u^X, is defined as*

$$\tau_u^X \quad \triangleq \quad \inf\{t \geq 0 | u \in [m_t, M_t]\}, \tag{2.1}$$

where $M_t \triangleq \sup_{s\in[0,t]} X_s$ and $m_t \triangleq \inf_{s\in[0,t]} X_s$ are the running maximum and *running minimum processes. By convention, we assume that* $\inf \phi = \infty$.

Note that if $X.$ is a continuous (skip-free) process, then (2.1) can be rewritten as:

$$\tau_u^X \triangleq \inf\{t \geq 0 | X_t = u\}. \tag{2.1'}$$

The drawdown, drawup and range processes are defined in terms of the running maximum and running minimum:

Definition 2.2. *Let* $X. = \{X_t; t \geq 0\}$ *be a real-valued stochastic process. Then the drawdown, drawup and range processes of* $X.$*, which are denote by* $D_t,\ U_t,\ R_t$*, are defined respectively as,*

$$D_t \triangleq M_t - X_t, \tag{2.2}$$

$$U_t \triangleq X_t - m_t, \tag{2.3}$$

$$R_t \triangleq M_t - m_t. \tag{2.4}$$

We adopt notations in definitions 2.1 and 2.2 throughout the rest of the paper. In particular, for $a, b, r > 0$, the first time to a drawdown (drawup, resp.) of a (b, resp.) units is denoted by τ_a^D (τ_b^U, resp.), and the first range time to r is denoted by τ_r^R, etc.

It is interesting to point out, the first range time τ_a^R is closely related to τ_a^D and τ_a^U. This property is used over and over again. We present it in the following lemma.

Lemma 2.1. *For any $a, T > 0$ we have*

$$\tau_a^R = \tau_a^D \wedge \tau_a^U, \tag{2.5}$$

$$\{\tau_a^D \leq \tau_a^U \wedge T\} = \{\tau_a^R \leq T, X_{\tau_a^R} < X_0\}. \tag{2.6}$$

Proof. It is easily seen that

$$\begin{aligned} \tau_a^R \leq T \Leftrightarrow \sup_{t\in[0,T]} (M_t - m_t) \geq a \quad &\Leftrightarrow \quad \max(\sup_{t\in[0,T]} D_t, \sup_{t\in[0,T]} U_t) \geq a \\ &\Leftrightarrow \quad \tau_a^D \wedge \tau_a^U \leq T. \end{aligned} \tag{2.7}$$

And (2.6) is a direct consequence of (2.5). $\square$

2.2 The Case of $a = b$

In this section, we first derive the probability that a drawdown of a units precedes a drawup of equal units in a finite time horizon T. That is,

$$P(\tau_a^D < \tau_a^U \wedge T). \tag{2.8}$$

The assumed underlying model considered is a random walk model. For this model we provide a closed-form formula for this probability both in the case of a symmetric random walk and in the case of a non-symmetric random walk. We then derive a closed-form formula for this probability in the case of a drifted Brownian motion model.

2.2.1 A random walk model

We begin by considering the random walk $X. = \{X_n; n \geq 0\}$:

$$X_n = \sum_{i=1}^{n} Z_i, \;\; X_0 = x, \tag{2.9}$$

where

$$Z_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } q. \end{cases}$$

That is, the process $\{X_n\}_{n\geq 1}$ is a simple random walk with parameter p.

In the next theorem we compute the probability that a drawdown of a units precedes a drawup of equal units in a pre-specified finite time-horizon T, where $T > a$.

Theorem 2.1. *For $a, T \in \mathbb{N}^*$, define*

$$\wp(T; a, p) \;\; \triangleq \;\; P(\tau_a^D < \tau_a^U \wedge T). \tag{2.10}$$

The probability that a drawdown of a units proceeds a drawup of equal units before time $T > a$ is given by

1. *for $a = 1$,*

$$\wp(T; 1, p) = q. \tag{2.11}$$

2. *for $a = 2$,*

$$\wp(T; 2, p) = q^2 + pq^2 + qpq^2 + \ldots + \underbrace{\ldots pqpq^2}_{(T-1) terms}. \tag{2.12}$$

3. for $a \geq 3$,

$$\wp(T;a,p) = q^a + \sum_{L=a+2}^{T} \sum_{i=1}^{a} \sum_{k=0}^{L-a-1} c_{i,1}^{a,L-a-k-1} \cdot c_{1,a-2}^{a-1,a+k-3} \cdot q^{\frac{L+a-i}{2}} p^{\frac{L-a+i-2}{2}}, \tag{2.13}$$

where for $m,k,i,j \in \mathbb{N}$,

$$c_{i,j}^{m,k} = \frac{2^{k+1}}{m+1} \sum_{\iota=1}^{m} \left(\cos\frac{\pi\iota}{m+1}\right)^k \sin\frac{i\pi\iota}{m+1} \sin\frac{j\pi\iota}{m+1}. \tag{2.14}$$

In order to proceed with the proof of this theorem, we will need to make use of two preliminary lemmas. In the first lemma we compute the probability that a random walk, which starts at 0 reaches a specific level $-1 \leq v \leq B$ in N steps, while remaining within a positive strip of a pre-specified height A.

Lemma 2.2. *For* $u, v, A, N \in \mathbb{N}$ *and* $0 \leq u, v \leq A$, *we have*

$$P_u(X_N = v, 0 \leq X_k \leq A \text{ for } \forall k \leq N) = c_{u+1,v+1}^{A+1,N} \cdot p^{\frac{N-u+v}{2}} q^{\frac{N+u-v}{2}}, \tag{2.15}$$

where $c_{u+1,v+1}^{A+1,N}$ *is defined in (2.14).*

Proof. The 1-step transition matrix of a simple random walk in $[0, A]$ is the Toeplitz matrix M_{A+1} generated by column vector c and row vector r, where

$$\mathrm{c} = (\underbrace{0, q, 0, \ldots, 0}_{A+1}) \quad \mathrm{r} = (\underbrace{0, p, 0, \ldots, 0}_{A+1}).$$

The N-step transition matrix is the N-th power of that matrix. The probability in (2.15) is the $(u+1, v+1)$-th entry of this N-step transition matrix. Using Theorem 2.3 on page 1064 of Salkuyeh [70], the result follows. $\square$

In the second lemma we compute the probability that a random walk, which starts at 0 reaches a specific level v in N steps while its minimum reaches the exact level $v - B$ and its maximum never exceeds $v + 1$. We denote this probability by $g(N, v; B)$.

Lemma 2.3. *For $B, N \in \mathbb{N}$ with $B \leq N$, and $v = -1, 0, \ldots, B$, define*

$$g_p(N, v; B) \triangleq P(X_N = v, \max_{1 \leq k \leq N} X_k \leq v + 1, \min_{1 \leq k \leq N} X_k = v - B). \quad (2.16)$$

We have

$$g_p(N, v; B) = \sum_{k=0}^{N-B} c_{B-v+1,1}^{B+2, N-B-k} \cdot c_{1,B}^{B+1, B+k-1} \cdot p^{\frac{N+v}{2}} q^{\frac{N-v}{2}}, \quad (2.17)$$

with coefficient $c_{i,j}^{m,k}$ defined in (2.14).

Proof. With $g_p(N, v; B)$ as in (2.16) we notice that

$$g_p(N, -1; B) \ = \ q \cdot g_p(N-1, 0; B), \quad (2.18)$$

$$g_p(N, B; B) = p \cdot g_p(N-1, B-1; B) + p \cdot g_p(N-1, B-1; B-1), \quad (2.19)$$

and for $-1 < v < B$ that,

$$g_p(N, v; B) \ = \ p \cdot g_p(N-1, v-1; B) + q \cdot g_p(N-1, v+1; B). \quad (2.20)$$

To see (2.19), we observe that $g(N, B; B)$ is the probability of an event that only includes paths on which the process remains non-negative. Equation (2.19) represents the decomposition of these paths into the ones on which the process stays strictly positive after the first upward step, and the ones on which it does not. Equation (2.20) follows by conditioning on the first step being up or down respectively.

Equations (2.18), (2.19), and (2.20) can be summarized by

$$G_N^{(B)} = M_{B+2} \cdot G_{N-1}^{(B)} + Y_{N-1}^{(B)}, \tag{2.21}$$

where M_{B+2} is the 1-step transition matrix of a simple random walk in $[-1, B+1]$ which appears in the proof of Lemma 2.2, $G_N^{(B)}$ and $Y_N^{(B)}$ are the $(B+2) \times 1$ vectors

$$G_N^{(B)} = (g_p(N, B; B), g_p(N, B-1; B), \ldots, g_p(N, -1; B))^\tau, \tag{2.22}$$

and

$$Y_N^{(B)} = (p \cdot g_p(N, B-1; B-1), 0, \ldots, 0)^\tau, \tag{2.23}$$

respectively, while

$$G_B^{(B)} = Y_{B-1}^{(B)} = (p^B, 0, \ldots, 0)^\tau, \tag{2.24}$$

We can now use (2.21) recursively to obtain

$$\begin{aligned} G_N^{(B)} &= [M_{B+2}]^{N-B} \cdot G_B^{(B)} + \sum_{k=0}^{N-B-1} [M_{B+2}]^{N-B-k-1} \cdot Y_{B+k}^{(B)} \\ &= \sum_{k=0}^{N-B} [M_{B+2}]^{N-B-k} \cdot Y_{B+k-1}^{(B)}. \end{aligned} \tag{2.25}$$

Equation (2.17) now follows from (2.25), Theorem 2.3 on page 1064 of Salkuyeh [70], and Lemma 2.2. □

We can now proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. Equations (2.11) and (2.12) are easy to see. For $a \geq 3$ it is also easy to see that

$$\wp(a+1; a, p) = q^a. \tag{2.26}$$

In order to establish (2.13), it suffices to determine

$$\begin{aligned} \Delta(T; a, p) &= \wp(T; a, p) - \wp(T-1; a, p) \\ &= P(\tau_a^D = T-1, \max_{k \leq T-1} U_k \leq a-1), \end{aligned} \tag{2.27}$$

for any $a, T \in \mathbb{N}^*$ and $T > a+1 \geq 4$.

We begin by examining the properties of all paths which are included in the event of (2.27). For convenience, let us reflect all such paths about the initial value $X_0 = 0$, and denote the reflected paths by $\overline{X}$. It is easily seen that

1. For all the reflected paths,

$$\overline{X}_{T-1} \in \{1, 2, \ldots, a\},$$

   for otherwise, a drawdown of a units precedes a drawup of equal size, or the range is less than a at time $T-1$.

2. Let us assume $\overline{X}_{T-1} = u \in \{1, 2, \ldots, a\}$, then

$$\min_{k \leq T-1} \overline{X}_k = u - a.$$

3. Assume $\overline{X}_{T-1} = u \in \{1, 2, \ldots, a\}$, then

$$\overline{X}_{T-2} = u - 1, \overline{X}_{T-3} = u - 2, \max_{k \leq T-3} \overline{X}_k \leq u - 1.$$

   This is because the drawup (which precedes the drawdown) is achieved by an upward move of the random walk $\{\overline{X}_n\}_{n \geq 1}$; moreover, the highest position of the random walk before $T-1$ can be at most $u-1$.

These properties give rise to the following representation

$$\Delta(T; a, p) \quad = \quad q^2 \cdot \sum_{v=-1}^{a-2} g_q(T-3, v; a-2). \tag{2.28}$$

Using Lemma 2.3, the result follows. This completes the proof of Theorem 2.1. $\square$

In the case that an investor is not restricted by a finite time horizon, the probability that his/her wealth makes a rally of a units before a drawdown of equal units is summarized in the following corollary. This result is easier derived by using martingale arguments (see Hadjiliadis [34]) and is displayed for completeness.

Corollary 2.1. *In the case of an infinite time-horizon we have*

$$P(\tau_a^D < \tau_a^U) \quad = \quad \frac{\left(\frac{p}{q}\right)^{a+1} - (a+1)\left(\frac{p}{q}\right) + a}{\left[1 - \left(\frac{p}{q}\right)^a\right]\left[\left(\frac{q}{p}\right)^{a+1} - 1\right]}, \tag{2.29}$$

The next corollary draws a connection of our result to the range process which is defined to be the difference of the running maximum and the running minimum.

Corollary 2.2. *Let $R.$ be the range process*[1] *of $X.$. Then for $T > a$, we have*

1. *for $a = 2$,*

$$\begin{aligned} P(R_{T-1} < 2) = 1 - p^2(1 + q + pq + \ldots + &\underbrace{\ldots qpq}_{(T-3) terms}) \\ &- q^2(1 + p + qp + \ldots + \underbrace{\ldots pqp}_{(T-3) terms}). \end{aligned} \tag{2.30}$$

[1] See Definition 2.2.

2. for $a \geq 3$,

$$P(R_{T-1} < a) = 1 - p^a - q^a - \sum_{L=a+2}^{T} \sum_{i=1}^{a} \sum_{k=0}^{L-a-1} \left\{ c_{i,1}^{a,L-a-k-1} \cdot c_{1,a-2}^{a-1,a+k-3} \right.$$
$$\left. \times (pq)^{\frac{L-2}{2}} \left[p \left(\frac{p}{q} \right)^{\frac{a-i}{2}} + q \left(\frac{q}{p} \right)^{\frac{a-i}{2}} \right] \right\}. \quad (2.31)$$

Proof. Using Lemma 2.1 we have that

$$P(R_{T-1} \geq a) = P(\tau_a^R < T) = P(\tau_a^D < \tau_a^U \wedge T) + P(\tau_a^U < \tau_a^D \wedge T), \quad (2.32)$$

where the first term of the right hand side is given in Theorem 2.1 and the second term of the right hand side is given in Theorem 2.1 when p is replaced by q. □

Remark 2.1. *In the case of a symmetric random walk ($p = q = \frac{1}{2}$) we notice that we can write*

$$P(\tau_a^D < \tau_a^U \wedge T) = \frac{1}{2} P(\tau_a^R < T), \quad (2.33)$$

where τ_a^R is the first range time. It is now easy to deduce that as $T \to \infty$, (2.33) reduces to $\frac{1}{2}$ as expected.

Finally, the case of a symmetric random walk ($p = q = \frac{1}{2}$) is summarized in the following corollary for any pre-specified time horizon T.

Corollary 2.3. *Let $a, T \in \mathbb{N}^*$. For the symmetric random walk the probability that a rally of a units proceeds a drawdown of equal units before time T is given by*

1. *for $a = 1$,*

$$\wp(T; 1, \frac{1}{2}) \quad = \quad \frac{1}{2}. \tag{2.34}$$

2. *for $a = 2$,*

$$\wp(T; 2, \frac{1}{2}) \quad = \quad \frac{1}{2}\left(1 - \frac{1}{2^{T-1}}\right). \tag{2.35}$$

3. *for $a \geq 3$,*

$$\wp(T; a, \frac{1}{2}) = \frac{1}{2^a} + \frac{1}{2} \sum_{L=a+2}^{T} \sum_{i=1}^{a} \sum_{k=0}^{L-a-1} d_{i,1}^{a,L-a-k-1} \cdot d_{1,a-2}^{a-1,a+k-3}, \tag{2.36}$$

where for $m, k, i, j \in \mathbb{N}$,

$$d_{i,j}^{m,k} = \frac{1}{m+1} \sum_{\iota=1}^{m} \left(\cos \frac{\pi \iota}{m+1}\right)^k \sin \frac{i \pi \iota}{m+1} \sin \frac{j \pi \iota}{m+1}. \tag{2.37}$$

Proof. The proof is seen by substituting $p = q = \frac{1}{2}$. □

In Tables 2.1 and 2.2 we calculate the probability of (2.10) for specific values of the parameters p, a, and T. We notice that both Tables 2.1 and 2.2 increase across rows reflecting the fact that as p increases so does the probability of (2.10). On the other hand, as the threshold a increases, the

Table 2.1: The probability of (2.10) for $T = 30$.

$a \downarrow$	$p = 0.3$	$p = 0.5$	$p = 0.7$
5	0.6382	0.4684	0.0630
10	0.3772	0.1040	0.0012
20	0.0272	1.6319×10^{-4}	1.0945×10^{-8}

Table 2.2: The probability of (2.10) for $T = 50$.

$a \downarrow$	$p = 0.3$	$p = 0.5$	$p = 0.7$
5	0.6413	0.4981	0.0640
10	0.4595	0.2609	0.0023
20	0.2586	0.0064	2.3012×10^{-7}

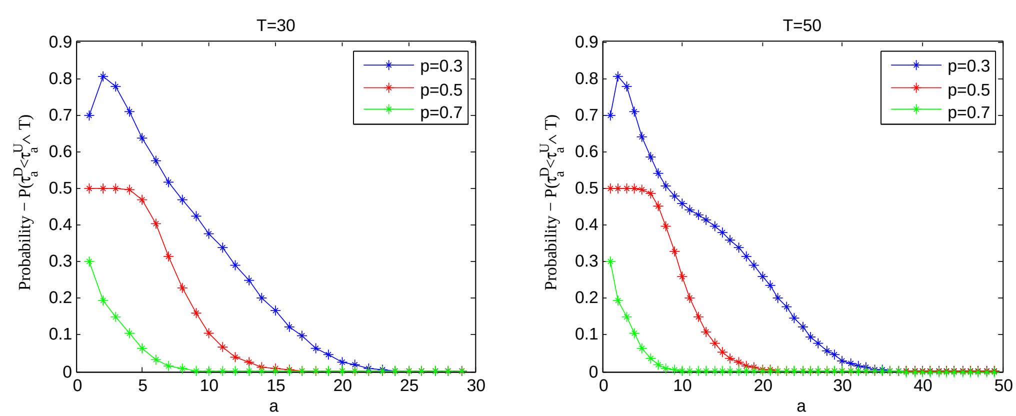


Figure 2.1: A graph of the probability of (2.10) for (Left) $T = 30$ and (Right) $T = 50$.

probability of (2.10) typically decreases. However, in the case that $p = 0.7 (> 0.5)$ the probability of (2.10) experiences a slight increase from $a = 1$ to $a = 2$ followed by a dramatic decrease. This is seen in Fig. 2.1. Finally, as the time-horizon T increases the probability of (2.10) increases as well. However, for small values of a, the increase is not as dramatic as for larger values of a.

We now proceed to the continuous time case.

2.2.2 A Brownian motion model

Let us consider the case of a continuous time Brownian motion $X_{\cdot} = \{X_t; t \geq 0\}$. In particular, let

$$dX_t = \nu dt + \sigma dW_t, \; X_0 = 0, \tag{2.38}$$

where ν is the drift coefficient and $\sigma > 0$ is the diffusion coefficient.

In the theorem that follows we compute the probability that a drawdown of a units precedes a drawup of equal units in a pre-specified finite time-horizon T.

Theorem 2.2. *Let $dX_t = \nu dt + \sigma dW_t$ be the Brownian motion with drift coefficient ν and diffusion coefficient σ. Define*

$$\wp(T; a, \nu) \triangleq P(\tau_a^D < \tau_a^U \wedge T). \tag{2.39}$$

Then,

$$\wp(T;a,\nu) = \sum_{n=1}^{\infty} \frac{2n^2\pi^2}{C_n^2} \Bigg\{ (1-(-1)^n e^{-\frac{\nu a}{\sigma^2}}) \left(1 - \frac{4\nu^2 a^2}{\sigma^4 C_n}\right) + (-1)^n \frac{\nu a}{\sigma^2} e^{-\frac{\nu a}{\sigma^2}}$$
$$- e^{-\frac{\sigma^2 C_n}{2a^2}T} \left[(1+(-1)^n e^{-\frac{\nu a}{\sigma^2}}) \left(1 + \frac{n^2\pi^2\sigma^2 T}{a^2} - \frac{4\nu^2 a^2}{\sigma^4 C_n}\right) + (-1)^n \frac{\nu a}{\sigma^2} e^{-\frac{\nu a}{\sigma^2}} \right] \Bigg\}, \tag{2.40}$$

where $C_n = n^2\pi^2 + \nu^2 a^2/\sigma^4$, $n \in \mathbb{N}$.

The proof of the above theorem makes use of the following proposition:

Proposition 2.1. *For* $t > 0$ *and* $-a < x \leq 0$, *we have*

$$P(\tau_a^D \in dt, \tau_a^U > t, X_t \in dx) = g(t,x;a,\nu)dtdx, \tag{2.41}$$

where

$$\begin{aligned} & g(t,x;a,\nu) \\ = \; & \frac{\sigma^2}{a^5} \sum_{n=1}^{\infty} n\pi e^{-\frac{\sigma^2 C_n}{2a^2}t + \frac{\nu x}{\sigma^2}} \left\{ (2a^2 - n^2\pi^2\sigma^2 t) \sin\left(\frac{n\pi x}{a}\right) + n\pi a x \cos\left(\frac{n\pi x}{a}\right) \right\}, \end{aligned} \tag{2.42}$$

with C_n , $n \in \mathbb{N}$ *defined as above.*

In order to prove Proposition 2.1 and Theorem 2.2, we will need the following lemma.

Lemma 2.4. *For any* $u \in [-a, 0)$, *the first hitting time* τ_u^X *satisfies,*

$$P(\tau_u \in dt, \sup_{s \leq t} X_s \leq u + a) = h(t,u;a,\nu)dt, \tag{2.43}$$

where

$$h(t,u;a,\nu) = \frac{1}{\sigma t^{\frac{3}{2}}} e^{\frac{\nu}{\sigma^2}u-\frac{\nu^2}{2\sigma^2}t} \sum_{k=-\infty}^{\infty} (2ka-u)\phi\left(\frac{2ka-u}{\sigma\sqrt{t}}\right) \tag{2.44}$$
$$= -\frac{\sigma^2}{a^2} e^{\frac{\nu}{\sigma^2}u-\frac{\nu^2}{2\sigma^2}t} \sum_{n=1}^{\infty}(n\pi)\exp\left(-\frac{n^2\pi^2\sigma^2}{2a^2}t\right)\sin\left(\frac{n\pi u}{a}\right).$$

Proof. The proof follows by recognizing that, $h(t,u;a,\nu)$ appears in Anderson (1960), Theorem 5.1. In particular, $h(t,u;a,\nu)$ is $dP_2(t)/dt$ of (5.3) with parameters $\gamma_1 = u/\sigma, \gamma_2 = (u+a)/\sigma$ and $\delta_1 = \delta_2 = -\nu/\sigma$. More specifically, after substitution and some algebra, we obtain

$$\frac{1}{t^{\frac{3}{2}}}\phi\left(\frac{\delta_1 t+\gamma_1}{\sqrt{t}}\right)\sum_{k=0}^{\infty} e^{-(2k/t)[(k+1)\gamma_1-k\gamma_2][\delta_1 t+\gamma_1-(\delta_2 t+\gamma_2)]}[(2k+1)\gamma_1-2k\gamma_2]$$
$$= \frac{1}{\sigma t^{\frac{3}{2}}}\exp\left(\frac{\nu}{\sigma^2}u-\frac{\nu^2}{2\sigma^2}t\right)\sum_{k=0}^{\infty}(2ka-u)\phi\left(\frac{2ka-u}{\sigma\sqrt{t}}\right),$$

while

$$\frac{1}{t^{\frac{3}{2}}}\phi\left(\frac{\delta_1 t+\gamma_1}{\sqrt{t}}\right)\sum_{k=0}^{\infty}\frac{(2k+1)\gamma_1-2(k+1)\gamma_2}{e^{[2(k+1)/t][k\gamma_1-(k+1)\gamma_2][\delta_1 t+\gamma_1-(\delta_2 t+\gamma_2)]}}$$
$$= \frac{1}{\sigma t^{\frac{3}{2}}}\exp\left(\frac{\nu}{\sigma^2}u-\frac{\nu^2}{2\sigma^2}t\right)\sum_{k=0}^{\infty}[u+2(k+1)a]\phi\left(\frac{-u-2(k+1)a}{\sigma\sqrt{t}}\right).$$

By combining the above two identities we obtain the upper expression in (2.45). The last expression in (2.45) is obtained by a Fourier transform. □

We now proceed to the proof of Proposition 2.1.

Proof of Proposition 2.1. Using Lemma 2.1, we have

$$\{\tau_a^D \in dt, \tau_a^U > t, X_t \in du\} \;=\; \{\tau_u^X \in dt, \sup_{s\in[0,t]} X_s \in a + du\}, \tag{2.45}$$

for any $u \in [-a, 0)$. It follows that

$$g(u, t; a, \nu) \;=\; \frac{\partial}{\partial a} h(t, u; a, \nu).$$

This completes the proof of Proposition 2.1. □

We can now proceed to the proof of Theorem 2.2.

Proof of Theorem 2.2. We use Proposition 2.1 to obtain

$$\wp(T; a, \nu) \;=\; \int_0^T \int_{-a}^0 P(\tau_a^D \in dt, \tau_a^U > t, X_t \in du),$$

which completes the proof of Theorem 2.2. □

In the case that an investor is not restricted by a finite time horizon, the probability that his/her wealth makes a rally of a units before a drawdown of equal units in the model of (2.38) is summarized in the following corollary. This result is easier derived by using martingale arguments (see Hadjiliadis [34], Hadjiliadis & Vecer [38]) and is displayed here for completeness.

Corollary 2.4. *In the case of an infinite time-horizon we have*

$$P(\tau_a^D < \tau_a^U) \;=\; \frac{e^{-\frac{2\nu}{\sigma^2}a} + \frac{2\nu}{\sigma^2}a - 1}{e^{-\frac{2\nu}{\sigma^2}a} + e^{\frac{2\nu}{\sigma^2}a} - 2}.$$

The next corollary draws a connection of our result to the range process of a Brownian motion.

Corollary 2.5. *Let R_t be the range process*[2] *of (2.38). Then*

$$P(R(T) \leq a) = \sum_{n=1}^{\infty} \frac{4n^2\pi^2}{C_n^2} \exp\left(-\frac{\sigma^2 C_n}{2a^2}T\right) \Big\{\big(1-(-1)^n \cosh(\nu a/\sigma^2)\big) \times \left(1+\frac{n^2\pi^2\sigma^2}{a^2}T - \frac{4\nu^2 a^2}{\sigma^4 C_n}\right) - (-1)^n \frac{\nu a}{\sigma^2}\sinh(\nu a/\sigma^2)\Big\} \quad (2.46)$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned} P(R(T) \leq a) &= P(\tau_a^R \geq T) = 1 - P(\tau_a^R < T) \\ &= 1 - P(\tau_a^D < \tau_a^U \wedge T) - P(\tau_a^U < \tau_a^D \wedge T) \\ &= 1 - \wp(T; a, \nu) - \wp(T; a, -\nu). \end{aligned} \quad (2.47)$$

The result follows from Theorem 2.2. □

The result in Corollary 2.5 is also seen in Tanré and Vallois [77].

The case of a Brownian motion without a drift is summarized in the following corollary:

Corollary 2.6.

$$\wp(T; a, 0) = \frac{1}{2} - \sum_{n\geq 1, odd} \frac{4}{n^2\pi^2} e^{-\frac{n^2\pi^2\sigma^2}{2a^2}T} \cdot \left(1 + \frac{n^2\pi^2\sigma^2}{a^2}T\right). \quad (2.48)$$

[2]See Definition 2.2

We notice that (2.48) of Corollary 2.6 reduces to $\frac{1}{2}$ as $T \to \infty$ as expected.

Let us now proceed to treat the general linear diffusion dynamics and the cases of $a \neq b$.

2.3 General Cases: Path Decomposition and Laplace Transform

In this section we extend the results in the previous section in two directions. First, we consider more general diffusion models. Second, we will treat the general cases in which the thresholds of the drawdown and the drawup are different, i.e. the cases $a \neq b$. In order to address the above generalizations, we adopt the Laplace transform approach.

Let $I = (l, r)$ be a non-empty open interval of the real line. Consider a linear diffusion $X_\cdot = \{X_t; t \geq 0\}$ on I with continuous infinitesimal parameters and natural (or entrance[3]) boundaries (see, for example, Itô & McKean [42]). Its evolution is governed by the equation

$$dX_t \quad = \quad \mu(X_t)dt + \sigma(X_t)dB_t, \;\; X_0 = x \in I, \tag{2.49}$$

on a filtered probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{F} = \{\mathcal{F}_t\}$. The process $B_\cdot$ is a standard Brownian motion with respect to $\mathcal{F}$, and $(\mu(\cdot), \sigma^2(\cdot))$ is a pair of real-valued continuous functions. To reflect the dependence of probability

[3]The process can start from l or/and r if they are entrance boundaries.

measure on the initial value x, we define $P_x(\cdot) = P(\cdot|X_0 = x)$.

In the following sections, we derive the main results in this paper. We need the following fundamental lemma to finish the proofs.

Lemma 2.5. *For $y \leq x \leq z$ and $\lambda \geq 0$, define*

$$E_x\Big\{e^{-\lambda\tau_y^X} \cdot \mathbb{1}_{\{\tau_y^X < \tau_z^X\}}\Big\} \;\triangleq\; \ell_x^{X,\lambda}(y,z). \tag{2.50}$$

Then

$$\ell_x^{X,\lambda}(y,z) \;=\; \frac{g^\lambda(x)h^\lambda(z) - g^\lambda(z)h^\lambda(x)}{g^\lambda(y)h^\lambda(z) - g^\lambda(z)h^\lambda(y)} \tag{2.51}$$

with $g^\lambda(\cdot)$ and $h^\lambda(\cdot)$ being any two independent solutions of the ordinary differential equation

$$\frac{1}{2}\sigma^2(u)\frac{\partial^2 f}{\partial u^2} + \mu(u)\frac{\partial f}{\partial u} \;=\; \lambda f. \tag{2.52}$$

Proof. See Lehoczky [49], page 603. □

In the following sections we derive formulas for the Laplace transform of the probability density of the drawdown of a units when it precedes the drawup of b units, for any $a, b > 0$ satisfying $x \pm a, x \pm b \in I$. We define

$$E_x\Big\{e^{-\lambda\tau_a^D} \cdot \mathbb{1}_{\{\tau_a^D < \tau_b^U\}}\Big\} \;\triangleq\; \int_0^\infty e^{-\lambda t} P_x(\tau_a^D \in dt, \tau_b^U > t). \tag{2.53}$$

In the sequel we denote the Laplace transform (2.53) by $L_x^{X,\lambda}(a,b)$.

2.3.1 The case of $a = b$

We determine $L_x^{X,\lambda}(a,a)$ for $a > 0$ in this paragraph.

Theorem 2.3. *For $a > 0$ and $\lambda > 0$, we have*

$$L_x^{X,\lambda}(a,a) \;=\; \int_{x-a}^{x} \frac{\partial}{\partial a} \ell_x^{X,\lambda}(u, u+a)du. \tag{2.54}$$

Proof. Using Lemma 2.1, we have that for $t > 0$ and $a > 0$,

$$\{\tau_a^D \in dt, \tau_a^U > t\} \;=\; \{\tau_a^R \in dt, x-a < X_t < x\}. \tag{2.55}$$

Following the proof of Proposition 2.1, we have

$$P_x(\tau_a^D \in dt, \tau_a^U > t) \;=\; \int_{x-a}^{x} \frac{\partial}{\partial a} P_x(\tau_u^X \in dt, \tau_{u+a}^X > t)du, \tag{2.56}$$

which implies (2.54) and completes the proof of the theorem. $\square$

2.3.2 The case of $a > b$

We determine $L_x^{X,\lambda}(a,b)$ for $a > b > 0$ in this paragraph. To prove the main result we need the following proposition.

Proposition 2.2. *For $b > 0$, $c < u$ such that $c, u+b \in I$, and $\lambda > 0$, define*

$$H_u^{X,\lambda}(b,c) \;\triangleq\; E_u\big\{e^{-\lambda\tau_c^X} \cdot 1\!\!1_{\{\tau_c^X < \tau_b^U\}}\big\}. \tag{2.57}$$

Then

$$H_u^{X,\lambda}(b,c) \;=\; \exp\bigg(\int_c^u \frac{\partial}{\partial w}\bigg|_{w=v} \ell_w^{X,\lambda}(v, v+b)dv\bigg). \tag{2.58}$$

Proof. We follow the idea of Hadjiliadis, Hernandez & Stamos [35], Lehoczky [49], and partition the interval $[c, u]$ into k subintervals $\{[v_i, v_{i-1}]; 1 \leq i \leq k\}$ with $u = v_0 > v_1 > \ldots > v_k = c$. Let $\Delta_k = \max_{1 \leq i \leq k}(v_{i-1} - v_i)$ and assume $\Delta_k \to 0$ as $k \to \infty$. As a discrete approximation to $H_u^{X,\lambda}(b, c)$ defined by (2.57), compute

$$
\begin{aligned}
& E_u\Big\{e^{-\lambda \sum_{i=1}^{k}(\tau_{v_i}^X - \tau_{v_{i-1}}^X)} \cdot \mathbb{1}_{\left\{\text{after } \tau_{v_{i-1}}^X, X_t \text{ hits } v_i \text{ before increasing to } v_i+b, 1\leq i\leq k\right\}}\Big\} \\
= \; & \prod_{i=1}^{k} E_{v_{i-1}}\Big\{e^{-\lambda \tau_{v_i}^X} \cdot \mathbb{1}_{\{\tau_{v_i}^X < \tau_{v_i+b}^X\}}\Big\},
\end{aligned}
$$

where the last equality follows from the strong Markov property and the continuity of paths.

It will be shown that as $k \to \infty$ and $\Delta_k \to 0$, the limit of the above expression exists and does not depend on the particular sequence of partition chosen. Moreover, let

$$
Y_k \;=\; e^{-\lambda \sum_{i=1}^{k}(\tau_{v_i}^X - \tau_{v_{i-1}}^X)} \cdot \mathbb{1}_{\left\{\text{after } \tau_{v_{i-1}}^X, X_t \text{ hits } v_i \text{ before increasing to } v_i+b, 1\leq i\leq k\right\}}.
$$

By the continuity of paths, we have $Y_k \to e^{-\lambda \tau_c^X}\, \mathbb{1}_{\{\tau_c^X < \tau_b^U\}}$, *a.s.*. Furthermore, it is also the case that $|Y_k| \leq 1$. Therefore, by the Lebesgue dominated convergence theorem, $\lim_{k\to\infty} E_u[Y_k] = E_u[\lim_{k\to\infty} Y_k] = H_u^{X,\lambda}(b, c)$.
By Lemma 2.5,

$$
\prod_{i=1}^{k} E_{v_{i-1}}\Big\{e^{-\lambda \tau_{v_i}^X} \cdot \mathbb{1}_{\left\{\tau_{v_i}^X < \tau_{v_i+b}^X\right\}}\Big\} \;=\; \prod_{i=1}^{k} \ell_{v_{i-1}}^{X,\lambda}(v_i, v_i + b).
$$

Taking log gives us

$$\begin{aligned}
& \sum_{i=1}^{k} \log \ell_{v_{i-1}}^{X,\lambda}(v_i, v_i+b) \\
= \quad & \sum_{i=1}^{k} \log \left(\ell_{v_i}^{X,\lambda}(v_i, v_i+b) + \ell_{v_{i-1}}^{X,\lambda}(v_i, v_i+b) - \ell_{v_i}^{X,\lambda}(v_i, v_i+b) \right) \\
= \quad & \sum_{i=1}^{k} \log \left(1 + \ell_{v_{i-1}}^{X,\lambda}(v_i, v_i+b) - \ell_{v_i}^{X,\lambda}(v_i, v_i+b) \right) \\
= \quad & \sum_{i=1}^{k} \frac{\partial}{\partial w}\bigg|_{w=v_i} \ell_{w}^{X,\lambda}(v_i, v_i+b) \cdot (v_{i-1} - v_i) + O(\Delta_k) \\
\to \quad & \int_c^u \frac{\partial}{\partial w}\bigg|_{w=v} \ell_{w}^{X,\lambda}(v, v+b) dv, \qquad \text{as } \Delta_k \to 0^+,
\end{aligned}$$

from which we obtain

$$H_u^{X,\lambda}(b,c) \quad = \quad \exp\left(\int_c^u \frac{\partial}{\partial w}\bigg|_{w=v} \ell_{w}^{X,\lambda}(v, v+b) dv \right).$$

This completes the proof of Proposition 1. □

Remark 2.2. *In the case that $X. = B.$ is a Brownian motion, then Proposition 2.2 is related to a Laplace transform of the inverse of local time. More specifically, from Lévy isomorphism, the pair (M_t, D_t) has the same law as the pair $(L_t, |B_t|)$, where L_t is the local time of Brownian motion $B.$ at zero. We thus have*

Corollary 2.7. *Let $B. = \{B_t; t \geq 0\}$ be a standard Brownian motion starting at zero, and $\varrho(t) \stackrel{\triangle}{=} \inf\{s \geq 0 | L_s \geq t\}$ be the inverse of the local time L_t, then*

$$E_0\left\{ e^{-\lambda \varrho(t)} \cdot \mathbb{1}_{\{\sup_{s\in[0,\varrho(t)]} |B_s| < b\}} \right\} \quad = \quad H_0^{B,\lambda}(b, -t). \tag{2.59}$$

Now let us state and prove the main result in this paragraph.

Theorem 2.4. *For $a > b > 0$ and $\lambda > 0$, we have*

$$L_x^{X,\lambda}(a,b) = \int_{x-b}^{x} \frac{\partial}{\partial b} \ell_x^{X,\lambda}(u,u+b) \cdot H_u^{X,\lambda}(b,u-a+b)du. \quad (2.60)$$

Proof. Any path in the event $\{\tau_a^D < \tau_b^U\}$ has the decomposition

1. $\{X_t; 0 \le t \le \tau_b^D\}$;
2. $\{X_{t+\tau_b^D}; 0 \le t \le \tau_a^D - \tau_b^D\}$.

Conditioning on $\{X_{\tau_b^D} = u\}$, the process in 2 starts at u, and decreases to $u-a+b$ before it incurs the drawup of b units occurs. This gives rise to the representation

$$\tau_a^D = \tau_b^D + \tau_{u-a+b}^X \circ \theta_{\tau_b^D}. \quad (2.61)$$

Therefore, for $x - b < u < x$,

$$\begin{aligned} & e^{-\lambda\tau_a^D} \cdot \mathbb{1}_{\left\{\tau_a^D<\tau_b^U, X_{\tau_b^D}\in du\right\}} \\ = \; & e^{-\lambda[\tau_b^D+\tau_{u-a+b}\cdot\theta_{\tau_b^D}]} \cdot \mathbb{1}_{\left\{\tau_a^D<\tau_b^U, X_{\tau_b^D}\in du\right\}} \\ = \; & \underbrace{e^{-\lambda\tau_b^D} \cdot \mathbb{1}_{\{\tau_b^D<\tau_b^U, X_{\tau_b^D}\in du\}}}_{\text{before } \tau_b^D} \cdot \underbrace{e^{-\lambda\tau_{u-a+b}\circ\theta_{\tau_b^D}} \cdot \mathbb{1}_{\left\{\tau_{u-a+b}^X\circ\theta_{\tau_b^D}<\tau_b^U\circ\theta_{\tau_b^D}\right\}}}_{\text{after } \tau_b^D}. \end{aligned} \quad (2.62)$$

To get the expectation of the above expression under E_x, we first compute its conditional expectation given $\{X_{\tau_b^D} = u\}$. By the strong Markov property, the factor "before τ_b^D" is $\mathcal{F}_{\tau_b^D}$-measurable, and the factor "after τ_b^D" has conditional expectation

$$\begin{aligned} & E_x\Big\{e^{-\lambda\tau^X_{u-a+b}\circ\theta_{\tau_b^D}} \cdot \mathbb{1}_{\{\tau^X_{u-a+b}\circ\theta_{\tau_b^D}<\tau_b^U\circ\theta_{\tau_b^D}\}} \mid X_{\tau_b^D}=u\Big\} \\ = \;& E_u\Big\{e^{-\lambda\tau^X_{u-a+b}} \cdot \mathbb{1}_{\{\tau^X_{u-a+b}<\tau_b^U\}}\Big\}, \end{aligned}$$

which, by Proposition 1, is equal to $H_u^{X,\lambda}(b, u-a+b)$. Taking the expectation of (2.62) under E_x, and using (2.3), we obtain

$$\begin{aligned} & E_x\Big\{e^{-\lambda\tau_a^D} \cdot \mathbb{1}_{\{\tau_a^D<\tau_b^U, X_{\tau_b^D}\in du\}}\Big\} && (2.63) \\ = \;& E_x\Big\{e^{-\lambda\tau_b^D} \cdot \mathbb{1}_{\{\tau_b^D<\tau_b^U, X_{\tau_b^D}\in du\}}\Big\} \cdot H_u^{X,\lambda}(b, u-a+b) \\ = \;& \frac{\partial}{\partial b}\ell_x^{X,\lambda}(u, u+b) \cdot H_u^{X,\lambda}(b, u-a+b)du. \end{aligned}$$

The integration of the above identity over the interval $(x-b, x)$ in u yields (2.60) and completes the proof of Theorem 2.4. □

2.3.3 The case of $b > a$

We determine $L_x^{X,\lambda}(a, b)$ for $b > a > 0$ in this paragraph. To prove the main result we need the following proposition.

Proposition 2.3. *For any $a > 0$ and $x \in I$ satisfying $x - a \in I$, and $\lambda > 0$, define*

$$J_x^{X,\lambda}(a) \;\triangleq\; E_x e^{-\lambda\tau_a^D}. \qquad (2.64)$$

Then

$$J_x^{X,\lambda}(a) = -\int_x^r \frac{\partial}{\partial w}\Big|_{w=u} \ell_w^{X,\lambda}(u-a,u) \cdot e^{-\int_x^u \frac{\partial}{\partial w}\big|_{w=v} \ell_w^{X,\lambda}(v,v-a)dv} du.$$

Proof. See Lehoczky [49], page 602. □

Now let us state and prove the main result in this paragraph.

Theorem 2.5. *For $b > a > 0$ and $\lambda > 0$, we have*

$$\begin{aligned} L_x^{X,\lambda}(a,b) &= J_x^{X,\lambda}(a) - \int_x^{x+a} dv \frac{\partial}{\partial a} \ell_x^{2x-X,\lambda}(2x-v, 2x-v+a) \\ &\times H_{2x-v}^{2x-X,\lambda}(a, 2x-v-b+a) \cdot J_{v+b-a}^{X,\lambda}(a). \end{aligned} \tag{2.65}$$

Proof. First, it is easily seen that for $b \geq a > 0$,

$$L_x^{X,\lambda}(a,b) = J_x^{X,\lambda}(a) - E_x\big\{e^{-\lambda\tau_a^D} \cdot 1\!\!1_{\{\tau_a^D > \tau_b^U\}}\big\}.$$

Therefore, to prove (2.65), it suffices to show that

$$\begin{aligned} E_x\big\{e^{-\lambda\tau_a^D} \cdot 1\!\!1_{\{\tau_a^D > \tau_b^U\}}\big\} = &\int_x^{x+a} dv \frac{\partial}{\partial a} \ell_x^{2x-X,\lambda}(2x-v, 2x-v+a) \\ &\times H_{2x-v}^{2x-X,\lambda}(a, 2x-v-b+a) \cdot J_{v+b-a}^{X,\lambda}(a). \end{aligned} \tag{2.66}$$

Consider the path decomposition for any path in the event $\{\tau_a^D > \tau_b^U\}$. We have

1. $\{X_t; 0 \leq t \leq \tau_b^U\}$;

2. $\{X_{t+\tau_b^U}; 0 \le t \le \tau_a^D - \tau_b^U\}$.

Intuitively, before time τ_b^U, the process experiences no drawdown of a units and the first drawup of b units occurs at τ_b^U, when the process also reaches a new maximum; thereafter, the process has a drawdown of a units at time τ_a^D. Thus for any path in the event $\{\tau_a^D > \tau_b^U\}$ we have

$$\tau_a^D \;=\; \tau_b^U + \tau_a^D \circ \theta_{\tau_b^U}. \tag{2.67}$$

Therefore, for $b \ge a$ and $x < v < x + a$,

$$\begin{aligned}
& E_x\Big\{e^{-\lambda\tau_a^D} \cdot \mathbb{1}_{\{\tau_a^D > \tau_b^U, X_{\tau_a^U} \in dv\}}\Big\} \qquad (2.68)\\
=\;& E_x\Big\{e^{-\lambda\tau_b^U + \tau_a^D \circ \theta_{\tau_b^U}} \cdot \mathbb{1}_{\{\tau_a^D > \tau_b^U, X_{\tau_a^U} \in dv\}}\Big\}\\
=\;& E_x\Big\{\underbrace{e^{-\lambda\tau_b^U} \cdot \mathbb{1}_{\{\tau_a^D > \tau_b^U, X_{\tau_a^U} \in dv\}}}_{\text{before } \tau_b^U} \times \underbrace{e^{-\lambda\tau_a^D \circ \theta_{\tau_b^U}}}_{\text{after } \tau_b^U}\Big\}\\
=\;& E_x\Big\{e^{-\lambda\tau_b^U} \cdot \mathbb{1}_{\{\tau_a^D > \tau_b^U, X_{\tau_a^U} \in dv\}} E_{v+b-a}\Big\{e^{-\lambda\tau_a^D}\Big\}\Big\}\\
=\;& E_x\Big\{e^{-\lambda\tau_b^U} \cdot \mathbb{1}_{\{\tau_a^D > \tau_b^U, X_{\tau_a^U} \in dv\}} \cdot J_{v+b-a}^{X,\lambda}(a)\Big\}\\
=\;& E_x\Big\{e^{-\lambda\tau_b^U} \cdot \mathbb{1}_{\{\tau_a^D > \tau_b^U, X_{\tau_a^U} \in dv\}}\Big\} \cdot J_{v+b-a}^{X,\lambda}(a),
\end{aligned}$$

where the third equality follows from the strong Markov property. The expectation in the last line can be determined as follows. Note that for the process $\{Y_t = 2x - X_t; t \ge 0\}$,

$$dY_t \;=\; -\mu(2x - Y_t)dt + \sigma(2x - Y_t)dB_t^{'},\; Y_0 = x,$$

with $B'_t = -B_t$, the vector of random variables $\left(T_U^Y(a), T_D^Y(b), 2x - Y_{T_D^Y(a)}\right)^4$ has the same law as the vector of random variables $\left(\tau_a^D, \tau_b^U, X_{\tau_a^U}\right)$ for X. under P_x. So we know from (2.63) that

$$
\begin{aligned}
& E_x\Big\{e^{-\lambda\tau_b^U} \cdot 1\!\!1_{\{\tau_a^D > \tau_b^U, X_{\tau_a^U} \in dv\}}\Big\} \qquad (2.69)\\
= \;& E_x\Big\{e^{-\lambda T_D^Y(b)} \cdot 1\!\!1_{\{T_D^Y(b) < T_U^Y(a), Y_{T_D^Y(a)} \in 2x - dv\}}\Big\} \\
= \;& \frac{\partial}{\partial a} \ell_x^{2x-X,\lambda}(2x - v, 2x - v + a) \cdot H_{2x-v}^{2x-X,\lambda}(a, 2x - v - b + a)dv.
\end{aligned}
$$

The integration of (2.68) over the interval $(x, x + a)$ in v yields (2.66) and completes the proof of Theorem 2.5. □

We now proceed to treat the special case of a drifted Brownian motion model.

2.4 Brownian Motion Revisited

In theorem 2.2, we have studied the drifted Brownian motion in the case $a = b$. In this section, we apply the more general results in Theorem 2.3, Theorem 2.4 and Theorem 2.5 to a drifted Brownian motion model and calculate the probability density of the drawdown of a units when it precedes the drawup of b units.

[4] $T_D^Y(b)$ and $T_U^Y(a)$ are the drawdown and drawup of the process $\{Y_t; t \geq 0\}$ respectively.

First it is easily seen that $I = (-\infty, \infty)$ in a drifted Brownian motion model. The function $\ell_x^{X,\lambda}(y,z)$ for $X_t = x + \mu t + \sigma W_t$ can be found in Borodin & Salminen [12] page 295:

$$\ell_x^{X,\lambda}(y,z) = \frac{\sinh[(z-x)S_{\mu,\sigma}^{\lambda}]}{\sinh[(z-y)S_{\mu,\sigma}^{\lambda}]} e^{\frac{\mu(y-x)}{\sigma^2}}, \tag{2.70}$$

where $S_{\mu,\sigma}^{\lambda} = \sqrt{(2\lambda/\sigma^2) + (\mu^2/\sigma^4)}$. Thus the Laplace transforms in Theorem 2.3, Theorem 2.4 and Theorem 2.5 can be calculated explicitly as:

1. $a = b > 0$:

$$L_0^{X,\lambda}(a) = \frac{S_{\mu,\sigma}^{\lambda}}{(2\lambda/\sigma^2)} \left\{ \frac{e^{-\frac{\mu a}{\sigma^2}} \left(S_{\mu,\sigma}^{\lambda} \coth[aS_{\mu,\sigma}^{\lambda}] + \frac{\mu}{\sigma^2} \right)}{\sinh[aS_{\mu,\sigma}^{\lambda}]} - \frac{S_{\mu,\sigma}^{\lambda}}{\sinh^2[aS_{\mu,\sigma}^{\lambda}]} \right\}; \tag{2.71}$$

2. $a > b > 0$:

$$L_0^{X,\lambda}(a,b) = L_0^{X,\lambda}(b,b) \cdot \exp\left(T_{\mu,\sigma}^{\lambda}(b)(a-b) \right), \tag{2.72}$$

where

$$T_{\mu,\sigma}^{\lambda}(b) = -\frac{\mu}{\sigma^2} - S_{\mu,\sigma}^{\lambda} \coth[bS_{\mu,\sigma}^{\lambda}]; \tag{2.73}$$

3. $b > a > 0$:

$$L_0^{X,\lambda}(a,b) = \left(1 - L_0^{-X,\lambda}(a,a) \cdot e^{T_{-\mu,\sigma}^{\lambda}(a)(b-a)} \right) \cdot J_0^{X,\lambda}(a), \tag{2.74}$$

where

$$\begin{aligned} J_0^{X,\lambda}(a) &= \frac{S_{\mu,\sigma}^{\lambda} e^{-\frac{\mu a}{\sigma^2}}}{S_{\mu,\sigma}^{\lambda} \cosh[aS_{\mu,\sigma}^{\lambda}] - (\mu/\sigma^2)\sinh[aS_{\mu,\sigma}^{\lambda}]} \\ &= -\frac{S_{\mu,\sigma}^{\lambda}(a) e^{-\frac{\mu a}{\sigma^2}}}{\sinh[aS_{\mu,\sigma}^{\lambda}(a)]} \cdot \frac{1}{T_{-\mu,\sigma}^{\lambda}(a)}. \end{aligned} \tag{2.75}$$

One can easily obtain several known results from (2.71), (2.72) and (2.74). First, by letting $\lambda \to 0^+$, the formulas coincide with the probability results in Hadjiliadis & Vecer [38]. Second, by letting $b \to \infty$ in (2.74), one obtains the Laplace transform of τ_a^D, $J_0^{X,\lambda}(a)$.

Moreover, we can invert (2.72) analytically to obtain the density $P(\tau_a^D \in dt, \tau_b^U > t)$ for any $a \geq b > 0$. In fact we have

Theorem 2.6. *Define* $p^{(\mu)}(t;a,b)dt = P(\tau_a^D \in dt, \tau_b^U > t)$ *for* $a \geq b > 0$, *then*

$$
\begin{aligned}
p^{(\mu)}(t;a,b) \quad = \quad & \frac{2}{t} e^{-\frac{\mu^2 t}{2\sigma^2} - \frac{\mu(a-b)}{\sigma^2}} \sum_{m,n=0}^{\infty} \frac{(m+n+1)!}{(m+1)!m!n!} \left(\frac{2(a-b)}{\sigma\sqrt{t}} \right)^m \\
& \times \left\{ 2F_{m,n}^{(1)}(t) - e^{-\frac{\mu b}{\sigma^2}} F_{m,n}^{(2)}(t) - \delta_n e^{-\frac{\mu b}{\sigma^2}} F_m^{(3)}(t) \right\} \\
& + \frac{2\mu^2}{\sigma^2} e^{-\frac{\mu(a-b)}{\sigma^2}} \sum_{m,n=0}^{\infty} \frac{(m+n+1)!}{(m+1)!m!n!} \left(\frac{2\mu(a-b)}{\sigma^2} \right)^m \times \\
& \left\{ G_{m+n+\frac{1}{2}}^{(\mu)}(t) + (-1)^m G_{m+n+\frac{1}{2}}^{(-\mu)}(t) - e^{-\frac{\mu b}{\sigma^2}} G_{m+n+1}^{(\mu)}(t) \right. \\
& \left. -(-1)^m e^{-\frac{\mu b}{\sigma^2}} G_{m+n}^{(-\mu)}(t) \right\}, \qquad (2.76)
\end{aligned}
$$

where δ_n is the Kronecker delta and

$$\begin{aligned}
F^{(1)}_{m,n}(t) =& \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \left[\frac{\mu\sqrt{t}}{\sigma}\right]^{2k} \phi^{(m+1-2k)}\left(\frac{(2m+2n+1)b+a}{\sigma\sqrt{t}}\right) \\
F^{(2)}_{m,n}(t) =& \sum_{k=0}^{m+1} \left[\frac{\mu\sqrt{t}}{\sigma}\right]^{k} \left[1+(-1)^k \frac{m+n+2}{n+1}\right] \phi^{(m+1-k)}\left(\frac{2(m+n+1)b+a}{\sigma\sqrt{t}}\right) \\
F^{(3)}_{m}(t) =& \sum_{k=0}^{m+1} \left[-\frac{\mu\sqrt{t}}{\sigma}\right]^{k} \phi^{(m+1-k)}\left(\frac{2mb+a}{\sigma\sqrt{t}}\right) \\
G^{(\mu)}_{m}(t) =& e^{\frac{\mu(2mb+a)}{\sigma^2}} \Phi\left(\frac{2mb+a+\mu t}{\sigma\sqrt{t}}\right),
\end{aligned}$$

with ϕ and Φ being the standard normal probability density and cumulative distribution respectively. $\phi^{(k)}$ is the k-th derivative of ϕ.

Proof. We start by rewriting (2.72) in a more tractable way,

$$L^{X,\lambda}_0(a,b) \quad = \quad \int_{-b}^{0} due^{\frac{\mu u}{\sigma^2}} \frac{S^\lambda_{\mu,\sigma} \sinh[(-u)S^\lambda_{\mu,\sigma}]}{\sinh^2[bS^\lambda_{\mu,\sigma}]} \cdot \exp\left(T^\lambda_{\mu,\sigma}(b)(a-b)\right). \quad (2.77)$$

By using the first formula on page 643 of Borodin & Salminen [12], in their notation, we obtain the inverse Laplace transform of the integrand in (2.77)

$$\frac{\sigma^2}{2} e^{-\frac{\mu^2 t}{2\sigma^2} + \frac{\mu(u+b-a)}{\sigma^2}} [es_{\sigma^2 t}(1,2,b,u,a-b) - es_{\sigma^2 t}(1,2,b,-u,a-b)]. \quad (2.78)$$

After some simple manipulation, the above expression becomes

$$\frac{2e^{-\frac{\mu^2 t}{2\sigma^2} + \frac{\mu(u+b-a)}{\sigma^2}}}{\sigma\sqrt{t^3}} \sum_{m,n=0}^{\infty} \frac{(m+n+1)!}{(m+1)!m!n!} \left(\frac{2(a-b)}{\sigma\sqrt{t}}\right)^m \times$$
$$\left\{\phi^{(m+2)}\left(\frac{(2(m+n)+1)b+a+u}{\sigma\sqrt{t}}\right) - \phi^{(m+2)}\left(\frac{(2(m+n)+1)b+a-u}{\sigma\sqrt{t}}\right)\right\}.$$

Formula (2.76) follows from integration of the above expression over $(-b, 0)$ in u. $\square$

One can let $a = b$ in (2.78) to get a similar joint probability density as that in Proposition 2.1.

Moreover, for $a < b$ observe that

$$P_x(\tau_a^D \in dt, \tau_a^D > \tau_b^U) \quad = \quad P_x(\tau_a^D \in dt, \sup_{s \leq \tau_a^D} U_s \geq b),$$

and, hence, the interest is focused on the computation of the joint density

$$P(\tau_a^D \in dt, \sup_{s \leq t} U_s \in a + dz) \quad = \quad \frac{\partial}{\partial z} p^{(\mu)}(t; a, a+z) dt dz \qquad \forall a, z > 0.$$

In particular, we have

Theorem 2.7. *For a Brownian motion with constant drift μ and constant volatility σ, any $a, z > 0$, we have*

$$\begin{aligned} \frac{\partial}{\partial z} p^{(\mu)}(t; a, a+z) \quad = \quad & -\frac{4e^{-\frac{\mu^2 t}{2\sigma^2} + \frac{\mu(z-a)}{\sigma^2}}}{\sigma\sqrt{t^3}} \sum_{m,n=0}^{\infty} \frac{(m+n+2)!}{(m+2)!m!n!} \left(\frac{2z}{\sigma\sqrt{t}}\right)^m \\ & \times \left\{ 2F_{m,n}^{(1)}(t,z) - e^{\frac{\mu a}{\sigma^2}} F_{m,n}^{(2)}(t,z) - \delta_n e^{\frac{\mu a}{\sigma^2}} F_m^{(3)}(t,z) \right\} \\ & -\frac{4\mu^3}{\sigma^4} e^{\frac{\mu(z-a)}{\sigma^2}} \sum_{m,n=0}^{\infty} \frac{(m+n+2)!}{(m+2)!m!n!} \left(\frac{2\mu z}{\sigma^2}\right)^m \times \\ & \left\{ G_{m+n+\frac{1}{2}}^{(\mu)}(t,z) - (-1)^m G_{m+n+\frac{1}{2}}^{(-\mu)}(t,z) - e^{\frac{\mu a}{\sigma^2}} G_{m+n}^{(\mu)}(t,z) \right. \\ & \left. +(-1)^m e^{\frac{\mu a}{\sigma^2}} G_{m+n+1}^{(-\mu)}(t,z) \right\}, \end{aligned} \tag{2.79}$$

where

$$
\begin{aligned}
F^{(1)}_{m,n}(t,z) =& \sum_{k=0}^{\lfloor\frac{m+2}{2}\rfloor}\left[\frac{\mu\sqrt{t}}{\sigma}\right]^{2k}\phi^{(m+2-2k)}\left(\frac{(2m+2n+3)a+z}{\sigma\sqrt{t}}\right)\\
F^{(2)}_{m,n}(t,z) =& \sum_{k=0}^{m+2}\left[\frac{\mu\sqrt{t}}{\sigma}\right]^{k}\left[(-1)^k+\frac{m+n+3}{n+1}\right]\phi^{(m+2-k)}\left(\frac{2(m+n+2)a+z}{\sigma\sqrt{t}}\right)\\
F^{(3)}_{m}(t,z) =& \sum_{k=0}^{m+2}\left[\frac{\mu\sqrt{t}}{\sigma}\right]^{k}\phi^{(m+2-k)}\left(\frac{(2m+2)a+z}{\sigma\sqrt{t}}\right)\\
G^{(\mu)}_{m}(t,z) =& e^{\frac{\mu[2(m+1)a+z]}{\sigma^2}}\Phi\left(\frac{2(m+1)a+z+\mu t}{\sigma\sqrt{t}}\right).
\end{aligned}
$$

Proof. We start from the equality

$$
\begin{aligned}
& L_0^{X,\lambda}(a,b)\\
&= J_0^{X,\lambda}(a) - L_0^{-X,\lambda}(a,a)e^{T^{\lambda}_{-\mu,\sigma}(a)(b-a)}J_0^{X,\lambda}(a)\\
&= J_0^{X,\lambda}(a) - L_0^{-X,\lambda}(a)J_0^{X,\lambda}(a) + \frac{S^{\lambda}_{\mu,\sigma}(a)L_0^{-X,\lambda}(a)}{\sinh[aS^{\lambda}_{\mu,\sigma}(a)]e^{\frac{\mu a}{\sigma^2}}}\int_0^{b-a}e^{T^{\lambda}_{-\mu,\sigma}(a)z}dz\\
&= L_0^{X,\lambda}(a) + \int_{-a}^{0}du\int_0^{b-a}dz\frac{[S^{\lambda}_{\mu,\sigma}(a)]^2\sinh[(-u)S^{\lambda}_{\mu,\sigma}(a)]}{\sinh^3[aS^{\lambda}_{\mu,\sigma}(a)]e^{\frac{\mu(u+a)}{\sigma^2}}}e^{T^{\lambda}_{-\mu,\sigma}(a)z},
\end{aligned}
$$

By using the first formula on page 643 of Borodin & Salminen [12], the integrand in the last line has inverse Laplace transform

$$
\frac{\sigma^2}{2}e^{-\frac{\mu^2 t}{2\sigma^2}-\frac{\mu(u-z+a)}{\sigma^2}}[es_{\sigma^2 t}(2,3,a,u,z)-es_{\sigma^2 t}(2,3,a,-u,z)]. \tag{2.80}
$$

After some simple manipulation, the above expression becomes

$$
\frac{4e^{-\frac{\mu^2 t}{2\sigma^2}-\frac{\mu(u-z+a)}{\sigma^2}}}{\sigma^2 t^2}\sum_{m,n=0}^{\infty}\frac{(m+n+2)!}{(m+2)!m!n!}\left(\frac{2z}{\sigma\sqrt{t}}\right)^m\times
$$

$$
\left\{\phi^{(m+3)}\left(\frac{(2m+2n+3)a+z-u}{\sigma\sqrt{t}}\right)-\phi^{(m+3)}\left(\frac{(2m+2n+3)a+z+u}{\sigma\sqrt{t}}\right)\right\}.
$$

The integration of the above expression over $(-a, 0)$ in u yields (2.79) and completes the proof. □

2.5 Application

In this section we present two applications of the results in previous sections in finance and in the problem of quickest detection.

2.5.1 Relative drawdowns and relative drawups of stock prices

Consider the case of a stock with geometric Brownian motion dynamics under a probability measure P:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \; S_0 = 1. \tag{2.81}$$

Using Theorem 2.6 and Theorem 2.7, we are in the position to address the following question:

What is the probability that this stock would drop by $(100\times\boldsymbol{\alpha})$% from its historical high before it incurs a rise of $(100 \times \boldsymbol{\beta})$% from its historical low in a pre-specified plan horizon T?

First observe that

$$d\log S_t = \nu dt + \sigma dW_t, \; \log S_0 = 0, \tag{2.82}$$

where $\nu = \mu - \frac{1}{2}\sigma^2$ represents the logarithm of the return of the stock.

We let $U_D(\alpha)$ be the first time the stock drops by $(100 \times \alpha)\%$ from its historical high and $U_R(\beta)$ the first time that the stock rises by an amount equal to $(100 \times \beta)\%$ from its historical low. That is,

$$\begin{aligned} U_D(\alpha) &= \inf\{t \geq 0 |\, S_t = (1-\alpha) \times \sup_{s \in [0,t]} S_s\}, \qquad (2.83) \\ U_R(\beta) &= \inf\{t \geq 0 |\, S_t = (1+\beta) \times \inf_{s \in [0,t]} S_s\}. \qquad (2.84) \end{aligned}$$

Thus, it is possible to calculate the exact expression for the probability that a percentage relative drop of $(100 \times \alpha)\%$ precedes a relative rise of $(100 \times \beta)\%$ by noticing that

$$\begin{cases} U_D(\alpha) &= \tau^D_{-\log(1-\alpha)} \\ U_R(\beta) &= \tau^U_{\log(1+\beta)} \end{cases}. \qquad (2.85)$$

And this probability can be calculated explicitly as

$$P(U_D(\alpha) < U_R(\beta) \wedge T) \;=\; \int_0^T p^{(\nu)}(t; -\log(1-\alpha), \log(1+\beta))dt.$$

Moreover, a digital option on the event that the relative drawdown precedes the relative drawup can also be perceived as a means of protection against adverse movements in the market. In particular, the discounted payoff of this digital option can be written as

$$PO(\alpha, \beta) \;=\; e^{-rt} \cdot \mathbb{1}_{\{U_D(\alpha) \in dt, U_R(\beta) > t\}} \cdot \mathbb{1}_{\{t \leq T\}}, \qquad (2.86)$$

where $r > 0$ is the risk-free interest rate and T is the maturity of the option.

Under a risk-neutral measure Q, the stock price and its logarithm have the following dynamics respectively,

$$dS_t = rS_t dt + \sigma S_t dW_t, \ S_0 = 1, \tag{2.87}$$

$$d\log S_t = \nu' dt + \sigma dW_t, \ \log S_0 = 0, \tag{2.88}$$

where $\nu' = r - \frac{1}{2}\sigma^2$.

Using (2.85) and our results we are able to derive the risk-neutral price at time 0 of this digital option:

In the case of a perpetual option (see Karatzas & Shreve [44]), the risk-neutral price of the digital option is already given by the Laplace transform (2.71), (2.72) and (2.74). In particular,

$$Q\{PO(\alpha,\beta)\} = L_0^{\log S, r}(-\log(1-\alpha), \log(1+\beta)).$$

In the case of a finite life option maturing at time $T < \infty$, we can apply the densities (2.76) and (2.79) to calculate the risk-neutral price.

1. $(1-\alpha)(1+\beta) \le 1$:

$$Q\{PO(\alpha,\beta)\} = \int_0^T e^{-rt} p^{(\nu')}(t; -\log(1-\alpha), \log(1+\beta))dt; \tag{2.89}$$

2. $\delta = (1-\alpha)(1+\beta) > 1$:

$$\begin{aligned} &Q\{PO(\alpha,\beta)\} - Q\{PO(\alpha, \alpha/(1-\alpha))\} \\ &= \int_0^T e^{-rt} \int_0^{\log\delta} \frac{\partial}{\partial z} p^{(\nu')}(t; -\log(1-\alpha), -\log(1-\alpha)+z)dzdt. \end{aligned} \tag{2.90}$$

2.5.2 The problem of transient signal detection and identification of two-sided changes

In this example, we consider the problem of signal detection and identification of two-sided changes described in Pospisil, Vecer & Hadjiliadis [65], when the signal is transient with an exponential or a deterministic lifetime.

In particular, let $X_{\cdot} = \{X_t; t \geq 0\}$ be a diffusion process with the initial value $X_0 = x$ and the following dynamics up to a deterministic time τ:

$$dX_t \quad = \quad \sigma(X_t)dW_t, \quad t \leq \tau. \tag{2.91}$$

For $\tau + T > t > \tau$, the process evolves according to one of the following stochastic differential equations:

$$dX_t \quad = \quad \mu(X_t)dt + \sigma(X_t)dW_t \quad \tau + T > t > \tau, \tag{2.92}$$

$$dX_t \quad = \quad -\mu(X_t)dt + \sigma(X_t)dW_t \quad \tau + T > t > \tau, \tag{2.93}$$

with initial condition $y = X_\tau$. The lifetime of the signal T is assumed to be deterministic, or exponentially distributed with parameter $\lambda > 0$ and independent of the process $X_{\cdot}$. The time of the regime change, τ, is deterministic but unknown. We observe the process $X_{\cdot}$ sequentially and our goal is to detect the time of onset of the signal, as well as possibly identify its direction, before the lifetime of the signal T.

Using the notation and setup set forth in Pospisil, Vecer & Hadjiliadis [65], Theorems 2.3, 2.4 and 2.5 can be used to compute the probability of sequential misidentification of the signal in the case that the onset of the signal occurs at time 0 and the lifetime of the signal T is exponentially distributed with parameter $\lambda > 0$. More specifically, let $\{X_t^{0,+}; t \geq 0\}$ denote a process that follows (2.92) when $\tau = 0$. Then,

$$\begin{aligned} P_x^{0,+}(\tau_a^D < \tau_b^U \wedge T) &= \int_0^\infty P_x^{0,+}(\tau_a^D < \tau_b^U \wedge t) \cdot \lambda e^{-\lambda t} dt \\ &= \int_0^\infty e^{-\lambda t} P_x^{0,+}(\tau_a^D \in dt, \tau_b^U > t) dt \\ &= L_x^{X^{0,+},\lambda}(a,b), \end{aligned} \tag{2.94}$$

expresses the probability that an alarm indicating that the regime switched to (2.93) will occur before T while in fact (2.92) is the true regime. Thus, (2.94) can be seen as the probability of a misidentification. Moreover, in the case that the density of the random variable X_τ admits an analytical representation, we can also compute

$$\begin{aligned} & \int P_y^{\tau,+}(\tau_a^D \circ \theta_\tau < \tau_b^U \circ \theta_\tau \wedge T) f_{X_\tau}(y|x) dy \\ = & \int L_y^{X^{0,+},\lambda}(a,b) f_{X_\tau}(y|x) dy, \end{aligned} \tag{2.95}$$

which can be interpreted as the aggregate probability (or unconditional probability) of a misidentification for any given change-point τ.

On the other hand, if the lifetime of the signal T is deterministic, using Theorem 2.6 we are still able to compute the probability of misidentification for Brownian motion ($\sigma(\cdot) = \sigma > 0, \mu(\cdot) = \mu$). More specifically,

$$P_x^{\tau,+}(\tau_a^D \circ \theta_\tau < \tau_a^U \circ \theta_\tau \wedge T) \quad = \quad \int_0^T p^{(\mu)}(t; a, a)dt, \tag{2.96}$$

expresses the probability of misidentification for any given change-point τ.

2.6 Conclusion

In this work we characterize the probability that the drawdown of a units precedes the drawup of b units for a general diffusion process. We derive the probability density of a drawdown when it precedes a drawup in the special cases of a simple random walk and a drifted Brownian motion model. Although several authors in the literature have studied drawdowns and drawups [85, 49, 38, 65, 71], this work summarizes the probabilistic properties of a drawdown on the event that it precedes a drawup for a general diffusion process. These results are of practical interest in two main areas: financial risk-management and transient signal detection and identification.

Chapter 3

Static and Semi-static Replications of Digital Options on Drawdowns and Drawups

In this chapter we study two new financial products on drawdowns and drawups, and replication strategies of these claims. We assume no frictions and no arbitrage in all that follows. Let S_t denote the spot price of some asset which can be monitored continuously over the fixed time interval $[0,T]$. Let $M_t \triangleq \max_{s\in[0,t]} S_s$ and $m_t \triangleq \min_{s\in[0,t]} S_s$ be the continuously-monitored maximum and minimum of this asset price over $[0,t]$. We follow the notations introduced in Definitions 2.1 and 2.2. In particular, recall that for $K>0$, τ_K^D (τ_K^U, resp.) is the time at which the drawdown (drawup, resp.) process D (U, resp.) first reaches K. Then a digital call on maximum drawdown is a digital option which pays $\$1\!\!1_{\{\tau_K^D \leq T\}}$ at maturity. Similarly, a digital call on the K-drawdown preceding a K-drawup is a digital option which pays $\$1\!\!1_{\{\tau_K^D \le \tau_K^U \wedge T\}}$ at maturity. Both of these instruments clearly provide protection against adverse movements in the market. It is easy to notice that the latter claim is cheaper than the former since

$$1\!\!1_{\{\tau_K^D \le \tau_K^U \wedge T\}} = 1\!\!1_{\{\tau_K^D \le T\}} - 1\!\!1_{\{\tau_K^U \le \tau_K^D \wedge T\}}.$$

In the last chapter we derived analytic results for the price of these two claims at time 0. In this work, we develop replication strategies of both claims using double barrier options and their spreads, respectively. Since these instruments are relatively illiquid at present, we also derive semi-static hedges using single barrier one-touches and vanilla options under symmetry and continuity assumptions.

The remainder of this chapter is structured in the following way. In Section 3.1, after introducing all the instruments we need, we develop a model-free static replication of the digital call on the K-drawdown preceding a K-drawup using one-touch knockouts. In Section 3.2, we impose an assumption of continuity and symmetry to develop a semi-static replication of the digital call on maximum drawdown with one-touch knockouts. This symmetry assumption is reinforced in Sections 3.3 and 3.4 in order to develop a semi-static portfolio of one-touches and binary options to replicate the payoffs of both target claims. In Section 3.5, we proceed to geometric models and present a static replication strategy for the latter digital call with one-touch knockouts. In Sections 3.6 through 3.7, under appropriate geometric symmetry assumptions, we develop semi-static replication of both target digital calls with consecutively more liquid instruments. In Section 3.8, we discuss how to extend previous results to certain stochastic processes with discrete state space. Finally, we summarize the paper with some closing remarks in Section 3.9.

3.1 Model-free Static Replication

Let $B_t(T)$ be the price of a default-free zero coupon bond paying one dollar with certainty at T. We assume that $B_t(T) > 0$ for all $t \in [0, T]$ and hence no arbitrage implies the existence of a probability measure $\mathbb{Q}^T$ associated with this numeraire. The measure $\mathbb{Q}^T$ is equivalent to the statistical probability measure and hence is usually referred to as an equivalent martingale measure. Under $\mathbb{Q}^T$, the ratios of non-dividend paying asset prices to B are martingales. We will use $\mathbb{Q}^T$ to describe the arbitrage-free values of options in this paper.

Let us denote by $DC_t^{MD}(K, T)$ the value at time $t \in [0, T]$ of a digital call on maximum drawdown, and by $DC_t^{D<U}(K, T)$ the value at time $t \in [0, T]$

of a digital call on the K-drawdown preceding a K-drawup. That is,

$$DC_t^{MD}(K,T) \triangleq B_t(T)\mathbb{Q}_t^T(\tau_K^D \leq T), \tag{3.1}$$

$$DC_t^{D<U}(K,T) \triangleq B_t(T)\mathbb{Q}_t^T(\tau_K^D \leq \tau_K^U \wedge T). \tag{3.2}$$

In this section, we will replicate $DC_t^{D<U}(K,T)$ using bonds, one-touch knockouts and their spreads.

Before describing the payoffs of one-touch knockouts and their spreads, it will be helpful to introduce terminology that indicates exactly where the spot price is when a barrier option knocks in or knocks out. For concreteness, we will focus on a lower barrier L. Then the payoff $\$\mathbb{1}_{\{\tau_L^S \leq T\}}$ is the same as the payoff $\$\mathbb{1}_{\{m_T \leq L\}}$.

If a barrier L is assumed to be skipfree, then when $\tau_L^S \leq T$:

$$S_{\tau_L^S} = L. \tag{3.3}$$

When condition (3.3) holds, we say that a barrier has been touched. When we instead have $S_{\tau_L^S} < L$ we say that a barrier has been crossed. While when we have $S_{\tau_L^S} \leq L$, we say that a barrier has been hit. When we have both $S_{\tau_L^S} = L$ and $m_T = L$, we say that a barrier has been grazed. Let us now define one-touch knockouts and their spread.

Definition 3.1. *An one-touch knockout is a double barrier option with an in-barrier V, an out-barrier W, and a fixed expiry date T. The payoff of an* *one-touch knockout at maturity is* $\$1\!\!1_{\{\tau_V^S \le \tau_W^S \wedge T\}}$. *The price of the option at any time* $t \in [0,T]$ *is denoted by*

$$OTKO_t(V,W,T) \stackrel{\triangle}{=} B_t(T)\mathbb{Q}_t^T(\tau_V^S \le \tau_W^S \wedge T). \tag{3.4}$$

Remark 3.1. *We assume that the spot stays in between* V *and* W *when the one-touch knockout is issued. For concreteness, we will focus on the case in which the out-barrier* W *is the higher barrier. Then it is easily seen that*

$$\{\tau_V^S \le \tau_W^S \wedge T\} \quad = \quad \{\tau_V^S \le T, M_{\tau_V^S} < W\}. \tag{3.5}$$

It follows that the one-touch knockout pays one dollar at its expiry date T *if and only if the spot price hits the in-barrier* V *before hitting the out-barrier* W *and this first hitting time to* V *occurs before the expiry* T. *Notice that the one-touch knockout also pays one dollar at* T *if* $\tau_V^S \le \tau_W^S \le T$. *In words, the out-barrier* W *is extinguished when the in-barrier* V *is first hit.*

Sometimes it is convenient to modify the knockout condition of an one-touch knockout. For example, we consider the following payoff

$$OTKO_t(V,W^+,T) \stackrel{\triangle}{=} B_t(T)\mathbb{Q}_t^T(\tau_V^S \le T, M_{\tau_V^S} \le W). \tag{3.6}$$

This claim pays out one dollar at expiry if and only if the spot price S *hits the in-barrier* V *before* <u>*crossing*</u> *the out-barrier* W *and this first hitting time to* V *occurs before the expiry* T.

The last claim which we want to use is a sequential double-touch whose payoff is the result of differentiating the payoff of an one-touch knockout in (3.4) with respect to its higher out-barrier W. This claim has a positive payoff if and only if the underlying spot price first touches W and then hits V from above before maturity. We accordingly refer to this claim as a ricochet-upper-first down-and-in:

Definition 3.2. *A ricochet-upper-first down-and-in is a double barrier option with an in-barrier V, a graze-barrier W, and a fixed expiry date T. The price of the option at any time $t \in [0, T]$ is denoted by*

$$RUFDI_t(V, W, T) \quad = \quad B_t(T) E_t^{\mathbb{Q}^T} \{ \mathbb{1}_{\{\tau_V^S \leq T\}} \delta(M_{\tau_V^S} - W) \}. \qquad (3.7)$$

Notice that a ricochet-upper-first down-and-in is itself a spread of two one-touch knockouts with slightly different upper out-barriers and identical lower in-barriers set at V.

In the next theorem, we present a replication of the payoff the digital call on the K-drawdown preceding a K-drawup with a portfolio, which is replicating, non-anticipating, self-financing, and robust.

Theorem 3.1 (Robust Replication: I)**.** *Under frictionless markets, no arbitrage implies that the digital call on the K-drawdown preceding a K-drawup can be valued relative to the prices of bonds, one-touch knockouts and their* *spreads:*

$$\begin{aligned}DC_t^{D<U}(K,T) =& 1\!\!1_{\{\tau_K^D \le \tau_K^U \wedge t\}} B_t(T) + 1\!\!1_{\{t<\tau_K^D \wedge \tau_K^U\}} \cdot \Big\{ OTKO_t(M_t - K, M_t^+, T) \\ &+ \int_{M_t^+}^{(m_t+K)^-} RUFDI_t(H-K, H, T) dH \Big\}. \end{aligned} \quad (3.8)$$

for any $t \in [0,T]$ and $K > 0$.

Proof. Suppose that a digital call on the K-drawdown preceding a K-drawup has been sold at time 0. In order to develop a static hedge, we condition on being at some time t before expiry and before a drawdown or drawup of size K has been realized:

$$t \in [0, \tau_K^D \wedge \tau_K^U \wedge T].$$

Then the maximum-to-date M_t and the minimum-to-date m_t are both known constants that bracket the current spot S_t. The fact that neither a drawdown nor a drawup of size K has yet occurred implies that $M_t - m_t < K$. As a result, we have:

$$M_t - K < m_t \le S_t \le M_t < m_t + K.$$

Let us focus on the running maximum at time τ_K^D. Since the running maximum is an increasing process, we must have:

$$\begin{aligned}\{\tau_K^D \le \tau_K^U \wedge T\} &= \{\tau_K^D \le \tau_K^U \wedge T, M_{\tau_K^D} \ge M_t\} \\ &= \{\tau_K^D \le \tau_K^U \wedge T, M_{\tau_K^D} \in [M_t, m_t + K)\}.\end{aligned}$$

This is because, if $M_{\tau_K^D} = M$ for some $M \geq m_t + K$, then either $\tau_M^S < t$ and hence $\tau_K^D \wedge \tau_K^U \leq t$, or else $\tau_M^S \in [t, \tau_K^D)$ in which case $\tau_K^U \leq \tau_K^D$. Moreover, by restricting $M_{\tau_K^D}$ to the interval $[M_t, m_t + K)$, we can't have a K-drawup precede a K-drawdown, since if $\tau_K^U \leq \tau_K^D \leq T$, then $M_{\tau_K^D} > m_t + K$. So we can further obtain that

$$\begin{aligned}\{\tau_K^D \leq \tau_K^U \wedge T\} &= \{\tau_K^D \leq \tau_K^U \wedge T, M_{\tau_K^D} \in [M_t, m_t + K)\} \\ &= \{\tau_K^D \leq T, M_{\tau_K^D} \in [M_t, m_t + K)\},\end{aligned}$$

We now present a key result that allows the digital call to be replicated with one-touch knockouts. Observe that if and when the unit payoff of the digital call is realized, the stock price has to be visiting a new low level:

$$\begin{aligned}\{\tau_K^D \leq \tau_K^U \wedge T\} &= \{\tau_K^D \leq T, M_{\tau_K^D} \in [M_t, m_t + K)\} \\ &= \{\tau_K^D \leq T, \tau_K^D = \tau^S_{M_{\tau_K^D} - K}, M_{\tau_K^D} \in [M_t, m_t + K)\}.\end{aligned}$$

As a consequence of (3.9), the payoff of a digital call has the following representation:

$$\begin{aligned}1\!\!1_{\{\tau_K^D \leq \tau_K^U \wedge T\}} &= 1\!\!1_{\{\tau_K^D \leq T, \tau_K^D = \tau^S_{M_{\tau_K^D} - K}, M_{\tau_K^D} = M_t\}} \\ &\quad + \int_{M_t^+}^{(m_t+K)^-} 1\!\!1_{\{\tau_K^D \leq T, \tau_K^D = \tau^S_{H-K}\}} \delta(M_{\tau_K^D} - H) dH \\ &= 1\!\!1_{\{\tau^S_{M_t - K} \leq T, M_{\tau^S_{M_t - K}} = M_t\}} + I,\end{aligned}$$

where:

$$I \triangleq \int_{M_t^+}^{(m_t+K)^-} \mathbb{1}_{\{\tau^S_{H-K} \leq T\}} \delta(M_{\tau^S_{H-K}} - H) dH.$$

Under no arbitrage assumption, taking expectations of (3.9) under $\mathbb{Q}^T$ implies that:

$$DC_t^{D<U}(K,T) =$$
$$= OTKO_t(M_t - K, M_t^+, T) + \int_{M_t^+}^{(m_t+K)^-} RUFDI_t(H-K, H, T) dH,$$

for all $t \in [0, \tau_K^D \wedge \tau_K^U \wedge T]$.

If and when $\tau_K^D \wedge \tau_K^U \leq T$, then at that time, we do not hold any sequential double-touches, the one-touch knockout in the portfolio either knocks into a bond if $\tau_K^D \leq \tau_K^U$, or knocks out if $\tau_K^D \geq \tau_K^U$. As a consequence, the digital call can be valued at any $t \in [0, T]$. Hence, we have (3.8). $\square$

We have shown a robust hedge of the digital call on K-drawdown preceding a K-drawup. This hedge portfolio (3.8) can be set up with one-touch knockouts and their spreads, which do trade liquidly in the OTC currency option market. However, to obtain a replicating portfolio of the digital call on maximum drawdown with tradeable assets, we need to place structure on the spot price process. We proceed to develop this in the next section.

3.2 Semi-static Replication of a Digital Call on Maximum Drawdown with OTKO

In this section we place structure on S, the stochastic process governing the spot price of the underlying asset. In particular, we assume that the running maximum can only increase continuously whenever $\tau_K^D > t$. Of course, this condition is already met if the process is continuous or spectrally negative. We also impose a symmetry condition on the process between the first time that a new maximum M_t is established and the first exit time of the corridor $(M_t - K, M_t + K)$. To be more specific, recall that τ_B^S denotes the first hitting time of the spot price process S to a barrier B. Let $\tau(M, K)$ be the first exit time of a corridor centered at M with lower barrier $M - K$ and higher barrier $M + K$. Then whenever the underlying spot price process is at its maximum to date M_t with $MD_t < K$, we have

$$\mathbb{Q}_t^T(\tau(M_t, K) = \tau_{M_t - K}^S \wedge T) = \mathbb{Q}_t^T(\tau(M_t, K) = \tau_{M_t + K}^S \wedge T). \tag{3.9}$$

In words, first exiting on the left before T has the same risk-neutral probability as first exiting on the right before T. This condition is met by symmetric Lévy processes such as symmetric stable processes which includes standard Brownian motion. It is also met by the Ocone martingales (see Ocone [59]) as well as any process constructed as the difference of two independent identically distributed processes.

We will need to impose both of our assumptions in order to replicate a digital call on maximum drawdown using just bonds and one-touch knock-outs. The set of stochastic processes that satisfy both assumptions are said to satisfy **A1**:

A1: Continuity of the Maximum and Exit Symmetry *While $MD_t < K$, the running maximum is continuous. Moreover, at times $\tau(u) \triangleq \tau_u^S \wedge \tau_K^D \wedge T$ for all $u > S_0$, the risk-neutral probability of first exiting at $M_{\tau(u)} - K$ before T is the same as the risk-neutral probability of first exiting at $M_{\tau(u)} + K$ before T.*

Remark 3.2. *It is possible to construct a positive martingale that satisfies assumption* **A1**. *For example, consider a continuous process $S. = \{S_t \triangleq S_0 + W(\tan(\frac{\pi t}{2T}) \wedge T_D^W(K)); 0 \leq t \leq T\}$, where $W(t)$ is a standard Brownian motion starting at zero and $0 < K < S_0$, and $T_D^W(K)$[1] is defined as*

$$T_D^W(K) = \inf\{t > 0 | \sup_{s \in [0,t]} W_s - W_t \geq K\}.$$

Then $S.$ is obviously positive and satisfies the symmetry in **A1** *whenever $S_t = M_t$. Moreover, using the fact that, conditioning on $\{S_t = M_t\}$[2], $M_{\tau_K^D}$ is exponentially distributed with parameter $1/K$ on $[M_t, \infty)$ (see Lehoczky [49]),*

[1] It is easily seen that $\tan(\pi\tau_K^D/2T) = T_D^W(K)$.

[2] If $S_t < M_t$, then $P(M_{\tau_K^D} = M_t | \mathcal{F}_t) > 0$ and $M_{\tau_K^D}$ is not a continuous random variable.

we can prove that $S.$ *is indeed a martingale. In fact, for a given* $t \in [0, T)$, *let us define*

$$\eta_t \stackrel{\triangle}{=} \inf\{s > t | S_s \notin (M_t - K, M_t)\}.$$

Then at $t < \tau_K^D$, *given* $\mathcal{F}_t = \sigma\{S_s; s \leq t\}$,

$$\begin{aligned} E\{S_{\tau_K^D} | \mathcal{F}_t\} &= P(S_{\eta_t} = M_t - K | \mathcal{F}_t) \cdot (M_t - K) \\ &\quad + P(S_{\eta_t} = M_t | \mathcal{F}_t) \cdot E\{E\{S_{\tau_K^D} | S_{\eta_t} = M_t\} | \mathcal{F}_t\} \\ &= \frac{M_t - S_t}{K}(M_t - K) + \frac{K + S_t - M_t}{K}\left(M_t + \int_0^\infty e^{-\frac{x}{K}} dx - K\right) = S_t. \end{aligned}$$

Therefore,

$$E\{S_T | \mathcal{F}_t\} = E\{S_{\tau_K^D} | \mathcal{F}_t\} = 1\!\!1_{\{\tau_K^D \leq t\}} S_t + 1\!\!1_{\{t < \tau_K^D\}} S_t = S_t.$$

A possible replication of digital calls on maximum drawdown is done with one-touches:

Definition 3.3. *An one-touch is a single barrier option with a barrier* B *and a fixed expiry date* T. *The payoff of the option at* T *is* $1\!\!1_{\{\tau_B^S \leq T\}}$. *The arbitrage-free price of the option at time* $t \in [0, T]$ *is given by*

$$OT_t(B, T) = B_t(T) \mathbb{Q}_t^T (\tau_B^S \leq T).$$

Suppose that we attempt to replicate the payoff of a digital call on maximum drawdown. At time t when $\tau_K^D \leq t$, we simply hold a bond, but while

$\tau_K^D > t$, we attempt a semi-dynamic strategy by holding an one-touch with barrier at $M_t - K$ and rolling up this barrier each time the running maximum increases. No other instruments are held. While this strategy is replicating , it is not yet self-financing as it costs money to move up the lower barrier of an one-touch closer to the spot price. To finance the rollup of the barriers of this one-touch until $\tau_K^D \wedge T$, we assume that **A1** holds, i.e. we rely on the continuity of the running maximum and the exit symmetry assumed present when the maximum ticks up. For $t \in [0, \tau_K^D \wedge T]$, suppose that we also hold an upper barrier one-touch struck K dollars above the maximum-to-date. While this augmentation finances the rollup of the lower barrier one-touch being held, it no longer replicates the desired payoff, since a path that first hits $M_{\tau_K^D} - K$ and then hits $M_{\tau_K^D} + K$ will trigger payoffs from both one-touches. For $t \in [0, \tau_K^D \wedge T]$, suppose we further alter the strategy by imposing a knockout barrier at the lower level $M_t - K$ on the one-touch struck at $M_t + K$, and a knockout barrier at the higher level $M_t + K$ on the one-touch struck at $M_t - K$. Then we are using two one-touch knockouts. It is easily seen that, when the underlying satisfies **A1**, the latest strategy self-finances and replicates the payoff of a digital call on maximum drawdown. In particular, we have:

Theorem 3.2 (Semi-robust Pricing using OTKO)**.** *Under frictionless mar-*

kets and assumption **A1***, no arbitrage implies that the digital call on maximum drawdown can be valued relative to the prices of bonds and one-touch knockouts as:*

$$\begin{aligned} DC_t^{MD}(K,T) &= 1\!\!1_{\{\tau_K^D > t\}} B_t(T) + 1\!\!1_{\{\tau_K^D \le t\}} \times \{OTKO_t\,(M_t - K, M_t + K, T) \\ &\quad + OTKO_t\,(M_t + K, M_t - K, T)\}\,, \end{aligned} \tag{3.10}$$

for $t \in [0, T]$ *and* $K > 0$*.*

Proof. Suppose the digital call on maximum drawdown has been sold at time 0. In order to hedge this position, consider a strategy of always holding two one-touch knockouts whose barriers are each K units away from the maximum to date. This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting time of running drawdown to the strike K. If the first hitting time of the running drawdown to K occurs before T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum can never increase by a jump, rolling up double-touches never yields a payout due to a cross of the upper barrier being held. When the running maximum increases continuously, assumption **A1** implies that the cost of rolling up both barriers is zero. Hence, the only way to get a cash flow from the portfolio of one-touch knockouts is if the spot price crosses the lower barrier of the one-touch knockouts being held. Let us denote $\tau \triangleq \tau_K^D$. Then if $\tau > T$, the stock price was always within K of its running maximum and hence the one-touch knockouts expire worthless, as does the target claim. In contrast, if $\tau \leq T$, then at time τ, the stock price is at least K units below its maximum to date, hence, the one-touch knockout with the lower in-barrier converts into a bond at this time, and the one-touch knockout with the upper in-barrier knocks out.

We conclude that in all cases, the payoff of the target claim is replicated by trading one-touch knockouts and bonds. Furthermore, the right hand side of (3.10) is the cost of setting up the replicating strategy at time t. Hence, no arbitrage implies that this cost is also the price of a digital call on maximum drawdown. □

3.3 Semi-static Replication with One-touches

In the last two sections, we derived static and semi-static hedges of the target digital calls with one-touch knockouts and their spreads. Since one-touch knockouts are relatively illiquid at present, this section presents an alternative semi-static hedge which just uses single-barrier one-touches. The replication only succeeds under some symmetry and continuity assumptions, which we will make precise. The next section shows that under further conditions, each one-touch can also be replicated with vanilla options. It follows that the payoff on the target digital calls can also be replicated with vanilla options. We present this replicating portfolio in the next section.

As the first step, suppose that the spot starts inside the corridor between V and W, where V and W are the in-barrier and out-barrier of an one-touch knockout respectively. Let τ be the first exit time of the above corridor, then we impose the following assumption:

A2: Skip-freedom and Hitting Symmetry *The spot S cannot exit the corridor between V and W by a jump. If the first exit time $\tau \leq T$, then we have*

$$\mathbb{Q}_\tau^T(\tau^S_{S_\tau - \Delta} \leq T) = \mathbb{Q}_\tau^T(\tau^S_{S_\tau + \Delta} \leq T), \ \forall \Delta > 0. \tag{3.11}$$

Under our assumptions, we claim that the payoff of an one-touch knockout with in-barrier V and out-barrier W is replicated by a portfolio of one-touches:

Proposition 3.1 (Semi-static Pricing of One-touch Knockouts: I)**.** *Under frictionless markets and assumption* **A2***, no arbitrage implies that* $t \in [0, \tau_V^S \wedge \tau_W^S \wedge T]$

$$OTKO_t(V, W, T) = OT_t(V, T) + \sum_{n=1}^{\infty} [OT_t(V - 2n\triangle, T) - OT_t(V + 2n\triangle, T)], \tag{3.12}$$

where $\triangle = W - V$.

Proof. Suppose an one-touch knockout with in-barrier V and out-barrier W has been sold at time 0. In order to hedge this position, an investor takes a long position on a series of one-touches with barriers at V, $V-2\triangle$, $V-4\triangle, \ldots$ and also takes a short position on a series of one-touches with barriers at $V+2\triangle$, $V+4\triangle, \ldots$. If neither barrier is hit by T, then all one touches expire worthless. If $\tau_V^S \leq \tau_W^S \wedge T$, then at τ_V^S, the one-touch with barrier V becomes a bond, while **A2** implies that all of the other one-touches can be costlessly liquidated. The reason is that for each $n = 1, 2, \ldots$, the long position in the one-touch with barrier $V - 2n\triangle$, is canceled by the short position in the one-touch with barrier $V + 2n\triangle$. On the other hand, if $\tau_W^S \leq \tau_V^S \wedge T$, then at τ_W^S, **A2** implies that all of the one-touches can be costlessly liquidated. The reason is that since $V = W - \triangle$, the portfolio can also be considered as long a series of one-touches with barriers at $W - \triangle$, $W - 3\triangle$, $W - 5\triangle, \ldots$, while also being short a series of one-touches with barriers at $W + \triangle$, $W + 3\triangle$, $W + 5\triangle \ldots$. Hence, for each $n = 1, 2, \ldots$, the long position in the one-touch with barrier $W - (2n-1)\triangle$, is canceled by the short position in the one-touch with barrier $W + (2n-1)\triangle$. Since the value of the one-touch portfolio matches the payoff of the one-touch knockout when (S, t) exits $(V \wedge W, V \vee W) \times [0, T]$, no arbitrage forces the values prior to exit to be the same. $\square$

Recall that Theorem 3.1 stated that the payoff of a digital call on the K-drawdown preceding a K-drawup can be statically replicated by one-touch knockouts, and Theorem 3.2 stated that under **A1**, the payoff of a digital call on maximum drawdown can be dynamically replicated by rolling up the barriers of one-touch knockouts. If **A2** holds for all barriers of one-touch knockouts being held, then the target digital calls can be replicated just by rolling up the barriers of a portfolio of single barrier one-touches.

In subsection 3.3.1 and 3.3.2, we will separately develop portfolios of one-touches which can be used to replicate the payoff of a digital call on maximum drawdown and the payoff of a digital call on the K-drawdown preceding a K-drawup, respectively.

3.3.1 Hedging digital call on maximum drawdown with one-touches

In this subsection we develop a semi-static replication of a digital call on maximum drawdown using one-touches. By Theorem 3.2 and Proposition 3.1, we need to ensure **A2** holds for all barriers of one-touch knockouts being held. For this purpose we impose structure on the spot price process:

A3: Continuity of the Maximum, Drawdown, and Hitting Symmetry *While $t < \tau_K^D$, the running maximum is continuous, and the drawdown cannot jump up by more than $K - D_t$. Moreover, at times $\tau(u) \triangleq \tau_u^S \wedge \tau_K^D \wedge T$*

for all $u > S_0$,

$$\mathbb{Q}_t^T(\tau^S_{S_{\tau(u)}-\Delta} \leq T) = \mathbb{Q}_t^T(\tau^S_{S_{\tau(u)}+\Delta} \leq T),\ \forall \Delta > 0. \tag{3.13}$$

Note that the positive continuous martingale introduced in Remark 3.2 does not satisfy **A3** in that, the maximum at T, $M_T = M_{\tau_K^D}$ can take any positive value that is greater than or equal to $M_{\tau(u)}$, whereas the minimum at T, m_T can only take value that is greater than or equal to $M_{\tau(u)} - K$.

From Proposition 3.1, it is not difficult to see that **A3** also implies **A1**. In fact, under **A3**, at times $\tau(u) \triangleq \tau_u^S \wedge \tau_K^D \wedge T$ for $u > S_0$, evaluating (3.12) at $V = M_{\tau(u)} \mp K$ and $W = M_{\tau(u)} \pm K$, we obtain

$$\begin{aligned} &OTKO_{\tau(u)}(M_{\tau(u)} \mp K, M_{\tau(u)} \pm K, T) = \\ &= \sum_{n=0}^{\infty} OT_{\tau(u)}(M_{\tau(u)} \mp (4n+1)K, T) - \sum_{n=1}^{\infty} OT_{\tau(u)}(M_{\tau(u)} \pm (4n-1)K, T), \end{aligned} \tag{3.14}$$

which implies that

$$OTKO_{\tau(u)}(M_{\tau(u)} - K, M_{\tau(u)} + K, T) = OTKO_{\tau(u)}(M_{\tau(u)} + K, M_{\tau(u)} - K, T).$$

As a result, we have:

Theorem 3.3 (Semi-robust Pricing using One-touches: I)**.** *Under frictionless markets and assumption* **A3**, *no arbitrage implies that the digital call* *on maximum drawdown can be valued relative to the prices of bonds and one-touches as:*

$$
\begin{aligned}
DC_t^{MD}(K,T) =& \mathbb{1}_{\{\tau_K^D > t\}} B_t(T) + \mathbb{1}_{\{\tau_K^D \leq t\}} \Big\{ \sum_{n=0}^{\infty} OT_t(M_t + (4n \pm 1)K, T) \\
&+ \sum_{n=1}^{\infty} OT_t(M_t - (4n \pm 1)K, T) \Big\}, \qquad (3.15)
\end{aligned}
$$

for any $t \in [0,T]$ and $K > 0$.

Proof. Suppose the digital call on maximum drawdown has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of one-touches on the right hand side of (3.15). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting time of running drawdown to the strike K. If the running drawdown increase to K before T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum is continuous, the above replicating portfolio never yields a payout due a hit of barriers higher than M_t. When the running maximum increases continuously with $t < \tau_K^D \wedge T$, assumption **A3** guarantees that it costs nothing to move the barriers of one-touches being held. Hence, the first time to receive a cash flow from the above portfolio is at time $\tau \triangleq \tau_K^D$. If $\tau > T$, then all one-touches expire worthless, as does the target claim. If $\tau \leq T$, then at τ, $S_\tau = M_\tau - K$, Proposition 3.1 and assumption **A3** imply that, the portfolio of one-touches has the same value as

$$OTKO_\tau(M_\tau - K, M_\tau + K, T) + OTKO_\tau(M_\tau + K, M_\tau - K, T) = B_\tau(T).$$

We conclude that in all cases, the payoff the digital call is matched by the liquidation value of a non-anticipating self-financing portfolio of bonds and one-touches. Furthermore, the right hand side of (3.15) is the cost of setting up the replicating portfolio at time t. Hence, no arbitrage implies that this cost is also the price of the target claim. □

3.3.2 Hedging digital call on the K-drawdown preceding a K-drawup with one-touches

In this subsection we develop a semi-static replication of a digital call on the K-drawdown preceding a K-drawup using one-touches. By Theorem 3.1 and Proposition 3.1, we need to ensure **A2** holds for all barriers of one-touch knockouts being held. For this purpose we impose structure on the spot price process:

A3': Continuity of the Maximum, Minimum, and Hitting Symmetry *While* $t \leq \tau_K^D \wedge \tau_K^U \wedge T$*, the running maximum and the running minimum are continuous. Moreover, at times* $\theta(u) \triangleq \tau_u^D \wedge \tau_u^U \wedge T$ *for all* $u \in (0, K]$*,*

$$\mathbb{Q}_t^T(\tau^S_{S_{\theta(u)}-\Delta} \leq T) \;\; = \;\; \mathbb{Q}_t^T(\tau^S_{S_{\theta(u)}+\Delta} \leq T), \; \forall \Delta > 0. \qquad (3.16)$$

Assumption **A3'** is sufficient for applying Proposition 3.1. Evaluating (3.12) at $V = M_t - K$ and $W = M_t$, we obtain

$$\begin{aligned} & OTKO_t(M_t - K, M_t, T) \\ = \; & \sum_{n=0}^{\infty} OT_t(M_t - (2n+1)K, T) - \sum_{n=1}^{\infty} OT_t(M_t + (2n-1)K, T), \end{aligned} \quad (3.17)$$

for $K > 0$ and $t \in [0, \tau^S_{M_t - K} \wedge \tau^S_{M_t} \wedge T]$. Differentiating (3.12) with respect to W, and evaluating at $V = H - K$ and $W = H$ implies that for $K > 0$ and $t \in [0, \tau^S_{H-K} \wedge \tau^S_H \wedge T]$:

$$\begin{aligned} & RUFDI_t(H - K, H, T) \\ & = -2 \sum_{n=1}^{\infty} n \left(\frac{\partial}{\partial B} OT_t(B, T) \Big|_{B = H - (2n+1)K} + \frac{\partial}{\partial B} OT_t(B, T) \Big|_{B = H + (2n-1)K} \right) \\ & = -2 \sum_{n=1}^{\infty} n \left(\frac{\partial}{\partial H} OT_t(H - (2n+1)K, T) + \frac{\partial}{\partial H} OT_t(H + (2n-1)K, T) \right), \end{aligned} \quad (3.18)$$

since K is a constant.

Substituting (3.17) and (3.18) in (3.8), and ignoring the left and right limits, we obtain

$$\begin{aligned} DC_t^{D<U}(K, T) = \; & 1\!\!1_{\{\tau_K^D \le t \wedge \tau_K^U \wedge T\}} B_t(T) + 1\!\!1_{\{t < \tau_K^D \wedge \tau_K^U\}} \times \\ & \Bigg\{ \sum_{n=0}^{\infty} (2n+1) [OT_t(M_t - (2n+1)K, T) + OT_t(M_t + (2n+1)K, T)] \\ & \qquad - \sum_{n=1}^{\infty} 2n \left[OT_t(m_t - 2nK, T) + OT_t(m_t + 2nK, T) \right] \Bigg\}, \end{aligned}$$

which gives rise to:

Theorem 3.4 (Semi-robust Pricing using One-touches: II). *Under frictionless markets and assumption* **A3'**, *no arbitrage implies that the digital call on the K-drawdown preceding a K-drawup can be valued relative to the price of bonds and one-touches as:*

$$\begin{aligned} DC_t^{D<U}(K,T) &= \mathbb{1}_{\{\tau_K^D \le t \wedge \tau_K^U \wedge T\}} B_t(T) + \mathbb{1}_{\{t < \tau_K^D \wedge \tau_K^U\}} \times \\ &\left\{ \sum_{n=0}^{\infty} (2n+1)[OT_t(M_t - (2n+1)K,T) + OT_t(M_t + (2n+1)K,T)] \right. \\ &\left. \qquad - \sum_{n=1}^{\infty} 2n\,[OT_t(m_t - 2nK,T) + OT_t(m_t + 2nK,T)] \right\}, \end{aligned} \tag{3.19}$$

for any $t \in [0,T]$ and $K > 0$.

Proof. Suppose that the digital call on the K-drawdown preceding a K-drawup has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of one-touches on the right hand side of (3.19). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting times of the running drawdown/drawup to the strike K. If the running drawdown increases to K before τ_K^U and T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum and the running minimum are continuous, the above replicating portfolio never yields a payout due to a hit of barriers outside the corridor $[m_t, M_t]$. When the running maximum increase or the running minimum decreases continuously with $t < \tau_K^D \wedge \tau_K^U \wedge T$, assumption **A3'** guarantees that it cost nothing to move the barriers of one-touches in the above portfolio. Hence, the first time to get a cash flow from the above portfolio is when $M_t - m_t = K$. Let us denote by τ the first time that $M_t - m_t \geq K$, then clearly $\tau = \tau_K^D \wedge \tau_K^U$. If $\tau > T$, then the one-touches expire worthless, as does the target claim. If $\tau \leq T$, then at τ, $M_\tau = m_\tau + K$, Proposition 3.1 and assumption **A3'** imply that, the portfolio of one-touches has the same value as the one-touch knockout $OTKO_\tau(M_\tau - K, M_\tau, T)$, whose payoff matches the target option, with value zero of the price of a bond. In the former case, $\tau = \tau_K^U$, the one-touches are liquidated for zero; while in the latter case, $\tau = \tau_K^D$, the liquidation proceeds are used to buy the bond. We conclude that in all cases, the payoff of the target digital call is matched by the liquidation value of a non-anticipating self-financing portfolio of bonds and one-touches. Furthermore, the right hand side of (3.19) is the cost of setting up the replicating strategy at time t. Hence, no arbitrage implies that this cost is also the price of the target claim. $\square$

The hedging strategies in Theorem 3.3 and Theorem 3.4 would be easier to implement in practice than the hedges using one-touch knockouts because they do not involve integrating over barriers. If we enforce the symmetry assumption of the underlying spot price process, then it is possible to develop hedging strategies with only digital options on the underlying. We present these results in the next section.

3.4 Semi-static Replication with Vanilla Options

In the previous section, we developed two semi-static hedges with a series of co-terminal single-barrier one-touches of the target digital calls. Since barrier options are not so liquid for most underlyings, this section presents another semi-static hedge which uses digital options on the underlying. The replication only succeeds under some symmetry and continuity assumptions, which we will make precise.

We first give the definition of digital options on the underlying:

Definition 3.4. *Let $B \in \mathbb{R}$ be the strike of a digital option on the underlying in effect from $t = 0$ to $t = T$. For $t \in [0, T]$, let $DP_t(B, T)$ and $DC_t(B, T)$ denote the prices at time t of a digital put and a digital call on spot respectively,*

$$DP_t(B,T) \triangleq B_t(T)\mathbb{Q}_t^T(S_T < B) + \frac{1}{2}B_t(T)E_t^{\mathbb{Q}^T}\{\delta(S_T - B)\}, \quad (3.20)$$

$$DC_t(B,T) \triangleq B_t(T)\mathbb{Q}_t^T(S_T > B) + \frac{1}{2}B_t(T)E_t^{\mathbb{Q}^T}\{\delta(S_T - B)\}. \quad (3.21)$$

Notice that if S_T turn out to be at B, then both digital options pay 50 cents at expiry. We will make use of these digital options to replicate the payoff of an one-touch knockout. To this end, we develop semi-static replication of the target digital calls with vanilla options.

Consider a spot price process starting inside the corridor between V and W, where V and W are the in-barrier and out-barrier of an one-touch knockout respectively. Let $\tau \triangleq \tau_V^S \wedge \tau_W^S$ be the first exit time of the above corridor, then we impose the following assumption:

A4: Skip-freedom and Symmetry *The spot S cannot exit the corridor between V and W by a jump. If the first exit time $\tau \leq T$, then at time τ, the conditional risk-neutral probability distribution of S_T is symmetric about S_τ.*

The above assumption is obviously met by all continuous symmetric Lévy processes. Tehranchi [86] proves that a continuous martingale always (not only at time τ) satisfies the symmetry in **A4** if and only if, the conditional distribution of S_T, given the σ-algebra $\mathcal{F}_t = \sigma\{S_s; s \leq t\}$ and the quadratic variation $\langle S \rangle_t$, is normally distributed with mean S_t and variance $\langle S \rangle_T - \langle S \rangle_t$. In particular, such a martingale can not be always positive.

Under our assumptions, we claim that the payoff of an one-touch knockout with skip-free in-barrier V and out-barrier W is replicated by a portfolio of digital options:

Proposition 3.2 (Semi-static Pricing of One-touch Knockouts: II)**.** *Under frictionless markets and assumption* **A4***, no arbitrage implies that for* $t \in [0, \tau_V^S \wedge \tau_W^S \wedge T]$,

1. *If* $V < W$*:*

$$\begin{aligned} & OTKO_t(V, W, T) \\ = \; & 2\sum_{n=0}^{\infty} DP_t(V - 2n\triangle, T) - 2\sum_{n=1}^{\infty} DC_t(V + 2n\triangle, T); \end{aligned} \tag{3.22}$$

2. *If* $V > W$*:*

$$\begin{aligned} & OTKO_t(V, W, T) \\ = \; & 2\sum_{n=0}^{\infty} DC_t(V - 2n\triangle, T) - 2\sum_{n=1}^{\infty} DP_t(V + 2n\triangle, T); \end{aligned} \tag{3.23}$$

where $\triangle = W - V$*.*

Proof. We will prove the result in the case $V < W$. The other case can be proven with a similar argument. Suppose an one-touch knockout with in-barrier V and out-barrier W has been sold at time 0. In order to hedge this position, an investor takes a long position on a series of digital puts struck at V, $V - 2\triangle$, $V - 4\triangle, \ldots$ and also takes a short position on a series of digital calls struck at $V + 2\triangle$, $V + 4\triangle, \ldots$. If neither barrier is hit by T, then $S_T \in (V, W)$ and hence, all digital options expire worthless. Otherwise, let us denote by τ the first exit time of the corridor (V, W). If $\tau_V^S \leq \tau_W^S \wedge T$, then at time τ_V^S, one can sell the digital puts struck at V and with the premium obtained to buy a bond with the same maturity.

$$2DP_\tau(V,T) = DP_\tau(V,T) + DC_\tau(V,T) = B_\tau(T).$$

Moreover, **A4** implies that all of the other digital options can be costlessly liquidated. The reason is that for each $n = 1, 2, \ldots$, the long position in the digital puts with barrier $V - 2n\triangle$, is canceled by the short position in the digital calls with barrier $V + 2n\triangle$. On the other hand, if $\tau_W^S \leq \tau_V^S \wedge T$, then at τ_W^S, **A4** implies that all of the digital options can be costlessly liquidated. The reason is that since $V = W - \triangle$, the portfolio can also be considered as long a series of digital puts with strikes at $W - \triangle$, $W - 3\triangle$, $W - 5\triangle, \ldots$, while also being short a series of digital calls with strikes at $W + \triangle$, $W + 3\triangle$, $W + 5\triangle \ldots$. Hence, for each $n = 1, 2, \ldots$, the long position in the digital puts with barrier $W - (2n-1)\triangle$, is canceled by the short position in the digital calls with barrier $W + (2n-1)\triangle$. Since the value of the digital option portfolio matches the payoff of the one-touch knockout when (S, t) exits $(V, W) \times [0, T]$, no arbitrage forces the values prior to exit to be the same. $\square$

In Subsections 3.4.1 and 3.4.2, we will separately develop portfolios of digital options which can be used to replicate the payoff of a digital call on maximum drawdown and the payoff of a digital call on the K-drawdown preceding a K-drawup, respectively.

3.4.1 Hedging digital call on maximum drawdown with vanilla options

In this subsection we develop a semi-static replication of a digital call on maximum drawdown using digital options on the underlying. By Theorem 3.1 and Proposition 3.2, we need to ensure **A4** holds for all barriers of one-touch knockouts being held. For this purpose we impose structure on the spot price process:

A5: Continuity of the Maximum, Drawdown, and Symmetry *While $t < \tau_K^D$, the running maximum is continuous, and the drawdown cannot jump up by more than $K - D_t$. Moreover, at times $\tau(u) \triangleq \tau_u^S \wedge \tau_K^D \wedge T$ for all $u > S_0$, the conditional risk-neutral probability distribution of S_T, is symmetric about $S_{\tau(u)}$.*

From Proposition 5.1, it is not difficult to see that **A5** also implies **A1**. In fact, under **A5**, whenever the maximum increases continuously with $t < \tau_K^D$, evaluating (3.22) and (3.23) at $V = M_t \mp K$ and $W = M_t \pm K$:

$$
\begin{aligned}
& OTKO_t(M_t - K, M_t + K, T) \\
= \; & 2\sum_{n=0}^{\infty}\{DP_t(M_t - (4n+1)K, T) - DC_t(M_t + (4n+3)K, T)\}, \\
& OTKO_t(M_t + K, M_t - K, T) \\
= \; & 2\sum_{n=0}^{\infty}\{DC_t(M_t + (4n+1)K, T) - DP_t(M_t - (4n+3)K, T)\},
\end{aligned}
$$

which implies that

$$
OTKO_t(M_t - K, M_t + K, T) \; = \; OTKO_t(M_t + K, M_t - K, T).
$$

As a result, we have:

Theorem 3.5 (Semi-robust Pricing using Vanilla Options: I)**.** *Under frictionless markets and assumption* **A5***, no arbitrage implies that the digital call on maximum drawdown can be valued relative to the prices of bonds and digital options as:*

$$
\begin{aligned}
DC_t^{MD}(K,T) = \mathbb{1}_{\{\tau_K^D \le t\}} B_t(T) + \mathbb{1}_{\{t < \tau_K^D\}} \times & \\
\Bigg\{ 2\sum_{n=0}^{\infty}[DP_t(M_t - (4n+1)K, T) + DC_t(M_t + (4n+1)K, T)] & \\
- 2\sum_{n=1}^{\infty}[DC_t(M_t + (4n-1)K, T) + DP_t(M_t - (4n-1)K, T)] \Bigg\}, & \quad (3.24)
\end{aligned}
$$

for any $t \in [0, T]$ and $K > 0$.

Proof. Suppose the digital call on maximum drawdown has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of vanilla digital options on the right hand side of (3.24). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting time of running drawdown to the strike K. If the running drawdown increase to K before T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum is continuous, the above replicating portfolio never yields a payout from vanilla digital options with strikes higher than M_t. When the running maximum increases continuously with $t < \tau_K^D \wedge T$, assumption **A5** guarantees that it costs nothing to move the barriers of one-touches being held. Hence, the first time to receive a cash flow from the above portfolio is at time $\tau \triangleq \tau_K^D$. If $\tau > T$, then $S_T \in (M_T - K, M_T)$, all vanilla digital options expire worthless, as does the target claim. If $\tau \leq T$, then at τ, $S_\tau = M_\tau - K$, Proposition 5.1 and assumption **A5** imply that, the portfolio of one-touches has the same value as

$$OTKO_\tau(M_\tau - K, M_\tau + K, T) + OTKO_\tau(M_\tau + K, M_\tau - K, T) = B_\tau(T).$$

We conclude that in all cases, the payoff the digital call is matched by the liquidation value of a non-anticipating self-financing portfolio of bonds and vanilla digital options. Furthermore, the right hand side of (3.24) is the cost of setting up the replicating portfolio at time t. Hence, no arbitrage implies that this cost is also the price of the target claim. □

3.4.2 Hedging digital call on the K-drawdown preceding a K-drawup with vanilla options

In this subsection we develop a semi-static replication of a digital call on the K-drawdown preceding a K-drawup using one-touches. By Theorem 3.1 and Proposition 3.2, we need to ensure **A4** holds for all barriers of one-touch knockouts being held. For this purpose we impose structure on the spot price process:

A5': Continuity of the Maximum, Minimum, and Symmetry *While $t \leq \tau_K^D \wedge \tau_K^U \wedge T$, the running maximum and the running minimum are continuous. Moreover, at times $\theta(u) \stackrel{\triangle}{=} \tau_u^D \wedge \tau_u^U \wedge T$ for all $u \in (0, K]$, the conditional risk-neutral probability distribution of hitting S_T is symmetric about $S_{\theta(u)}$.*

Assumption **A5'** is sufficient for applying Proposition 5.1. Evaluating

(3.22) at $V = M_t - K$ and $W = M_t$, we obtain:

$$\begin{aligned} & OTKO_t(M_t - K, M_t, T) = \\ & \qquad = 2\sum_{n=0}^{\infty}\{DP_t(M_t - (2n+1)K, T) - DC_t(M_t + (2n+1)K, T)\}, \quad (3.25) \end{aligned}$$

for $K > 0$ and $t \in [0, \tau^S_{M_t-K} \wedge \tau^S_{M_t} \wedge T]$. Differentiating (3.22) with respect to W, and evaluating at $V = H - K$ and $W = H$ implies that for $K > 0$ and $t \in [0, \tau^S_{H-K} \wedge \tau^S_H \wedge T]$:

$$\begin{aligned} & RUFDI_t(H - K, H, T) \\ & = -4\sum_{n=1}^{\infty} n\left(\frac{\partial}{\partial B}DP_t(B,T)\bigg|_{B=H-(2n+1)K} + \frac{\partial}{\partial B}DC_t(B,T)\bigg|_{B=H+(2n-1)K}\right) \\ & = -4\sum_{n=1}^{\infty} n\left(\frac{\partial}{\partial H}DP_t(H-(2n+1)K,T) + \frac{\partial}{\partial H}DC_t(H+(2n-1)K,T)\right), \end{aligned} \quad (3.26)$$

since K is a constant.

Substituting (3.25) and (3.26) in (3.8), and ignoring the left and right limits, we obtain

$$\begin{aligned} DC_t^{D<U}(K,T) = {} & 1\!\!1_{\{\tau_K^D \leq t \wedge \tau_K^U\}} B_t(T) + 1\!\!1_{\{t < \tau_K^D \wedge \tau_K^U\}} \times \\ & \left\{ \sum_{n=0}^{\infty} (4n+2)[DP_t(M_t - (2n+1)K, T) + DC_t(M_t + (2n+1)K, T)] \right. \\ & \left. \qquad - 4\sum_{n=1}^{\infty} n\left(DP_t(m_t - 2nK, T) + DC_t(m_t + 2nK, T)\right) \right\}, \end{aligned}$$

which gives rise to:

Theorem 3.6 (Semi-robust Pricing using Vanilla Options: II)**.** *Under frictionless markets and assumption* **A5'**, *no arbitrage implies that the digital call on the K-drawdown preceding a K-drawup can be valued relative to the price of bonds and digital options as:*

$$DC_t^{D<U}(K,T) = 1\!\!1_{\{\tau_K^D \le t \wedge \tau_K^U\}} B_t(T) + 1\!\!1_{\{t < \tau_K^D \wedge \tau_K^U\}} \times \left\{ \sum_{n=0}^{\infty} (4n+2)[DP_t(M_t - (2n+1)K, T) + DC_t(M_t + (2n+1)K, T)] \right.$$
$$\left. - 4 \sum_{n=1}^{\infty} n \left[DP_t(m_t - 2nK, T) + DC_t(m_t + 2nK, T)\right] \right\}, \quad (3.27)$$

for $t \in [0,T]$ and $K > 0$.

Proof. Suppose that the digital call on the K-drawdown preceding a K-drawup has been sold at time 0. In order to hedge this position, consider a strategy of always holding the portfolio of digital options on the right hand side of (3.27). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting times of running relative drawdown/drawup to the strike K. If the running relative drawdown increases to K before τ_K^U and T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum and the running minimum are continuous, the above replicating portfolio never yields a payout from vanilla options with strikes outside the corridor $[m_t, M_t]$. When the running maximum increases or the running minimum decreases continuously with $t < \tau_K^D \wedge \tau_K^U \wedge T$, assumption **A5'** implies that it costs nothing to move the strikes of digital options in the above portfolio. Let us denote by τ the first time that $M_t - m_t \geq K$, then clearly $\tau = \tau_K^D \wedge \tau_K^U$. If $\tau > T$, then $M_T - K < m_T \leq S_T \leq M_T < m_T + K$, hence, all vanilla options in the replicating portfolio expire worthless, as does the target claim. If $\tau \leq T$, then at time τ, $M_\tau = m_\tau + K$, Proposition 3.2 and assumption **A5'** imply that the portfolio of digital options have the same value as the one-touch knockout $OTKO_\tau(M_\tau - K, M_\tau, T)$, whose value matches the target digital call, with value either zero or the price of a bond. In the former case, $\tau = \tau_K^U$, the one touches are liquidated for zero; while in the latter case, $\tau = \tau_K^D$, the liquidation proceeds are used to buy the bond. We conclude that in all cases, the payoff of a digital call can be replicated by trading bonds and vanilla options. The right hand side of (3.27) is the cost of setting up the replicating strategy at time t. Hence, no arbitrage implies that this cost is also the price of the target claim. $\square$

3.5 Static Replication with OTKO in Geometric Models

In the previous sections we developed static and semi-static replications under certain arithmetic symmetry assumptions. However, there are obvious financial drawbacks of this setup. For example, it requires no carrying cost for the underlying asset; the price of the underlying can be negative with positive probability. In what follows we will consider a more complicated setup to supersede these limitations.

As the spot price is always positive, it is much more convenient to consider the percentage drawdown and the percentage drawup:

Definition 3.5. *For any $t \in [0, T]$, we define the relative drawdown and the relative drawup processes respectively as*

$$D_t^r \triangleq M_t/S_t, \tag{3.28}$$

$$U_t^r \triangleq S_t/m_t. \tag{3.29}$$

For a fixed $K > 1$, let $\tau_K^{D^r}$ ($\tau_K^{U^r}$, resp.) be the time at which the relative drawdown (drawup, resp.) process D^r (U^r, resp.) first reaches K. As usual, if D^r (U^r, resp.) never reaches K, then we set $\tau_K^{D^r} = \infty$ ($\tau_K^{U^r} = \infty$, resp.).

We are interested in digital calls on maximum relative drawdown and digital calls on the K-relative drawdown preceding a K-relative drawup. A digital call on maximum relative drawdown pays $\$\mathbb{1}_{\{\tau_K^{D^r}\le T\}}$ at expiry. We denote the price of the option at time t by

$$DC_t^{MD^r}(K,T) \triangleq B_t(T)\mathbb{Q}_t^T(MD_T^r \ge K). \tag{3.30}$$

A digital call on the K-relative drawdown preceding a K-drawup pays $\$\mathbb{1}_{\{\tau_K^{D^r}\le\tau_K^{U^r}\wedge T\}}$ at expiry. We denote the price of this option at time t by

$$DC_t^{D^r<U^r}(K,T) \triangleq B_t(T)\mathbb{Q}_t^T(\tau_K^{D^r} \le \tau_K^{U^r} \wedge T). \tag{3.31}$$

Analogous to the absolute drawdown setting in section 3.1, we can replicate the payoff of the digital call on K-relative drawdown preceding a K-relative drawup with one-touch knockouts and their spreads. The argument is exact the same as in Theorem 3.1. We present the following theorem without proof.

Theorem 3.7 (Robust Replication: II)**.** *Under frictionless markets, no arbitrage implies that the digital call on the K-relative drawdown preceding a K-relative drawup can be valued relative to the prices of bonds, one-touch knockouts and their spreads:*

$$\begin{aligned} DC_t^{D^r<U^r}(K,T) &= \mathbb{1}_{\{\tau_K^D\le t\wedge\tau_K^U\}}B_t(T) + \mathbb{1}_{\{t<\tau_K^{D^r}\wedge\tau_K^{D^r}\}}\times \\ &\left\{OTKO_t(M_tK^{-1},M_t,T) + \int_{M_t^+}^{(m_tK)^-} RUFDI_t(HK^{-1},H,T)dH\right\}, \end{aligned} \tag{3.32}$$

for any $t\in[0,T]$ and $K>1$.

In the rest of the Chapter, we will develop semi-robust replications of the above two digital options under continuity and certain geometric symmetry assumptions on the dynamics of the spot price process.

3.6 Semi-static Replication with Single-Barrier One-touches in Geometric Models

In this section, we present semi-static hedges which just use single-barrier one-touches and lookbacks. The replications only succeed under certain symmetry and continuity assumptions. More specifically, suppose that the spot starts inside the corridor between V and W, where V and W are the in-barrier and out-barrier of an one-touch knockout respectively. Let $\tau \triangleq \tau^S_V \wedge \tau^S_W$ be the first exit time of the above corridor, we assume that:

G1: Skip-freedom and Geometric Hitting Symmetry *The spot price process $S_\cdot$ cannot exit the corridor between V and W by a jump. Moreover, there exist a constant q, such that if the first exit time of the above corridor $\tau \leq T$, we have*

$$\mathbb{Q}^T_\tau(\tau^S_{S_\tau \Delta^{-1}} \leq T) = \Delta^q \cdot \mathbb{Q}^T_\tau(\tau^S_{S_\tau \Delta} \leq T),\ \forall \Delta > 0. \tag{3.33}$$

Under the above assumption, an one-touch knockout with in-barrier V and out-barrier W is replicated by a portfolio of one-touches. In particular, we have:

Proposition 3.3 (Semi-static Pricing of One-touch Knockouts: III)**.** *Under frictionless markets and assumption* **G1***, no arbitrage implies that, for any* $t \in [0, \tau_V^S \wedge \tau_W^S \wedge T]$

$$OTKO_t(V, W, T) = \\ = OT_t(V, T) + \sum_{n=1}^{\infty} [\triangle^{-nq} OT_t(V\triangle^{-2n}, T) - \triangle^{nq} OT_t(V\triangle^{2n}, T)], \quad (3.34)$$

where $\triangle = W/V \neq 1$*.*

Proof. Suppose an one-touch knockout with in-barrier V and out-barrier W has been sold at time 0. In order to hedge this position, consider a strategy of being long a series of one-touches with barriers at V, $V\triangle^{-2}$, $V\triangle^{-4}, \ldots$, and also being short a series of one-touches with barriers at $V\triangle^2$, $V\triangle^4, \ldots$. If neither barrier is hit by T, then all one touches expire worthless. If $\tau_V^S \leq \tau_W^S \wedge T$, then at τ_V^S, the one-touch with payoff with barrier at V knocks in, while assumption **G1** implies that all of the other one-touches can be costlessly liquidated. The reason is that for each $n = 1, 2, \ldots$, the long position in the one-touches with barrier at $V\triangle^{-2n}$, is canceled by the short position in the one-touches with barrier at $V\triangle^{2n}$. Similarly, if $\tau_W^S \leq \tau_V^S \wedge T$, then at τ_W^S, assumption **G1** implies that all of the one-touches can be costlessly liquidated. The reason is that since $V = W\triangle^{-1}$, the portfolio can also be considered as long a series of one-touches with barriers at $W\triangle^{-1}$,

$W\triangle^{-3}$, $W\triangle^{-5}, \ldots$, while also being short a series of one-touches with barriers at $W\triangle$, $W\triangle^3$, $W\triangle^5 \ldots$:

$$\sum_{n=0}^{\infty}[\triangle^{-nq}OT_t(W\triangle^{-2n-1},T) - \triangle^{(n+1)q}OT_t(W\triangle^{2n+1},T)].$$

Hence, for each $n = 0, 1, 2, \ldots$, the long position in the one-touches with barrier at $W\triangle^{-2n-1}$, is canceled by the short position in the one-touch with barrier at $W\triangle^{2n+1}$. Since the value of the one-touch portfolio matches the payoff of the one-touch knockout when (S,t) exits $(V \wedge W, V \vee W) \times [0,T]$, no arbitrage forces the values prior to exit to be the same. $\square$

Remark 3.3. *Sbuelz [72] uses decomposition of Laplace transforms to obtain a similar result at $t = 0$ under geometric Brownian motion model. However, our proof does not involve analytic results and therefore is more robust.*

In virtue of Theorem 3.7 and discussion in Section 3.3, Proposition 3.3 plays a crucial role to develop replicating strategies with one-touches for the digital call on K-relative drawdown preceding K-relative drawup. Moreover, we will see that, under a similar assumption, the digital call on maximum drawdown can also be replicated with one-touches and lookbacks. In Subsections 3.6.1 and 3.6.2, we will separately develop portfolios to replicate the payoff of a digital call on maximum drawdown, the payoff of a digital call on the K-relative drawdown preceding a K-relative drawup, respectively.

3.6.1 Hedging digital call on maximum drawdown with one-touches and lookbacks in geometric models

In this subsection we develop a semi-static replication of a digital call on maximum drawdown using one-touches and lookbacks. For this purpose we impose the following assumption:

G2: Continuity of the Maximum, Drawdown, and Hitting Symmetry *While $t < \tau_K^{D^r}$, the running maximum is continuous, and the relative drawdown cannot jump up by more than $K - D_t^r$. Moreover, there exists a constant q, so that at times $\tau(u) \triangleq \tau_u^S \wedge \tau_K^{D^r} \wedge T$ for all $u > S_0$, we have*

$$\mathbb{Q}^T_{\tau(u)}(\tau^S_{S_{\tau(u)}\Delta^{-1}} \leq T) \quad = \quad \Delta^q \cdot \mathbb{Q}^T_{\tau(u)}(\tau^S_{S_{\tau(u)}\Delta} \leq T), \ \forall \Delta > 0. \tag{3.35}$$

The above assumption is clearly satisfied by geometric Brownian motion and its independent time-changes. The following result provides a semi-static replication for the digital call on maximum relative drawdown.

Theorem 3.8 (Semi-robust Pricing using One-touches: III)**.** *Under frictionless markets and assumption* **G2***, no arbitrage implies that the digital call on maximum relative drawdown can be valued relative to the prices of bonds,* *one-touches, and lookback options as:*

$$DC_t^{MD^r}(K,T) = \mathbb{1}_{\{\tau_K^{D^r}\le t\}}B_t(T) + \mathbb{1}_{\{t<\tau_K^{D^r}\}}\times\Bigg\{\sum_{n=0}^{\infty}K^{-2nq}OT_t(M_tK^{-4n-1},T)$$
$$+\sum_{n=0}^{\infty}\left(K^{(2n+1)q}OT_t(M_tK^{4n+1},T) - K^{2(n+1)q}OT_t(M_tK^{4n+3},T)\right)$$
$$-\sum_{n=0}^{\infty}K^{-(2n+1)q}OT_t(M_tK^{-4n-3},T)+q\left[LBP_t(M_t,K,T) - LBC_t(M_t,K,T)\right]\Bigg\}, \tag{3.36}$$

for $t\in[0,T]$ and $K>1$. Here the prices of the lookback put/call are given by,

$$LBP_t(M,K,T)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{K^{(n+1)q}}\int_0^{MK^{-(2n+3)}}\left(\frac{K^{2n+3}}{M/H}\right)^q P_n\left(\frac{q}{2}\log\frac{K^{2n+3}}{M/H}\right)OT_t(H,T)\frac{dH}{H}, \tag{3.37}$$

$$LBC_t(M,K,T)$$
$$=\sum_{n=0}^{\infty}(-1)^nK^{(n+1)q}\int_{MK^{2n+1}}^{\infty}P_n\left(\frac{q}{2}\log\frac{MK^{2n+1}}{H}\right)OT_t(H,T)\frac{dH}{H}, \tag{3.38}$$

where $P_n(x)$ is a polynomial of degree n, satisfying

$$P_0(x)=1, \quad P_n(0)=n+1, \tag{3.39}$$

$$P'_{n+1}(x) = P'_n(x) + 2P_n(x). \tag{3.40}$$

Proof. Suppose the digital call on maximum relative drawdown has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio on the right hand side of (3.36). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting time of running relative drawdown to the strike K. If the running relative drawdown increases to K before T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum can never increase by a jump, the above replicating portfolio never yields a payout due to a cross of the barriers higher than M_t. When the running maximum increases continuously with the maximum relative drawdown less than K, assumption **G2** can guarantee that it costs nothing to move the barriers of options in the above portfolio[3]. Hence, the first time to receive a cash flow from the above portfolio is at time $\tau \triangleq \tau_K^{D^r}$. If $\tau > T$, then the spot price S_t is always within $(M_t/K, M_t]$, so all the one-touches expiry worthless, as does the target claim. On the other hand, if $\tau \leq T$, then at time τ, $S_\tau = M_\tau/K$, assumption **G2** and (A.6) in Appendix A.1 imply that,

$$LBP_\tau(M_\tau, K, T) = LBC_\tau(M_\tau, K, T).$$

Moreover, at time τ, Proposition 3.3 and assumption **G2** imply that the

[3]Please refer to Appendix A.1 for a proof.

portfolio of one-touches has the same value as

$$OTKO_\tau(M_\tau/K, M_\tau K, T) + K^q \cdot OTKO_\tau(M_\tau K, M_\tau/K, T) = B_\tau(T).$$

We conclude that in all cases, the payoff of the target claim can be replicated by trading one-touches, lookbacks and bonds. The right hand side of (3.36) is the cost of setting up the replicating strategy at time t. Hence, no arbitrage implies that this cost is also the price of the call on maximum drawdown. $\square$

3.6.2 Hedging digital call on the K-relative drawdown preceding a K-relative drawup with one-touches in geometric models

In this subsection we develop a semi-static replication of a digital call on the K-relative drawdown preceding a K-relative drawup using one-touches. The following assumption ensures the validity of the replication.

G2': Continuity of the Maximum, Minimum, and Hitting Symmetry *While* $t \leq \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$, *the running maximum and the running minimum are continuous. Moreover, there exists a constant* q, *so that at times* $\theta(u) \triangleq \tau_u^{D^r} \wedge \tau_u^{U^r} \wedge T$ *for all* $u \in (1, K]$, *we have that*

$$\mathbb{Q}^T_{\theta(u)}(\tau^S_{S_{\theta(u)}\Delta^{-1}} \leq T) = \Delta^q \cdot \mathbb{Q}^T_{\theta(u)}(\tau^S_{S_{\theta(u)}\Delta} \leq T), \ \forall \Delta > 0. \tag{3.41}$$

Assumption **G2'** is sufficient for applying Proposition 3.3. Evaluating

(3.34) at $V = M_t/K$ and $W = M_t^+$, we obtain,

$$\begin{aligned}
&OTKO_t(M_t/K, M_t^+, T) = \\
&\quad = \sum_{n=0}^{\infty} \left\{ \frac{1}{K^{nq}} OT(M_t K^{-2n-1}, T) - K^{(n+1)q} OT_t(M_t K^{2n+1}, T) \right\}, \quad (3.42)
\end{aligned}$$

for $K > 0$ and $t \in [0, \tau^S_{M_t/K} \wedge \tau^S_{M_t^+} \wedge T]$. Differentiating (3.34) with respect to W, and evaluating at $V = H/K$ and $W = H$ implies that for $K > 1$ and $t \in [0, \tau^S_{H/K} \wedge \tau^S_H \wedge T]$:

$$\begin{aligned}
&RUFDI_t(H/K, H, T) \\
=&\frac{-2}{K} \sum_{n=1}^{\infty} n \left(\frac{1}{K^{(q+2)n}} \frac{\partial}{\partial B} OT_t(B, T) \Big|_{B=\frac{H}{K^{2n+1}}} + K^{(q+2)n} \frac{\partial}{\partial B} OT_t(B, T) \Big|_{B=\frac{H}{K^{-2n+1}}} \right) \\
&- \frac{q}{H} \sum_{n=1}^{\infty} n \left(\frac{1}{K^{nq}} OT_t(HK^{-2n-1}, T) + K^{nq} OT_t(HK^{2n-1}, T) \right) \\
=& -2 \sum_{n=1}^{\infty} n \left(\frac{1}{K^{nq}} \frac{\partial}{\partial H} OT_t(HK^{-2n-1}, T) + K^{nq} \frac{\partial}{\partial H} OT_t(HK^{2n-1}, T) \right) \\
&- \frac{q}{H} \sum_{n=1}^{\infty} n \left(\frac{1}{K^{nq}} OT_t(HK^{-2n-1}, T) + K^{nq} OT_t(HK^{2n-1}, T) \right), \qquad (3.43)
\end{aligned}$$

since K is a constant.

Substituting (3.42) and (3.43) in (3.32), and ignoring the left and right limits, we obtain

$$DC_t^{Dr}(K,T) = \mathbb{1}_{\{\tau_K^D \le t \wedge \tau_K^U\}} B_t(T) + \mathbb{1}_{\{t<\tau_K^D \wedge \tau_K^U\}} \times$$
$$\Bigg\{ \sum_{n=0}^{\infty} (2n+1) \left(\frac{1}{K^{nq}} OT(M_t K^{-2n-1}, T) + K^{(n+1)q} OT_t(M_t K^{2n+1}, T) \right)$$
$$- \sum_{n=1}^{\infty} 2n \left(\frac{1}{K^{nq}} OT_t(m_t K^{-2n}, T) + K^{nq} OT_t(m_t K^{2n}, T) \right)$$
$$- q \int_{M_t}^{m_t K} \sum_{n=1}^{\infty} n \left(\frac{OT_t(HK^{-2n-1}, T)}{K^{nq}} + \frac{OT_t(HK^{2n-1}, T)}{K^{-nq}} \right) \frac{dH}{H} \Bigg\},$$

which gives rise to:

Theorem 3.9 (Semi-robust Pricing using One-touches: IV)**.** *Under frictionless markets and assumption* **G2'***, no arbitrage implies that the digital call on the K-relative drawdown preceding a K-relative drawup can be valued relative to the prices of bonds and one-touches as:*

$$DC_t^{Dr}(K,T) = \mathbb{1}_{\{\tau_K^D \le t \wedge \tau_K^U\}} B_t(T) + \mathbb{1}_{\{t<\tau_K^D \wedge \tau_K^U\}} \times$$
$$\Bigg\{ \sum_{n=0}^{\infty} (2n+1) \left(\frac{1}{K^{nq}} OT(M_t K^{-2n-1}, T) + K^{(n+1)q} OT_t(M_t K^{2n+1}, T) \right)$$
$$- \sum_{n=1}^{\infty} 2n \left(\frac{1}{K^{nq}} OT_t(m_t K^{-2n}, T) + K^{nq} OT_t(m_t K^{2n}, T) \right)$$
$$- q \int_{M_t}^{m_t K} \sum_{n=1}^{\infty} n \left(\frac{OT_t(HK^{-2n-1}, T)}{K^{nq}} + \frac{OT_t(HK^{2n-1}, T)}{K^{-nq}} \right) \frac{dH}{H} \Bigg\}, \quad (3.44)$$

for any $t \in [0,T]$ and $K > 1$.

Proof. Suppose a digital call on the K-relative drawdown preceding a K-relative drawup has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of one-touches on the right hand side of (3.44). This semi-dynamic trading strategy is followed until the earlier of expiry and the first time at which the running relative drawdown or drawup reaches the strike K. If the running relative drawdown increases to K before T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum and the running minimum are continuous, the above replicating portfolio never yields a payout due to a hit of barriers outside the corridor $[m_t, M_t]$. When the running maximum increases or the running minimum decreases continuously with $t < \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$, assumption **G2'** guarantees that it costs nothing to move the barriers of one-touches in the above portfolio[4]. Hence, the first time to get a cash flow from the above portfolio is when $M_t/m_t = K$. Let us denote by τ the first time that $M_t/m_t \geq K$, then clearly $\tau = \tau_K^{D^r} \wedge \tau_K^{U^r}$. If $\tau > T$, then the one-touches expire worthless, as does the target claim. If $\tau \leq T$, then at τ, $M_\tau = m_\tau K$, by Proposition 3.3 and assumption **G2'**, the portfolio of one-touches has the same value as the one-touch knockout $OTKO_\tau(M_\tau/K, M_\tau, T)$, whose value matches the target option, with value either zero or the price of a bond. In the former case, $\tau = \tau_K^{U^r}$, the one touches are liquidated for zero; while in the latter case, $\tau = \tau_K^{D^r}$, the liquidation proceeds are used to buy the bond.

[4]Please refer to Appendix A.1 for a proof.

We conclude that in all cases, the payoff of the target digital call is matched by the liquidation value of a non-anticipating self-financing portfolio of bonds and one-touches. Furthermore, the right hand side of (3.44) is the cost of setting up the replicating strategy at time t. Hence, no arbitrage implies that this cost is also the price of the target claim. $\square$

3.7 Semi-static Replication with Vanilla Options in Geometric Models

In the previous section we developed semi-static hedges with a series of co-terminal single-barrier options of the target calls. In this section, we present another semi-static hedge which just uses more liquid vanilla options. The replications only succeed under some symmetry and continuity assumptions, which we will make precise.

Suppose that the spot starts inside the corridor between V and W, where V and W are the in-barrier and the out-barrier of an one-touch knockout respectively. Let τ be the first exit time of this corridor, we impose the following assumption:

G3: Skip-freedom and Geometric Symmetry *The spot S cannot exit the corridor between V and W by a jump. Moreover, there exist a constant*

q, such that if the first exit time of the above corridor $\tau \leq T$, we have

$$E_\tau^{\mathbb{Q}^T}\{\delta(S_T - S_\tau\Delta^{-1})\} = S_\tau^{-q} \cdot E_\tau^{\mathbb{Q}^T}\{S_T^q \delta(S_T - S_\tau\Delta)\},\ \forall\Delta > 0. \tag{3.45}$$

The symmetry in **G3** is often seen in finance literature. (Bowie and Carr [13]; Carr and Chou [17]; Carr, Ellies & Gupta [18]; Carr [16].) In particular, geometric Brownian motions and their independent time-changes all satisfy this assumption[5]. The characterization of continuous martingales that satisfy this symmetry conditions can be found in Tehranchi [86].

Remark 3.4. *If we alternatively assume that a barrier B is skip-free and (3.45) holds at the first hitting time τ_B^S, then an one-touch with barrier at B can be replicated with vanilla options. This is the reflection principle, which we present below for completeness.*

Lemma 3.1 (Reflection Principle)**.** *Under frictionless market, an one-touch with skip-free barrier $B > 0$ can be replicated with vanilla options, provided that (3.45) holds at τ_B^S. In particular, for any $t \in [0, \tau_B^S \wedge T]$,*

1. *If $B < m_t$,*

$$OT_t(B,T) = DP_t(B,T) + B^{-q}P_{q,t}(B,T); \tag{3.46}$$

[5] Please refer to Appendix A.3 for a proof.

2. If $B > M_t$,

$$OT_t(B,T) = DC_t(B,T) + B^{-q}C_{q,t}(B,T), \tag{3.47}$$

where the vanilla put/call prices $P_{q,t}/C_{q,t}$ are given by

$$P_{q,t}(B,T) \triangleq B_t(T)E_t^{\mathbb{Q}^T}\{S_T^q[1(S_T < B) + 0.5\delta(S_T - B)]\}, \tag{3.48}$$

$$C_{q,t}(B,T) \triangleq B_t(T)E_t^{\mathbb{Q}^T}\{S_T^q[1(S_T > B) + 0.5\delta(S_T - B)]\}. \tag{3.49}$$

Proof. We will only prove (3.46). (3.47) can be proven with a similar argument. Suppose an one-touch with barrier V has been sold at time 0. In order to hedge this position, consider a strategy of being long the two vanilla puts on the right hand side of (3.46). If the barrier V has not been hit by time T, then $S_T > B$, and hence, both vanilla puts expire worthless, as does the one-touch. Otherwise, let $\tau \triangleq \tau_B^S$, then at time τ, by assumption **G3**,

$$\begin{aligned}
&DP_\tau(B,T) + B^{-q}P_{q,\tau}(B,T)\\
=&B_\tau(T)E_\tau^{\mathbb{Q}^T}\{1(S_T < B) + 0.5\delta(S_T - B) + \frac{S_T^q}{B^q}[1(S_T < B) + 0.5\delta(S_T - B)]\}\\
=&B_\tau(T)E_\tau^{\mathbb{Q}^T}\{1(S_T < B) + 0.5\delta(S_T - B) + 1(S_T > B) + 0.5\delta(S_T - B)\}\\
=&B_\tau(T),
\end{aligned}$$

where the second equality follows from (3.45). Since the value of the vanilla puts matches the payoff of the one-touch when (S,t) exits $(B,\infty) \times [0,T]$, no arbitrage forces the value prior to exit to be the same. □

Remark 3.5. *If the spot price process is skip-free and satisfies* **G3***, it is easily seen that the condition (3.33) in assumption* **G2** *is also satisfied. In other words, for a skip-free process, the condition (3.45) in* **G3** *is stronger than (3.33) in* **G1**.

It is interesting to point out that, under **G3**, an one-touch knockout with in-barrier V and out-barrier W can be replicated by a portfolio of vanilla options:

Proposition 3.4 (Semi-static Pricing of One-touch Knockouts: IV)**.** *Under frictionless markets and assumption* **G3***, no arbitrage implies that, for* $t \in [0, \tau_V^S \wedge \tau_W^S \wedge T]$,

1. *If* $V < W$*:*

$$OTKO_t(V,W,T) = \sum_{n=0}^{\infty} \left(\frac{1}{\triangle^{nq}} DP_t(V\triangle^{-2n}, T) + \frac{\triangle^{nq}}{V^q} P_{q,t}(V\triangle^{-2n}, T) \right) - \sum_{n=1}^{\infty} \left(\triangle^{nq} DC_t(V\triangle^{2n}, T) + \frac{1}{V^q \triangle^{nq}} C_{q,t}(V\triangle^{2n}, T) \right), \quad (3.50)$$

2. *If* $V > W$*:*

$$OTKO_t(V,W,T) = \sum_{n=0}^{\infty} \left(\frac{1}{\triangle^{nq}} DC_t(V\triangle^{-2n}, T) + \frac{\triangle^{nq}}{V^q} C_{q,t}(V\triangle^{-2n}, T) \right) - \sum_{n=1}^{\infty} \left(\triangle^{nq} DP_t(V\triangle^{2n}, T) + \frac{1}{V^q \triangle^{nq}} P_{q,t}(V\triangle^{2n}, T) \right), \quad (3.51)$$

where $\triangle = W/V \neq 1$ *and* $P_{q,t}/C_{q,t}$ *are defined in (3.48) and (3.49).*

Proof. We will only prove (3.50) here. (3.51) can be proven with a similar argument. Suppose an one-touch knockout with lower in-barrier V and upper out-barrier W has been sold at time 0. In order to hedge this position, consider a strategy of being long a series of vanilla puts with strikes at $V, V\triangle^{-2}, V\triangle^{-4}, \ldots$, and also being short a series of calls with strikes at $V\triangle^{2}, V\triangle^{4}, \ldots$. If neither barrier is hit by T, then $S_T \in (V, W)$, and hence, all vanilla options expire worthless, as does the one-touch knockout. Otherwise, if $\tau_V^S \leq \tau_W^S \wedge T$, then at τ_V^S, assumption **G3** implies that the two puts struck at V can be traded in order to guarantee a unit payoff at expiry, as is seen in (3.46). Moreover, all the other vanilla options can be costlessly liquidated. The reason is that for each $n = 1, 2, \ldots$, the long position in the puts with strike at $V\triangle^{-2n}$, is canceled by the short position in the calls with strike at $V\triangle^{2n}$. On the other hand, if $\tau_W^S \leq \tau_V^S \wedge T$, then at this time, **G3** implies that all the vanilla options can be costlessly liquidated. The reason is that since $V = W\triangle^{-1}$, the portfolio can also be considered as long a series of puts with strikes at $W\triangle^{-1}, W\triangle^{-3}, W\triangle^{-5}$, while also being short a series of calls with strikes at $W\triangle, W\triangle^3, W\triangle^5$.

$$\sum_{n=0}^{\infty} \frac{1}{\triangle^{nq}} \left(DP_t(W\triangle^{-2n-1}, T) - \frac{1}{W^q} C_{q,t}(W\triangle^{2n+1}, T) \right) \\ + \sum_{n=0}^{\infty} \triangle^{(n+1)q} \left(\frac{1}{W^q} P_{q,t}(W\triangle^{-2n-1}, T) - DC_t(W\triangle^{2n+1}, T) \right).$$

Hence, for each $n = 0, 1, 2, \ldots$, the long position in the puts with strikes at $W\triangle^{-2n-1}$, is canceled by the short position in the calls with strike at $W\triangle^{2n+1}$. Since the value of the target option portfolio matches the payoff of the one-touch knockout when (S, t) exits $(V, W) \times [0, T]$, no arbitrage forces the values prior to exit to be the same. □

Lemma 3.1 and Proposition 3.4 provide fundamentals of our replication results in this section. In Subsections 3.7.1 and 3.7.2 we will separately develop portfolios of vanilla options to replicate the payoff of a digital call on maximum relative drawdown and the payoff of a digital call on the K-relative drawdown preceding a K-relative drawup, respectively.

3.7.1 Hedging digital call on maximum relative drawdown with vanilla options in geometric models

In this subsection we develop a semi-static replication of a digital call on maximum relative drawdown using vanilla options. Let us first state the necessary assumptions regarding the dynamics of the spot price process.

G4: Continuity of the Maximum, Drawdown, and Symmetry *While $t < \tau_K^{D^r}$, the running maximum is continuous, and the relative drawdown cannot jump up by more than $K - D_t^r$. Moreover, there exists a constant q, so that at times $\tau(u) \triangleq \tau_u^S \wedge \tau_K^{D^r} \wedge T$ for all $u > S_0$, we have that*

$$E_{\tau(u)}^{\mathbb{Q}^T}\{\delta(S_T - S_{\tau(u)}\Delta^{-1})\} = E_{\tau(u)}^{\mathbb{Q}^T}\left\{\frac{S_T^q}{S_{\tau(u)}^q}\delta(S_T - S_{\tau(u)}\Delta)\right\}, \ \forall\Delta > 0. \quad (3.52)$$

If the spot price process is always continuous, then using Theorem 3.8 and Lemma 3.1, we can develop a replicating portfolio of vanilla options to hedge the digital call on maximum relative drawdown. However, we will show in the next theorem that, under the weaker assumption **G4**, such a portfolio is also possible.

Theorem 3.10 (Semi-robust Pricing using Vanilla Options: III)**.** *Under frictionless markets and assumption* **G4***, no arbitrage implies that the digital call on maximum relative drawdown can be valued relative to the prices of* *bonds and vanilla options as:*

$$\begin{aligned}
DC_t^{MD^r}(K,T) = \mathbb{1}_{\{\tau_K^{D^r} \le t\}} B_t(T) + \mathbb{1}_{\{t < \tau_K^{D^r}\}} \times \\
\Bigg\{ \sum_{n=0}^{\infty} K^{-2nq} [DP_t(M_t K^{-4n-1}, T) + M_t^{-q} C_{q,t}(M_t K^{4n+1}, T)] \\
+ \sum_{n=0}^{\infty} K^{(2n+1)q} [DC_t(M_t K^{4n+1}, T) + M_t^{-q} P_{q,t}(M_t K^{-4n-1}, T)] \\
- \sum_{n=0}^{\infty} K^{2(n+1)q} [DC_t(M_t K^{4n+3}, T) + M_t^{-q} P_{q,t}(M_t K^{-4n-3}, T)] \\
- \sum_{n=0}^{\infty} K^{-(2n+1)q} [DP_t(M_t K^{-4n-3}, T) + M_t^{-q} C_{q,t}(M_t K^{4n+3}, T)] \\
+ q[VP_t(M_t, K, T) - VC_t(M_t, K, T)] \Bigg\},
\end{aligned} \tag{3.53}$$

for $t \in [0,T]$ *and* $K > 1$. *Here the prices of the vanilla put/call are given by,*

$$\begin{aligned}
VP_t(M,K,T) = \sum_{n=0}^{\infty} \frac{(-1)^n}{K^{(n+1)q}} \int_0^{MK^{-(2n+3)}} \left(\frac{K^{2n+3}}{M/H}\right)^q P_n\left(\frac{q}{2}\log\frac{K^{2n+3}}{M/H}\right) \times \\
[DP_t(H,T) + H^{-q} P_{q,t}(H,T)] \frac{dH}{H},
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
VC_t(M,K,T) = \sum_{n=0}^{\infty} (-1)^n K^{(n+1)q} \int_{MK^{2n+1}}^{\infty} P_n\left(\frac{q}{2}\log\frac{MK^{2n+1}}{H}\right) \times \\
[DP_t(H,T) + H^{-q} C_{q,t}(H,T)] \frac{dH}{H},
\end{aligned} \tag{3.55}$$

where $P_{q,t}/C_{q,t}$ *are given in (3.48) and (3.49), and polynomials* $\{P_n(x)\}$ *are defined in (3.39) and (3.40).*

Proof. Suppose a digital call on maximum relative drawdown has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of vanilla options in the right hand side of (3.53). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting time of running relative drawdown to the strike K. If the running relative drawdown increases to K before T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum can never increase by a jump, the above replicating portfolio never yields a payout from vanilla options with strikes higher than M_t. When the running maximum increases continuously with the maximum relative drawdown less than K, assumption **G4** guarantees that it costs nothing to move the strikes of vanilla options in the above portfolio[6]. Hence, the first time to receive a cash flow from the above portfolio is at time $\tau \triangleq \tau_K^{D^r}$. If $\tau > T$, since the running maximum cannot increase by a jump, the spot price at expiry $S_T \in (M_t/K, M_t]$, so all vanilla options being held expire worthless, as does the target claim. On the other hand, if $\tau \leq T$, then at time τ, $S_\tau = M_\tau/K$, assumption **G4**, (A.16) and (A.17) in Appendix B.1 imply that,

$$VP_\tau(M_\tau, K, T) = VC_\tau(M_\tau, K, T).$$

Moreover, at time τ, Proposition 3.4 and assumption **G4** imply that the

[6]Please refer to Appendix A.2 for a proof.

portfolio of DP/DC and P_q/C_q has the same value as

$$OTKO_\tau(M_\tau/K, M_\tau K, T) + K^q \cdot OTKO_\tau(M_\tau K, M_\tau/K, T) = B_\tau(T).$$

We conclude that in all cases, the payoff of a digital call can be replicated by trading bonds and vanilla options. The right hand side of (3.53) is the cost of setting up the replicating strategy at time t. Hence, no arbitrage implies that this cost is also the price of the target call on maximum drawdown. $\square$

3.7.2 Hedging digital call on the K-relative drawdown preceding a K-relative drawup with vanilla options in geometric models

In this subsection we develop a semi-static replication of a digital call on the K-relative drawdown preceding a K-relative drawup using vanilla options. We strengthen assumption **G4** in last subsection in order to meet the self-financing requirement of our replication portfolio.

G4': Continuity of the Maximum, Minimum, and Symmetry *While $t < \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$, the running maximum and the running minimum are continuous. Moreover, there exists a constant q, so that at times $\theta(u) \triangleq \tau_u^{D^r} \wedge \tau_u^{U^r} \wedge T$ for all $u \in (1, K]$, we have that*

$$E_{\theta(u)}^{\mathbb{Q}^T}\{\delta(S_T - S_{\theta(u)}\Delta^{-1})\} = E_{\theta(u)}^{\mathbb{Q}^T}\left\{\frac{S_T^q}{S_{\theta(u)}^q}\delta(S_T - S_{\theta(u)}\Delta)\right\}, \ \forall \Delta > 0. \quad (3.56)$$

Assumption **G4'** is sufficient for applying Proposition 8.1. Evaluating (3.50) at $V = M_t/K$ and $W = M_t$, we obtain,

$$\begin{aligned} & OTKO_t(M_t/K, M_t, T) \\ = & \sum_{n=0}^{\infty} \Bigg\{ \frac{1}{K^{nq}} [DP_t(M_t K^{-2n-1}, T) - M_t^{-q} C_{q,t}(M_t K^{2n-1}, T)]) \\ & + K^{(n+1)q} [M_t^{-q} P_{q,t}(M_t K^{-2n-1}, T) - DC_t(M_t K^{2n+1}, T)] \Bigg\}, \qquad (3.57) \end{aligned}$$

for $K > 1$.

Differentiating (3.50) with respect to W, and evaluating at $V = H/K$ and $W = H$ implies that for $K > 1$ and $t \in [0, \tau^S_{H/K} \wedge \tau^S_H \wedge T]$:

$$\begin{aligned} & RUFDI_t(H/K, H, T) \\ = & -2 \sum_{n=1}^{\infty} n \Bigg(\frac{1}{K^{nq}} \frac{\partial}{\partial H} DP_t(HK^{-2n-1}, T) + \frac{K^{(n+1)q}}{H^q} \frac{\partial}{\partial H} P_{q,t}(HK^{-2n-1}, T) \\ & + K^{nq} \frac{\partial}{\partial H} DC_t(HK^{2n-1}, T) + \frac{H^{-p}}{K^{(n-1)q}} \frac{\partial}{\partial H} C_{q,t}(HK^{2n-1}, T) \Bigg) \\ & - \frac{q}{H} \sum_{n=1}^{\infty} n \Bigg(\frac{1}{K^{nq}} DP_t(HK^{-2n-1}, T) - \frac{K^{(n+1)q}}{H^q} P_{q,t}(HK^{-2n-1}, T) \\ & + K^{nq} DC_t(HK^{2n-1}, T) - \frac{H^{-q}}{K^{(n-1)q}} C_{q,t}(HK^{2n-1}, T) \Bigg), \qquad (3.58) \end{aligned}$$

since K is a constant.

Substituting (3.57) and (3.58) in (3.32) we obtain that,

$$
\begin{aligned}
DC_t^{D^r}(K,T) = {} & \mathbb{1}_{\{\tau_K^D \le t \wedge \tau_K^U\}} B_t(T) + \mathbb{1}_{\{t < \tau_K^D \wedge \tau_K^U\}} \times \\
& \left\{ \sum_{n=0}^{\infty} \frac{(2n+1)}{K^{nq}} \left(DP_t(M_t K^{-2n-1}, T) + M_t^{-q} C_{q,t}(M_t K^{2n+1}, T) \right) \right. \\
& + \sum_{n=0}^{\infty} \frac{(2n+1)}{K^{-(n+1)q}} \left(DC_t(M_t K^{2n+1}, T) + M_t^{-q} P_{q,t}(M_t K^{-2n-1}, T) \right) \\
& - \sum_{n=1}^{\infty} \frac{2n}{K^{nq}} \left(DP_t(m_t K^{-2n}, T) + m_t^{-q} C_{q,t}(m_t K^{2n}, T) \right) \\
& - \sum_{n=1}^{\infty} \frac{2n}{K^{-nq}} \left(DC_t(m_t K^{2n}, T) + m_t^{-q} P_{q,t}(m_t K^{-2n}, T) \right) \\
& - q \int_{M_t}^{m_t K} \sum_{n=1}^{\infty} n \left(\frac{DP_t(HK^{-2n-1}, T)}{K^{nq}} + \frac{P_{q,t}(HK^{-2n-1}, T)}{K^{-(n+1)q} H^q} \right. \\
& \left. \left. + \frac{DC_t(HK^{2n-1}, T)}{K^{-nq}} + \frac{C_{q,t}(HK^{2n-1}, T)}{K^{(n-1)q} H^q} \right) \frac{dH}{H} \right\},
\end{aligned}
$$

which gives rise to:

Theorem 3.11 (Semi-robust Pricing using Vanilla Options: IV)**.** *Under frictionless markets and assumption* **G4'***, no arbitrage implies that the digital call on the K-relative drawdown preceding a K-relative drawup can be valued* *relative to the prices of bonds and vanilla options as:*

$$
\begin{aligned}
DC_t^{D^r}(K,T) = {} & \mathbb{I}_{\{\tau_K^D \le t \wedge \tau_K^U\}} B_t(T) + \mathbb{I}_{\{t < \tau_K^D \wedge \tau_K^U\}} \times \\
& \left\{ \sum_{n=0}^{\infty} \frac{(2n+1)}{K^{nq}} \left(DP_t(M_t K^{-2n-1}, T) + M_t^{-q} C_{q,t}(M_t K^{2n+1}, T) \right) \right. \\
& + \sum_{n=0}^{\infty} \frac{(2n+1)}{K^{-(n+1)q}} \left(DC_t(M_t K^{2n+1}, T) + M_t^{-q} P_{q,t}(M_t K^{-2n-1}, T) \right) \\
& - \sum_{n=1}^{\infty} \frac{2n}{K^{nq}} \left(DP_t(m_t K^{-2n}, T) + m_t^{-q} C_{q,t}(m_t K^{2n}, T) \right) \\
& - \sum_{n=1}^{\infty} \frac{2n}{K^{-nq}} \left(DC_t(m_t K^{2n}, T) + m_t^{-q} P_{q,t}(m_t K^{-2n}, T) \right) \\
& - q \int_{M_t}^{m_t K} \sum_{n=1}^{\infty} n \left(\frac{DP_t(HK^{-2n-1}, T)}{K^{nq}} + \frac{P_{q,t}(HK^{-2n-1}, T)}{K^{-(n+1)q} H^q} \right. \\
& \left. \left. + \frac{DC_t(HK^{2n-1}, T)}{K^{-nq}} + \frac{C_{q,t}(HK^{2n-1}, T)}{K^{(n-1)q} H^q} \right) \frac{dH}{H} \right\}, \qquad (3.59)
\end{aligned}
$$

for any $t \in [0,T]$ and $K > 1$.

Proof. Suppose a digital call on the K-relative drawdown preceding a K-relative drawup has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of vanilla options on the right hand side of (3.59). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting times of running relative drawdown/drawup to the strike K. If the running relative drawdown increases to K before T, then a bond of maturity T is held afterwards.

Since we assume that the running maximum and the running minimum are continuous, the above replicating portfolio never yields a payout from vanilla options with strikes outside the corridor $[m_t, M_t]$. When the running maximum increases or the running minimum decreases continuously with $t < \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$, assumption **G4'** guarantees that it costs nothing to move the strikes of vanilla options in the above portfolio[7]. Let us denote by τ the first time that $M_t/m_t \geq K$, then clearly $\tau = \tau_K^{D^r} \wedge \tau_K^{U^r}$. If $\tau > T$, then $M_T/K < m_T \leq S_T \leq M_T < m_T K$, hence, all vanilla options in the replicating portfolio expire worthless, as does the target claim. If $\tau \leq T$, then at time τ, $M_\tau = m_\tau K$, by Proposition 3.4 and assumption **G4'**, the portfolio of one-touches has the same value as the one-touch knockout $OTKO_\tau(M_\tau/K, M_\tau, T)$, whose value matches the target digital call, with value either zero or the price of a bond. In the former case, $\tau = \tau_K^{U^r}$, the one touches are liquidated for zero; while in the latter case, $\tau = \tau_K^{D^r}$, the liquidation proceeds are used to buy the bond. We conclude that in all cases, the payoff of a digital call can be replicated by trading bonds and vanilla options. The right hand side of (3.59) is the cost of setting up the replicating strategy at time t. Hence, no arbitrage implies that this cost is also the price of the target claim. $\square$

[7]Please refer to Appendix A.2 for a proof.

3.8 Poisson Jump Processes

In Sections 3.1-3.7 we developed static and semi-static replications of both digital options under certain continuity and symmetry assumptions. As it is pointed out earlier, the notion of continuity can be extended to skip-freedom so that purely jump models can be considered. In this section, we consider two different skip-free dynamical setups, increasing both complexity and financial realism. The first setup requires no carrying cost for the underlying asset and symmetry in the risk neutral price process. The second setup allows carrying costs and keeps prices positive. We refer to the two setups as the arithmetic case and the geometric case, respectively. In what follows we will develop replicating portfolio in both cases.

3.8.1 Arithmetic case

In this section, we require that the underlying has no carrying cost. This arises if the option we are concerned about is written on a forward price, or is written on a spot price, but only under stringent conditions (see Carr [16]). To cast the results of this section in their most favorable light, we will assume in this section that the barrier option is written on a forward price. The next section allows for nonzero carrying cost on the underlying asset.

Let F_t be the forward price at time $t \in [0, T]$. We assume that F is a continuous-time process. Under the risk-neutral measure $\mathbb{Q}^T$, F has representation

$$F_t = F_0 + a(N_{1,t} - N_{2,t}), \ t \in [0, T], \tag{3.60}$$

where $a > 0$ is a constant, N_1 and N_2 are independent identically distributed doubly stochastic processes (see Brémaud [15]), with jump intensity λ_t, which is independent of N_1 and N_2. In words, the forward price F starts at $F_0 > 0$ and jumps up or down by the amount a according to an independent clock. Clearly, F will satisfy all arithmetic symmetry conditions **A1**-**A5'**, if we extend the notion of continuity to skip-freedom. It follows that[8] we can construct replicating portfolios of one-touches or vanilla digital options once we have a replication with one-touch knockouts and their spreads in our hands.

Without loss of generality, let us assume that K is a positive integer multiple of a, so that overshoots are avoided. Since the replicating portfolio in Theorem 3.1 is purely static, one can easily extend (3.8) to the case in which the underlying is a skip-free process. More specifically, when the underlying process follows (3.60), a ricochet-upper-first down-and-in claim is a real

[8]This a consequence of Propositions 3.1 and 3.2.

spread of one-touch knockouts

$$\begin{aligned} RUFDI_t(H-K,H,T) &= B_t(T)E_t^{\mathbb{Q}^T}\{1(m_T \leq H-K)\delta(M_{\tau^F_{H-K}} - H)\} \\ &= OTKO_t(H-K,H+a,T) - OTKO_t(H-K,H,T), \end{aligned} \quad (3.61)$$

from which one immediately obtain the following counterpart of Theorem 3.1:

$$\begin{aligned} DC_t^{D<U}(K,T) = \mathbb{1}_{\{\tau_K^D \leq \tau_K^U \wedge t\}} B_t(T) + \mathbb{1}_{\{t < \tau_K^D \wedge \tau_K^U\}} \times \Bigg\{ OTKO_t(M_t - K, M_t + a, T) \\ + \sum_{i=1}^{\frac{m_t+K-M_t}{a}-1} RUFDI_t(M_t + ai - K, M_t + ai, T) \Bigg\}, \end{aligned} \quad (3.62)$$

for any $t \in [0,T]$ and $K > 0$.

Similarly, one can modify (3.10) slightly to obtain a replication of digital call on maximum drawdown:

$$\begin{aligned} DC_t^{MD}(K,T) = 1(MD_t \geq K)B_t(T) + 1(MD_t < K) \times \\ \{OTKO_t\,(M_t - K, M_t + K + a, T) + OTKO_t\,(M_t + K + a, M_t - K, T)\}, \end{aligned} \quad (3.63)$$

for $t \in [0,T]$ and $K > 0$. The portfolio on the right hand side of (3.63) obviously replicates the payoff of the digital call on maximum drawdown. Moreover, it is self-financing. This is because, when the maximum drawdown is less than K, using the symmetry of the underlying one can show that, whenever the maximum has an increase from M_{t^-} to $M_t = M_{t^-} + a$,

$$
\begin{aligned}
&OTKO_t\left(M_t - K, M_t + K + a, T\right) + OTKO_t\left(M_t + K + a, M_t - K, T\right) \\
=&OTKO_t(M_t + K, M_t - K - a, T) + OTKO_t(M_t - K - a, M_t + K, T) \\
=&OTKO_t(M_{t^-} + K + a, M_{t^-} - K, T) + OTKO_t(M_{t^-} - K, M_{t^-} + K + a, T).
\end{aligned}
$$

Let us now proceed to treat the complications that arise if we allow carrying costs on the underlying and if we further require that the underlying price process stays positive.

3.8.2 Geometric case

In this section, we will assume that all options are written on the spot price of some underlying asset. Let us consider a filtered risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{Q}^T)$, $\mathcal{F} = \cup_{t\in[0,T]}\mathcal{F}_t$. Let us denote by N_1 and N_2 two independent standard doubly stochastic processes, with positive jump arrival rates λ_1 and λ_2 under the risk-neutral measure $\mathbb{Q}^T$. We require that the trajectories of the intensities λ_1 and λ_2 are $\mathcal{F}_0$-measurable[9], and the ratio λ_1/λ_2 is a constant.

For given positive constants g and S_0, we assume the stochastic process governing the spot price of the underlying asset is given by

$$S_t = S_0 e^{g(N_{1,t} - N_{2,t})}, \ t \in [0, T]. \tag{3.64}$$

[9]However, we do not need to specify the trajectories.

In words, the spot price S starts at $S_0 > 0$ and jumps up by the amount $S_{t^-}(e^g - 1) > 0$ or down by the amount $S_{t^-}(e^{-g} - 1) < 0$ at independent exponential times. Let r_t and d_t be the instantaneous risk-free rate and the instantaneous dividend yield of the underlying respectively. Then under a frictionless market and no arbitrage, we must always have

$$\lambda_{1,t}(e^g - 1) + \lambda_{2,t}(e^{-g} - 1) = r_t - d_t, \ t \in [0, T]. \tag{3.65}$$

Before developing any replication portfolio, let us first examine the symmetry properties of the spot price process. Under the risk neutral measure $\mathbb{Q}^T$, the log price is a difference of two independent Poisson processes.

$$d \log S_t = g(dN_{1,t} - dN_{2,t}), \ t \in [0, T]. \tag{3.66}$$

One could employ Esscher transform (see Brémaud [15]; Shiryaev [74]) to construct a new probability measure equivalent to $\mathbb{Q}^T$, under which the log price $\log S$ is a symmetric martingale. More specifically, let us define a constant

$$\pi = \frac{1}{2g} \log \frac{\lambda_{2,0}}{\lambda_{1,0}}. \tag{3.67}$$

Then we have a positive martingale

$$\begin{aligned} Y_t &= \exp\left(\pi g(N_{1,t} - N_{2,t}) - \int_0^t \left(\lambda_{1,s}(e^{\pi g} - 1) + \lambda_{2,s}(e^{-\pi g} - 1)ds\right)\right) \\ &= \left(\frac{S_t}{S_0}\right)^{\pi} \cdot \phi(t), \ t \in [0, T], \end{aligned} \tag{3.68}$$

where $\phi(t) = \exp\left(-\int_0^t [\lambda_{1,s}(e^{\pi g} - 1) + \lambda_{2,s}(e^{-\pi g} - 1)]ds\right)$. Define a new measure $\mathbb{P}^T$ by

$$E_t^{\mathbb{P}^T}\{Z\} = \frac{1}{Y_t} E_t^{\mathbb{Q}^T}\{ZY_T\}, \tag{3.69}$$

for any $\mathcal{F}_T$-measurable random variable Z. Under $\mathbb{P}^T$, the log price $\log S$ is a difference of two independent identically distributed doubly stochastic processes with jump intensity $e^{\pi g}\lambda_1$. Thus, at any time $t \in [0, T]$, for any $\Delta > 0$

$$E_t^{\mathbb{P}^T}\{\delta(S_T - S_t\Delta^{-1})\} = E_t^{\mathbb{P}^T}\{\delta(S_T - S_t\Delta)\}. \tag{3.70}$$

It follows that,

$$\begin{aligned} E_t^{\mathbb{Q}^T}\{\delta(S_T - S_t\Delta^{-1})\} &= E_t^{\mathbb{P}^T}\left\{\left(\frac{S_T}{S_t}\right)^{-\pi} \cdot \frac{\phi(t)}{\phi(T)}\delta(S_T - S_t\Delta^{-1})\right\} \\ = \quad E_t^{\mathbb{P}^T}\left\{\left(\frac{S_T}{S_t}\right)^{\pi} \cdot \frac{\phi(t)}{\phi(T)}\delta(S_T - S_t\Delta)\right\} &= E_t^{\mathbb{Q}^T}\left\{\left(\frac{S_T}{S_t}\right)^{2\pi}\delta(S_T - S_t\Delta)\right\}, \end{aligned}$$

for all $\Delta > 0$. In other words, the spot price process will satisfy **G4'**, if we extend the notion of continuity to skip-freedom.

By the discussion in Remark 3.4, and the fact that the spot price process is skip-free, it follows that S will satisfy all geometric symmetry conditions in **G1-G4'**. Therefore, it suffices to develop the counterparts of Theorem 3.7 and Theorem 3.8 for the model in (3.64).

Without lose of generality, let us assume that $\log K$ is a positive integer multiple of g, so that overshoots are avoided. Since the result in Theorem 3.7 is purely static, it can be easily extended to the model in (3.64). More specifically, a ricochet-upper-first down-and-in claim is a real spread of one-touch knockouts

$$\begin{aligned} RUFDI_t(H/K, H, T) &= B_t(T) E_t^{\mathbb{Q}^T} \{1(m_T \leq H/K)\delta(M_{\tau^S_{H/F}} - H)\} \\ &= OTKO_t(H/K, He^g, T) - OTKO_t(H/K, H, T), \quad (3.71) \end{aligned}$$

from which one immediately obtains

$$\begin{aligned} DC_t^{D^r<U^r}(K,T) \;=\; & \mathbb{1}_{\{\tau_K^{D^r} \leq \tau_K^{U^r}\}} B_t(T) + \mathbb{1}_{\{t<\tau_K^{D^r} \wedge \tau_K^{U^r}\}} \times \\ & \left\{ OTKO_t(M_t/K, M_t e^g, T) + \sum_{i=1}^{\frac{1}{g}\log\frac{m_t K}{M_t} - 1} RUFDI_t(M_t e^{ig}/K, M_t e^{ig}, T) \right\}, \end{aligned} \quad (3.72)$$

for any $t \in [0, T]$ and $K > 0$.

Similarly, the result in Theorem 3.8 can be extended. In fact, one can show that, a digital call on maximum relative drawdown can be replicated with bonds, one-touch knockouts and lookbacks:

$$\begin{aligned} DC_t^{MD^r}(K,T) \;=\; & \mathbb{1}_{\{\tau_K^{D^r} \leq t\}} B_t(T) + \mathbb{1}_{\{t<\tau_K^{D^r}\}} \times \\ & \{OTKO_t(M_t/K, M_t K e^g, T) + K^q \cdot OTKO_t(M_t K e^g, M_t/K, T) \\ & + (1 - e^{-qg})[LBP_t(M_t, K, T) - LBC_t(M_t, K, T)]\}, \quad (3.73) \end{aligned}$$

for any $t \in [0, T]$ and $K > 0$. Here the prices of lookback put/call are given by

$$LBP_t(M, K, T) = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-\lceil \frac{n}{2} \rceil qg}}{K^{(n+1)q}} \sum_{i \le \frac{1}{g} \log \frac{M}{K^{2n+3}}} \left(\frac{K^{2n+3}}{Me^{-ig}} \right)^q P_n \left(\log \frac{K^{2n+3}}{Me^{-ig}} \right) \times OT_t(e^{(i-2\lfloor \frac{n+1}{2} \rfloor -1)g}, T), \quad (3.74)$$

$$LBC_t(M, K, T) = \sum_{n=0}^{\infty} \frac{(-1)^n}{K^{-(n+1)q}} e^{\lceil \frac{n+1}{2} \rceil qg} \sum_{i > \frac{1}{g} \log(MK^{2n+1})} P_n \left(\log \frac{MK^{2n+1}}{e^{(i-1)g}} \right) \times \times OT_t(e^{(i+2\lfloor \frac{n}{2} \rfloor +1)g}, T), \quad (3.75)$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the floor and the ceiling functions (Graham et al. 1994), and P_n is a function on the lattice $\mathbb{Z} \cdot g$, satisfying

$$P_0 = 1, \; P_n(0) = n + 1, \quad (3.76)$$

$$P_{n+1}((i+1) \cdot g) - P_{n+1}(i \cdot g) = e^{qg} P_n((i+1) \cdot g) - P_n(i \cdot g). \quad (3.77)$$

We omit the proof here. The interested reader can verify this result following the argument appearing in Appendix A.1.

3.9 Conclusion

In this work we developed static replications of a digital call on the K-drawdown preceding a K-drawup. We then developed semi-static replications of these options using consecutively more liquid instruments under appropriate symmetry and continuity assumptions. We considered two different dynamical setups, increasing in complexity and financial realism. In both cases, our portfolio is self-financing, and only needs occasional trading, typically when the maximum or the minimum changes. Finally, we extend the replication results to the case in which the underlying process is driven by the difference of two independent Poisson processes. We showed that the previous semi-static trading strategies continue to replicate the payoffs of these claims with slight modifications.

Chapter 4

Quickest Detection of Abrupt Changes with Multi-Source Observations

In this Chapter we study the applications of drawup processes in the problem of quickest detection with multi-source observations. We consider the situation in which the onset of a signal occurs at different times in the observations originating from N different sources. We also consider the case of equal-strength and unequal-strength signals across all sources, which in discrete-time models corresponds to the cases of the same and different out-of-control distributions. We adopt an N-dimensional extension of the CUSUM stopping rule, namely the N-CUSUM stopping rule. We assume that the N observed processes are independent, which constitutes an assumption consistent with the fact that the N change-points can be different.

In this work we consider the problem of detecting the earliest change observed in the system. We start by considering the problem in a Brownian motion model in Section 4.1. As our problem involves multiple source of observations, we extend Lorden's criterion (see [51]) in a min-max way as described in Section 4.1.1. Properties of the single source observations are presented in Section 4.1.2. In Section 4.1.3, the N-CUSUM rule is introduced for detecting the earliest change in a Brownian motion model. It is shown that, under the extended Lorden's criterion, the difference between the N-CUSUM stopping rule with the unknown optimal stopping rule tends to a constant, as the mean time to the first false alarm tends to infinity. In Section 4.2, we extend these optimality results of N-CUSUM rule to discrete-time models. Finally in Section 4.3, we close with concluding remarks and suggestions for future work.

4.1 The Brownian motion model

In this section we consider a continuous-time Brownian motion model.

4.1.1 Mathematical formulation of the problem

We sequentially observe the processes $\{\xi_t^{(i)}; t \geq 0\}$ for all $i = 1, \ldots, N$ with the following dynamics:

$$d\xi_t^{(i)} = \begin{cases} dw_t^{(i)} & t \leq \tau_i \\ \mu_i \, dt + dw_t^{(i)} & t > \tau_i, \end{cases} \tag{4.1}$$

where positive constants $\{\mu_i\}$ are known and represent the signal strengths, $\{w_t^{(i)}\}$ are independent standard Brownian motions, and the τ_i's are unknown constants, with τ_i representing the time point of onset of the signal from source S_i.

An appropriate measurable space is $\Omega = C[0, \infty) \times C[0, \infty) \times \ldots \times C[0, \infty)$ and $\mathcal{F} = \cup_{t>0} \mathcal{F}_t$, where $\{\mathcal{F}_t\}$ is the filtration of the observations with $\mathcal{F}_t = \sigma\{(\xi_s^{(1)}, \ldots, \xi_s^{(N)}); s \leq t\}$. Notice that in the case of centralized detection the filtration consists of the totality of the observations that have been received up until the specific point in time t.

On this space, we have the following family of probability measures $\{P_{\tau_1, \ldots, \tau_N}\}$, where $P_{\tau_1, \ldots, \tau_N}$ corresponds to the measure generated on Ω by the processes $(\xi_t^{(1)}, \ldots, \xi_t^{(N)})$ when the change in the N-tuple process occurs at time point τ_i, $i = 1, \ldots, N$. Notice that the measure $P_{\infty, \ldots, \infty}$ corresponds to the measure generated on Ω by N independent Brownian motions without drifts.

Our objective is to find a stopping rule T that balances the trade-off between a small detection delay subject to a lower bound on the mean-time between false alarms and will ultimately detect $\min\{\tau_1, \ldots, \tau_N\}$ [1].

As a performance measure we consider the following generalization of Lorden's performance index (see Lorden [51]):

$$J^{(N)}(T) \quad = \quad \sup_{\tau_1,\ldots,\tau_N} \operatorname{essup} E_{\tau_1,\ldots,\tau_N} \left\{ (T - \tau_1 \wedge \ldots \wedge \tau_N)^+ | \mathcal{F}_{\tau_1 \wedge \ldots \wedge \tau_N} \right\}, \tag{4.2}$$

where the supremum over $\tau_1, \ldots, \tau_N$ is taken over the set in which their minimum is finite. That is, we consider the worst detection delay over all possible realizations of paths of the N-tuple of stochastic processes $(\xi_t^{(1)}, \ldots, \xi_t^{(N)})$ up to $\min\{\tau_1, \ldots, \tau_N\}$ and then consider the worst detection delay over all possible N-tuples $\{\tau_1, \ldots, \tau_N\}$ over a set in which at least one of them is forced to take a finite value. This is because T is a stopping rule meant to detect the minimum of the N change-points and therefore if one of the N processes undergoes a regime change, any unit of time by which T delays in reacting, should be counted towards the detection delay.

The performance index presented in (4.2) results in the corresponding stochastic optimization problem of the form:

$$\begin{aligned} &\inf_{T} J^{(N)}(T) \\ \text{subject to} \quad &E_{\infty,\ldots,\infty}\{T\} \geq \gamma. \end{aligned} \tag{4.3}$$

[1] In what follows we will use $\tau_1 \wedge \ldots \wedge \tau_N$ to denote $\min\{\tau_1, \ldots, \tau_N\}$.

We notice that the expectation in the above constraint is taken under the measure $P_{\infty,\ldots,\infty}$. This is the measure generated on the space Ω in the case that none of the N processes $(\xi_t^{(1)},\ldots,\xi_t^{(N)})$ changes regime. Therefore, $E_{\infty,\ldots,\infty}\{T\}$ is the mean time to the first false alarm, and γ is the minimal acceptable value for this quantity. And it is easily seen that, in seeking solutions to the above problem, we can restrict our attention to stopping rules that satisfy the false alarm constraint with equality (see Moustakides [56]). To this effect, we introduce the following definition:

Definition 4.1. *Define $\mathcal{K}_\gamma$ to be set all $\mathcal{F}_t$-adapted stopping rules T that satisfy*

$$E_{\infty,\ldots,\infty}\{T\} = \gamma. \tag{4.4}$$

4.1.2 The 1D CUSUM stopping rule

In the case of only a single observation process (say $\{\xi_t^{(1)}\}$), the problem becomes one of detecting a one-sided change in a sequence of Brownian observations, whose optimal solution was found in Beibel [9] and Shiryaev [73]. The optimal solution is the continuous-time version of Page's CUSUM stopping rule. More specifically, the CUSUM stopping rule is the drawup of the log-likelihood ratio process.

Definition 4.2. *Define the following processes:*

$$y_t^{(1)} = \sup_{0\le\tau_1\le t} \log \frac{dP_{\tau_1}}{dP_\infty}\bigg|_{\mathcal{F}_t} = u_t^{(1)} - \inf_{0\le s\le t} u_s^{(1)}, \text{ where} \tag{4.5}$$

$$u_t^{(1)} = \mu_1 \xi_t^{(1)} - \frac{1}{2}\mu_1^2 t. \tag{4.6}$$

Then the CUSUM stopping rule is defined as the first hitting time

$$T_\nu^1 = \inf\{t \ge 0 | y_t^{(1)} \ge \nu\}, \tag{4.7}$$

where ν *is chosen so that* $E_\infty\{T_\nu\} = \frac{2}{\mu_1^2} f(\nu) = \gamma$, *with* $f(\nu) = e^\nu - \nu - 1$.

The one dimensional CUSUM stopping rule is optimal under Lorden's criterion. We present this property in the following lemma:

Lemma 4.1. *In the one dimensional case, the optimal stopping rule to question (4.3) is the CUSUM stopping rule. Moreover,*

$$\inf_{T\in\mathcal{K}_\gamma} J^{(1)}(T) = J^{(1)}(T_\nu) = E_0\{T_\nu\} = \frac{2}{\mu_1^2} f(-\nu), \tag{4.8}$$

with $f(\nu) = e^\nu - \nu - 1$. *Moreover, as* $\gamma = E_\infty\{T_\nu\} \to \infty$,

$$E_0\{T_\nu\} = \frac{2}{\mu^2}\log(\gamma) + o(1). \tag{4.9}$$

Proof. See Shiryaev [73]. □

The fact that the worst detection delay is the same as that incurred in the case in which the change-point is exactly 0 is a consequence of the strong

Markov property of the CUSUM process, from which it follows that the worst detection delay occurs when the CUSUM process at the time of the change is at 0 (see Hadjiliadis & Moustakides [36]).

Remark 4.1. *If the N change-points were the same, then the problem (4.3) is equivalent to observing only one stochastic process which is now N-dimensional. Thus, in this case, the solution is the same as that given in the above paragraph with $y_t^{(1)}$ replaced by the projection of $(y_t^{(1)}, \ldots, y_t^{(N)})$ onto the N-vector of all 1's.*

Let us now proceed to treat the general case when $N > 1$.

4.1.3 Equalizer rules and the N-CUSUM stopping rule I

In the general cases when $N > 1$, no optimal solution is known for problem (4.3). However, it can be shown that the optimal solution, T^*, must be an equalizer rule. That is, it must display the same detection delay regardless of which of the processes $\{\xi_t^{(i)}; t \geq 0\}$, $i = 1, \ldots, N$ undergoes a change first. This property is summarized in the following lemma:

Lemma 4.2. *For any $T \in \mathcal{K}_\gamma$, define partial detection delay indices:*

$$J_i^{(N)}(T) \quad \stackrel{\triangle}{=} \quad \sup_{\tau_i \leq \tau_j, j \neq i} \text{essup} E_{\tau_1,\ldots,\tau_N}\{(T - \tau_i)^+ | \mathcal{F}_t\},$$

for $i = 1, \ldots, N$. Then the optimal solution to (4.3), T^*, satisfies

$$J_1^{(N)}(T^*) \;=\; J_2^{(N)}(T^*) \;=\; \ldots \;=\; J_N^{(N)}(T^*). \qquad (4.10)$$

Proof. Please refer to Hadjiliadis, Zhang & Poor [39] for a proof. $\square$

Returning to problem (4.3), the optimality of the CUSUM stopping rule in the presence of only one observation process suggests that a CUSUM type of stopping rule might display similar optimality properties in the case of multiple observation processes. In particular, an intuitively appealing rule, when the detection of $\min\{\tau_1, \ldots, \tau_N\}$ is of interest, is $T_h = T_h^1 \wedge \ldots \wedge T_h^N$, where T_h^i is the CUSUM stopping rule for the process $\{\xi_t^{(i)}; t \geq 0\}$ for $i = 1, \ldots, N$. In particular, we employ a general -threshold N-CUSUM stopping rule $T_\hbar \in \mathcal{K}_\gamma$:

$$T_\hbar \;=\; \inf\left\{t \geq 0 | \max\left\{\frac{y_t^{(1)}}{h_1}, \ldots, \frac{y_t^{(N)}}{h_N}\right\} \geq 1\right\}, \qquad (4.11)$$

where $\{y_t^{(i)}\}$ is the CUSUM statistic process of $\{\xi_t^{(i)}\}$, and $\hbar \stackrel{\triangle}{=} (h_1, h_2, \ldots, h_N)$ is the vector of thresholds, such that $E_{\infty,\ldots,\infty}\{T_\hbar\} = \gamma$.

The N-CUSUM stopping rules share some striking properties, one of which is that its partial performance measures $\{J_i(T_\hbar)\}$ have simple representations:

Lemma 4.3. *For the N-CUSUM stopping rule defined in (4.11), we have*

$$J_1(T_\hbar) = E_{0,\infty,\dots,\infty}\{T_\hbar\},\ J_2(T_\hbar) = E_{\infty,0,\dots,\infty}\{T_\hbar\}$$
$$\dots,\ J_N(T_\hbar) = E_{\infty,\dots,\infty,0}\{T_\hbar\}.$$

Proof. This is because the worst detection delay occurs when only one of the N processes changes regime. The reason for this lies in the fact that the CUSUM process is a monotone function of μ, resulting in a longer on average passage time if $\mu = 0$ (see Hadjiliadis & Moustakides [36]). Thus, the worst detection delay will occur when none of the other processes changes regime, and due to the non-negativity of the CUSUM process the worst detection delay will occur when the CUSUM process of the remaining one process is at 0. $\square$

In virtue of Lemma 4.2, the optimal choice of the thresholds for the N-CUSUM stopping rule $T_\hbar \in \mathcal{K}_\gamma$ should satisfy

$$J^{(N)}(T_h) = E_{0,\infty,\dots,\infty}\left\{T_\hbar\right\} = E_{\infty,0,\infty,\dots,\infty}\left\{T_\hbar\right\} = \dots = E_{\infty,\dots,\infty,0}\left\{T_\hbar\right\}. \tag{4.12}$$

However, the exact optimal N-CUSUM stopping rule requires solving the implicit system (4.12), which is very hard task. When only asymptotic performance is concerned, it suffices to give an asymptotic characterization of the optimal choice of thresholds for large γ. In the following lemma we present an explicit condition on thresholds such that (4.12) holds asymptotically.

Lemma 4.4. *For $\hbar = (h_1, h_2, \ldots, h_N)$ such that*

$$\frac{1}{\mu_1^2}(h_1 - 1) \;=\; \frac{1}{\mu_2^2}(h_2 - 1) \;=\; \ldots \;=\; \frac{1}{\mu_N^2}(h_N - 1), \tag{4.13}$$

(4.12) holds asymptotically, and as $h_1 \to \infty$,

$$J^{(N)}(T_\hbar) \;=\; \frac{2}{\mu_1^2}(h_1 - 1) + o(1). \tag{4.14}$$

Proof. Please refer to Appendix B.1 for the proof. □

Not surprisingly, Lemma 4.4 suggests common thresholds across all components in the case of equal drifts after the changes. In the case of unequal drifts after the changes, we need to adjust the thresholds according to the drifts, or more specifically, equation (4.13).

Without loss of generality, let us assume that

$$\mu_1 = \mu_2 = \ldots = \mu_k < \min_{i>k}\{\mu_i\}. \tag{4.15}$$

We note that $J^{(N)}(T^*)$ is bounded from below by the detection delay of the one CUSUM when there is only one observation process, say only the first one, in view of the fact that

$$\begin{aligned} &\sup\nolimits_{\tau_1,\ldots,\tau_N} \operatorname{essup} E_{\tau_1,\ldots,\tau_N}\left\{(T - \tau_1 \wedge \ldots \wedge \tau_N)^+ | \mathcal{F}_{\tau_1 \wedge \ldots \wedge \tau_N}\right\} \\ &\quad\geq\; \sup\nolimits_{\tau_1} \operatorname{essup} E_{\tau_1}\left\{(T - \tau_1)^+ | \mathcal{F}^{(1)}_{\tau_1}\right\}, \end{aligned}$$

where $\mathcal{F}^{(1)}_{\tau_1} = \sigma\{\xi^{(1)}_s; s \le \tau_1\}$. Notice that the above inequality holds for all stopping rule T adapted to the filtration $\{\mathcal{F}^{(1)}_t\}$. The stopping rule that minimizes $\sup_{\tau_1} \operatorname{essup} E_{\tau_1}\left\{(T-\tau_1)^+|\mathcal{F}^{(1)}_{\tau_1}\right\}$ is the CUSUM stopping rule $T^1_{\nu_1}$ of (2.22), with ν_1 chosen so as to satisfy

$$E^1_\infty\{T^1_{\nu_1}\} \;=\; \gamma. \tag{4.16}$$

We begin by bounding the detection delay $J^{(N)}$ of the unknown optimal stopping rule T^* both above and below by

$$J^{(N)}(T_\hbar) \;\ge\; J^{(N)}(T^*) \;\ge\; \max_{1\le i\le N}\left\{E_0\{T^i_{\nu_i}\}\right\}, \tag{4.17}$$

where $\{\nu_i\}_{i=1}^N$ are chosen so that

$$E_\infty\{T^i_{\nu_i}\} \;=\; \gamma,\ i=1,\dots,N. \tag{4.18}$$

We will demonstrate that the difference between the upper and the lower bounds tends to zero as $\gamma\to\infty$, with $\hbar$ and ν_i satisfying (4.4), (4.13) and (4.18).

More specifically, we have

Proposition 4.1. *Under (4.15), for $\hbar = (h_1, h_2, \dots, h_N)$ satisfying (4.4) and (4.13),*

$$J^{(N)}(T_\hbar) \;=\; \frac{2}{\mu_1^2}\left[\log\gamma + \log\frac{k\mu_1^2}{2} - 1 + o(1)\right], \tag{4.19}$$

as $\gamma\to\infty$.

Proof. Please refer to Appendix B.1 for the proof. □

It is worth pointing out that Proposition 4.1 justifies us in ignoring signals with stronger strength as long as only asymptotic behavior is concerned. By examining the asymptotic difference of the upper and the lower bounds in (4.17), we obtain

Theorem 4.1. *When the number of signals with weakest strengths is k, the difference in detection delay $J^{(N)}$ of the unknown optimal stopping rule T^* and the detection delay of $T_\hbar$ of (4.11) with $\hbar$ satisfying (4.4) and (4.13) is bounded above by $(2/\mu_1^2)\log k$, as $\gamma \to \infty$.*

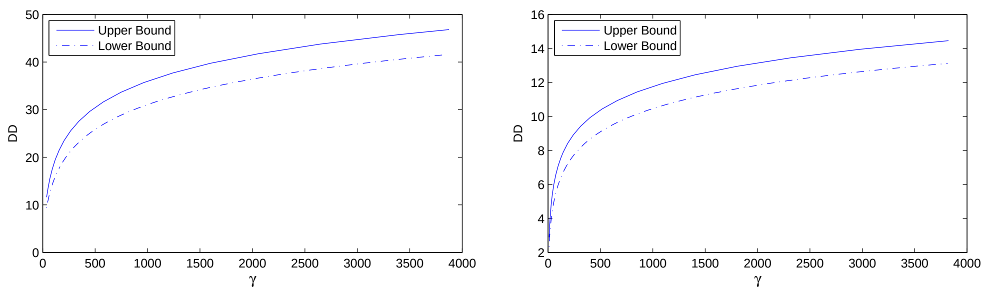
Proof. The asymptotic lower bound in (4.17) is $E_0\{T^1_{\nu_1}\}$. From (4.9) and Lemma 4 we obtain

$$J^{(N)}(T_\hbar) - J^{(N)}(T^*) \quad \leq \quad J^{(N)}(T_\hbar) - E_0\{T^1_{\nu_1}\} \leq \frac{2}{\mu_1^2}\log k + o(1),$$

as $\gamma \to \infty$. □

The consequence of Theorem 4.1, is the asymptotic optimality of (4.11) in detecting the first change of the system. We notice however that this asymptotic optimality holds for any finite number of sources N. Moreover, the more diverse the signal strengths are, the better the asymptotic optimality we achieve.

The upper and the lower bounds on the detection delay (DD) for the optimal stopping rule Symmetric Case



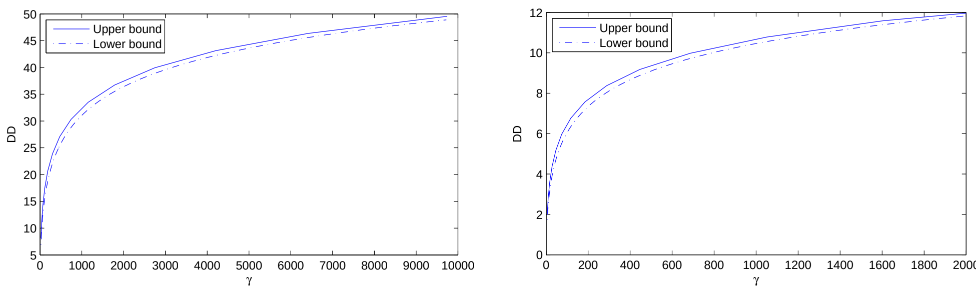
(a) $\mu = 0.5$ (b) $\mu = 1$

Figure 4.1: The upper and the lower bounds on detection delay for the optimal stopping rule: (Left) Case of $\mu = 0.5$. (Right) Case of $\mu = 1$.

The upper and the lower bounds on detection delay for the optimal stopping rule, when $\mu_1 = \mu_2 = 0.5$, $\mu_1 = \mu_2 = 1$, for the case $N = 2$ are shown in Figure 4.1. Note that the differences between the upper and the lower bounds are all bounded as γ increases. The upper and the lower bounds on detection delay for the optimal stopping rule, when $\mu_1 = 0.5$ and $\mu_2 = 1.2\mu_2$, $\mu_1 = 1$ and $\mu_2 = 1.2\mu_1$, for the case $N = 2$ are shown in Figure 4.2. Note that the differences between the upper and the lower bounds converge to zero as γ increases. An important observation is that, the convergence of the upper and the lower bounds is faster for stronger signal strength, and for larger ratio between the stronger signal strength and weaker signal strength.

We now discuss the results under discrete observation.

The upper and the lower bounds on the detection delay (DD) for the optimal stopping rule Non-symmetric Case



(a) $\mu_1 = 0.5, \mu_2 = 1.2\mu_1$ (b) $\mu_1 = 1, \mu_2 = 1.2\mu_1$

Figure 4.2: The upper and the lower bounds on detection delay for the optimal stopping rule: (Left) Case of $\mu_1 = 0.5, \mu_2 = 1.2\mu_1$. (Right) Case of $\mu_1 = 1, \mu_2 = 1.2\mu_1$.

4.2 The discrete-time model

In this section we consider a discrete-time model. It is assumed that the in-control distributions of the observations are the same across sources. The out-of-control distributions, however, can be different across all sources.

4.2.1 Mathematical formulation of the problem

We sequentially observe N mutually independent processes $\{\xi_n^{(i)}; n \geq 1\}$, $i = 1, \ldots, N$, with the following probability density functions (with respect to a σ-finite measure λ):

$$\xi_n^{(i)} \sim \begin{cases} g_\infty(x) & n < \tau_i \\ g_0^{(i)}(x) & n \geq \tau_i \end{cases}, \tag{4.20}$$

where $g_\infty(x)$ and $g_0^{(i)}(x)$, represent the distributions of the observations before and after the onset of the change in source S_i, and the τ_i's are unknown positive integers, with τ_i representing the time point of onset of the change in source S_i.

An appropriate measurable space is $\Omega = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \ldots \mathbb{R}^\infty$ and $\mathcal{F} = \cup_{n\geq 1}\mathcal{F}_n$, where $\{\mathcal{F}_n\}$ is the filtration of the observations with

$$\mathcal{F}_n = \sigma\{(\xi_k^{(1)}, \ldots, \xi_k^{(N)}); k \leq n\}.$$

Analogous to the Brownian motion observation model, on this space, we can define the family of probability measures $\{P_{\tau_1,\ldots,\tau_N}\}$ as before. In order to appropriately formulate this problem in discrete-time we need to specify assumptions regarding the probability density functions $g_0(x)$ and $g_\infty(x)$. To this effect let us consider the projection of $P_{\tau_1,\ldots,\tau_N}$ on the i-th component of Ω, with special attention to $P_1^{(i)}$ and P_∞, for all $i = 1, \ldots, N$. Let us also define the log-likelihood ratio

$$Z_n^{(i)} \;=\; \log \frac{g_0^{(i)}(\xi_n^{(i)})}{g_\infty(\xi_n^{(i)})}, \tag{4.21}$$

for which we assume that for all $i = 1, \ldots, N$,

$$-\infty \;<\; E_\infty\{Z_n^{(i)}\} \;<\; 0 \;<\; E_1^{(i)}\{Z_n^{(i)}\} \;<\; \infty, \tag{4.22}$$

$$E_1^{(i)}\{|Z_n^{(i)}|^2\} \;<\; \infty, \tag{4.23}$$

and that the $Z_n^{(i)}$'s are non-arithmetic with respect to $P_1^{(i)}$ and P_∞. We note that $E_1^{(i)}\{Z_n^{(i)}\}$ is the Kullback-Leibler divergence $D(g_0^{(i)}||g_\infty)$, which can also be written as

$$I_{g_0}^{(i)} \;=\; D(g_0^{(i)}||g_\infty) \;=\; \int \log \frac{g_0^{(i)}(x)}{g_\infty(x)} g_0^{(i)}(x)\lambda(dx). \tag{4.24}$$

Our objective is to find a stopping rule T that balances the trade-off between a small detection delay subject to a lower bound on the mean-time between false alarms and will ultimately detect $\min\{\tau_1, \ldots, \tau_N\}$.

As a performance measure we consider the following generalization of Lorden's performance index (see Lorden [51]):

$$J_D^{(N)}(T) = \sup_{\tau_1,\ldots,\tau_N} \operatorname{essup} E_{\tau_1,\ldots,\tau_N}\{(T - \tau_1 \wedge \ldots \wedge \tau_N + 1)^+ | \mathcal{F}_{\tau_1 \wedge \ldots \wedge \tau_N}\}, \tag{4.25}$$

where the supremum over $\tau_1, \ldots, \tau_N$ is taken over the set in which their minimum is finite. The performance index presented in (4.25) results in the corresponding stochastic optimization problem of the form:

$$\begin{array}{rl} & \inf_T J_D^{(N)}(T) \\ \text{subject to} & E_{\infty,\ldots,\infty}\{T\} \geq \gamma. \end{array} \tag{4.26}$$

Then similar arguments as before apply. In particular, the optimal solution to (4.26), T^*, still satisfies (4.10).

4.2.2 1D discrete CUSUM stopping rule

In the case of only a single observation process (say $\{\xi_n^{(1)}\}$), the problem becomes one of detecting a one-sided change in the distribution of a sequence of discrete observations, whose optimal solution was found in Moustakides [56]. The optimal solution is Page's CUSUM stopping rule, namely the drawup of the log-likelihood ratio process.

Definition 4.3. *Define the following processes:*

$$y_n^{(1)} = \sup_{1\le\tau_1\le n} \log \left.\frac{dP_{\tau_1}^{(1)}}{dP_\infty}\right|_{\mathcal{F}_n} = u_n^{(1)} - \min_{1\le k\le n} u_k^{(1)}, \text{ where} \tag{4.27}$$

$$u_n^{(1)} = \sum_{k=1}^{n} Z_k^{(1)}, \tag{4.28}$$

Then the CUSUM stopping rule is defined as the first hitting time

$$T_\nu^1 = \inf\{n \ge 1; y_n^{(1)} \ge \nu\}, \tag{4.29}$$

where ν is chosen so that $E_\infty\{T_\nu^1\} = \gamma$.

Similar as in the Brownian motion model, the detection delay of the CUSUM stopping rule under Lorden's criterion is given by the expectation $E_1^{(1)}\{T_\nu^1\}$. However, stopping rules involving likelihood ratios of discrete-time models of the type described in (4.20), are usually characterized by overshoot of the threshold ν. For this reason we give the following definition.

Definition 4.4. *We define the following quantities to characterize the limiting behavior of overshoots*[2].

$$\kappa_i = \lim_{\nu\to\infty} E_1^{(i)}\{y_{T_\nu^i} - \nu\}, \tag{4.30}$$

$$\beta_i = E_1^{(i)}\{m_\infty^{(i)}\}, \tag{4.31}$$

and

$$R_i = \lim_{\nu\to\infty} E_1^{(i)}\{\exp[-(u_{\eta_\nu^i}^{(i)} - \nu)]\}, \tag{4.32}$$

where $\eta_\nu^i = \inf\{n \geq 1; u_n^{(i)} \geq \nu\}$.

The above quantities characterize the detection delay and the meantime to the first false alarm of the CUSUM stopping rule. In particular, we have

Lemma 4.5. *As* $\nu \to \infty$,

$$E_\infty\{T_\nu^i\} = E_\infty\{T_\nu^1\} = \frac{1}{I_{g_0}^{(1)}(R_1)^2} e^\nu [1 + o(1)] \tag{4.33}$$

$$E_1^{(i)}\{T_\nu^i\} = E_1^{(1)}\{T_\nu^1\} = \frac{1}{I_{g_0}^{(1)}}(\nu + \beta_1 + \kappa_1) + o(1). \tag{4.34}$$

Moreover, as $\gamma = E_\infty\{T_\nu^1\} \to \infty$,

$$E_1^{(1)}\{T_\nu^1\} = \frac{1}{I_{g_0}^{(1)}}\{\log(\gamma I_{g_0}^{(1)}(R_1)^2) + \beta_1 + \kappa_1\} + o(1). \tag{4.35}$$

Proof. See Tartakovsky [79]. □

[2] κ_i is also the limiting expectation of overshoots of the one-sided sequential probability ratio test (SPRT), i.e., $\kappa_i = \lim_{\nu\to\infty} E_1^{(i)}\{u_{\eta_\nu^i}^{(i)} - \nu\}$; see page 323 of Tartakovsky [79] and Theorem 4.1 of Woodroofe [92] for details.

4.2.3 Equalizer rules and the N-CUSUM stopping rule II

Returning to problem (4.26), we will focus on the performance of the N-CUSUM stopping rule (4.11) with $\hbar = (h_1, h_2, \dots, h_N)$ satisfying (4.4) and

$$E_{1,\infty,\dots,\infty}\{T_\hbar\} = E_{\infty,1,\dots,\infty}\{T_\hbar\} = \dots = E_{\infty,\dots,\infty,1}\{T_\hbar\}. \tag{4.36}$$

We provide an explicit condition on thresholds such that (4.36) holds.

Lemma 4.6. *For $\hbar = (h_1, h_2, \dots, h_N)$ such that*

$$\frac{1}{I_{g_0}^{(1)}}(h_1+\beta_1+\kappa_1) \quad = \quad \frac{1}{I_{g_0}^{(2)}}(h_2+\beta_2+\kappa_2) = \dots = \frac{1}{I_{g_0}^{(N)}}(h_N+\beta_N+\kappa_N),$$

(4.36) holds asymptotically, and as $h_1 \to \infty$,

$$J_D^{(N)}(T_\hbar) \quad = \quad \frac{1}{I_{g_0}^{(1)}}(h_1+\beta_1+\kappa_1) + o(1). \tag{4.37}$$

Proof. Please refer to Appendix B.2 for the proof. □

It is easily seen that, Lemma 4.6 suggests common thresholds across sources in the case of common out-of-control distributions. In the case of different out-of-control distributions, we discuss the optimality of the N-CUSUM with thresholds determined by (4.4) and (4.37).

Without loss of generality, let us assume that

$$I_{g_0}^{(1)} = I_{g_0}^{(2)} = \dots = I_{g_0}^{(k)} < \min_{i>k}\{I_{g_0}^{(i)}\}, \tag{4.38}$$

with $1 < k \le N$ [3]. Without loss of generality, we also assume that

$$(R_1)^2 e^{\beta_1+\kappa_1} = \max_{1\le i\le k}\{(R_i)^2 e^{\beta_i+\kappa_i}\}. \tag{4.39}$$

Thus by (4.35),

$$\max_{1\le i\le N}\{E_1^{(i)}\{T_{\nu_i}^i\}\} = \max_{1\le i\le k}\{E_1^{(i)}\{T_{\nu_i}^i\}\} = E_1^{(1)}\{T_{\nu_1}^1\}. \tag{4.40}$$

In such cases, we have

Proposition 4.2. *Under (4.38) and (4.39), for $\hbar = (h_1, h_2, \dots, h_N)$ satisfying (4.4) and (4.37), as $\gamma \to \infty$,*

$$J^{(N)}(T_\hbar) = \frac{1}{I_{g_0}^{(1)}}\left[\log\gamma + \log\left(I_{g_0}^{(1)}\sum_{i=1}^{k}(R_i)^2 r_i\right) + \beta_1 + \kappa_1\right] + o(1), \tag{4.41}$$

where

$$r_i = e^{(\beta_i-\beta_1)+(\kappa_i-\kappa_1)}.$$

Proof. Please refer to Appendix B.2 for the proof. □

Just as in the Brownian motion case, we have

$$J_D^{(N)}(T_\hbar) > J_D^{(N)}(T^*) > \max_{1\le i\le N}\{E_1^{(1)}\{T_{\nu_i}^i\}\}, \tag{4.42}$$

where $\{\nu_i\}_{i=1}^N$ are chosen according to (4.18). We will demonstrate that the difference between the upper and the lower bounds tends to zero as $\gamma \to \infty$,

[3] The case of $k = 1$ is already treated in Theorem 5.

with $\hbar$ and ν_i satisfying (4.4), (4.37) and (4.18). By examining the asymptotic difference of the upper and the lower bounds in (4.42), we obtain

Theorem 4.2. *When (4.38) and (4.39) hold, the difference in detection delay* $J_D^{(N)}$ *of the unknown optimal stopping rule* T^* *and the detection delay of* $T_\hbar$ *of (4.11) with* $\hbar$ *satisfying (4.4) and (4.37) is bounded above by*

$$\frac{1}{I_{g_0}^{(1)}} \log \left[\sum_{i=1}^{k} \left(\frac{R_i}{R_1} \right)^2 r_i \right],$$

as $\gamma \to \infty$.

Proof. The asymptotic lower bound in (4.42) is $E_1^{(1)}\{T_{\nu_1}^1\}$. From (4.35) and Proposition 4.2 we obtain

$$\begin{aligned} J_D^{(N)}(T_\hbar) - J_D^{(N)}(T^*) &\leq J^{(N)}(T_\hbar) - E_1^{(1)}\{T_{\nu_1}^1\} \\ &\leq \frac{1}{I_{g_0}^{(1)}} \log \left[\sum_{j=1}^{k} \left(\frac{R_j}{R_1} \right)^2 r_j \right] + o(1), \end{aligned}$$

as $\gamma \to \infty$. □

The consequence of Theorem 4.2, is the asymptotic optimality of (4.11) in the discrete-time models described in (4.20). We notice however that this asymptotic optimality holds for any finite number of sources N. Moreover, the more diverse the out-of-control distributions are, the better the asymptotic optimality we achieve.

4.3 Conclusion

The main contribution of this work is that it demonstrates the asymptotic optimality of the N-CUSUM stopping rule, both in the case of continuous-time models and in the case of discrete-time models. The applications of this set-up are numerous. In particular, our setup arises in the detection of a change in the magnitude of the individual components of a vector parameter corresponding to the eigenstructure of linear dynamical state-space models. Such models have been extensively used for modeling and monitoring the health of mechanical, civil and aeronautical structures [29, 41, 43, 61]. The assumption of across-source independence is realistic at least in the particular examples which are described in detail in Basseville et. al. [8]. In this chapter we give explicit formulas for the optimal CUSUM threshold selection which becomes particularly relevant in the general case in which the out-of-control distributions or the signal strengths are different across sources.

Appendix A

Proofs of Results in Chapter 3

A.1 Proofs of Self-financing (One-touches)

In appendix A we prove the replicating strategies in Theorems 3.8 and 3.9 are self-financing. The proofs are based on the following fundamental lemma.

Lemma A.1. *Under the condition of* **G1***, we have for any* $\Delta > 0$ *that,*

$$OT_{\tau}^{'}(S_{\tau}\Delta^{-1}, T) + \Delta^{q+2} OT_{\tau}^{'}(S_{\tau}\Delta, T) = -q\frac{\Delta}{S_{\tau}} OT_{\tau}(S_{\tau}\Delta^{-1}, T), \qquad \text{(A.1)}$$

where $OT_{t}^{'}(K, T)$ *is the derivative of the price of the one-touch with respect to the barrier* K*.*

Proof. It directly follows from (3.33). $\square$

The portfolio (3.36) is self-financing

Proof. Let us denote

$$P(M_t,t)=\sum_{n=0}^{\infty}\left(K^{-2nq}OT_t(M_tK^{-4n-1},T)+K^{(2n+1)q}OT_t(M_tK^{4n+1},T)\right.$$
$$\left.-K^{2(n+1)q}OT_t(M_tK^{4n+3},T)-K^{-(2n+1)q}OT_t(M_tK^{-4n-3},T)\right). \quad (A.2)$$

Then from Lemma A.1, it is easily seen that at any time $t\le\tau_K^{D^r}\wedge T$,

$$P(M_t,t)-P_t(M_{t^-},t)$$
$$=-q\sum_{n=0}^{\infty}\left(K^{-2nq}OT_t(M_tK^{-4n-1},T)-K^{2(n+1)q}OT_t(M_tK^{4n+3},T)\right)\frac{dM_t}{M_t}. \quad (A.3)$$

On the other hand, from (3.37) we obtain that

$$\frac{\partial}{\partial M}LBP_t(M,K,T)=$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{K^{(n+1)q}}\left\{(n+1)\frac{OT_t(MK^{-2n-3},T)}{M}-\int_0^{MK^{-2n-3}}q\left(\frac{K^{2n+3}}{M/H}\right)^q\times\right.$$
$$\left.\left(2P_n\left(\frac{q}{2}\log\frac{K^{2n+3}}{M/H}\right)+P_n'\left(\frac{q}{2}\log\frac{K^{2n+3}}{M/H}\right)\right)\frac{OT_t(H,T)}{2M}\frac{dH}{H}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{K^{(n+1)q}}\left\{(n+1)\frac{OT_t(MK^{-2n-3},T)}{M}-\int_0^{MK^{-2n-3}}q\left(\frac{K^{2n+3}}{M/H}\right)^q\times\right.$$
$$\left.P_{n+1}'\left(\frac{q}{2}\log\frac{K^{2n+3}}{M/H}\right)\frac{OT_t(H,T)}{2M}\frac{dH}{H}\right\}$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{K^{nq}}\left\{n\frac{OT_t(MK^{-2n-1},T)}{M}-\int_0^{HK^{-2n-1}}q\left(\frac{K^{2n+1}}{M/H}\right)^q\times\right.$$
$$\left.P_n'\left(\frac{q}{2}\log\frac{K^{2n+1}}{M/H}\right)\frac{OT_t(H,T)}{2M}\frac{dH}{H}\right\}, \quad (A.4)$$

by (3.39) and (3.40). Similarly, from (3.38) we obtain that

$$\begin{aligned}
&\frac{\partial}{\partial M} LBC_t(M,K,T) = \\
&\qquad = \sum_{n=0}^{\infty} (-1)^n K^{(n+1)q} \Bigg\{ -(n+1)\frac{OT_t(MK^{2n+1},T)}{M} + \\
&\qquad\qquad + \int_{MK^{2n+1}}^{\infty} qP_n{}'\left(\frac{q}{2}\log\frac{MK^{2n+1}}{H}\right)\frac{OT_t(H,T)}{2M}\frac{dH}{H}\Bigg\}. \quad \text{(A.5)}
\end{aligned}$$

Moreover, from (3.33) and **G2** we have that, at times $\tau(u) \triangleq \tau_u^S \wedge \tau_K^{D^r} \wedge T$ for any $u > S_0$,

$$OT_{\tau(u)}(H,T) = \left(\frac{S_{\tau(u)}}{H}\right)^q OT_{\tau(u)}(S^2_{\tau(u)}/H,T). \tag{A.6}$$

Using (A.6) and the fact that $P_0(x) = 1$ we have from (A.4) and (A.5) that, at $\tau(M)$ with $M > S_0$,

$$\begin{aligned}
&\frac{\partial}{\partial M} LBP_{\tau(M)}(M,K,T) - \frac{\partial}{\partial M} LBC_{\tau(M)}(M,K,T) \\
&\qquad = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{K^{nq}} \frac{OT_{\tau(M)}(MK^{-2n-1},T)}{M} + \\
&\qquad + \sum_{n=0}^{\infty} (-1)^n K^{(n+1)q}(n+1)\frac{OT_{\tau(M)}(MK^{2n+1},T)}{M} \\
&= \frac{1}{M}\sum_{n=0}^{\infty}\left(\frac{1}{K^{2nq}} OT_{\tau(M)}(MK^{-4n-1},T) - K^{2(n+1)q} OT_{\tau(M)}(MK^{4n+3},T)\right),
\end{aligned} \tag{A.7}$$

by (3.33) and **G2**. Combine (A.3) with (A.7), we have proved the portfolio in (3.36) is self-financing before $\tau_K^{D^r} \wedge T$. □

The portfolio (3.44) is self-financing

Proof. Let us denote

$$\begin{aligned}
&P(M_t, m_t, t) = \\
&= \sum_{n=0}^{\infty}(2n+1)\left(\frac{1}{K^{nq}}OT_t(M_tK^{-2n-1}, T) + K^{(n+1)q}OT_t(M_tK^{2n+1}, T)\right) \\
&\quad - \sum_{n=1}^{\infty} 2n\left(\frac{1}{K^{nq}}OT_t(m_tK^{-2n}, T) + K^{nq}OT_t(m_tK^{2n}, T)\right) \\
&\quad - q\int_{M_t}^{m_tK}\sum_{n=0}^{\infty} n\left(\frac{1}{K^{nq}}OT_t(HK^{-2n-1}, T) + K^{nq}OT_t(HK^{2n-1}, T)\right)\frac{dH}{H}.
\end{aligned} \tag{A.8}$$

Then from Lemma A.1, it is easily seen that at any time $t \leq \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$,

$$\begin{aligned}
&P(M_t, m_t, t) - P(M_{t^-}, m_{t^-}, t) \\
&\quad = q\sum_{n=0}^{\infty}\left\{n\left(\frac{1}{K^{nq}}OT_t(M_tK^{-2n-1}, T) + K^{nq}OT_t(M_tK^{2n-1})\right)\right. \\
&\qquad\qquad \left. - \frac{(2n+1)}{K^{nq}}OT_t(M_tK^{-2n-1}, T) + \right\}\frac{dM_t}{M_t} \\
&\quad - q\sum_{n=1}^{\infty}\left\{n\left(\frac{1}{K^{nq}}OT_t(m_tK^{-2n}, T) + K^{nq}OT_t(m_tK^{2n}, T)\right)\right. \\
&\qquad\qquad \left. - \frac{2n}{K^{nq}}OT_t(m_tK^{-2n}, T)\right\}\frac{dm_t}{m_t}.
\end{aligned} \tag{A.9}$$

Since the running maximum and the running minimum cannot increase simultaneously, if $dM_t \neq 0$ then $dm_t = 0$ and $S_t = M_t$. In this case, (A.9) is zero by assumption **G2'**. Similarly, if $dm_t \neq 0$ then $dM_t = 0$ and $S_t = m_t$.

We can still use assumption **G2'** to show (A.9) is zero. Therefore, the portfolio in (3.44) is self-financing before $\tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$. $\square$

A.2 Proofs of Self-financing (Vanilla options)

In Appendix B we prove the replicating strategies in Theorems 3.10 and 3.11 are self-financing. The proof s are based on the following fundamental lemma.

Lemma A.2. *Under the conditional of* **G3***, we have for any* $\Delta > 0$ *that,*

$$DC'_{\tau}(S_{\tau}\Delta^{-1}, T) + S_{\tau}^{-q}\Delta^{2}P'_{q,\tau}(S_{\tau}\Delta, T) = 0, \tag{A.10}$$

$$DP'_{\tau}(S_{\tau}\Delta^{-1}, T) + S_{\tau}^{-q}\Delta^{2}C'_{q,\tau}(S_{\tau}\Delta, T) = 0, \tag{A.11}$$

where $DC'_t(K, T)$ *is the derivative of the price of the digital call with respect to its strike* K*, etc.*

Proof. It directly follows from (3.45). $\square$

The portfolio (3.53) is self-financing

Proof. Let us denote

$$\begin{aligned}P(M_t,t) = \sum_{n=0}^{\infty} \big\{ & K^{-2nq}[DP_t(M_t K^{-4n-1},T) + M_t^{-q} C_{q,t}(M_t K^{4n+1},T)] \\ & + K^{(2n+1)q}[DC_t(M_t K^{4n+1},T) + M_t^{-q} P_{q,t}(M_t K^{-4n-1},T)] \\ & - K^{2(n+1)q}[DC_t(M_t K^{4n+3},T) + M_t^{-q} P_{q,t}(M_t K^{-4n-3},T)] \\ & - K^{-(2n+1)q}[DP_t(M_t K^{-4n-3},T) + M_t^{-q} C_{q,t}(M_t K^{4n+3},T)] \big\} . \qquad \text{(A.12)}\end{aligned}$$

Then from Lemma B.1 and (3.45), it is easily seen that at any time $t \leq \tau_K^{D^r} \wedge T$,

$$\begin{aligned}& P(M_t,t) - P(M_{t^-},t) = \\ & \quad = -q \sum_{n=0}^{\infty} \left\{ \frac{1}{K^{2nq}} DP_t(M_t K^{-4n-1},T) + K^{(2n+1)q} DC_t(M_t K^{4n+1},T) \right. \\ & \left. - K^{2(n+1)q} DC_t(M_t K^{4n+3},T) - \frac{1}{K^{(2n+1)q}} DP_t(M_t K^{-4n-3},T) \right\} \frac{dM_t}{M_t}. \qquad \text{(A.13)}\end{aligned}$$

Moreover, analogous to the proof in Appendix A.1, we can obtain from (3.54) that

$$
\begin{aligned}
&\frac{\partial}{\partial M} VP_t(M,K,T) = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{K^{(n+1)q}} \Bigg\{ \frac{(n+1)}{M} \left(DP_t(MK^{-2n-3},T) + \left(\frac{K^{2n+3}}{M}\right)^q P_{q,t}(MK^{-2n-3},T) \right) \\
&\quad - \int_0^{MK^{-2n-3}} q \left(\frac{K^{2n+3}}{M/H}\right)^q \left(2P_n\left(\frac{q}{2}\log\frac{K^{2n+3}}{M/H}\right) + P_n'\left(\frac{q}{2}\log\frac{K^{2n+3}}{M/H}\right)\right) \times \\
&\qquad \times \frac{DP_t(H,T) + H^{-q}P_{q,t}(H,T)}{2M}\frac{dH}{H} \Bigg\} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{K^{nq}} \Bigg\{ \frac{n}{M} \left(DP_t(MK^{-2n-1},T) + \left(\frac{K^{2n+1}}{M}\right)^q P_{q,t}(MK^{-2n-1},T) \right) \\
&\quad - \int_0^{MK^{-2n-1}} q \left(\frac{K^{2n+1}}{M/H}\right)^q P_n'\left(\frac{q}{2}\log\frac{K^{2n+1}}{M/H}\right) \times \\
&\qquad \frac{DP_t(H,T) + H^{-q}P_{q,t}(H,T)}{2M}\frac{dH}{H} \Bigg\},
\end{aligned}
\tag{A.14}
$$

by (3.39) and (3.40). Similarly, from (3.55) we obtain that

$$
\begin{aligned}
&\frac{\partial}{\partial M} VC_t(M,K,T) = \\
&= \sum_{n=0}^{\infty} (-1)^n K^{(n+1)q} \Bigg\{ -\frac{(n+1)}{M} \left(DC_t(MK^{2n+1},T) + \frac{C_{q,t}(MK^{2n+1},T)}{(MK^{2n+1})^q} \right) \\
&+ \int_{MK^{2n+1}}^{\infty} q{P_n}'\left(\frac{q}{2}\log\frac{MK^{2n+1}}{H}\right) \frac{DC_t(H,T) + H^{-q}C_{q,t}(H,T)}{2M}\frac{dH}{H} \Bigg\}.
\end{aligned}
\tag{A.15}
$$

Moreover, from (3.45) and **G4** we have that, at times $\tau(u) \triangleq \tau_u^S \wedge \tau_K^{D^r} \wedge T$ for any $u > S_0$,

$$
DC_{\tau(u)}(H,T) = S_{\tau(u)}^{-q} P_{q,\tau(u)}(S_{\tau(u)}^2/H,T), \tag{A.16}
$$

$$
DP_{\tau(u)}(H,T) = S_{\tau(u)}^{-q} C_{q,\tau(u)}(S_{\tau(u)}^2/H,T). \tag{A.17}
$$

Using (A.16), (A.17) and the fact that $P_0(x) = 1$ we have from (A.14) and (A.15) that, at $\tau(M)$ with $M > S_0$,

$$\begin{aligned}
&\frac{\partial}{\partial M} VP_{\tau(M)}(M,K,T) - \frac{\partial}{\partial M} VC_{\tau(M)}(M,K,T) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{K^{nq} M} \left(DP_{\tau(M)}(MK^{-2n-1},T) + \left(\frac{K^{2n+1}}{M}\right)^q P_{q,\tau(M)}(MK^{-2n-1},T) \right) \\
&\qquad + \sum_{n=0}^{\infty} (-1)^n K^{(n+1)q} \frac{(n+1)}{M} \times \\
&\qquad \times \left(DC_{\tau(M)}(MK^{2n+1},T) + \frac{1}{(MK^{2n+1})^q} C_{q,\tau(M)}(MK^{2n+1},T) \right) \\
&= \frac{1}{M} \sum_{n=0}^{\infty} \left\{ \frac{1}{K^{2nq}} DP_{\tau(M)}(MK^{-4n-1},T) + K^{(2n+1)q} DC_{\tau(M)}(MK^{4n+1},T) \right. \\
&\qquad \left. - K^{2(n+1)q} DC_{\tau(M)}(MK^{4n+3},T) - \frac{1}{K^{(2n+1)q}} DP_{\tau(M)}(MK^{-4n-3},T) \right\},
\end{aligned} \tag{A.18}$$

by (3.45) and **G4**. Combine (A.13) with (A.18), we have proved the portfolio in (3.53) is self-financing before $\tau_K^{D^r} \wedge T$. □

The portfolio (3.59) is self-financing

Proof. Let us denote

$$
\begin{aligned}
&P(M_t, m_t, t) = \\
&\quad = \sum_{n=0}^{\infty}(2n+1)\Big\{\frac{1}{K^{nq}}\left(DP_t(M_tK^{-2n-1},T) + M_t^{-q}C_{q,t}(M_tK^{2n+1},T)\right) \\
&\qquad + K^{(n+1)q}\left(DC_t(M_tK^{2n+1},T) + M_t^{-q}P_{q,t}(M_tK^{-2n-1},T)\right)\Big\} \\
&\qquad - \sum_{n=1}^{\infty}2n\Big\{\frac{1}{K^{nq}}\left(DP_t(m_tK^{-2n},T) + m_t^{-q}C_{q,t}(m_tK^{2n},T)\right) + \\
&\qquad\quad + K^{nq}\left(DC_t(m_tK^{2n},T) + m_t^{-q}P_{q,t}(m_tK^{-2n},T)\right)\Big\} \\
&\quad - q\int_{M_t}^{m_tK}\sum_{n=1}^{\infty}n\Bigg(\frac{1}{K^{nq}}DP_t(HK^{-2n-1},T) + \frac{K^{(n+1)q}}{H^q}P_{q,t}(HK^{-2n-1},T) \\
&\qquad + K^{nq}DC_t(HK^{2n-1},T) + \frac{H^{-q}}{K^{(n-1)q}}C_{q,t}(HK^{2n-1},T)\Bigg)\frac{dH}{H}. \qquad \text{(A.19)}
\end{aligned}
$$

Then from Lemma B.1, it is easily seen that at any time $t \leq \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$,

$$
\begin{aligned}
&P(M_t, m_t, t) - P(M_{t^-}, m_{t^-}, t) = \\
&= -q\sum_{n=0}^{\infty}(2n+1)\left\{\frac{1}{K^{nq}}DP_t(M_tK^{-2n-1},T)+K^{(n+1)q}DC_t(M_tK^{2n+1},T)\right\}\frac{dM_t}{M_t} \\
&\quad + q\sum_{n=1}^{\infty}(2n)\left\{\frac{1}{K^{nq}}DP_t(m_tK^{-2n},T) + K^{nq}DC_t(M_tK^{2n},T)\right\}\frac{dm_t}{m_t} \\
&\quad - q\sum_{n=1}^{\infty}n\left\{\left(\frac{DP_t(m_tK^{-2n},T)}{K^{nq}} + \frac{K^{nq}}{m_t^q}P_{q,t}(m_tK^{-2n},T)+\right.\right. \\
&\quad\quad \left. + K^{nq}DC_t(m_tK^{2n},T) + \frac{C_{q,t}(m_tK^{2n},T)}{m_t^qK^{nq}}\right)\frac{dm_t}{m_t} \\
&\quad\quad - \left(\frac{DP_t(M_tK^{-2n-1},T)}{K^{nq}} + \frac{K^{(n+1)q}}{M_t^q}P_{q,t}(M_tK^{-2n-1},T)+\right. \\
&\quad\quad \left.\left. + K^{nq}DC_t(M_tK^{2n-1},T) + \frac{C_{q,t}(M_tK^{2n-1},T)}{M_t^qK^{(n-1)q}}\right)\frac{dM_t}{M_t}\right\}, \quad \text{(A.20)}
\end{aligned}
$$

Since the running maximum and the running minimum cannot increase simultaneously, if $dM_t \neq 0$ then $dm_t = 0$ and $S_t = M_t$. In this case, (A.20) is zero by assumption **G4'**. Similarly, if $dm_t \neq 0$ then $dM_t = 0$ and $S_t = m_t$. We can still use assumption **G4'** to show (A.20) is zero. Therefore, the portfolio in (3.59) is self-financing before $\tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$. $\square$

A.3 Geometric Brownian Motion and Independent Time-changes

In this section we prove that a spot price process driven by geometric Brownian motion with constant adjustment coefficient (see Asmussen [4]; Luenberger [52]) satisfies all geometric symmetry conditions in this paper. In particular, we assume that, under a filtered risk-neutral measure space $(\Omega, \mathcal{F}, \mathbb{Q}^T)$, $\mathcal{F} = \cup_{t\in[0,T]}\mathcal{F}_t$, the spot price process S has initial value $S_0 > 0$ and follows

$$d\log S_t = \nu_t dt + \sigma_t dW_t,\ \ t \in [0,T], \tag{A.21}$$

where W_t is a standard Brownian motion with respect to $\mathcal{F}$, ν_t and σ_t are $\mathcal{F}_t$-adapted processes, independent of W, and satisfy

$$\nu_t/\sigma_t^2 \text{ is a constant}, \tag{A.22}$$

$$E_0^{\mathbb{Q}^T} e^{\frac{1}{2}\int_0^T (\nu_t^2/\sigma_t^2)dt} < \infty. \tag{A.23}$$

Solving the stochastic differential equation (A.21), one can easily obtain that at any time $t \in [0,T]$,

$$S_t = S_0 \exp\left(\int_0^t \nu_s ds + \int_0^t \sigma_s dW_s\right). \tag{A.24}$$

So the spot price process is always continuous. By the discussion in Remark 3.4, it suffices to prove that S satisfies the symmetry conditions in **G4'**, which we will prove in the following paragraphs.

Under condition (A.23), we have a martingale

$$Y_t = \exp\left(-\int_0^t \frac{\nu_s}{\sigma_s}dW_s - \frac{1}{2}\int_0^t \frac{\nu_s^2}{\sigma_s^2}ds\right) = \phi(t)\cdot\left(\frac{S_t}{S_0}\right)^{-\frac{\nu_0}{\sigma_0^2}},\ \ t \in [0,T] \tag{A.25}$$

where $\phi(t) = \exp\left(\int_0^t \nu_s^2/\sigma_s^2 ds\right)$. Using Girsanov's theorem (see Revuz and Yor [68]), we can use the martingale Y to change the risk neutral measure

$\mathbb{Q}^T$ to another measure $\mathbb{P}^T$ as

$$E_t^{\mathbb{P}^T}\{Z\} = \frac{1}{Y_t} E_t^{\mathbb{Q}^T}\{ZY_T\}, \tag{A.26}$$

for any $\mathcal{F}_T$-measurable random variable Z. Under $\mathbb{P}^T$, the log spot price process $\log S$ is an Ocone martingale (see Ocone [59])

$$d\log S_t = \sigma_t d\widetilde{W}_t,\ t \in [0, T], \tag{A.27}$$

where $\widetilde{W}$ is a standard Brownian motion under $\mathbb{P}^T$. At any time $t \in [0, T]$, for any $\Delta > 0$

$$E_t^{\mathbb{P}^T}\{\delta(S_T - S_t\Delta^{-1})\} = E_t^{\mathbb{P}^T}\{\delta(S_T - S_t\Delta)\}, \tag{A.28}$$

which implies that

$$\begin{aligned} E_t^{\mathbb{Q}^T}\{\delta(S_T - S_t\Delta^{-1})\} &= E_t^{\mathbb{P}^T}\{Y_T^{-1}\delta(S_T - S_t\Delta)\} = \\ &= E_t^{\mathbb{P}^T}\left\{\frac{\phi(t)}{\phi(T)}\left(\frac{S_T}{S_t}\right)^{\frac{\nu_0}{\sigma_0^2}}\delta(S_T - S_t\Delta)\right\} = \\ &= E_t^{\mathbb{P}^T}\left\{\frac{\phi(t)}{\phi(T)}\left(\frac{S_T}{S_t}\right)^{-\frac{\nu_0}{\sigma_0^2}}\delta(S_T - S_t\Delta^{-1})\right\} = \\ &= E_t^{\mathbb{Q}^T}\left\{\left(\frac{S_T}{S_t}\right)^{-\frac{2\nu_0}{\sigma_0^2}}\delta(S_T - S_t\Delta^{-1})\right\}. \end{aligned} \tag{A.29}$$

Thus, (3.56) is satisfied with $q = -2\nu_0/\sigma_0^2$.

Appendix B

Proofs of Results in Chapter 4

B.1 The Continuous-time Brownian Motion Model

As an illustration for the general case, let us prove the results for $N = 2$. The general case for $N \geq 2$ will be discussed afterwards.

We begin by writing down the probability distributions of CUSUM stopping rule for single observation process appearing in Magdon et. al. [53]. For $h_i > 2$, $i = 1, 2$, we have

$$P_0(T^i_{h_i} > t) = 2e^{\frac{h_i}{2}} \sum_{n \geq 1} u(\phi^{(i)}_n) e^{-\frac{\mu_i^2 t}{8 \cos^2 \phi^{(i)}_n}}, \tag{B.1}$$

and

$$\begin{aligned} P_\infty(T^i_{h_i} > t) &= 2e^{-\frac{h_i}{2}} \sum_{n \geq 1} u(\theta^{(i)}_n) e^{-\frac{\mu_i^2 t}{8 \cos^2 \theta^{(i)}_n}} + 2e^{-\frac{h_i}{2}} v(\eta^{(i)}) e^{-\frac{\mu_i^2 t}{8 \cosh^2 \eta^{i)}}} \\ &= A(h_i, t) + B(h_i) e^{-\frac{\mu_i^2 t}{8 \cosh^2 \eta^{(i)}}}, \end{aligned} \tag{B.2}$$

where

$$u(x) = \frac{\sin^3 x}{x - \sin x \cos x} \tag{B.3}$$

$$v(x) = \frac{\sinh^3 x}{\sinh x \cosh x - x}, \tag{B.4}$$

and

$$\tan \phi_n^{(i)} = -\frac{2}{h_i}\phi_n^{(i)} < 0, \tag{B.5}$$

$$\tan \theta_n^{(i)} = \frac{2}{h_i}\theta_n^{(i)} > 0, \tag{B.6}$$

$$\tanh \eta^{(i)} = \frac{2}{h_i}\eta^{(i)} > 0. \tag{B.7}$$

Using the above notation, by the independence of $T_{h_1}^1$ and $T_{h_2}^2$, to derive expressions for $E_{0,\infty}\{T_\hbar\}, E_{\infty,0}\{T_\hbar\}$ and $E_{\infty,\infty}\{T_\hbar\}$, where $\hbar = (h_1, h_2)$. In particular, we have

$$\begin{aligned} E_{0,\infty}\{T_\hbar\} &= \int_0^\infty P_0(T_{h_1}^1 > t)P_\infty(T_{h_2}^2 > t)dt \\ &= \int_0^\infty P_0(T_{h_1}^1 > t)\Big(A(h_2,t) + B(h_2)e^{-\frac{\mu_2^2 t}{8\cosh^2 \eta^{(2)}}}\Big)dt \\ &= I_1(h_1,h_2) + I_2(h_1,h_2), \end{aligned} \tag{B.8}$$

$$\begin{aligned} E_{\infty,0}\{T_\hbar\} &= \int_0^\infty P_\infty(T_{h_1}^1 > t)P_0(T_{h_2}^2 > t)dt \\ &= \int_0^\infty \Big(A(h_1,t) + B(h_1)e^{-\frac{\mu_1^2 t}{8\cosh^2 \eta^{(1)}}}\Big)P_0(T_{h_2}^2 > t)dt \\ &= I_1(h_2,h_1) + I_2(h_2,h_1). \end{aligned} \tag{B.9}$$

Moreover,

$$\begin{aligned}
E_{\infty,\infty}\{T_{\hbar}\} &= \int_0^\infty P_\infty(T_{h_1}^1 > t)P_\infty(T_{h_2}^2 > t)dt \\
&= \int_0^\infty \Big(A(h_1,t)A(h_2,t) + B(h_2)A(h_1,t)e^{-\frac{\mu_2^2 t}{8\cosh^2\eta^{(2)}}}\Big)dt \\
&\quad + \int_0^\infty B(h_1)A(h_2,t)e^{-\frac{\mu_1^2 t}{8\cosh^2\eta^{(1)}}}dt \\
&\quad + B(h_1)B(h_2)\int_0^\infty e^{-\frac{\mu_1^2 t}{8\cosh^2\eta^{(1)}}-\frac{\mu_1^2 t}{8\cosh^2\eta^{(1)}}}dt \\
&= I_3(h_1,h_2) + I_4(h_1,h_2) + I_4(h_2,h_1) + I_5(h_1,h_2). \qquad \text{(B.10)}
\end{aligned}$$

Let us examine the asymptotic behavior of $I_1(h_1,h_2)$ through $I_5(h_1,h_2)$ as $h_1, h_2 \to \infty$. We have four preliminary lemmas to finish the proofs of Lemma 4.4 and Proposition 4.1:

Lemma B.1.

$$\left|\sum_{m,n\geq 1} u(\theta_m^{(1)})u(\theta_n^{(2)})\frac{\cos^2\theta_m^{(1)}\cos^2\theta_n^{(2)}}{\mu_1^2\cos^2\theta_n^{(2)} + \mu_2^2\cos^2\theta_m^{(1)}}\right| \leq C, \qquad \text{(B.11)}$$

where

$$C = \int_0^\infty\int_0^\infty \frac{\pi^{-2}dxdy}{\sqrt{(1+x^2)(1+y^2)}(\mu_1^2+\mu_2^2+\mu_2^2x^2+\mu_1^2y^2)}.$$

Proof. To simplify notation, let us denote $p_i = 2/h_i$, $i = 1, 2$. Then

$$\begin{aligned}
&\left|\sum_{m,n\geq 1} u(\theta_m^{(1)})u(\theta_n^{(2)})\frac{\cos^2\theta_m^{(1)}\cos^2\theta_n^{(2)}}{\mu_1^2\cos^2\theta_n^{(2)} + \mu_2^2\cos^2\theta_m^{(1)}}\right| \\
\leq & \sum_{m,n\geq 1} |u(\theta_m^{(1)})u(\theta_n^{(2)})|\frac{\cos^2\theta_m^{(1)}\cos^2\theta_n^{(2)}}{\mu_1^2\cos^2\theta_n^{(2)} + \mu_2^2\cos^2\theta_m^{(1)}} \leq \sum_{m,n\geq 1} w_1(p_1\theta_m^{(1)}, p_2\theta_n^{(2)})p_1p_2,
\end{aligned}$$

where

$$w_1(x,y) = \frac{1}{\sqrt{(1+x^2)(1+y^2)}(\mu_1^2+\mu_2^2+\mu_2^2x^2+\mu_1^2y^2)}.$$

Since $(\theta_m^{(1)}, \theta_n^{(2)}) \in \big((2m-1)\frac{\pi}{2}, (2m+1)\frac{\pi}{2}\big) \times \big((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\big)$, we have

$$\sum_{m,n\geq 1} w_1(p_1\theta_m^{(1)}, p_2\theta_n^{(2)})p_1p_2 \leq \frac{1}{\pi^2}\int_0^\infty\int_0^\infty w_1(x,y)dxdy,$$

by the monotone decreasing property of w_1 in both variables in the first quadrant. $\square$

Lemma B.2. *For $\alpha_n^{(i)} = \theta_n^{(i)}$, ($\alpha_n^{(i)} = \phi_n^{(i)}$, resp.), $i = 1, 2$,*

$$\lim_{h_i\to\infty}\left|\sum_{n\geq 1} u(\alpha_n^{(i)})\cos^2\alpha_n^{(i)}\right| \leq \frac{1}{\pi}. \tag{B.12}$$

Proof. Let us denote $p_i = 2/h_i$, $i = 1, 2$. Then

$$\left|\sum_{n\geq 1} u(\alpha_n^{(i)})\cos^2\alpha_n^{(i)}\right| \leq \sum_{n\geq 1}|u(\alpha_n^{(i)})|\cos^2\alpha_n^{(i)} \leq \sum_{n\geq 1} w_2(p_2\alpha_n^{(i)})p_i,$$

where

$$w_2(x) = \frac{1}{(1+x^2)^{3/2}}.$$

Because $\alpha_n^{(i)} \in \big((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\big)$, and w_2 is decreasing on the positive half axis, we have

$$\sum_{m,n\geq 1} w_2(p_2\alpha_n^{(i)})p_i \leq \frac{1}{\pi}\int_{-\frac{p_i\pi}{2}}^\infty w_2(x)dx \to \frac{1}{\pi},$$

as $p_i \to 0^+$. $\square$

Lemma B.3. *Asymptotically,*

$$e^{2\eta^{(i)}-h_i} = 1 - 4\eta^{(i)}e^{-2\eta^{(i)}} + o(e^{-3\eta^{(i)}}), \tag{B.13}$$

and as $h_i \to \infty$*,*

$$B(h_i) = 1 + 2\eta^{(i)}e^{-2\eta^{(i)}} - 3e^{-2\eta^{(i)}} + O(e^{-2\eta^{(i)}}). \tag{B.14}$$

Proof. Equation (B.13) is easily verified. By (B.13),

$$\begin{aligned} B(h_i) &= 2e^{-\frac{h_i}{2}}\frac{\sinh^2\eta^{(i)}}{\cosh\eta^{(i)}}\left(1 - \frac{\eta^{(i)}}{\sinh\eta^{(i)}\cosh\eta^{(i)}}\right)^{-1} \\ &= e^{\eta^{(i)}-\frac{h_i}{2}}\frac{(1-e^{-2\eta^{(i)}})^2}{1+e^{-2\eta^{(i)}}}\left(1 - \frac{4\eta^{(i)}e^{-2\eta^{(i)}}}{1-e^{-4\eta^{(i)}}}\right)^{-1} \\ &= 1 + 2\eta^{(i)}e^{-2\eta^{(i)}} - 3e^{-2\eta^{(i)}} + O(e^{-2\eta^{(i)}}). \end{aligned} \tag{B.15}$$

as $h_i \to \infty$. □

Lemma B.4. *If there exists an* $\alpha > 0$ *such that* $h_1 - \alpha h_2 = O(1)$ *holds asymptotically as* $h_1, h_2 \to \infty$*, then we have*

$$\lim_{h_1,h_2\to\infty} I_1(h_1,h_2) = \lim_{h_1,h_2\to\infty} I_1(h_2,h_1) = 0, \tag{B.16}$$

$$\lim_{h_1,h_2\to\infty} \left| I_2(h_1,h_2) - \frac{2}{\mu_1^2}(h_1 - 1)\right| = 0, \tag{B.17}$$

$$\lim_{h_1,h_2\to\infty} \left| I_2(h_2,h_1) - \frac{2}{\mu_2^2}(h_2 - 1)\right| = 0. \tag{B.18}$$

Proof. Applying the Schwartz inequality to $I_1(h_1, h_2)$, we have

$$\begin{aligned}
|I_1(h_1,h_2)| &\leq \sqrt{\int_0^\infty [P_0(T^1_{h_1} > t)]^2 dt} \sqrt{\int_0^\infty [A(h_2,t)]^2 dt} \\
&\leq \sqrt{\int_0^\infty P_0(T^1_{h_1} > t) dt} \sqrt{\int_0^\infty [A(h_2,t)]^2 dt} \\
&\leq \sqrt{E_0\{T^1_{h_1}\}} \sqrt{\frac{32}{\mu_2^2} e^{-h_2} \sum_{m,n\geq 1} \frac{u(\theta_m^{(2)})u(\theta_n^{(2)})}{\sec^2\theta_m^{(2)} + \sec^2\theta_n^{(2)}}} \\
&\leq \frac{8}{\mu_1\mu_2} \sqrt{e^{-h_2}[h_1 + e^{-h_1} - 1]\cdot C},
\end{aligned}$$

where we used (4.8) and Lemma B.1 in the last line. Clearly, with linear dependence between h_1 and h_2,

$$|I_1(h_1,h_2)| = o(1), \text{ as } h_1, h_2 \to \infty.$$

So (B.16) is done.

To prove (B.17), note that

$$\begin{aligned}
&\left| I_2(h_1,h_2) - E_0\{T^1_{h_1}\} \right| \\
&\leq E_0\{T^1_{h_1}\}|B(h_2) - 1| + B(h_2) \left| \int_0^\infty P_0(T^1_{h_1} > t)(e^{-\frac{-\mu_2^2 t}{8\cosh^2\eta^{(2)}}} - 1)dt \right|.
\end{aligned} \tag{B.19}$$

By (4.8) and (B.14), the first term in (B.19) converges to zero as $h_1, h_2 \to \infty$. We need to show the integral in the second absolute value tends to zero as

$h_1, h_2 \to \infty$. We have

$$\begin{aligned} 0 &\leq \int_0^\infty P_0(T_{h_1}^1 > t)(1 - e^{-\frac{\mu_2^2 t}{8\cosh^2\eta^{(2)}}})dt \\ &= \left(\int_0^{\frac{8}{\mu_1^2}h_1} + \int_{\frac{8}{\mu_1^2}h_1}^\infty\right) P_0(T_{h_1}^1 > t)(1 - e^{-\frac{\mu_2^2 t}{8\cosh^2\eta^{(2)}}})dt \\ &= H(h_1, h_2) + T(h_1, h_2). \end{aligned} \tag{B.20}$$

By using the fact that $1 - e^{-x} \leq x$, $H(h_1, h_2)$ can be bounded as follows,

$$\begin{aligned} 0 &\leq H(h_1, h_2) \leq \int_0^{\frac{8}{\mu_1^2}h_1} P_0(T_{h_1}^1 > t) \cdot \frac{\mu_2^2}{\mu_1^2}\frac{h_1}{\cosh^2\eta^{(2)}}dt \\ &\leq \frac{\mu_2^2}{\mu_1^2}\frac{h_1}{\cosh^2\eta^{(2)}}\int_0^\infty P_0(T_{h_1}^1 > t)dt = \frac{\mu_2^2}{\mu_1^2}\frac{h_1}{\cosh^2\eta^{(2)}}E_0\{T_{h_1}^1\} \\ &= \frac{\mu_2^2}{\mu_1^4}\frac{h_1(h_1 + e^{-h_1} - 1)}{\cosh^2\eta^{(2)}} \leq \frac{4\mu_2^2}{\mu_1^4}h_1^2 e^{-2\eta^{(2)}}, \end{aligned} \tag{B.21}$$

which goes to zero as $h_1, h_2 \to \infty$ due to (B.13).

Moreover,

$$\begin{aligned} 0 &\leq T(h_1, h_2) \leq \int_{\frac{8}{\mu_1^2}h_1}^\infty P_0(T_{h_1}^1 > t)dt \\ &= \frac{16}{\mu_1^2}\sum_{n\geq 1} u(\phi_n^{(1)})\cos^2\phi_n^{(1)} e^{-h_1(\sec^2\phi_n^{(1)} - \frac{1}{2})} \\ &\leq \frac{16}{\mu_1^2}e^{-\frac{h_1}{2}}\sum_{n\geq 1}|u(\phi_n^{(1)})|\cos^2\phi_n^{(1)} = O(e^{-\frac{h_1}{2}}), \end{aligned} \tag{B.22}$$

where the last line is because of Lemma B.2. So (B.17) and (B.18) (by similar argument) are done. □

In the following paragraph we shall prove Lemma 4.4 in the case $N = 2$.

Then we discuss the asymptotic behavior of the N-CUSUM for $N \geq 2$, and prove Proposition 4.1 at the end.

Proof of Lemma 4.4. By Lemma B.4, we have under the constraint (4.13) that,

$$J^{(2)}(T_{\hbar}) = E_{0,\infty}\{T_{\hbar}\} + o(1) = E_{\infty,0}\{T_{\hbar}\} + o(1) = \frac{2}{\mu_1^2}(h_1 - 1) + o(1), \quad \text{(B.23)}$$

as $h_1, h_2 \to \infty$. So Lemma 4.4 is proven for $N = 2$. $\square$

Proof of Proposition 4.1. We will prove the result in the case $N = 2$, and then extend it to general cases.

First, we show that $I_3(h_1, h_2)$, $I_4(h_1, h_2)$ and $I_4(h_2, h_1)$ all converge to zero as $h_1, h_2 \to \infty$ without any constraint on dependence of thresholds, and then examine how $I_5(h_1, h_2)$ behaves as $h_1, h_2 \to \infty$ under constraint (4.13).

First, Lemma B.1 implies that

$$|I_3(h_1, h_2)| = O(e^{-\frac{h_1+h_2}{2}}), \text{ as } h_1, h_2 \to \infty; \quad \text{(B.24)}$$

Lemma B.2 and (B.14) in Lemma B.3 imply that

$$\begin{aligned} |I_4(h_1, h_2)| &\leq 16 e^{-\frac{h_1}{2}} B(h_2) \sum_{n \geq 1} |u(\theta_n^{(1)})| \frac{\cos^2 \theta_n^{(1)} \cosh^2 \eta^{(2)}}{\mu_1^2 \cosh^2 \eta^{(2)} + \mu_2^2 \cos^2 \theta_n^{(1)}} \\ &\leq \frac{16}{\mu_1^2} e^{-\frac{h_1}{2}} B(h_2) \sum_{n \geq 1} |u(\theta_n^{(1)})| \cos^2 \theta_n^{(1)} = O(e^{-\frac{h_1}{2}}), \quad \text{(B.25)} \end{aligned}$$

as $h_1, h_2 \to \infty$. Similarly,

$$|I_4(h_2, h_1)| = O(e^{-\frac{h_2}{2}}), \text{ as } h_1, h_2 \to \infty. \tag{B.26}$$

Now let us assume $\mu_1 < \mu_2$ (i.e., $k = 1$) and we choose h_1, h_2 according to (4.13). By Lemma B.3, as $h_1, h_2 \to \infty$,

$$\begin{aligned} I_5(h_1, h_2) &= \frac{8B(h_1)B(h_2)}{\mu_1^2/\cosh^2\eta^{(1)} + \mu_2^2/\cosh^2\eta^{(2)}} \\ &= \frac{2B(h_1)B(h_2)e^{h_1}e^{2\eta^{(1)}-h_1}}{\mu_1^2\left((1+e^{-2\eta^{(1)}})^{-2} + \frac{\mu_2^2}{\mu_1^2}e^{-2(\eta^{(1)}-\eta^{(2)})}(1+e^{-2\eta^{(2)}})^{-2}\right)} \\ &= \frac{2}{\mu_1^2}(e^{h_1} + \text{"lower exponents"}). \end{aligned} \tag{B.27}$$

Formulas (B.23), (B.24), (B.25), (B.26) and (B.27) imply the asymptotic formula in Proposition 4.1 for $N = 2$ and $k = 1$.

On the other hand, if we assume $\mu_1 = \mu_2$ (i.e., $k = 2$), we need only to change the computation of $I_5(h_1, h_2)$ in (B.27) to get that, as $h_1, h_2 \to \infty$,

$$\begin{aligned} I_5(h_1, h_2) &= \frac{4}{\mu_1^2}\left[B(h_1)\right]^2\cosh^2\eta^{(1)} \\ &= \frac{4e^{2\eta^{(1)}-h_1}}{\mu_1^2(1+e^{-2\eta^{(1)}})^{-2}}e^{h_1}\left(1 + 2\eta^{(1)}e^{-2\eta^{(1)}} - 3e^{-2\eta^{(1)}} + o(e^{-3\eta^{(1)}})\right)^2 \\ &= \frac{1}{\mu_1^2}e^{h_1}\left(1 + 4\eta^{(1)}e^{-2\eta^{(1)}} - 6e^{-2\eta^{(1)}} + o(e^{-3\eta^{(1)}})\right) \\ &\quad \times\left(1 - 4\eta^{(1)}e^{-2\eta^{(1)}} + o(e^{-3\eta^{(1)}})\right)(1+e^{-2\eta^{(1)}})^2 \\ &= \frac{1}{\mu_1^2}e^{h_1}\left(1 - 4e^{-2\eta^{(1)}} + o(e^{-3\eta^{(1)}})\right) \\ &= \frac{1}{\mu_1^2}\left(e^{h_1} - 4 + o(e^{-\frac{h_1}{2}})\right). \end{aligned} \tag{B.28}$$

Formulas (B.23), (B.24), (B.25), (B.26) and (B.28) imply the asymptotic formula in Proposition 4.1 for $N = k = 2$.

Now let us consider the N-CUSUM with $N \geq 2$. With similar derivation as above, we can extend our Lemma B.1, Lemma B.2 and Lemma B.4 to deal with the general case. In this manner we can determine the asymptotic formula for the detection delay $J^{(N)}$ (Lemma 4.4 for $N \geq 2$) to be

$$J^{(N)}(T_\hbar) = \frac{2}{\mu_1^2}(h_1 - 1) + o(1), \tag{B.29}$$

and the mean time to the first false alarm to be

$$E_{\infty,\ldots,\infty}\{T_\hbar\} = \frac{8B(h_1)\ldots B(h_N)}{\mu_1^2/\cosh^2\eta^{(1)} + \ldots + \mu_N^2/\cosh^2\eta^{(N)}} + o(1). \tag{B.30}$$

Using Lemma B.3, we can compare h_i with $\eta^{(i)}$ and obtain the asymptotic formulas in the cases $k = 1$ and $k = N$ for any $N \geq 2$. In the general case when $1 < k < N$, from the above discussion we need only to get the asymptotic formula of (B.30). This can be seen as follows

$$\begin{aligned}
&\frac{8B(h_1)\ldots B(h_N)}{\mu_1^2/\cosh^2\eta^{(1)} + \ldots + \mu_N^2/\cosh^2\eta^{(N)}} \\
= \; &\frac{8B(h_1)\ldots B(h_N)}{k\mu_1^2/\cosh^2\eta^{(1)} + \mu_{k+1}^2/\cosh^2\eta^{(k+1)} + \ldots + \mu_N^2/\cosh^2\eta^{(N)}} \\
= \; &\frac{2}{k\mu_1^2}\left(e^{h_1} + \text{“lower exponents”}\right).
\end{aligned} \tag{B.31}$$

Equations (B.29), (B.30) and (B.31) imply the asymptotic formula in Proposition 4.1 and finish the proof. □

B.2 The Discrete-time Model

As before we prove the results for $N = 2$. The general case for $N \geq 2$ will be discussed afterwards. We have the following preliminary lemma to help us:

Lemma B.5. *: If there exists an $\alpha > 0$ such that $h_1 - \alpha h_2 = O(1)$ holds asymptotically as $h_1, h_2 \to \infty$, then we have*

$$\lim_{h_1,h_2\to\infty} \left| E_{1,\infty}\{T_\hbar\} - E_1^{(1)}\{T^1_{h_1}\} \right| = 0, \tag{B.32}$$

$$\lim_{h_1,h_2\to\infty} \left| E_{\infty,1}\{T_\hbar\} - E_1^{(2)}\{T^1_{h_2}\} \right| = 0. \tag{B.33}$$

Proof. Without loss of generality we will only give the proof of (B.32). We observe that[1]

$$\begin{aligned}
& E_{1,\infty}\{T_\hbar\} = e^{h_2} E_{1,\infty}\left\{\frac{T^1_{h_1}}{e^{h_2}} \wedge \frac{T^2_{h_2}}{e^{h_2}}\right\} \\
= \;& e^{h_2}\int_0^\infty P_1^{(1)}\left(\frac{T^1_{h_1}}{e^{h_2}} \geq t\right) P_\infty\left(\frac{T^2_{h_2}}{e^{h_2}} \geq t\right) dt \\
= \;& e^{h_2}\int_0^\infty P_1^{(1)}\left(\frac{T^1_{h_1}}{e^{h_2}} \geq t\right) dt - e^{h_2}\int_0^\infty P_1^{(1)}\left(\frac{T^1_{h_1}}{e^{h_2}} \geq t\right)\left(1 - P_\infty\left(\frac{T^2_{h_2}}{e^{h_2}} \geq t\right)\right)dt \\
= \;& \int_0^\infty P_1^{(1)}\left(T^1_{h_1} \geq u\right) du - e^{h_2}\int_0^\infty P_1^{(1)}\left(\frac{T^1_{h_1}}{e^{h_2}} \geq t\right)\left(1 - P_\infty\left(\frac{T^2_{h_2}}{e^{h_2}} \geq t\right)\right)dt \\
= \;& E_1^{(1)}\{T^1_{h_1}\} - I_6(h_1,h_2).
\end{aligned} \tag{B.34}$$

To prove (B.32), it suffices to show $I_6(h_1, h_2)$ tends to zero as $h_1, h_2 \to \infty$. By using Lemma 1 of Tartakovsky [79] (or Theorem 3 of Khan [47]), we have

[1]The integral representation is used for convenience. However, it should be realized that every integral is actually a summation.

for large h_2,

$$
\begin{aligned}
I_6(h_1,h_2) &= e^{h_2}\int_0^\infty P_1^{(1)}\left(\frac{T_{h_1}^1}{e^{h_2}} \geq t\right)\left(1-e^{-t}(1+o(1))\right)dt \\
&= (1+o(1))\int_0^\infty P_1^{(1)}\left(T_{h_1}^1 \geq u\right)\left(1-e^{-\frac{u}{e^{h_2}}}\right)du. \quad \text{(B.35)}
\end{aligned}
$$

By using the fact that $1-e^{-x}\leq x$, we further have

$$
\begin{aligned}
0 \leq I_6(h_1,h_2) &\leq (1+o(1))\int_0^\infty P_1^{(1)}\left(T_{h_1}^1 \geq u\right)\frac{u}{e^{h_2}}du \\
&= (1+o(1))e^{-h_2}\int_0^\infty uP_1^{(1)}\left(T_{h_1}^1 \geq u\right)du \\
&= \frac{1+o(1)}{2}e^{-h_2}E_1^{(1)}\{\left(T_{h_1}^1\right)^2\}. \quad \text{(B.36)}
\end{aligned}
$$

However, it is easily seen from the proof of Theorem 1 of Tartakovsky [79] (also Theorem 4.1 of [28]) that

$$E_1^{(1)}\{\left(T_{h_1}^1\right)^2\} = O((h_1)^2).$$

Therefore,

$$0 \leq I_6(h_1,h_2) = O(e^{-h_2}(h_1)^2) \to 0,$$

as $h_1, h_2 \to \infty$. This completes the proof of Lemma B.5. □

In the following paragraph we shall prove our Lemma 4.6 in the case $N = 2$. Then we discuss the asymptotic behavior of the N-CUSUM for $N \geq 2$, and prove Proposition 4.2 at the end.

Proof of Lemma 4.6. Lemma B.5 and (4.34), we have under the constraint (4.37) that,

$$\begin{aligned} J_D^{(2)}(T_\hbar) &= E_{1,\infty}\{T_\hbar\} + o(1) = E_{\infty,1}\{T_\hbar\} + o(1) \\ &= \frac{1}{I_{g_0}^{(1)}}\left(h_1 + \beta_1 + \kappa_1\right) + o(1), \end{aligned} \tag{B.37}$$

as $h_1, h_2 \to \infty$. So Lemma 6 is proven for $N = 2$. $\square$

Proof of Proposition 4.2. We begin by using Lemma 1 of Tartakovsky Tartakovsky [79] (or Theorem 3 of Khan [47]) to obtain

$$E_{\infty,\infty}\{T_\hbar\} = \frac{1}{I_{g_0}^{(1)}(R_1)^2 e^{-h_1} + I_{g_0}^{(2)}(R_2)^2 e^{-h_2}}(1 + o(1)), \tag{B.38}$$

as $h_1, h_2 \to \infty$.

Now let us assume $I_{g_0}^{(1)} < I_{g_0}^{(2)}$ (i.e., $k = 1$), and choose h_1 and h_2 according to (4.37). Then

$$E_{\infty,\infty}\{T_\hbar\} = \frac{1}{I_{g_0}^{(1)}(R_1)^2} e^{h_1}(1 + o(1)), \tag{B.39}$$

as $h_1, h_2 \to \infty$. Formulas (B.37) and (B.39) imply the asymptotic formula in Proposition 4.2 for $N = 2$ and $k = 1$.

On the other hand, let us alternatively assume that $I_{g_0}^{(1)} = I_{g_0}^{(2)}$, $R_1 = R_2$ and $h_1 = h_2$ in (B.38). Then we can obtain

$$E_{\infty,\infty}\{T_\hbar\} = \frac{1}{2I_{g_0}^{(1)}(R_1)^2} e^{h_1}(1 + o(1)), \tag{B.40}$$

as $h_1, h_2 \to \infty$. Formulas (B.37) and (B.40) imply the asymptotic formula in Proposition 4.2 for $N = k = 2$.

For the N-CUSUM with $N \geq 2$, we can easily extend Lemma B.5 to address the general case. And by using Lemma 1 of Tartakovsky [79] (or Theorem 3 of Khan [47]), (B.38) becomes

$$E_{\infty,\ldots,\infty}\{T_{\hbar}\} = \left(\sum_{i=1}^{N} I_{g_0}^{(i)} R_i^2 e^{-h_i}\right)^{-1} (1+o(1)), \tag{B.41}$$

as $h_i \to \infty$, $i = 1, \ldots, N$. Then Lemma 4.6 and Proposition 4.2 are proven in the cases $k = 1$ and $k = N$ for any $n \geq 2$.

In the general cases when $1 < k < N$, from the above discussion we just need to get the asymptotic formula of (B.41) when $\hbar$ satisfies (4.37). This can be seen as follows:

$$\begin{aligned}
\left(\sum_{i=1}^{N} I_{g_0}^{(i)} (R_i)^2 e^{-h_i}\right)^{-1} &= \left(\sum_{i=1}^{k} I_{g_0}^{(i)} (R_i)^2 e^{-h_i} + \sum_{i=k+1}^{N} I_{g_0}^{(i)} (R_i)^2 e^{-h_i}\right)^{-1} \\
&= \left(\sum_{i=1}^{k} I_{g_0}^{(i)} (R_i)^2 e^{-h_i}\right)^{-1} (1+o(1))^{-1} \\
&= \frac{e^{h_1}}{I_{g_0}^{(1)} \sum_{i=1}^{k} (R_i)^2 e^{h_1-h_i}} (1+o(1)) \\
&= \frac{e^{h_1}}{I_{g_0}^{(1)} \sum_{i=1}^{k} (R_i)^2 e^{(\beta_i-\beta_1)+(\kappa_i-\kappa_1)}} (1+o(1)) \\
&= \frac{e^{h_1}}{I_{g_0}^{(1)} \sum_{i=1}^{k} (R_i)^2 r_i} (1+o(1)), \qquad \text{(B.42)}
\end{aligned}$$

as $h_i \to \infty$, $i = 1, \ldots, N$. Formulas (B.37) and (B.42) imply the asymptotic formula in Proposition 4.2 and complete the proof. □

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