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THE COMBINATORICS OF CHESSBOARDS

by

KAIYAN ZHAO

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1998

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

9/14/98
Date

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Chair of Examining Committee

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THE CITY UNIVERSITY OF NEW YORK

Abstract

THE COMBINATORICS OF CHESSBOARDS

by

Kaiyan Zhao

Adviser: Professor Michael Anshel

The classic combinatorial problem know as *The n-Queens Problem* is to find the number of arrangements of n queens on an n×n chessboard such that no queen attacks another. In addition to numerous papers on the topic, the problem has many extensions. Examples include *The Toroidal n-Queens Problem*: To find the number of arrangements of n queens on a toroidal n×n chessboard such that no queen attacks another; *The Cylinder n-Queens Problem*: To find a similar solution for a cylindrical

n×n chessboard: *The Minimum Queens Problem*: To place fewer than n queens on an n×n chessboard so that none attacks another, but so that they also together attack every unoccupied cell: *The Reflecting Queens Problem*: *The Queens on an Infinite Chessboard Problem*: and many others. The classic problem and each of its variations contains unsolved problems.

In this paper, I present a new method for generating solutions to the classic problem using quasi-groups and I offer yet another extension to the problem. *The Queens Problem on a Partial Chessboard*, which is to arrange more than n queens on an n×n chessboard with m cells blocked such that no queen attacks another. Under what conditions do such arrangements exist? How many blocked cells are needed to yield solutions? I also present a computer simulation of the n-Queens Problem and of the Queens Problem on a Partial Chessboard that is a useful tool for mathematicians who study these problems.

Thanks to all of the people who have helped me do this.

Theo, who didn't mind going to daycare eight hours a day so that Mommy could work.

Aaron, for everything.

Professor Anshel, for pointing me in the right direction and guiding me all along the way.

And Kaihui for being the best brother in the world.

1. The Classic n-Queens Problem

1.1. Introduction

The n-Queens problem is to find the number of arrangements of n queens on an n×n chessboard so that no queen attacks another. Stated another way, the problem is to find the number of ways that n queens can be put on a chessboard so that no two queens share a row, column, or diagonal. Each arrangement is defined to as a solution.

Two arrangements of queens are considered equivalent if one can be obtained from the other by one of the eight symmetries of the chessboard: identity, rotation by π radians, rotation by $\pm\pi/2$ radians, reflections about either of the main diagonals, and reflection about the horizontal or vertical midline. (Example: The arrangements in Figure 1.1.1 are equivalent because each can be obtained from the other by a rotation of $\pm\pi/2$ radians.)

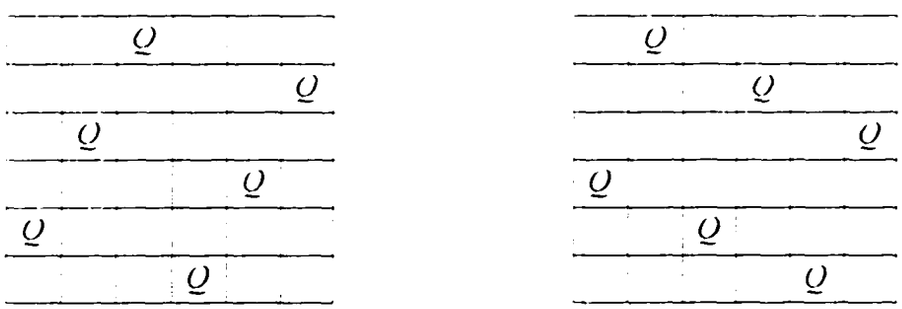
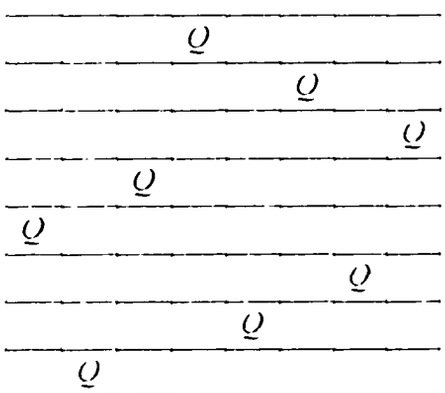


Figure 1.1.1

The problem was first posed for the ordinary 8×8 chessboard in the mid 1800's and is quite a bit more difficult than it might appear. Gauss, himself, offered the first conjecture, that there are 76 solutions [**24**]. The Schachzeitung, the chess journal of Berlin, in 1854 published only 40 positions, which had been discovered by various players. There are, in fact, a total of 92 arrangements, which are now commonly represented by 12 inequivalent solutions (Figure 1.1.2). These 12 inequivalent solutions are also called fundamental solutions.



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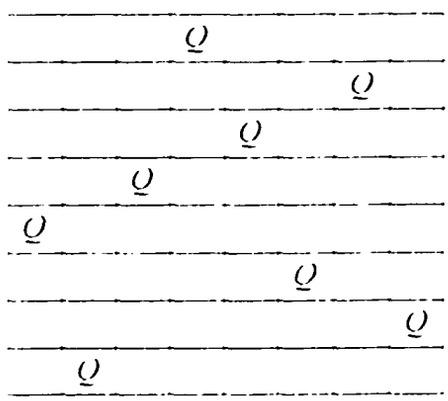

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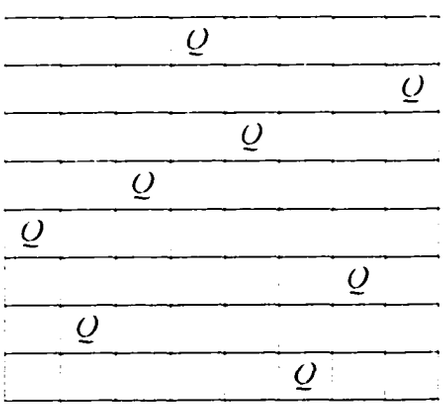

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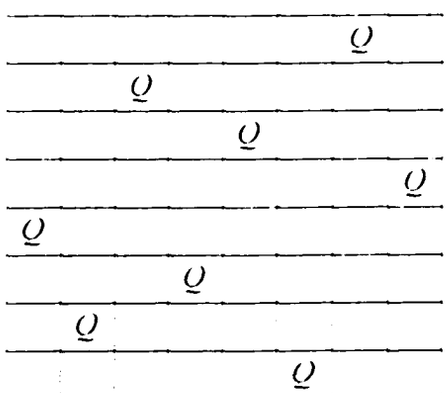

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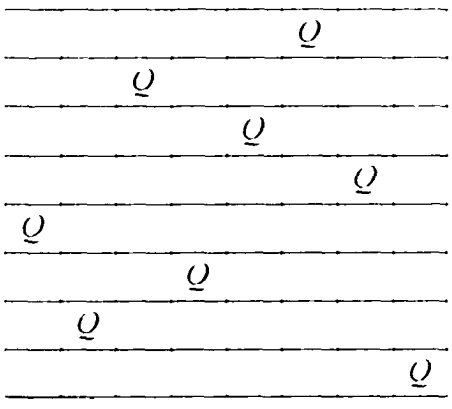
Figure 1.1.2



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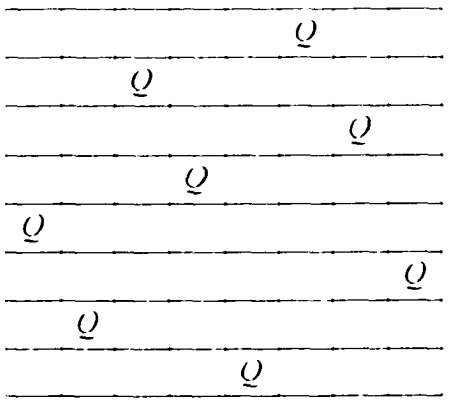

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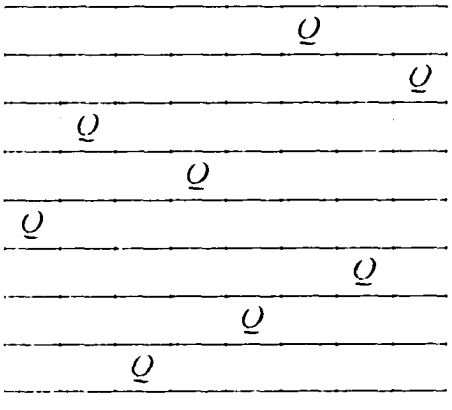

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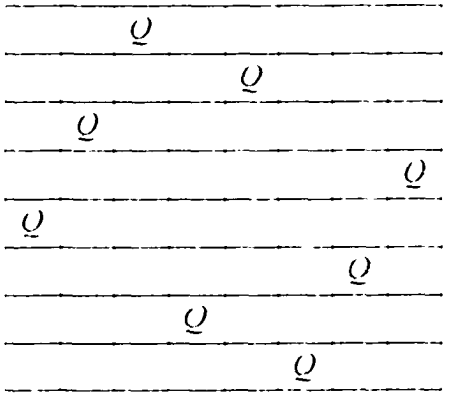

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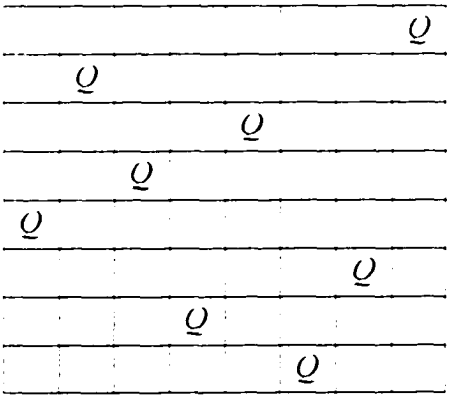

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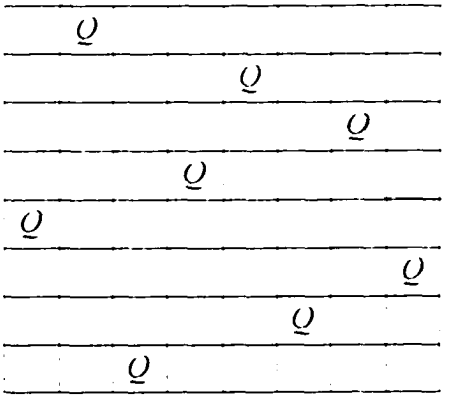

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Figure 1.1.2 (continued)

From 11 of these 12 diagrams we can obtain eight arrangements by rotation and reflection. However, the permutation $(4682^{-}7135)$ yields only four distinct solutions, because it is symmetric.

From now on, we mainly discuss the number of arrangements, or solutions, other than the fundamental solutions.

Let $Q(n)$ represent the number of arrangements of n queens on an n×n chessboard. We have $Q(1) = 1$, $Q(2) = Q(3) = 0$, and $Q(n) > 1$ for $n > 3$. To show this, write a solution for placing queens as a function $f(k)$ where $k = 0, 1, 2, \ldots, n-1$. The k-th queen is placed at the $(k, f(k))$ coordinate (see Figure 1.1.2).

Theorem 1.1.1. *$f(s)$ represents a queens solution if and only if $s \longmapsto f(s)$, $s \longmapsto f(s) + s$, and $s \longmapsto f(s) - s$ are one-to-one.*

Proof. For an n×n chessboard, the one-to-one mapping $s \longrightarrow f(s)$, where $0 \le s < n$, guaranties that there is only one queen on each row and column. It suffices to show that there are no two queens on the same diagonal.

There are $2n - 1$ positive diagonals (Figure 1.1.3 a). For any k, $1-n \le k \le n-1$, the elements in the k-th positive diagonal are elements which have coordinate $(s, f(s))$, where $k = s - f(s)$ (the 0^{th} positive diagonal is called the principal

diagonal). There is only one queen on each positive diagonal if and only if $s \longmapsto s - f(s)$ is one-to-one.

Similarly, an n×n square board has $2n - 1$ negative diagonals (Figure 1.1.3 b). For any k, $0 \le k \le 2n-2$, the elements on the k-th negative diagonal have *coordinates* $(s, f(s))$ where $s - f(s) = k$. There is only one queen on each negative diagonal if and only if $s \longrightarrow s - f(s)$ is one to one.

a b

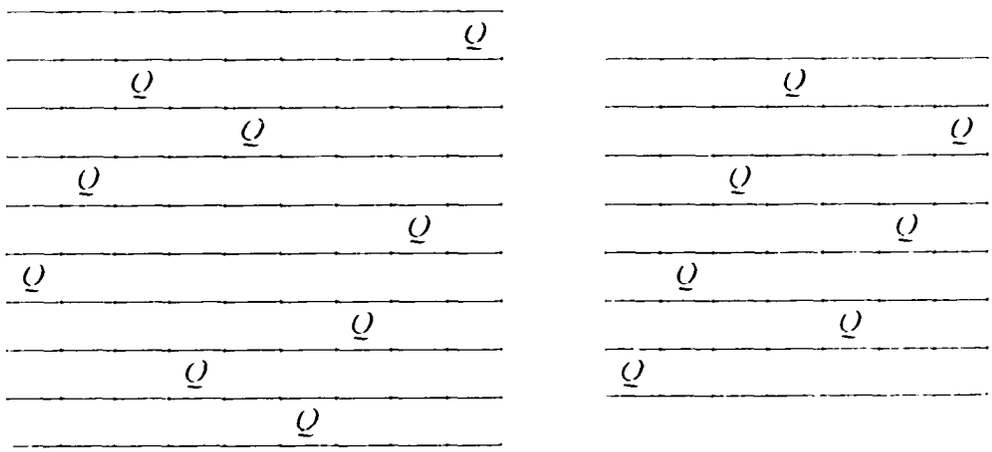
Figure 1.1.3

We now present a constructive proof [**8**] of the existence of solutions to the n-queens problem.

Theorem 1.1.2. *$Q(n) > 0$, for any $n \ge 3$.*

Proof. The proof divides according to the residue class of $n \pmod 6$.

a) Case $n = 6m - 1$, or $n = 6m - 5$. Set $f(k) = 2k$. All the queens are one knight's move apart. Figure 1.1.4 b gives an example for the case $n = 7$.



a b

Figure 1.1.4

b) Case $n = 6m$ or $n = 6m - 4$. We can obtain a solution from $6m - 1$ and $6m - 5$ by removing the queen in the $(0, 0)$ position. (see Figure 1.1.4 b for the case where $n = 6$)

c) Case $n = 6m - 2$.

Define $f(k) = \begin{cases} 2k + (n-2), 2 \pmod{n} & \text{if } (n-2), 2 \geq k \geq 0, \\ (n-1) - f(n-1-k) \pmod{n} & \text{if } (n-1) \geq k \geq n - 2. \end{cases}$

See Figure 1.1.2, (46827135) for $n = 8$.

d) Case $n = 6m - 3$.

Since this solution does not have a queen on the principal diagonal, we can construct a $6m + 3$ solution by adding a row and column to the edge of the board and putting a queen on the new corner.

Figure 1.1.4 a gives an example for the case $n = 9$.

The above proof shows that there is a solution for $n \geq 3$, but how many solutions are there for a large n? The precise nature of $Q(n)$ seems very difficult to understand. Table 1.1.1 gives the values for small n's that were computed using a backtracking algorithm. ($Q(19)$ and $Q(20)$ were computed by J. Shapira [**6**] , the remainder are commonly known.)

N	Q(n)
4	2
5	10
6	4
7	40
8	92
9	352
10	724
11	2.680
12	14.200
13	73.712
14	365.596
15	2.279.184
16	14.772.512
17	95.815.104
18	666.090.624
19	4.968.057.848
20	39.029.188.884

Table 1.1.1

1.2. p-Quasigroup and p-Queens Solution

In this section, a new method of generating solutions for the n-Queens problem is developed based on p-quasigroups. Most of the notions in this section are from [2].

Definition 1.2.1. *Let S be a set of n elements. A latin square of order n based on S is an $n \times n$ array with entries from S such that each element of S occurs once in each row and once in each column.*

As usual, the latin squares of order n in this paper will be based on $\{1, 2, 3, \ldots, n\}$.

Definition 1.2.2. *A quasigroup is a binary operation $(\cdot)$ defined on a set S such that $\forall a, b \in S$, the equations $ax = b$ and $ya = b$ each have exactly one solution.*

Theorem 1.2.1. *The multiplication table of a quasigroup is a latin square.*

Proof. Let $a_1, a_2, \ldots, a_n$ be the elements of the quasigroup and let its multiplication table be as shown in Figure 1.2.1, where the entry a_{rs} which occurs in the r^{th} row of the s^{th} column is the product $a_r a_s$ of the elements a_r and a_s. If the same entry occurred twice in the r^{th} row in the s^{th} and t^{th} columns so that $a_{rs} = a_{rt} = b$, then we would have two solutions to the equation $a_r x = b$, which

would contradict the quasigroup axioms. Similarly, if the same entry occurred twice in the s^{th} column, then we would have two solutions to the equation $y \cdot a = c$ for some c. We conclude that each element of the quasigroup occurs exactly once in each row and once in each column. So, the multiplication table (a square array of n^2 rows and n^2 columns) is a latin square.

	a_1	a_2	$\cdots$	a_r	$\cdots$	a_s	$\cdots$	a_n
a_1	a_{11}							
a_2								
$\vdots$								
a_r						a_{rs}		
$\vdots$								
a_s								
$\vdots$								
a_n								a_{nn}

Figure 1.2.1

Definition 1.2.3. *A partition groupoid (denoted by p-groupoid) is a groupoid $(V, \bullet)$ satisfying*

1) $a \bullet a = a$ *for all* $a \in V$

2) $a = b$ *implies* $a = a \bullet b = b$ *for* $\forall a, b \in V$

3) $a \bullet b = c$ *implies and is implied by* $c \bullet b = a$ *for* $\forall a, b, c \in V$

Definition 1.2.4. *A p-groupoid that is also a quasigroup will be called a p-quasigroup.*

Theorem 1.2.2. *If n is any odd positive integer, the set $V = \{1, 2, 3, \ldots, n\}$ forms a p-quasigroup under the operation $(\bullet)$ defined by $r \bullet s = 2s - r \pmod n$.*

Proof. To show that $(V, \bullet)$ is a quasigroup, assume that $\exists\, x, y \in V = \{1, 2, \ldots, n\}$ such that $ax = b$ and $ay = b \pmod n$. For some integer m, we have $2y - a = 2x - a + mn$. So, $2(y - x) = mn$. Since n is odd, m has to be divisible by 2. Say $m = 2k$ for some integer k, therefor k is 0. Hence, $x = y$. Similarly, if $xa = ya = b \pmod n$ then $x = y + kn$ for some integer k, which means that $x = y$.

To show that $(V, \bullet)$ is a p-quasigroup, we have $r \bullet r = r$ and $r = r \bullet s = s$ for $r = s$ by definition. If $r \bullet s = t$, then $t \bullet s = (2s - r) \bullet s = 2s - 2s + r = r$.

Theorem 1.2.3. *Let p be a prime number that is greater than 3. Let $(V, \bullet)$ be a p-quasigroup that is defined on the set $V = \{1, 2, 3, \ldots, p\}$, under the operation $(\bullet)$ defined by $r \bullet s = 2s - r \pmod n$. Let $K(p)$ be the multiplication table of $(V, \bullet)$. If $K(p)$ is superimposed onto a $p \times p$ chessboard and queens are placed on the squares that contain the integer k where $k \in V$, then the resultant board configuration corresponds to a queen's solution.*

Proof. It is known that the multiplication table of a quasigroup is a latin square. It suffices to check that each diagonal contains at most one queen.

Positive diagonals:

For the k-th diagonal where $1 - p \le k \le p - 1$, suppose that $r \bullet s$ and $i \bullet j$ lie on the same diagonal k. We have

$$\begin{cases} r - s = i - j \pmod{n} \\ j - i = s - r = k \pmod{n} \end{cases} \Rightarrow \begin{cases} 2s - r = 2j - i - m_1 n \\ j - i = k + m_2 n \\ s - r = k - m_3 n \end{cases} \quad \text{where } m \text{ and } m \in Z$$

$$\Rightarrow \begin{cases} -i = k - m_2 n - j \\ -r = k + m_3 n - s \end{cases} \Rightarrow s - j = m n \Rightarrow s - j \text{ and } r - i.$$

Therefore there is none of the numbers appear twice on the same positive diagonal.

Negative diagonals:

For the k-th diagonal where $2 \le k \le 2n$, suppose that $r \bullet s$ and $i \bullet j$ lie on the same diagonal k. Then $\begin{cases} 2s - r = 2j - i \\ r + s = i + j = k \end{cases} \pmod{n} \Rightarrow$

$$\begin{cases} 2s - r = 2j - i + m.n \\ r - s = k + m.n \\ i + j = k + m.n \end{cases} \quad \text{where } m \text{ and } m \in Z \Rightarrow \begin{cases} r = k + m.n - s \\ i = k + m.n - j \end{cases} \Rightarrow$$

$3s \quad k \quad m.n \quad 3j \quad k \quad m.n \quad m.n \quad \Rightarrow \quad 3s \quad 3j \quad m.n.$

Since n is prime m has to be divisible by 3. So, m has to be 0. Therefore, $r \quad i$ and $s \quad j$. Therefore there is none of the numbers appear twice on the same negative diagonal.

The following Figure gives an example for case $p \quad 5$.

(·)	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

1.2 = 3	1.3 = 5	1.4 = 2	1.5 = 4
2.3 = 4	3.5 = 2	4.2 = 5	5.4 = 3
3.4 = 5	5.2 = 4	2.5 = 3	4.3 = 2
4.5 = 1	2.4 = 1	5.3 = 1	3.2 = 1
5.1 = 2	4.1 = 3	3.1 = 4	2.1 = 5

Figure 1.2.2

If we place queens by following any of the integers *1, 2, 3, 4* or *5*, we will get a queen's solution.

1.3. Algorithms For The n-Queens Problem

One method for solving the n-queens problem that systematically generates all possible solutions is a backtracking search. Backtracking uses a depth-first search of subconfigurations, returning to the most recent branch point when it decides the current path will not produce a solution. Due to the exponential growth of the search load in backtracking, this type of search is not a practical method for large n. Recent results indicate that we may only solve the n-queens problem in this way with n up to *100* [**13**] [**14**]. It is desirable to investigate some alternative approaches for which there is no backtracking overhead. In the next two sections, I will summarize two new methods.

1) A probabilistic local search algorithm by Rok Sosic and Jun Gu[**25**], is a polynomial time algorithm .

2) A neural network solution by James Freeman [**28**] is an application of Hopfield network to constraint-satisfaction problems.

Each of the following methods finds a solution to the problem, but does not necessarily find all solutions. These methods are acceptable as long as you do not care which solution you find.

1.3.1 A Polynomial Time Algorithm for The n-Queens Problem

Rok Sosic and Jun Gu[25] found a new probabilistic local search algorithm that is a polynomial time.

QueenArray[] is the n positions of n queens on a chessboard. *QueenArray[i]* = *f(i)*, (*i* = 1, 2, ... *n*) where *f* is a random permutation of *i* = 1, 2, ... *n*. It guaranties that no two queens will be on the same row or the same column.

The number of collisions can be counted by tracing each diagonal line. Let *pdArray[]* and *ndArray[]* be arrays of 2*n* − 1 positive and negative diagonals respectively. p*dArray[]* and *ndArray[]* are used to keep track of the number of queens, i.e., the number of collisions, on each of diagonal. If *pdArray[m]* = *k*, then there are *k* − 1 queens on *m-th* positive diagonal line and there are *k* collisions on this diagonal line.

The *swap()* function incrementally updates *pdArray[]* and *ndArray[]*. Since one queen can affect at most two diagonals, correspondingly at most two values in *pdArray[]* and *ndArray[]* are affected. A swap of two queens can affect at most eight diagonals: four for both "source" queens and four for both "destination" queens. In order to determine whether a swap reduces the number of collisions,

we need only to check these eight diagonals. This test operation and a possible subsequent swap operation are repeated for all possible pairs of queens until a solution is found. If no more swaps can be performed and collisions still exist, a new permutation is invoked.

For an initial permutation, if no solution is found after the completion of the loop, a new permutation is generated and a new search process is started.

The following figure (Figure 1.3.1) is a flowchart of this algorithm. The final returned array *QueenArray[]* is a solution to the n-queens problem.

This algorithm is capable of finding a solution for extremely large n within a reasonably short time period because it runs in polynomial time.

The estimated running time of this algorithm is found in [**25**].

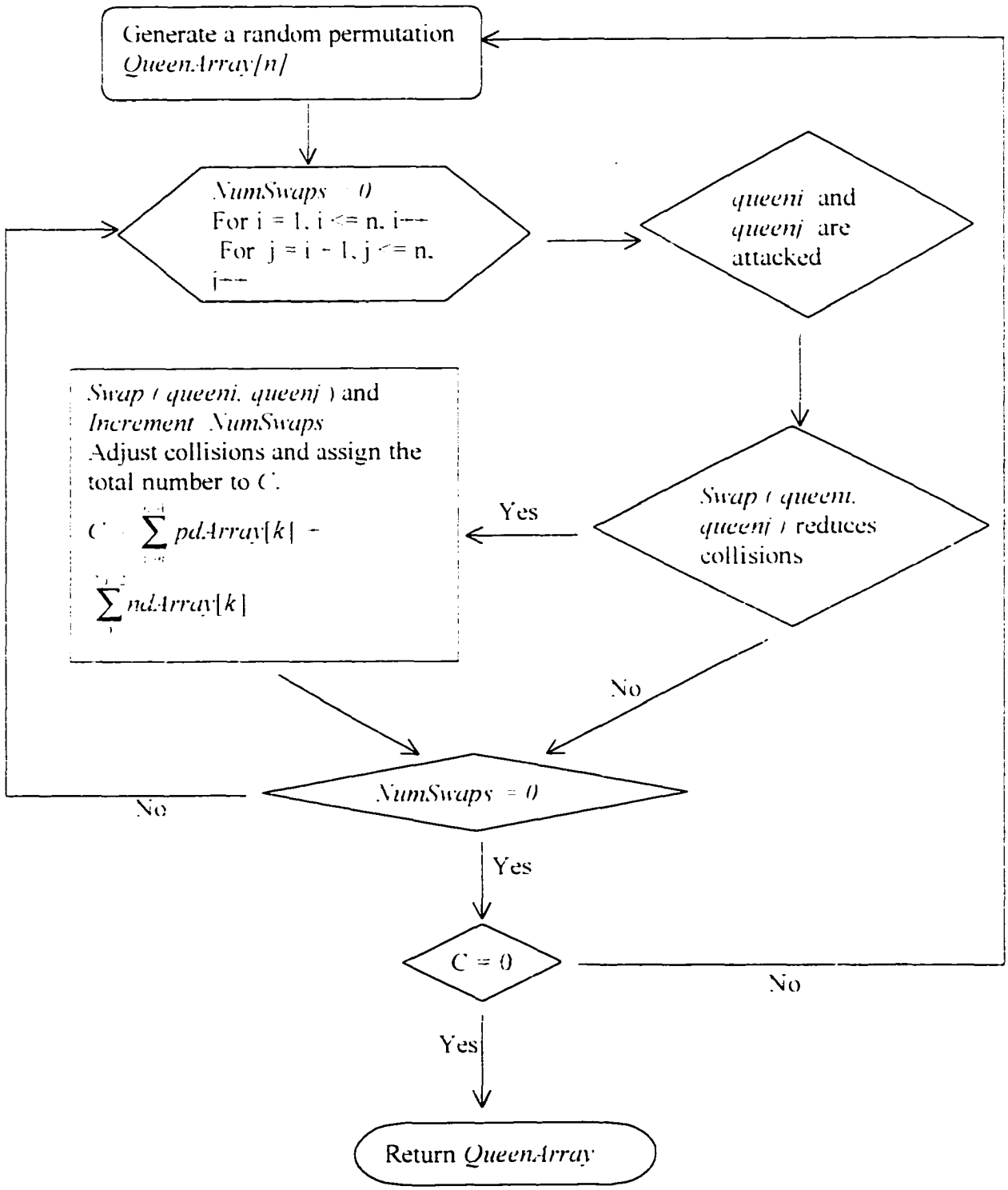


Figure 1.3.1

1.3.2. A neural network solution

In this section, we will examine a method for representing certain types of constraint-satisfaction or optimization problems in a neural-network paradigm. James Freeman mapped the n-queens problem to the Hopfield Network [17]. Each square of the chessboard is identified with a unit in a Hopfield network. The output of the unit is 1 if the square is occupied by a queen and 0 otherwise.

The four constraints of the n-queens problem are

1) Exactly n queens must occupy squares on the board.

2) No two queens may occupy squares on the same row

3) No two queens may occupy squares on the same column.

4) No two queens may occupy squares on the same diagonal.

Each of the constraints of the n-queens problem can be expressed as an n-out-of-n problem [15].

1) Exactly n queens on the board, is an n-out-of-n^2 problem.

2) One queen per row, is a 1-out-of-n problem.

3) One queen per column, is a 1-out-of-n problem.

4) One queen per diagonal, is a 1-out-of –d problem (where d is the number of squares on the diagonal).

Let's define I_i to be the external inputs, v_i be the output of the i-th unit, and T_i

to be the weight from the j-th unit to the i-th unit. Since for the moment v is either 0 or 1, the constraint that exactly n outputs have the value 1 is represented by $\sum_{i}^{n} v = n$. This condition will be satisfied when the function $(n - \sum_{i}^{n} v)^2$ (called the "energy function"), is at its minimum. In order to guarantee the minimum occurs for values of v that are either 0 or 1, we add the term $\sum_{i}^{n} v(1-v)$ to the energy function, expand the sum, and ignore the constant n^2 term. We get the new energy function.

$E = -1/2 \sum_{i}^{n} \sum_{j \neq i}^{n} (-2) v_i v_j - \sum_{i}^{n} v_i (2n-1)$. Comparing this energy function with Hopfield network's energy function $E = -1/2 \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} v_i v_j - \sum_{i}^{n} I_i v_i$ we find the Hopfield net weights $T_{ij} = \begin{cases} -2, & i \neq j \\ 0, & otherwise \end{cases}$, and external inputs $I_i = (2n-1)$.

In order to represents a position on the chessboard as a position of each Hopfield network unit, we arrange the n^2 units of the Hopfield network in an n×n array.

We denote the outputs of the units in the array by v (corresponding to the j-th position on the i-th row of the board). The weight matrix becomes an array with four indices: $T_{\ \ ,\ ,}$ is the weight from unit $v_{,\ ,}$ to unit v . The constraints

1) *There are exactly n queens on the board* is realized by connecting each unit to all the others with a weight of -2. $T_{\ \ ,\ ,}=-2(1-\delta_{,}\delta$) The external input to each unit is $(2n-1)$.

2) *There must be exactly one queen on each row* translates to n separate 1-out-of –n problems. Each unit on a row should inhibit the other units on the same row with strength of -2. We can represent this constraint for all of the rows by the term $-2\delta_{}(1-\delta_{,})$. The first delta selects units on the same row. The second factor is zero if $j=i$ (a unit does not inhibit itself), but is one if $j\neq i$ (a unit inhibits all other units on the same row). Since all units participate in a 1-out-of-n problem, each receives an additional external input of 1.

3) There must be one and only one queen per column contributes the term $-2\delta_{}(1-\delta_{,})$ to the weight array and adds 1 to the input to each unit, bringing the total external input per unit to $2n-1$.

4) The final constraint concerns the units on the various diagonals of the board. If we adopt the approach that each individual diagonal represents a

1-out-of-d problem, where d is the number of units on the particular diagonal, we can represent the contribution to the weight array from all the diagonals on the board as $-2(\delta_{\ldots}+\delta_{\ldots})(1-\delta_{\ldots})$ and add 2 to each external input.

This brings the total external input to each unit to $2n-3$.

Each entry in the T matrix is the sum of appropriate row, column, diagonal and global inhibitions and is constructed as follows:

$$T_{\ldots}=A*(1-\delta(j,l))*\delta(i,k)-B*\delta(j,l)*(1-\delta(i,k))-c$$
$$+D*(\delta(i+j,k+l)+\delta(i-j,k-l))*(1-\delta(i,k))$$

where $\delta(x,y)=\begin{cases}1, & x=y\\ 0, & otherwith\end{cases}$. A and B are connection weights corresponding to the row and column inhibitions. c is the global inhibition which enforces the requirement that exactly n queens are selected. D is the connection weight corresponding to a diagonal inhibition.

1.4. Related Problems

1.4.1. The Problem of Reflecting Queens

Klarner [9] posed the following problem: For which n is it possible to form pairs $(1, b_1), (2, b_2), \ldots, (n, b_n)$ with $\{b_1, b_2, \ldots, b_n\} = \{n+1, n+2, \ldots, 2n\}$ so that all of the numbers $b_i - i$, $b_i + i$, $i = 1,2,\ldots,n$ are distinct?

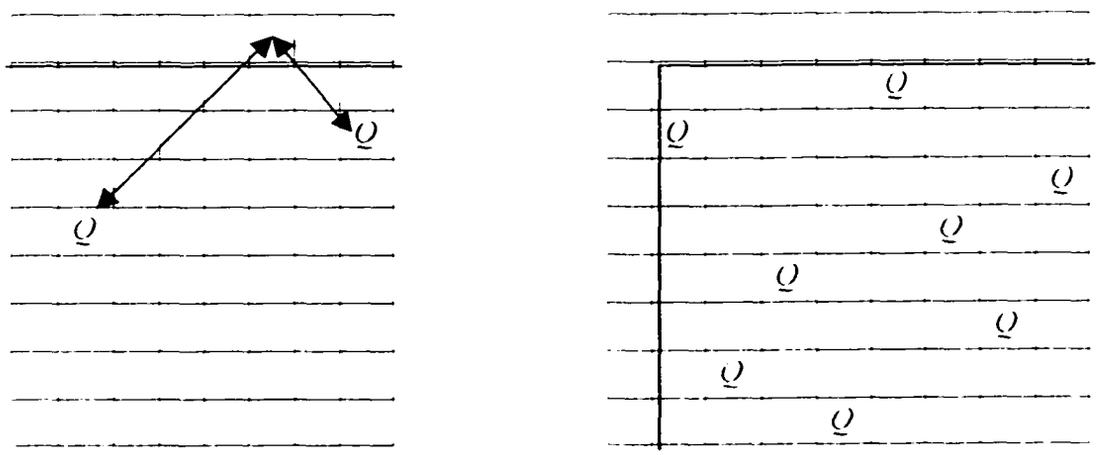
To see this problem in terms of the queen problem, we start with an n×n chessboard with a 1×n strip of squares added along one side, which we call a reflecting strip. A reflecting queen is defined to be a standard chess queen with the added power of being able to reflect her path via the reflecting strip.

From Figure 1.4.1 a, we can see that the reflecting queen is the regular queen problem with additional condition: no two queens share a reflected diagonal.

We number the rows of the n×n square $1, 2, \ldots, n$ beginning with the first row below the reflecting strip, and number the columns from left to right $n+1, n+2, \ldots, 2n$. A reflecting queen on the (i, j) cell attacks the following cells:

Column cells: $(i \pm 1, j)$, $(i \pm 2, j)$, ...

Row cells: $(i, j \pm 1)$, $(i, j \pm 2)$, ...



a b

Figure 1.4.1

Diagonal cells: $(i \pm 1, j \pm 1)$, $(i \pm 2, j \pm 2)$, ...

$(i \pm 1, j \mp 1)$, $(i \pm 2, j \mp 2)$, ..

Reflected diagonal cells: $(1, j \pm (i-1))$, $(2, j \pm (i-2))$, ..

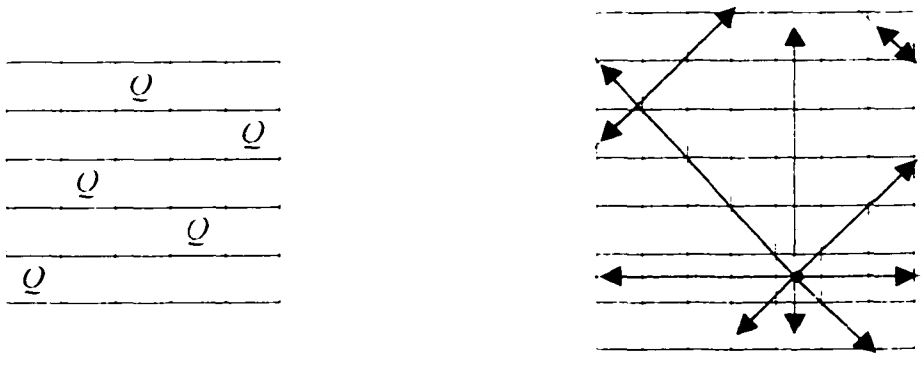
Unlike the regular chessboard, the reflecting board has only one symmetry which is the reflecting about the center line that is perpendicular to the reflecting stripe, since a rotation shifts the reflecting strip to another edge. Figure 1.4.1 b yields two solution for $n = 8$. He conjectured that the problem of reflecting queens has at least one solution for every $n > 6$, and Sebastian [**10**] verified this for $9 \le n \le 27$. He published some of his computer solutions to the problem of reflecting queens by using a backtracking method two years later. The following table gives an example solution on the problem of reflecting queens for each $n \le 28$. With the notation $(h_1, h_2, \ldots, h_n)$ being used as an abbreviation of $\{(1,h_1),(2,h_2),\ldots,(n,h_n)\}$.

n	Number of solutions	Some example solutions
4	1	(7, 5, 8, 6)
5	1	(10, 8, 6, 9, 7)
6	0	None
7	1	(11, 13, 8, 10, 12, 14, 9)
8	4	(13, 9, 16, 14, 11, 15, 10, 12)
9	≥ 1	(11, 16, 14, 17, 10, 13, 15, 12, 18)
10	≥ 1	(12, 17, 20, 14, 11, 18, 15, 13, 16, 19)
11	≥ 1	(13, 18, 21, 15, 12, 19, 22, 14, 17, 20, 16)
12	≥ 1	(14, 19, 22, 16, 23, 20, 15, 24, 18, 21,13, 17)
13	≥ 1	(15, 20, 23, 25, 22, 18, 26, 17, 14, 21, 24, 16, 19)
14	≥ 1	(16, 21, 24, 26, 23, 19, 27, 18, 15, 22, 25, 28, 20, 17)
15	≥ 1	(17, 22, 25, 27, 30, 23, 19, 29, 24, 20, 16, 26, 21, 18, 28)
16	≥ 1	(18, 23, 26, 28, 31, 24, 20, 30, 25, 21, 17, 32, 27, 29, 22, 19)
17	≥ 1	(19, 24, 27, 29, 32, 25, 22, 20, 30, 26, 34, 23, 21, 18, 28, 33, 31)
18	≥ 1	(20, 25, 28, 30, 33, 26, 36, 22,3 1, 34, 24, 29, 23, 19, 27, 32, 35, 21)
19	≥ 1	(21, 26, 29, 31, 34, 37, 30, 38, 27, 35, 22, 28, 25, 20, 36, 33, 24, 32, 23)
20	≥ 1	(22, 27, 30, 32, 35, 38, 31, 39, 28, 36, 23, 29, 26, 21, 37, 34, 25, 33, 24, 40)
21	≥ 1	(23, 28, 31, 33, 36, 39, 42, 24, 30, 37, 29, 32, 25, 22, 40, 27, 34, 41, 38, 26, 35)
22	≥ 1	(24, 29, 32, 34, 37, 40, 43, 41, 35, 27, 30, 28, 26, 38, 36, 44, 31, 25, 39, 42, 33, 23)
23	≥ 1	(25, 30, 33, 35, 38, 41, 44, 46, 36, 31, 45, 32, 42, 28, 37, 24, 29, 43, 34, 39, 27, 40, 26)
24	≥ 1	(26, 31, 34, 36, 39, 42, 45, 47, 37, 32, 46, 33, 43, 29, 38, 25, 30, 44, 35, 40, 28, 41, 27, 48)
25	≥ 1	(27, 32, 35, 37, 40, 43, 46, 48, 45, 41, 33, 36, 30, 28, 31, 34, 38, 47, 42, 39, 26, 49, 29, 44, 50)
26	≥ 1	(28, 33, 36, 38, 41, 44, 47, 49, 52, 42, 34, 37, 31, 51, 43, 40, 30, 35, 45, 50, 27, 29, 32, 39, 46, 48)
27	≥ 1	(29, 34, 37, 39, 42, 45, 48, 50, 53, 43, 49, 38, 33, 31, 46, 32, 40, 36, 30, 44, 35, 41, 52, 28, 47, 51, 54)

Table 1.4.1

1.4.2. Toroidal n-Queens problem

How many ways can one place n-queens on an n×n chessboard so that no two queens share a row, column, or extended diagonal. (Figure 1.4.2 b and Figure 1.4.2 a is a toroidal 5-queens solution.)



a b

Figure 1.4.2

Where the vertical and horizontal lines wraps around: diagonal line $(i,1)$ $(1, j)$ connect to the diagonal line $(i+1,n)$ $(n, j+1)$; diagonal line $(i,1)$ (n, j) connects to the diagonal line $(i-1,n)$ $(1, j+1)$.

If we denote $T(n)$ as the number of n×n toroidal queens solutions, then $T(n)$ $Q(n)$. Polya [11] first studied this problem and showed that $T(n) > 0$ if and only if $(n, 6) = 1$.

Rivin [**12**] has showed that for a prime number p such that $(p-1)/2$ is not prime, $T(p) > 2^{(p-1)/d}$, where d is the smallest nontrivial divisor of $(p-1)/2$. They also Conjectured that $\lim_{n \to \infty} \frac{\log T(n)}{n \log n} = \alpha > 0$, and $\lim_{n \to \infty} \frac{\log Q(n)}{n \log n} = \beta > 0$. $T(n)$ was found for n up to 23 (see Table 1.4.2) in [**12**].

N	5	7	11	13	17	19	23
T(n)	10	28	88	4.524	140.692	1.820.496	128.850.048

Table 1.4.2

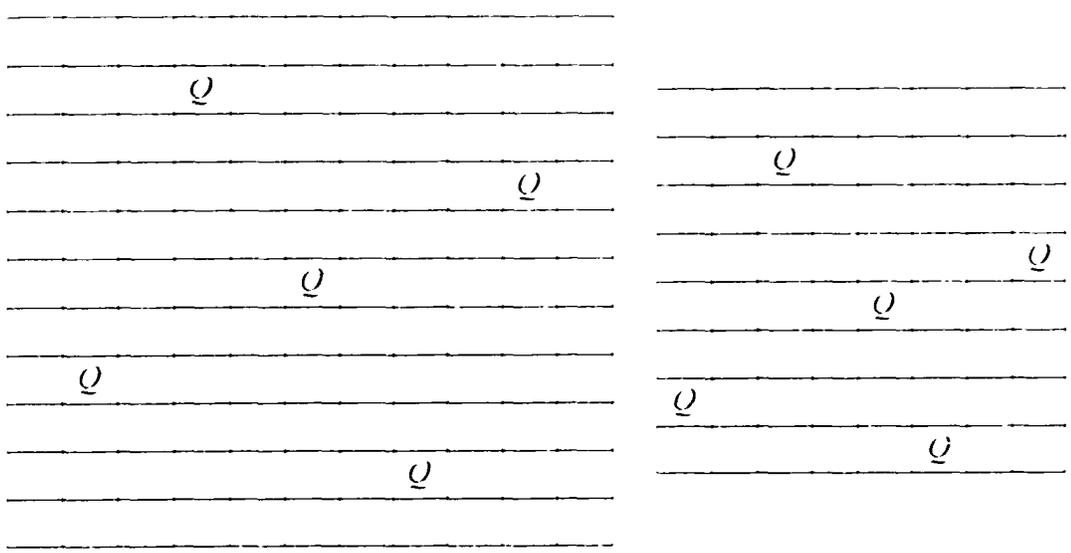
$T(23) = 128,850,048$ was computed using backtracking in LeLisp as a distributed computation over a network of 20 Suns at INRIA, Rocquencourt, and took 26 days of CPU time.

1.4.3. Domination of Queens

a) *What is the minimum number of queens that can be placed on an n×n chessboard so that every square is either occupied or attacked by a queen?*

This is the same as finding the minimum externally stable set for a graph on n^2 vertices with two vertices joined only if they are on the same rank, file or diagonal. For size 8, 9, 10, and 11 chessboards, 5 queens are sufficient.

(See Figure 1.4.3. a)



a b

Figure 1.4.3

b) *Placing n queens on an n×n board to leave a maximum number of unattacked vacant cells.*

See Figure 1.4.3 b for an example on an 8×8 board. There are 11 unattacked squares, which are denoted by 0's. Is 11 maximum? What is the maximum number for n in general?

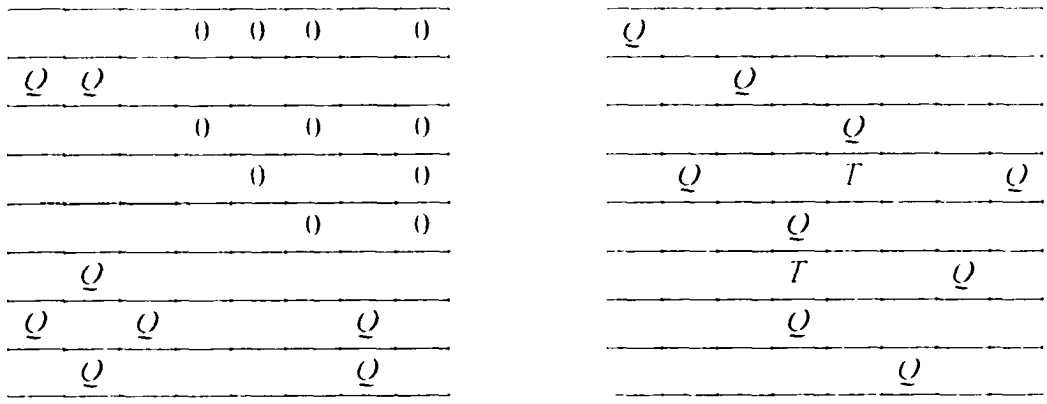


Figure 1.4.3

1.4.4. The Three-Dimensional nxn Queens Problem

McCarty posed the Three-Dimensional n×n Queens Problem in 1978. Gilles Brassard and Paul Bratley[**26**] defined the problem as follows:

Investigate the problem of placing k queens on an nxnxn three dimensional board. Assume that a queen in three dimensions threatens positions in the same rows, columns, or diagonals as herself in the obvious way. Clearly k cannot exceed nxn (read as n-squared). Not counting the trivial case $n = 1$, what is the smallest value of n for which a solution with nxn queens is possible?

The problem was solved by Allison,Yee,and McGaughey in 1989. They found that there are no solutions for $n = 11$ but 264 solutions exist for $n = 11$. [**26**, p.327] The details of these 264 solutions are found in [**27**].

2. The Queens Problem On A Partial Chessboard

2.1. Introduction

In this section we will discuss The Queens Problem on a Partial Chessboard in greater detail. My mentor, Prof. Michael Anshel first posed this problem in 1995 in the course of an Artificial Intelligence seminar. At that time, our group wrote a backtracking program to generate partial boards (an 8×8 chessboard with some squares blocked), and solutions on these boards. We found some results that were interesting from a probabilistic perspective. I have since continued work on this problem.

The Queens Problem on a Partial Chessboard: *Place more than n queens on an n×n chessboard with some cells blocked so that no queen attacks another. A solution for an nxn chessboard with m cells removed is an arrangement of more than n queens on the board such that no queen attacks another.*

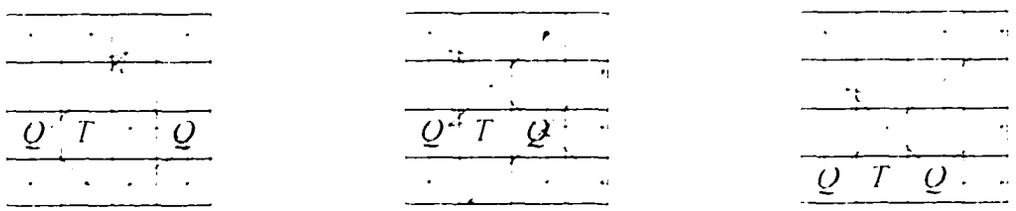
The maximum queen problem on a partial chessboard: *What is the maximum number of queens one can place on an n×n board if one can block as many cells as he needs?*

For example, for 3×3 board, one can place 4 queens on the board, if 5 cells are blocked (see Figure 2.1.2).

How do the blocks affect the classical n-queens solution: *Does a blocked cell increase the number of solutions to the classical n-queens solution? And what is the relationship between the number of blocked cells and the number of increased solutions?*

Theorem 2.1.1. *There is no solution to the queens problem on a partial chessboard for n×n boards where* $n = 1, 2,$ *or* 4.

The case when n is 1 and 2 is easy to verify. For the case $n = 4$: Let $T = \{a_{ij}\}$ be the collection of blocked tiles, and $t(a_{ij})$ be the cardinality of T. Let $att(a_{ij})$ be the number of times that a_{ij} has been attacked. $occ(a_{ij})$ be the number of a_{ij} that has been occupied.



a b c

Figure 2.1.1

Proof. The only way to place 5 queens on a 4×4 chessboard is to let at least two queens share a row or column. The shared rows or columns must contain blocked squares.

If $T = a_{00}$ ($T = a_{01}$, $T = a_{20}$, $T = a_{02}$, $T = a_{12}$, $T = a_{22}$, $T = a_{32}$ and $T = a_{32}$ cases will be similar). If we place queens on a_{00} and a_{20} (a_{01} and a_{21} have the same result), then $att(a_{ij}) = 0$, where the i's are 0, 1, 2, 3 (see Figure 2.1.1 c). We can place two more queens on a_{13} and a_{33} without blocking any more squares or we can block the entire a_1 row and repeat row a_0 in row a_2 which does not increase the total number of queens that can be placed.

If $T = a_{11}$ (the $T = a_{12}$, $T = a_{21}$ and $T = a_{22}$ cases are similar), one can place two queens in the pattern of Figure 2.1.1 a or Figure 2.1.1 c. Both cases result in row $att(a_{ij}) = 0$. If we follow Figure 2.1.1 a, we can block a_{23} and one of a_{03} or a_{33} to place two more queens. If we follow Figure 2.1.1 b, we can block square a_{23} and place two queens on a_{03} and a_{33}.

Therefore, on a 4×4 blocked board, the maximum $occ(a_{ij})$ is 4, and incrementing $t(a_{ij})$ has no affect on $occ(a_{ij})$.

Q	T	Q
T	T	T
Q	T	Q

Figure 2.1.2

Even though there is no solution for the 4×4 partial board, there is a solution for the 3×3 board. As the left figure shows, 5 cells need to be blocked in order to place 4 queens on the board.

This is also an example solution to the maximum queen problem on a partial chessboard.

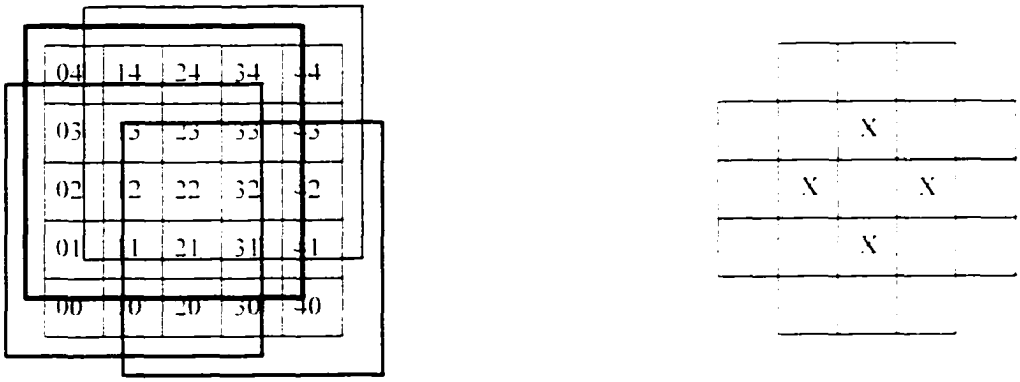
Theorem 2.1.2. *The minimum number of squares that need to be blocked in order to place 6 queens on a 5×5 board is 3.*

Proof. We will show that blocking two squares is insufficient, and that blocking three yields a solution.

The proof that it is impossible to place 6 queens on a board with two squares blocked has four parts:

i) *Any solution would require that at least 4 queens be placed on the edges.*

A 5×5 board contains four 4×4 boards: *(00, 03, 33, 30)*, *(01, 04, 34, 31)*, *(10, 13, 40, 43)*, and *(11, 14, 44, 41)* (see Figure 2.1.2 a). By Theorem 2.1.1, we know that none of the 4×4 boards can contain more than 4 queens. This means that the four "*L*" shaped regions, *(04, 44, 40)*, *(00, 40, 44)*, *(00, 04, 44)*, and *(04, 00, 40)* must each contain at least two queens, and every edge, *(00, 04)*, *(04, 44)*, *(44, 40)*, and *(00, 40)* must contain at least one queen. If a queen is placed in a corner, then that prevents any queen on the two adjacent sides. Therefore, the shape shown in Figure 2.1.2 b must have at least one queen on each edge. A queen placed on any "X" cells will result in the exclusion of the adjacent edge. So, a queen cannot be placed on any of the "X" squares.

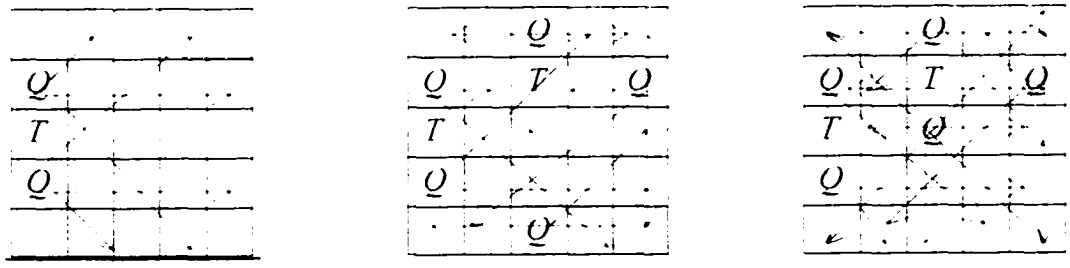


a b

Figure 2.1.3

ii) *If we place five queens on the edges (00, 04), (04, 44), (44, 40), and (00, 40), then* $att(a_{ij}) > 0$ *for all the cells who's* $occ(a_{ij}) = 0$.

If there are five queens on four edges, then one of the edges has to contain two queens. Say this edge is *(00, 04)*. That will require a blocked square at a_{02} (see Figure 2.1.3 a). Since $att(a_{02}) = 2$ where

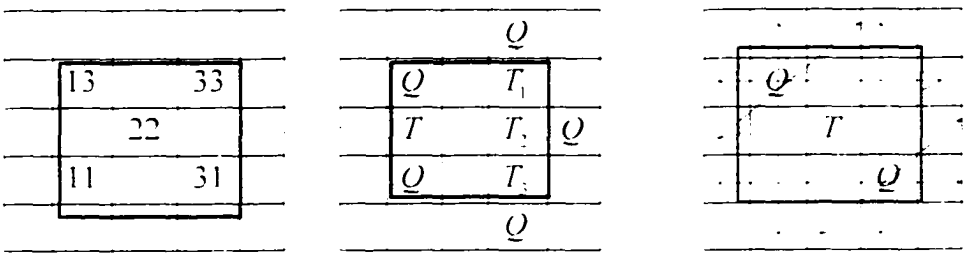


a b c

Figure 2.1.4

i and j are *12, 21* and *23*, if we block one of them, say a_{23}, then that will yield two different arrangements: Figure 2.1.3 b and Figure 2.1.3 c . In either case, we have $att(a_{ij}) = 0$ for all the cells who's $occ(a_{ij}) = 0$

iii) *If four queens are placed on the edges, then no more than one queen can be placed on the center.* $\{a_{11}, a_{13}, a_{22}, a_{31}, a_{33}\}$. (see Figure 2.1.4 a)



a b c

Figure 2.1.5

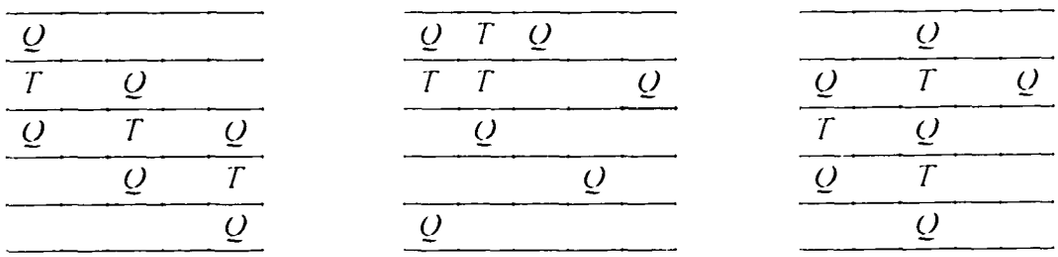
Assume this is false. Four queens have been placed on the edges and two queens are placed in the region $\{a_{11}, a_{13}, a_{22}, a_{31}, a_{33}\}$. There are two ways to place two queens on $\{a_{11}, a_{13}, a_{22}, a_{31}, a_{33}\}$ as Figure 2.1.4 a shows. One is to place two queens on one line say a_{11} and a_{13} with a_{12} blocked. This makes it impossible for edges $(00, 04) \cup (04, 44)$ and $(00, 40) \cup (00, 04)$ to contain two queens. Four edges can contain maximum three queens as Figure 2.1.4 b showing. Another is to place two queens on principal diagonal say a_{13} and a_{31}

with a_{33} blocked, which results no more queens in the four edge's area Figure 2.2.4 c showing. Both cases will contradict to four queens on the edges.

ii) and iii) showed that one can not place more than five queens on a 5x5 board with two squares blocked.

iv) *Three blocked squares result solutions.*

Figure 2.1.5 gives some solutions for the 5x5 board with three squares blocked.



a b c

Figure 2.1.6

Therefore, Theorem 2.1.2 is true.

Theorem 2.1.3. *For an n×n board where n ≥ 5, the minimum number of squares that need to be removed in order to put n − 1 on queens on the board is one, when n = 6m, or n = 6m − 2.*

We construct one solution from $n = 6m - 1$, or $n = 6m - 5$. Add one column to the left, and one row to the top of the structure (see Figure 2.1.1 for the case $n = 6$). Block square $(0, n-2)$. Place a queen on $(-1, n-2)$ and a queen on $(0, n)$. The rest of the queens follow the knight's move $f(k) = 2k \bmod (n+1)$.

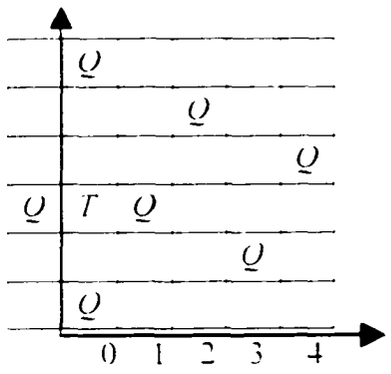


Figure 2.1.7

During the study of a paper covering coalgebras and bialgebras applied to combinatorics [**1**] by Joni and Rota, I found that point-lattice coalgebras give the chessboard a unique structure.

The coalgebra's structure is very interesting and helpful for understanding the structure of a partial board. In the following sections, I would like to summarize the application of point-lattice coalgebra to chessboards. In order for this to be self-contained, related definitions are given in the next section. In section 2.3 more results will be given.

2.2. Notation and Terminology

Definition 2.2.1. A partial ordering relation (denoted by $\leq$) on a set P is one which is reflexive ($a \leq a$), transitive ($a \leq b$, $b \leq c$ then $a \leq c$), and anti-symmetric ($a \leq b$, $b \leq a$ then $a = b$). A set P together with a partial ordering relation is a partially ordered set, or PO set for short.

A PO set is said to be locally finite if every segment is finite.

Definition 2.2.2. A lattice is a PO set where the maximal and minimal of two elements (we call them join $\vee$ and meet $\wedge$) are defined.

Definition 2.2.3. An element y is said to cover x if the segment $/x, y/$ has two elements. An atom of P is an element that covers a minimal element.

Definition 2.2.4. The incidence algebra $\varphi(P)$ of a locally finite PO set P over a field: Assume that K has characteristic zero. The members of $\varphi(P)$ are functions of two variables $f: P \times P \to K$ such that $f(x, y) = 0$ unless $x \leq y$. The sum of two functions and multiplication by scalars are defined as usual. The product $f \otimes g = h$ is defined by

$h(x, y) = \sum_{z \in P} f(x, z) g(z, y)$. The unit element δ is defined by

$$\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & otherwise \end{cases}.$$

To show that $\varphi(P)$ is an algebra is trivial:

$$M^2(I\otimes M)(f\otimes g\otimes h)(x,y) = M(f(x,y)(\sum_{x\le z\le y} g(x,z)h(z,y))$$

$$= \sum_{x\le w\le y} f(x,w)(\sum_{w\le z\le y} g(w,z)h(z,y)) = \sum_{x\le w\le y}\sum_{w\le z\le y} f(x,w)g(w,z)h(z,y)$$

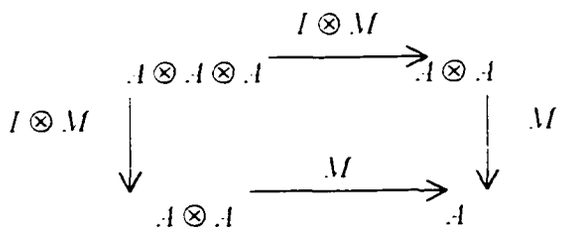
and

$$M^2(M\otimes I)(f\otimes g\otimes h)(x,y) = M((\sum_{x\le z\le y} f(x,z)g(z,y))h(x,y))$$

$$= \sum_{x\le w\le y}(\sum_{w\le z\le y} f(x,z)g(z,w))h(w,y) = \sum_{x\le w\le y}\sum_{w\le z\le y} f(x,z)g(z,w)h(w,y)$$

z and w

range over the finite segment $[x,y]$. so M satisfies the associativity . namely



(associativity of M)

Figure 2.2.1

since

$$M\circ(I\otimes\delta)(f(x,y)\otimes g(x,y)) = M\circ(f(x,y)\otimes 1_{(x,y)}) = \sum_{x\le z\le y} f(x,z)1_{(z,y)} = f(x,y)$$

it satisfies the unitary property:

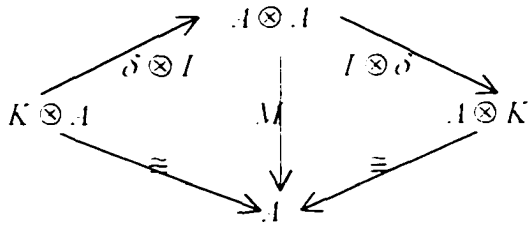
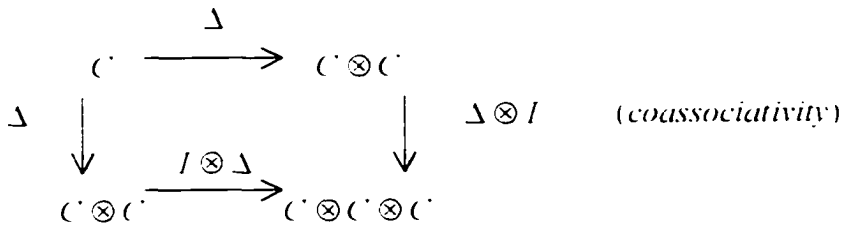


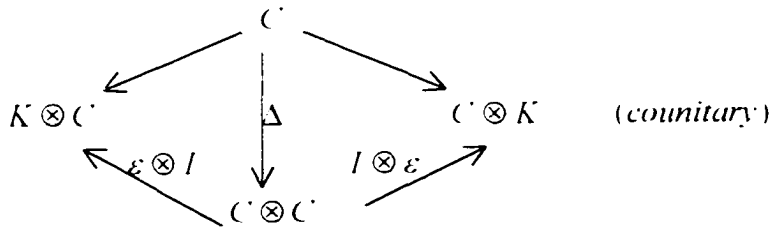
Figure 2.2.2

Definition 2.2.5 A coalgebra is a triple (C, Δ, ε) with C a K-vector space, $\Delta : C \to C \otimes C$ a map called diagonalization, and $\varepsilon : C \to K$ a map called the counit, where Δ and ε satisfy the following commutative diagrams:



(*coassociativity*)

Figure 2.2.3



(*counitary*)

Figure 2.2.4

Definition 2.2.6. A point lattice L is a lattice in which every element is the supremum of a set of atoms.

We can show that L is isomorphic to the lattice of closed sets relative to the closure operation defined on subsets of the set A of atoms by $\bar{A} = \{p \in \Lambda \mid p \leq \sup A\}$, for A.

The closure operation has the properties $A \subseteq \bar{A}$, $\bar{\bar{A}} = \bar{A}$, if $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$. The complements of closed sets, called open sets, can be characterized by

1) The union of any family of open sets is an open set.

2) Every open set is the union of the minimal nonempty open sets it contains.

Thus, every point lattice can be represented as the family of all open sets in a closure relation where the join in the lattice is set-theoretic union. Let's assume that L is represented by a fixed set Λ, and has a unique minimal element, which is the empty set. This representation of L allows us to define a coalgebra structure on Λ. As a vector space, this coalgebra $C(L)$ is isomorphic to the vector space over K with basis consisting of all open sets of Λ. For each open set $A \subseteq \Lambda$, the diagonalization is $\Delta A = \sum_{\substack{A_1, A_2 \text{ open} \\ A_1 \cap A_2 = \Phi \\ A_1 \cup A_2 = A}} A_1 \otimes A_2$ and the counit is

$$\varepsilon(A) = \begin{cases} 1 & A = \Phi \\ 0 & otherwise \end{cases}$$

Since the union of open sets is again an open set, it follows immediately that the above diagonalization is coassociative.

In the next section, an example of an application of a point-lattice coalgebra will be given.

2.3. An Application of Coalgebras to the N×N Board

Let Λ denote the collection of the n^2 square cells $\{a_{ij}\}_{i,j=1}^{n}$ on an n×n chessboard. Our point lattice L is represented by the following family of open subsets of Λ: The minimal nonempty open sets are the columns and rows, we call them *forbidden sets*. Each open set A is uniquely determined by $R(A) = \{i \ \ row \ \ i \ \ is \ \ in \ \ A\}$ and $C(A) = \{j \ \ column \ \ j \ \ is \ \ in \ \ A\}$. Two open sets A_1 and A_2 can have $A_1 \cap A_2 = \Phi$ if and only if either $R(A_1) = R(A_2) = 0$ (A_1 and A_2 are different columns) or $C(A_1) = C(A_2) = 0$ (A_1 and A_2 are different rows).

Let's construct a new lattice $St(L)$, the lattice of stars of L as follows: $\forall p \in \Lambda$, the star of p, $st(p)$, is the union of all forbidden sets containing p. If $A \subseteq \Lambda$, we set $st(A) = \bigcup_{p \in A} st(p)$. Thus $S \in L$ is a star if and only if $S = st(A)$ for some $A \subseteq \Lambda$ and if $A \subseteq B$ then $st(A) \subseteq st(B)$. We say that S is generated by A, if A is the smallest set such that $st(A) = S$. The Lattice $St(L)$ consists of all stars of L.

ordered by inclusion, where the join is set-theoretic union. In our case, the minimal nonempty stars of L are the unions of the row and the column through each square $\{a_{ij}\}$. Therefore, the number of atoms of $St(L)$ is n^2, and since every star is a union of these minimal stars, $St(L)$ is also a point lattice. If we define a non-taking subset A of a board as a collection of squares $\{a_{ij}\}$ such that no two squares have the same row or column index, then a non-taking set $A \subseteq A$ is a set such that each $a_{ij} \neq a_{kl}$ in A implies $a_{kl} \notin st(a_{ij})$ and $a_{ij} \notin st(a_{kl})$. Let $T = \{a_{ij}\}$ be the collection of blocked squares and $t(a_{ij})$ be the cardinality of T. Furthermore, let $at(a_{ij})$ be the number of times that a_{ij} has been attacked and let $oc(a_{ij})$ be the number of a_{ij} that have been occupied, we expect $at(T)$, and $oc(\{a_{ij} \mid a_{ij} \in st(T)\})$ to be as large as possible. The following theorems give some necessary conditions for the existence of the partial board queens' solutions.

Theorem 2.3.0. *For an n×n regular board, if $S = \{a_{ij}\}$ is a solution set for the n queens problem, then $at(\{a_{ij} \mid a_{ij} \in \bar{S}\}) > 1$, $oc(st\{a_{ij} \mid a_{ij} \in S\}) = 0$, $oc(st\{a_{ij} \mid a_{ij} \in \bar{S}\}) = 2$, and for $\forall a_{ij} \in \bar{S}$, we have $at(a_{ij}) \geq oc(st(a_{ij}))$.*

Proof. $at(\{a_{ij} \mid a_{ij} \in \bar{S}\}) > 1$ Implies that some queen on a standard board attacks every cell $oc(st\{a_{ij} \mid a_{ij} \in S\}) = 0$ implies that no two queens attack each other. Since $oc(st\{a_{ij} \mid a_{ij} \in \bar{S}\}) \leq 2$ contains one column and one row, it follows

that $st\{a_i \mid a_i \in \bar{S}\}$ contains a maximum of two non-attacking queens a_i can be attacked by the queens lying on the same diagonals, and diagonals are not forbidden sets. So, $at(a_i) \geq oc(st(a_i))$ is true. If $at(a_i) \geq oc(st(a_i)) = 2$, then a_i is attacked by the two queens on $st(a_i)$.

Definition 2.3.1. *A k partial n×n board is defined as an n×n board with k squares blocked.*

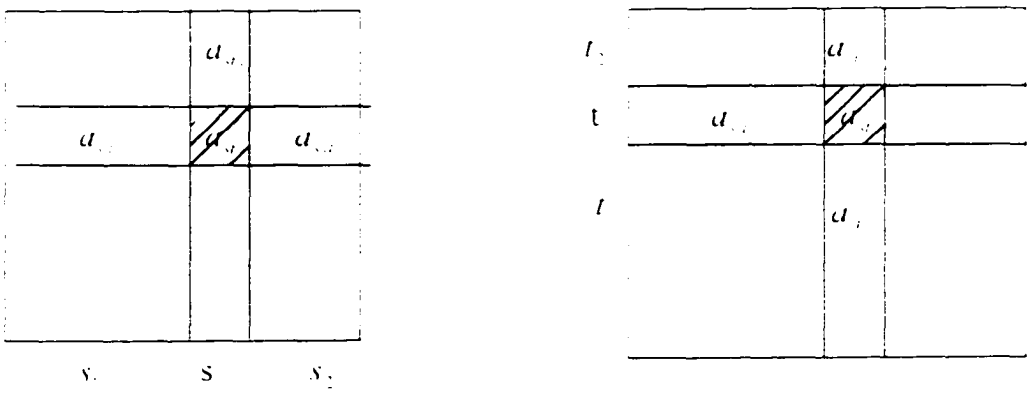
Theorem 2.3.1. *For an n×n partial board queens' solution, where $n \geq 5$, $T = \{a_i\}$ where a_i are the blocked cells, }*

a) If $t(T) = 1$, then $oc(st(a_i)) = 4$.

b) If $t(T) = 1$, then at least one of $a_i \in T$ has the property that $oc(st(a_i)) \geq 3$.

c) If $t(T) = 2$, then $oc(st(T)) \geq 5$.

Proof. a) Since the maximum number of queens that can be placed on $st(a_i)$, such that no two queens attack each other is 4 (see Figure 2.3.1 a), $oc(st(a_i))$ cannot be larger than 4. We have $oc(st(a_i)) \leq 4$. Assume that $oc(st(a_i)) < 4$. Say $oc(st(a_i)) = 3$ and three queens are on columns s_1, s_2 and s_3 (see Figure 2.3.1 a). Then, there are $n-3$ columns left to fill in with $n-2$ queens without any more queens on $st(a_i)$.



a b

Figure 2.3.1

So. by the pigeonhole principal. one column has to contain 2 queens. Contradiction. Similarly. we can handle the case in which three rows are occupied as in Figure 2.3.1 b. so $oct(st(a_{ij})) = 4$.

b) Assume that none of the $a_{ij} \in T$ has the property that $oct(st(a_{ij})) \geq 3$. which says $oct(st(a_{ij})) \leq 2$ for all $a_{ij} \in T$. There are two possible arrangements. as show below. First. two queens belong to the same forbidden set (see Figure 2.3.2). Say two queens are in column i and none are in row j. Then. no more than $n - 1$ queens can be placed on the remaining $n - 1$ rows and $n - 1$ columns.

Second. two queens are in different forbidden sets (see Figure 2.3.2 b). Say one is in row i and the other in column j. Then. the remaining $n-1$ rows and $n-1$ columns can be filled maximally by $n-1$ queens. This contradicts

the condition of being a partial chessboard solution. Therefore $oc(st(a_{ij})) \geq 3$ is true for some $a_{ij} \in T$.

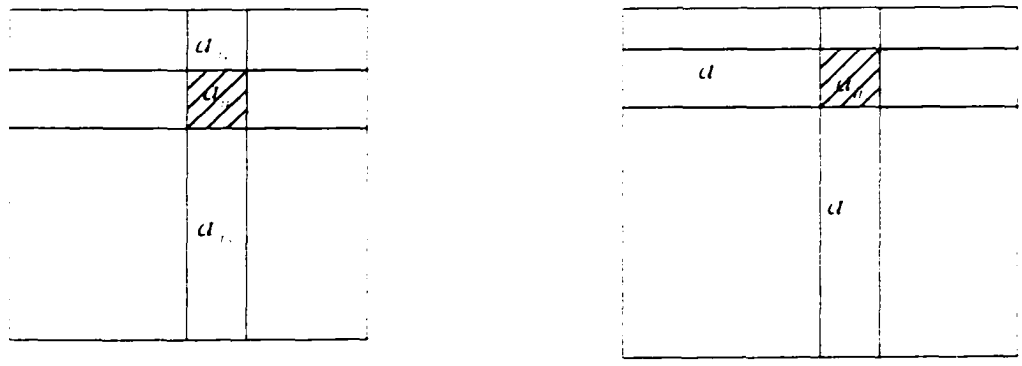


Figure 2.3.2

c) From part b, $\exists a_{ij} \in T$ such that $oc(st(a_{ij})) \geq 3$. From Theorem 2.3.0, we know that $oc(st(a_{ij})) \geq 2$ for any other cells. Therefore, $oc(st(T)) \geq 5$.

Theorem 2.3.2. *For an n×n chessboard with $n \geq 5$, let $C = T \cap \{a_{11}, a_{1n}, a_{n1}, a_{nn}\}$; there is no solution, if $|C| = t(T)$.*

Proof. Since $\max(oc(st(a_{ij}))) = 2$ for $\forall a_{ij} \in C$, $C = T \cap \{a_{11}, a_{1n}, a_{n1}, a_{nn}\}$ and $|C| = t(T) = |C| = T$. Thus, none of the elements in T have stars containing more than 2 queens. By a previous theorem, this implies no solution.

Theorem 2.3.3. *If the only two blocked cells belong to edge forbidden sets, then there is no solution if the two edges forbidden sets are parallel.*

Proof. There are three cases as shown in Figure 2.3.3 for $n = 6$.

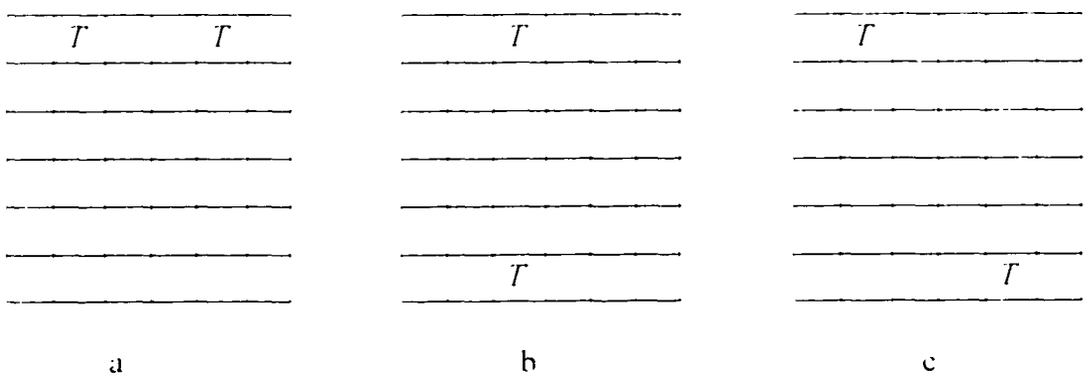


Figure 2.3.3

Since every queen being placed on the board attacks the entire vertical forbidden set, one can't place more than n queens on n vertical forbidden sets.

Theorem 2.3.4. *For an n×n board with $n \geq 6$ and two squares blocked, the maximum number of queens that can be placed on the board is $n - 2$.*

Proof. Since $\max(oc(st\{a_{ij} \mid a_{ij} \in T\})) = 4$, these four queens are on three rows and three columns. With two squares blocked, $oc(st(T)) = 8$ and there are 6 rows and 6 columns occupied. So, we can at most place $n - 6 - 8$ queens on the board.

Figure 2.3.4 is an example of the 8×8 board with two squares blocked and ten queens on it.

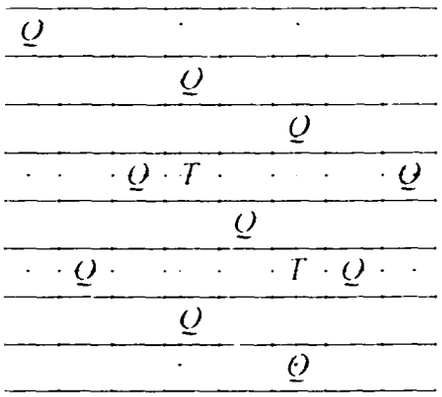


Figure 2.3.4

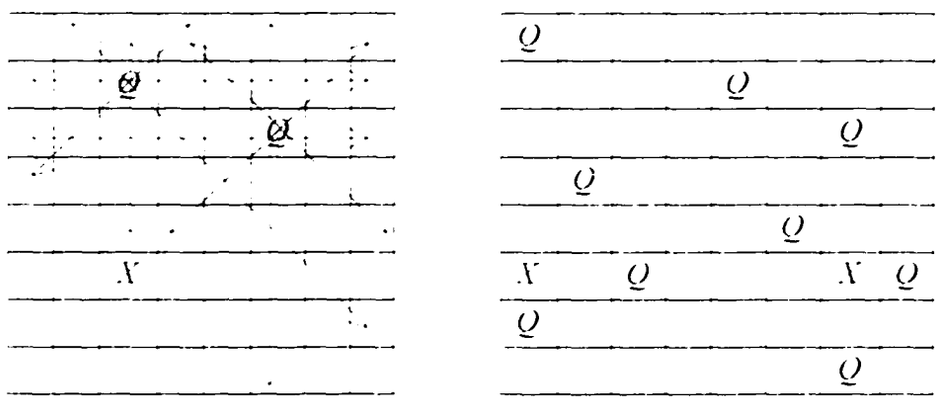
Theorem 2.3.5. *For an 8×8 board with two squares blocked, there are always solutions if the two blocked squares are two rows and two columns away, provided that they don't fall in the exceptions.*

The proof of this theorem will be presented in next section.

2.4. A Simulation of The Queens Problem on a Partial Chessboard

During my research on *St(L)*, I wrote a program to keep track of *att( )* numbers and *oct( )* numbers of each cells on the n×n board, which provided new insight. The simulation is also an interesting game. It lets the user place queens on an n×n

chessboard with two or more squares blocked. A queen's attack does not extend beyond any blocked squares (see Figure 2.4.1 a where the block is denoted by X). The simulation enforces the rule that none of these queens attack each other. This means that no two queens are on the same row, column or diagonal unless the row, column or diagonal contains a blocked square. Figure 2.4.1 b is an example solution.



a b

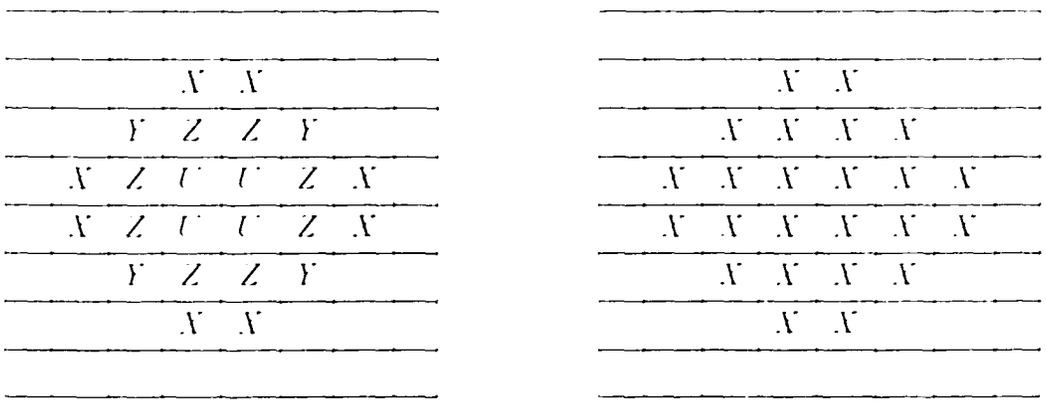
Figure 2.4.1

The user has a choice of

1) The size of the board. (how big n is)

2) Having squares blocked randomly or choosing blocked squares by himself.

Theorem 2.4.1. *If the block is within the area showed below (Figure 2.4.2 b), then one blocked square is sufficient to allow the placing of 9 queens on the 8×8 board.*

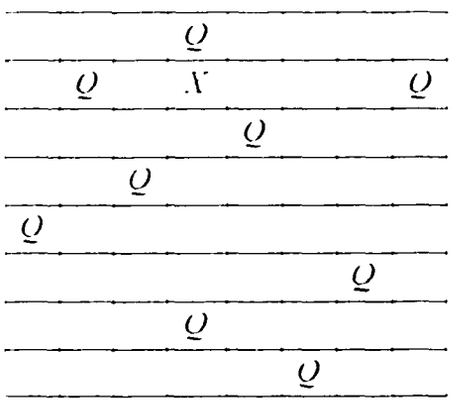


a b

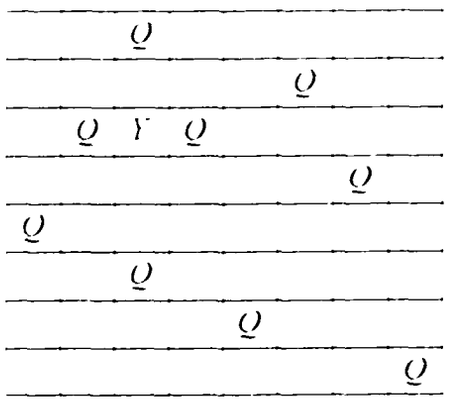
Figure 2.4.2

We divide the boards into four classes, called X, Y, Z, and U, because boards with one X square blocked can be obtained by rotation or reflection of each other, so class X is a collection of boards with an X square blocked. Class Y (Z or U) is a collection of boards with a Y (Z or U) square blocked. In each class, all of the members are rotations or reflections of others (see Figure 2.4.2).

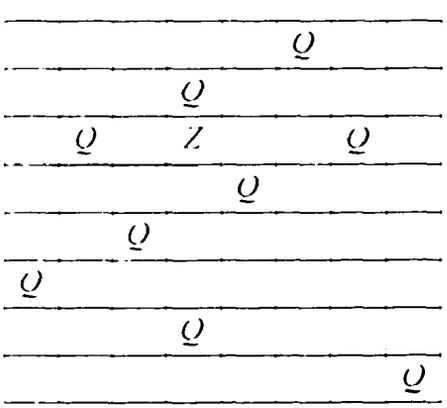
In order to show that Theorem 2.4.1 is true, we will give an example solution for one member in each class. Following are example solutions for each class (see Figure 2.4.3).



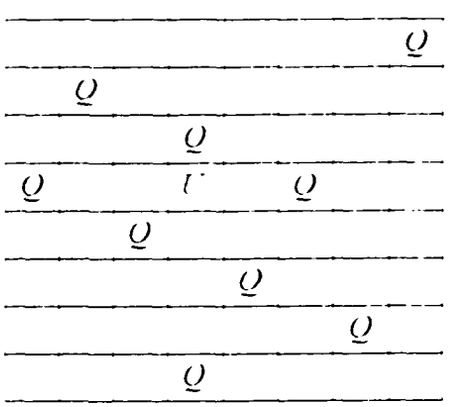
Class X



Class Y



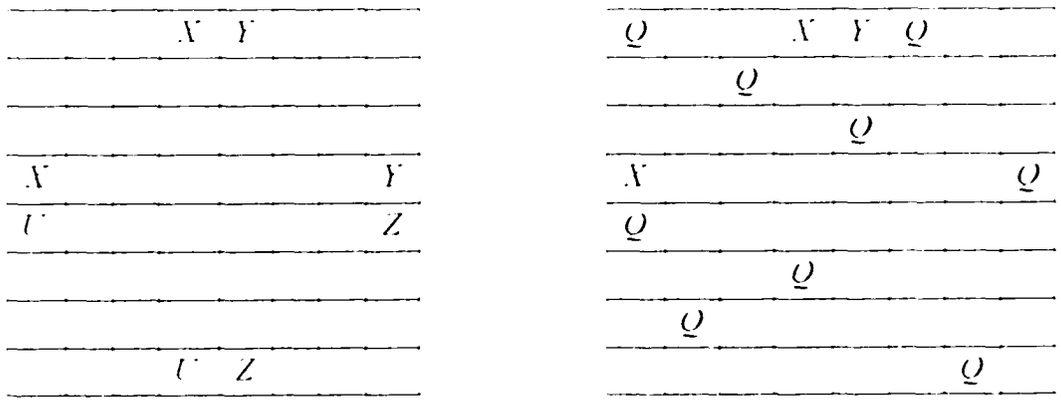
Class Z



Class U

Figure 2.4.3

Theorem 2.4.2. *If two blocked cells are chosen from X's (Y's, Z's, or U's) region in Figure 2.4.4 a, then there is at least one solution.*



a b

Figure 2.4.4

This fact is showed by giving a solution to the board that has two "X" blocks as Figure 2.4.4 b shows. "Y" blocks is the reflection of "X" along vertical center line. "I" blocks is the reflection of "X" along horizontal center line and "Z" blocks is the reflection of "X" along negative diagonal.

Combining Theorem 2.4.1 and Theorem 2.4.2, we can show that Theorem 2.3.4 is true.

A board with two blocked cells, which are two rows and columns away, falls into two cases (see Figure 2.4.6.):

1) At least one blocked cell is on a "X" square. In this case, Theorem 2.4.1 tells us that there is a solution;
2) None of the blocks is in "X" area;

In this case, we need to consider:

a) Both blocks are in "Y" area.

b) Both blocks are in blank area.

c) One block is in "Y" area and one in blank area.

Theorem 2.4.2. gives solutions for all three cases.

Theorem 2.4.3. *If two blocked cells are* $a_{\cdot,\cdot}$ *and* $a_{\cdot,\cdot}$, *so far the only solution has been found is for* $n = 6m - 3$.

In order to have $oct(st(T)) = 5$, one has to place two queens at corners $a_{\cdot,\cdot}$ and $a_{\cdot,\cdot}$. The sub board $a_{\cdot,\cdot}\, a_{\cdot,\cdot}\, a_{\cdot,\cdot}\, a_{\cdot,\cdot}$ of $a_{\cdot,\cdot}\, a_{\cdot,\cdot}\, a_{\cdot,\cdot}\, a_{\cdot,\cdot}$ is an $(6m-2)\times(6m-2)$ board.

and $$f(k-1)=\begin{cases} 2k+(n-2)\ 2 \pmod{n} & \text{if } (n-2)\ 2 \geq k-1 \geq 0. \\ (n-1)-f(n-1-k) \pmod{n} & \text{if } (n-1) \geq k-1 \geq n\ 2. \end{cases}$$

gave a $6m - 2$ queens solution without any queen on the principal diagonals, which means these $6m - 2$ queens do not threaten $a_{\cdot,\cdot}$ and $a_{\cdot,\cdot}$. Therefore we have constructed a solution for two partial $(6m-3)\times(6m-3)$ board queen problem(see Figure 2.4.5 for $n = 9$).

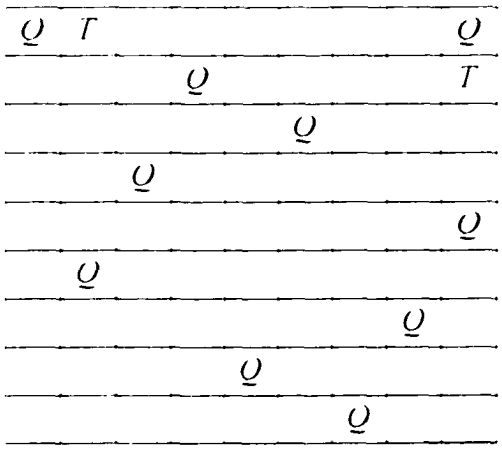


Figure 2.4.5

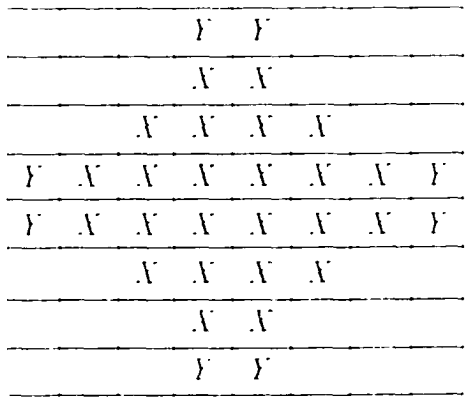


Figure 2.4.6

Theorem 2.4.4. *There is always a solution for any two partial n×n board.*

if $n \geq 5$

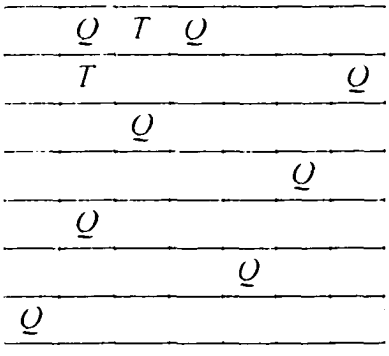


Figure 2.4.7

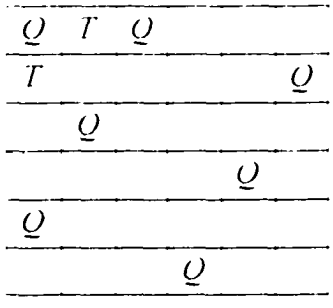


Figure 2.4.8

Proof. In order to show this theorem, we divide n into three cases:

a) $n = 6m - 1$ or $n = 6m - 5$:

Cell $a_{1,n-1}$ is on positive diagonal $n - 2$ ($= n - 1 - 1$) and negative diagonal n ($= 1 - n - 1$). If we place queens by $f(k) = 2k \pmod n$, meaning place queen on cell $(k, 2k)$ then $at(a_{1,n-1}) = 2$. If $2k - k = n - 2$, then $2(n - 2)$ will be

off the board: if $2k - k \quad n$ then $3k \quad 6m - 1$ or $3k \quad 6m - 5$. but neither $6m - 1$ no $6m - 5$ is divisible by 3. so there is no such k. Therefore $at(a_{\quad}) = 2$. and we can block $a_{1\,n-2}$, $a_{2\,n-1}$ and place a queen on $a_{\quad}$ to construct a solution (see Figure 2.4.7).

b) $n \quad 6m$ or $n \quad 6m - 4$:

We can obtain a solution from $6m - 1$ and $6m - 5$ by removing the queen in the $(0, 0)$ position (see Figure 2.4.8).

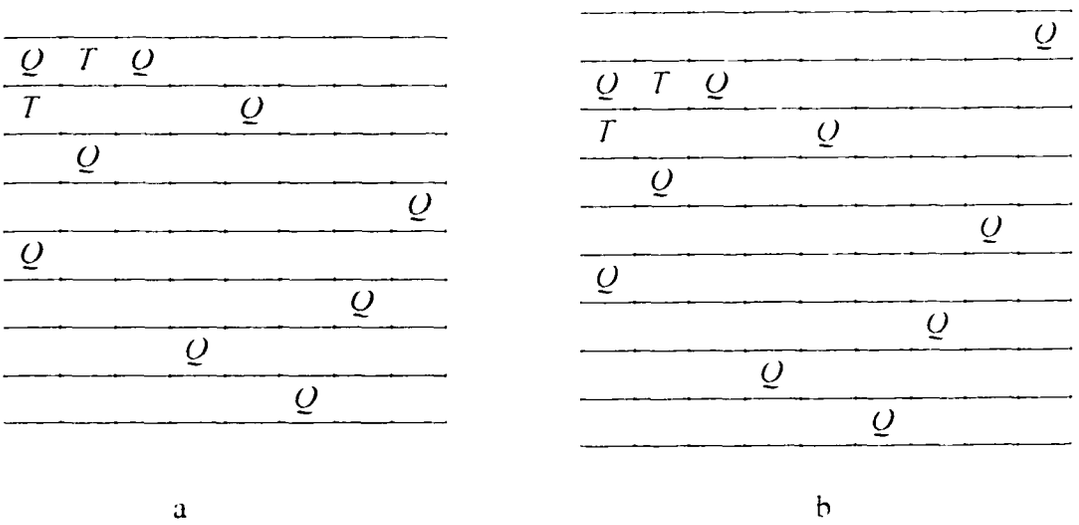
c) $n = 6m - 2$ or $n = 6m - 3$:

For $n = 6m - 2$. we place queens as following

$$f(k) = \begin{cases} 2k + (n-2), 2 \pmod{n} & if\ (n-2), 2 \geq k \geq 0. \\ (n-1) - f(n-1-k) \pmod{n} & if\ (n-1) \geq k \geq n, 2. \end{cases}$$

Since this solution does not have a queen on the principal diagonal(see Figure 2.4.9 a for $n = 8$), we can block cells $a_{1\,n-1}$, $a_{0\,n-2}$ and place the $(n+1)^{th}$ queen on cell $a_{0\,n-1}$.

For $n = 6m - 3$. we add a row and column to the $n = 6m - 2$ board and place a queen on the new corner cell $a_{n-1\,n-1}$ (see Figure 2.4.9 b for $n = 9$).



a b

Figure 2.4.9

2.5. Three-Dimensional Partial Board Queens

Suppose the queen is endowed with added powers and allowed to roam over a cubical board with n^3 cells, call it an n-cube. Let the n-cube be constructed by stacking n levels of an n×n partial boards so that the left hand corners of each level alternate color. Furthermore, let the queen be allowed to move diagonally up and down the cube as well as directly above and below the cell (not blocked) that it occupies.

To visualize this: The problem may be reformulated by considering the n-cube as an n×n grid. Each cell contains zero or more integers from 1 to n inclusive representing the level of the cube at which a queen has been placed (letting 1 represent the bottommost level). For example:

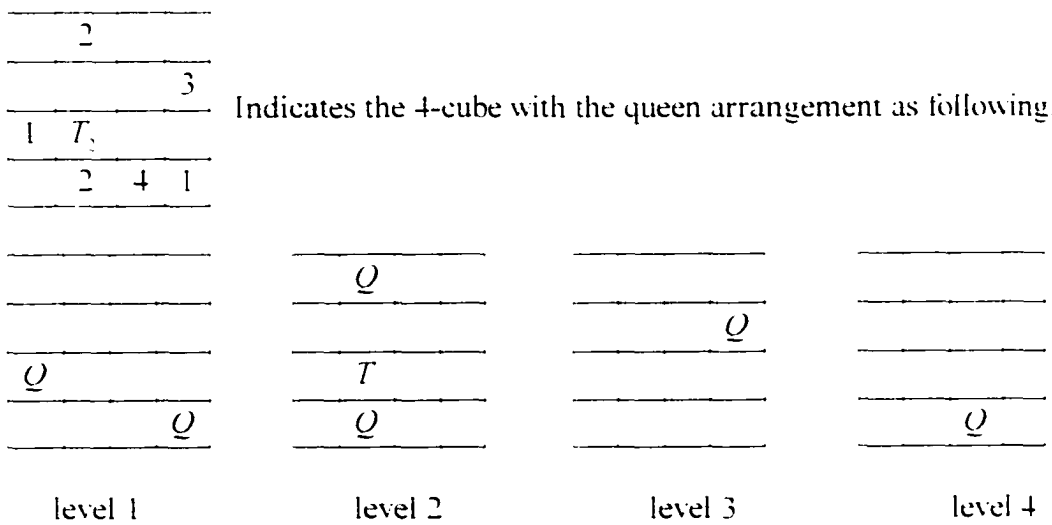


Figure 2.5.1

The placing of the k non-attacking queens into an partial n-cube is now reduced to the filling of an n×n grid with k copies of the set $\{1, 2, \ldots, n\}$ such that: on the same level, if there are no blocked cells in a row, column or diagonal then no two element are in the same row, column, diagonal; and on different levels, if there are no blocked cells in the diagonal that runs cross the different levels, then no two elements are on the same such diagonal.

Definition:

A solutions to three dimensional partial n-cube is a square arrangement of the elements from the set $S = \{1, 2, \ldots, n\}$ and empty squares such that

1) No element $i \in S$ may appear twice in the same row, column, or diagonal unless there are cells blocked in that row, column or diagonal, and

2) The elements i *and* $j \in S$ may not be placed i j squares apart from each other along any row, column or diagonal, unless there are blocked squares in that row, column or diagonal.

The problems to consider are:

1) Which cells can be blocked to effect the number of queens that can be place on the n-cube?

2) What is the maximum number of queens that can be placed on a certain n-cube with minimum cells blocked?

3) Does an algorithm exist for the maximal placement of the queens into the n-cube?

3. Rooks, Bishops And Knights

3.1. Rooks Problem

There are n! ways to put n rooks on an n×n chessboard so that none of them share a row or column. But, how many fundamental solutions are there? The problem of the rooks is to find the number of inequivalent arrangements of n rooks on an n×n chessboard such that no rook attacks another. Each such arrangement is said to be a solution to the problem of the rooks.

The symmetries of the chessboard are the identity (denoted by e), rotation by π radians (denoted by c), rotation by $\pm\pi/2$ radians (denoted by q and q'), the reflections about the horizontal and vertical mid-lines (denoted by m and m'), and the reflections about either main diagonal (denoted by d and d'). We denote the group of eight symmetries by A. In 1890, Lucas [5] found the number of inequivalent solutions to the problem of the rooks. 40 years later, Kraitchik [3] rediscovered the problems of rooks, queens, and bishops and determined some of the inequivalent solutions up to automorphism. The details of those results will be given in the following of this section.

In the Table 3.1.1, the multiplication table of the group A is given. Each solution to the problem of the rooks has some subgroup of A as its group of automorphism.

e	c	q	q'	d	d'	m	m'
c	e	q'	q	d'	d	m'	m
q	q'	c	e	m'	m	d	d'
q'	q	e	c	m	m'	d'	d
d	d'	m'	m	c	e	q	q'
d'	d	m	m'	e	c	q'	q
m	m'	d	d'	q	q'	e	c
m'	m	d'	d	q'	q	c	e

Table 3.1.1

From the multiplication table we can see that for n = 1 there are 10 subgroups:

1) The identity: e.

2) Rotations through π radians: c.

3) Reflections about the vertical midline: m'.

4) Reflections about the horizontal midline: m.

5) Reflections about the negative main diagonal: d'.

6) Reflections about the positive main diagonal: d.

7) Rotations through $\pi/2$: q.

8) Reflections about either midline: m, m'.

9) Reflections about either main diagonal: d, d'.

10) The whole group: A.

Every solution to the problem of the rooks has its corresponding automorphism group (the largest subgroup of the given group that leaves it invariant).

Burnside's lemma [18]. *The number of orbits of a finite permutation group is the average of the number of fixed points of the permutations.*

Let t_n be the number of inequivalent n×n rook solutions. Then, it is also the number of orbits of A on this object set. We let E_n, C_n, Q_n, and D_n be the number of n×n solutions to the problem of the rooks that are left fixed by e, c, q, and d respectively. There are no solutions left fixed by m or m'. By Burnside's lemma,

$$t_n = \tfrac{1}{8} E_n + \tfrac{1}{8} C_n + \tfrac{1}{4} Q_n + \tfrac{1}{4} D_n \qquad \text{(Equation 3.1.1)}$$

In the equation above, we have $E_n = n!$. Lucas [5, p. 66] has shown that the number of $n \times n$ solutions invariant under c is $2k(2k-2)\ldots 2$ for $n = 2k$ or $2k-1$. Thus,

$$C_{2k} = C_{2k-1} = 2^k k! \qquad \text{(Equation 3.1.2)}$$

For n×n solutions invariant under q, Lucas [5, p. 67] has $(4m-2)(4m-6)\ldots 2$ when $n = 4m$ or $4m-1$ and none otherwise.

Thus

$$Q_n = \begin{cases} 0 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4. \\ \dfrac{(2m)!}{m!} & \text{if } n = 4m \text{ or } 4m-1. \end{cases} \qquad \text{(Equation 3.1.3)}$$

He also shows that D_n satisfies the recurrence [5. p.215]

$$D_{n+1} = D_n + nD_{n-1} \qquad \text{where } n > 0 \text{ and we set } D_0 = 1. \qquad \text{(Equation 3.1.4)}$$

Lucas calculated the values of L_n [5. p.222] for $1 \le n \le 13$ on the basis of Equations 3.1.1 – 3.1.4. These values are shown in Table 3.1.2.

n	L_n	E_n
2	1	2
3	2	6
4	7	24
5	23	120
6	115	720
7	694	5.040
8	5.282	40.320
9	46.066	362.880
10	456.454	3.628.800
11	4.999.004	39.916.800
12	59.916.028	479.001.600

Table 3.1.2

3.2. Bishop Problems

3.2.1. Maximum bishop problem

What is the maximum number of bishops that can be put on an nxn chessboard such that none attacks another?

The answer to this question in $2n - 2$. To prove this, note that the number of diagonals running in one direction is $2n - 1$. The two diagonals that consist of single squares, however, cannot both be occupied because two bishops would then be on the main diagonals running the other way. This reduces the maximum to $2n - 2$. Thus, on the standard 8x8 board, no more than fourteen bishops can be placed so that no two attack each other. One method of placing the maximum number of bishops on a board is to put n bishops along one edge, and $n - 2$ bishops along the opposite edge(see Figure 3.2.1).

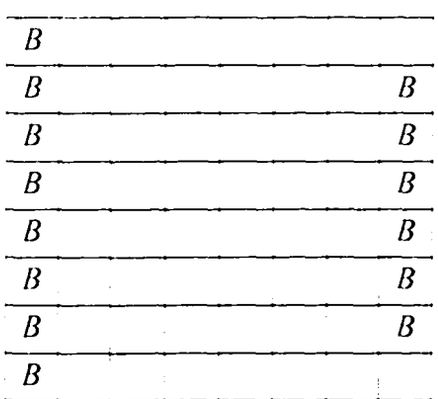


Figure 3.2.1

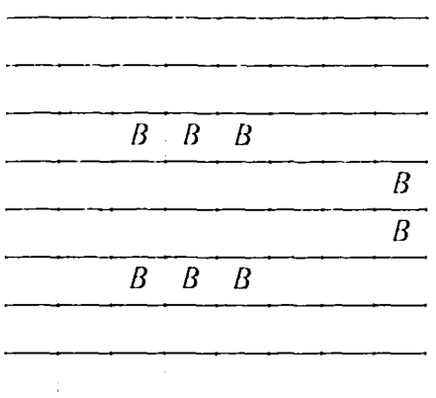


Figure 3.2.2

Theorem 3.2.1. *The number of $n \times n$ solutions to the maximum bishop's problem is 2^n.*

Proof. Consider the positive main diagonal, $pd = 0$. There are two possible places for a bishop, $(1, 1)$ and (n, n). Suppose $(1, 1)$ is occupied by a bishop (without loss of generality because the (n, n) case is similar). There will be two possible places for a bishop on each successive positive diagonal $pd = 1, 2, \ldots, n - 1$, namely $(1, 2)$ or $(2, 1)$, $(1, 3)$ or $(3, 1)$, $\ldots$, $(1, n-1)$ or $(n-1, 1)$. The placement of bishops on the negative diagonals will be completely determined by the positive diagonal choices. Therefore, there are 2^n placements.

3.2.2. The Minimum Bishops Problem

What is the minimum number of bishops that can be placed on an $n \times n$ chessboard such that every square is either occupied or attacked by a bishop?

In graph theory language, this is the same as finding the minimum externally stable set for a graph on n^2 vertices with two vertices joined only if they are on the same diagonal.

Given a graph $G=(X,U)$, a set $T(\subset X)$ is externally stable if for every vertex $x \notin T$, we have $\Gamma x \cap T \neq \Phi$; in other words, we have $\Gamma^{-1}T \supset X-T$.

If φ denotes the family of externally stable sets of a graph, we have $X \in \varphi$

$T \in \varphi$, $A \supset T \Rightarrow A \in \varphi$

By difinition, the coefficient of external stability of the graph G is

$$\beta(G) = \min_{T \in \varphi} T$$

In this case, vertices $(x,y) \in U$ if and only if the cells x and y are on the same diagonal and $\beta(G)=n$.

Figure 3.2.2 gives an example placement of minimum bishop problem for an 8×8 regular chessboard.

3.2.3. The m Bishops Problem

For any number m, where $n \leq m \leq 2n-2$ (for $n \geq 1$), Can m bishops be placed on an n×n chessboard so that every square is either occupied or attacked by a bishop? How many solutions are there? How many of them are inequivalent solutions?

Theorem 3.2.2. *There are always solutions to the m Bishops problem for $n \leq m \leq 2n-2$.*

Proof. The $m \quad n$ and $m \quad 2n \quad 2$ cases have already been discussed. For the $n \quad m \quad 2n \quad 2$ cases, let $k \quad (2n \quad 2) \quad m$ and consider the solution in Figure 3.2.1. Remove k bishops $(1, 2), (1, 3), \ldots, (1, k - 1)$ from the first column. Move k bishops $(n, n - 1)$ to $(n \quad 1, n), (n, n \quad 2)$ to $(n - 2, n), \ldots, (n, n \quad k \quad 1)$ to $(n \quad k \quad 1, n)$. This will give a solution because the new location of each bishop takes care of both the diagonals that the removed bishop and moved bishops had occupied (see Figure 3.2.3 for a solution to 11 bishops on 8×8 board).

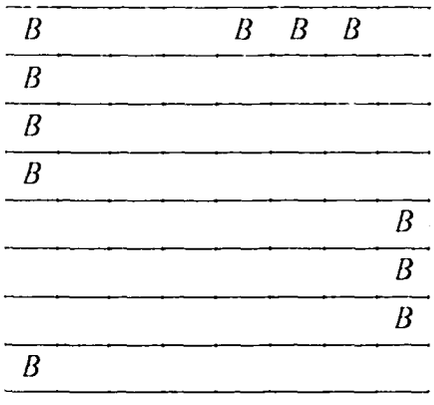


Figure 3.2.3

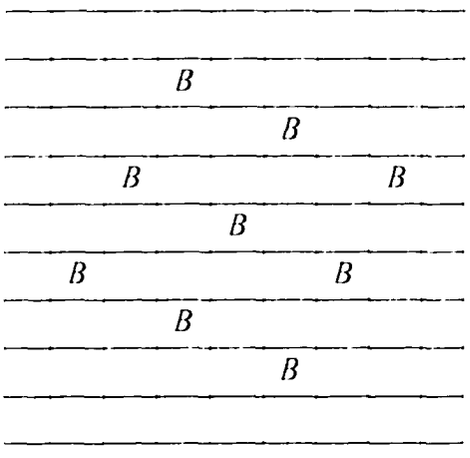


Figure 3.2.4

No other general results on the numbers of solutions to this problem are currently known.

3.2.4. The inequivalent solutions to the problem of the bishops

What is the number of inequivalent arrangements of n bishops on an n × n chessboard such that no bishop threatens another and every unoccupied square is threatened by at least one bishop?

One can always place n bishops on an n×n board so that the bishops dominate the board (Figure 3.2.2 and 3.2.3 are two example solutions to this problem). But, how many inequivalent solutions are there? Two solutions are considered to be equivalent if one can be obtained from the other by rotations or reflections. Robert W. Robinson [7] solved this problem. He evaluated the number of solutions inequivalent up to automorphism by building on Lucas' results.

Let e be identity, c be the rotation by π, q and q' be the rotation by $\pm\pi/2$, d and d' be the reflections about the main diagonals, m and m' be the reflections about the horizontal and vertical midlines. Let E_n, C_n, Q_n, M_n, and S_n be the number of n×n solutions left invariant by e, c, q, m, and both m and m', respectively. Then Q_n is also the number left fixed by q', and M_n is the number left fixed by m'. For $n > 1$ none will be left fixed by d or d'.

Thus if we let r_n be the total number of inequivalent solution on the n×n board.

then $r_n = 1/8 E_n + 1/8 C_n + 1/4 Q_n + 1/4 M_n$.

He found both exact and asymptotic (as $n \to \infty$) solutions. The following table [Table 3.2.1] shows the numbers for $1 \le n \le 17$

n	B (n)
2	1
3	2
4	7
5	23
6	115
7	694
8	5.282
9	46.066
10	456.454
11	4.999.004
12	59.916.028
13	778.525.516
14	10.897.964.660
15	163.461.964.024
16	2.615.361.578.344

Table 3.2.1

3.3. The Problem of Knights

3.3.1. Introduction

From the problems associated with the other chess pieces, one might expect attention to be focused on the problem of board domination by knights, namely the problem of finding arrangements of knights on a chessboard such that every square of the chessboard is either occupied or attacked by a knight.

How would we determine the minimum number of knights needed to dominate an n×n chessboard? How many solutions are there?

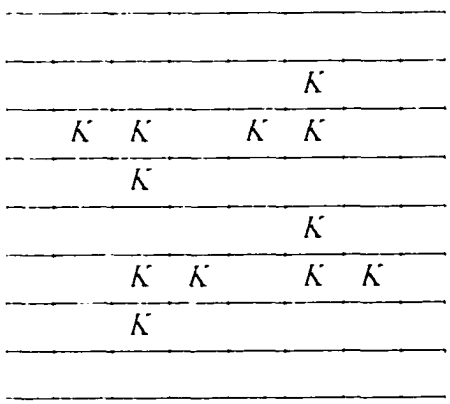


Figure 3.3.1

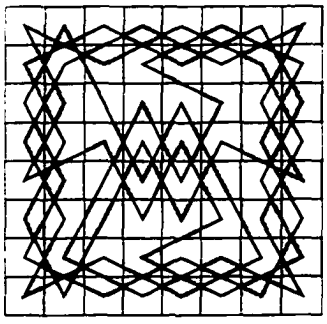
Figure 3.3.1 gives an example solution for an ordinary board. In this case, a minimum of 9 knights are needed to cover the 8×8 ordinary chessboard. Despite these potentially interesting questions,

however, most attention is focused on the more interesting problem of knight tours, in which the question is whether a knight, by repeated movement, can reach every square on the board exactly once.

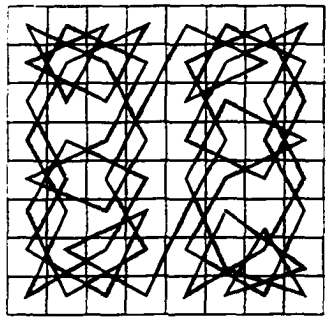
3.3.2. The Knight's Tour Problem

A solution to the knight's tour problem is a tour of the chessboard: a path of knight movements that visits each square exactly once. When do solutions exist and how many are there?

A solution is said to be closed if the knight can be brought back to his initial position with one final move. Otherwise the solution is open. Such routes are simply Hamiltonian cycles and Hamiltonian paths in the graph of order n^2 corresponding to all possible knight moves on the chessboard. Figure 3.3.2 gives two examples of a closed knight's tour on 8×8 regular chessboard. And Figure 3.3.3 is an example of an open knight's tour on a 3×4 board.



a



b

Figure 3.3.2

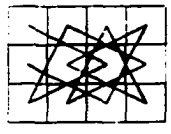


Figure 3.3.3

Knight's tours have long been studied and the literature includes such famous contributors as Euler. An entertaining and widely available discussion of some of the classical methods and solutions was given by Rouse Ball [**23**]. Many alterations

of the size and shape of the board have already been considered. One may also form a chessboard of unusual shape, such as n×m rectangular boards, infinite boards or the board shaped as in Figure 3.3.4. Schwenk [**22**] determined all rectangular chessboards on which a knight's tour exists. Eggleton [**24**] studied the rectilinear chessboards and compound chessboards, made from suitable combinations of rectilinear boards. It is shown that any finite compound chessboard comprising two or more 4×4 sub-boards has a knight's tour.

The Knight's Tour Puzzle: *Is it possible for a knight to have a closed tour on the chessboard?* This problem dates back to the time of Euler. Wilson and Watkins [**19**, p. 145] report that the question was fully resolved by Euler in 1759 and 12 years later, independently, by Vandermonde. In 1859 William R. Hamilton marketed a puzzle called A Voyage Round the World based on the concept of the knight's tour. Accounts of Hamilton's puzzle can be found in *Graphs, an Introductory Approach* [**19**].

The Knight's Tour Puzzle can be represented by a graph G, where the vertices v correspond to the squares S of the chessboard, and v is adjacent to v in G if and only if it is possible for a knight to proceed from S to S in a single move. Since the color of the square cells on which the knight lands changes at each move. It follows that if a board of a particular shape has one more cell of one color than of the other, a solution (if it exists) must be open. But if the difference in the numbers of cells of the two colors is greater than one, then no solution is possible. For

example, it is not possible to visit all the cells of the board in Figure 3.3.4, which contains 32 black and 25 white cells. If the cells are not colored it is often more difficult to tell whether a tour is possible.

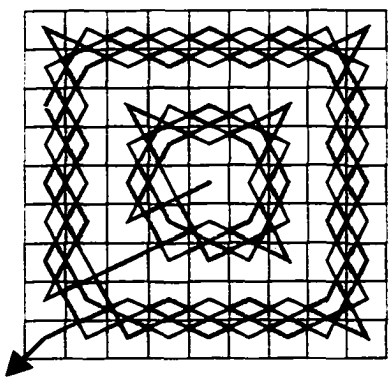


Figure 3.3.8

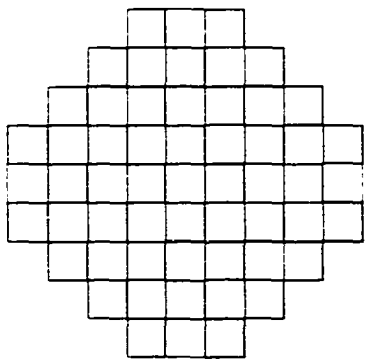


Figure 3.3.4

3.3.3. Knight's Cycle and Tour

Definition 3.3.1. *A hamiltonian tour of a knight on an m×n board is a sequence of successive knight moves in which the knight visits every cell exactly once. A hamiltonian cycle is a closed hamiltonian tour.*

Theorem 3.3.1. *If G is a hamiltonian graph and S is a proper subset of $V(G)$, then $k(G-S) \leq |S|$.*

Proof. Since G is hamiltonian, G contains a hamiltonian cycle C. Suppose that $k(G-S)=n$, where $G_1, G_2, \ldots, G_n$ are the components of $G-S$. Let v_i, where $1 \le i \le n$, denote the last vertex of C that belongs to G_i, and let v_{i+1} be the vertex that immediately follows v_i on C. Necessarily, $v_{i+1} \in S$ for each i $(1 \le i \le n)$, and $v_j \ne v_k$ for $j \ne k$. Thus, there are at least as many vertices in S as there are components in $G-S$. That is, $k(G-S) \le |S|$.

Theorem 3.3.2. *There are no Hamiltonian tours or cycles for $1 \times n$, $2 \times n$, $m \times 1$ and $m \times 2$ boards.*

Proof. Consider cell $(1, 1)$. Since it does not have two edges to be used in the cycle, it does not have a hamiltonian cycle.

Theorem 3.3.3. *An $m \times n$ chessboard has no Hamiltonian cycle if m and n are both odd numbers. But, it may have Hamiltonian tour.*

Proof. If we start at cell (i, j), we can complete the move on one of eight cells: $(i \pm 2, j \pm 1)$ or $(i \pm 1, j \pm 2)$. Cell (i, j) and all cells that are one knight's move away have different colors. So, the graph is a bipartite graph. Every cycle of a bigraph must be even, but if n and m are both odd, then so is mn. Therefore, no hamiltonian cycle exists.

Theorem 3.3.4. *There are no hamiltonian cycles for $4\times n$ and $n\times 4$ boards, but these boards may have a Hamiltonian tour.*

Proof. Let's consider a $4\times n$ board. (The $n\times 4$ case is similar.) Assume that there is a Hamiltonian cycle $v_1v_2\ldots v_{4n}v_1$. We recolor the vertices in row 1 red, row 2 blue, row 3 blue and row 4 red (see Figure 3.3.5). Since there are n red and n blue vertices, and every

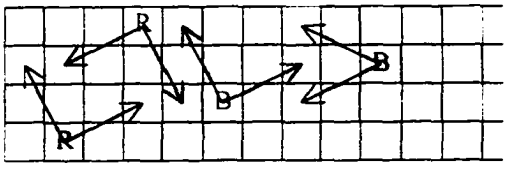


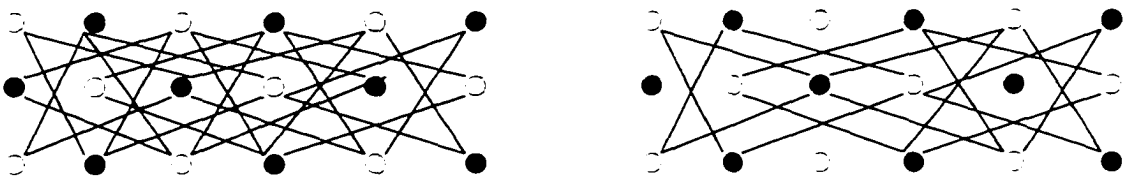
Figure 3.3.5

red vertex is adjacent only to blue vertices (for example, vertices $(1, 2)$ and $(4, 4)$), red and blue vertices must alternate around the cycle. If we start from a black cell $(1, 2)$, then adjunct vertices should be blue, and these blue vertices should land on white cells. Thus, all blues are whites. This contradicts the two different colorings. The assumption that there is a Hamiltonian cycle must be false.

Therefore, there is no Hamiltonian cycle. This elegant proof is due to Louis Pósa [**20**]. Figure 3.3.3 is a tour on the 3×4 board.

Schwenk defined $G(m, n)$ as a graph on $m\cdot n$ vertices by replacing each cell of the board with a vertex and joining two vertices by an edge when they are one knight's move apart (see Figure 3.3.6 a). He proved the following two theorems.

Theorem 3.3.5. *There are no hamiltonian cycles for $G(3, 6)$ and $G(6, 3)$.*



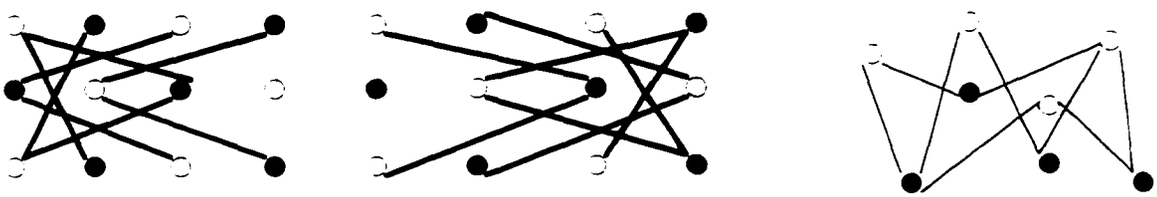
a b

Figure 3.3.6

Proof. For $G(3, 6)$. If we remove the set of vertices $S = \{(1, 3), (3, 3)\}$ from G, that will leave three components (see Figure 3.3.6 a), namely $(1, 2)$, $(5, 2)$ and the rest of $G - S$. By Theorem 3.3.1, we conclude that there is no hamiltonian cycle.

Theorem 3.3.6. *There is no hamiltonian cycle for $G(3, 8)$ and $G(8, 3)$.*

Proof. Assume that there is a Hamiltonian cycle C. C has to contain the 16 edges shown below (Figure 3.3.7 a). Because vertices $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 2)$, $(3, 2)$, $(6, 2)$, $(7, 2)$, $(8, 2)$, $(8, 1)$ and $(8, 3)$ have degree 2, they have to be included in C. We divide those 16 into 6 parts. Each connected path is treated as a



a(graph G) b(graph G')

Figure 3.3.7

vertex in a new graph G' along with the two additional vertices $(4, 2)$ and $(5, 2)$. We join each vertex in G' when there is an edge in G connecting one end of a path to one end of another path (see Figure 3.3.7 b). If G has a Hamiltonian cycle, then G' has a Hamiltonian cycle. But, G' contains two degree 3 vertices, which, if removed, will leave three components. Therefore, G' has no Hamiltonian cycle and therefore, G has no Hamiltonian cycle either.

Schwenk excluded all board sizes that do not have Hamiltonian cycles and developed a constructive way to generate solutions for the remaining boards. For details on how to generate a solution, see his paper [**21**].

3.3.4. Generalizations

Roger Eggleton and Abdul Eid [23] investigated the knight's tour on infinite chessboards. They defined a rectilinear chessboard R as a collection of cells $\{(x, y) : \alpha < x < \beta, \gamma < y < \delta\}$, where $\alpha, \beta, \gamma, \delta$ are members of the extended integers $Z \cup \{-\infty, \infty\}$, chosen so that R is nonempty. In particular, rectangular chessboards are rectilinear chessboards in which $\alpha, \beta, \gamma, \delta$ are all integers. If exactly one of $\alpha, \beta, \gamma, \delta$ is infinite, R is a one-way infinite strip chessboard. If exactly one of the sets $\{\alpha, \beta\}$, $\{\gamma, \delta\}$ is infinite, R is a quarter lattice chessboard. If exactly one of $\alpha, \beta, \gamma, \delta$ is an integer, R is a half lattice chessboard. If all of $\alpha, \beta, \gamma, \delta$ are infinite, R is the full lattice chessboard. Eggleton and Eid also defined a hamiltonian one-way chain (one-way knight's tour) in an infinite graph G as a route $v_0 v_1 v_2 \ldots v_{i-1} v_i \ldots$ in the graph such that the vertices v_{i-1} and v_i are adjacent for each positive integer i and every vertex of G appears in the route precisely once.

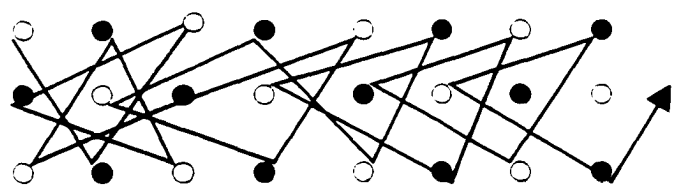


Figure 3.3.8

A hamiltonian two-way chain (two-way knight's tour) in G is a route $\ldots v_i v_j v_k \ldots v_r v \ldots$ such that the vertices v_r and v are adjacent for each integer i and every vertex of G appears in the route precisely once. Figure 3.3.8 is one-way and Figure 3.3.9 is two-way.

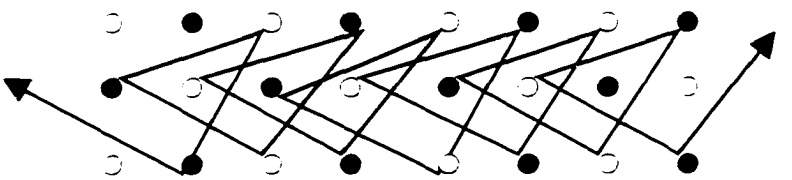


Figure 3.3.9

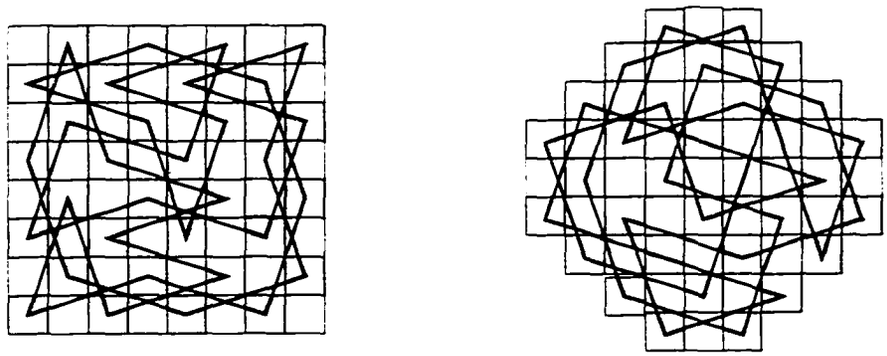
They consider the analogue of a knight's tour to be a one-way knight's tour, while the analogue of a knight's cycle is a two-way knight's tour. It is trivial to derive a knight's tour from a knight's cycle, but there is no such relation between two-way and one-way knight's tours.

The fact is: for an infinite generalized, the existence of one-way knight's tours is independent of the existence of two-way knight's tours.

One can find the proof in [**23**].

Not only have the shape and size of the board been generalized, but also the movement of the knight has been generalized. Instead of $(i \pm 1, j \pm 2)$ or $(i \pm 2, j \pm 1)$, the knight uses moves $(i \pm 1, j \pm 3)$ or $(i \pm 3, j \pm 1)$. Figure 3.3.10 a

is one such tour on an ordinary board, and Figure 3.3.10 b is such a tour on a board on which no tour can be made by an ordinary knight.



a b

Figure 3.3.10

The problems and solutions developed in section 3.3 suggest many questions: we have been able to give serious consideration to only a few of them.

If we replace knights by other chess pieces, another family of questions results.

If a knight is allowed to tour on higher dimensional chessboards, another new direction results.

As Kraitchik said: "Combinatorics owes many interesting problems to the game of chess. Indeed the game itself is a single enormously complicated combinatorial problem that has never been – and probably never will be –completely solved."

4. Future Research Topics

In chapter two, I primarily studied cases in which the number of blocked cells is one, two, or three. What happens when the number of blocked cells get large is still unknown. I would like to keep doing my research on this subject.

We define a solution to three dimensional partial n-cube is a square arrangement of the elements from the set $S = \{1, 2, \ldots, n\}$ and empty squares such that no element $i \in S$ may appear twice in the same row, column, or diagonal unless there are cells blocked in that row, column or diagonal, and the elements $i, j \in S$ may not be placed $|i - j|$ squares apart from each other along any row, column or diagonal, unless there are blocked squares in that row, column or diagonal.

There are remaining topics that I would like to keep doing research on, such as:

1) Which cells removed will effect the number of queens that can be place on the n-cube?
2) Does there are exist an algorithm to generate a solution for a partial n-cube queen problem?
3) What is the maximum number of queens that can be placed on a certain n-cube with some cells removed?
4) Does there exist an algorithm for the maximal placement of the queens into partial n-cube?

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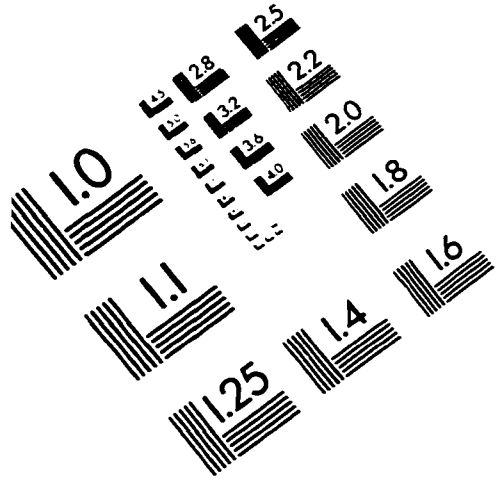
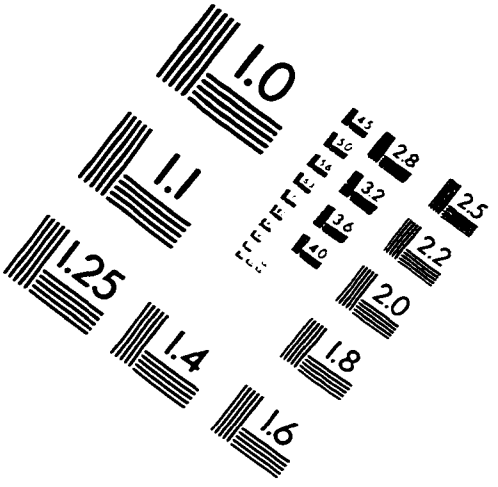
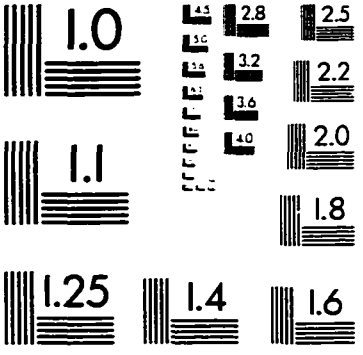
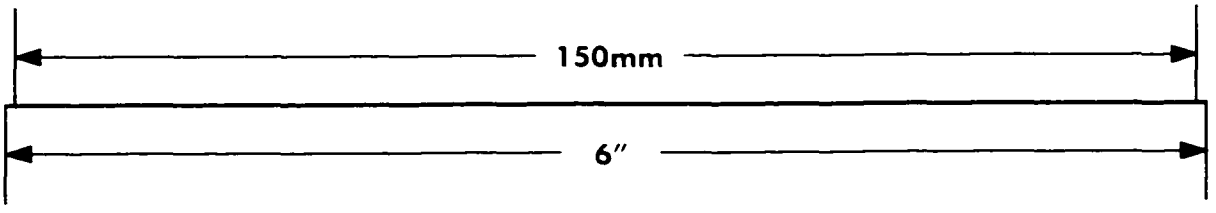
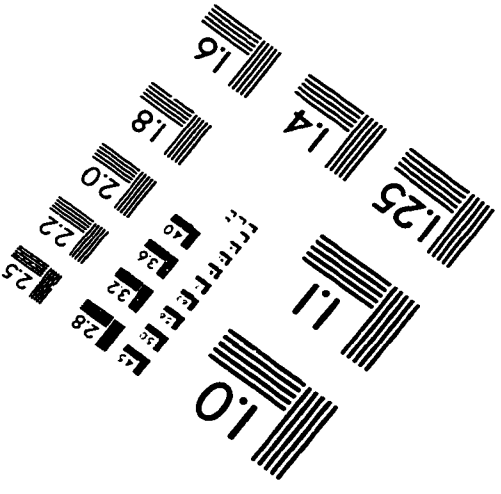
IMAGE EVALUATION
TEST TARGET (QA-3)











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