ESSAYS ON THRESHOLD AUTOREGRESSIVE MODELING OF BOND TIME SERIES

by

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Abstract

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The linear Gaussian models such as AR models, ARMA models and ARIMA models have been proved by many previous research that they are not ideally suited for modeling some financial time series that exhibiting asymmetry, limit cycles and jump phenomena, in other words, nonlinearity. Researchers therefore resort to nonlinear models to interpret financial time series. The threshold autoregressive model (TAR) introduced by Tong and Lim (1980), among the family of nonlinear models, has been found to best capture asymmetries, limit cycles and jump phenomena in the dynamic structure of economic and financial time series. Nonlinearity in stock prices and exchange rates has been often detected by various statistical tests. However, only few attempts have been made to subsequently model the nonlinearity explicitly. My dissertation investigates nonlinearity in bond yields and prices, and model the nonlinearity explicitly. The thesis consists of two essays.

In the first essay, I explore the presence of nonlinearities for the daily series of 10-year Japanese government bond (JGB) yields by using the Tsay (1989) test, I find the threshold nonlinearity due to a significant change in Japanese debt management policy. I test to find the sixth lag of the series as the threshold variable, then locate the threshold, and estimate a 2-regime self-exciting threshold autoregressive model (SETAR) for this time series.

In the second essay, I investigate the presence of nonlinearities for the daily series of 10-year American Treasury note (T-note) prices by using the Tsay (1989) test, I analyze the threshold nonlinearity due to government intervention and the price protection pursued by investors. I test to find the first lag of the series as the threshold variable, then I estimate two SETAR models for this time series based on different lag lengths and compare these two models.

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Essay I

Threshold Autoregressive Modeling of Bond Series ---- Japanese Case

1 Introduction

The linear Gaussian models such as AR models, ARMA models and ARIMA models have been proved by many previous research that they are not ideally suited for modeling some financial time series that exhibiting asymmetry, limit cycles and jump phenomena, in other words, nonlinearity.

Econometricians and statisticians therefore resort to nonlinear models to interpret financial time series. The threshold autoregressive model (TAR) introduced by Tong and Lim (1980), among the family of nonlinear models, has been found to best capture asymmetries, limit cycles and jump phenomena in the dynamic structure of economic and financial time series.

Besides, TAR models are pretty popular in the nonlinear time-series literature, and they are relatively simple to specify, estimate, and interpret comparing with many other nonlinear time-series models.

Nonlinearity in stock prices and exchange rates has been often detected by various statistical tests. (Hinich and Patterson 1985; Scheinkmann and LeBaron 1989; Hsieh 1989, 1991; Crato and de Lima 1994; Brooks 1996). However, only few attempts have been made to subsequently model the nonlinearity explicitly.

In this paper, I explore the threshold nonlinearity for the daily series of 10-year JGB yields due to the extraordinary change in Japanese debt management policy and estimate a 2-regime SETAR model for the time series.

I find strong evidence for a TAR model using the sixth lag of the series as the threshold variable, and estimate the threshold. Finally, I conclude that the autoregressive structure of JGB yields changes once during the studied time period.

The remainder of the paper is organized as follows. The next section introduces TAR models. Section 3 describes the characteristics of the daily series of 10-year JGB yields. In section 4, I apply a SETAR model to the series. I first introduce both the theory and application of the Tsay (1989) test of nonlinearity against linear autoregressive (AR) models hypothesis, then I estimate a 2-regime SETAR model, I also do model checking by calculating the autocorrelation function (ACF) and the partial autocorrelation function (PACF) of standardized residuals of the model. The final section contains a brief conclusion.

2 The TAR Models

The TAR models use threshold space to improve linear approximation. The basic idea of threshold models is the introduction of regimes via thresholds, in other words, the local approximation over states. If I name this idea as the threshold principle, I may group a number of finite parametric nonlinear time series models under the threshold principle. The principle allows the analysis of a complex stochastic system by decomposing it into simpler subsystems. A TAR model is a generalization of an AR model which permits for different regimes for the series depending on its past values.

TAR models have been successfully applied to model nonlinearities in financial variables such as exchange rates, volatility of return and arbitrage trading. For example, a TAR model of exchange rates explains an inner regime of sluggish adjustment for small disequilibria — or small deviations from some long run equilibrium path or attractor and an outer regime of mean reversion comprising large deviations.

The important application of TAR models in volatility is to handle the asymmetric responses in volatility between positive and negative returns.

TAR models can also be used to study arbitrage trading in index futures and cash prices.

Besides financial variables, TAR models have also been used successfully to explore asymmetries in macroeconomic variables such as unemployment, GNP, etc., over the course of the business cycle. In this respect there is a question of whether the apparent persistence in an economic time series such as GNP or unemployment provides evidence of asymmetries that standard Gaussian linear parameter models cannot accommodate (Neftci 1984).

A TAR model has several characteristics: first, geometrically ergodic and stationary. Second, the series exhibits an asymmetric increasing and decreasing pattern (Tsay 2002).

3 The Characteristics of the Daily Series of 10-year JGB Yields

Some characteristics of JGB yields time series include:

(a) They tend to move cyclically with Japanese business cycles.

As I can observe, when economy is growing rapidly, investors dump bonds and invest in stocks, so bonds prices go down, therefore bonds yields go up; when economy is in down turn, the demand for bonds is high, this pushes bonds prices up, so bonds yields go down.

(b) The rates rise slowly, but decay quickly.

In the JGB market, there is one bond issue that is designated as the liquid bond at any given time. The liquid bond is usually called the benchmark bond. The benchmark bond is chosen from 10-year government bonds. Moreover, the Ministry of Finance and the Bank of Japan have a strong enough negotiating position to impose terms on the underwriting syndicate of government bonds.

When the economy is over-heated, the benchmark yield is raised gradually by the action that Japanese debt management authority undertake in order to give the economy a soft landing. While when economy needs to be heated up, the benchmark yield is pressed down relatively quickly serving as a stimulating tool.

The latter characteristic suggests that the dynamic structure of the 10-year bond yields series exhibits an asymmetric increasing and decreasing pattern, in other words, nonlinearity.

Besides, by checking the time plot of the bond yields series (figure 1), I find that the conditional mean of the bond yields changes over time. This is an evidence of regime change.

Furthermore, the histogram of the series (figure 2) confirms the asymmetry of the distribution of the series.

Last, by comparing the standard deviation, skewness and kurtosis of the JGB yields series with those of the standard normal distribution (table 1), I conclude that standard Gaussian linear parameter models cannot fully explain the behavior of the JGB yields series.

4 SETAR Modeling of the 10-year JGB Yields Series

A time series Y_t is a SETAR process if it follows the model

$$\mathrm{Y}_t = \beta_0^{(j)} + \textstyle\sum_{i=1}^{p} \beta_i^{(j)} Y_{t-i} + \varepsilon_t^{(j)}, \tag{1}$$

$$\text{if } \gamma_{j-1} \leq \mathrm{Y}_{t-d} < \gamma_j$$

Where

$$j = 1, \ldots, k$$

and d are positive integers. The thresholds are

$$-\infty = \gamma_0 < \gamma_1 < \ldots < \gamma_k = \infty$$

The superscript j is used to signify the regime, $\varepsilon_t^{(j)}$ are i.i.d. sequences with mean 0 and variance σ_j^2 and are mutually independent for different j. The parameter d is referred to as the delay parameter or the threshold lag. The SETAR model is nonlinear provided that $k > 1$.

Such a process partitions the one-dimensional Euclidean space into K regimes and follows a linear AR model in each regime. When there are at least two regimes with different linear models, the overall process Y_t is nonlinear.

4.1 A Test for Threshold Nonlinearity

In order to apply a threshold model to the bond yields time series, it is necessary to assess the need for a threshold model for the data.

4.1.1 Tsay's (1989) test

In this section, I test threshold nonlinearity of the JGB yields time series.

In order to test linearity against nonlinearity of switching type, I can apply many techniques developed for testing parameter constancy against structural change. For example, a linear model whose parameters change once at a given point of time is piecewise linear in the same way as a switching regression model with two regimes. The available observation vectors (assuming i.i.d. errors or a martingale difference error process) may be rearranged in the ascending or descending order according to the threshold variable or the switching variable. If this is done, linearity tests may be obtained using ideas previously applied to detecting structural change. (Granger and Tersvirta 1993)

The proposed test is a combined version of the nonlinearity tests of Keenan (1985), Tsay (1986), and Petruccelli and Davies (1986). It is simple and widely applicable. Its asymptotic distribution under the linear model assumption is nothing but the usual F distribution (Tsay 1989).

There is a general agreement on the nonexistence of a global optimal test, since the number and locations of the thresholds are unknown before the estimation. However, according to Tsay (1989), the test that he proposed is more powerful and simpler than other tests available in the literature such as the portmanteau test proposed by Petruccelli and Davies (1986), etc. The Tsay (1989) test also avoids the problem of nuisance parameters encountered by the likelihood ratio test.

Tsay (1989) makes use of arranged autoregression and recursive estimation to derive a test for threshold nonlinearity. The null hypothesis is that the model is a linear AR process. While the alternative hypothesis is that the model exhibits threshold nonlinearity. The arranged autoregression seeks to transfer the SETAR model into a model change problem with the thresholds serving as the change points.

4.1.2 The Arranged Autoregression

This section discusses the arranged autoregression concept introduced by Tsay(1989) which facilitates efficient estimation of TAR models.

Under the null hypothesis of linearity, residuals of a properly specified linear model should be independent. Any violation of independence in the residuals indicates inadequacy of the entertained model, including the linearity assumption. The idea behind the Tsay (1989) test is that under the null hypothesis there is no model change in the arranged autoregression so that the standardized predictive residuals should be close to i.i.d. with mean zero and variance 1. In this case, they should also have no correlation with the regressors (Tsay 2002).

Let me show the arranged autoregression by matrix. I denote the threshold variable as ν_{t-d}.

$$(Y|\nu) = (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p}|\nu_{t-d}) =$$

$$\begin{pmatrix} Y_{11} & Y_{12} & \ldots & Y_{1p} & | & \nu_1 \\ Y_{21} & Y_{22} & \ldots & Y_{2p} & | & \nu_2 \\ \vdots & \vdots & \vdots & \vdots & | & \vdots \\ Y_{n1} & Y_{n2} & \ldots & Y_{np} & | & \nu_n \end{pmatrix}$$

Rearrange the rows of Y, let $(Y|\nu)$ follow the order of the last column of $(Y|\nu)$. This yields:

$$(Y^\nu|\nu) = (Y^\nu_{t-1}, Y^\nu_{t-2}, \ldots, Y^\nu_{t-p}|\nu_{(i)}) =$$

$$\begin{pmatrix} Y^\nu_{11} & Y^\nu_{12} & \ldots & Y^\nu_{1p} \\ Y^\nu_{21} & Y^\nu_{22} & \ldots & Y^\nu_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ Y^\nu_{n1} & Y^\nu_{n2} & \ldots & Y^\nu_{np} \end{pmatrix}$$

where ν_i denotes the ith smallest observation of ν_{t-d} and the superscript ν denotes ordering according to ν_{t-d}. A crucial property of this arranged form is that by reordering rows or cases of the initial matrix-form setup, it preserves the dynamics of Y_t. Furthermore, the ordinary regression residuals of the arranged autoregression are not correlated over time.

4.1.3 The Test

There are two steps to do the test. The first step is to run an arranged autoregression. Secondly, use the predictive residuals to calculate the associated F statistic. If the F statistic is larger than the critical F value, I reject the null of linearity.

For the first step, an important practical matter is the question of how to appropriately choose the order of autoregressive approximation, p. A rough rule is to take p in the range of 4 to 8, although even bigger p values seem to do as well. (Keenan 1985)

(1) Running an Arranged Autoregression

Write an AR (p) regression with n observations as

$$\mathrm{Y}_t = (1, Y_{t-1}, \ldots, Y_{t-p})\alpha + \varepsilon_t \tag{2}$$

for

$$t = p+1, \ldots, n,$$

where α is the (p+1)-dimensional vector of coefficients and ε_t is the noise. While $(Y_t, 1, Y_{t-1}, \ldots, Y_{t-p})$ is a case of data for the AR(p) model. Then, an arranged autoregression is an autoregression with case rearranged, based on the values of a particular regressor (in my test, I arranged cases based on the values of $Y_{t-1}, \ldots, Y_{t-p}$, one regressor one time). For example, if I sort the series by Y_{t-1}, I place the smallest Y_{t-1} first and the largest Y_{t-1} last. This gives an arranged autoregression,

$$\mathrm{Y}_t^* = (1, Y_{t-1}^*, \ldots, Y_{t-p}^*)\beta + \gamma_t \tag{3}$$

If there are nonlinearities of the TAR type, then the β-vector associated with the small and large values of Y^*_{t-1} should be different from that associated with medium sized Y^*_{t-1}. This hypothesis can be checked by testing the arranged autoregression for structural breaks.

For a self-exciting model, arranged autoregression becomes useful if it is arranged according to the threshold variable. But the problem is that I do not know the threshold variable before I do the test. So, I assume that $Y_{t-1}, \ldots, Y_{t-p}$ are all candidates of threshold variable, then run arranged autoregression based on each threshold variable candidate, finally pick the candidate that exhibits the most significant nonlinearity as the threshold variable.

After I find the threshold variable, I consider a simple case, $k = 2$, i.e., the TAR model has two regimes, separated by a threshold γ_1, and for each regime, follows an AR model of different p. Note that the separation does not require knowing the precise value of γ_1. Only the number of observations in each group depends on γ_1.

If the thresholds γ_1 were known, then consistent estimates of the parameters could easily be obtained. But note that the actual value of γ_1 is not required in order to perform Tsay (1989) test; all that is needed is the existence of a nontrivial threshold. While according to Tsay (1989) test, if I get a test result of rejecting the null of linearity, I proof the existence of a nontrivial threshold.

(2) Performing the F test

The F test does not require knowing the thresholds, it simply tests that the predictive residuals have no correlations with regressors if the null of linearity holds.

Therefore I first pick out the recursive predictive residuals of arranged autoregression,$\hat{\gamma}_t$, then perform the following regression

$$\hat{\gamma}_t = (1, Y^*_{t-1}, \ldots, Y^*_{t-p})\omega + \epsilon_t \tag{4}$$

and test if $\omega \neq 0$ with a F test. And denote the F statistic F_1. To increase the power of the test, sort the cases in reversed order, i.e., the largest regressor first and the small last, then repeat the entire procedure with the reversed series and denote the corresponding F statistic F_2. Pick the larger F statistic of the two and use it as the test statistic. If it is larger than the critical value, I reject the null of linearity.

Because I do not know which lag of Y_t is the threshold variable, I assume that $Y_{t-1}, \ldots, Y_{t-d}$ (where $d = 1, 2, 3, \ldots, 8$ for the JGB yields series) are all candidates of threshold variable, then do the F test based on each threshold variable candidate, finally pick the candidate that exhibits the most significant nonlinearity as the threshold variable.

4.2 The Estimation method

Several statistical assumptions are the foundation of ordinary regression analysis. One key assumption is that the ordinary regression residuals are independent of each other. However, the regression errors of time series usually are correlated over time. Since the assumptions of which the classical linear regression model are based will usually be violated, it is not proper to use ordinary regression analysis for time series data. However, the ordinary regression residuals of the arranged autoregression do not violate the key assumption.

Therefore for the arranged regression, least squares estimates and maximum likelihood estimates are the same. Since the least squares estimation is much easier to apply than the maximum likelihood estimation, Tsay (1989) used least squares estimates. I also choose to use least square estimates.

4.3 The data sets

The data sets are the daily close yield series of 10-year JGB, from November 17, 2002 to July 18, 2004, in total 610 observations (source: Reuters). The data are in percentages.

The reason why I choose 10-year maturity is because original issue with 10-year maturities account for the majority of new issues of JGBs. Furthermore, according to the research by Boudoukh and Whitelaw (1991), trading in 10-year issues constitutes over 99% of the total trading in Japanese government issues that also include other issues such as 20-year issues, short maturity bonds and municipal bonds.

Another reason why I choose to study 10-year maturity bond is that the 10-year JGB is usually designated as the liquid bond called benchmark bond. I believe that the result on the benchmark bond might have significant application on studying other bonds.

4.4 The Test Results

The test results are summarized in table 2. Based on the test results and comparing with the critical F value, I observe that Y_{t-6} series shows the most significant nonlinearity. So I conclude that Y_{t-6} is the threshold variable.

I run five arranged AR models such as AR(4), AR(5), AR(6), AR(7) and AR(8). For each AR process, I perform Tsay (1989) test for the variable Y_{t-d}, where d is the threshold lag. In this case, $d = 1, 2, 3, \ldots, 8$. From the test result, I observe that, for AR(6) process, Y_{t-6} has the largest F value, i.e., 3.80, with the most significant p value of 0.1%, comparing with other lags of Y_t such as Y_{t-1}, Y_{t-2}, Y_{t-3}, Y_{t-4} and Y_{t-5}. For AR(7) process, again, Y_{t-6} has the largest F value of 3.11 with the most significant p value that is 0.2%, among other lags: Y_{t-1}, Y_{t-2}, Y_{t-3}, Y_{t-4}, Y_{t-5}, Y_{t-7}. Last, for AR(8) process, Y_{t-6} also has the largest F value that is 3.01 with the most significant p value of 0.2%, comparing with Y_{t-1}, Y_{t-2}, Y_{t-3}, Y_{t-4}, Y_{t-5}, Y_{t-7}, Y_{t-8}.

4.5 Finding the Best Lag Length

In order to find the best lag length, I can calculate the Akaike Information Criterion (AIC) or the Schwartz Bayesian Criterion (SBC) from each equation and then to exam the output of the model with the smallest AIC and/or SBC.

It is also common to determine a lag length based on the outcome of t-tests. This methodology picks the lag length such that the t-statistic for the last lag is significant at some pre-specified level.

I estimate twelve AR models such as AR(1), AR(2), AR(3),..., AR(12), then I calculate the AICs, SBCs and t-tests for these 12 models. The results are shown in table 3.

The AIC selects the model with 6 lags while the SBC selects the model with 2 lags. In this case, the choice is unclear since t-statistic on lag 6 has a prob-value that is even greater than 10%.

I decide to select the model with 6 lags, because generally speaking, the longer the lag, the better the fitting of the AR model.

4.6 Locating the Thresholds and Estimating the SETAR Model

Sometimes economic theory is helpful in choosing a particular model, but more often it is not. Choosing models involve subjective judgment. While for nonlinear time series modeling, there are still some general guidelines to follow besides subjective judgment. Nonlinear time series modeling starts with building an adequate linear model on which nonlinearity tests are based. If nonlinearity is statistically significant, then one chooses a class of nonlinear models to entertain. The selection here may depend on the experience of the analyst and the substantive matter of the problem under study. For TAR models, one may use the procedures given in Chan (1993), Tong (1983, 1990) and Tsay (1989, 1998) to build an adequate model.

Tong (1983) developed a two-regime version of the SETAR model as follow:

$$Y_t = I_t[\alpha_0 + \textstyle\sum_{i=1}^{p} \alpha_i y_{t-i}] + (1 - I_t)[\beta_0 + \textstyle\sum_{i=1}^{p} \beta_i y_{t-i}] + \varepsilon_t \tag{5}$$

Where: Y_t is the series of interest, the α_i and β_i are coefficients to be estimated, τ is the value of the threshold, p is the order of the SETAR model and I_t is the Heaviside indicator function:

$$I_t = \{^{1\ if\ Y_{t-1} \geq \tau}_{0\ if\ Y_{t-1} < \tau} \tag{6}$$

Where Y_{t-1} is the threshold variable, τ is the threshold.

Of course, for different time series, Y_{t-1} is not necessarily always the threshold variable. The threshold variable could be any lag of Y_t such as

Y_{t-1}, Y_{t-2}, Y_{t-3}, etc.

The nature of the system is that there are two states of the world. In one state of the world, Y_{t-1} exceeds the value of the threshold τ so that $I_t = 1$ and $(1 - I_t) = 0$. As such, Y_t follows the autoregressive process: $\alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}$. Similarly, in the other state, Y_t falls short of the threshold τ, so that $I_t = 0$, $(1 - I_t) = 1$ and Y_t follows the autoregressive process: $\beta_0 + \sum_{i=1}^{p} \beta_i y_{t-i}$. It seems that there are two attractors or potential equilibrium values. In the 'high' state, the system is drawn toward $\alpha_0/(1 - \sum \alpha_i)$; in the 'low' state, the system is drawn toward $\beta_0/(1 - \sum \beta_i)$. Moreover, the degree of autoregressive decay will differ across the two states if for any value of i, $\alpha_i \neq \beta_i$. The key feature of the SETAR model is that a sufficiently large shock denoted by ε_t can cause the system to switch between states.

When threshold τ is unknown, how to locate it? Chan (1993) shows how to obtain a super-consistent estimate of the threshold parameter. For a SETAR model, the procedure is to order the observations from smallest to largest such that:

$$Y^1 < Y^2 < Y^3 ... < Y^T \tag{7}$$

For each value of Y^j, let $\tau = Y^j$, set the Heaviside indicator according to this potential threshold and estimate a SETAR model. Among the group of different SETAR models based on each value of Y^j, the regression equation with the smallest residual sum of squares contains the consistent estimate of the threshold. In practice, the highest and lowest 15% of the $\{Y^j\}$ values are excluded from the grid search so as to ensure an adequate number of observations on each side of the threshold.

I do the grid search by running a loop program, and I take the associated value of threshold test and use it as a potential threshold. Then I estimate a SETAR model, and calculate the residual sum of squares associated with each regression. Finally, I plot a scatter diagram of residual sums of squares against potential thresholds, that is shown in figure 3.

By checking the plot of residual sums of squares, I find that the threshold 0.920 results in the lowest residual sum of squares. And the plot clearly shows that there is only one threshold. Therefore, it implies the existence of two states for the 10-year JGB yields series.

With the threshold $\tau = 0.920$, I estimate a SETAR model for the daily series of 10-year JGB yields from Nov 17, 2002 to July 18, 2004. The coefficients and their statistics are reported in table 4.

Because there are 2 regimes for the AR structure of the 10-year JGB yields series during the studied period, I estimate 2 different AR models for these two regimes. The AR orders are 6 for regime 1, 6 for regime 2; the number of observations are 226 in regime 1, 203 in regime 2. Details of the SETAR model are:

$$regime\ 1:$$

$$\mathrm{Y}_t = -0.003 + 1.025Y_{t-1} - 0.089Y_{t-2} - 0.132Y_{t-3}$$
$$+ 0.209Y_{t-4} + 0.042Y_{t-5} - 0.053Y_{t-6}$$

$$When$$

$$\mathrm{Y}_{t-6} \geq 0.920$$

$$regime\ 2:$$

$$\mathrm{Y}_t = 0.001 + 1.122Y_{t-1} + 0.139Y_{t-2} + 0.233Y_{t-3}$$
$$- 0.466Y_{t-4} - 0.437Y_{t-5} + 0.411Y_{t-6}$$

$$When$$

$$\mathrm{Y}_{t-6} < 0.920$$

4.7 Checking the Model

In table 5, I show the 2 regimes and the ACF and PACF of the standardized residuals of the 2-regime SETAR model.

In model checking, the ACF and PACF of the standardized residuals of the model all fail to suggest any model inadequacy, because the ACF and PACF plots (figure 4) show little evidence that the regression residuals of the 2-regime SETAR model are correlated over time. Moreover, based on my calculating the mean and variance of the residuals, I conclude that the residuals of the model are nearly i.i.d. and normally distributed.

Based on the model, I can observe that there are two states of the world for the 10-year JGB yields series. In one state of the world, Y_{t-6} exceeds the value of the threshold 0.920. As such the 10-year JGB yield follows the autoregressive process:

$$\mathrm{Y}_t = -0.003 + 1.025Y_{t-1} - 0.089Y_{t-2} - 0.132Y_{t-3}$$
$$+ 0.209Y_{t-4} + 0.042Y_{t-5} - 0.053Y_{t-6}$$

After calculate $\alpha_0/(1-\sum\alpha_i)$, I find that the yields are drawn toward 1.5. Similarly, in the other state of the world, Y_{t-6} falls short of the threshold 0.920. As such the 10-year JGB yield follows the autoregressive process:

$$\mathrm{Y}_t = 0.001 + 1.122Y_{t-1} + 0.139Y_{t-2} + 0.233Y_{t-3} \\ - 0.466Y_{t-4} - 0.437Y_{t-5} + 0.411Y_{t-6}$$

and the system is drawn toward -0.5 based on the calculation of $\beta_0/(1-\sum\beta_i)$. Moreover, the degree of the autoregressive decay differs across the two states, since the coefficients of the AR process in one state are different with those of the AR process in the other state. A sufficiently large shock can cause the system to switch between the two states.

In this case, the large shock is the significant change in Japanese debt management policy during the end of 2002 to the beginning of 2003. I will come back to this issue in detail in the next section.

5 Conclusions

It is well known that in Japan, a bond issue is authorized by the Ministry of Finance but implemented by the Bank of Japan. Both the Ministry of Finance and Bank of Japan constitute Japanese debt management authority. They have a strong enough negotiating position to impose terms on the underwriting syndicate of government bonds. (Boudoukh and Whitelaw 1991)

The 2-regime SETAR model of the daily series of 10-year JGB yields indicates that the AR structure of the series changes once during the studied period.

One major reason why the AR structure changes is because the shock of interventions by Japanese debt management authority around the end of 2002 to the beginning of 2003.

During December 2002 to March 2003, there was an extraordinary move in Japanese debt management policy due to an expected redemption rush of mainly 10-year bonds in fiscal year 2008: a new government bond issuance plan was enacted; there was also a launch of various JGB-related measures stipulated in the settlement system reform laws; a buy-back program was introduced, too. All in one, in order to accommodate the funds needed for the redemption rush, the bond yield was raised.

As I discuss in the previous section, the rates rise slowly, but decay quickly. The threshold variable Y_{t-6} indicates that the yield raising is done slowly, taking about a week, in order words, the intervention of Japanese debt management authority on the bond yields is reflected in reactions from the agents about a week later. The reactions include the market digesting the ripple of intervention and adjusting investment portfolio allocations.

I also can conclude that Japanese bond market is not perfectly efficient. The market does not respond to news instantly. Adjustment costs do exist.

Further research could be done on refining upon SETAR models and integrating two key properties of financial time series — nonlinear and long-memory properties.

Figure 1: The Time Plot of the Daily Series of 10-year JGB Yields (November 17, 2002 - July 18, 2004)

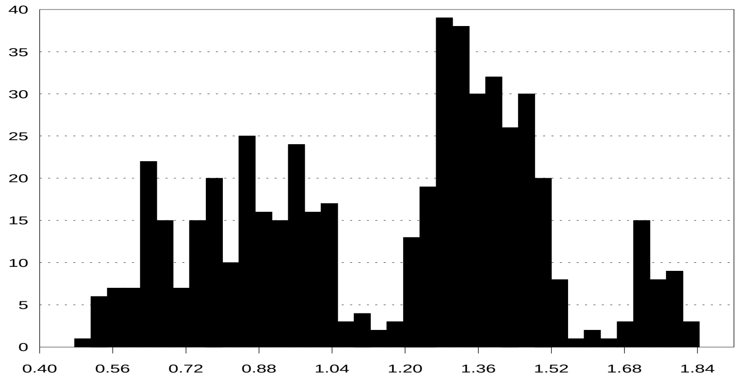

Figure 2: The Histogram of the 10-year JGB Yields Series (November 17, 2002 - July 18, 2004)

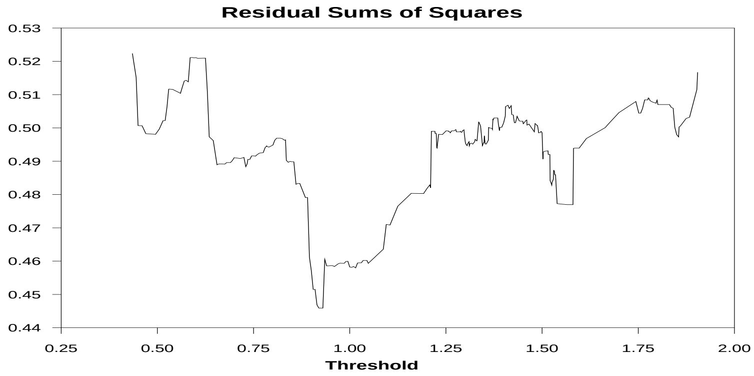

Figure 3: The Plot of Residual Sums of Squares of SETAR Models of the JGB Yields Series against Potential Thresholds (November 17, 2002 - July 18, 2004)

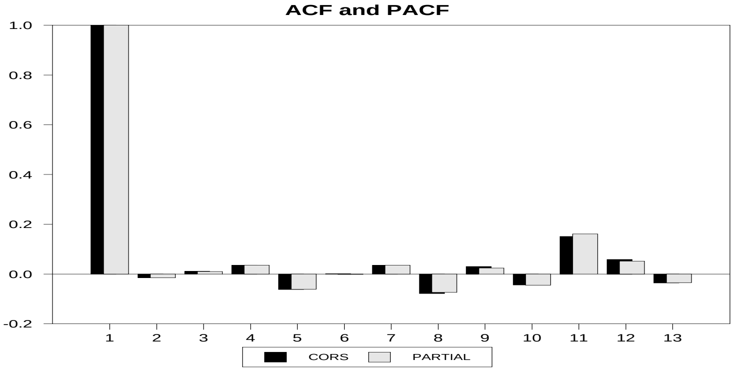

Figure 4: The Autocorrelation Function (ACF) and the Partial Autocorrelation Function (PACF) Plots of Standardized Residuals of a 2-Regime SETAR Model for the JGB Yields Series (November 17, 2002 - July 18, 2004)

	The Distribution of the JGB Yields Series	The Standard Normal Distribution
Standard Deviation	0.366	1.000
Skewness	-0.132	0.000
Kurtosis	-0.978	3.000

Table 1. Comparison of the Distribution of the JGB Yields Series with the Standard Normal Distribution

Note: The skewness is a measure of the asymmetry of the distribution of a series. The negative skewness implies that the left tail of the distribution is fatter than the right tail, or that large negative yields tend to occur more often than large positive ones. The kurtosis measures the fatness of tails of the distribution of a series.

			Threshold	Lags, d				
F Statistics	1	2	3	4	5	6	7	8
			AR(4)					
$F_{5,333}$	2.41	2.42	2.51	3.06				
p value	0.036	0.036	0.030	0.010				
			AR(5)					
$F_{6,315}$	2.30	2.44	2.97	2.74	2.81			
p value	0.034	0.026	0.021	0.013	0.011			
			AR(6)					
$F_{7,301}$	2.70	2.97	3.02	3.04	3.29	3.80		
p value	0.010	0.005	0.004	0.002	0.001	0.001		
			AR(7)					
$F_{8,287}$	2.37	2.78	2.71	2.53	2.92	3.11	2.93	
p value	0.018	0.006	0.007	0.011	0.004	0.002	0.004	
			AR(8)					
$F_{9,273}$	2.29	2.70	2.63	2.38	2.92	3.01	2.77	2.62
p value	0.017	0.005	0.006	0.013	0.003	0.002	0.004	0.006

Table 2. Tsay (1989) Test Results of the JGB Yields Series

Note: $F_{a,b}$ denotes the proposed F statistic with a and b degrees of freedom. The AR orders used are those commonly employed in the literature.

Lags	AIC	SBC	t
1	-170.660	-162.763	178.936
2	-177.658	-165.950	-1.961
3	-172.206	-156.774	0.061
4	-168.054	-148.983	0.048
5	-182.525	-159.915	-0.895
6	-187.063	-160.930	0.066
7	-185.243	-155.666	-1.555
8	-184.491	-151.556	1.368
9	-177.569	-141.365	-0.141
10	-173.026	-133.649	0.330
11	-175.991	-133.543	-2.571
12	-170.837	-125.427	-0.650

Table 3. The Akaike Information Criterion (AIC), the Schwartz Bayesian Criterion (SBC) and t-test for 12 AR Models

Note: These 12 AR models are AR(1), AR(2),..., AR(12).

Independent Variables	Coefficients	
α_0	-0.003	(0.016)
Y_{t-1}	1.025^{**}	(0.068)
Y_{t-2}	-0.089	(0.098)
Y_{t-3}	-0.132	(0.096)
Y_{t-4}	0.209^{*}	(0.097)
Y_{t-5}	0.042	(0.097)
Y_{t-6}	-0.053	(0.063)
β_0	0.001	(0.020)
Y_{t-1}	1.122^{**}	(0.099)
Y_{t-2}	0.139	(0.149)
Y_{t-3}	0.233	(0.147)
Y_{t-4}	-0.466^{**}	(0.149)
Y_{t-5}	-0.437^{**}	(0.161)
Y_{t-6}	0.411^{**}	(0.119)

Table 4. The Estimation Results of a 2-Regime SETAR Model

Note: The number in the parenthesis is standard error,$'*'$ indicates 5% level of significance,$'**'$ indicates 1% level of significance. The Centered R^2 of the SETAR model is 0.990395, the $\bar{R}^2$ of the model is 0.989980 and the Uncentered R^2 is 0.999061.

				Lags				
Regimes	0	1	2	3	4	5	6	7
				AR	Coefficients			
1	-0.003	1.025	-0.089	-0.132	0.209	0.042	-0.053	
2	0.001	1.122	0.139	0.233	-0.466	-0.437	0.411	
				ACF of	Standardized	Residuals		
		-0.015	0.010	0.035	-0.062	0.001	0.035	-0.079
				PACF of	Standardized	Residuals		
		-0.015	0.010	0.036	-0.061	-0.001	0.035	-0.074

Table 5. The Autocorrelation Function (ACF) and the Partial Autocorrelation Function (PACF) of the Standardized Residuals of the 2-Regime SETAR Model

Appendix

1. A stationary linear time series always has a unique and stable equilibrium which is equal to its mean. A nonlinear time series can have a single (stable or unstable) equilibrium, multiple equilibria or no equilibrium at all. Furthermore, even if the equilibrium is unique and stable, it is not necessarily equal to the mean of the time series. (Franses and Dijk, 2000)

2. limit cycle: A k-period limit cycle is defined as a set of k points. If the time series started in one of the points and no shocks occurred, the series would cycle among the k points. For high-order SETAR models, it can be quite difficult to establish the existence of equilibria, attractors and/or limit cycles analytically. A pragmatic way to investigate the properties of the skeleton of a high-order SETAR model is to use what might be called deterministic simulation.

3. Tong (1990, p.379) defines an alternative AIC for a 2-regime SETAR model as the sum of the AICs for the AR models in the two regimes, that is,

$$p_1, p_2 = n_1 ln\hat{\sigma}_1^2 + n_2 ln\hat{\sigma}_2^2 + 2(p_1+1) + 2(p_2+1) \tag{8}$$

where n_j, $j = 1, 2$, is the number of observations in the regime, and $\hat{\sigma}_j^2$, $j = 1, 2$, is the variance of the residuals in the regime. The SBC for a 2-regime SETAR model can be defined analogously as

$$p_1, p_2 = n_1 ln\hat{\sigma}_1^2 + n_2 ln\hat{\sigma}_2^2 + (p_1+1) ln \mathrm{n}_1 + (\mathrm{p}_2+1) ln \mathrm{n}_2 \tag{9}$$

The selected lag orders in the two regimes are those for which the information criterion is minimized.

4. The new government bond issuance plan: in designing the new government bond issuance plan, Japanese debt management authority decided to increase super-long-term issues in response to market trends and needs, while maintaining an appropriate balance among different maturity zones — short-term, medium-term, long-term and super-long-term. Consequently, the average maturity of JGBs to be issued in the market in this plan will be extended to 5 years and 8 months — 2 months longer than 5 years and 6 months which is the initial budget base for fiscal year 2002.

5. The launch of various JGB-related measures stipulated in the settlement system reform laws: in January 2003, Japan started to implement various JGB-related measures stipulated in the settlement system reform Laws, which was enacted June 2002. A new settlement system for government bonds was put into operation on January 27 2003, prior to the corporate bonds, as the first step to make settlement of government and corporate bonds paperless — a scheme provided for by the Laws. As a result, all government bonds issued thereafter, will be fully paperless and managed only by the records kept in the transfer accounts at financial institutions. This should increase transaction speed and efficiency for JGBs, thus improving the government bond market even further.

6. The buy-back program: to implement the proactive government debt management policies not only at the issuance stage, but also for outstanding JGBs, a buy-back program was introduced, too. While the program can be used for various purposes, for the time being, Japan debt management authority use it to level the amount of redemption of mainly 10-year bonds, which are expected to reach a redemption rush in fiscal year 2008.

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Essay II

Threshold Autoregressive Modeling of Bond Series ---- American Case

1 Introduction

The linear Gaussian models such as AR models, ARMA models and ARIMA models have been proved by many previous research that they are not ideally suited for modeling some financial time series that exhibiting asymmetry, limit cycles and jump phenomena, in other words, nonlinearity.

Econometricians and statisticians therefore resort to nonlinear models to interpret financial time series. The threshold autoregressive model (TAR) introduced by Tong and Lim (1980), among the family of nonlinear models, has been found to best capture asymmetries, limit cycles and jump phenomena in the dynamic structure of economic and financial time series.

Besides, TAR models are pretty popular in the nonlinear time-series literature, and they are relatively simple to specify, estimate, and interpret comparing with many other nonlinear time-series models.

Nonlinearity in stock prices and exchange rates has been often detected by various statistical tests. (Hinich and Patterson 1985; Scheinkmann and LeBaron 1989; Hsieh 1989, 1991; Crato and de Lima 1994; Brooks 1996). However, only few attempts have been made to subsequently model the nonlinearity explicitly.

While considerable work has concentrated on threshold nonlinearity in exchange rates, GNP, little study has been done on threshold modeling of bond prices.

In this paper, I explore the threshold nonlinearity for the daily series of 10-year US T-note prices due to government intervention and the price protection pursued by investors, and estimate two 2-regime SETAR models for the time series based on different lag lengths and compare these two models.

I find strong evidence for a TAR model using the first lag of the series as the threshold variable, and estimate the threshold. Finally, I conclude that the autoregressive structure of 10-year US T-note prices changes once during the studied time period.

The remainder of the paper is organized as follows. The next section introduces TAR models. Section 3 describes the characteristics of the daily series of 10-year US T-note prices. In section 4, I apply a SETAR model to the series. I first introduce both the theory and application of the Tsay (1989) test of nonlinearity against linear AR models hypothesis, then I estimate two 2-regime SETAR models based on different lag lengths and compare these two models, I also do model checking by calculating the autocorrelation function (ACF) and the partial autocorrelation function (PACF) of standardized residuals of the better SETAR model. The final section contains a brief conclusion.

2 The TAR Models

The TAR models use threshold space to improve linear approximation. The basic idea of threshold models is the introduction of regimes via thresholds, in other words, the local approximation over states. If I name this idea as the threshold principle, I may group a number of finite parametric nonlinear time series models under the threshold principle. The principle allows the analysis of a complex stochastic system by decomposing it into simpler subsystems. A TAR model is a generalization of an AR model which permits for different regimes for the series depending on its past values.

TAR models have been successfully applied to model nonlinearities in financial variables such as exchange rates, volatility of return and arbitrage trading. For example, a TAR model of exchange rates explains an inner regime of sluggish adjustment for small disequilibria — or small deviations from some long run equilibrium path or attractor and an outer regime of mean reversion comprising large deviations.

The important application of TAR models in volatility is to handle the asymmetric responses in volatility between positive and negative returns.

TAR models can also be used to study arbitrage trading in index futures and cash prices.

Besides financial variables, TAR models have also been used successfully to explore asymmetries in macroeconomic variables such as unemployment, GNP, etc., over the course of the business cycle. In this respect there is a question of whether the apparent persistence in an economic time series such as GNP or unemployment provides evidence of asymmetries that standard Gaussian linear parameter models cannot accommodate (Neftci 1984).

A TAR model has several characteristics: first, geometrically ergodic and stationary. Second, the series exhibits an asymmetric increasing and decreasing pattern (Tsay 2002).

3 The Characteristics of the Daily Series of 10-year US T-note Prices

Some characteristics of 10-year US T-note prices time series include:

(a) They tend to move counter - cyclically with US business cycles.

As I can observe, when economy is growing rapidly, investors dump bonds and invest in stocks, so the note prices go down; when economy is in down turn, the demand for bonds is high, this pushes the note prices up.

(b) The distribution of the T-note prices series exhibits nonlinearity.

First by checking the time plot of the T-note prices series (figure 1), I find that the conditional mean of the T-note prices changes over time. This is an evidence of regime change.

Secondly, the histogram of the series (figure 2) confirms the asymmetry of the distribution of the series. As I observe from the graph, the right tail of the distribution is fatter than the left tail.

Last, by comparing the standard deviation, skewness and kurtosis of the T-note prices series with those of the standard normal distribution (table 1), I conclude that standard Gaussian linear parameter models cannot fully explain the behavior of 10-year US T-note prices series.

4 SETAR Modeling of the 10-year US T-note Prices Series

A time series Y_t is a SETAR process if it follows the model

$$\mathrm{Y}_t = \beta_0^{(j)} + \textstyle\sum_{i=1}^{p} \beta_i^{(j)} \mathrm{Y}_{t-i} + \varepsilon_t^{(j)}, \tag{1}$$

$$\text{if } \gamma_{j-1} \leq \mathrm{Y}_{t-d} < \gamma_j$$

Where

$$j = 1, \ldots, k$$

and d are positive integers. The thresholds are

$$-\infty = \gamma_0 < \gamma_1 < \ldots < \gamma_k = \infty$$

The superscript j is used to signify the regime, $\varepsilon_t^{(j)}$ are i.i.d. sequences with mean 0 and variance σ_j^2 and are mutually independent for different

j. The parameter d is referred to as the delay parameter or the threshold lag. The SETAR model is nonlinear provided that $k > 1$.

Such a process partitions the one-dimensional Euclidean space into K regimes and follows a linear AR model in each regime. When there are at least two regimes with different linear models, the overall process Y_t is nonlinear.

4.1 A Test for Threshold Nonlinearity

In order to apply a threshold model to 10-year US T-note prices time series, it is necessary to assess the need for a threshold model for the data.

4.1.1 Tsay's (1989) test

In this section, I test threshold nonlinearity of the T-note prices time series.

In order to test linearity against nonlinearity of switching type, I can apply many techniques developed for testing parameter constancy against structural change. For example, a linear model whose parameters change once at a given point of time is piecewise linear in the same way as a switching regression model with two regimes. The available observation vectors (assuming i.i.d. errors or a martingale difference error process) may be rearranged in the ascending or descending order according to the threshold variable or the switching variable. If this is done, linearity tests may be obtained using ideas previously applied to detecting structural change. (Granger and Tersvirta 1993)

The proposed test is a combined version of the nonlinearity tests of Keenan (1985), Tsay (1986), and Petruccelli and Davies (1986). It is simple and widely applicable. Its asymptotic distribution under the linear model assumption is nothing but the usual F distribution (Tsay 1989).

There is a general agreement on the nonexistence of a global optimal test, since the number and locations of the thresholds are unknown before the estimation. However, according to Tsay (1989), the test that he proposed is more powerful and simpler than other tests available in the literature such as the portmanteau test proposed by Petruccelli and Davies (1986), etc. The Tsay (1989) test also avoids the problem of nuisance parameters encountered by the likelihood ratio test.

Tsay (1989) makes use of arranged autoregression and recursive estimation to derive a test for threshold nonlinearity. The null hypothesis is that the model is a linear AR process. While the alternative hypothesis is that the model exhibits threshold nonlinearity. The arranged autoregression seeks to transfer the SETAR model into a model change problem with the thresholds serving as the change points.

4.1.2 The Arranged Autoregression

This section discusses the arranged autoregression concept introduced by Tsay(1989) which facilitates efficient estimation of TAR models.

Under the null hypothesis of linearity, residuals of a properly specified linear model should be independent. Any violation of independence in the residuals indicates inadequacy of the entertained model, including the linearity assumption. The idea behind the Tsay (1989) test is that under the null hypothesis there is no model change in the arranged autoregression so that the standardized predictive residuals should be close to i.i.d. with mean zero and variance 1. In this case, they should also have no correlation with the regressors (Tsay 2002).

Let me show the arranged autoregression by matrix. I denote the threshold variable as ν_{t-d}.

$$(Y|\nu) = (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p}|\nu_{t-d}) =$$

$$\begin{pmatrix} Y_{11} & Y_{12} & \ldots & Y_{1p} & | & \nu_1 \\ Y_{21} & Y_{22} & \ldots & Y_{2p} & | & \nu_2 \\ \vdots & \vdots & \vdots & \vdots & | & \vdots \\ Y_{n1} & Y_{n2} & \ldots & Y_{np} & | & \nu_n \end{pmatrix}$$

Rearrange the rows of Y, let $(Y|\nu)$ follow the order of the last column of $(Y|\nu)$. This yields:

$$(Y^{\nu}|\nu) = (Y^{\nu}_{t-1}, Y^{\nu}_{t-2}, \ldots, Y^{\nu}_{t-p}|\nu_{(i)}) =$$

$$\begin{pmatrix} Y^{\nu}_{11} & Y^{\nu}_{12} & \ldots & Y^{\nu}_{1p} \\ Y^{\nu}_{21} & Y^{\nu}_{22} & \ldots & Y^{\nu}_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ Y^{\nu}_{n1} & Y^{\nu}_{n2} & \ldots & Y^{\nu}_{np} \end{pmatrix}$$

where ν_i denotes the ith smallest observation of ν_{t-d} and the superscript ν denotes ordering according to ν_{t-d}. A crucial property of this arranged form is that by reordering rows or cases of the initial matrix-form setup, it preserves the dynamics of Y_t. Furthermore, the ordinary regression residuals of the arranged autoregression are not correlated over time.

4.1.3 The Test

There are two steps to do the test. The first step is to run an arranged autoregression. Secondly, use the predictive residuals to calculate the associated F statistic. If the F statistic is larger than the critical F value, I reject the null of linearity.

For the first step, an important practical matter is the question of how to appropriately choose the order of autoregressive approximation, p. A rough rule is to take p in the range of 4 to 8, although even bigger p values seem to do as well. (Keenan 1985)

(1) Running an Arranged Autoregression

Write an AR (p) regression with n observations as

$$\mathrm{Y}_t = (1, Y_{t-1}, \ldots, Y_{t-p})\alpha + \varepsilon_t \tag{2}$$

for

$$t = p + 1, \ldots, n,$$

where α is the $(p+1)$-dimensional vector of coefficients and ε_t is the noise. While $(Y_t, 1, Y_{t-1}, \ldots, Y_{t-p})$ is a case of data for the AR(p) model. Then, an arranged autoregression is an autoregression with case rearranged, based on the values of a particular regressor (in my test, I arranged cases based on the values of $Y_{t-1}, \ldots, Y_{t-p}$, one regressor one time). For example, if I sort the series by Y_{t-1}, I place the smallest Y_{t-1} first and the largest Y_{t-1} last. This gives an arranged autoregression,

$$\mathrm{Y}^*_t = (1, Y^*_{t-1}, \ldots, Y^*_{t-p})\beta + \gamma_t \tag{3}$$

If there are nonlinearities of the TAR type, then the β-vector associated with the small and large values of Y^*_{t-1} should be different from that associated with medium sized Y^*_{t-1}. This hypothesis can be checked by testing the arranged autoregression for structural breaks.

For a self-exciting model, arranged autoregression becomes useful if it is arranged according to the threshold variable. But the problem is that I do not know the threshold variable before I do the test. So, I assume that $Y_{t-1}, \dots, Y_{t-p}$ are all candidates of threshold variable, then run arranged autoregression based on each threshold variable candidate, finally pick the candidate that exhibits the most significant nonlinearity as the threshold variable.

After I find the threshold variable, I consider a simple case, $k = 2$, i.e., the TAR model has two regimes, separated by a threshold γ_1, and for each regime, follows an AR model of different p. Note that the separation does not require knowing the precise value of γ_1. Only the number of observations in each group depends on γ_1.

If the thresholds γ_1 were known, then consistent estimates of the parameters could easily be obtained. But note that the actual value of γ_1 is not required in order to perform Tsay (1989) test; all that is needed is the existence of a nontrivial threshold. While according to Tsay (1989) test, if I get a test result of rejecting the null of linearity, I proof the existence of a nontrivial threshold.

(2) Performing the F test

The F test does not require knowing the thresholds, it simply tests that the predictive residuals have no correlations with regressors if the null of linearity holds.

Therefore I first pick out the recursive predictive residuals of arranged autoregression,$\hat{\gamma}_t$, then perform the following regression

$$\hat{\gamma}_t = (1, Y^*_{t-1}, \ldots, Y^*_{t-p})\omega + \epsilon_t \tag{4}$$

and test if $\omega \neq 0$ with a F test. And denote the F statistic F_1. To increase the power of the test, sort the cases in reversed order, i.e., the largest regressor first and the small last, then repeat the entire procedure with the reversed series and denote the corresponding F statistic F_2. Pick the larger F statistic of the two and use it as the test statistic. If it is larger than the critical value, I reject the null of linearity.

Because I do not know which lag of Y_t is the threshold variable, I assume that $Y_{t-1}, \ldots, Y_{t-d}$ (where $d = 1, 2, 3, \ldots, 8$ for the US T-note prices series) are all candidates of threshold variable, then do the F test based on each threshold variable candidate, finally pick the candidate that exhibits the most significant nonlinearity as the threshold variable.

4.2 The Estimation method

Several statistical assumptions are the foundation of ordinary regression analysis. One key assumption is that the ordinary regression residuals are independent of each other. However, the regression errors of time series usually are correlated over time. Since the assumptions of which the classical linear regression model are based will usually be violated, it is not proper to use ordinary regression analysis for time series data. However, the ordinary regression residuals of the arranged autoregression do not violate the key assumption.

Therefore for the arranged regression, least squares estimates and maximum likelihood estimates are the same. Since the least squares estimation is much easier to apply than the maximum likelihood estimation, Tsay (1989) used least squares estimates. I also choose to use least square estimates.

4.3 The data sets

The data sets are the daily closing price series of 10-year US T-note, from July 4, 2000 to October 29, 2004, in total 1579 observations (source: Reuters). The data are quoted as a percent of the T-note's face value.

The reason why I choose 10-year maturity is because, 10-year T-note, replacing 30-year treasury bond, becomes the benchmark bond in US bond market, and it is the most liquid bond. I believe that the result on the benchmark bond might have significant application on studying other bonds.

4.4 The Test Results

The test results are summarized in table 2. Based on the test results and comparing with the critical F value, I observe that Y_{t-1} series shows the most significant nonlinearity. So I conclude that Y_{t-1} is the threshold variable.

I run five arranged AR models such as AR(4), AR(5), AR(6), AR(7) and AR(8). For each AR process, I perform Tsay (1989) test for the variable Y_{t-d}, where d is the threshold lag. In this case, $d = 1, 2, 3, \ldots, 8$. From the test results, I observe that, for AR(4) process, Y_{t-1} has the largest F value, i.e., 9.05, comparing with other lags of Y_t such as Y_{t-2}, Y_{t-3} and Y_{t-4}. For AR(5) process, again, Y_{t-1} has the largest F value of 7.617 , among other lags: Y_{t-2}, Y_{t-3}, Y_{t-4}, Y_{t-5}. For AR(6) process, again, Y_{t-1} has the largest F value of 7.736, comparing with other lags of Y_t such as Y_{t-2}, Y_{t-3}, Y_{t-4}, Y_{t-5} and Y_{t-6}. The same happens for AR(7) and AR(8). For AR(7) process, Y_{t-1} has the largest F value that is 5.602, comparing with Y_{t-2}, Y_{t-3}, Y_{t-4}, Y_{t-5}, Y_{t-6} and Y_{t-7}; for AR(8) process, Y_{t-1} also has the largest F value of 7.744, comparing with Y_{t-2}, Y_{t-3}, Y_{t-4}, Y_{t-5}, Y_{t-6} Y_{t-7} and Y_{t-8}.

4.5 Finding the Best Lag Length

In order to find the best lag length, I can calculate the Akaike Information Criterion (AIC) or the Schwartz Bayesian Criterion (SBC) from each equation and then to exam the output of the model with the smallest AIC and/or SBC.

It is also common to determine a lag length based on the outcome of t-tests. This methodology picks the lag length such that the t-statistic for the last lag is significant at some pre-specified level.

I first estimate twelve AR models such as AR(1), AR(2), AR(3),..., AR(12), then I calculate the AICs, SBCs and t-tests for these 12 models. The results are shown in table 3.

The AIC selects the model with 12 lags and the SBC also selects the model with 12 lags. Moreover, as the order of AR model p increases, the AIC and SBC decrease. Therefore, I also calculate the AICs, SBCs and t-tests for AR(13), AR(14), AR(15),..., AR(24). I observe again that as the order of AR model p increases, the AIC and SBC decrease. However, it is not practical to estimate a AR model with too big p value. In this case, I use the t-statistic to find the last lag that is significant at 5%. I find that the t-statistics on lag 1 and lag 5 both have a prob-value that is less than 5%.

Next I estimate the SETAR model based on the best lag lengths of 1 and 5, then I compare the residuals sums of squares of these two SETAR models, the model with the smaller residuals sum of squares is more favored than the other one.

4.6 Locating the Thresholds and Estimating the SETAR Model

Sometimes economic theory is helpful in choosing a particular model, but more often it is not. Choosing models involve subjective judgment. While for nonlinear time series modeling, there are still some general guidelines to follow besides subjective judgment. Nonlinear time series modeling starts with building an adequate linear model on which nonlinearity tests are based. If nonlinearity is statistically significant, then one chooses a class of nonlinear models to entertain. The selection here may depend on the experience of the analyst and the substantive matter of the problem under study. For TAR models, one may use the procedures given in Chan (1993), Tong (1983, 1990) and Tsay (1989, 1998) to build an adequate model.

Tong (1983) developed a two-regime version of the SETAR model as follow:

$$Y_t = I_t[\alpha_0 + \textstyle\sum_{i=1}^{p} \alpha_i y_{t-i}] + (1 - I_t)[\beta_0 + \textstyle\sum_{i=1}^{p} \beta_i y_{t-i}] + \varepsilon_t \tag{5}$$

Where: Y_t is the series of interest, the α_i and β_i are coefficients to be estimated, τ is the value of the threshold, p is the order of the SETAR model and I_t is the Heaviside indicator function:

$$I_t = \{ {}^{1 \ if \ Y_{t-1} \geq \tau}_{0 \ if \ Y_{t-1} < \tau} \tag{6}$$

Where Y_{t-1} is the threshold variable, τ is the threshold.

Of course, for different time series, Y_{t-1} is not necessarily always the threshold variable. The threshold variable could be any lag of Y_t such as Y_{t-1}, Y_{t-2}, Y_{t-3}, etc.

The nature of the system is that there are two states of the world. In one state of the world, Y_{t-1} exceeds the value of the threshold τ so that $I_t = 1$ and $(1 - I_t) = 0$. As such, Y_t follows the autoregressive process: $\alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}$. Similarly, in the other state, Y_t falls short of the threshold τ, so that $I_t = 0$, $(1 - I_t) = 1$ and Y_t follows the autoregressive process: $\beta_0 + \sum_{i=1}^{p} \beta_i y_{t-i}$. It seems that there are two attractors or potential equilibrium values. In the 'high' state, the system is drawn toward $\alpha_0/(1 - \sum \alpha_i)$; in the 'low' state, the system is drawn toward $\beta_0/(1 - \sum \beta_i)$. Moreover, the degree of autoregressive decay will differ across the two states if for any value of i, $\alpha_i \neq \beta_i$. The key feature of the SETAR model is that a sufficiently large shock denoted by ε_t can cause the system to switch between states.

When threshold τ is unknown, how to locate it? Chan (1993) shows how to obtain a super-consistent estimate of the threshold parameter. For a SETAR model, the procedure is to order the observations from smallest to largest such that:

$$Y^1 < Y^2 < Y^3 ... < Y^T \tag{7}$$

For each value of Y^j, let $\tau = Y^j$, set the Heaviside indicator according to this potential threshold and estimate a SETAR model. Among the group of different SETAR models based on each value of Y^j, the regression equation with the smallest residual sum of squares contains the consistent estimate of the threshold. In practice, the highest and lowest 15% of the $\{Y^j\}$ values are excluded from the grid search so as to ensure an adequate number of observations on each side of the threshold.

At first I estimate a SETAR model assuming that the best lag length is 1. I do the grid search by running a loop program, and I take the associated value of threshold test and use it as a potential threshold, then I estimate a SETAR model, and calculate the residual sum of squares associated with each regression. I plot a scatter diagram of residual sums of squares against potential thresholds, that is shown in figure 3.

By checking the plot of residual sums of squares, I find that the threshold 93.810 results in the lowest residual sum of squares. And the plot clearly shows that there is only one threshold. Therefore, it implies the existence of two states for the T-note prices series.

With the threshold $\tau = 93.810$, I estimate a SETAR model for the daily series of 10-year US T-note prices from July 4, 2000 to October 29, 2004. The coefficients and their statistics are reported in table 4.

Because there are 2 regimes for the AR structure of the T-note prices series during the studied period, I estimate 2 different AR models for these two regimes. The AR orders are 1 for regime 1; 1 for regime 2. Details of the SETAR model are:

$$regime\ 1:$$

$\mathrm{Y}_t = 3.357 + 0.967Y_{t-1}$

$$When$$

$\mathrm{Y}_{t-1} \geq 93.810$

$$regime\ 2:$$

$\mathrm{Y}_t = 2298.402 - 23.545Y_{t-1}$

$$When$$

$\mathrm{Y}_{t-1} < 93.810$

Then, I estimate a SETAR model under assumption of that the best lag length is 5. By repeating the same procedure as before, I find that the threshold 94.020 results in the lowest residual sum of squares, that is shown in figure 4. And the plot clearly shows that there is only one threshold. Therefore, it implies the existence of two states for the T-note prices series.

With the threshold $\tau = 94.020$, I estimate a SETAR model for the daily series of 10-year T-note prices from July 4, 2000 to October 29, 2004. The coefficients and their statistics are reported in table 5.

The AR orders are 5 for regime 1; 5 for regime 2. Details of the SETAR model are:

$$regime\ 1:$$

$\mathrm{Y}_t = 2.667 + 0.979Y_{t-1} - 0.016Y_{t-2} - 0.029Y_{t-3}$
$+ 0.105Y_{t-4} - 0.066Y_{t-5}$

$$When$$

$\mathrm{Y}_{t-6} \geq 94.020$

$$regime\ 2:$$

$$\mathrm{Y}_t = 812.205 - 5.768Y_{t-1} - 0.364Y_{t-2} + 0.128Y_{t-3}$$
$$+ 0.961Y_{t-4} - 2.573Y_{t-5}$$
$$When$$
$$\mathrm{Y}_{t-6} < 94.020$$

After I compare the residual sums of squares of these two SETAR models, I pick the second model since it has smaller residual sum of squares than the first model.

4.7 Checking the Second 2-Regime SETAR Model

In table 6, I show the 2 regimes and the ACF and PACF of the standardized residuals of the second 2-regime SETAR model.

In model checking, the ACF and PACF of the standardized residuals of the model all fail to suggest any model inadequacy, because the ACF and PACF plots (figure 5) show little evidence that the regression residuals of the 2-regime SETAR model are correlated over time. Furthermore, based on my calculating the mean and variance of the residuals, I conclude that the residuals of the model are nearly i.i.d. and normally distributed.

Based on the model, I can observe that there are two states of the world for the 10-year US T-note prices series. In one state of the world (the "high" state), Y_{t-1} exceeds the value of the threshold 94.020. As such the T-note prices follow the autoregressive process:

$$Y_t = 2.667 + 0.979Y_{t-1} - 0.016Y_{t-2} - 0.029Y_{t-3} + 0.105Y_{t-4} - 0.066Y_{t-5}$$

After calculate $\alpha_0/(1-\sum\alpha_i)$, I find that the prices are drawn toward 99.861. Similarly, in the other state of the world (the "low" state), Y_{t-1} falls short of the threshold 94.020. As such the T-note prices series follows the autoregressive process:

$$Y_t = 812.205 - 5.768Y_{t-1} - 0.364Y_{t-2} + 0.128Y_{t-3} + 0.961Y_{t-4} - 2.573Y_{t-5}$$

and the system is drawn toward 94.262 based on the calculation of $\beta_0/(1-\sum\beta_i)$. Moreover, the degree of the autoregressive decay differs across the two states, since the coefficients of the AR process in one state are different with those of the AR process in the other state. A sufficiently large shock can cause the system to switch between the two states.

In this case, the large shock is government intervention and the price protection pursued by investors during the studied period. I will come back to this issue in detail in the next section.

5 Conclusions

The 2-regime SETAR model of the daily series of 10-year US T-note prices indicates that the AR structure of the series changes once during the studied period. I believe that it is government intervention and the price protection pursued by investors that causes the change.

But how government intervention and the price protection cause the T-note prices to switch between the "high" and "low" states?

First let me explain the role that bonds play in investments. Bonds are the cornerstone of a well diversified portfolio. Bonds offer investors fixed-income payments, portfolio diversification and a hedge against an economic slowdown. As the largest securities market available, bonds offer a plethora of choices for investors seeking price protection.

On Halloween 2001, the US Treasury said it would no longer issue a 30-year bond. Without a 30-year bond, the average maturity of the debt held by foreigners has been falling to 54 months in 2004 from 60 months in 2003. Such short maturities together with a depreciating dollar make financial markets fearful that in the near term, foreign lenders such as Asian central banks may choose to invest in other currencies' assets when their dollar-based debt matures, forcing the United States to raise rates to attract the loans it needs. Eventually the raising rate pushes bonds' prices down. Thus, the T-note prices switch to the "low" state.

In the other hand, bonds investors seek price protection by adjusting their portfolio. This protection behavior forces the T-note prices switch to the "high" state.

I also conclude that even though US bond market is not perfectly efficient, it is much more efficient than Japanese bond market based on the fact that the threshold variable for the US T-note series is the first lag of Y_t while the threshold variable for the Japanese government bond series is the sixth lag of Y_t.

Further research could be done on integrating two key properties of financial time series — nonlinear and long-memory properties.

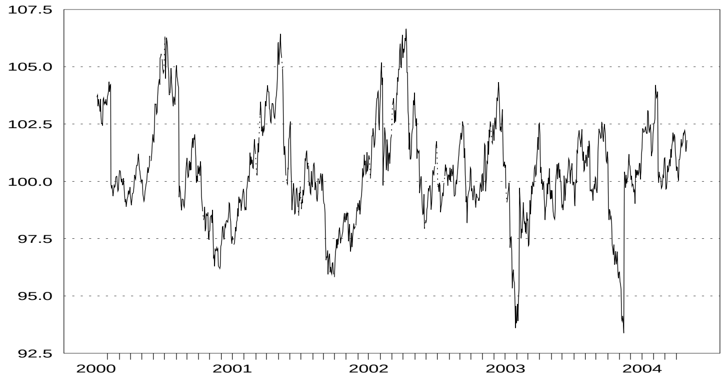

Figure 1: The Time Plot of the Daily Closing Prices of the 10-year US T-note (July 4, 2000 - October 29, 2004)

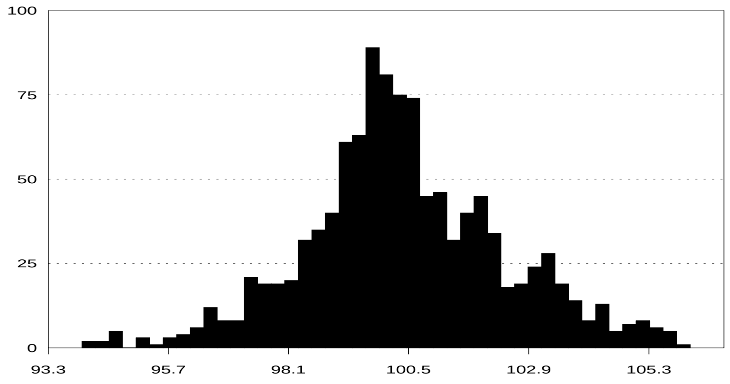

Figure 2: The Histogram of the 10-year US T-note Closing Prices Series (July 4, 2000 - October 29, 2004)

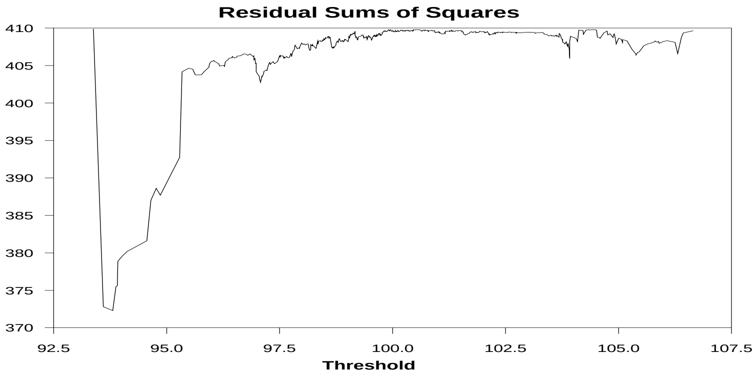

Figure 3: The Plot of Residual Sums of Squares of SETAR Models of the 10-year US T-note Closing Prices Series against Potential Thresholds Assuming the Best Lag Length is 1 (July 4, 2000 - October 29, 2004)

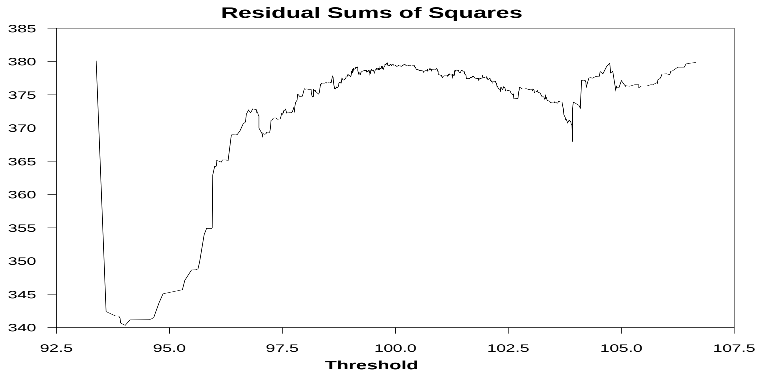

Figure 4: The Plot of Residual Sums of Squares of SETAR Models of the 10-year US T-note Closing Prices Series against Potential Thresholds Assuming the Best Lag Length is 5 (July 4, 2000 - October 29, 2004)

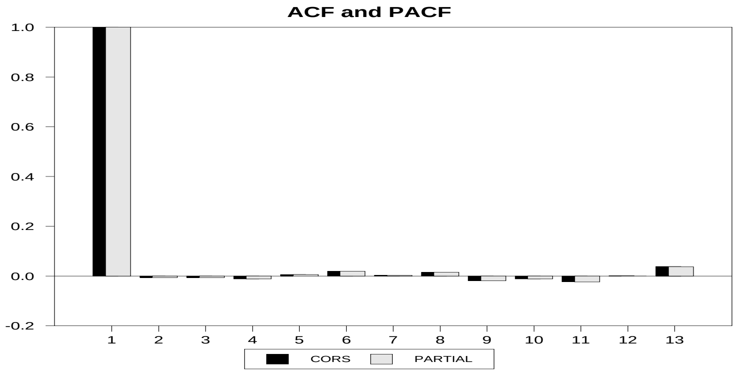

Figure 5: The Autocorrelation Function (ACF) and the Partial Autocorrelation Function (PACF) Plots of Standardized Residuals of the Second SETAR Model for the 10-year US T-note Closing Prices Series (July 4, 2000 - October 29, 2004)

	The Distribution of the US T-note Prices Series	The Standard Normal Distribution
Standard Deviation	2.187	1.000
Skewness	0.048	0.000
Kurtosis	0.508	3.000

Table 1. Comparison of the Distribution of the US T-note Prices Series with the Standard Normal Distribution

Note: The skewness is a measure of the asymmetry of the distribution of a series. The positive skewness implies that the right tail of the distribution is fatter than the left tail, or that large positive values tend to occur more often than large negative ones. The kurtosis measures the fatness of tails of the distribution of a series.

			Threshold	Lags, d				
F Statistics	1	2	3	4	5	6	7	8
			AR(4)					
$F_{5,970}$	9.050	4.280	5.870	3.640				
p value	0.119	0.001	0.000	0.003				
			AR(5)					
$F_{6,941}$	7.617	3.809	6.988	4.980	7.368			
p value	0.150	0.001	0.000	0.000	0.000			
			AR(6)					
$F_{7,912}$	7.736	3.550	5.954	4.898	6.140	2.649		
p value	0.000	0.001	0.000	0.000	0.000	0.010		
			AR(7)					
$F_{8,884}$	5.602	5.118	4.412	5.356	2.355	0.957	1.593	
p value	0.000	0.001	0.000	0.000	0.000	0.017	0.468	
			AR(8)					
$F_{9,857}$	7.744	4.466	6.068	5.326	6.401	3.004	1.479	2.028
p value	0.000	0.000	0.000	0.000	0.000	0.002	0.151	0.034

Table 2. Tsay (1989) Test Results of US T-note Prices Series

Note: $F_{a,b}$ denotes the proposed F statistic with a and b degrees of freedom. The AR orders used are those commonly employed in the literature.

Lags	AIC	SBC	t
1	6371.704	6381.634	111.115
2	6178.459	6193.270	-0.198
3	5983.356	6002.991	0.433
4	5794.688	5819.085	0.949
5	5629.665	5658.778	-2.007
6	5459.158	5492.928	0.335
7	5293.289	5331.663	0.114
8	5142.062	5184.988	-0.772
9	4984.003	5031.420	0.425
10	4837.259	4889.115	-0.528
11	4684.175	4740.406	0.802
12	4540.482	4601.019	0.142

Table 3. The AIC, the SBC and t-test for 12 AR Models

Note: These 12 AR models are AR(1), AR(2),..., AR(12).

Independent Variables	Coefficients	
α_0	3.357^{**}	(0.824)
Y_{t-1}	0.967^{**}	(0.008)
β_0	2298.402^{**}	(355.154)
Y_{t-1}	-23.545^{**}	(3.799)

Table 4. The Estimation Results of a SETAR Model Assuming the Best Lag Length is 1

Note: The number in the parenthesis is standard error, $'**'$ indicates 1% level of significance. The Centered R^2 of the SETAR model is 0.929267, the $\bar{R}^2$ of the model is 0.929068 and the Uncentered R^2 is 0.999966.

Independent Variables	Coefficients	
α_0	2.667**	(0.950)
Y_{t-1}	0.979**	(0.031)
Y_{t-2}	-0.016	(0.043)
Y_{t-3}	-0.029	(0.043)
Y_{t-4}	0.105*	(0.043)
Y_{t-5}	-0.066*	(0.031)
β_0	812.205**	(160.047)
Y_{t-1}	-5.768**	(1.544)
Y_{t-2}	-0.364	(0.798)
Y_{t-3}	0.128	(0.497)
Y_{t-4}	0.961	(0.767)
Y_{t-5}	-2.573**	(0.835)

Table 5. The Estimation Results of a SETAR Model Assuming the Best Lag Length is 5

Note: The number in the parenthesis is standard error,$'*'$ indicates 5% level of significance,$'**'$ indicates 1% level of significance. The Centered R^2 of the SETAR model is 0.927647, the $\bar{R}^2$ of the model is 0.926801 and the Uncentered R^2 is 0.999965.

				Lags			
Regimes	0	1	2	3	4	5	6
				AR	Coefficients		
1	2.667	0.979	-0.016	-0.029	0.105	-0.066	
2	812.205	-5.768	-0.364	0.128	0.961	-2.573	
				ACF of	Standardized	Residuals	
		-0.007	-0.006	-0.012	0.006	0.019	0.002
				PACF of	Standardized	Residuals	
		-0.007	-0.006	-0.012	0.006	0.019	0.003

Table 6. The ACF and the PACF of the Standardized Residuals of the Second SETAR Model

Appendix

1. The easiest way to understand bond prices is to add a zero to the price quoted in the market.

2. A stationary linear time series always has a unique and stable equilibrium which is equal to its mean. A nonlinear time series can have a single (stable or unstable) equilibrium, multiple equilibria or no equilibrium at all. Furthermore, even if the equilibrium is unique and stable, it is not necessarily equal to the mean of the time series. (Franses and Dijk, 2000)

3. limit cycle: A k-period limit cycle is defined as a set of k points. If the time series started in one of the points and no shocks occurred, the series would cycle among the k points. For high-order SETAR models, it can be quite difficult to establish the existence of equilibria, attractors and/or limit cycles analytically. A pragmatic way to investigate the properties of the skeleton of a high-order SETAR model is to use what might be called deterministic simulation.

4. Tong (1990, p.379) defines an alternative AIC for a 2-regime SETAR model as the sum of the AICs for the AR models in the two regimes, that is,

$$p_1, p_2 = n_1 ln\hat{\sigma}_1^2 + n_2 ln\hat{\sigma}_2^2 + 2(p_1 + 1) + 2(p_2 + 1) \tag{8}$$

where n_j, $j = 1, 2$, is the number of observations in the regime, and $\hat{\sigma}_j^2$, $j = 1, 2$, is the variance of the residuals in the regime. The SBC for a 2-regime SETAR model can be defined analogously as

$$p_1, p_2 = n_1 ln\hat{\sigma}_1^2 + n_2 ln\hat{\sigma}_2^2 + (p_1 + 1) ln \mathrm{n}_1 + (\mathrm{p}_2 + 1) ln \mathrm{n}_2 \tag{9}$$

The selected lag orders in the two regimes are those for which the information criterion is minimized.

5. In the US treasury bonds market, the 10-year T-note replaced the 30-year Treasury bond as the benchmark bond in determining interest rate trends. As a group, Treasuries are regarded as the safest bond investments, because they are backed by ”full faith and credit” of the U.S. government.

6. Protection against economic slowdown or deflation: Bonds can help protect investors against an economic slowdown for several reasons. Recall that the price of a bond depends on how much investors value the income that bonds provide. Most bonds pay a fixed income that doesn’t change. When the prices of goods and services are rising, an economic condition known as ”inflation”, a bond’s fixed income becomes less attractive because that income buys fewer goods and services. Inflation is usually caused by faster economic growth, which increases demand for goods and services. On the other hand, slower economic growth usually leads to lower inflation, which makes bond income more attractive. An economic slowdown is also typically bad for corporate profits and stock returns, adding to the attractiveness of bond income as a source of return. If the slowdown becomes bad enough that consumers stop buying things and prices in the economy begin to fall - a dire economic condition known as ”deflation” - then bond income becomes even more attractive because you can buy more goods and services (due to their deflated prices) with the same bond income. As demand for bonds increases, so do bond prices and bondholder return.

The unique characteristics of the many bond issuers in today’s market create opportunities for investors with a broad spectrum of risk or return objectives.

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