Late Points of Projections of Planar Symmetric Random Walks on the Lattice Torus

by

Michael J. Carlisle

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	Jay Rosen
Date	Chair of Examining Committee
	Józef Dodziuk
Date	Executive Officer

Elena Kosygina

Olympia Hadjiliadis

Supervisory committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Advisor: Professor Jay Rosen

We examine the *cover time* and set of *late points* of a symmetric random walk on $\mathbb{Z}^2$ projected onto the torus $\mathbb{Z}_K^2$. This extends the work done for the simple random walk in [9] to a large class of random walks. The approach uses comparisons between planar and toral hitting times and distributions on annuli, and uses only random walk methods. There are also generalizations of Green's functions, hitting times, and hitting distributions on $\mathbb{Z}^2$ and $\mathbb{Z}_K^2$ which are of independent interest.

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... and, Leah, success is possible without ruin, it might just be a bit late. Tell 'em what we said 'bout "Paint It Black".

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1 Introduction

I will be bold enough to suggest this solution to the ancient problem:
The Library is unlimited and periodic.

- Jorge Luis Borges, "The Library of Babel", *Ficciones*

Wilf, in [25], describes watching a simple random walk on a computer screen, where, on each time step, a dark pixel turns (and remains) bright if the walk visits it for the first time. How many steps, he wonders, will it take on average for the nearest neighbor walk's path (wrapping at the edges of the screen, making a discrete two-dimensional torus) to fill the screen? He refers to this as the "white screen time" problem.

He gives solutions of the white screen problem for the one dimensional path and cycle, and the complete graph $\mathcal{K}_n$ (known as the *coupon collector*'s problem), and refers to research related to the white screen problem under the name of *covering times.* Leaving the original problem unresolved, Wilf points to a 1989 work of Zuckerman which gives bounds on the two-dimensional square lattice torus $\mathbb{Z}^2_K := \mathbb{Z}^2/K\mathbb{Z}^2$. Denoting the cover time of the graph G by a random walk as $\mathcal{T}_{cov}(G) := \sup_{x\in G}\mathcal{T}(x)$, where $\mathcal{T}(x)$ is the first hitting time of x, then, for the simple random walk on $\mathbb{Z}^2_K$,

$$C_1(K\log K)^2 \leq \mathcal{T}_{cov}(\mathbb{Z}^2_K) \leq C_2(K\log K)^2$$

for some positive constants C_1, C_2.

Over the course of the next 20 years, closely related problems were solved by Aldous ([2]), Dembo, Peres, Rosen, & Zeitouni ([7], [8], [9]), Lawler ([15], [16], [18]), Rosen ([22]), and Rosen & Bass ([3]). This paper builds on these works to examine the structure of the so-called *late points* (those not hit until "soon" before the cover time) which Wilf refers to as allowing the viewer of a slowly-filling white screen to "safely go read *War and Peace* without missing any action."

We are interested in the number of late points on the square torus $\mathbb{Z}_K^2$ for large, increasing K, and will investigate this for a class of projected planar lattice, *i.e.*, $\mathbb{Z}^2$, random walks $S_t = S_0 + \sum_{j=0}^{t} X_j$, for $X = \{X_j\}_{j\in\mathbb{N}\cup\{0\}}$ with the following properties: S is symmetric recurrent, X_1 has finite covariance matrix equal to a scalar times the identity, *i.e.*, $\Gamma := cov(X_1) = cI$, $c > 0$, and X is strongly aperiodic.* X_1 has, for some $\beta > 0$ and $M := 4 + 2\beta$,

$$\mathbb{E}|X_1|^M = \sum_{x\in\mathbb{Z}^2} |x|^M p_1(x) < \infty, \tag{1.1}$$

where, as usual in the literature,

$$p_1(x, y) = p_1(y - x) = P^x(X_1 = y)$$

is the one-step transition probability. The random walk methods used in this paper require $M > 4$; this seems to be necessary for certain Harnack inequalities which we develop (whereas, in [3], $M = 3 + 2\beta$ sufficed for frequent points on the plane).

*[3] requires the covariance matrix of X_1 to be equal to $\frac{1}{2}I$, but this is a convenience for three technical points (on pages 9, 12, and 42), relating only to rotations. It is worthy (if not elementary) to note that the simple random walk on $\mathbb{Z}^d$'s X_1 covariance matrix is $\text{cov}(X_1) = \frac{1}{d}I$. If K is odd, this walk projects to a strongly aperiodic simple random walk on $\mathbb{Z}_K^d$.

X satisfies **Condition A** if either p_1 has bounded support, or, from any point "just outside" a disc, we will enter the disc with positive probability; *i.e.*, for any $s \le n$, for large enough n,

$$\inf_{y: n \le |y| < n+s} \sum_{z \in D(x,n)} p_1(y,z) = \inf_{y \in \partial D(x,n)_s} P^y(X_1 \in D(x,n)) \ge c e^{-\beta s^{1/4}}, \tag{1.2}$$

where the (Euclidean) s-annulus around the disc $D(x,n)$ (also called an x-*band*) is defined as

$$\partial D(x,n)_s := D(x,n+s) \setminus D(x,n). \tag{1.3}$$

In particular, if X_1 has infinite range, then for any $y \in \partial D(0,n)_s$, there exists $x \in D(0,n)$ such that $p_1(y,x) > 0$.

We will switch between the planar and toral representations of the random walk and corresponding stopping times, hitting distributions, etc. Define the projections, for $x = (x_1, x_2) \in \mathbb{Z}^2$, by

$$\begin{aligned}
&\pi_K : \mathbb{Z}^2 \to [-K/2, K/2)^2 \cap \mathbb{Z}^2, \\
&\pi_K(x) = \big((x_1 + \lfloor \tfrac{K}{2} \rfloor)(\text{mod } K) - \lfloor \tfrac{K}{2} \rfloor, (x_2 + \lfloor \tfrac{K}{2} \rfloor)(\text{mod } K) - \lfloor \tfrac{K}{2} \rfloor\big)\,; \\
&\hat{\pi}_K : \mathbb{Z}^2 \to \mathbb{Z}_K^2, \quad \hat{\pi}_K(x) = (\pi_K x) + (K\mathbb{Z})^2.
\end{aligned}$$

(For example, if $x = (-12, 6)$ and $K = 11$, then $\pi_{11}(\mathbb{Z}^2) = \{-5, \ldots, 5\}^2$, $\pi_{11}(x) = (-1, -5)$, and $\hat{\pi}_{11}(x) = (-1, -5) + (11\mathbb{Z})^2$.)

We call the set of lattice points $\pi_K(\mathbb{Z}^2) = [-K/2, K/2)^2 \cap \mathbb{Z}^2$ the **primary copy** in $\mathbb{Z}^2$, and for $x \in \pi_K(\mathbb{Z}^2)$, $\hat{x} := \hat{\pi}_K x$ is its corresponding element in $\mathbb{Z}_K^2$. Any $z \in \pi_K^{-1} x$, $z \neq \pi_K x$, is called a **copy** of x. Likewise, for a set $A \subset \mathbb{Z}^2$, $\hat{A} := \hat{\pi}_K A$ is the toral projection of A, and the set of all copies of A is

$$\pi_K^{-1}\pi_K A = \hat{\pi}_K^{-1}\hat{A} := \{z \in \mathbb{Z}^2 : z = x + (iK, jK),\ i, j \in \mathbb{Z}, x \in A\}.$$

Figure 1.1 displays the projection of a planar set A onto the torus as $\hat{A}$, and its pullback onto $\pi_K^{-1}A$. (If $A \subset \pi_K\mathbb{Z}^2$, then of course, $A = \pi_K A$.)

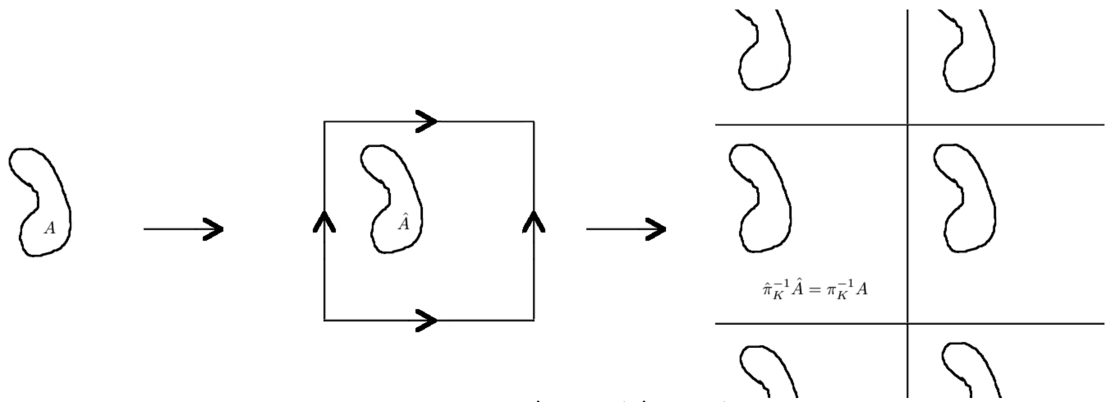

Figure 1.1: $A \to \hat{A} \to \hat{\pi}_K^{-1}\hat{A} = \pi_K^{-1}A$

For a given $\hat{x} \in \mathbb{Z}_K^2$, we define x to be the (planar) primary copy of that element; $x := \pi_K\hat{\pi}_K^{-1}\hat{x}$.

While X_j is the jth step of the planar walk and S_j its position at time j, we use $\hat{S}_j$ to denote the position of the toral walk at time j. The distance between two points $x, y \in \mathbb{Z}^2$ will be the Euclidean distance $|x - y|$; on the torus, the distance between two points $\hat{x}, \hat{y} \in \mathbb{Z}_K^2$ will be the minimum Euclidean distance $|\hat{x} - \hat{y}| \leq K\sqrt{2}/2$. To limit the issues regarding this distance, we will restrict any discs on $\mathbb{Z}_K^2$ to have radius $n < K/4$ (sometimes written as a diameter constraint: $2n < K/2$).

To bound our functions, we need a precise notion of bounding distance on the lattice torus $\mathbb{Z}_K^2$. As in [9], a function $f(x)$ is said to be $O(x)$ if $f(x)/x$ is bounded, uniformly in all implicit geometry-related quantities (such as K). That is, $f(x) = O(x)$ if there exists a universal constant C (not depending on K) such that $|f(x)| \leq Cx$. Thus $x = O(x)$ but Kx is *not* $O(x)$. A similar convention applies to $o(x)$.

Next, we will define a few terms describing the distance of a random walk step, relative to a reference disc of radius n and an s-annulus around the disc. A **small** jump refers to a step that is short enough to possibly (but not necessarily) stay inside a disc of radius n (*i.e.*, $|X_1| < 2n$). A **baby** jump refers to a small jump that is too short to hop over an s-annulus from inside a disc (*i.e.*, $|X_1| < s$). A **medium** jump refers to a step that is sufficiently large to hop out of a disc and past an s-annulus, but with magnitude strictly less than K, and cannot land near a toral copy of its launching point (*i.e.*, $s < |X_1| < K - 2n$). A **large** jump is a step which, in the toral setting, would be considered "wrapping around" in one step (*i.e.*, $|X_1| > K - 2n$). A **targeted** jump is a large jump which lands directly in a copy of the disc or annulus just launched from (*i.e.*, $j(K - 2n) \leq |X_1| \leq j(K + 2n)/\sqrt{2}$ for some j). These terms will aid in dealing with differences between planar and toral hitting and escape times.[†]

As in [7], Section 5, set $\pi_\Gamma := 2\pi\sqrt{\det \Gamma}$, and let $\alpha \in (0, 1)$. (For simple random walk, $\Gamma = \frac{1}{2}I$, so $\pi_\Gamma = \pi$.) We call $\hat{x}$ an α, K*-late point* of the random walk $\hat{S}$ on $\mathbb{Z}^2_K$ if the first hitting time of $\hat{x}$, $\mathcal{T}_K(\hat{x})$, is such that $\mathcal{T}_K(\hat{x}) \geq \frac{4\alpha}{\pi_\Gamma}(K \log K)^2$. Set $\mathcal{L}_K(\alpha)$ to be the set of α, K-late points in $\mathbb{Z}^2_K$, *i.e.*,

$$\mathcal{L}_K(\alpha) := \left\{ \hat{x} \in \mathbb{Z}^2_K : \frac{\mathcal{T}_K(\hat{x})}{(K \log K)^2} \geq \frac{4\alpha}{\pi_\Gamma} \right\}.$$

We prove the following, generalizing [9, Proposition 1.1]:

[†]We have distinguished between three types of jumps on the torus that in the planar-only case (as in *e.g.*, [3]) are referred to only as large jumps.

Theorem 1.1. *For any* $0 < \alpha < 1$,

$$\lim_{K \to \infty} \frac{\log |\mathcal{L}_K(\alpha)|}{\log K} = 2(1-\alpha) \quad \textit{in probability.} \tag{1.4}$$

As $\alpha \to 1$, a corollary of (1.4) is that we can generalize the cover time result of [8, Theorem 1.1] to our class of random walks:

Corollary 1.2.

$$\lim_{K \to \infty} \frac{\mathcal{T}_{cov}(\mathbb{Z}_K^2)}{(K \log K)^2} = \frac{4}{\pi_\Gamma} \quad \textit{in probability.} \tag{1.5}$$

The paper is structured as follows. In Chapter 2, we prove results about probabilities of exiting a disc in the plane and torus. Chapter 3 contains results involving entering a disc. In Chapter 4, we develop a general framework for analyzing moving between three sets that partition a sample space, and discuss the application of these ideas to hitting an annulus just outside a disc, and gambler's ruin estimates in that case. With this knowledge, in Chapter 5 we build Harnack inequalities for disc escape and disc entry, with fine-tuned results when the landing point is a nearby annulus. These Harnack inequalities are applied in Chapter 6 to examine excursions between consecutive concentric annuli. Finally, in Chapter 7 we estimate the rarity of traveling between these annuli without ever visiting their common center point (thereby deeming the path "late" in visiting the center), and reveal our results.

2 Disc Escape

In this chapter we develop the notions of hitting time and Green's function on the plane and torus, and find relationships between the two with respect to the timing of the random walk's escape from a disc.

2.1 Disc escape time

The *hitting time* of a random walk to a set A is defined as the stopping time $T_A = \inf\{t \geq 0 : S_t \in A\}$. Likewise, the *escape time* of the walk from A is the stopping time T_{A^c}. For a recurrent, strongly aperiodic, irreducible random walk on $\mathbb{Z}^2$, $T_{A^c} < \infty$ a.s. We denote $T_{\hat{A}}$ to be the hitting time of $\hat{A} \subset \mathbb{Z}_K^2$. We will examine several relationships between planar and toral hitting times.

An immediate observation on hitting times (*e.g.*, from [24]) is that, the larger the set to hit, the quicker it will be hit. If $A \subset B$, then obviously $T_B \leq T_A$. It is clear, then, that $\hat{\pi}_K^{-1}\hat{A}$, as an infinite number of copies of $A \subset \mathbb{Z}^2$, has a quicker hitting time than just one copy of A. In fact, we have

$$T_{\pi_K^{-1}A} = T_{\hat{\pi}_K^{-1}\hat{A}} = T_{\hat{A}}. \tag{2.1}$$

Let n, s be such that $n+s < K/4$, and $D(0,n) = \pi_K D(0,n)$ the primary copy of $D(0,n) \subset \mathbb{Z}^2$. Define the primary copy's portion of the complement of $D(0,n)$ to be $D(0,n)^c_K := D(0,n)^c \cap \pi_K \mathbb{Z}^2$. (2.2) and Figure 2.1 describe the nestedness of sets from the planar annulus $\partial D(0,n)_s$ up to the planar disc complement $D(0,n)^c$:

$$\begin{aligned} \partial D(0,n)_s \subset \quad & \pi_K^{-1}(\partial D(0,n)_s) = \hat{\pi}_K^{-1} \hat{\pi}_K (\partial D(0,n)_s) \\ \subset \quad & \hat{\pi}_K^{-1} \hat{\pi}_K (D(0,n)^c_K) = \pi_K^{-1}(D(0,n)^c_K) \subset D(0,n)^c. \end{aligned} \tag{2.2}$$

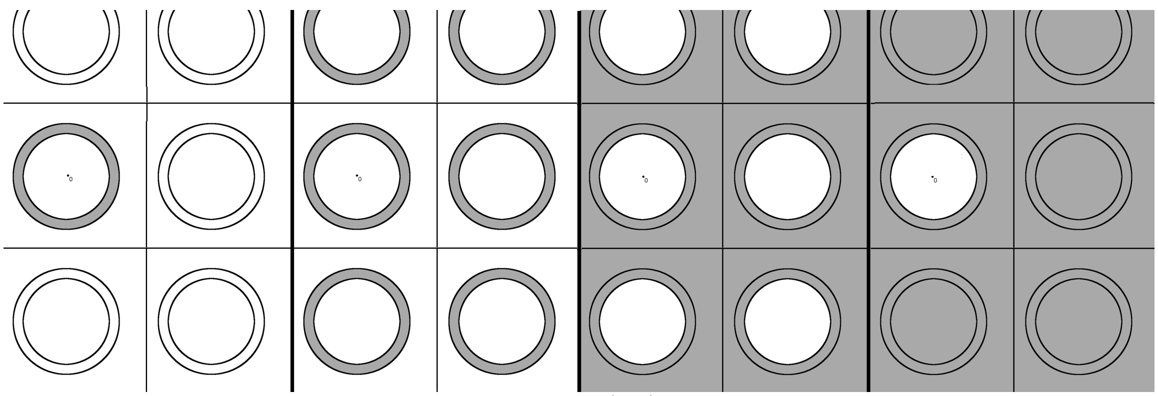

Figure 2.1: Comparison of planar sets listed in (2.2), on the plane. Labeled sets are shaded.

By (2.1), (2.2) yields, starting at any $x \in D(0,n)$, the *disc escape time inequalities*

$$\begin{aligned} T_{\partial D(0,n)_s} \geq T_{\pi_K^{-1} \partial D(0,n)_s} &= T_{\hat{\pi}_K^{-1} \hat{\pi}_K (\partial D(0,n)_s)} \\ \geq T_{\hat{\pi}_K^{-1} \hat{\pi}_K (D(0,n)^c_K)} &= T_{\pi_K^{-1}(D(0,n)^c_K)} \geq T_{D(0,n)^c} \geq 1. \end{aligned} \tag{2.3}$$

We shall take planar starting points from the primary copy ($x = \pi_K x$). The probabilities of these inequalities being strict (*e.g.*, $P^x(T_{D(0,n)^c} < T_{\hat{\pi}_K (D(0,n)^c_K)})$) and the means of the stopping times will be of interest to us. We start with estimating the mean of the planar escape time from $D(0,n)$ (which improves on [17, Prop. 6.2.6]), and then use this probability to estimate the toral escape time from $\hat{\pi}_K(D(0,n))$.

Lemma 2.1. *Let $S_t = S_0 + \sum_{j=1}^t X_j$ be a random walk in $\mathbb{Z}^2$ with $E|X_1|^2 < \infty$, and covariance matrix Γ such that $tr(\Gamma) = \gamma^2 > 0$. Then, uniformly for $x \in D(0,n)$, and for sufficiently large n,*

$$\frac{n^2 - |x|^2}{\gamma^2} \leq \mathbb{E}^x(T_{D(0,n)^c}) \leq \frac{n^2 - |x|^2}{\gamma^2} + 2n + 1. \tag{2.4}$$

Proof By [17, Exercise 1.4], the process $M_t := |S_t|^2 - \gamma^2 t$ is a martingale.

For any given k, $k \wedge T_{D(0,n)^c}$ is a bounded stopping time, and $T_{D(0,n)^c} < \infty$ a.s., so by the monotone convergence theorem,

$$\lim_{k\to\infty} \mathbb{E}^x(k \wedge T_{D(0,n)^c}) = \mathbb{E}^x(T_{D(0,n)^c}). \tag{2.5}$$

Hence, by the optional stopping theorem, uniformly for $x \in D(0,n)$,

$$\mathbb{E}^x(M_{k\wedge T_{D(0,n)^c}}) = \mathbb{E}^x(M_0) = |x|^2. \tag{2.6}$$

Decompose $|S_{k\wedge T_{D(0,n)^c}}|^2$ along the time k:

$$|S_{k\wedge T_{D(0,n)^c}}|^2 = 1_{\{k\geq T_{D(0,n)^c}\}}|S_{T_{D(0,n)^c}}|^2 + 1_{\{k<T_{D(0,n)^c}\}}|S_k|^2. \tag{2.7}$$

Its expectation, then, is

$$E(|S_{k\wedge T_{D(0,n)^c}}|^2) = E\left(1_{\{k\geq T_{D(0,n)^c}\}}|S_{T_{D(0,n)^c}}|^2\right) + E\left(1_{\{k<T_{D(0,n)^c}\}}|S_k|^2\right). \tag{2.8}$$

Then by the MCT again, since $T_{D(0,n)^c} < \infty$ a.s.,

$$\lim_{k\to\infty} E\left(1_{\{k\geq T_{D(0,n)^c}\}}|S_{T_{D(0,n)^c}}|^2\right) = E\left(|S_{T_{D(0,n)^c}}|^2\right). \tag{2.9}$$

For the second term, note that $1_{\{k<T_{D(0,n)^c}\}}|S_k|^2 \leq n^2$, and also $1_{\{k<T_{D(0,n)^c}\}}|S_k|^2 \to 0$ a.s. since, again, $T_{D(0,n)^c} < \infty$ a.s. Thus by the dominated convergence theorem,

$$\lim_{k\to\infty} E\left(1_{\{k<T_{D(0,n)^c}\}}|S_k|^2\right) = 0. \tag{2.10}$$

Combining (2.5)-(2.10) yields the expected time

$$\begin{aligned} |x|^2 = \mathbb{E}^x(M_{T_{D(0,n)^c}}) &= \mathbb{E}^x(|S_{T_{D(0,n)^c}}|^2) - \gamma^2\mathbb{E}^x(T_{D(0,n)^c}) \\ \implies \mathbb{E}^x(T_{D(0,n)^c}) &= \frac{\mathbb{E}^x(|S_{T_{D(0,n)^c}}|^2) - |x|^2}{\gamma^2}. \end{aligned} \tag{2.11}$$

We can bound $|S_{k\wedge T_{D(0,n)^c}}|^2$ by decomposing along its escape jump: if $X_j = (X_j^{(1)}, X_j^{(2)})$ is the jth step, then for any $j \leq T_{D(0,n)^c}$,

$$|S_j|^2 = |S_{j-1}|^2 + 2S_{j-1}\cdot X_j + |X_j|^2 \leq n^2 + 2n(|X_j^{(1)}| + |X_j^{(2)}|) + |X_j|^2. \tag{2.12}$$

It is clear that $tr(\Gamma) = \gamma^2 = \mathbb{E}(|X_j|^2)$, and, since $X_j^{(i)} \in \mathbb{Z}$, $i = 1, 2$, then

$$|X_j^{(1)}| + |X_j^{(2)}| \leq |X_j^{(1)}|^2 + |X_j^{(2)}|^2 = |X_j|^2.$$

Therefore, (2.12) becomes, substituting $k \wedge T_{D(0,n)^c} = j \leq T_{D(0,n)^c}$,

$$|S_{k\wedge T_{D(0,n)^c}}|^2 \leq n^2 + (2n+1)|X_{k\wedge T_{D(0,n)^c}}|^2.$$

By taking expectations,

$$\mathbb{E}^x(|S_{k\wedge T_{D(0,n)^c}}|^2) \leq n^2 + (2n+1)\mathbb{E}^x(|X_{k\wedge T_{D(0,n)^c}}|^2) \leq n^2 + (2n+1)\gamma^2 < \infty.$$

Since, at $T_{D(0,n)^c}$, we have escaped the disc, we have a lower bound as well. By (2.8)-(2.10),

$$n^2 \leq \mathbb{E}^x(|S_{T_{D(0,n)^c}}|^2) \leq n^2 + (2n+1)\gamma^2. \tag{2.13}$$

Combining (2.13) with (2.11) yields (2.4). □

For $\Gamma = cI$, $\gamma^2 = 2c$ and so (2.4) becomes*

$$\frac{n^2 - |x|^2}{2c} \leq \mathbb{E}^x(T_{D(0,n)^c}) \leq \frac{n^2 - |x|^2}{2c} + 2n + 1. \tag{2.14}$$

Spitzer, in [24], defines the *truncated Green's function*, for $x, y \in A$ of a random walk from x to y before exiting A as the total expected number of visits to y, starting from x:

$$G_A(x,y) := \mathbb{E}^x\left[\sum_{j=0}^{\infty} 1_{\{S_j = y; j < T_{A^c}\}}\right] = \sum_{j=0}^{\infty} P^x(S_j = y; j < T_{A^c}) \tag{2.15}$$

and 0 if x or $y \notin A$. (Since the walk is recurrent and aperiodic, there is no "all-time" Green's function to count the total number of visits to x from $j = 0$ to ∞.) An elementary result for any random walk (found, for example, in [24], or [15, Sect. 1.5]) is that, for $x, y \in A \subset B$, there are more possible visits inside B than inside A:

$$G_A(x,y) \leq G_B(x,y). \tag{2.16}$$

*For simple random walk on $\mathbb{Z}^2$, $c = 1/2$, which yields [9, (2.3)].

Also of interest is the expected hitting time identity

$$\mathbb{E}^x(T_{A^c}) = \sum_{z\in A} G_A(x,z). \tag{2.17}$$

Starting at a point $x \in A^c$, the *hitting distribution* of A is defined as

$$H_A(x,y) := P^x(S_{T_A} = y).$$

The *last exit decomposition* of a hitting distribution is based on the Green's function: for A a proper subset of $\mathbb{Z}^2$, $x \in A^c$, $y \in A$,

$$H_A(x,y) = \sum_{z\in A^c} G_{A^c}(x,z)p_1(z,y). \tag{2.18}$$

An immediate result follows from (2.16): If $y \in A \subset B$, then for $x \in B^c \subset A^c$, we have by (2.16) the monotonicity result

$$\begin{aligned} H_A(x,y) &= \textstyle\sum_{z\in A^c} G_{A^c}(x,z)p_1(z,y) \\ &\geq \textstyle\sum_{z\in B^c} G_{B^c}(x,z)p_1(z,y) = H_B(x,y) \end{aligned} \tag{2.19}$$

and the subset hitting time relations (assuming a recurrent random walk)

$$\begin{aligned} &P^x(T_A = T_B) = \sum_{z\in A} H_B(x,z); \\ &P^x(T_A \neq T_B) = P^x(T_A > T_B) = \sum_{z\in B\setminus A} H_B(x,z) \end{aligned} \tag{2.20}$$

which we will revisit in Chapter 4.

By Markov's inequality, large jumps are rare: if $C_M = \mathbb{E}(|X_1|^M) < \infty$, then since

$2n < K/2$,

$$P(|X_1| > K - 2n) \leq \frac{C_M}{(K-2n)^M} < \frac{2^M C_M}{K^M} = O(K^{-M}). \tag{2.21}$$

Recall that, when given a toral element $\hat{x} \in \mathbb{Z}_K^2$, we define x to be the (planar) primary copy of that element; $x := \pi_K \hat{\pi}_K^{-1} \hat{x}$. A toral step $\hat{x} \to \hat{y}$ must take into account large jumps that, on the plane, would land on a copy of y (*i.e.*, in $\hat{\pi}_K^{-1}\hat{y}$). All of these positions, together, are a small addition to the planar jump probability. By (2.21) we have, for $\hat{x}, \hat{y} \in \mathbb{Z}_K^2$, the targeted jump estimate

$$\begin{aligned} \hat{p}_1(\hat{x}, \hat{y}) := P^{\hat{x}}(\hat{S}_1 = \hat{y}) &= P^x(S_1 = y) + P^x\left(|X_1| > K - 2n; S_1 \in \hat{\pi}_K^{-1}\hat{y} \setminus \{y\}\right) \\ &\leq p_1(x, y) + O(K^{-M}). \end{aligned} \tag{2.22}$$

By (2.18), (2.21), and then (2.4) and (2.17), for some $c < \infty$ and any $x \in D(0, n)$,

$$\begin{aligned} P^x(T_{\hat{\pi}_K(D(0,n)^c_K)} > T_{D(0,n)^c}) &= \sum_{z \in \left(\hat{\pi}_K^{-1}\hat{\pi}_K(D(0,n)) \setminus D(0,n)\right)} \sum_{y \in D(0,n)} G_{D(0,n)}(x, y) p_1(y, z) \\ &\leq cK^{-M} \sum_{y \in D(0,n)} G_{D(0,n)}(x, y) = O(K^{-M} n^2). \end{aligned} \tag{2.23}$$

We now find that the mean of the disc escape time on the torus is larger than on the plane, but only by a small factor (induced by the rarity of targeted jumps).

Lemma 2.2. *For $n < K/4$, $x \in D(0, n)$, and n and K sufficiently large,*

$$\mathbb{E}^{\hat{x}}[T_{\hat{\pi}_K(D(0,n)^c_K)}] \leq \mathbb{E}^x[T_{D(0,n)^c}] + O(K^{-M} n^2) \max_{y \in D(0,n)} \mathbb{E}^y[T_{D(0,n)^c}]. \tag{2.24}$$

Proof To bound the disc escape time above, consider a "worst case" scenario (making the $\hat{\pi}_K(D(0,n))$-escape time as long as possible) where every large jump targets the same point inside the disc. (Example 2.3 gives an explicit case of such a "worst case" scenario.)

Let y^* be a point on $D(0,n)$ such that $\mathbb{E}^{y^*}(T_{D(0,n)^c}) = \max_{y\in D(0,n)} \mathbb{E}^y(T_{D(0,n)^c})$. Define the times σ_i and τ, and index variable N, by

$$\sigma_0 = T_{D(0,n)^c};\ \sigma_{i+1} = \inf\{j > \sigma_i : S^*_{j-1} + X_j \in D(0,n)^c\},\ i \geq 0 \tag{2.25}$$
$$\tau = \inf\left\{j > 0 : |X_j| \leq K - 2n, S^*_{j-1} + X_j \in D(0,n)^c\right\}$$
$$N = j \iff \sigma_j = \tau$$

where σ_0 is the original walk S's planar disc escape time, and the modified walk S^* is defined as the walk whose large jumps (of size $> K - 2n$) target y^*, until the walk escapes $D(0,n)$ via a nonlarge jump:

$$S^*_t := \begin{cases} x, & t = 0 \\ x + \sum_{k=1}^t X_k, & 0 < t < \sigma_0 \\ x + \sum_{k=1}^\tau X_k, & t \geq \tau \text{ on } \{\tau = \sigma_0\} \\ y^*, & t = \sigma_i,\ |X_{\sigma_i}| > K - 2n,\ 0 \leq i < N \\ y^* + \sum_{k=\sigma_i+1}^t X_k, & \sigma_i < t < \sigma_{i+1},\ |X_{\sigma_i}| > K - 2n,\ 0 \leq i < N \\ y^* + \sum_{k=\sigma_{N-1}+1}^\tau X_k, & t \geq \tau \text{ on } \{\tau = \sigma_N,\ N > 0\}. \end{cases} \tag{2.26}$$

σ_i, $i \geq 0$, are the successive would-be escape times from $D(0,n)$, if y^*-targeting was not "enabled". τ is the smallest σ_i such that escape from $D(0,n)$ actually occurs, and N is the number of large jumps before this escape occurs. Note that, considering

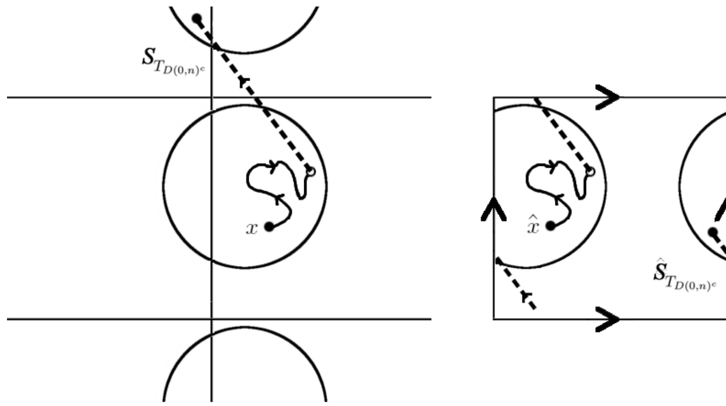

Figure 2.2: An example of a path in $\{T_{\hat{\pi}_K(D(0,n)^c_K)} > T_{D(0,n)^c}\}$, where a targeted jump of planar distance $\geq O(K)$ keeps the walk in $\hat{\pi}_K(D(0,n))$.

times on the original walk S,

$$\{N = 0\} = \left\{\tau = T_{\hat{\pi}_K(D(0,n)^c_K)} = T_{D(0,n)^c}\,,\, |X_{T_{D(0,n)^c}}| \leq K - 2n\right\}$$
$$\{N > 0\} = \left\{\tau \geq T_{\hat{\pi}_K(D(0,n)^c_K)} > T_{D(0,n)^c}\,,\, |X_{T_{D(0,n)^c}}| > K - 2n\right\},$$

and, conditioned on $\{N > 0\}$, N is a geometric random variable with success parameter $p = P^{y^*}\left(T_{D(0,n)^c} = T_{\hat{\pi}_K(D(0,n)^c_K)}\,,\, |X_{T_{D(0,n)^c}}| \leq K - 2n\right) = 1 + O(K^{-M}n^2)$ by (2.23) and (2.21) (where a "failure" is a targeted jump back to $y^* \in D(0,n)$). Thus, $\mathbb{E}^{\hat{x}}[T_{\hat{\pi}_K(D(0,n)^c_K)}] \leq \mathbb{E}^{\hat{x}}[\tau]$, since τ is the escape time of $\hat{\pi}_K(D(0,n))$, with targeting back to y^*. Conditioning on $\{N > 0\}$, and by (2.23) and (2.21) and the strong Markov property on σ_0, we have

$$\begin{aligned}
\mathbb{E}^x[\tau] &= \mathbb{E}^x[\tau|N = 0]P^x(N = 0) + \mathbb{E}^x[\tau|N > 0]P^x(N > 0)\\
&= \mathbb{E}^x[\sigma_0|N = 0]P^x(N = 0) + \mathbb{E}^x[\sigma_0 + \sigma_N - \sigma_0|N > 0]P^x(N > 0)\\
&\leq \mathbb{E}^x[T_{D(0,n)^c}] + \mathbb{E}^x[\sigma_N - \sigma_0|N > 0]P^x(N > 0) \qquad (2.27)\\
&\leq \mathbb{E}^x[T_{D(0,n)^c}] + O(K^{-M}n^2)\mathbb{E}^{y^*}[\sigma_N - \sigma_0|N > 0].
\end{aligned}$$

On $\{N > 0\}$, the time of the jth excursion from y^* until attempted disc escape is $\nu_j := \sigma_j - \sigma_{j-1}$, for $1 \le j \le N$, are IID with mean $\mathbb{E}[\nu_j] = \mathbb{E}^{y^*}[T_{D(0,n)^c}]$. Since $P(N < \infty) = 1$, by Wald's identity we have

$$\begin{aligned}\mathbb{E}^{y^*}[\sigma_N - \sigma_0 | N > 0] = \mathbb{E}^{y^*}\left[\sum_{j=1}^{N} \nu_j \middle| N > 0\right] = \mathbb{E}^{y^*}[N|N>0]\,\mathbb{E}^{y^*}[\nu_1] \\ = \frac{1}{p}\,\mathbb{E}^{y^*}[T_{D(0,n)^c}] = (1 + O(K^{-M}n^2))\mathbb{E}^{y^*}[T_{D(0,n)^c}].\end{aligned}$$

Therefore, (2.27) becomes

$$\begin{aligned}\mathbb{E}^x[\tau] &\le \mathbb{E}^x[T_{D(0,n)^c}] + O(K^{-M}n^2)\mathbb{E}^{y^*}[\sigma_N - \sigma_0 | N > 0] \\ &\le \mathbb{E}^x[T_{D(0,n)^c}] + O(K^{-M}n^2)\mathbb{E}^{y^*}[T_{D(0,n)^c}]. \quad \square\end{aligned}$$

Example 2.3. *Let $A = D(0, \sqrt{2}) = \{0, +e_1, -e_1, +e_2, -e_2\} \subset \mathbb{Z}^2$, where e_i is the ith unit vector in $\mathbb{Z}^2$, and K odd and fixed. Let X be the symmetric random walk on $\mathbb{Z}^2$ starting at $X_0 = 0$ defined by the probabilities*

$$p_1(K^j e_i) = P^0(X_1 = K^j e_i) = \frac{1}{4} e^{-\lambda} \frac{\lambda^j}{j!}, j = 0, 1, 2, \ldots; \;\; i = 1, 2.$$

$\frac{\log|X_1|}{\log K}$ is a Poisson random variable with parameter λ, and moving any of the four primary lattice directions is equally likely. S_t is strongly aperiodic recurrent and has infinite range, $E(|X_1|^m) < \infty$ for all $m < \infty$ (and, in particular, cov$(|X_1|) = \Gamma = \frac{1}{2}e^{(K^2-1)\lambda}I$), and every large jump causes a landing in a new copy of A. The only way to escape $\pi_K^{-1}A = \hat{\pi}_K^{-1}\hat{A}$ is a step of size $K^0 = 1$.

Computational bounds on $\mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(D(0,n)^c_K)})$, by (2.24) and (2.4), are

$$\frac{n^2-|x|^2}{\gamma^2} \le \mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(D(0,n)^c_K)}) \le \frac{n^2-|x|^2}{\gamma^2} + 2n + 1 + O(K^{-M}n^4). \tag{2.28}$$

Example 2.4. *Define the ε-lazy simple random walk on $\mathbb{Z}^d$, for $0 \le \varepsilon < 1$, to be the walk with steps $p_1(e_j) = p_1(-e_j) = \frac{1-\varepsilon}{2d}$, $j = 1, ..., d$; $p_1(0) = \varepsilon$,* i.e., *the walk stands still for a step with probability ε, and acts "simply" otherwise. Then $\Gamma = \left(\frac{1-\varepsilon}{d}\right) I$, and so for $d = 2$, $\mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(D(0,n)^c_K)}) = \frac{n^2-|x|^2}{1-\varepsilon} + O(n)$.*

We will next see that, from inside a disc, the probability of hitting zero before escaping is nearly the same on the torus as on the plane. Recall that, for $\hat{x} \in \mathbb{Z}^2_K$, $x := \pi_K \hat{\pi}_K^{-1} \hat{x}$.

Lemma 2.5. *For all $\hat{x} \in \hat{\pi}_K(D(0,n))$ and n sufficiently large with $2n < K/2$,*

$$P^{\hat{x}}(T_{\hat{0}} < T_{\hat{\pi}_K(D(0,n)^c_K)}) = P^x(T_0 < T_{D(0,n)^c}) + O(K^{-M}n^2). \tag{2.29}$$

Proof The event $\{T_{\hat{0}} < T_{\hat{\pi}_K(D(0,n)^c_K)}\}$ can occur in two ways:

- The walk hits $\hat{0}$ after a small jump, never leaving the disc. This is equivalent to the planar event $\{T_{\hat{0}} = T_0 < T_{D(0,n)^c}\}$.
- The planar walk (wlog starting from $\pi_K x$) does not hit 0, and exits $D(0,n)$ via a targeted jump into $\hat{\pi}_K^{-1}\hat{\pi}_K(D(0,n))$. It may do this multiple times before finally hitting $\pi_K^{-1}0$ (via a small or large jump).

We can represent this event as the disjoint union

$$\begin{aligned}\{T_{\hat{0}} < T_{\hat{\pi}_K(D(0,n)^c_K)}\} = \ &\{T_{\hat{0}} = T_0 < T_{D(0,n)^c} \le T_{\hat{\pi}_K(D(0,n)^c_K)}\} \\ &\sqcup \{T_{D(0,n)^c} < T_{\hat{0}} < T_{\hat{\pi}_K(D(0,n)^c_K)}\}.\end{aligned}$$

The first case contains $\{T_0 < T_{D(0,n)^c}\}$, so a lower bound on the toral probability is the planar result. An upper bound on the second case is found in the event $\{T_{D(0,n)^c} < T_{\hat{\pi}_K(D(0,n)^c_K)}\}$, which by (2.23) is rare. Hence,

$$\begin{aligned} P^x(T_0 < T_{D(0,n)^c}) &\leq P^{\hat{x}}(T_{\hat{0}} < T_{\hat{\pi}_K(D(0,n)^c_K)}) \\ &\leq P^x(T_0 < T_{D(0,n)^c}) + P^x(T_{D(0,n)^c} < T_{\hat{\pi}_K(D(0,n)^c_K)}) \\ &\leq P^x(T_0 < T_{D(0,n)^c}) + O(K^{-M}n^2). \quad \square \end{aligned}$$

Finally, we calculate bounds for hitting time probabilities of a small disc around zero before escaping the n-disc. Let $\rho(\hat{x}) := n - |\hat{x}|$ be the distance between $\hat{x}$ and $\hat{\pi}_K(D(0,n))$.

Lemma 2.6. *Let $0 < \delta < \varepsilon < 1$. Then there exist $0 < c_1 < c_2 < \infty$ such that for all $\hat{x} \in \hat{\pi}_K(D(0,n)) \setminus \hat{\pi}_K(D(0,\varepsilon n))$, for n sufficiently large,*

$$c_1 \frac{\rho(\hat{x}) \vee 1}{n} \leq P^{\hat{x}}(T_{\hat{\pi}_K(D(0,\delta n))} < T_{\hat{\pi}_K(D(0,n)^c_K)}) \leq c_2 \frac{\rho(\hat{x}) \vee 1}{n}. \tag{2.30}$$

Proof From (2.3), it is clear that $T_{D(0,n)^c} \leq T_{\hat{\pi}_K(D(0,n)^c_K)}$. Note that $T_{D(0,\delta n)} < T_{D(0,n)^c}$ only if the walk enters $D(0,\delta n)$ via a small jump (of distance no more than $(1-\delta)n$), and in this case $T_{D(0,\delta n)} = T_{\hat{\pi}_K(D(0,\delta n))}$. A large jump automatically causes planar exit of $D(0,n)$, regardless of where in the torus the walk lands. Breaking down the sets of paths involved, we have the planar case

$$\{T_{D(0,\delta n)} < T_{D(0,n)^c}\} = \{T_{\hat{\pi}_K(D(0,\delta n))} = T_{D(0,\delta n)} < T_{D(0,n)^c} \leq T_{\hat{\pi}_K(D(0,n)^c_K)}\}$$

which covers all small-jump entrances to $\hat{\pi}_K^{-1}\hat{\pi}_K(D(0,\delta n))$; for the toral case, we have

$$\begin{aligned}
&\{T_{\hat{\pi}_K(D(0,\delta n))} < T_{\hat{\pi}_K(D(0,n)^c_K)}\} \\
= \ &\{T_{\hat{\pi}_K(D(0,\delta n))} = T_{D(0,\delta n)} < T_{D(0,n)^c} \leq T_{\hat{\pi}_K(D(0,n)^c_K)}\} \\
\sqcup \ &\{T_{D(0,n)^c} \leq T_{\hat{\pi}_K(D(0,\delta n))} < T_{\hat{\pi}_K(D(0,n)^c_K)}, T_{D(0,\delta n)}\},
\end{aligned} \tag{2.31}$$

where the second set contains all paths where a large jump occurs at or before entry to the inner disc. Hence,

$$P^x(T_{D(0,\delta n)} < T_{D(0,n)^c}) \leq P^{\hat{x}}(T_{\hat{\pi}_K(D(0,\delta n))} < T_{\hat{\pi}_K(D(0,n)^c_K)}),$$

and so we get the lower bound from [3, Lemma 2.1].

The upper bound simply bounds the second set in (2.31). By (2.23),

$$\begin{aligned}
P^{\hat{x}}(T_{\hat{\pi}_K(D(0,\delta n))} < T_{\hat{\pi}_K(D(0,n)^c_K)}) &\leq P^x(T_{D(0,\delta n)} < T_{D(0,n)^c}) \\
&\quad + P^x(T_{D(0,n)^c} < T_{\hat{\pi}_K(D(0,n)^c_K)}) \\
&\leq P^x(T_{D(0,\delta n)} < T_{D(0,n)^c}) + O(K^{-M}n^2),
\end{aligned}$$

and the error term is absorbed by the upper bound on $P^x(T_{D(0,\delta n)} < T_{D(0,n)^c})$. □

2.2 Internal Green's function

Here we will examine *internal* Green's functions on the plane (*i.e.*, from inside a disc; Green's functions external to a disc will be analyzed in Chapter 3). We extend some results of [17] for symmetric random walks on $\mathbb{Z}^2$ to projections of these random walks onto $\mathbb{Z}_K^2$.

We define the Green's function in the usual way for $\hat{x}, \hat{y} \in \hat{\pi}_K(A) = \hat{A} \in \mathbb{Z}_K^2$ to be, in comparison to (2.15),

$$\hat{G}_{\hat{\pi}_K(A)}(\hat{x}, \hat{y}) := \sum_{j=0}^{\infty} P^{\hat{x}}(\hat{S}_j = \hat{y}; j < T_{\hat{\pi}_K(A_K^c)}) \tag{2.32}$$

and 0 else. In the planar case, the stopping time T_{A^c} for a bounded set A has a clear meaning, as a sufficiently large jump (one with magnitude greater than the diameter of A, for example) will certainly exit A. Jumps targeting A land, in $\mathbb{Z}^2$, in $\pi_K^{-1}A = \hat{\pi}_K^{-1}\hat{A}$; on $\mathbb{Z}_K^2$, they land in $\hat{A}$. This means that planar estimates must be adjusted to reach similar results on the torally-projected walk.

Please note that (2.32) is different from the planar Green's function on the periodic planar set $\pi_K^{-1}A$:

$$G_{\pi_K^{-1}A}(x, y) := \sum_{j=0}^{\infty} P^{x}(S_j = y; j < T_{\pi_K^{-1}(A_K^c)}),\ x, y \in \pi_K^{-1}A. \tag{2.33}$$

We will explore this distinction in Chapter 3.

Note that $S_j \in \hat{\pi}_K^{-1}\hat{S}_j$ for every j. By (2.3) it is clear that planar escape happens at or before toral escape. Hence, the number of planar visits is less than or equal to the number of toral visits; for any $x, y \in A \subset \pi_K \mathbb{Z}^2$,

$$\begin{aligned} G_A(x,y) &= \sum_{j=0}^{\infty} P^x(S_j = y;\ j < T_{A^c}) \\ &= \sum_{j=0}^{\infty} P^x(S_j \in \pi_K^{-1} y;\ j < T_{A^c}) = \sum_{j=0}^{\infty} P^{\hat{x}}(\hat{S}_j = \hat{y};\ j < T_{A^c}) \\ &\le \sum_{j=0}^{\infty} P^{\hat{x}}(\hat{S}_j = \hat{y};\ j < T_{\hat{\pi}_K(A_K^c)}) = \hat{G}_{\hat{\pi}_K(A)}(\hat{x},\hat{y}), \end{aligned} \tag{2.34}$$

where equality occurs between the first and second lines because, of all the copies of y in $\pi_K^{-1} y$, only the primary copy $y = \pi_K y$ can be hit before the planar escape time T_{A^c}.

We start by giving bounds on the number of visits to $\hat{0}$ before escaping a disc.

Lemma 2.7. *For n sufficiently large (with $2n < K/2$),*

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0}) = G_{D(0,n)}(0,0)[1 + O(K^{-M} n^2)]. \tag{2.35}$$

Proof Our lower bound $G_{D(0,n)}(0,0) \le \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0})$ is clear from (2.34). To achieve the upper bound, first decompose the count, noting that the toral event $\{\hat{S}_j = \hat{0}; j < T_{\hat{\pi}_K(D(0,n)_K^c)}; T_{D(0,n)^c} = T_{\hat{\pi}_K(D(0,n)_K^c)}\}$ equals the planar event $\{S_j = 0; j <$

$T_{D(0,n)^c}; T_{D(0,n)^c} = T_{\hat{\pi}_K(D(0,n)^c_K)}\}$. Applying the strong Markov property at $T_{D(0,n)^c}$,

$$\begin{aligned} \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0}) &= \sum_{j=0}^{\infty} P^{\hat{0}}\left(\hat{S}_j = \hat{0}; j < T_{\hat{\pi}_K(D(0,n)^c_K)}\right) \\ &= \sum_{j=0}^{\infty} P^{\hat{0}}\left(\hat{S}_j = \hat{0}; j < T_{\hat{\pi}_K(D(0,n)^c_K)}; T_{D(0,n)^c} = T_{\hat{\pi}_K(D(0,n)^c_K)}\right) \\ &\quad + \sum_{j=0}^{\infty} P^{\hat{0}}\left(\hat{S}_j = \hat{0}; j < T_{\hat{\pi}_K(D(0,n)^c_K)}; T_{D(0,n)^c} < T_{\hat{\pi}_K(D(0,n)^c_K)}\right) \\ &= G_{D(0,n)}(0,0) + \mathbb{E}^0\left[\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{S}_{T_{D(0,n)^c}}, \hat{0});\ T_{D(0,n)^c} < T_{\hat{\pi}_K(D(0,n)^c_K)}\right], \end{aligned} \tag{2.36}$$

where, on $\{T_{D(0,n)^c} < T_{\hat{\pi}_K(D(0,n)^c_K)}\}$, $\hat{S}_{T_{D(0,n)^c}}$ is the point in $\hat{\pi}_K(D(0,n))$ that our walk lands once escaping the planar disc $D(0,n)$ via a targeted jump into a copy.

By (2.23), we know $P^0(T_{D(0,n)^c} < T_{\hat{\pi}_K(D(0,n)^c_K)}) = O(K^{-M}n^2)$. The strong Markov property applied at T_0 gives us the planar equality

$$G_{D(0,n)}(x,0) = P^x(T_0 < T_{D(0,n)^c})\, G_{D(0,n)}(0,0) \tag{2.37}$$

which implies $G_{D(0,n)}(x,0) \leq G_{D(0,n)}(0,0)$ for all $x \in D(0,n)$. This equality has a clear analog on the torus, by applying the strong Markov property at $T_{\hat{0}}$:

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{0}) = P^{\hat{x}}(T_{\hat{0}} < T_{\hat{\pi}_K(D(0,n)^c_K)})\, \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0}), \tag{2.38}$$

which, with (2.28) implies, for all $\hat{x} \in \hat{\pi}_K(D(0,n))$,

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{0}) \leq \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0}) \leq \mathbb{E}^{\hat{0}}(T_{\hat{\pi}_K(D(0,n)^c_K)}) < \infty. \tag{2.39}$$

Thus, $\max_{\hat{x}\in\hat{\pi}_K(D(0,n))} \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{S}_{T_{D(0,n)^c}}, \hat{0}) = \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0})$, and by combining

(2.36), (2.39), and (2.23), we have

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0}) \le G_{D(0,n)}(0,0) + O(K^{-M}n^2)\, \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0}),$$

which, when substituted back into itself gives, for some $c < \infty$,

$$\begin{aligned} \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0}) &\le G_{D(0,n)}(0,0) + \textstyle\sum_{j=1}^{\infty} (cK^{-M}n^2)^j\, G_{D(0,n)}(0,0) \\ &= G_{D(0,n)}(0,0)\left(1 + O(K^{-M}n^2)\right). \quad \square \end{aligned}$$

Define the *potential kernel* for X on $\mathbb{Z}^2$ as follows: for $x \in \mathbb{Z}^2$,

$$a(x) := \lim_{n\to\infty} \sum_{j=0}^{n} [p_j(0) - p_j(x)]. \tag{2.40}$$

Combining the generality of rotation of [24, Ch. III, Sec. 12, P3] and [17, Theorem 4.4.6] and the infinite-range argument of [3, Prop. 9.2] gives, for covariance matrix Γ and norm $\mathcal{J}^*(x) := |x \cdot \Gamma^{-1} x|$, as $|x| \to \infty$,

$$a(x) = \frac{2}{\pi_\Gamma} \log \mathcal{J}^*(x) + C(p_1) + o(|x|^{-1}), \tag{2.41}$$

where $C(p_1)$ is a constant depending on p_1 but not x, and $\pi_\Gamma = 2\pi\sqrt{\det \Gamma}$. For $\Gamma = cI$, this reduces to

$$\begin{aligned} a(x) &= \frac{1}{c\pi} \log\left(\frac{|x|}{\sqrt{c}}\right) + C(p_1) + o(|x|^{-1}) \\ &= \frac{1}{c\pi} \log|x| + C'(p_1) + o(|x|^{-1}), \end{aligned} \tag{2.42}$$

where $C'(p_1) = C(p_1) - \frac{1}{2c\pi}\log c$. For simple random walk on $\mathbb{Z}^2$, $c = \frac{1}{2}$, and so this is, from [17, Theorem 4.4.4],

$$a(x) = \frac{2}{\pi} \log |x| + \frac{2\gamma + \log 8}{\pi} + o(|x|^{-1}), \tag{2.43}$$

where γ is Euler's constant. From here on, we will write (2.42) with the form

$$a(x) = \frac{2}{\pi_\Gamma} \log |x| + C'(p_1) + o(|x|^{-1}). \tag{2.44}$$

By the argument in [3, (2.8)-(2.12)] (which calculates the overshoot estimate of $O(n^{-1/4})$ mentioned in the note after [17, Prop. 6.3.1]), and using (2.44), we get a computational result for (2.35) if $\Gamma = cI$:

$$G_{D(0,n)}(0,0) = \frac{2}{\pi_\Gamma} \log n + C' + O(n^{-1/4}) \tag{2.45}$$

which implies the toral Green's function

$$\begin{aligned} \implies \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0}) &= G_{D(0,n)}(0,0)(1 + O(K^{-M}n^2)) \\ &= \left(\frac{2}{\pi_\Gamma} \log n + C' + O(n^{-1/4})\right)(1 + O(K^{-M}n^2)) \\ &= \frac{2}{\pi_\Gamma} \log n + C' + O(n^{-1/4}). \end{aligned} \tag{2.46}$$

For $x, y \in \mathbb{Z}^2$ such that $|x| \ll |y|$, we have, by a Taylor expansion around y,

$$\log |y - x| = \log |y| + O\left(\frac{|x|}{|y|}\right). \tag{2.47}$$

In particular, if $x \in D(0, 2r)$ and $y \in D(0, R/2)^c$, with $R = 4mr$, we have

$$\log |y - x| = \log |y| + O\left(m^{-1}\right). \tag{2.48}$$

Note that (2.47) and (2.48) hold in the toral case without adjustment.

Let $\eta = \inf\{t \geq 1 : S_t \in \{0\} \cup D(0,n)^c\}$. Then, following the argument of [3, (2.14)-(2.15)], since $a(x)$ is harmonic with respect to p, $a(S_{t\wedge\eta})$ is a bounded martingale. Hence, $|a(S_{t\wedge\eta})|^2$ is a submartingale, so $\mathbb{E}|a(S_{t\wedge\eta})|^2 \leq \mathbb{E}|a(S_\eta)|^2 < \infty$, meaning $\{a(S_{t\wedge\eta})\}$ are uniformly integrable. Hence, by the optional stopping and bounded convergence theorems, (2.44), and (2.48),

$$\begin{aligned} a(x) &= \lim_{t\to\infty} \mathbb{E}^x(a(S_{t\wedge\eta})) = \mathbb{E}^x(a(S_\eta)) = \mathbb{E}^x(a(S_\eta);\, S_\eta \neq 0) \\ &= \sum_{y\in\partial D(0,n)_{n^{3/4}}} a(y)P^x(S_\eta = y) + \sum_{y\in D(0,n+n^{3/4})^c} a(y)P^x(S_\eta = y) \\ &= \left(\frac{2}{\pi_\Gamma}\log n + C'(p_1) + o(|x|^{-1}) + O(n^{-1/4})\right) P^x(S_\eta \neq 0) + O(n^{-1/4}), \end{aligned}$$

which, combining the error terms into $O(|x|^{-1/4})$, matches [17, Prop. 6.4.3]:

$$\begin{aligned} P^x(T_0 < T_{D(0,n)^c}) &= P^x(S_\eta = 0) = 1 - \frac{a(x) - O(n^{-1/4})}{\frac{2}{\pi_\Gamma}\log n + C' + O(|x|)^{-1/4}} \qquad (2.49) \\ &= 1 - \frac{\frac{2}{\pi_\Gamma}\log|x| + C' + O(|x|^{-1/4})}{\frac{2}{\pi_\Gamma}\log n + C' + O(n^{-1/4})} = \left(\frac{\log(n/|x|) + O(|x|^{-1/4})}{\log n}\right)(1 + O((\log n)^{-1})). \end{aligned}$$

With (2.29), we move this to the torus:

$$\begin{aligned} P^{\hat{x}}(T_{\hat{0}} < T_{\hat{\pi}_K(D(0,n)^c_K)}) &= \frac{\log(n/|\hat{x}|) + O(|\hat{x}|^{-1/4})}{\log(n)}\left(1 + O((\log n)^{-1})\right) + O(K^{-M}n^2) \\ &= \frac{\log(n/|\hat{x}|) + O(|\hat{x}|^{-1/4})}{\log(n)}\left(1 + O((\log n)^{-1})\right). \qquad (2.50) \end{aligned}$$

Next, we examine $\hat{x} \in \hat{\pi}_K(D(0,R)) \setminus \hat{\pi}_K(D(0,r))$. By the fact that a large targeted jump may land a planar walk into $\hat{\pi}_K^{-1}\hat{\pi}_K(D(0,r)) \setminus D(0,r)$ (the set of any copy of $D(0,r)$ that is not the primary copy), we may transfer the planar results [3, (2.20), (2.21)]

$$P^x(T_{D(0,r)} > T_{D(0,R)^c}) = \frac{\log(|x|/r) + O(r^{-1/4})}{\log(R/r)} \tag{2.51}$$

$$P^x(T_{D(0,r)} < T_{D(0,R)^c}) = \frac{\log(R/|x|) + O(r^{-1/4})}{\log(R/r)} \tag{2.52}$$

uniformly for $r < |x| < R$ to the toral results

$$\begin{aligned} P^{\hat{x}}(T_{\hat{\pi}_K(D(0,r))} > T_{\hat{\pi}_K(D(0,R)^c_K)}) &= \frac{\log(|\hat{x}|/r) + O(r^{-1/4})}{\log(R/r)} + O(K^{-M}R^2) \\ &= \frac{\log(|\hat{x}|/r) + O(r^{-1/4})}{\log(R/r)} \end{aligned} \tag{2.53}$$

$$\begin{aligned} P^{\hat{x}}(T_{\hat{\pi}_K(D(0,r))} < T_{\hat{\pi}_K(D(0,R)^c_K)}) &= \frac{\log(R/|\hat{x}|) + O(r^{-1/4})}{\log(R/r)} + O(K^{-M}R^2) \\ &= \frac{\log(R/|\hat{x}|) + O(r^{-1/4})}{\log(R/r)}. \end{aligned} \tag{2.54}$$

By (2.37), (2.45), (2.46), (2.38), (2.49), and (2.50), we get as corollaries calculations and bounds for $G_{D(0,n)}(x,0)$, $\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{0})$, $G_{D(0,n)}(x,z)$, and $\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{z})$: for $x \in D(0,n)$ and $\hat{x} \in \hat{\pi}_K(D(0,n))$, for some $C = C(p) < \infty$,

$$\begin{aligned} G_{D(0,n)}(x,0) &= P^x(T_0 < T_{D(0,n)^c})\, G_{D(0,n)}(0,0) \\ &= \frac{\log(n/|x|) + O(|x|^{-1/4})}{\log(n)} \left(1 + O((\log n)^{-1}\right) \left(\frac{2}{\pi_\Gamma}\log n + C' + O(n^{-1/4})\right) \\ &= \frac{2}{\pi_\Gamma}\log\left(\frac{n}{|x|}\right) + C + O(|x|^{-1/4}), \end{aligned} \tag{2.55}$$

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{0}) = \frac{2}{\pi_\Gamma}\log\left(\frac{n}{|\hat{x}|}\right) + C + O(|\hat{x}|^{-1/4}) \tag{2.56}$$

$$G_{D(0,n)}(x,z) \le G_{D(x,2n)}(0,z-x) \le c\log n. \tag{2.57}$$

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{z}) = G_{D(0,n)}(x,z) + O(K^{-M}n^2\log n) \le c\log n. \tag{2.58}$$

Finally, we have the following result paralleling (2.30). Recall that $\rho(\hat{x}) = n - |\hat{x}|$.

Lemma 2.8. *For any* $0 < \delta < \varepsilon < 1$ *we can find* $0 < c_1 < c_2 < \infty$*, such that for all* $\hat{x} \in \hat{\pi}_K(D(0,n)) \setminus \hat{\pi}_K(D(0,\varepsilon n))$*,* $\hat{y} \in \hat{\pi}_K(D(0,\delta n))$ *and all* n *sufficiently large such that* $2n < K/2$*,*

$$c_1\frac{\rho(\hat{x}) \vee 1}{n} \le \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{y},\hat{x}) \le c_2\frac{\rho(\hat{x}) \vee 1}{n}. \tag{2.59}$$

Proof We follow our standard technique. [3, Lemma 2.2, (2.38)] gives bounds for the planar random walk's Green's function. The toral version of this Green's function has a lower bound of the planar version:

$$c_1\frac{\rho(x) \vee 1}{n} \le G_{D(0,n)}(y,x) \le \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{y},\hat{x}). \tag{2.60}$$

For the upper bound, use (2.58) and [3, Lemma 2.2] again, for some constant c_2:

$$\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{y},\hat{x}) \le G_{D(0,n)}(y,x) + O(K^{-M}n^2\log n) \le c_2\frac{\rho(x) \vee 1}{n}. \quad \square$$

3 Disc Entry

Here we will examine paths starting outside a disc. Since, on $\mathbb{Z}^2$,

$$\partial D(0,n)_s \subset \left\{ \begin{array}{c} \pi_K^{-1}\partial D(0,n)_s \\ D(0,n+s) \end{array} \right\} \subset \pi_K^{-1} D(0,n+s), \tag{3.1}$$

then starting at any $y \in \pi_K^{-1}(D(0,n+s)^c \cap \pi_K \mathbb{Z}^2)$ (as seen in Figure 2.1) yields the *disc entrance time inequalities*

$$T_{\pi_K^{-1} D(0,n+s)} \leq \left\{ \begin{array}{c} T_{\pi_K^{-1}\partial D(0,n)_s} \\ T_{D(0,n+s)} \end{array} \right\} \leq T_{\partial D(0,n)_s}. \tag{3.2}$$

These relationships will be exploited in this and the next chapter.

3.1 External Green's function

To supplement the internal Green's functions of Chapter 2 are external Green's functions: those counting the number of visits to a point outside of a set before entering that set. Wlog x and $D(0,n)$ are in the primary copy. We will find bounds on three different external Green's functions:

Green's function	scope	starting at	counts visits to	before...
$G_{D(0,n)^c}(x,y)$	planar	x	y	$T_{D(0,n)}$
$G_{\pi_K^{-1}(D(0,n)_K^c)}(x,y)$	planar	x	y	$T_{\pi_K^{-1}D(0,n)} = T_{\hat{\pi}_K(D(0,n))}$
$\hat{G}_{\hat{\pi}_K(D(0,n)_K^c)}(\hat{x},\hat{y})$	toral	$\hat{x}$	$\hat{y}$	$T_{\pi_K^{-1}D(0,n)} = T_{\hat{\pi}_K(D(0,n))}$

Note that, similar to (2.37), for any $x, y \in D(0,n)^c$, by the symmetry of G_A and the strong Markov property at T_x,

$$G_{D(0,n)^c}(x,y) = P^y(T_x < T_{D(0,n)})\, G_{D(0,n)^c}(x,x), \tag{3.3}$$

so, assuming $|x| < |y|$, we only need $G_{D(0,n)^c}(x,x)$ for an upper bound. Fix $j > 2$ and let

$$\begin{aligned} U_0 &= 0, \\ V_i &= \min\{t > U_i : |S_t| < n \text{ or } |S_t| > |x|^j\}, \\ U_{i+1} &= \min\{t > V_i : S_t = x\}. \end{aligned}$$

U_i is the ith visit to x after visiting $D(0,n)$ or $D(0,|x|^j)^c$ (noting that there can be multiple visits to x before V_i, but none in the interval $V_i \le t < U_{i+1}$). Hence,

$$\begin{aligned} G_{D(0,n)^c}(x,x) &= \mathbb{E}^x\left(\sum_{i=1}^{\infty} 1_{\{x\}}(S_t)\, 1_{\{t<T_{D(0,n)}\}}\right) \\ &\le \sum_{i=1}^{\infty} \mathbb{E}^x\left(\sum_{U_i<t<V_i} 1_{\{x\}}(S_t)\,;\, U_i < T_{D(0,n)}\right) \\ &\le \sum_{i=1}^{\infty} \mathbb{E}^x\left(G_{D(0,|x|^j)}(S_{U_i},x)\,;\, U_i < T_{D(0,n)}\right) \end{aligned}$$

by the strong Markov property at U_i. Since $S_{U_i} = x$, and setting $a_i = a_i(x) := P^x(U_i < T_{D(0,n)})$, we have

$$G_{D(0,n)^c}(x,x) \le \sum_{i=1}^{\infty} \mathbb{E}^x \left(G_{D(0,|x|^j)}(S_{U_i}, x)\,;\, U_i < T_{D(0,n)} \right) = G_{D(0,|x|^j)}(x,x) \left(\sum_{i=1}^{\infty} a_i \right).$$

To sum the a_i, note that by strong Markov at U_i again,

$$\begin{aligned} a_{i+1} &\le \mathbb{E}^x \left(P^{S_{U_i}}(T_{D(0,|x|^j)^c} < T_{D(0,n)})\,;\, U_i < T_{D(0,n)} \right) \\ &= P^x(U_i < T_{D(0,n)})\, P^x(T_{D(0,|x|^j)^c} < T_{D(0,n)}) \\ \implies a_{i+1} &\le a_i P^x(T_{D(0,|x|^j)^c} < T_{D(0,n)}), \end{aligned}$$

where, by (2.51), and for sufficiently large n,

$$\begin{aligned} P^x(T_{D(0,|x|^j)^c} < T_{D(0,n)}) &= \frac{\log(|x|/n) + O(n^{-1/4})}{\log(|x|^j/n)} \\ &\le \frac{1 + O(n^{-1/4})}{j} \le \frac{2}{j}. \end{aligned} \tag{3.4}$$

Hence, $a_{i+1} \le \frac{2}{j} a_i$, which implies $a_i \le (2/j)^i$, and so by (2.45),

$$\begin{aligned} G_{D(0,n)^c}(x,x) &\le G_{D(0,|x|^j)}(x,x) \sum_{i=1}^{\infty} \left(\frac{2}{j} \right)^i \\ &\le \frac{2/j}{1 - 2/j} G_{D(0,|x|^j + |x|)}(0,0) \\ &\le \frac{2}{j-2} G_{D(0,2|x|^j)}(0,0) \le \frac{2j}{j-2} G_{D(0,2|x|)}(0,0). \end{aligned} \tag{3.5}$$

Moving the external Green's function to the torus is a bit trickier: we must examine the conflict between counting visits to an infinite number of planar copies of x in $\hat{\pi}_K^{-1}\hat{x} = \pi_K^{-1}x$ versus avoiding an infinite number of copies of $D(0,n)$ in

$\hat{\pi}_K^{-1}\hat{\pi}_K(D(0,n)) = \pi_K^{-1}D(0,n)$. Also, the size of j (via $|\hat{x}|^j$) is restricted relative to K. We use the same argument as before, with the adjustment of (2.54) applied to the argument of (3.4), yielding

$$\begin{aligned} P^{\hat{x}}(T_{\hat{\pi}_K(D(0,|\hat{x}|^j)^c_K)} < T_{\hat{\pi}_K(D(0,n))}) &= P^x(T_{D(0,|x|^j)^c} < T_{D(0,n)}) + O(K^{-M}n^2) \\ &\le \frac{2}{j} + O(K^{-M}n^2). \end{aligned}$$

This gives the toral analog $\hat{a}_i \le \left(\frac{2}{j} + O(K^{-M}n^2)\right)^i$, and the sum $\sum_{i=1}^{\infty} \hat{a}_i$ gives the slightly different bound of $\sum_{i=1}^{\infty} \hat{a}_i \le \dfrac{2}{j-2} + O(K^{-M}n^2)$. This gives a toral upper bound of

$$\begin{aligned} \hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x},\hat{x}) &\le \hat{G}_{\hat{\pi}_K(D(0,|x|^j))}(\hat{x},\hat{x}) \left(\sum_{i=1}^{\infty} \hat{a}_i\right) \\ &\le \left(\frac{2}{j-2} + O(K^{-M}n^2)\right) j\hat{G}_{\hat{\pi}_K(D(0,2|x|))}(0,0). \end{aligned} \tag{3.6}$$

Easy lower bounds for both the planar and toral cases are found by merely considering the visits to x in the disc $D(x,|x|-n)$, whose boundary rests just outside $D(0,n)$. This bound, (3.5), and (3.6) give, for $j > 2$,

$$G_{D(0,|x|-n)}(0,0) \le G_{D(0,n)^c}(x,x) \le \frac{2j}{j-2} G_{D(0,2|x|)}(0,0) \tag{3.7}$$

$$\begin{aligned} \hat{G}_{\hat{\pi}_K(D(0,|x|-n))}(\hat{0},\hat{0}) &\le \hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x},\hat{x}) \\ &\le \left(\frac{2}{j-2} + O(K^{-M}n^2)\right) j\hat{G}_{\hat{\pi}_K(D(0,2|x|))}(\hat{0},\hat{0}). \end{aligned} \tag{3.8}$$

Thus, for any $x, y \in \pi_K(D(0,n)^c_K)$ such that $|x| \le |y|$, by (2.58) and (2.46), (3.7) and (3.8) become the computational bounds

$$G_{D(0,n)^c}(x,y) \le \frac{2j}{j-2}\left[\frac{2}{\pi_\Gamma}\log(2|x|) + C + O(|x|^{-1/4})\right] \le c_j \log|x|, \quad (3.9)$$

$$\hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x},\hat{y}) \le \frac{2j}{j-2}\left[\frac{2}{\pi_\Gamma}\log(2|x|) + C + O(|x|^{-1/4})\right] \le \hat{c}_j \log|\hat{x}|, (3.10)$$

where $c_j, \hat{c}_j$ depend on $j > 2$, $c_j \ge \hat{c}_j$, and in the toral case, such that $|\hat{x}| < (\frac{K}{2})^{1/j}$ (there is no such restriction on the planar case). From here on, we consider $j = 3$.

For $\left(\frac{K}{2}\right)^{1/3} < |x| < \frac{K}{2}$, first note that $\log|x| \asymp \log K$. By (3.9) and the fact that $\pi_K^{-1}(D(0,n)^c \cap \pi_K\mathbb{Z}^2) = \hat{\pi}_K^{-1}\hat{\pi}_K(D(0,n)^c_K) \subset D(0,n)^c$, on $\mathbb{Z}^2$ we have

$$G_{\hat{\pi}_K^{-1}\hat{\pi}_K(D(0,n)^c_K)}(x,x) \le G_{D(0,n)^c}(x,x) \le c\log|x| \quad (3.11)$$

for any $x \in \pi_K(D(0,n)^c_K)$. We can relate $G_{D(0,n)^c}(x,x)$ to $\hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x},\hat{x})$ by the following inductive strategy: define

$$\begin{aligned} T_0 &= 0 \\ T_1 &= \inf\left\{t > 0 \,|\, S_t \in \hat{\pi}_K^{-1}\hat{x} \setminus \{x\}\right\} \\ &\vdots \\ T_{j+1} &= \inf\left\{t > 0 \,|\, S_t \in \hat{\pi}_K^{-1}\hat{x} \setminus \{S_{T_j}\}\right\},\ j = 1,2,3,\ldots \end{aligned}$$

as the hitting times of distinct copies of x. Let $U = T_{\hat{\pi}_K(D(0,n))}$. Then

$$\hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x},\hat{x}) = \sum_{j=0}^{\infty}\mathbb{E}^x\left(\sum_{T_j \le t < T_{j+1}\wedge U} 1_{\{S_t = S_{T_j}\}}\right).$$

By the Markov property, translation invariance for different points of $\hat{\pi}_K^{-1}\hat{x}$, and (3.11),

$$\begin{aligned}\mathbb{E}^x\left(\sum_{T_j\le t<T_{j+1}\wedge U} 1_{\{S_t=S_{T_j}\}}\right) &= \mathbb{E}^x\left(\sum_{T_j\le t<T_{j+1}\wedge U} 1_{\{S_t=S_{T_j}\}}, T_j<U\right)\\ &= P^x(T_j<U)\mathbb{E}^x\left(\sum_{0\le t<T_1\wedge U} 1_{\{S_t=x\}}\right)\\ &\le P^x(T_j<U)G_{D(0,n)^c}(x,x).\end{aligned}$$

Hence,

$$\hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x},\hat{x}) \le G_{D(0,n)^c}(x,x)\sum_{j=0}^{\infty} P^x(T_j<U). \tag{3.12}$$

Let $\rho = P^x(T_1<U)$. By the Markov property, $P^x(T_j<U)\le \rho^j$, so since $\rho<1$, by (3.11) and (3.12), we have, for some $C<\infty$,

$$\hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x},\hat{x}) \le \frac{C\log K}{1-\rho}. \tag{3.13}$$

Examining ρ, note that

$$\begin{aligned}\{T_1<U\} &= \left\{T_{\pi_K^{-1}x\setminus\{x\}} < T_{\hat{\pi}_K(D(0,n))}\right\}\\ &\subset \left\{T_{D(0,K)^c} < T_{\hat{\pi}_K(D(0,n))}\right\} \subset \left\{T_{D(0,K)^c} < T_{D(0,n)}\right\},\end{aligned}$$

which, by (2.52) gives the bound

$$\rho \le 1 - P^x\left(T_{D(0,n)} < T_{D(0,K)^c}\right) \le 1 - \frac{\log(K/|x|)+O(n^{-1/4})}{\log(K/n)} \le 1-\frac{c}{\log K} \tag{3.14}$$

for some $c < \infty$. Therefore, for $(\frac{K}{2})^{1/3} \leq |x| < \frac{K}{2}$, combining (3.13) and (3.14) we have the upper bound

$$\hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x}, \hat{x}) \leq C(\log K)^2. \tag{3.15}$$

Finally, for $|x| \geq \frac{K}{2}$, and considering $x = \pi_K x$, condition on the quadrant containing S_{T_1} via a gambler's ruin from the opposite quadrant. For example, as in Figure 3.1, if S_{T_1} is in Quadrant 1, let $y = (-K, -K)$ (opposite, in $Q3$). Then there exists c, $\left(\sqrt{2} + \frac{1}{2}\right) K \leq c < 2\sqrt{2}$ (say, $c = \sqrt{2} + \frac{1}{2}$ if $x \notin Q1$, and $c = \frac{7}{4}\sqrt{2}$ if $x \in Q1$), such that, again by (2.52), we have the bound

$$\begin{aligned} P^x(T_1 < U \mid S_{T_1} \in Qi) &\leq 1 - P^x\left(T_{D(y,n)} < T_{D(y,cK)^c}\right) \\ &\leq 1 - \frac{\log(cK/|x-y|) + O(n^{-1/4})}{\log(cK/n)} \leq 1 - \frac{c'}{\log K} \end{aligned} \tag{3.16}$$

for some $c' < \infty$.

Thus, we have a bound on ρ of

$$\rho = \sum_{i=1}^{4} P^x(T_1 < U \mid S_{T_1} \in Qi) P(S_{T_1} \in Qi) \leq 1 - \frac{c^*}{\log K} \tag{3.17}$$

for some $c^* < \infty$. Therefore, by (3.13) and (3.17), (3.15) holds for $|x| > \frac{K}{2}$. Collecting these results and applying them to the torus, we have proven

Lemma 3.1. *For $\hat{x} \in \hat{\pi}_K(D(0,n)^c_K)$,*

$$\hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x}, \hat{x}) \leq \begin{cases} C\log|\hat{x}| & n < |\hat{x}| < \left(\frac{K}{2}\right)^{1/3} \\ C\log^2|\hat{x}| & \left(\frac{K}{2}\right)^{1/3} \leq |\hat{x}|. \end{cases} \tag{3.18}$$

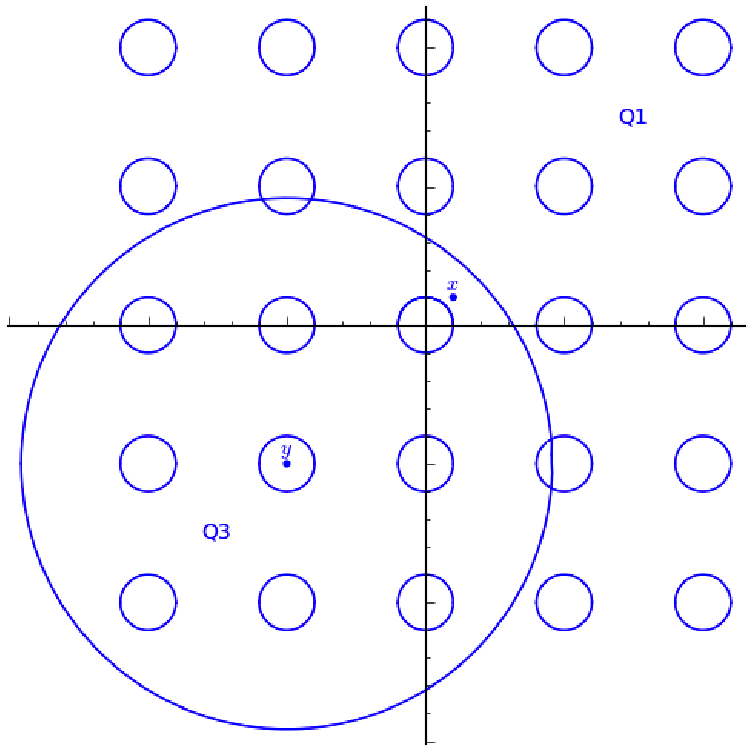

Figure 3.1: $x \in Q1 \cap D(0,n)^c$, $y = (-K,-K)$.

3.2 Disc entrance time

We will now approach disc entrance times. Our first planar result mirrors (2.4), with a very different end result, which is hinted by the first passage time result for SRW on $\mathbb{Z}$ (in, for example, [23]).

Lemma 3.2. *For any* $y \in D(0,n)^c$,

$$\mathbb{E}^y(T_{D(0,n)}) = \infty. \tag{3.19}$$

Proof As in Lemma 2.1, we use the martingale $M_t = |S_t|^2 - \gamma^2 t$, only this time we stop it at the stopping time $T_N := T_{D(0,n)} \wedge T_{D(0,N)^c}$, for $n \leq |y| < N$. Thus, the martingale stopped at $T_N \wedge k$ has expected value

$$\mathbb{E}^y(M_{T_N \wedge k}) = |y|^2 = \mathbb{E}^y\left(|S_{T_N \wedge k}|^2 - \gamma^2(T_N \wedge k)\right).$$

It is clear that $T_N \leq T_{D(0,N)^c}$, so by the argument given in the proof of Lemma 2.1, we have $\mathbb{E}^y\left(|S_{T_N \wedge k}|^2\right) \to \mathbb{E}^y\left(|S_{T_N}|^2\right)$ and $\mathbb{E}^y(T_N \wedge k) \to \mathbb{E}^y(T_N)$ as $k \to \infty$. Hence, as in Lemma 2.1, letting $\mathbb{E}^y(M_{T_N \wedge k}) \to \mathbb{E}^y(M_{T_N})$, we have

$$\mathbb{E}^y(T_N) = \frac{\mathbb{E}^y\left(|S_{T_N}|^2\right) - |y|^2}{\gamma^2}.$$

Using (2.52), we decompose $\mathbb{E}^y\left(|S_{T_N}|^2\right)$ and achieve the lower bound

$$\begin{aligned}\mathbb{E}^y\left(|S_{T_N}|^2\right) &= \mathbb{E}^y\left(|S_{T_N}|^2 \,|\, T_N = T_{D(0,n)}\right) P^y(T_{D(0,n)} < T_{D(0,N)^c}) \\ &\quad + \mathbb{E}^y\left(|S_{T_N}|^2 \,|\, T_N = T_{D(0,N)^c}\right) P^y(T_{D(0,n)} > T_{D(0,N)^c}) \\ &\geq 0 + N^2 P^y(T_{D(0,n)} > T_{D(0,N)^c}) \\ &\geq \frac{N^2\left[\log\left(|y|/n\right) + O(n^{-1/4})\right]}{\log\left(N/n\right)} \geq cN\end{aligned}$$

for some $c < \infty$. This gives us a lower bound on the expected entrance time of

$$\mathbb{E}^y(T_N) \geq \frac{cN - |y|^2}{\gamma^2},$$

which clearly goes to ∞ as $N \to \infty$. $\square$

Next, we find finite bounds on the expected time to enter a toral disc.

Lemma 3.3. *For any $n < \frac{K}{6}$ and $\hat{y} \in \hat{\pi}_K(D(0,n)^c_K)$, there exists $c < \infty$ such that*

$$\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,n))}) \leq \begin{cases} cK^2 \log(n) & n < |\hat{y}| < n^2 \\ cK^2 \log\left(\frac{|\hat{y}|}{n}\right) & n^2 \leq |\hat{y}| < \left(\frac{K}{2}\right)^{1/3} \\ cK^2 (\log|\hat{y}|)^2 & \left(\frac{K}{2}\right)^{1/3} \leq |\hat{y}|. \end{cases} \tag{3.20}$$

Also, we have the lower bound

$$\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,n))}) \geq \begin{cases} \frac{(|\hat{y}|-n)^2}{\gamma^2} & |\hat{y}| < \frac{K}{3} \\ \frac{c(K-n)^2}{\gamma^2} & |\hat{y}| \geq \frac{K}{3} \end{cases} \tag{3.21}$$

where γ^2 is as in the proof of Lemma 3.2.

Proof For the upper bound, let $\hat{y} \in \hat{\pi}_K(D(0,n)^c_K)$. We have the decomposition

$$\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,n))}) = \sum_{\hat{z} \in \hat{\pi}_K(D(0,n)^c_K)} \hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{y}, \hat{z}),$$

which, for $|\hat{y}| > \left(\frac{K}{2}\right)^{1/3}$, by (3.18) is clearly bounded above by $cK^2(\log K)^2$. For closer y, we further decompose to

$$\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,n))}) = \sum_{\hat{z} \in \hat{\pi}_K(D(0,|\hat{y}|)) \setminus \hat{\pi}_K(D(0,n))} \hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{y}, \hat{z})$$
$$+ \sum_{z \in \hat{\pi}_K(D(0,|\hat{y}|)^c)} \hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{y}, \hat{z}),$$

which, by (3.18), is bounded by

$$\begin{aligned}\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,n))}) &\leq \sum_{\hat{z}:\, n\leq|\hat{z}|<|\hat{y}|} c\log|\hat{z}| \;+\; \sum_{\hat{z}:\,|\hat{y}|<|\hat{z}|} c\log|\hat{y}| \\ &\leq c\int_0^{2\pi}\int_n^{|\hat{y}|} w\log w\,dw\,d\theta \;+\; c(K^2-\pi|\hat{y}|^2)\log|\hat{y}| \\ &\leq 2\pi c\left(\frac{1}{2}|\hat{y}|^2\log|\hat{y}| - \frac{|\hat{y}|^2}{4} - n^2\log n + \frac{n^2}{4}\right) \;+\; c(K^2-\pi|\hat{y}|^2)\log|\hat{y}| \\ &\leq cK^2\log|\hat{y}| \leq cK^2\log\left(\frac{|\hat{y}|}{n}\right) + cK^2\log n - 2\pi cn^2\log n.\end{aligned}$$

If $|\hat{y}| < \frac{K}{3}$, then $\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,n))}) \geq \mathbb{E}^{\hat{y}}\left(T_{\hat{\pi}_K(D(\hat{y},|\hat{y}|-n)^c_K)}\right)$, and (3.21) follows directly from (2.28). The far-off $|\hat{y}| \geq \frac{K}{3}$ follows directly from the nearby case, since by the strong Markov property at $T_{\hat{\pi}_K(D(0,K/3))}$,

$$\begin{aligned}\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,n))}) &\geq \mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,K/3))};\, T_{\hat{\pi}_K(D(0,K/3))} = T_{\hat{\pi}_K(\partial D(0,K/6)_{K/6})}) \\ &\quad + \mathbb{E}^{\hat{y}}(\mathbb{E}^{\hat{S}_{T_{\hat{\pi}_K(\partial D(0,K/6)_{K/6})}}}(T_{\hat{\pi}_K(D(0,n))})) \geq \frac{c(K-n)^2}{\gamma^2}. \quad \square\end{aligned}$$

(3.20) hints at the cover time (and so, late points) results of Chapter 7. We will improve on these bounds in our discussion on excursions.

4 Annulus Entry

If you can't solve it, generalize it.

- Herbert Wilf

In this chapter we will find general Green's functions, hitting times, and hitting distributions by a symmetric recurrent random walk X on a set partitioned into three pieces. We then apply these results to the partition of disc, annulus, and "outside" to relate our results from Chapters 2 and 3 to the annulus. We conclude by finding tailored gambler's ruin-based probabilities and hitting distribution bounds for annuli.

4.1 Bounds on a three-partitioned set

Let $A \sqcup B \sqcup C$ partition our sample space (for our purposes, this is $\mathbb{Z}^2$ or $\mathbb{Z}_K^2$, but everything in this section generalizes). We find estimates for the Green's function $G_{A\cup B}$ and the hitting time $E^x(T_C)$ for $x, y \in A \cup B$, with interest in the case where C "separates" A and B in a sense (*i.e.*, the probability of jumping from A to B, or vice versa, without hitting C, is small). This gives a notion for how probabilistically "separate" they are.

4.1.1 Green's functions

Simple lower bounds for the Green's function $G_{A\cup B}$ are obvious; to find upper bounds for these cases, we analyze excursions between A and B before hitting C.

Lemma 4.1. *For $a, a' \in A$ and $b, b' \in B$, with θ_t the usual shift operators,*

$$T_B^* := \inf\{t > T_A : X_t \in B\} = T_A + T_B \circ \theta_{T_A},$$
$$T_A^* := \inf\{t > T_B : X_t \in A\} = T_B + T_A \circ \theta_{T_B},$$

and defining

$$\psi_a := \sum_{b' \in B} H_{B\cup C}(a, b') \qquad = P^a(T_B < T_C) \tag{4.1}$$
$$\sigma_b := \sum_{a' \in A} H_{A\cup C}(b, a') \qquad = P^b(T_A < T_C) \tag{4.2}$$
$$\rho_a := \sum_{b' \in B} H_{B\cup C}(a, b')\sigma_{b'} \qquad = P^a(T_B, T_A^* < T_C) \tag{4.3}$$
$$\phi_b := \sum_{a' \in A} H_{A\cup C}(b, a')\psi_{a'} \qquad = P^b(T_A, T_B^* < T_C), \tag{4.4}$$

we have the Green's function bounds

$$G_A(a, a') \le G_{A\cup B}(a, a') \le G_A(a, a') + \frac{\rho_a}{1 - \rho_{a'}} G_A(a', a') \tag{4.5}$$
$$G_B(b, b') \le G_{A\cup B}(b, b') \le G_B(b, b') + \frac{\phi_b}{1 - \phi_{b'}} G_B(b', b') \tag{4.6}$$
$$0 \le G_{A\cup B}(a, b) \le \min\left\{ \frac{\sigma_b}{1 - \rho_a} G_A(a, a), \frac{\psi_a}{1 - \phi_b} G_B(b, b) \right\}. \tag{4.7}$$

Recall that G is symmetric, so the inputs can be swapped in any of these bounds. Also, by their definitions, $\psi_a \ge \rho_a$ for every $a \in A$ and $\sigma_b \ge \phi_b$ for every $b \in B$.

Proof We will prove this for (4.5) and (4.7) (the proof for (4.6) matches (4.5)'s proof). By (2.15), for $a, a' \in A$,

$$\begin{aligned} G_{A\cup B}(a,a') &= \sum_{i=0}^{\infty} P^a(X_i = a', i < T_C) \\ &= \sum_{i=0}^{\infty}[P^a(X_i = a', i < T_C, i < T_B) + P^a(X_i = a', T_B < i < T_C)] \\ &= G_A(a,a') + \sum_{i=0}^{\infty} P^a(X_i = a', T_B < i < T_C). \end{aligned} \tag{4.8}$$

Since $a' \in A$, once the walk enters B it must return to A before hitting a' again. By splitting and switching sums and applying the strong Markov property at T_B,

$$\begin{aligned} G_{A\cup B}(a,a') &= G_A(a,a') + \sum_{i=0}^{\infty}\sum_{b\in B} P^a(X_{T_B} = b, X_i = a', T_B < i < T_C) \\ &= G_A(a,a') + \sum_{b\in B} H_{B\cup C}(a,b) G_{A\cup B}(b,a'). \end{aligned} \tag{4.9}$$

We now switch from (4.5) to (4.7): for $G_{A\cup B}(b,a')$, with $b \in B$ and $a' \in A$, decomposing over A, and using the strong Markov property at T_A,

$$\begin{aligned} G_{A\cup B}(b,a') &= \sum_{i=0}^{\infty} P^b(X_i = a', i < T_C) \\ &= \sum_{i=0}^{\infty}\sum_{a''\in A} P^b(X_i = a', T_A \le i < T_C, X_{T_A} = a'') \\ &= \sum_{a''\in A} H_{A\cup C}(b,a'') G_{A\cup B}(a'',a'). \end{aligned} \tag{4.10}$$

We thus have a recurrence relation between (4.5) and (4.7).

By the symmetry of G_A and a generalization of (2.37), we have the upper bound

$$G_A(a'', a') = P^{a''}(T_{a'} < T_{A^c})G_A(a', a') \leq G_A(a', a') \tag{4.11}$$

which yields, by (4.2),

$$G_{A\cup B}(b, a') = \sum_{a'' \in A} H_{A\cup C}(b, a'')G_{A\cup B}(a'', a') \leq \sigma_b G_{A\cup B}(a', a'). \tag{4.12}$$

Combining (4.9), (4.12), and (4.3) gives us

$$\begin{aligned} G_{A\cup B}(a, a') &= G_A(a, a') + \sum_{b\in B} H_{B\cup C}(a, b)G_{A\cup B}(b, a') \\ &\leq G_A(a, a') + G_{A\cup B}(a', a')\sum_{b\in B} H_{B\cup C}(a, b)\sigma_b \\ &= G_A(a, a') + G_{A\cup B}(a', a')\rho_a. \end{aligned} \tag{4.13}$$

In particular, (4.13) gives us

$$G_{A\cup B}(a', a') \leq \frac{G_A(a', a')}{1 - \rho_{a'}}. \tag{4.14}$$

(4.14) used again in (4.13) yields (4.5). Proving (4.6) similarly, (4.6) and (4.14) applied to (4.12) yields (4.7). □

4.1.2 Hitting times

We now find the expected time of hitting the set C, starting from A, in terms of hitting $B \cup C$. Lower bounds are simple: just tack the other set on for a quicker hitting time. The upper bounds will require a recursive excursion treatment similar to the proof of Lemma 4.1.

Lemma 4.2. *For $a \in A$ and $b \in B$, defining via* (4.1) *and* (4.2),

$$f_A := \sup_{a \in A} E^a(T_{B\cup C}), \quad f_B := \sup_{b \in B} E^b(T_{A\cup C}), \quad \psi := \sup_{a \in A} \psi_a, \quad \sigma := \sup_{b \in B} \sigma_b, \tag{4.15}$$

we have the expected hitting time bounds

$$E^a(T_{B\cup C}) \le E^a(T_C) \le E^a(T_{B\cup C}) + \psi_a \left[\frac{f_B + \sigma f_A}{1 - \psi\sigma}\right] \tag{4.16}$$

$$E^b(T_{A\cup C}) \le E^b(T_C) \le E^b(T_{A\cup C}) + \sigma_b \left[\frac{f_A + \psi f_B}{1 - \psi\sigma}\right] \tag{4.17}$$

Proof We will prove (4.16) (the proof of (4.17) is the same). First, decompose T_C along the two possibilities for $T_{B\cup C}$. Recall that $T_{B\cup C} = T_C \iff T_C < T_B$. By the strong Markov property at T_B,

$$\begin{aligned} E^a(T_C) &= E^a(T_C 1_{\{T_{B\cup C} = T_C\}}) + E^a(T_C 1_{\{T_{B\cup C} = T_B\}}) \\ &\le E^a(T_{B\cup C}) + \sum_{b \in B} H_{B\cup C}(a,b) E^b(T_C). \end{aligned} \tag{4.18}$$

Likewise, for $b \in B$,

$$E^b(T_C) \le E^b(T_{A\cup C}) + \sum_{a' \in A} H_{A\cup C}(b,a') E^{a'}(T_C). \tag{4.19}$$

By combining (4.18) and (4.19), recursing on itself, keeping the first couple terms in terms of a, and maximizing the rest via (4.1), (4.2), and (4.15), we get

$$E^a(T_C) \le E^a(T_{B\cup C}) + \sum_{b\in B} H_{B\cup C}(a,b)\left(E^b(T_{A\cup C}) + \sum_{a'\in A} H_{A\cup C}(b,a')\left[E^{a'}(T_C)\right]\right)$$
$$\le E^a(T_{B\cup C}) + \sum_{b\in B} H_{B\cup C}(a,b)\left(E^b(T_{A\cup C}) + \sum_{a'\in A} H_{A\cup C}(b,a')\left[f_A + \psi(f_B + \sigma[...])\right]\right),$$

which is bounded by

$$E^a(T_C) \le E^a(T_{B\cup C}) + \psi_a\left(f_B + \sigma[f_A + \psi(f_B + \sigma[...])]\right)$$
$$= E^a(T_{B\cup C}) + \psi_a\left(f_B + \sigma f_A + (\sigma\psi)f_B + (\sigma\psi)\sigma f_A + (\sigma\psi)^2 f_B + (\sigma\psi)^2\sigma f_A + ...\right)$$
$$= E^a(T_{B\cup C}) + \psi_a(f_B + \sigma f_A)\sum_{i=0}^{\infty}(\psi\sigma)^i = E^a(T_{B\cup C}) + \frac{\psi_a(f_B+\sigma f_A)}{1-\psi\sigma}. \quad \square$$

4.1.3 Hitting distributions

(2.19) and (2.20) hint at a relationship between the hitting distributions of two sets C and $C \cup A$. We find a bound on this relationship. Let $b \in B$ and $c \in C$. By (2.19), there is a probability $p(C, A)$ such that

$$H_C(b,c) = H_{C\cup A}(b,c) + p(C,A). \tag{4.20}$$

To bound $p(C, A)$, we rewrite using the definition of $H_C(b, c)$ and decompose along the event $\{T_C < T_A\}$ (whose probability is $1 - \sigma_b$ in (4.2)):

$$H_C(b,c) = P^b(S_{T_C} = c) \quad = P^b(S_{T_C} = c,\, T_C < T_A) + P^b(S_{T_C} = c,\, T_A < T_C);$$
$$H_{C\cup A}(b,c) = P^b(S_{T_{C\cup A}} = c) \quad = P^b(S_{T_{C\cup A}} = c,\, T_C < T_A) + P^b(S_{T_{C\cup A}} = c,\, T_A < T_C).$$

Note that

$$P^b(S_{T_C} = c,\, T_C < T_A) = P^b(S_{T_{C\cup A}} = c,\, T_C < T_A)$$

and

$$S_{T_{C\cup A}} = c \in C \implies T_C < T_A,$$

so clearly $P^b(S_{T_{C\cup A}} = c,\, T_A < T_C) = 0$ and we get the simple bound

$$p(C, A) = P^b(S_{T_C} = c,\, T_A < T_C) \le P^b(T_A < T_C) = \sigma_b. \tag{4.21}$$

Note that $p(C, A)$ is *not* symmetric; *e.g.*, $p(A, C) = 1 - p(C, A) = 1 - \sigma_b$.

4.2 Application: Internal-External-Annulus

Let the following sets partition $\mathbb{Z}_K^2$, with $s \le n < K \in \mathbb{N}$:

$$A = \hat{\pi}_K(D(0, n)),\;\; B = \hat{\pi}_K(D(0, n + s)_K^c),\;\; C = \hat{\pi}_K(\partial D(0, n)_s).$$

4.2.1 Hitting probabilities

Starting from deep inside a disc, we first prove a bound on the probability of escaping the disc beyond an annulus outside it.

Lemma 4.3.

$$\sup_{x \in D(0,n/2)} P^x(T_{\partial D(0,n)_s} > T_{D(0,n+s)^c}) \le c(s^{-M+2} \vee n^{-M+2}). \tag{4.22}$$

$$\psi = \sup_{\hat{x} \in \hat{\pi}_K(D(0,n/2))} P^{\hat{x}}(T_{\hat{\pi}_K(\partial D(0,n)_s)} > T_{\hat{\pi}_K(D(0,n+s)_K^c)}) \le c(s^{-M+2} \vee n^{-M+2}). \tag{4.23}$$

Proof To deal with targeting jumps that are possible on the torus, the proof of (4.23) is below. Replacing $\hat{p}_1(\hat{y},\hat{w})$ with $p_1(y,w)$, $\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{y})$ with $G_{D(0,n)}(x,y)$, (2.28) with (2.4), and (2.59) with its planar version yields the proof for (4.22).

For $\hat{x} \in \hat{\pi}_K(D(0,n/2))$, we begin, by the last exit decomposition and (2.22), with

$$\begin{aligned}
P^{\hat{x}}(T_{\hat{\pi}_K(\partial D(0,n)_s)} > T_{\hat{\pi}_K(D(0,n+s)^c_K)}) &= \sum_{\hat{w}\in\hat{\pi}_K(D(0,n+s)^c_K)} \sum_{\hat{y}\in\hat{\pi}_K(D(0,n))} \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{y})\hat{p}_1(\hat{y},\hat{w}) \\
&= \sum_{|\hat{y}|\le 3n/4} \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{y}) \sum_{|\hat{w}|\ge n+s} (p_1(y,w)+O(K^{-M})) \\
&\quad + \sum_{3n/4<|\hat{y}|<n} \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{y}) \sum_{|\hat{w}|\ge n+s} (p_1(y,w)+O(K^{-M})).
\end{aligned}$$

By (2.17), (2.28), and the facts that $|\hat{y}-\hat{w}| > n/4$ and $K > n$, the first sum has the bound

$$c \sum_{|\hat{y}|\le 3n/4} \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{y}) n^{-M} \le cn^{-M+2}.$$

By (2.59) and switching to a polar integral, the second sum is bounded by

$$\begin{aligned}
c \sum_{3n/4<|\hat{y}|<n} \frac{n-|\hat{y}|}{n}(n+s-|\hat{y}|)^{-M} &\le \frac{c}{n} \sum_{3n/4<|\hat{y}|<n} \frac{n-|\hat{y}|}{(n+s-|\hat{y}|)^M} \\
\le \frac{c}{n} \sum_{3n/4<|\hat{y}|<n} (n+s-|\hat{y}|)^{-M+1} &\le c\int_{3n/4}^{n} (n+s-u)^{-M+1}du \\
&\le c\int_0^{n/4} (s+v)^{-M+1}dv \le cs^{-M+2}. \quad \square
\end{aligned}$$

Note that for $\hat{x} \in \hat{\pi}_K(D(0,n))$, by (2.3),

$$\{T_{\hat{\pi}_K(\partial D(0,n)_s)} > T_{\hat{\pi}_K(D(0,n)^c_K)}\}^c = \{T_{\hat{\pi}_K(\partial D(0,n)_s)} = T_{\hat{\pi}_K(D(0,n)^c_K)}\}.$$

Hence, provided $\hat{x} \in \hat{\pi}_K(D(0,n/2))$, and $s \le n$, (4.23) is a bound for $\psi_{\hat{x}}$ from (4.1).

Also, (2.50) and (4.23) gives us the chance of escaping a disc, into its s-annulus, before visiting its center:

$$\begin{aligned} &P^{\hat{x}}(T_{\hat{0}} > T_{\hat{\pi}_K(D(0,n)^c_K)};\, T_{\hat{\pi}_K(D(0,n)^c_K)} = T_{\hat{\pi}_K(\partial D(0,n)_s)}) \\ &= 1 - \frac{\log(n/|\hat{x}|) + O(|\hat{x}|^{-1/4})}{\log n}(1 + O((\log n)^{-1}) + O(s^{-M+2}). \end{aligned} \tag{4.24}$$

By (4.23), (4.3), and (4.4), for $\hat{x} \in \hat{\pi}_K(D(0, n/2))$ and $\hat{y} \in \hat{\pi}_K(D(0, n+s)^c_K)$,

$$\begin{aligned} \rho_{\hat{x}} &= P^{\hat{x}}(T_{\hat{\pi}_K(D(0,n+s)^c_K)}, T^*_{\hat{\pi}_K(D(0,n))} < T_{\hat{\pi}_K(\partial D(0,n)_s)}) \\ &\leq c(s^{-M+2} \vee n^{-M+2}); \end{aligned} \tag{4.25}$$

$$\begin{aligned} \phi_{\hat{y}} &= P^{\hat{y}}(T_{\hat{\pi}_K(D(0,n))}, T^*_{\hat{\pi}_K(D(0,n+s)^c_K)} < T_{\hat{\pi}_K(\partial D(0,n)_s)}) \\ &\leq c(s^{-M+2} \vee n^{-M+2}). \end{aligned} \tag{4.26}$$

Next, we find a bound for $\sigma_{\hat{x}}$ from (4.2).

Lemma 4.4. *For n sufficiently large,*

$$\begin{aligned} \sigma &= \sup_{\hat{x} \in \hat{\pi}_K(D(0,n+s)^c_K)} P^{\hat{x}}(T_{\hat{\pi}_K(D(0,n))} < T_{\hat{\pi}_K(\partial D(0,n)_s)}) \\ &\leq cn^2 \log(n)^2 (s^{-M} + n^{-M}). \end{aligned} \tag{4.27}$$

Proof Apply (3.18) to the last exit decomposition to get

$$\begin{aligned}
&\sup_{\hat{x}\in\hat{\pi}_K(D(0,n+s)^c_K)} P^{\hat{x}}(T_{\hat{\pi}_K(D(0,n))} < T_{\hat{\pi}_K(\partial D(0,n)_s)}) \qquad (4.28)\\
&= \sup_{\hat{x}\in\hat{\pi}_K(D(0,n+s)^c_K)} \sum_{\substack{\hat{y}\in\hat{\pi}_K(D(0,n+s)^c)\\ \hat{w}\in\hat{\pi}_K(D(0,n))}} \hat{G}_{\hat{\pi}_K(D(0,n)^c_K)}(\hat{x},\hat{y})\hat{p}_1(\hat{y},\hat{w})\\
&\leq c\log(n)^2 \sum_{n+s\leq|\hat{y}|<2n} (|\hat{y}|-n)^{-M} + c\sum_{2n\leq|\hat{y}|} \log(|\hat{y}|)^2(|\hat{y}|-n)^{-M}\\
&\leq cn^2\log(n)^2 s^{-M} + c\log(n)^2 n^{-M+2}. \quad \square
\end{aligned}$$

In particular, if $s = O(n)$, since $M = 4 + 2\beta$, (4.27) is bounded above by cn^{-2}, and if $s = O(\sqrt{n})$, (4.27) is bounded above by $cn^{-\beta}$.

Combining (2.53) and (4.23), we find the probability that, starting far from a small disc $\hat{\pi}_K(D(0,r))$, the walk escapes a larger disc $\hat{\pi}_K(D(0,R))$ before entering $\hat{\pi}_K(D(0,r))$. If $r < R$ and $\hat{x} \in \hat{\pi}_K(D(0,R/2))$, we have

$$\begin{aligned}
&P^{\hat{x}}(T_{\hat{\pi}_K(D(0,R)^c_K)} < T_{\hat{\pi}_K(D(0,r))}; T_{\hat{\pi}_K(D(0,R)^c_K)} = T_{\hat{\pi}_K(\partial D(0,R)_s)})\\
&= \frac{\log(|\hat{x}|/r) + O(r^{-1/4})}{\log(R/r)} + O(s^{-M+2}). \qquad (4.29)
\end{aligned}$$

To enter a disc, we first quote the planar result [3, Lemma 2.4]: if $s < r < R$ sufficiently large with $R \leq r^2$ we can find $c < \infty$ and $\delta > 0$ such that for any $r < |x| < R$,

$$P^x(T_{D(0,r)} < T_{D(0,R)^c}; T_{D(0,r)} = T_{D(0,r-s)}) \leq cr^{-\delta} + cs^{-M+2}. \qquad (4.30)$$

We see the same result on $\mathbb{Z}^2_K$, with an extra toral term (which is absorbed).

Lemma 4.5. *For the conditions listed above,*

$$P^{\hat{x}}(T_{\hat{\pi}_K(D(0,r))} < T_{\hat{\pi}_K(D(0,R)^c_K)}; T_{\hat{\pi}_K(D(0,r))} = T_{\hat{\pi}_K(D(0,r-s))})$$
$$\leq cr^{-\delta} + cs^{-M+2}. \tag{4.31}$$

Proof Wlog we assume $\hat{x} \in \pi_K(D(0,R) \setminus D(0,r))$. Let

$$\hat{A} := \{T_{\hat{\pi}_K(D(0,r))} < T_{\hat{\pi}_K(D(0,R)^c_K)}; T_{\hat{\pi}_K(D(0,r))} = T_{\hat{\pi}_K(D(0,r-s))}\}.$$

Decompose $\hat{A}$ along the event $\{T_{\hat{\pi}_K(D(0,R)^c_K)} = T_{D(0,R)^c}\}$: by (2.3) and Figure 2.1,

$$P^{\hat{x}}(\hat{A}) = P^{\hat{x}}(\hat{A}; T_{\hat{\pi}_K(D(0,R)^c_K)} = T_{D(0,R)^c}) + P^{\hat{x}}(\hat{A}; T_{\hat{\pi}_K(D(0,R)^c_K)} > T_{D(0,R)^c}). \tag{4.32}$$

The first probability in (4.32) accounts for all walks that have no large jumps before the planar time $T_{D(0,r)}$, since

$$\hat{A} \cap \{T_{\hat{\pi}_K(D(0,R)^c_K)} = T_{D(0,R)^c}\} \subset \{T_{\hat{\pi}_K(D(0,r))} = T_{D(0,r)}\} \cap \{T_{\hat{\pi}_K(D(0,r))} = T_{\hat{\pi}_K(D(0,r-s))}\}.$$

Thus, (4.30) bounds the first probability. The second probability is bounded by (2.23), which, as $cK^{-M}n^2$, is absorbed by cs^{-M+2} since $s \leq R < K$. $\square$

We use (4.31) along with (2.54) to get the toral gambler's ruin-via-annulus estimate:

$$P^{\hat{x}}(T_{\hat{\pi}_K(D(0,r))} < T_{\hat{\pi}_K(D(0,R)^c_K)}; T_{\hat{\pi}_K(D(0,r))} = T_{\hat{\pi}_K(\partial D(0,r-s)_s)})$$
$$= \frac{\log(R/|\hat{x}|) + O(r^{-\delta})}{\log(R/r)} + O(s^{-M+2}). \tag{4.33}$$

We now give results on these probabilities for a finely-tuned set of radii and annuli which will appear in later chapters. For n large and $c > 0$ and set the following:*

$$r_{n,k} = e^n n^{3k}, s_k = n^4, \qquad r'_{n,k} = r_{n,k} + s_k, \qquad k = 0, 1, \dots, n;$$
$$s^{n\downarrow}_{n-1} = \sqrt{r_{n,n-1}}. \tag{4.34}$$

For large enough n, $n^4 < r^{\delta}_{n,l}$ for any $1/2 \le \delta < 1$, so for any $\hat{x} \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})$ and $1 \le l \le n-1$,

$$\begin{aligned} n^3 = \frac{r_{n,l}}{r_{n,l-1}} < \frac{|\hat{x}|}{r_{n,l-1}} < \frac{r_{n,l} + r^{\delta}_{n,l}}{r_{n,l-1}} < n^3 + e^{-n(1-\delta)} n^{-3l(1-\delta)+3} < n^3 + n^{-1} \\ \implies \log\left(\frac{|\hat{x}|}{r_{n,l-1}}\right) = 3\log n + O(n^{-4}), \end{aligned} \tag{4.35}$$

so by (4.29) and (4.35) we have

$$\begin{aligned} a_{l+1} &:= P^{\hat{x}}\left(T_{\hat{\pi}_K(D(0,r_{n,l+1})^c)} < T_{\hat{\pi}_K(D(0,r'_{n,l-1}))}; T_{\hat{\pi}_K(D(0,r_{n,l+1})^c)} = T_{\hat{\pi}_K(\partial D(0,r_{n,l+1})_{s_{l+1}})}\right) \\ &= \frac{3\log n + O(n^{-4}) + O(r^{-1/4}_{n,l-1})}{\log(r_{n,l+1}/r_{n,l-1})} + O(s_l^{-M+2}) \\ &= \frac{3\log n + O(n^{-4})}{6\log n} + O(s_l^{-M+2}) = \frac{1}{2} + o(n^{-4}), \end{aligned} \tag{4.36}$$

Likewise, using (4.35),

$$\begin{aligned} n^{-3} = \frac{r_{n,l}}{r_{n,l+1}} < \frac{|\hat{x}|}{r_{n,l+1}} < \frac{r_{n,l} + r^{\delta}_{n,l}}{r_{n,l+1}} < n^{-6}(n^3 + e^{-n(1-\delta)} n^{-3l(1-\delta)+3}) < n^{-3} + n^{-7} \\ \implies n^3 - \frac{n^3}{n^4+1} = n^3 - O(n^{-1}) < \frac{r_{n,l+1}}{|\hat{x}|} < n^3 \\ \implies \log\left(\frac{r_{n,l+1}}{|\hat{x}|}\right) = 3\log n + O(n^{-4}), \end{aligned} \tag{4.37}$$

*The use of different thicknesses of s_{n-1} depending on direction is due to the entry probability from level n in the lower bound argument of Chapter 7; see Section 6.2 and (7.22) for details.

so by (4.33) and (4.37) we have

$$b_l := P^{\hat{x}}\left(T_{\hat{\pi}_K(D(0,r'_{n,l-1}))} < T_{\hat{\pi}_K(D(0,r_{n,l+1})^c)}; T_{\hat{\pi}_K(D(0,r'_{n,l-1}))} = T_{\hat{\pi}_K(\partial D(0,r_{n,l-1})_{s_{l-1}})}\right)$$
$$= \frac{1}{2} + o(n^{-4}). \tag{4.38}$$

4.2.2 Green's functions

We start calculating bounds for the external Green's function with $\hat{x} \in \hat{\pi}_K(D(0,n/2))$, $\hat{y} \in \hat{\pi}_K(D(0,n))$: by (4.5) with $A = \hat{\pi}_K(D(0,n))$, (2.58), and (4.25),

$$\hat{G}_{\hat{\pi}_K((\partial D(0,n)_s)^c_K)}(\hat{x},\hat{y}) \leq \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{y}) + \frac{\rho_{\hat{x}}}{1-\rho_{\hat{y}}}\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{y},\hat{y}). \tag{4.39}$$

In particular, if $\hat{y} = \hat{0}$ and $s = O(n)$, then $\rho_{\hat{x}} \leq cn^{-2}$ and by (2.55),

$$\hat{G}_{\hat{\pi}_K((\partial D(0,n)_s)^c_K)}(\hat{x},\hat{0}) \leq \hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{0}) + \frac{\rho_{\hat{x}}}{1-\rho_{\hat{0}}}\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{0},\hat{0})$$
$$\implies \hat{G}_{\hat{\pi}_K((\partial D(0,n)_s)^c_K)}(\hat{x},\hat{0}) = \frac{2}{\pi_\Gamma}\log\left(\frac{n}{|\hat{x}|}\right) + C(\hat{p}_1) + O(|\hat{x}|^{-1/4}). \tag{4.40}$$

By (4.6), (3.18), and (4.26), for $\hat{x},\hat{y} \in \hat{\pi}_K(D(0,n+s)^c_K)$,

$$\hat{G}_{\hat{\pi}_K((\partial D(0,n)_s)^c_K)}(\hat{x},\hat{y}) \leq \hat{G}_{\hat{\pi}_K(D(0,n+s)^c_K)}(\hat{x},\hat{y}) + \frac{\phi_{\hat{x}}}{1-\phi_{\hat{y}}}\hat{G}_{\hat{\pi}_K(D(0,n+s)^c_K)}(\hat{y},\hat{y})$$
$$\leq c(\log(|\hat{x}| \wedge |\hat{y}|))^2. \tag{4.41}$$

Finally, for $\hat{x} \in \hat{\pi}_K(D(0,n/2))$ and $\hat{y} \in \hat{\pi}_K(D(0,n+s)^c_K)$, by (4.7), (4.23), (4.27), and the above,

$$\begin{aligned}
&\hat{G}_{\hat{\pi}_K((\partial D(0,n)_s)^c_K)}(\hat{x},\hat{y}) \\
&\quad \le \min\left\{\frac{\sigma_x}{1-\rho_y}\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x},\hat{x}), \frac{\psi_x}{1-\phi_y}\hat{G}_{\hat{\pi}_K(D(0,n+s)^c_K)}(\hat{y},\hat{y})\right\} \\
&\quad \le c \min\left\{n^2(\log n)^3(s^{-M}+n^{-M}), (\log(|\hat{y}|))^2(s^{-M+2} \vee n^{-M+2})\right\}.
\end{aligned} \tag{4.42}$$

In particular, if $s = O(n)$, then in this case $\hat{G}_{\hat{\pi}_K((\partial D(0,n)_s)^c_K)}(\hat{x},\hat{y}) \le cn^{-2}$, and if $s = O(\sqrt{n})$, the bound is $cn^{-\beta}$.

4.2.3 Hitting times

By (3.2) and (3.19), for $y \in D(0,n+s)^c \subset \mathbb{Z}^2$, the external planar annulus hitting time $E^y(T_{\partial D(0,n)_s}) = \infty$. Since, starting from inside the disc $x \in D(0,n)$, there is positive probability of hopping over an s-width annulus, then by the strong Markov property on $T_{D(0,n+s)^c}$, the internal planar annulus hitting time $E^x(T_{\partial D(0,n)_s}) = \infty$ as well. This is not the case for the toral analogues of these times.

Torally, our walk can make small or targeted jumps before the disc escape time. To bound the annulus hitting times, we employ (2.28), (3.20), and (4.15). These yield, for some $c, c' < \infty$,

$$f_{\hat{\pi}_K(D(0,n))} = \sup_{\hat{x}\in\hat{\pi}_K(D(0,n))} E^{\hat{x}}(T_{\hat{\pi}_K(D(0,n)^c_K)}) \le cn^2, \tag{4.43}$$

$$f_{\hat{\pi}_K(D(0,n+s)^c_K)} = \sup_{\hat{y}\in\hat{\pi}_K(D(0,n+s)^c_K)} E^{\hat{y}}(T_{\hat{\pi}_K(D(0,n+s))}) \le c'(K\log K)^2. \tag{4.44}$$

By (4.16), (4.17), (4.43), (4.44), (4.23), and (4.27), the expected annulus hitting time is bounded above: if $\hat{x} \in \hat{\pi}_K(D(0,n/2))$ and $\hat{y} \in \hat{\pi}_K(D(0,n+s)^c_K)$,

$$\begin{aligned}
\mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(\partial D(0,n)_s)}) &\leq \mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(D(0,n)^c_K)}) + \psi_{\hat{x}} \left[\frac{f_{\hat{\pi}_K(D(0,n+s)^c_K)} + \sigma f_{\hat{\pi}_K(D(0,n))}}{1-\psi\sigma} \right] \\
&\leq \mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(D(0,n)^c_K)}) + c\left(s^{-M+2} \vee n^{-M+2}\right)(K\log K)^2; \qquad (4.45) \\
\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(\partial D(0,n)_s)}) &\leq \mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(0,n+s))}) + \sigma_{\hat{y}} \left[\frac{f_{\hat{\pi}_K(D(0,n))} + \psi f_{\hat{\pi}_K(D(0,n+s)^c_K)}}{1-\psi\sigma} \right] \\
&\leq c(K\log K)^2. \qquad (4.46)
\end{aligned}$$

In particular, if $s, n = O(K)$, then for K sufficiently large, note that by (2.28),

$$\mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(D(0,n)^c_K)}) = \frac{K^2 - |\hat{x}|^2}{\gamma^2} + O(K),$$

which, with $M = 4 + 2\beta$, reduces (4.45) to

$$\mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(\partial D(0,n)_s)}) = \left(1 + O(K^{-2-\beta})\right) \mathbb{E}^{\hat{x}}(T_{\hat{\pi}_K(D(0,n)^c_K)}). \qquad (4.47)$$

5 Harnack Inequalities

Here we will generalize Harnack inequality results from [3] by expanding their range and moving them to the torus.

5.1 Interior Harnack inequalities

Our first interior Harnack inequality is flexible enough to be useful on its own, and can be fine-tuned to our applications. We find the planar version first, then move it to the torus.

Lemma 5.1. *Uniformly for $1 \leq m \ll r$, with $s \ll \frac{r}{4m}$, $x, x' \in D(0, 2r)$, $R = 4mr$, and $y \in D(0, R)^c$,*

$$H_{D(0,R)^c}(x, y) = (1 + O(m^{-1}))H_{D(0,R)^c}(x', y) + O(R^{-M} \log R), \tag{5.1}$$

where the error term is completely absorbed, i.e.,

$$H_{D(0,R)^c}(x, y) = (1 + O(m^{-1}))H_{D(0,R)^c}(x', y), \tag{5.2}$$

if $s \leq (\log R)^4$ and $y \in \partial D(0, R)_s$.

Furthermore, if $x \in \partial D(0,r)_r$ *and* $y \in D(0,R)^c$,

$$P^x\left(S_{T_{D(0,R)^c}} = y,\, T_{D(0,R)^c} < T_{D\left(0,\frac{r}{4m}+s\right)}\right) \tag{5.3}$$
$$= (1+O(m^{-1}))P^x\left(T_{D(0,R)^c} < T_{D\left(0,\frac{r}{4m}+s\right)}\right) H_{D(0,R)^c}(x,y) + O(R^{-M}\log R),$$

with a similar loss of the error term if $y \in \partial D(0,R)_s$.

Proof (In this proof, we switch freely between R and $4mr$.) First, we decompose $D(0,R)$ and examine $H_{D(0,R)^c}(x,y)$:

$$H_{D(0,R)^c}(x,y) = \left(\sum_{z\in D(0,2mr)} + \sum_{\substack{z\in D(0,3mr)\\ \setminus D(0,2mr)}} + \sum_{\substack{z\in D(0,4mr)\\ \setminus D(0,3mr)}}\right) G_{D(0,R)}(x,z)p_1(z,y). \tag{5.4}$$

If X is finite range, then for r sufficiently large, the first two sums of (5.4) are zero. Otherwise, we bound the Green's function via (2.55) and (2.57), and by Markov's inequality, $\sum_{z\in D(0,2mr)} p_1(z,y) \le c(mr)^{-M} \le cR^{-M}$. Together, these yield, for some $c<\infty$,

$$G_{D(0,R)}(x,z) \le G_{D(0,2R)}(0,z) \le c\log R$$
$$\implies \sum_{z\in D(0,2mr)} G_{D(0,R)}(x,z)p_1(z,y) \le cR^{-M}\log R. \tag{5.5}$$

By (2.44) and (2.47), uniformly in $x \in D(0,2r)$ and $y \in D(0,2mr)^c$,

$$a(y-x) = \frac{2}{\pi_\Gamma}\log|y-x| + C' + O(|y-x|^{-1})$$
$$= \frac{2}{\pi_\Gamma}\log|y| + C' + O(m^{-1}) = a(y) + O(m^{-1}). \tag{5.6}$$

For $z \in D(0,4mr) \setminus D(0,2mr)$, by the symmetry of the Green's function, the fact that H is a probability, [17, (4.28)], and (5.6), we have

$$\begin{aligned} G_{D(0,R)}(x,z) &= G_{D(0,R)}(z,x) \\ &= \left(\sum_{w \in D(0,R)^c} H_{D(0,R)^c}(z,w) a(w-x) \right) - a(z-x) \qquad (5.7) \\ &= \left(\sum_{w \in D(0,R)^c} H_{D(0,R)^c}(z,w) a(w) \right) - a(z) + O(m^{-1}) \\ &= G_{D(0,4mr)}(z,0) + O(m^{-1}). \end{aligned}$$

By (2.55), $G_{D(0,R)}(z,0) \geq c > 0$ uniformly for $z \in D(0,3mr) \setminus D(0,2mr)$, yielding

$$G_{D(0,R)}(x,z) = G_{D(0,R)}(0,z)(1 + O(m^{-1})). \qquad (5.8)$$

For $z \in D(0,4mr) \setminus D(0,3mr)$, by the strong Markov property at $T_{D(0,3mr)}$,

$$\begin{aligned} G_{D(0,R)}(z,x) &= \mathbb{E}^z(G_{D(0,R)}(S_{T_{D(0,3mr)}}, x); T_{D(0,3mr)} < T_{D(0,4mr)^c}) \qquad (5.9) \\ &= \mathbb{E}^z(G_{D(0,R)}(S_{T_{D(0,3mr)}}, x); T_{D(0,3mr)} < T_{D(0,4mr)^c}; |X_{T_{D(0,3mr)}}| > 2mr) \\ &\quad + \mathbb{E}^z(G_{D(0,R)}(S_{T_{D(0,3mr)}}, x); T_{D(0,3mr)} < T_{D(0,4mr)^c}; |X_{T_{D(0,3mr)}}| \leq 2mr). \end{aligned}$$

By (2.57) and (4.27), the last term here is bounded, for sufficiently large r, by

$$\begin{aligned} c(\log R) P^z(|X_{T_{D(0,3mr)}}| \leq 2mr) &\leq c(\log R) P^z(T_{D(0,2mr)} < T_{\partial D(0,2mr)_{mr}}) \\ &\leq c(\log R)(2mr)^2 \log(2mr)^2 \left[(mr)^{-M} + (2mr)^{-M}\right] \\ &\leq c(\log R)^3 R^{-M+2} \leq c R^{-M+2+\beta}. \end{aligned}$$

Applying (5.8) to the first term, then switching it back to its original form, yields, for $z \in D(0,4mr) \setminus D(0,3mr)$,

$$G_{D(0,R)}(z,x) = (1+O(m^{-1}))G_{D(0,R)}(z,0) + O(R^{-M+2+\beta}). \tag{5.10}$$

The planar version of (2.59) gives us $G_{D(0,R)}(z,x) \geq \frac{c}{mr}$ for $z \in D(0,4mr)\setminus D(0,3mr)$. This reduces (5.10) to

$$G_{D(0,R)}(z,x) = (1+O(m^{-1}))G_{D(0,R)}(z,0). \tag{5.11}$$

Combining (5.4), (5.5), (5.8), and (5.11) yields (5.1).

For (5.2), let $y \in \partial D(0,R)_s$. The only thing we need to do here is show that the error terms are absorbed, *i.e.*, for some $c > 0$, with $M = 4 + 2\beta$,

$$m^{-1}H_{D(0,R)^c}(x,y) \geq cR^{-M}\log R. \tag{5.12}$$

Wlog, we can show this for $x = 0$. First note that, for $|z| \leq \frac{R}{100}$, by (2.55) we have

$$G_{D(0,R)}(z,0) \geq c\log\frac{R}{(R/100)} = c\log 100 \geq c \geq \frac{c}{R}$$

for some $c > 0$, and for $z \in D(0,R) \setminus D(0,R/100)$, by the planar version of (2.59), $G_{D(0,R)}(z,0) \geq \frac{c}{R}$ as well. Hence, by this, a last exit decomposition, and (1.2),

$$\begin{aligned} m^{-1}H_{D(0,R)^c}(0,y) &= m^{-1}\sum_{z\in D(0,R)} G_{D(0,R)}(0,z)p_1(z,y) \geq \frac{c}{mR}\sum_{z\in D(0,R)} p_1(z,y) \qquad (5.13)\\ &\geq \frac{c}{mR}e^{-\beta s^{1/4}} = c(mR)^{-1}e^{-\beta\log R} = cm^{-1}R^{-1-\beta} > cR^{-M}\log R. \end{aligned}$$

To show (5.3), we start with the decomposition

$$\begin{aligned} &P^x\left(S_{T_{D(0,R)^c}} = y,\, T_{D(0,R)^c} < T_{D\left(0,\frac{r}{4m}+s\right)}\right) \\ &\quad = H_{D(0,R)^c}(x,y) - P^x\left(S_{T_{D(0,R)^c}} = y,\, T_{D(0,R)^c} > T_{D\left(0,\frac{r}{4m}+s\right)}\right). \end{aligned} \tag{5.14}$$

By the strong Markov property at $T_{D\left(0,\frac{r}{4m}+s\right)}$,

$$\begin{aligned} &P^x\left(S_{T_{D(0,R)^c}} = y,\, T_{D(0,R)^c} > T_{D\left(0,\frac{r}{4m}+s\right)}\right) \\ &\quad = \mathbb{E}^x\left[H_{D(0,R)^c}\left(S_{T_{D\left(0,\frac{r}{4m}+s\right)}}, y\right);\, T_{D(0,R)^c} > T_{D\left(0,\frac{r}{4m}+s\right)}\right]. \end{aligned} \tag{5.15}$$

By (5.1), uniformly in $w \in D\left(0, 2r\right)$,

$$H_{D(0,R)^c}\left(S_{T_{D\left(0,\frac{r}{4m}+s\right)}}, y\right) = \left(1 + O\left(m^{-1}\right)\right) H_{D(0,R)^c}(w,y) + O(R^{-M}\log R).$$

By (2.51) and (2.52), with $m \gg 1$, uniformly for $x \in \partial D(0,r)_r$ (say $|x| = cr$, $1 < c < 2$), $\exists c', c'' > 0$ such that

$$\begin{aligned} P^x\left(T_{D(0,R)^c} < T_{D\left(0,\frac{r}{4m}+s\right)}\right) &= \frac{\log(c'm) + O\left(\left(\frac{r}{m}\right)^{-1/4}\right)}{\log(c''m^2)} = \frac{1}{2} + o\left(\left(\frac{r}{m}\right)^{-1/4}\right), \\ P^x\left(T_{D(0,R)^c} > T_{D\left(0,\frac{r}{4m}+s\right)}\right) &= \frac{1}{2} + o\left(\left(\frac{r}{m}\right)^{-1/4}\right), \end{aligned} \tag{5.16}$$

so the probabilities are both bounded below by a constant. (The small m case operates similarly, but due to the small constants involved, the lower bound must be reduced;

$\frac{1}{4}$ for one of them suffices.) Combining these and (5.15) into (5.14) yields

$$\begin{aligned}
&P^x\left(S_{T_{D(0,R)^c}} = y,\, T_{D(0,R)^c} < T_{D\left(0,\frac{r}{4m}+s\right)}\right) \\
&\quad = H_{D(0,R)^c}(x,y) - \mathbb{E}^x\left[H_{D(0,R)^c}\left(S_{T_{D(0,\frac{r}{4m}+s)}}, y\right);\, T_{D(0,R)^c} > T_{D\left(0,\frac{r}{4m}+s\right)}\right] \\
&\quad = H_{D(0,R)^c}(x,y)\Big[P^x\left(T_{D(0,R)^c} < T_{D\left(0,\frac{r}{4m}+s\right)}\right) + P^x\left(T_{D(0,R)^c} > T_{D\left(0,\frac{r}{4m}+s\right)}\right) \\
&\quad - \left(1+O\left(m^{-1}\right)\right) P^x\left(T_{D(0,R)^c} > T_{D\left(0,\frac{r}{4m}+s\right)}\right) + O(R^{-M}\log R)\Big] \\
&\quad = H_{D(0,R)^c}(x,y)\left(1+O\left(m^{-1}\right)\right) P^x\left(T_{D(0,R)^c} < T_{D\left(0,\frac{r}{4m}+s\right)}\right) + O(R^{-M}\log R). \quad \square
\end{aligned}$$

Here is a focused result for our applications which follows directly.

Lemma 5.2. *Let* $e^n \le r$, $R = n^3 r$ (i.e., $m = \frac{n^3}{4}$ *for* $R = 4mr$). *Uniformly for* $x, x' \in D(0, r+\sqrt{r})$ *and* $y \in \partial D(0,R)_{n^4}$,

$$H_{D(0,R)^c}(x,y) = \left(1+O\left(n^{-3}\right)\right) H_{D(0,R)^c}(x',y). \tag{5.17}$$

Furthermore, uniformly in $x \in \partial D(0,r)_{\sqrt{r}}$ *and* $y \in \partial D(0,R)_{n^4}$,

$$\begin{aligned}
&P^x(S_{T_{D(0,R)^c}} = y, T_{D(0,R)^c} < T_{D(0,\frac{r}{n^3}+n^4)}) \\
&\quad = \left(1+O\left(n^{-3}\right)\right) P^x(T_{D(0,R)^c} < T_{D(0,\frac{r}{n^3}+n^4)}) H_{D(0,R)^c}(x,y).
\end{aligned} \tag{5.18}$$

We now move these results to the torus.

Lemma 5.3. *For large* r *and* $1 \le m \ll r$ *such that* $R = 4mr < K/6$ *and* $s \le (\log R)^4$, *uniformly for* $\hat{x}, \hat{x}' \in \hat{\pi}_K(D(0,2r))$ *and* $\hat{y} \in \hat{\pi}_K(D(0,R)^c_K)$,

$$\hat{H}_{\hat{\pi}_K(D(0,R)^c_K)}(\hat{x},\hat{y}) = \left(1 + O\left(m^{-1}\right)\right)\hat{H}_{\hat{\pi}_K(D(0,R)^c_K)}(\hat{x}',\hat{y}) + O(R^{-M}\log R \vee K^{-M}R^2). \tag{5.19}$$

Furthermore, uniformly in $\hat{x} \in \hat{\pi}_K(\partial D(0,r)_r)$ *and* $\hat{y} \in \hat{\pi}_K(D(0,R)^c_K)$,

$$\begin{aligned} &P^{\hat{x}}(\hat{S}_{T_{\hat{\pi}_K(D(0,R)^c_K)}} = \hat{y}, T_{\hat{\pi}_K(D(0,R)^c_K)} < T_{\hat{\pi}_K(D(0,\frac{r}{4m}+s))}) \\ &= \left(1 + O\left(m^{-1}\right)\right) P^{\hat{x}}(T_{\hat{\pi}_K(D(0,R)^c_K)} < T_{\hat{\pi}_K(D(0,\frac{r}{4m}+s))})\hat{H}_{\hat{\pi}_K(D(0,R)^c_K)}(\hat{x},\hat{y}) \\ &+ O(R^{-M}\log R \vee K^{-M}R^2). \end{aligned} \tag{5.20}$$

If $\hat{y} \in \hat{\pi}_K(\partial D(0,R)_s)$, *the error term is absorbed in both of these statements.*

Lemma 5.4. *Let* $n > 13$, $e^n \le r$, $R = n^3 r$ *(*i.e., $m = \frac{n^3}{4}$ *for* $R = 4mr$*). Uniformly for* $\hat{x}, \hat{x}' \in \hat{\pi}_K(D(0,2r))$, $K > 4(R+n^4)$, *and* $\hat{y} \in \hat{\pi}_K(\partial D(0,R)_{n^4})$,

$$\hat{H}_{\hat{\pi}_K(D(0,R)^c_K)}(\hat{x},\hat{y}) = \left(1 + O\left(n^{-3}\right)\right)\hat{H}_{\hat{\pi}_K(D(0,R)^c_K)}(\hat{x}',\hat{y}). \tag{5.21}$$

Furthermore, uniformly in $\hat{x} \in \hat{\pi}_K(\partial D(0,r)_{\sqrt{r}})$ *and* $\hat{y} \in \hat{\pi}_K(\partial D(0,R)_{n^4})$,

$$\begin{aligned} &P^{\hat{x}}(\hat{S}_{T_{\hat{\pi}_K(D(0,R)^c_K)}} = \hat{y}, T_{\hat{\pi}_K(D(0,R)^c_K)} < T_{\hat{\pi}_K(D(0,\frac{r}{n^3}+n^4))}) \\ &= \left(1 + O\left(n^{-3}\right)\right) P^{\hat{x}}(T_{\hat{\pi}_K(D(0,R)^c_K)} < T_{\hat{\pi}_K(D(0,\frac{r}{n^3}+n^4))})\hat{H}_{\hat{\pi}_K(D(0,R)^c_K)}(\hat{x},\hat{y}). \end{aligned} \tag{5.22}$$

Proof As before, wlog, we can take $\hat{x}' = \hat{0}$. Let s be the size of the annulus for $\hat{y}$ (R

for Lemma 5.3, n^4 for Lemma 5.4). For brevity, set

$$\begin{aligned} A_p &:= \{S_{T_{D(0,R)^c}} = y\}, & d_p &:= |S_{T_{D(0,R)^c}} - S_{T_{D(0,R)^c}-1}|, \\ A_t &:= \{\hat{S}_{T_{\hat{\pi}_K(D(0,R)^c_K)}} = \hat{y}\}, & d_t &:= |S_{T_{\hat{\pi}_K(D(0,R)^c_K)}} - S_{T_{\hat{\pi}_K(D(0,R)^c_K)}-1}|. \end{aligned}$$

Note that x and y are the primary copies of $\hat{x}$ and $\hat{y}$, and so $|x-y| \leq \frac{K}{\sqrt{2}}$, but that d_p is a planar distance using a planar escape time and d_t is a planar distance using a toral escape time; hence, both can exceed $\frac{K}{\sqrt{2}}$, the maximum distance between two points in $\mathbb{Z}^2_K$.

To prove (5.19), first we re-label (5.1) as

$$H_{D(0,R)^c}(x,y) = P^x(A_p) = \left(1 + O\left(\frac{r}{R}\right)\right) P^{x'}(A_p) + O\left(R^{-M}\log R\right). \tag{5.23}$$

We have the decomposition

$$P^x(A_p) = P^x(A_p; d_p < K - 2R) + P^x(A_p; d_p \geq K - 2R). \tag{5.24}$$

On the plane, the second term of (5.24) is zero for all but the furthest-away y in the primary copy ($i.e.$, $K - 2R \leq |x-y| \leq \frac{K}{\sqrt{2}}$); for those y, we have, by (2.21), (2.17), and (2.4),

$$\begin{aligned} P^x(A_p;\, d_p \geq K - 2R) &= \sum_{z \in D(0,R)} G_{D(0,R)}(x,z) P^z(|X_1 - z| > K - 2R) \\ &\leq cK^{-M}R^2. \end{aligned} \tag{5.25}$$

The toral version can be written using planar distances as a decomposition, but using the toral disc escape time means a further decomposition comparing the planar and toral escape times a la (2.23). We decompose $\hat{H}_{\hat{\pi}_K(D(0,R)^c_K)}(\hat{x},\hat{y}) = P^{\hat{x}}(A_t)$ as

$$\begin{aligned} P^{\hat{x}}(A_t) = {} & P^{\hat{x}}(A_t;\, d_t < K-2R;\, T_{D(0,R)^c} = T_{\hat{\pi}_K(D(0,R)^c_K)}) \\ & + P^{\hat{x}}(A_t;\, d_t \geq K-2R;\, T_{D(0,R)^c} = T_{\hat{\pi}_K(D(0,R)^c_K)}) \\ & + P^{\hat{x}}(A_t;\, T_{D(0,R)^c} < T_{\hat{\pi}_K(D(0,R)^c_K)}). \end{aligned} \tag{5.26}$$

In the torus, the first term of (5.26) equals the first term of (5.24), plus a large jump error which contains some paths from the second term of (5.24) (if y is far): by (5.25),

$$\begin{aligned} & P^{\hat{x}}(A_t; d_t < K-2R;\, T_{D(0,R)^c} = T_{\hat{\pi}_K(D(0,R)^c_K)}) \\ & \quad = P^x(A_p; d_p < K-2R) + P^x(A_p; d_p \geq K-2R; d_t < K-2R) \\ & \quad = P^x(A_p) + O(K^{-M}R^2). \end{aligned}$$

The second term of (5.26) only occurs if the final, escaping jump is large: by (2.21), (2.17), and (2.28), just as in (5.25),

$$\begin{aligned} & P^{\hat{x}}(A_t;\, d_t \geq K-2R;\, T_{D(0,R)^c} = T_{\hat{\pi}_K(D(0,R)^c_K)}) \leq P^{\hat{x}}(A_t;\, d_t \geq K-2R) \\ & \quad = \sum_{\hat{z}\in\hat{\pi}_K(D(0,R))} \hat{G}_{\hat{\pi}_K(D(0,R))}(\hat{x},\hat{z}) P^z(|X_1 - z| > K-2R) \leq cK^{-M}R^2. \end{aligned}$$

The last term of (5.26) requires a large jump to have occured. Hence, by (2.23),

$$P^{\hat{x}}(A_t;\, T_{D(0,R)^c} < T_{\hat{\pi}_K(D(0,R)^c_K)}) \leq cK^{-M}R^2.$$

Therefore, (5.26) reduces to

$$P^{\hat{x}}(A_t) = P^x(A_p) + O(K^{-M}R^2). \tag{5.27}$$

(Due to targeting, this is generalizable to any planar set $B \subset D(0, R)$ for $R < K/4$.) Combining this with (5.23) gives us (5.19):

$$\begin{aligned} P^{\hat{x}}(A_t) &= P^{x}(A_p) + O(K^{-M}R^2) \\ &= \left(1 + O\left(\tfrac{r}{R}\right)\right) P^{x'}(A_p) + O(K^{-M}R^2) + O\left(R^{-M}\log R\right) \\ &= \left(1 + O\left(\tfrac{r}{R}\right)\right) P^{\hat{x}'}(A_t) + O(K^{-M}R^2) + O\left(R^{-M}\log R\right). \end{aligned}$$

The proof of (5.20) follows from the Markov property argument for (5.3), using the appropriate toral identities: (5.19) for (5.1), and (2.53)-(2.54) for (2.51)-(2.52). (5.21) and (5.22) are applications. $\square$

5.2 Exterior Harnack inequality

To aid our construction of an exterior Harnack inequality, we first establish uniform bounds on external Green's functions and probabilities in the torus and plane. Fix $\delta < 1$ and use $r \geq e^n$ for some $n > 13$, $R = 4mr$ for some $1 \leq m \ll r$, and $s \leq (\log R)^4$. First, for $\hat{x} \in \hat{\pi}_K(\partial D(0, R)_{R/100})$ and $\hat{y} \in \hat{\pi}_K(D(0, R)^c_K)$, we show that

$$\hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{x}, \hat{y}) \geq c > 0. \tag{5.28}$$

Pick some $\hat{x}_1 \in \hat{\pi}_K(\partial D(0, R))$, and, proceeding clockwise, choose points $\hat{x}_2, ..., \hat{x}_{36} \in \hat{\pi}_K(\partial D(0, R))$ whose rays beginning at $\hat{0}$ divide $\hat{\pi}_K(\partial D(0, R))$ into 36 approximately equal arcs. The distance between any two adjacent such $\hat{x}_j$ is, for sufficiently large R, approximately $2R\sin(\pi/36) \approx 0.174R$. Thus, using discs of radius $R/5$ (so adjacent circles contain their neighbor's centers), and by (2.54) we have for any $j = 1, ..., 36$

$$\inf_{\hat{x}\in\hat{\pi}_K(D(x_j,R/5))} P^{\hat{x}}(T_{\hat{\pi}_K(D(x_{j+1},R/5))} < T_{\hat{\pi}_K(D(0,r+s))}) \tag{5.29}$$
$$\geq \inf_{\hat{x}\in\hat{\pi}_K(D(x_{j+1},2R/5))} P^{\hat{x}}(T_{\hat{\pi}_K(D(x_{j+1},R/5))} < T_{\hat{\pi}_K(D(\hat{x}_{j+1},R/2)^c_K)}) \geq c_1 > 0,$$

for some c_1 independent of n, r, R, n large, and $\hat{x}_{37} = \hat{x}_1$.

Hence, by the strong Markov property, rotating through the arcs, we have

$$\inf_{j,k} \inf_{\hat{x}\in\hat{\pi}_K(D(x_j,R/5))} P^{\hat{x}}(T_{\hat{\pi}_K(D(x_k,R/5))} < T_{\hat{\pi}_K(D(0,r+s))}) \geq c_2 := c_1^{36}. \tag{5.30}$$

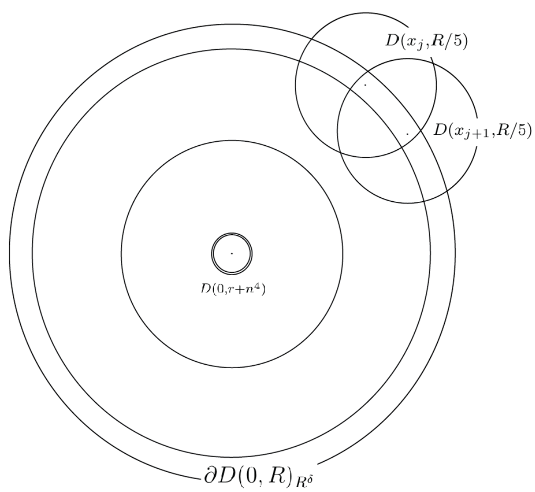

Figure 5.1: Possible selections of $\hat{x}_j$ and $\hat{x}_{j+1}$, with associated discs and annuli.

Furthermore, it follows from (2.50) that for any j,

$$\inf_{\hat{x},\hat{x}'\in\hat{\pi}_K(D(x_j,R/5))} P^{\hat{x}}(T_{\hat{x}'} < T_{\hat{\pi}_K(D(0,r+s))}) \tag{5.31}$$
$$\geq \inf_{\hat{x}\in\hat{\pi}_K(D(x',2R/5))} P^{\hat{x}}(T_{\hat{x}'} < T_{\hat{\pi}_K(D(x',R/2)^c_K)}) \geq \frac{c_3}{\log R}$$

for some independent $c_3 > 0$.

Since $\hat{\pi}_K(\partial D(0,R)_{R/100}) \subset \cup_{j=1}^{36} \hat{\pi}_K(D(x_j, R/5))$, combining (5.30) and (5.31) we have, for some independent $c_4 > 0$,

$$\inf_{\hat{x},\hat{x}' \in \hat{\pi}_K(\partial D(0,R)_{R/100})} P^{\hat{x}}(T_{\hat{x}'} < T_{\hat{\pi}_K(D(0,r+s))}) \geq \frac{c_4}{\log R}. \tag{5.32}$$

It then follows from (2.46) that

$$\begin{aligned} &\inf_{\hat{x},\hat{x}' \in \hat{\pi}_K(\partial D(0,R)_{R/100})} \hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{x}, \hat{x}') \\ =&\inf_{\hat{x},\hat{x}' \in \hat{\pi}_K(\partial D(0,R)_{R/100})} P^{\hat{x}}(T_{\hat{x}'} < T_{\hat{\pi}_K(D(0,r+s))}) \hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{x}', \hat{x}') \\ \geq& \frac{c_4}{\log R} \hat{G}_{\hat{\pi}_K(D(x',R/2))}(\hat{x}', \hat{x}') \geq c_5 > 0 \end{aligned} \tag{5.33}$$

for some independent $c_5 > 0$. Using the strong Markov property, (5.33), and (4.27), we see that

$$\begin{aligned} &\inf_{\substack{\hat{z} \in \hat{\pi}_K(D(0,1.01R)^c_K), \\ \hat{x} \in \hat{\pi}_K(\partial D(0,R)_{R/100})}} \hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{z}, \hat{x}) \\ &\geq \mathbb{E}^{\hat{z}} \left(\hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{S}_{T_{\hat{\pi}_K(D(0,1.01R))}}, \hat{x}); \hat{S}_{T_{\hat{\pi}_K(D(0,1.01R))}} \in \hat{\pi}_K(\partial D(0,R)_{R/100}) \right) \\ &\geq c > 0. \end{aligned} \tag{5.34}$$

This gives (5.28). Applying the same argument once more,

$$\inf_{\substack{\hat z\in\hat\pi_K(D(0,1.01R)^c_K)\\ \hat x\in\hat\pi_K(D(0,R)^c_K)}} \hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat z,\hat x) \tag{5.35}$$
$$\geq \mathbb{E}^{\hat z}\left(\hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat S_{T_{\hat\pi_K(D(0,1.01R))}},\hat x);\hat S_{T_{\hat\pi_K(D(0,1.01R))}}\in\hat\pi_K(\partial D(0,R)_{R/100})\right)$$
$$\geq c>0.$$

Hence, for all $\hat x,\hat y\in\hat\pi_K(D(0,R)^c_K)$,

$$\hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat x,\hat y)\geq c>0. \tag{5.36}$$

Next, we look at the external Green's function near the r-disc: uniformly for $\hat x\in\hat\pi_K(D(0,R)^c_K)$ and $\hat z\in\hat\pi_K(D(0,2r))\setminus\hat\pi_K(D(0,5r/4))$, we have by (5.36) and (2.53),

$$\hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat x,\hat z)=\hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat z,\hat x)$$
$$=\mathbb{E}^{\hat z}\left(\hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat S_{T_{\hat\pi_K(D(0,R)^c_K)}},\hat x);T_{\hat\pi_K(D(0,R)^c_K)}<T_{\hat\pi_K(D(0,r+s))}\right)$$
$$\geq cP^{\hat z}(T_{\hat\pi_K(D(0,R)^c_K)}<T_{\hat\pi_K(D(0,r+s))})\geq\frac{c}{\log m}. \tag{5.37}$$

Getting closer to the disc, for any $\varepsilon>0$, uniformly in $\hat x,\hat x'\in\hat\pi_K(\partial D(0,R)_{R/100})$ and $\hat z\in\hat\pi_K(D(0,2r))\setminus\hat\pi_K(D(0,r+(1+\varepsilon)s))$, we have by the strong Markov property and (5.32), for any $\hat x'\in\partial D(0,R)_{R/100}$,

$$\hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat x,\hat z)\geq P^{\hat x}(T_{\hat x'}<T_{\hat\pi_K(D(0,r+s))})\,\hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat x',\hat z)$$
$$\geq\frac{c\hat G_{\hat\pi_K(D(0,r+s)^c_K)}(\hat x',\hat z)}{\log R}. \tag{5.38}$$

In view of (2.59), if $\hat x'\in\hat\pi_K(\partial D(0,R)_{R/100})$ is chosen as close as possible to the ray from the origin which passes through $\hat{z}$, we have

$$\hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{x}', \hat{z}) \geq \hat{G}_{\hat{\pi}_K(D(\hat{x}',|\hat{x}'|-(r+s)))}(\hat{x}', \hat{z}) \geq \frac{c}{R}, \tag{5.39}$$

which, combined with (5.38), gives us

$$\inf_{\substack{\hat{x}\in\hat{\pi}_K(\partial D(0,R)_{R/100}) \\ \hat{z}\in\hat{\pi}_K(D(0,2r))\setminus\hat{\pi}_K(D(0,r+(1+\varepsilon)s))}} \hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{x}, \hat{z}) \geq \frac{c}{R\log R}. \tag{5.40}$$

Using the strong Markov property, (5.40), and (4.23), we see that

$$\begin{aligned} &\inf_{\substack{\hat{x}\in\hat{\pi}_K(D(0,1.01R)^c_K) \\ \hat{z}\in\hat{\pi}_K(D(0,2r))\setminus\hat{\pi}_K(D(0,r+(1+\varepsilon)s))}} \hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{z}, \hat{x}) \\ &\geq \mathbb{E}^{\hat{z}}\left(\hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{S}_{T_{\hat{\pi}_K(D(0,1.01R)^c_K)}}, \hat{x}); \hat{S}_{T_{\hat{\pi}_K(D(0,1.01R)^c_K)}} \in \hat{\pi}_K(\partial D(0,R)_{R/100})\right) \\ &\geq \frac{c}{R\log R}. \end{aligned} \tag{5.41}$$

Hence

$$\inf_{\substack{\hat{x}\in\hat{\pi}_K(D(0,R)^c_K) \\ \hat{z}\in\hat{\pi}_K(D(0,2r))\setminus\hat{\pi}_K(D(0,r+(1+\varepsilon)s))}} \hat{G}_{\hat{\pi}_K(D(0,r+s)^c_K)}(\hat{x}, \hat{y}) \geq \frac{c}{R\log R}. \tag{5.42}$$

By removing the (hidden) targeted jump error terms, the entire argument in (5.28)-(5.42) also applies to the plane. We now modify a planar Harnack inequality, for entering a small disc from far outside, that parallels Lemma 5.2.

Lemma 5.5. *Let $R = 4mr$ with $1 \leq m \ll r$ ($m = o(r^{1/4})$) and large enough r, and $s \leq (\log R)^4$. Then, uniformly for $x, x' \in D(0,R)^c$ and $y \in \partial D(0,r)_s$,*

$$H_{D(0,r+s)}(x, y) = \left(1 + O\left(m^{-1}\log m\right)\right) H_{D(0,r+s)}(x', y). \tag{5.43}$$

Furthermore, for $x, x' \in \partial D(0,R)_{\sqrt{R}}$,

$$\begin{aligned} P^x(S_{T_{D(0,r+s)}} = y;\, T_{D(0,r+s)} < T_{D(0,4mR)^c}) \qquad (5.44) \\ = \left(1 + O\left(m^{-1}\log m\right)\right) H_{D(0,r+s)}(x,y) P^x(T_{D(0,r+s)} < T_{D(0,4mR)^c}) \\ = \left(1 + O\left(m^{-1}\log m\right)\right) P^{x'}(S_{T_{D(0,r+s)}} = y; T_{D(0,r+s)} < T_{D(0,4mR)^c}). \end{aligned}$$

Proof For $x, x' \in D(0,R)^c$ and $y \in \partial D(0,r)_s$, we have the last exit decomposition

$$H_{D(0,r+s)}(x,y) = \left(\sum_{\substack{z \in D(0,5r/4) \\ \setminus D(0,r+s)}} + \sum_{\substack{z \in D(0,2r) \\ \setminus D(0,5r/4)}} + \sum_{z \in D(0,2r)^c} \right) G_{D(0,r+s)^c}(x,z) p_1(z,y). \quad (5.45)$$

Let $x, x' \in \partial D(0,R)_R$ and set $N \geq 4mR$. Uniformly for $z \in D(0,2r) \cup D(0,N)^c$, by (2.44) and (2.48),

$$a(x-z) = \frac{2}{\pi_\Gamma} \log|x-z| + C' + O(|x-z|^{-1}) = a(x'-z) + O(m^{-1}). \quad (5.46)$$

Using the same approach as in (5.7), (5.46) implies that, for $A(r+s,N) := D(0,N) \setminus D(0,r+s)$,

$$G_{A(r+s,N)}(x,z) = G_{A(r+s,N)}(x',z) + O(m^{-1}),$$

which, by letting $N \to \infty$ and applying the dominated convergence theorem,

$$G_{D(0,r+s)^c}(x,z) = G_{D(0,r+s)^c}(x',z) + O(m^{-1}). \quad (5.47)$$

Applying (5.37) to (5.47) yields, for $z \in D(0,2r) \setminus D(0,5r/4)$,

$$G_{D(0,r+s)^c}(x,z) = (1+O(m^{-1}\log m))G_{D(0,r+s)^c}(x',z). \tag{5.48}$$

Next, by the symmetry of the Green's function, the strong Markov property at $T_{D(0,5r/4)^c}$, (5.48) for $z \in D(0,5r/4) \setminus D(0,r+s)$, and decomposing, we have

$$\begin{aligned} &G_{D(0,r+s)^c}(x,z) = G_{D(0,r+s)^c}(z,x) \qquad (5.49)\\ &= \mathbb{E}^z(G_{D(0,r+s)^c}(S_{T_{D(0,5r/4)^c}},x)\,;\, T_{D(0,5r/4)^c} < T_{D(0,r+s)}),\\ &= \mathbb{E}^z(G_{D(0,r+s)^c}(S_{T_{D(0,5r/4)^c}},x)\,;\, T_{D(0,5r/4)^c} < T_{D(0,r+s)}\,,\, |S_{T_{D(0,5r/4)^c}}| \le 2r)\\ &\quad + \mathbb{E}^z(G_{D(0,r+s)^c}(S_{T_{D(0,5r/4)^c}},x)\,;\, T_{D(0,5r/4)^c} < T_{D(0,r+s)}\,,\, |S_{T_{D(0,5r/4)^c}}| > 2r). \end{aligned}$$

By (5.48) on the first term and (3.9) on the second term, (5.49) is bounded above:

$$\begin{aligned} G_{D(0,r+s)^c}(z,x) \le &(1+O(m^{-1}\log m)) \qquad (5.50)\\ &\mathbb{E}^z(G_{D(0,r+s)^c}(S_{T_{D(0,5r/4)^c}},x')\,;\, T_{D(0,5r/4)^c} < T_{D(0,r+s)})\\ &+ c\log(R)P^z(|S_{T_{D(0,5r/4)^c}}| > 2r). \end{aligned}$$

Applying the first two lines of (5.49) again, the first term here is

$$\begin{aligned} &(1+O(m^{-1}\log m))\mathbb{E}^z(G_{D(0,r+s)^c}(S_{T_{D(0,5r/4)^c}},x')\,;\, T_{D(0,5r/4)^c} < T_{D(0,r+s)})\\ &= (1+O(m^{-1}\log m))G_{D(0,r+s)^c}(z,x') = (1+O(m^{-1}\log m))G_{D(0,r+s)^c}(x',z). \end{aligned}$$

A last exit decomposition of $P^z(|S_{T_{D(0,5r/4)^c}}| > 2r)$, then (2.56) and (1.1) yield

$$\begin{aligned} G_{D(0,r+s)^c}(x,z) \le& (1+O(m^{-1}\log m))G_{D(0,r+s)^c}(x',z) \\ &+ c\log(R) \sum_{\substack{|y|<5r/4 \\ 2r<|w|}} G_{D(0,5r/4)}(z,y)p_1(y,w) \\ \le& (1+O(m^{-1}\log m))G_{D(0,r+s)^c}(x',z) + c\log(R)\log(r)r^{-M+2}. \end{aligned}$$

Since this argument is symmetric in x and x', then we have that for $z \in D(0,5r/4) \setminus D(0,r+s)$, and $c\log R = c(\log 4 + \log m + \log r) = O(\log r)$,

$$G_{D(0,r+s)^c}(x,z) = (1+O(m^{-1}\log m))G_{D(0,r+s)^c}(x',z) + O(r^{-M+2}(\log r)^2). \quad (5.51)$$

Finally, by (3.9), for $z \in D(0,2r)^c$, $G_{D(0,r+s)^c}(x,z) = O(\log R)$. Thus, for $y \in \partial D(0,r)_s$, since $\sum_{z\in D(0,2r)^c} p_1(z,y) \le O(r^{-M})$ by symmetry and Markov's inequality,

$$\sum_{z\in D(0,2r)^c} G_{D(0,r+s)^c}(x,z)p_1(z,y) = O(r^{-M}\log r). \quad (5.52)$$

Combining (5.48), (5.51), and (5.52) bounds the sums in (5.45) to

$$H_{D(0,r+s)}(x,y) = (1+O(m^{-1}\log m))H_{D(0,r+s)}(x',y) + O(r^{-M+2}(\log r)^2). \quad (5.53)$$

To complete the proof of (5.43) for $x, x' \in \partial D(0,R)_R$, we must show that, uniformly for $x \in \partial D(0,R)_R$ and $y \in \partial D(0,r)_s$,

$$r^{-M+2}(\log r)^2 \le c(m^{-1}\log m)H_{D(0,r+s)}(x,y). \quad (5.54)$$

With $A_r := D(0,2r)\setminus D(0,r+(1+\varepsilon)s)$, using a last exit decomposition and bounding with the planar version of (5.42),

$$\begin{aligned} H_{D(0,r+s)}(x,y) &= \sum_{z\in D(0,r+s)^c} G_{D(0,r+s)^c}(x,z)p_1(z,y) \\ &\geq \sum_{z\in A_r} G_{D(0,r+s)^c}(x,z)p_1(z,y) \geq \frac{c''}{R\log R}\sum_{z\in A_r} p_1(z,y) \end{aligned} \tag{5.55}$$

for any $\varepsilon > 0$. Note that the annulus A_r contains the disc $D(v, 2(1+\varepsilon)s)$, where $v := (r+3(1+\varepsilon)s)y/|y|$. Thus, $2(1+\varepsilon)s \leq |y-v| \leq 3(1+\varepsilon)s$, and (1.2) (where we consider $y \in \partial D(v, 2(1+\varepsilon)s)_{(1+\varepsilon)s}$), and with $s \leq (\log R)^4 \leq c(\log r)^4$,

$$\sum_{z\in A_r} p_1(z,y) \geq \sum_{z\in D(v,2(1+\varepsilon)s)} p_1(z,y) \geq ce^{-\beta((1+\varepsilon)s)^{1/4}} \geq cr^{-(1+\varepsilon)^{1/4}\beta}. \tag{5.56}$$

Hence, combining (5.55) and (5.56), and since $m \leq \sqrt{r}$, $R < r^2$, some $\varepsilon' > 0$, and $r^\beta > (\log r)^3$ for large enough r,

$$\begin{aligned} c(m^{-1}\log m)H_{D(0,r+s)}(x,y) &\geq \frac{c(m^{-1}\log m)r^{-(1+\varepsilon)^{1/4}\beta}}{R\log R} \\ &\geq cr^{-(1+\varepsilon)^{1/4}\beta}(m^{-1}\log m)(mr)^{-1}(2\log r)^{-1} \\ &\geq cr^{-1-(1+\varepsilon')\beta}m^{-2}(\log m)(\log r)^{-1} \\ &\geq cr^{-2-2\beta}(\log m)(\log r)^2 \geq cr^{-M+2}(\log r)^2, \end{aligned} \tag{5.57}$$

which proves (5.54), and hence (5.43), for $x, x' \in \partial D(0,R)_R$.

Next we show (5.43) for $x \in D(0,2R)^c$. Decompose the hitting distribution on whether or not we enter $D(0,2R)$ via the R-annulus: uniformly for $x \in D(0,2R)^c$

and $y \in \partial D(0,r)_s$,

$$H_{D(0,r+s)}(x,y) = P^x(S_{T_{D(0,r+s)}} = y\,,\, T_{\partial D(0,R)_R} > T_{D(0,R)}) + P^x(S_{T_{D(0,r+s)}} = y\,,\, T_{\partial D(0,R)_R} < T_{D(0,R)}).$$

We can bound the first term by (4.27):

$$P^x(S_{T_{D(0,r+s)}} = y\,,\, T_{\partial D(0,R)_R} > T_{D(0,R)}) \le P^x(T_{\partial D(0,R)_R} > T_{D(0,R)}) \le cR^2(\log R)^2(R^{-M} + R^{-M}) \le cR^{-M+2}(\log R)^2 < cr^{-M+2}(\log r)^2.$$

By the strong Markov property at $T_{\partial D(0,R)_R}$, the second term can be bounded, uniformly for $x' \in \partial D(0,R)_R$, by (5.43):

$$P^x(S_{T_{D(0,r+s)}} = y\,,\, T_{\partial D(0,R)_R} < T_{D(0,R)}) = \mathbb{E}^x(H_{D(0,r+s)}(S_{T_{\partial D(0,R)_R}}, y),\, T_{\partial D(0,R)_R} < T_{D(0,R)}) \le (1 + O(m^{-1}\log m))H_{D(0,r+s)}(x',y).$$

Thus, combining the two, we have for $x \in D(0,2R)^c$ and $x' \in \partial D(0,R)_R$,

$$H_{D(0,r+s)}(x,y) = \left(1 + O\left(m^{-1}\log m\right)\right) H_{D(0,r+s)}(x',y) + O(r^{-M+2}(\log r)^2), \quad (5.58)$$

which gives (5.43) for $x \in D(0,2R)^c$ and $x' \in \partial D(0,R)_R$. Applying (5.43) again for the same x' and $x'' \in D(0,2R)^c$ gives (5.43) for $x, x'' \in D(0,2R)^c$.

To prove (5.44) for $x, x' \in \partial D(0,R)_{\sqrt{R}}$, decompose $H_{D(0,r+s)}(x,y)$ over the event

$\{T_{D(0,r+s)} > T_{D(0,4mR)^c}\}$ to get

$$P^x(S_{T_{D(0,r+s)}} = y\,,\, T_{D(0,r+s)} < T_{D(0,4mR)^c}) \tag{5.59}$$
$$= H_{D(0,r+s)}(x,y) - P^x(S_{T_{D(0,r+s)}} = y\,,\, T_{D(0,r+s)} > T_{D(0,4mR)^c}).$$

By the strong Markov property at $T_{D(0,4mR)^c}$ and (5.43), the last term of (5.59) can be further decomposed to

$$P^x(S_{T_{D(0,r+s)}} = y\,,\, T_{D(0,r+s)} > T_{D(0,4mR)^c}) \tag{5.60}$$
$$= \mathbb{E}^x(H_{D(0,r+s)}(S_{T_{D(0,4mR)^c}}, y)\,;\, T_{D(0,r+s)} > T_{D(0,4mR)^c})$$
$$= (1 + O(m^{-1}\log m))H_{D(0,r+s)}(x,y)P^x(T_{D(0,r+s)} > T_{D(0,4mR)^c}),$$

which gives us the first equality in (5.44). The second follows from (2.52) and (2.47), since, for $x, x' \in \partial D(0,R)_{\sqrt{R}}$, if $|x| = R$ and $|x'| = R + \sqrt{R}$,

$$\frac{P^{x'}(T_{D(0,r+s)} < T_{D(0,4mR)^c})}{P^x(T_{D(0,r+s)} < T_{D(0,4mR)^c})} = \frac{\log\left(\frac{R+\sqrt{R}}{r+s}\right) + O(r^{-1/4})}{\log\left(\frac{R}{r+s}\right) + O(r^{-1/4})} \tag{5.61}$$
$$= 1 + O\left(\frac{\sqrt{R}}{R\log(\frac{R}{r+s})}\right) + o(r^{-1/4}) = 1 + o(m^{-1}\log m). \quad \Box$$

We now fine-tune this result for our applications.

Lemma 5.6. *As in Lemma 5.2, let* $e^n \le r$, $R = 4mr = n^3 r$. *Then, uniformly for* $x, x' \in D(0,R)^c$ *and* $y \in \partial D(0,r)_{n^4}$,

$$H_{D(0,r+n^4)}(x,y) = \left(1 + O\left(n^{-3}\log n\right)\right) H_{D(0,r+n^4)}(x',y). \tag{5.62}$$

Furthermore, for $x, x' \in \partial D(0,R)_{\sqrt{R}}$,

$$P^x(S_{T_{D(0,r+n^4)}} = y; T_{D(0,r+n^4)} < T_{D(0,n^3R)^c}) \tag{5.63}$$
$$= \left(1 + O\left(n^{-3}\log n\right)\right) H_{D(0,r+n^4)}(x,y) P^x(T_{D(0,r+n^4)} < T_{D(0,n^3R)^c})$$
$$= \left(1 + O\left(n^{-3}\log n\right)\right) P^{x'}(S_{T_{D(0,r+n^4)}} = y; T_{D(0,r+n^4)} < T_{D(0,n^3R)^c}).$$

Proof This is a direct application of Lemma 5.5. □

When attempting to move the planar exterior Harnack inequality to the torus, we run into difficulties in dealing with walks that wander and enter far-off copies of $D(0, r+s)$ instead of the primary copy. We modify the exterior Harnack inequality for the toral case to fit our requirements.

Lemma 5.7. *Let* $R = 4mr$ *with* $1 \leq m = o(r^{1/4})$ *and large enough* r, $4mR < K/4$, *and* $s \leq (\log R)^4$. *Then, uniformly for* $\hat{x}, \hat{x}' \in \hat{\pi}_K(\partial D(0,R)_{\sqrt{R}})$ *and* $\hat{y} \in \hat{\pi}_K(\partial D(0,r)_s)$,

$$P^{\hat{x}}(\hat{S}_{T_{\hat{\pi}_K(D(0,r+s))}} = \hat{y};\, T_{\hat{\pi}_K(D(0,r+s))} < T_{\hat{\pi}_K(D(0,4mR)^c_K)}) \tag{5.64}$$
$$= \left(1 + O\left(m^{-1}\log m\right)\right) P^{\hat{x}'}(\hat{S}_{T_{\hat{\pi}_K(D(0,r+s))}} = \hat{y}; T_{\hat{\pi}_K(D(0,r+s))} < T_{\hat{\pi}_K(D(0,4mR)^c_K)}).$$

As in Lemma 5.6, let $e^n \leq r$, $R = 4mr = n^3 r$. *Then, uniformly for* $\hat{x}, \hat{x}' \in \hat{\pi}_K(\partial D(0,R)_{\sqrt{R}})$ *and* $\hat{y} \in \hat{\pi}_K(\partial D(0,r)_{n^4})$,

$$P^{\hat{x}}(\hat{S}_{T_{\hat{\pi}_K(D(0,r+n^4))}} = \hat{y};\, T_{\hat{\pi}_K(D(0,r+n^4))} < T_{\hat{\pi}_K(D(0,n^3R)^c_K)}) \tag{5.65}$$
$$= \left(1 + O\left(n^{-3}\log n\right)\right) P^{\hat{x}'}(\hat{S}_{T_{\hat{\pi}_K(D(0,r+n^4))}} = \hat{y}; T_{\hat{\pi}_K(D(0,r+n^4))} < T_{\hat{\pi}_K(D(0,n^3R)^c_K)}).$$

Proof For brevity, set

$$D^* := D(0, r+s) \cup D(0, 4mR)^c, \qquad A_p := \{S_{T_{D^*}} = y\},$$

$$\hat{D}^* := \hat{\pi}_K(D(0, r+s)) \cup \hat{\pi}_K(D(0, 4mR)^c_K), \qquad A_t := \{\hat{S}_{T_{\hat{D}^*}} = \hat{y}\}.$$

We start our walk at the primary copy x, consider the planar landing at the primary copy y, and decompose $P^x(A_t) = \hat{H}_{\hat{D}^*}(x, y)$ along the planar large disc escape time $T_{D(0,4mR)^c}$ and the toral annulus escape time $T_{\hat{D}^*}$:

$$P^x(A_t) = P^x(A_t; T_{\hat{D}^*} < T_{D(0,4mR)^c}) + P^x(A_t; T_{\hat{D}^*} \geq T_{D(0,4mR)^c}). \tag{5.66}$$

Since $\hat{\pi}_K^{-1}\hat{D}^* \subset D^*$, $T_{D^*} \leq T_{\hat{D}^*}$ a.s. The first term of (5.66) happens in the event $\{T_{D^*} = T_{\hat{D}^*} = T_{D(0,r+s)}\}$, so the entirety of its action before the final step is inside the primary copy of $D(0, 4mR)$. Hence,

$$P^x(A_t; T_{\hat{D}^*} < T_{D(0,4mR)^c}) = P^x(A_t; T_{D^*} = T_{\hat{D}^*}) = P^x(A_p).$$

Note that $P^x(A_p)$ is (5.44). The second term of (5.66) only occurs if a targeted jump lands in a non-primary copy of $D(0, 4mR) \setminus D(0, r+s)$. Hence, by (2.23),

$$\begin{aligned} P^x(A_t; T_{\hat{D}^*} \geq T_{D(0,4mR)^c}) &\leq P^x(T_{D(0,4mR)^c} < T_{\hat{\pi}_K(D(0,4mR)^c_K)}) \\ &\leq O(K^{-M}(mR)^2). \end{aligned}$$

(5.66) thus reduces to $P^x(A_t) = P^x(A_p) + O(K^{-M}(mR)^2)$, which, by (5.44), is

$$P^x(A_t) = (1 + O(m^{-1}\log m))P^{x'}(A_t) + O(K^{-M}(mR)^2). \tag{5.67}$$

Since $M > 4$, the error term $O(K^{-M}(mR)^2) = o(K^{-2-\beta})$ is absorbed via (5.57) applied to the $P^x(A_p)$ term above, with (5.16) one "level" up ($D(0, 4mR)^c$ as the outer bound instead of $D(0, R)^c$, $D(0, r+s)$ instead of $D(0, \frac{r}{4m}+s)$, and $x, x' \in \partial D(0, R)_{\sqrt{R}}$ instead of $\partial D(0, r)_r$), which yields (5.64).

(5.65) is a direct application of (5.64). $\square$

6 Excursions

In this chapter we find bounds on times of excursions between concentric annuli. As in [9], for any hitting time $\hat{T}$ on the torus $\mathbb{Z}_K^2$, we set

$$||\hat{T}|| := \sup_{\hat{y}\in\mathbb{Z}_K^2} \mathbb{E}^{\hat{y}}(\hat{T}).$$

By Kac's moment formula for the strong Markov process $\hat{S}_t$ (see [14, (6)]), we have for any t and $\hat{y}$,

$$\mathbb{E}^{\hat{y}}(\hat{T}^k) \leq k!\mathbb{E}^{\hat{y}}(\hat{T})||\hat{T}||^{k-1}. \tag{6.1}$$

6.1 Between a small annulus and far out

Let $R = 4mr$. In this section, when considering visits to $\hat{x} \in \mathbb{Z}_K^2$, we will consider excursions between a small annulus and the complement of a large disc, both centered at $\hat{x}$. Define the times

$$\tau^{(0)} = \inf\{t \geq 0 : \hat{S}_t \in \hat{\pi}_K(\partial D(x,r)_s)\}, \tag{6.2}$$

$$\sigma^{(1)} = \inf\{t \geq \tau^{(0)} : \hat{S}_t \in \hat{\pi}_K(D(x,R)_K^c)\}, \tag{6.3}$$

and inductively for $j = 1, 2, \ldots$, let

$$\tau^{(j)} = \inf\{t \geq \sigma^{(j)} : \hat{S}_{t+\mathfrak{T}_{j-1}} \in \hat{\pi}_K(\partial D(x,r)_s)\}, \quad (6.4)$$

$$\sigma^{(j+1)} = \inf\{t \geq 0 : \hat{S}_{t+\mathfrak{T}_j} \in \hat{\pi}_K(D(x,R)_K^c)\}, \quad (6.5)$$

where $\mathfrak{T}_j = \sum_{i=0}^{j} \tau^{(i)}$ for $j = 0, 1, 2, \ldots$. Thus $\tau^{(j)}$ is the length of time of the jth excursion $\mathcal{E}_j$ from $\hat{\pi}_K(\partial D(x,r)_s) \to \hat{\pi}_K(D(x,R)_K^c) \to \hat{\pi}_K(\partial D(x,r)_s)$, and $\sigma^{(j)}$ is the amount of time it takes for the first leg of $\mathcal{E}_j$. From here on, set $\tau = \tau^{(1)}$.

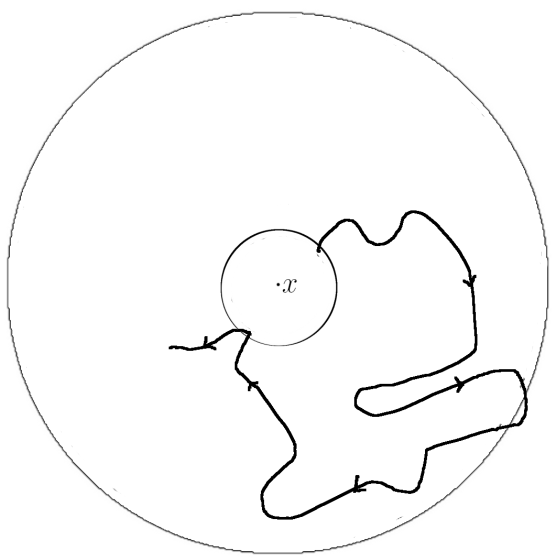

Figure 6.1: A sample excursion $\mathcal{E}_j$.

Our first lemma gives bounds on these excursion times, and shows their concentration near the asymptotic limit.

Lemma 6.1. *Uniformly for* $1 \leq m < r$, $R = 4mr$, $cK^{1-\epsilon} = R \leq \frac{K}{24}$ *for some small* $0 \leq \epsilon \ll \min\{\beta, \frac{1}{2}\}$, *and* $(\log K)^2 < s < (\log R)^4$, $\exists c_1 < \infty$ *such that* $\forall \eta$*:*

$1 \geq \eta \geq c_1 \left(\left(\frac{r}{R} \right) + s^{-1} + K^{-2\beta - 2\epsilon} (\log K)^2 \right),$

$$(1-\eta)\frac{2}{\pi_\Gamma}K^2 \log\left(\frac{R}{r}\right) \leq \min_{\hat{x},\hat{y}\in\mathbb{Z}_K^2} \mathbb{E}^{\hat{y}}(\tau) \leq \max_{\hat{x},\hat{y}\in\mathbb{Z}_K^2} \mathbb{E}^{\hat{y}}(\tau) \leq (1+\eta)\frac{2}{\pi_\Gamma}K^2 \log\left(\frac{R}{r}\right). \tag{6.6}$$

Proof Note that $\hat{x}$ is the center of the discs we will analyze. Let $\hat{S}_0$ be distributed uniformly on $\mathbb{Z}_K^2$. Then $\{\hat{S}_t\}$ is a stationary and ergodic stochastic process. By Birkhoff's ergodic theorem we then have that

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=0}^{t} 1_{\{\hat{x}\}}(\hat{S}_i) = \frac{1}{K^2} \quad \text{a.s.}$$

Thus, with $\mathfrak{T}_{-1} = 0$,

$$\lim_{t\to\infty} \frac{\frac{1}{t}\sum_{j=0}^{t}\sum_{i=0}^{\tau^{(j)}} 1_{\{\hat{x}\}}(\hat{S}_{i+\mathfrak{T}_{j-1}})}{\frac{1}{t}\sum_{j=0}^{t}\tau^{(j)}} = \frac{1}{K^2} \quad \text{a.s.} \tag{6.7}$$

Let ρ be uniform measure on $\mathbb{Z}_K^2$, and for $j \geq 1$, let

$$Z_j := \tau^{(j)} - \mathbb{E}^{\rho}(\tau^{(j)}|\mathcal{F}_{\mathfrak{T}_{j-1}}) = \tau^{(j)} - \mathbb{E}^{\hat{S}_{\mathfrak{T}_{j-1}}}(\tau).$$

By the strong Markov property, $\{Z_j\}$ is an orthogonal sequence. Since any irreducible, aperiodic Markov chain with finite state space is positive recurrent, we have that $||T_{\hat{\pi}_K(\partial D(x,r)_s)}||, ||T_{\hat{\pi}_K(D(x,R)^c_K)}|| < \infty$, and using (6.1) we see that the sequence $\{\tau^{(j)}\}$ and hence $\{Z_j\}$ has uniformly bounded second moments. It follows from Rajchman's strong law of large numbers that

$$\lim_{t\to\infty} \frac{1}{t}\sum_{j=1}^{t}[\tau^{(j)} - \mathbb{E}^{\hat{S}_{\mathfrak{T}_{j-1}}}(\tau)] = 0 \quad \text{a.s.} \tag{6.8}$$

Similarly, set $\sigma^{(0)} = \tau^{(0)}$ and for $j \geq 0$ let Y_j be the number of visits to $\hat{x}$ on the jth excursion $\hat{\pi}_K(\partial D(x,r)_s) \to \hat{\pi}_K(D(x,R)^c_K) \to \hat{\pi}_K(\partial D(x,r)_s)$:

$$Y_j := \sum_{i=0}^{\tau^{(j)}} 1_{\{\hat{x}\}}(\hat{S}_{i+\mathfrak{T}_{j-1}}) = \sum_{i=0}^{\sigma^{(j)}} 1_{\{\hat{x}\}}(\hat{S}_{i+\mathfrak{T}_{j-1}}) + \sum_{i=\sigma^{(j)}+1}^{\tau^{(j)}} 1_{\{\hat{x}\}}(\hat{S}_{i+\mathfrak{T}_{j-1}}). \tag{6.9}$$

Define

$$\tilde{Y}_j := Y_j - \mathbb{E}^{\rho}(Y_j|\mathcal{F}_{\mathfrak{T}_{j-1}}) = Y_j - \mathbb{E}^{\hat{S}_{\mathfrak{T}_{j-1}}}(Y_1).$$

By the strong Markov property, $\{\tilde{Y}_j\}$ is also an orthogonal sequence, and since $Y_j \leq \tau^{(j)}$, the sequence $\{\tilde{Y}_j\}$ also has uniformly bounded second moments. Thus, by Rajchman's strong law of large numbers,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{j=1}^{t} [Y_j - \mathbb{E}^{\hat{S}_{\mathfrak{T}_{j-1}}}(Y_1)] = 0 \quad \text{a.s.} \tag{6.10}$$

Let $\hat{y} \in \hat{\pi}_K(\partial D(x,r)_s)$. To bound $\mathbb{E}^{\hat{y}}(Y_1)$ we need to consider the two sums in (6.9). By (2.32), (6.9), and the strong Markov property at $\sigma^{(1)}$, we have

$$\mathbb{E}^{\hat{y}}(Y_1) = \hat{G}_{\hat{\pi}_K(D(x,R))}(\hat{y},\hat{x}) + \mathbb{E}^{\hat{y}}\left(\hat{G}_{\hat{\pi}_K((\partial D(x,r)^c_s)_K)}\left(\hat{S}_{T_{\hat{\pi}_K(D(x,R)^c_K)}},\hat{x}\right)\right).$$

By (2.56), for some constant $c^* = c^*(\hat{p}_1)$, and any $\hat{y} \in \hat{\pi}_K(\partial D(x,r)_s)$,

$$\hat{G}_{\hat{\pi}_K(D(x,R))}(\hat{y},\hat{x}) = \frac{2}{\pi_\Gamma}\log\left(\frac{R}{r}\right) + c^* + O(r^{-1/4}).$$

Also, $O(R) \leq |\hat{S}_{T_{\hat{\pi}_K(D(x,R)^c_K)}} - \hat{x}| \leq O(K)$, so by (4.42) and $(\log K)^2 < s$,

$$\mathbb{E}^{\hat{y}}\left(\hat{G}_{\hat{\pi}_K((\partial D(x,r)^c_s)_K)}\left(\hat{S}_{T_{\hat{\pi}_K(D(x,R)^c_K)}},\hat{x}\right)\right) \leq c(\log K)^2 s^{-M+2} \leq cs^{-M+3} = o(s^{-1}).$$

Hence, for some finite universal constant $c_0 > 0$ and all allowable s,

$$\frac{2}{\pi_\Gamma}\log\left(\frac{R}{r}\right) + c^* - c_0 s^{-1} \leq \min_{\hat{x}} \min_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(Y_1) \tag{6.11}$$
$$\leq \max_{\hat{x}} \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(Y_1) \leq \frac{2}{\pi_\Gamma}\log\left(\frac{R}{r}\right) + c^* + c_0 s^{-1}.$$

With $\tau^{(0)}$ finite, we get by combining (6.7), (6.8), and (6.10) that, a.s.,

$$\lim_{t\to\infty} \frac{\frac{1}{t}\sum_{j=1}^{t} \mathbb{E}^{\hat{S}_{\mathfrak{T}_{j-1}}}(\tau)}{\frac{1}{t}\sum_{j=1}^{t} \mathbb{E}^{\hat{S}_{\mathfrak{T}_{j-1}}}(Y_1)} = K^2. \tag{6.12}$$

Consequently, in view of (6.11), for some universal constant c_2 and all $1 \geq \eta \geq c_2\left(s^{-1} + \frac{r}{R}\right)$,

$$\min_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau) \leq \frac{2}{\pi_\Gamma}K^2\left(1+\frac{\eta}{3}\right)\log\left(\frac{R}{r}\right)$$
$$\max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau) \geq \frac{2}{\pi_\Gamma}K^2\left(1-\frac{\eta}{3}\right)\log\left(\frac{R}{r}\right) \tag{6.13}$$

For $\hat{y} \in \hat{\pi}_K(\partial D(x,r)_s)$ we have $\tau^{(0)} = 0$ and by the strong Markov property at $\sigma^{(1)}$,

$$\mathbb{E}^{\hat{y}}(\tau) = \mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(x,R)^c_K)}) + \sum_{\hat{z}\in\hat{\pi}_K(D(x,R)^c_K)} \hat{H}_{\hat{\pi}_K(D(x,R)^c_K)}(\hat{y},\hat{z})\,\mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)}). \tag{6.14}$$

By (2.28) and $R = cK^{1-\epsilon}$,

$$\mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(x,R)^c_K)}) = cK^{2-2\epsilon} + O(K^{1-\epsilon}) \tag{6.15}$$

for every $\hat{y} \in \hat{\pi}_K(\partial D(x,r)_s)$. Hence,

$$\max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(x,R)^c_K)}) \leq \left(1+O\left(\frac{r}{R}\right)\right) \min_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(x,R)^c_K)}). \tag{6.16}$$

For the sum in (6.14), the Harnack inequality (5.19) yields, for any $\hat{y}, \hat{y}' \in \hat{\pi}_K(\partial D(x,r)_s)$,

$$\begin{aligned}\sum_{\hat{z}\in\hat{\pi}_K(D(x,R)^c_K)} & \hat{H}_{\hat{\pi}_K(D(x,R)^c_K)}(\hat{y},\hat{z})\mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)}) \qquad (6.17)\\ &= \left(1+O\left(\frac{r}{R}\right)\right) \sum_{\hat{z}\in\hat{\pi}_K(D(x,R)^c_K)} \hat{H}_{\hat{\pi}_K(D(x,R)^c_K)}(\hat{y}',\hat{z})\,\mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)})\\ &\quad + O(R^{-M}\log R \vee K^{-M}R^2) \sum_{\hat{z}\in\hat{\pi}_K(D(x,R+s)^c_K)} \mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)}).\end{aligned}$$

The last term of (6.17) is zero if p_1 is finite range, by taking s large enough so, due to (5.19), the error term does not appear. Otherwise, the sum needs to be controlled: since $R = cK^{1-\epsilon}$ and $\epsilon \geq 0$ is small, the Harnack inequality error is bounded above by

$$cR^{-M}\log R = c'K^{-4-2\beta+\epsilon(4+2\beta)}\log K \ll cK^{-M}R^2 \;\; = cK^{-4-2\beta+2-2\epsilon} = cK^{-2-2\beta-2\epsilon}$$

and by (4.46) with $R = cK^{1-\epsilon}$, the sum is bounded by $cK^{4-2\epsilon}(\log K)^2$. Together these, with (6.15) and (6.16), bound the last term of (6.17):

$$\begin{aligned}c(R^{-M}\log R \vee K^{-M}R^2) & \sum_{\hat{z}\in\hat{\pi}_K(D(x,R)^c_K)} \mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)}) \qquad (6.18)\\ &\leq cK^{2-2\beta-4\epsilon}(\log K)^2 \leq cK^{-2\beta-2\epsilon}(\log K)^2 \min_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(x,R)^c_K)}).\end{aligned}$$

Hence, by (6.14)-(6.18),

$$\max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau) \le \Bigg(1 + O\left(\frac{r}{R}\right) + O(s^{-1}) + O\left(K^{-2\beta-2\epsilon}(\log K)^2\right)\Bigg) \min_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau). \tag{6.19}$$

Taking also $c_1 \ge 3c_0$, we get (6.6) by combining (6.13) and (6.19). □

The next corollary gives upper bounds for the hitting time of $\hat{\pi}_K(\partial D(x,r)_s)$, and improves on (3.20) for certain large radii.

Corollary 6.2. *With the same hypotheses as above,*

$$\max_{\hat{x}\in\mathbb{Z}_K^2} \max_{\hat{w}\in\hat{\pi}_K(\partial D(x,R)_R)} \mathbb{E}^{\hat{w}}(T_{\hat{\pi}_K(\partial D(x,r)_s)}) \le c_1 K^2 \log\left(\frac{R}{r}\right); \tag{6.20}$$

$$\max_{\hat{x}\in\mathbb{Z}_K^2} ||T_{\hat{\pi}_K(\partial D(x,r)_s)}|| \le c_1 K^2 \log\left(\frac{K}{r}\right). \tag{6.21}$$

Proof Consider (6.14) for $\hat{y} \in \hat{\pi}_K(\partial D(x,r)_s)$ escaping to $\hat{\pi}_K(D(x,4R)_K^c)$ instead of $\hat{\pi}_K(D(x,R)_K^c)$, before returning. Then, by (6.6),

$$\sum_{\hat{z}\in\hat{\pi}_K(D(x,4R)_K^c)} \hat{H}_{\hat{\pi}_K(D(x,4R)_K^c)}(\hat{y},\hat{z})\, \mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)}) \le cK^2\log(4R/r) \le c'K^2\log(R/r). \tag{6.22}$$

Using the strong Markov property at $T_{\hat{\pi}_K(D(x,4R)_K^c)}$, (2.28), (5.19), (6.22), (4.46), and

(6.18), we have for any $\hat{w} \in \hat{\pi}_K(\partial D(x,R)_R)$ and some universal $c < \infty$,

$$\begin{aligned}
\mathbb{E}^{\hat{w}}(T_{\hat{\pi}_K(\partial D(x,r)_s)}) &\leq \mathbb{E}^{\hat{w}}(T_{\hat{\pi}_K(D(x,4R)^c_K)}) \qquad (6.23)\\
&\quad + \mathbb{E}^{\hat{w}}(T_{\hat{\pi}_K(\partial D(x,r)_s)} - T_{\hat{\pi}_K(D(x,4R)^c_K)}; T_{\hat{\pi}_K(\partial D(x,r)_s)} > T_{\hat{\pi}_K(D(x,4R)^c_K)})\\
&\leq c\left[(4R+1)^2 + \sum_{\hat{z} \in \hat{\pi}_K(D(x,4R)^c_K)} \hat{H}_{\hat{\pi}_K(D(x,4R)^c_K)}(\hat{w},\hat{z})\, \mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)})\right]\\
&\leq c\Bigg[(4R+1)^2 + \sum_{\hat{z} \in \hat{\pi}_K(D(x,4R)^c_K)} \left[\left(1 + O\left(\frac{r}{R}\right)\right) \hat{H}_{\hat{\pi}_K(D(x,4R)^c_K)}(\hat{y},\hat{z})\right] \mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)})\\
&\quad + O(R^{-M}\log R \vee K^{-M}R^2) \sum_{\hat{z} \in \hat{\pi}_K(D(x,4R+s)^c_K)} \mathbb{E}^{\hat{z}}(T_{\hat{\pi}_K(\partial D(x,r)_s)})\Bigg] \leq cK^2 \log(R/r).
\end{aligned}$$

Setting $c_1 \geq c$, we have (6.20). (6.21) follows directly from (6.20), by considering S projected onto $\mathbb{Z}^2_{24K}$ instead of $\mathbb{Z}^2_K$ for the furthest-out points $\hat{w}$. Note that, for these $\hat{w}$ such that $|\hat{w} - \hat{x}| > \frac{K}{24}$ on $\mathbb{Z}^2_K$, (6.20) on $\mathbb{Z}^2_{24K}$ and the fact that annulus entrance takes longer on larger spaces,

$$\mathbb{E}^{\hat{w}}(T_{\hat{\pi}_K(\partial D(x,r)_s)}) \leq \mathbb{E}^{\hat{w}}(T_{\hat{\pi}_{24K}(\partial D(x,r)_s)}) \leq c(24K)^2 \log(24K/r) \leq c_1 K^2 \log(K/r). \quad \square$$

6.2 Decoupling an excursion from its endpoints

Let $n > 13$ and set the following variables as defined in (4.34):

$$r_{n,k} = e^n n^{3k}, s_k = n^4, \qquad r'_{n,k} = r_{n,k} + s_k, \qquad k = 0,1,\ldots,n;$$
$$s^{n\downarrow}_{n-1} = \sqrt{r_{n,n-1}}$$

and set $K_n := n^{\overline{\gamma}} r_{n,n}$, where $\overline{\gamma} \in [b, b+4]$ for some $b = b(p_1) \geq 10$, to be determined in Chapter 7.

We say that, for a point $\hat{x} \in \mathbb{Z}^2_K$, and a path ω starting at $\hat{x}_0 \in \mathbb{Z}^2_{K_n}$, $\hat{x}_0 \neq \hat{x}$, the path ω **does not skip $\hat{x}$-bands** if the path's entrances and exits from the $r_{n,k}$-sized concentric discs around $\hat{x}$ are made by small or annulus-targeted jumps, not by medium or large untargeted jumps. More formally, a path does not skip $\hat{x}$-bands for a specified period of time if, during that time, escapes from $\hat{\pi}_K(D(x, r_{n,k}))$ and entrances to $\hat{\pi}_K(D(x, r'_{n,k}))$ land in $\hat{\pi}_K(\partial D(x, r_{n,k})_{s_k})^*$.

By the strong Markov property, the only effect that one excursion between annuli has on another is via its beginning and ending points. In this section we build a structure in which to analyze the dependence on these endpoints for a special class of excursions.

The excursions we wish to examine are those from inside $\hat{\pi}_K(D(0, r'_{n,l-1}))$ out to $\hat{\pi}_K(D(0, r_{n,l})^c_K)$ prior to "one larger" disc escape at $T_{\hat{\pi}_K(D(0,r_{n,l+1})^c_K)}$. Consider a random path starting between these sets at $\hat{z} \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})$. Focusing on annulus-based excursion end points $\hat{w} \in \hat{\pi}_K(\partial D(0, r_{n,l+1})_{s_{l+1}})$ and l large, let $\mathcal{H}_{n,l-1\uparrow l}$ be the σ-algebra of outward excursions $\hat{\pi}_K(D(0, r'_{n,l-1})) \to \hat{\pi}_K(D(0, r_{n,l})^c_K)$ prior to $T_{\hat{\pi}_K(D(0,r_{n,l+1})^c_K)}$. Let $\tau_0 = 0$, and for $i = 0, 1, 2, \ldots$, define the excursion endpoint times

$$\begin{aligned}
\tau_{2i+1} &= \inf\{k \geq \tau_{2i} : \hat{S}_k \in \hat{\pi}_K(D(0, r'_{n,l-1})) \cup \hat{\pi}_K(D(0, r_{n,l+1})^c_K)\} \\
\tau_{2i+2} &= \inf\{k \geq \tau_{2i+1} : \hat{S}_k \in \hat{\pi}_K(D(0, r_{n,l})^c_K)\}.
\end{aligned}$$

Abbreviating $\overline{\tau} = T_{\hat{\pi}_K(D(0,r_{n,l+1})^c_K)}$, note that $\overline{\tau} = \tau_{2I+1}$ for some (unique) non-negative integer I. Then $\mathcal{H}_{n,l-1\uparrow l}$ is the σ-algebra generated by the excursions $\{\hat{e}_{(j)} : j =$

---

*That is, with the exception of level $n-1$: entrances to $\hat{\pi}_K(D(x, r_{n,n-1} + s^{n\downarrow}_{n-1}))$ land in the thicker band $\hat{\pi}_K(\partial D(x, r_{n,n-1})_{s^{n\downarrow}_{n-1}})$. This is for the purposes of re-entering the level structure from the outermost level n; see (7.22) for details, and assume this notation for excursions from level n down to level $n-1$ if it is not mentioned.

$1, \ldots, I\}$, where $\hat{e}_{(j)} = \{\hat{S}_k : \tau_{2j-1} \leq k \leq \tau_{2j}\}$ is the jth excursion $\hat{\pi}_K(D(0, r'_{n,l-1})) \to \hat{\pi}_K(D(0, r_{n,l})^c_K)$. (The event $\{I = 0\}$ is, of course, also included.)

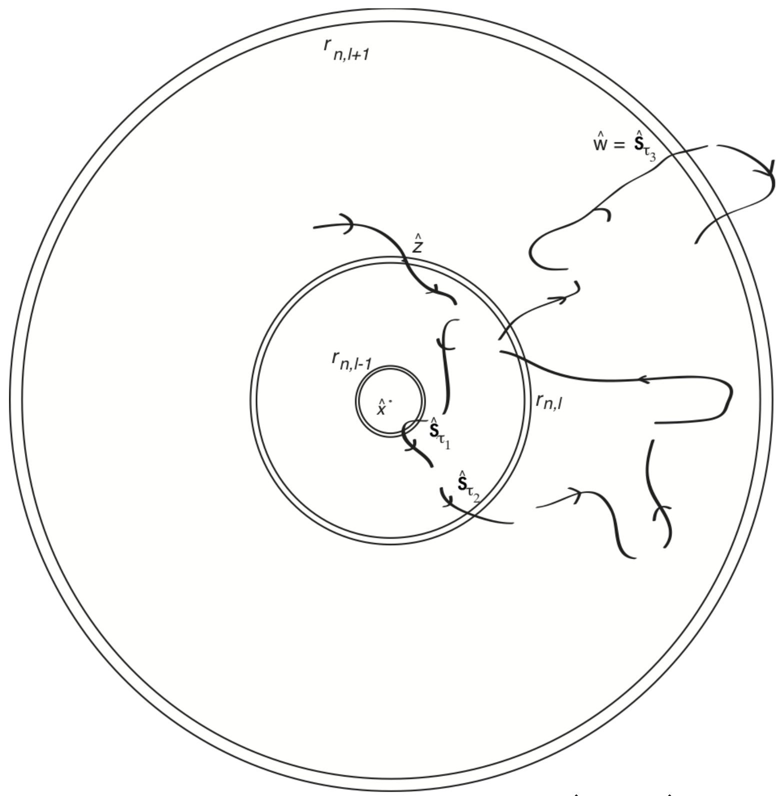

Figure 6.2: Sample excursions - $\hat{e}_{(1)}$ is between $\hat{S}_{\tau_1}$ and $\hat{S}_{\tau_2}$. $I = 2$ for this path.

Let $\mathcal{F}_j = \sigma(\hat{S}_k : k = 0, 1, \ldots, j)$, and for any stopping time τ, let $\mathcal{F}_\tau$ denote the collection of events A such that $A \cap \{\tau = j\} \in \mathcal{F}_j$ for all j.

We will focus on paths which do not skip $\hat{x}$-bands over a number of concentric annulus excursions. Let $\Omega^{i-1,\ldots,j}_{\hat{x},n,l+1,m}$ denote the set of paths which do not skip $\hat{x}$-bands on excursions between levels $k = i-1, i, \ldots, j$ until completion of the first m outward excursions from $\hat{\pi}_K(D(x, r'_{n,l})) \to \hat{\pi}_K(D(x, r_{n,l+1})^c_K)$, and $\Omega^A_{\hat{x},n,l+1,m}$ the same for the levels in the index set A. Our first lemma shows that excursion paths faithful to hitting $\hat{x}$-bands are "almost" independent of their beginning and ending points.

Lemma 6.3. *Uniformly in l, n, K_n, $B_n \in \mathcal{H}_{n,l-1\uparrow l}$, $\hat{z}, \hat{z}' \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})$, and $\hat{w} \in \hat{\pi}_K(\partial D(0, r_{n,l+1})_{s_{l+1}})$,*

$$P^{\hat{z}}\left(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1} \,\middle|\, \hat{S}_{T_{\hat{\pi}_K(D(0,r_{n,l+1})^c_K)}} = \hat{w}\right) = (1 + O(n^{-3}))P^{\hat{z}}\left(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}\right) \tag{6.24}$$

and

$$P^{\hat{z}}\left(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}\right) = (1 + O(n^{-3}\log n))P^{\hat{z}'}\left(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}\right). \tag{6.25}$$

Proof Fixing a starting point $\hat{z} \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})$, it suffices to consider $B_n \in \mathcal{H}_{n,l-1\uparrow l}$ such that $P^{\hat{z}}(B_n) > 0$. Fix such a set B_n and an ending point $\hat{w} \in \hat{\pi}_K(\partial D(0, r_{n,l+1})_{s_{l+1}})$. Using the notation just introduced, for any $i \geq 1$, we can write

$$\begin{aligned} &B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1} \cap \{I = i\} \\ &\quad = B_{n,i} \cap A_i \cap \{\tau_{2i} < \overline{\tau}\} \cap (\{I = 0, \hat{S}_{\overline{\tau}} \in \hat{\pi}_K(\partial D(0, r_{n,l+1})_{s_{l+1}})\} \circ \theta_{\tau_{2i}}) \end{aligned}$$

for some $B_{n,i} \in \mathcal{F}_{\tau_{2i}}$, where

$$A_i = \{\hat{S}_{\tau_{2j-1}} \in \hat{\pi}_K(\partial D(0, r_{n,l-1})_{s_{l-1}}),\, \hat{S}_{\tau_{2j}} \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})\,, \forall j \leq i\} \in \mathcal{F}_{\tau_{2i}},$$

so by the strong Markov property at τ_{2i},

$$P^{\hat{z}}(\{\hat{S}_{\overline{\tau}} = \hat{w}\} \cap B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1} \cap \{I = i\})$$
$$= \mathbb{E}^{\hat{z}}[P^{\hat{S}_{\tau_{2i}}}(\hat{S}_{\overline{\tau}} = \hat{w};\ I = 0); B_{n,i} \cap A_i \cap \{\tau_{2i} < \overline{\tau}\}];$$
$$P^{\hat{z}}(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1} \cap \{I = i\})$$
$$= \mathbb{E}^{\hat{z}}[P^{\hat{S}_{\tau_{2i}}}(\hat{S}_{\overline{\tau}} \in \hat{\pi}_K(\partial D(0, r_{n,l+1})_{s_{l+1}});\ I = 0); B_{n,i} \cap A_i \cap \{\tau_{2i} < \overline{\tau}\}].$$

Consequently, for all $i \geq 1$,

$$P^{\hat{z}}(\{\hat{S}_{\overline{\tau}} = \hat{w}\} \cap B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1} \cap \{I = i\}) \tag{6.26}$$
$$\geq P^{\hat{z}}(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1} \cap \{I = i\}) \min_{\hat{x} \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})} \frac{P^{\hat{x}}(\hat{S}_{\overline{\tau}} = \hat{w}; I = 0)}{P^{\hat{x}}(\hat{S}_{\overline{\tau}} \in \hat{\pi}_K(\partial D(0, r_{n,l+1})_{s_{l+1}}); I = 0)}.$$

Note that

$$\{I = 0\} = \{\tau = T_{\hat{\pi}_K(D(0, r_{n,l+1})^c_K)} < T_{\hat{\pi}_K(D(0, r'_{n,l-1}))}\}.$$

Necessarily, $P^{\hat{z}}(B_n | I = 0) \in \{0, 1\}$ and is independent of $\hat{z}$ for any $B_n \in \mathcal{H}_{n,l-1\uparrow l}$, implying that (6.26) applies for $i = 0$ as well. Hence, by (5.22) and (5.21), there exists $c < \infty$ such that for any $\hat{z}, \hat{x} \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})$ and $\hat{w} \in \hat{\pi}_K(\partial D(0, r_{n,l+1})_{s_{l+1}})$,

$$\frac{P^{\hat{x}}(\hat{S}_{\overline{\tau}} = \hat{w}; I = 0)}{P^{\hat{x}}(\hat{S}_{\overline{\tau}} \in \hat{\pi}_K(\partial D(0, r_{n,l+1})_{s_{l+1}}); I = 0)} \geq (1 - cn^{-3}) \hat{H}_{\hat{\pi}_K(D(0, r_{n,l+1})^c_K)}(\hat{z}, \hat{w}).$$

We note that, since (5.22) and (5.21) accommodate starting points up to a square root of the distance away from their level's starting radius of $r_{n,l}$, this bound is good for even the wide band $s^{n\downarrow}_{n-1} = \sqrt{r_{n,n-1}} \ll r_{n,n-1}$ as a starting point (this is the case $l = n - 1$).

Hence, summing (6.26) over $I = 0, 1, \ldots$, we get that

$$P^{\hat{z}}(\{\hat{S}_{\overline{\tau}} = \hat{w}\} \cap B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}) \geq (1 - cn^{-3}) P^{\hat{z}}(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}) \hat{H}_{\hat{\pi}_K(D(0,r_{n,l+1})^c_K)}(\hat{z}, \hat{w}).$$

A similar argument shows that

$$P^{\hat{z}}(\{\hat{S}_{\overline{\tau}} = \hat{w}\} \cap B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}) \leq (1 + cn^{-3}) P^{\hat{z}}(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}) \hat{H}_{\hat{\pi}_K(D(0,r_{n,l+1})^c_K)}(\hat{z}, \hat{w}),$$

and we obtain (6.24).

By the strong Markov property at τ_1, for any $\hat{z} \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})$,

$$\begin{aligned} P^{\hat{z}}(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}) &= P^{\hat{z}}(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1} \cap \{I = 0\}) \\ &+ \sum_{\hat{x} \in \hat{\pi}_K(\partial D(0,r_{n,l-1})_{s_{l-1}})} \hat{H}_{\hat{\pi}_K(D(0,r'_{n,l-1})) \cup \hat{\pi}_K(D(0,r_{n,l+1})^c)}(\hat{z}, \hat{x}) P^{\hat{x}}(B_n \cap \Omega^{l-1,l,l+1}_{\hat{0},n,l+1,1}). \end{aligned}$$

The first term is handled by (2.53). (6.25) then follows from (5.65). □

Next, we examine excursions going inward: let $\mathcal{G}^{\hat{x}}_{n,l+1\downarrow l}$ denote the σ-algebra of excursions from $\hat{\pi}_K(D(x, r_{n,l+1})^c_{K_n})$ into $\hat{\pi}_K(D(x, r'_{n,l}))$. To this end, let $\hat{x} \in \mathbb{Z}^2_{K_n}$, let $\overline{\tau}_0 = 0$ and for $i = 1, 2, \ldots$ define

$$\begin{aligned} \tau_i &= \inf\{k \geq \overline{\tau}_{i-1} : \hat{S}_k \in \hat{\pi}_K(D(x, r'_{n,l}))\}, \\ \overline{\tau}_i &= \inf\{k \geq \tau_i : \hat{S}_k \in \hat{\pi}_K(D(x, r_{n,l+1})^c_{K_n})\}. \end{aligned}$$

Then $\mathcal{G}^{\hat{x}}_{n,l+1\downarrow l}$ is the σ-algebra generated by the excursions $\{\hat{e}^{(j)} : j = 1, \ldots\}$, where $\hat{e}^{(j)} = \{\hat{S}_k : \overline{\tau}_{j-1} \leq k \leq \tau_j\}$ is the jth excursion $\hat{\pi}_K(D(x, r_{n,l+1})^c_{K_n}) \to \hat{\pi}_K(D(x, r'_{n,l}))$ (so for $j = 1$ we begin at $t = 0$).

Let $\mathcal{H}^{\hat{x}}_{n,l-1\uparrow l}(m)$ be the σ-algebra of excursions from $\hat{\pi}_K(D(x, r'_{n,l-1}))$ out to $\hat{\pi}_K(D(x, r_{n,l})^c_{K_n})$

during the first m excursions from $\hat\pi_K(D(x,r'_{n,l}))$ out to $\hat\pi_K(D(x,r_{n,l+1})^c_{K_n})$, *i.e.*, from τ_1 to $\overline{\tau}_m$. In more detail, for each $j=1,2,\ldots,m$, let $\overline{\zeta_{j,0}}=\tau_j$ and for $i=1,\ldots,$ define

$$\begin{aligned}\zeta_{j,i} &= \inf\{k\geq\overline{\zeta}_{j,i}:\hat S_k\in\hat\pi_K(D(x,r'_{n,l-1}))\},\\ \overline{\zeta}_{j,i} &= \inf\{k\geq\zeta_{j,i}:\hat S_k\in\hat\pi_K(D(x,r_{n,l})^c_{K_n})\},\\ v_{j,i} &= \{\hat S_k:\zeta_{j,i}\leq k\leq\overline{\zeta_{j,i}}\},\\ Z^j &= \sup\{i\geq 0:\overline{\zeta}_{j,i}<\overline{\tau}_j\}.\end{aligned}$$

Then $\mathcal{H}^{\hat x}_{n,l-1\uparrow l}(m)$ is the σ-algebra generated by the intersection of the σ-algebras $\mathcal{H}^{\hat x}_{n,l,j}=\sigma(v_{j,i}:i=1,2,\ldots,Z^j)$ of the excursions between τ_j and $\overline{\tau}_j$, for $j=1,2,\ldots,m$.

Lemma 6.4. *There exists $C<\infty$ such that, uniformly over all $m\leq(n\log n)^2$, $l,\hat x\in\mathbb{Z}^2_{K_n}$ and $\hat y_0,\hat y_1\in\mathbb{Z}^2_{K_n}\setminus\hat\pi_K(D(x,r'_{n,l}))$, and $H\in\mathcal{H}^{\hat x}_{n,l-1\uparrow l}(m)$,*

$$\begin{aligned}&(1-Cmn^{-3}\log n)P^{\hat y_1}(H\cap\Omega^{l-1,l,l+1}_{\hat x,n,l+1,m})\\ &\quad\leq P^{\hat y_0}(H\cap\Omega^{l-1,l,l+1}_{\hat x,n,l+1,m}|\mathcal{G}^{\hat x}_{n,l+1\downarrow l})\leq(1+Cmn^{-3}\log n)P^{\hat y_1}(H\cap\Omega^{l-1,l,l+1}_{\hat x,n,l+1,m}).\end{aligned}\tag{6.27}$$

Proof Applying the Monotone Class Theorem to the algebra of their finite disjoint unions, it suffices to prove (6.27) for the generators of the σ-algebra $\mathcal{H}^{\hat x}_{n,l-1\uparrow l}(m)$ of the form $H=H_1\cap H_2\cap\cdots\cap H_m$, with $H_j\in\mathcal{H}^{\hat x}_{n,l,j}$ for $j=1,\ldots,m$. Conditioned upon $\mathcal{G}^{\hat x}_{n,l+1\downarrow l}$, the events H_j are independent. Further, each H_j then has the conditional law of an event B_j in the σ-algebra $\mathcal{H}_{n,l-1\uparrow l}$ of Lemma 6.3, for some random end points $\hat z_j=\hat S_{\tau_j}-\hat x\in\hat\pi_K(\partial D(0,r_{n,l})_{s_l})$ and $\hat w_j=\hat S_{\overline{\tau}_j}-\hat x\in\hat\pi_K(\partial D(0,r_{n,l+1})_{s_{l+1}})$, both measurable on $\mathcal{G}^{\hat x}_{n,l+1\downarrow l}$. By our conditions, the uniform estimates (6.24) and (6.25) yield that for any fixed $\hat{z}' \in \hat{\pi}_K(\partial D(0, r_{n,l})_{s_l})$,

$$\begin{aligned} P^{\hat{y}_0}(H \cap \Omega^{l-1,l,l+1}_{\hat{x},n,l+1,m} | \mathcal{G}^{\hat{x}}_{n,l+1\downarrow l}) &= P^{\hat{y}_0}(\cap_{j=1}^m (H_j \cap \Omega^{l-1,l,l+1}_{\hat{x},n,l+1,1}) | \mathcal{G}^{\hat{x}}_{n,l+1\downarrow l}) \\ &= \prod_{j=1}^m P^{\hat{z}_j}(B_j \cap \Omega^{l-1,l,l+1}_{\hat{x},n,l+1,1} | \hat{S}_{T_{D(0,r_{n,l})^c}} = \hat{w}_j) \\ &= \prod_{j=1}^m (1 + O(n^{-3})) P^{\hat{z}_j}(B_j \cap \Omega^{l-1,l,l+1}_{\hat{x},n,l+1,1}) \\ &= (1 + O(n^{-3}\log n))^m \prod_{j=1}^m P^{\hat{z}'}(B_j \cap \Omega^{l-1,l,l+1}_{\hat{x},n,l+1,1}). \end{aligned}$$

Since $m \le (n \log n)^2$ and the last expression above neither depends on $\hat{y}_0 \in \mathbb{Z}^2_K$ nor on the extra information in $\mathcal{G}^{\hat{x}}_{n,l+1\downarrow l}$, we get (6.27). □

Now that we have control over the excursion structure of paths that do not skip $\hat{x}$-bands, we will control their layered excursion counts. Fix $0 < a < 2$, and define $v_k = v_k(a) := 3ak^2 \log k$ for $k = 2, 3, ..., n$, and $N^{\hat{x}}_{n,l}$, $l = 2, \dots, n-1$, as the number of excursions from $\hat{\pi}_K(D(x, r'_{n,l-1}))$ out to $\hat{\pi}_K(D(x, r_{n,l})^c_{K_n})$ until time $\mathcal{R}^{\hat{x}}_n(a)$, the time that v_n excursions from $\hat{\pi}_K(D(x, r_{n,n-1}))$ out to $\hat{\pi}_K(D(x, r_{n,n})^c_{K_n})$ have been completed. Let $m \overset{k}{\sim} v$ denote the bound $|m - v| \le k$. Finally, let $N^{\hat{x}}_{n,0}$ be the number of visits to $\hat{x}$ before $\mathcal{R}^{\hat{x}}_n(a)$.

Lemma 6.5. *Let* $\Gamma^{\hat{y}}_{n,l} := \{N^{\hat{y}}_{n,i} = m_i : i = 0, 2, \dots, l-1\} \cap \Omega^{1,\dots,l}_{\hat{y},n,l+1,m_l}$. *Then, for any* $1 < n_0 < n$, *uniformly over all* $n_0 \le l \le n-1$, $m_l \overset{l}{\sim} v_l$, $\{m_i : i = 0, 2, \dots, l\}$, $\hat{y} \in \mathbb{Z}^2_{K_n}$, *and* $\hat{x}_0, \hat{x}_1 \in \mathbb{Z}^2_{K_n} \setminus \hat{\pi}_K(D(y, r'_{n,l}))$,

$$\begin{aligned} &P^{\hat{x}_0}(\Gamma^{\hat{y}}_{n,l}, N^{\hat{y}}_{n,l} = m_l | \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}) \\ &\quad = (1 + O(n^{-1}(\log n)^2)) P^{\hat{x}_1}(\Gamma^{\hat{y}}_{n,l} | N^{\hat{y}}_{n,l} = m_l) 1_{\{N^{\hat{y}}_{n,l} = m_l\}}. \end{aligned} \tag{6.28}$$

Proof For $j = 1, 2, \dots$ and $i = 2, \dots, l$, let Z^j_i denote the number of excursions from $\hat{\pi}_K(D(x, r'_{n,i}))$ out to $\hat{\pi}_K(D(x, r_{n,i+1})^c_{K_n})$ by the random walk during the time interval

$[\tau_j, \overline{\tau_j}]$. The event

$$H = \left\{ \sum_{j=1}^{m_l} Z_i^j = m_i : i = 2, \ldots, l-1 \right\} \cap \Omega^{2,\ldots,l-1}_{\hat{y},n,l,m_{l-1}}$$

belongs to the σ-algebra $\mathcal{H}^{\hat{y}}_{n,l-1\uparrow l}(m_l)$ of Lemma 6.4. It is easy to verify that, starting at any $\hat{x}_0 \notin \hat{\pi}_K(D(y, r'_{n,l}))$, when the event $\{N^{\hat{y}}_{n,l} = m_l\} \in \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}$ occurs, it implies that $N^{\hat{y}}_{n,i} = \sum_{j=1}^{m_l} Z_i^j$ for $i = 2, \ldots, l$. Thus, setting $H' = H \cap \Omega^{l-1,l,l+1}_{\hat{y},n,l+1,m_l}$,

$$P^{\hat{x}_0}(\Gamma^{\hat{y}}_{n,l} | \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}) 1_{\{N^{\hat{y}}_{n,l} = m_l\}} = P^{\hat{x}_0}(H' | \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}) 1_{\{N^{\hat{y}}_{n,l} = m_l\}}. \tag{6.29}$$

With $m_l/(l^2 \log l)$ bounded above, by (6.27) we have, uniformly in $\hat{y} \in \mathbb{Z}^2_{K_n}$ and $\hat{x}_0, \hat{x}_1 \in \mathbb{Z}^2_{K_n} \setminus \hat{\pi}_K(D(y, r'_{n,l}))$,

$$P^{\hat{x}_0}(H' | \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}) = (1 + O(n^{-1}(\log n)^2)) P^{\hat{x}_0}(H'). \tag{6.30}$$

Hence,

$$P^{\hat{x}_0}(\Gamma^{\hat{y}}_{n,l} | \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}) 1_{\{N^{\hat{y}}_{n,l} = m_l\}} = (1 + O(n^{-1}(\log n)^2)) P^{\hat{x}_1}(H') 1_{\{N^{\hat{y}}_{n,l} = m_l\}}. \tag{6.31}$$

Setting $\hat{x}_0 = \hat{x}_1$ and taking expectations with respect to $P^{\hat{x}_0}$ yields

$$P^{\hat{x}_1}(\Gamma^{\hat{y}}_{n,l} | N^{\hat{y}}_{n,l} = m_l) = (1 + O(n^{-1}(\log n)^2)) P^{\hat{x}_1}(H') \tag{6.32}$$

$$\begin{aligned} \implies P^{\hat{x}_1}(\Gamma^{\hat{y}}_{n,l} | N^{\hat{y}}_{n,l} = m_l) 1_{\{N^{\hat{y}}_{n,l} = m_l\}} &= (1 + O(n^{-1}(\log n)^2)) P^{\hat{x}_1}(H') 1_{\{N^{\hat{y}}_{n,l} = m_l\}} \\ &= (1 + O(n^{-1}(\log n)^2)) P^{\hat{x}_0}(\Gamma^{\hat{y}}_{n,l} | \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}) 1_{\{N^{\hat{y}}_{n,l} = m_l\}} \end{aligned}$$

where we used (6.31) for the last equality. Using that $\{N^{\hat{y}}_{n,l} = m_l\} \in \mathcal{G}^{y}_{n,l\downarrow l-1}$, this is (6.28). $\square$

7 Late Points

We define the *cover time* of $\mathbb{Z}_K^2$ by the random walk $\hat{S}$ to be the maximum first visiting time over all points in $\mathbb{Z}_K^2$: if $\mathcal{T}_K(\hat{x}) = \inf\{t \geq 0 : \hat{S}_t = \hat{x}\}$ is the first time visiting $\hat{x}$, then the cover time of $\mathbb{Z}_K^2$ is

$$\mathcal{T}_{cov}(\mathbb{Z}_K^2) := \max_{\hat{x} \in \mathbb{Z}_K^2} \mathcal{T}_K(\hat{x}). \tag{7.1}$$

In [8], Dembo, Peres, Rosen, and Zeitouni showed that the cover time of $\mathbb{Z}_K^2$ for simple random walk is asymptotic to $\frac{4}{\pi}(K \log K)^2$ as $K \to \infty$. This result was found via strong approximation techniques to Brownian motion. The team reproduced this result via purely random walk methods in [9], along with a multifractal analysis of the late points of the torus. Here we generalize results from [3] and [9] to gain similar results for toral random walks with jumps of infinite range.

Let $\alpha \in (0, 1)$. Anticipating the result, we call $\hat{x}$ an α, K-*late point* of the random walk $\hat{S}$ on $\mathbb{Z}_K^2$ if $\mathcal{T}_K(\hat{x}) \geq \frac{4\alpha}{\pi_\Gamma}(K \log K)^2$. Set $\mathcal{L}_K(\alpha)$ to be the set of α, K-late points in $\mathbb{Z}_K^2$, *i.e.*,

$$\mathcal{L}_K(\alpha) := \left\{ \hat{x} \in \mathbb{Z}_K^2 : \frac{\mathcal{T}_K(\hat{x})}{(K \log K)^2} \geq \frac{4\alpha}{\pi_\Gamma} \right\}.$$

7.1 Upper bound of late point probabilities

First we show that excursion times are concentrated around their mean, and relate excursions to hitting times.

Lemma 7.1. *With the notation of Lemma 6.1, we can find $\delta_0 > 0$ and $C > 0$ such that, if $R \leq K/24$ and $\delta \leq \delta_0$ with $\delta \leq 6c_1(s^{-1} + r/R)$, then for all $\hat{x}, \hat{x}_0 \in \mathbb{Z}_K^2$,*

$$P^{\hat{x}_0}\left(\sum_{j=0}^{N} \tau^{(j)} \leq (1-\delta)N\frac{2K^2\log(R/r)}{\pi_\Gamma}\right) \leq e^{-C\delta^2 N(\log(R/r)/\log(K/r))} \tag{7.2}$$

and

$$P^{\hat{x}_0}\left(\sum_{j=0}^{N} \tau^{(j)} \geq (1+\delta)N\frac{2K^2\log(R/r)}{\pi_\Gamma}\right) \leq e^{-C\delta^2 N(\log(R/r)/\log(K/r))}. \tag{7.3}$$

Proof With $\tau = \tau^{(1)} = \left\{T_{\hat{\pi}_K(D(x,R)_K^c)} + T_{\hat{\pi}_K(\partial D(x,r)_s)} \circ \theta_{T_{\hat{\pi}_K(D(x,R)_K^c)}}\right\} \circ \theta_{T_{\hat{\pi}_K(\partial D(x,r)_s)}}$,

$$\begin{aligned}
\max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau^n) &\leq \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}\left(\left\{T_{\hat{\pi}_K(D(x,R)_K^c)} + T_{\hat{\pi}_K(\partial D(x,r)_s)} \circ \theta_{T_{\hat{\pi}_K(D(x,R)_K^c)}}\right\}^n\right) \\
&\leq \sum_{j=0}^{n} \binom{n}{j} \max_{y\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}\big(T^j_{\hat{\pi}_K(D(x,R)_K^c)}\,\big(T^{n-j}_{\hat{\pi}_K(\partial D(x,r)_s)} \circ \theta_{T_{\hat{\pi}_K(D(x,R)_K^c)}}\big)\big) \\
&\leq \sum_{j=0}^{n} \binom{n}{j} \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}\big(T^j_{\hat{\pi}_K(D(x,R)_K^c)}\big) \max_{\hat{z}\in\hat{\pi}_K(D(x,R)_K^c)} \mathbb{E}^{\hat{z}}\big(T^{n-j}_{\hat{\pi}_K(\partial D(x,r)_s)}\big).
\end{aligned}$$

Let $u = \frac{2K^2}{\pi_\Gamma}\log(K/r)$ and $u' = \frac{2K^2}{\pi_\Gamma}\log(R/r)$. Then, by (6.1), (6.20), (2.28), and (6.21), we can bound the moments of τ: there exist universal constants $c_1, c_2 < \infty$

such that for all $\hat{x} \in \mathbb{Z}_K^2$,

$$\begin{aligned} \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau^n) &\leq \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(T_{\hat{\pi}_K(D(x,R)_K^c)})||T_{\hat{\pi}_K(D(x,R)_K^c)}||^{n-1}n! \\ &\quad +2c_1 \textstyle\sum_{j=0}^{n-1} n!||T_{\hat{\pi}_K(D(x,R)_K^c)}||^j u'||T_{\hat{\pi}_K(\partial D(x,r)_s)}||^{n-j-1} \\ &\leq (n+1)!u'(c_2u)^{n-1}. \end{aligned} \tag{7.4}$$

Taking $\eta = \delta/6 > 0$, with our choice of r and R, it thus follows by (6.6) that for $\rho = c_4uu'$ and all $\theta > 0$,

$$\begin{aligned} \max_{\hat{x}} \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(e^{-\theta\tau}) &\leq 1 - \theta \min_{\hat{x}} \min_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau) \\ &\quad +\tfrac{\theta^2}{2} \max_{\hat{x}} \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau^2) \\ &\leq 1 - \theta(1-\eta)u' + \rho\theta^2 \\ &\leq \exp(\rho\theta^2 - \theta(1-\eta)u'). \end{aligned} \tag{7.5}$$

Since $\tau^{(0)} \geq 0$, using Markov's inequality, we bound the left-hand side of (7.2) by

$$\begin{aligned} P^{\hat{x}_0}\left(\sum_{j=1}^N \tau^{(j)} \leq (1-6\eta)u'N\right) &\leq e^{\theta(1-3\eta)u'N}\mathbb{E}^{\hat{x}_0}(e^{-\theta\sum_{j=1}^N \tau^{(j)}}) \qquad (7.6) \\ &\leq e^{-\theta u'N\delta/3}\left[e^{\theta(1-\eta)u'} \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(e^{-\theta\tau})\right]^N, \end{aligned}$$

where the last inequality follows by the strong Markov property of $\hat{S}_t$ on $\{\mathfrak{T}_j\}$. Combining (7.5) and (7.6) for $\theta = \delta u'/(6\rho)$ results in (7.2) for $C = 1/(36c_4)$.

Since $\tau^{(0)} = T_{\hat{\pi}_K(\partial D(x,r)_s)}$, by (6.1) and (6.21), there exist universal constants c_5, $c_6 < \infty$ such that

$$\max_{\hat{x},\hat{y}} \mathbb{E}^{\hat{y}}(e^{\tau^{(0)}/c_5u}) \leq c_6.$$

This implies

$$P^{\hat{x}_0}\left(\tau^{(0)} \geq \frac{\delta}{3}u'N\right) = P^{\hat{x}_0}\left(\frac{\tau^{(0)}}{c_5 u} \geq \frac{\delta}{3c_5}\frac{u'}{u}N\right) \leq c_6 e^{(-3c_5)^{-1}\delta(u'/u)N}.$$

Thus, the proof of (7.3), like in (7.2), comes down to bounding

$$P^{\hat{x}_0}\left(\sum_{j=1}^{N}\tau^{(j)} \geq (1+4\eta)u'N\right) \leq e^{-\theta u'N\delta/3}\left[e^{-\theta(1+2\eta)u'} \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{y}(e^{\theta\tau})\right]^N.$$

Noting that, by (7.4) and (6.6), there exists a universal constant $c_8 < \infty$ such that for $\rho = c_8 uu'$ and all $0 < \theta < 1/(2c_3 u)$,

$$\begin{aligned}\max_{\hat{x}} \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(e^{\theta\tau}) &\leq 1 + \theta \max_{\hat{y}\in\hat{\pi}_K(\partial D(x,r)_s)} \mathbb{E}^{\hat{y}}(\tau) + \sum_{n=2}^{\infty}\frac{\theta^n}{n!}\mathbb{E}^{\hat{y}}(\tau^n) \\ &\leq 1 + \theta(1+2\eta)u' + \rho\theta^2 \qquad (7.7) \\ &\leq \exp(\theta(1+2\eta)u' + \rho\theta^2).\end{aligned}$$

Taking $\delta_0 < 3c_8/c_3$, the proof of (7.3) now follows that of (7.2). □

Next we apply Lemma 7.1 to bound the upper tail of $\mathcal{T}_K(\hat{x})$, the first hitting time of $\hat{x} \in \mathbb{Z}_K^2$.

Lemma 7.2. *For any $\delta > 0$ we can find $c < \infty$ and $K_0 < \infty$ such that, for all $K \geq K_0$, $b \geq 0$, and $\hat{x}, \hat{x}_0 \in \mathbb{Z}_K^2$,*

$$P^{\hat{x}_0}\left(\mathcal{T}_K(\hat{x}) \geq b(K\log K)^2\right) \leq cK^{-(1-\delta)\pi_\Gamma b/2}. \qquad (7.8)$$

Proof Fix $\delta \in (0, \delta_0)$, where δ_0 is from Lemma 7.1. Let $R = \frac{K}{24}$ and $r = R/\log K$. Then Lemma 7.1 applies for all $K \geq K_0$ and some $K_0 = K_0(\delta) < \infty$. Fixing $b \geq 0$

and such K, let

$$n_K := (1-\delta)\frac{\pi_\Gamma b(\log K)^2}{2\log(R/r)} = (1-\delta)\frac{\pi_\Gamma b(\log K)^2}{2\log\log K}.$$

Then,

$$P^{\hat{x}_0}\left(\mathcal{T}_K(\hat{x}) \geq b(K\log K)^2\right) \leq \ P^{\hat{x}_0}\left(\mathcal{T}_K(\hat{x}) \geq \sum_{j=0}^{n_K}\tau^{(j)}\right) + P^{\hat{x}_0}\left(\sum_{j=0}^{n_K}\tau^{(j)} \geq b(K\log K)^2\right). \qquad (7.9)$$

The first probability in the sum in (7.9) is the probability of not hitting $\hat{x}$ during the first n_K consecutive $\hat{\pi}_K(\partial D(x,r)_s) \to \hat{\pi}_K(D(x,R)^c_K) \to \hat{\pi}_K(\partial D(x,r)_s)$ excursions. By (2.50),

$$P^{\hat{x}_1}\left(T_{\hat{x}} < T_{\hat{\pi}_K(D(x,R)^c_K)}\right) = \left[\frac{\log(R/r)+O(r^{-1/4})}{\log(R)}\right]\left(1+O(\log(R)^{-1})\right) \qquad (7.10)$$

uniformly for $\hat{x}_1 \in \hat{\pi}_K(\partial D(x,r)_s)$. For any $\hat{x}_2 \in \hat{\pi}_K(D(x,R)^c_K)$,

$$P^{\hat{x}_2}\left(T_{\hat{x}} < T_{\hat{\pi}_K(\partial D(x,r)_s)}\right) < 1. \qquad (7.11)$$

Hence, by (7.10) and (7.11), the first probability in (7.9) is bounded above by

$$\begin{aligned}
&\max_{\substack{\hat{x}_1\in\hat{\pi}_K(\partial D(x,r)_s)\\ \hat{x}_2\in\hat{\pi}_K(D(x,R)^c_K)}} \left[\left(1-P^{\hat{x}_1}\left(T_{\hat{x}} < T_{\hat{\pi}_K(D(x,R)^c_K)}\right)\right)\left(1-P^{\hat{x}_2}\left(T_{\hat{x}} < T_{\hat{\pi}_K(\partial D(x,r)_s)}\right)\right)\right]^{n_K} \\
&\leq \max_{\hat{x}_1\in\hat{\pi}_K(\partial D(x,r)_s)} \exp\left(-P^{\hat{x}_1}(T_{\hat{x}} < T_{\hat{\pi}_K(D(x,R)^c_K)})n_K\right) \\
&\leq e^{-\left[\left(\frac{\log(R/r)+O(r^{-1/4})}{\log(R)}\right)\left(1+O(\log(R)^{-1})\right)\right]n_K} \leq e^{-(1-\delta)\frac{\pi_\Gamma b(\log K)^2}{2\log(R/r)}\left(\frac{\log(R/r)}{\log(R)}\right)} \\
&= e^{-(1-\delta)\frac{\pi_\Gamma b(\log K)^2}{2\log(R)}} \leq e^{-(1-\delta)\pi_\Gamma b(\log K)/2} \leq K^{-(1-\delta)\pi_\Gamma b/2}.
\end{aligned} \qquad (7.12)$$

The second probability in (7.9) is bounded above by (7.3),

$$P^{\hat{x}_0}\left(\sum_{j=0}^{n_K}\tau^{(j)}\geq b(K\log K)^2\right)\leq P^{\hat{x}_0}\left(\sum_{j=0}^{n_K}\tau^{(j)}\geq(1+\delta)n_K\frac{2K^2\log(R/r)}{\pi_\Gamma}\right)$$
$$\leq e^{-C'(1-\delta)\pi_\Gamma b(\log(K))^2/\log(\log K)}, \tag{7.13}$$

for some $C'=C'(\delta)>0$. (7.12) and (7.13) combined with (7.9) gives us (7.8). □

The upper bound of (1.4) is as follows: For any $\alpha\in(0,1)$ and $\gamma>0$, we have by Lemma 7.2, that for $\gamma/(2\alpha)>\delta>0$ small enough,

$$P\left(\left|\left\{\hat{x}\in\mathbb{Z}_K^2:\frac{\mathcal{T}_K(\hat{x})}{(K\log K)^2}\geq\frac{4\alpha}{\pi_\Gamma}\right\}\right|\geq K^{2(1-\alpha)+\gamma}\right)$$
$$\leq K^{-2(1-\alpha)-\gamma}\,\mathbb{E}\left(\left|\left\{\hat{x}\in\mathbb{Z}_K^2:\frac{\mathcal{T}_K(\hat{x})}{(K\log K)^2}\geq\frac{4\alpha}{\pi_\Gamma}\right\}\right|\right)$$
$$=K^{-2(1-\alpha)-\gamma}\sum_{\hat{x}\in\mathbb{Z}_K^2}P\left(\frac{\mathcal{T}_K(\hat{x})}{(K\log K)^2}\geq\frac{4\alpha}{\pi_\Gamma}\right)$$
$$\leq K^{2\delta\alpha-\gamma}\underset{K\to\infty}{\longrightarrow}0. \tag{7.14}$$

7.2 Lower bound of late point probabilities

Fixing $0<\alpha<1$, we prove in this section the lower bound of (1.4): for any $\delta>0$, $K_n=e^n n^{3n+\bar{\gamma}}$, and some universal $n_0(\delta)<\infty$, there exists $f_n(\delta)\to 0$ as $n\to\infty$ such that

$$P\left(\left|\left\{\hat{x}\in\mathbb{Z}_{K_n}^2:\frac{\mathcal{T}_{K_n}(\hat{x})}{(K_n\log K_n)^2}\geq\frac{4\alpha}{\pi_\Gamma}\right\}\right|\geq K_n^{2(1-\alpha)-\delta}\right)\geq 1-f_n(\delta).$$

The sequence $\{K_n\}_{n\geq n_0}$ covers all integers sufficiently to imply

$$\lim_{m\to\infty} P\left(\left|\left\{\hat{x}\in\mathbb{Z}_m^2 : \frac{\mathcal{T}_m(\hat{x})}{(m\log m)^2}\geq\frac{4\alpha}{\pi_\Gamma}\right\}\right|\geq m^{2(1-\alpha)-\delta}\right) = 1. \tag{7.15}$$

Let $a=2\alpha$ and fix $\rho<\frac{2-a}{2}$. We call a pair $(\hat{x},\omega)$ n-**successful** if the path ω does not skip $\hat{x}$-bands and has the following excursion and visiting counts (where, recall, $v_k=3ak^2\log k$):

$$N_{n,0}^{\hat{x}}=0,\ |N_{n,k}^{\hat{x}}-v_k|\leq k,\ \textit{i.e.},\ N_{n,k}^{\hat{x}}\overset{k}{\sim} v_k,\ k=\rho n,\ldots,n-1.$$

Recall that $\mathcal{R}_n^{\hat{x}}$ is the time it takes for v_n excursions from $\hat{\pi}_K(D(x,r_{n,n-1}))$ out to $\hat{\pi}_K(D(x,r_{n,n})^c_{K_n})$ to complete, and note that $\{N_{n,0}^{\hat{x}}=0\}=\{\mathcal{T}_{K_n}(\hat{x})>\mathcal{R}_n^{\hat{x}}\}$. The next lemma relates the notions of n-success and first hitting times.

Lemma 7.3. *Let $\mathcal{S}_n=\{\hat{x}\in\mathbb{Z}_{K_n}^2 : \mathcal{T}_{K_n}(\hat{x})>\mathcal{R}_n^{\hat{x}}\}$. Then, for some $c>0$ and all $n\geq n_0$,*

$$P\left(\bigcup_{\hat{x}\in\mathcal{S}_n}\left\{\frac{\mathcal{T}_{K_n}(\hat{x})}{(K_n\log K_n)^2}\leq\frac{2a}{\pi_\Gamma}-\frac{2}{\log n}\right\}\right)\leq c^{-1}e^{-cn^2/\log n}. \tag{7.16}$$

Proof Set $r=r_{n,n-1}$, $R=r_{n,n}$, and $\delta=\frac{\pi_\Gamma}{2a\log n}$. Then $\log(R/r)=3\log n$, and by (7.2) under $N=v_n=3an^2\log n$ excursions, we have that, for some $C>0$, all $n\geq n_0$, and any $\hat{x},\hat{x}_0\in\mathbb{Z}_{K_n}^2$,

$$\begin{aligned}
P_{\hat{x}} &:= P^{\hat{x}_0}\left(\mathcal{T}_{K_n}(\hat{x})\leq\left(\frac{2a}{\pi_\Gamma}-\frac{2}{\log n}\right)(K_n\log K_n)^2, \mathcal{T}_{K_n}(\hat{x})>\mathcal{R}_n^{\hat{x}}\right)\\
&\leq P^{\hat{x}_0}\left(\sum_{j=0}^{v_n}\tau^{(j)}\leq\left(\frac{2a}{\pi_\Gamma}-\frac{1}{\log n}\right)K_n^2(3n\log n)^2\right)\\
&\leq P^{\hat{x}_0}\left(\sum_{j=0}^{v_n}\tau^{(j)}\leq(1-\delta)v_n\frac{2K_n^2\log(R/r)}{\pi_\Gamma}\right)\leq e^{-C\frac{n^2}{\log n}}.
\end{aligned}$$

Sum over $\hat{x} \in \mathbb{Z}^2_{K_n}$ and select $c < C/2$ so that $c^{-1}e^{-cn_0^2} \geq 1$ to get (7.16). □

Let $Y(n, \hat{x})$, $\hat{x} \in \mathbb{Z}^2_{K_n}$, be the indicator random variable for the event

$$\{\hat{x} \text{ is } n\text{-successful}\} = \{\omega : (\hat{x}, \omega) \text{ is } n\text{-successful}\}.$$

In view of Lemma 7.3, we have (7.15) (and hence (1.4)) as soon as we show that, for any $\delta > 0$, all n sufficiently large, there exists a sequence $f_n \to 0$ such that

$$P\left(\sum_{\hat{x} \in \mathbb{Z}^2_{K_n}} Y(n, \hat{x}) \geq K_n^{2-a-\delta}\right) \geq 1 - f_n(\delta). \tag{7.17}$$

First, we state [3, Lemma 6.1], a combinatorial result that will aid us in the proof of Lemma 7.5.

Lemma 7.4. *For some $C = C(a) < \infty$ and all $k \geq 2$, $|m - v_{k+1}| \leq k+1$, $|l+1-v_k| \leq k$,*

$$\frac{C^{-1}k^{-3a-1}}{\sqrt{\log k}} \leq \binom{m+l}{l}\left(\frac{1}{2}\right)^{m+l+1} \leq \frac{Ck^{-3a-1}}{\sqrt{\log k}}. \tag{7.18}$$

Lemma 7.5. *Fix $\rho < \rho' < \frac{2-a}{2}$. Then there exists $b \geq 10$ and $q_n \geq r_{n,n}^{-a+o(1_n)}$ such that for all n sufficiently large, uniformly in $\overline{\gamma} \in [b, b+4]$ and $\hat{x} \in S_{K_n} := \mathbb{Z}^2_{K_n} \setminus \hat{\pi}_K(D(0, r_{n,n}))$,*

$$P(\hat{x} \text{ is } n\text{-successful}) = (1 + o(1_n))q_n. \tag{7.19}$$

Proof We start by defining a way to examine excursions on a path. Let $\tau(1)$ be the time of the first visit to $\hat{\pi}_K(\partial D(x, r_{n,n-1})_{s^{n\downarrow}_{n-1}})$ (starting at $\hat{0}$, so coming from outside $\hat{x}$'s levels into $\hat{x}$'s large level $n-1$), and define $\tau(2)$, $\tau(3)$, ... to be the successive hitting times of different elements of $A_n := \bigcup_{k=\rho n}^{n} \hat{\pi}_K(\partial D(x, r_{n,k})_{s_k})$ until time $\mathcal{R}_n^{\hat{x}}$.

We can construct a path ω's "history" as follows: let $m = (m_{\rho n}, \ldots, m_{n-1}, m_n)$, where m_k is the number of upcrossing excursions of ω (candidate values for $N^{\hat{x}}_{n,k}$) from level $k-1$, *i.e.*, $\hat{\pi}_K(\partial D(x, r_{n,k-1})_{s_{k-1}})$, out to level k, *i.e.*, $\hat{\pi}_K(\partial D(x, r_{n,k})_{s_k})$ before $\mathcal{R}^{\hat{x}}_n$, and set $|\overline{m}| = 2\sum_{k=\rho n}^{n} m_k - 1$. Let $\Phi : A_n \mapsto \{\rho n - 1, \ldots, n-1, n\}$ label the points of A_n by their annulus: set $\Phi(\hat{y}) = k$ if $\hat{y} \in \hat{\pi}_K(\partial D(x, r_{n,k})_{s_k})$. Set $h(\omega, j) = \Phi(\omega(\tau(j)))$, the label of the annulus hit at time $\tau(j)$, where $\omega \in \Omega^{\rho n-1,\ldots,n}_{\hat{x},n,n-1,m_n}$. (Note that, since we are referring to upcrossings here, at level $n-1$ we use the thin band $s_{n-1} = n^4$ rather than the thick band $s^{n\downarrow}_{n-1} = \sqrt{r_{n,n-1}}$, which is reserved for the downcrossing $n \downarrow n-1$.) Since $\omega \in \Omega^{\rho n-1,\ldots,n}_{\hat{x},n,n-1,m_n}$, h satisfies

$$h(\omega, 1) = n-1; \;\; |h(\omega, j+1) - h(\omega, j)| = 1, j = 1, \ldots, |\overline{m}|-1; \;\; h(\omega, |\overline{m}|) = n. \quad (7.20)$$

Let $\mathcal{H}_n(|\overline{m}|)$ be the collection of all such maps

$$s : \{1, 2, \ldots, |\overline{m}|\} \mapsto \{\rho n - 1, \ldots, n-1, n\}$$

satisfying (7.20) for a given $\omega \in \Omega^{\rho n-1,\ldots,n}_{\hat{x},n,n-1,m_n}$. Note that the number of upcrossings from level $k-1$ to k is

$$u(k) := |\{(j, j+1) : (s(j), s(j+1)) = (k-1, k)\}| = m_k.$$

An upcrossing from $k-1$ to k can only occur before the last upcrossing from k to $k+1$. Hence, the number of ways to partition $u(k)$ upcrossings from $k-1$ to k among and before the $u(k+1)$ upcrossings from k to $k+1$ is

$$\binom{u(k+1) + u(k) - 1}{u(k)},$$

the number of ways to partition $u(k)$ identical objects into $u(k+1)$ sets. Since the mapping s is in one-to-one correspondence with the relative ordering of all its upcrossings, we have

$$|\mathcal{H}_n(\overline{m})| = \prod_{k=\rho n}^{n-1} \binom{m_{k+1} + m_k - 1}{m_k}.$$

Let $h|_k$ be the first k coordinates of the sequence h. Applying the strong Markov property at the times $\tau(1)$, $\tau(2)$, ..., $\tau(|\overline{m}|-1)$, we have, uniformly for $s \in \mathcal{H}_n(\overline{m})$ and $\hat{x} \in S_{K_n}$,

$$P(h|_{|\overline{m}|} = s; \Omega^{\rho n-1,\ldots,n}_{\hat{x},n,n-1,m_n}; \mathcal{T}_{K_n}(\hat{x}) > \tau(|\overline{m}|)) = \prod_{k=\rho n}^{n} a_k^{m_k} b_k^{m_k}, \tag{7.21}$$

where a_l and b_l are described below.

We wish to examine the probabilities of excursions between annuli. For the outermost level, from level n (*i.e.*, the $\hat{x}$-band of width $s_n = n^4$ at radius $r_{n,n}$), the probability that the toral walk crosses back down to $r_{n,n-1}$ via the thick $\hat{x}$-band (which is of width $s^{n\downarrow}_{n-1} = \sqrt{r_{n,n-1}}$, unlike all other bands) can be estimated by the bound below (4.27). Uniformly for $\hat{w} \in \hat{\pi}_K(\partial D(x, r_{n,n})_{s_n})$, and for large enough n, there exists $c, c' > 0$

such that

$$
\begin{aligned}
b_n &= P^{\hat{w}}\left(T_{\hat{\pi}_K(D(x,r_{n,n-1}+s^{n\downarrow}_{n-1}))} = T_{\hat{\pi}_K(\partial D(x,r_{n,n-1})_{s^{n\downarrow}_{n-1}})}\right) \\
&= 1 - P^{\hat{w}}\left(T_{\hat{\pi}_K(D(x,r_{n,n-1}))} < T_{\hat{\pi}_K(\partial D(x,r_{n,n-1})_{s^{n\downarrow}_{n-1}})}\right) \\
&\geq 1 - c r^2_{n,n-1}\log^2(r_{n,n-1}) r^{-M/2}_{n,n-1} \\
&\geq 1 - c r^{2-M/2}_{n,n-1}\log(r_{n,n-1})^2 \\
&\geq 1 - c' r^{-\beta}_{n,n-1} n^2 (\log n)^2 \\
&\geq 1 - c' e^{-\beta n} n^{-3\beta(n-1)+2}(\log n)^2 = 1 + o(n^{-4}).
\end{aligned} \tag{7.22}
$$

From the innermost level $\rho n - 1$, applying (4.24), we will avoid visiting $\hat{x}$ and cross back up to level ρn via its $s_{\rho n} = n^4$-band, uniformly in $\hat{w} \in \hat{\pi}_K(\partial D(x, r_{n,\rho n-1})_{s_{\rho n-1}})$, with probability

$$
\begin{aligned}
a_{\rho n} &= P^{\hat{w}}\left(T_{\hat{\pi}_K(D(x,r_{n,\rho n})^c_K)} < T_{\hat{x}};\, T_{\hat{\pi}_K(D(x,r_{n,\rho n})^c_K)} = T_{\hat{\pi}_K(\partial D(x,r_{n,\rho n})_{s_{\rho n}})}\right) \\
&= 1 - \frac{\log\left(\frac{r_{n,\rho n}}{r_{n,\rho n-1}}\right) + O(r^{-1/4}_{n,\rho n-1})}{\log r_{n,\rho n}}\left(1 + O((\log r_{n,\rho n})^{-1})\right) + o(n^{-8}) \qquad (7.23) \\
&= 1 - \frac{3\log n + o(e^{-n/4})}{n + 3\rho n \log n}(1 + O((\rho n \log n)^{-1})) + o(n^{-8}) \\
&= 1 - \frac{1}{\rho n} + O\left((\rho n^2 \log n)^{-1}\right).
\end{aligned}
$$

For the middle levels, set a_l to the probability in (4.36) for upcrossings for $l = \rho n, \ldots, n$, and b_l to (4.38) for downcrossings:

$$
a_l,\, b_l = \frac{1}{2} + o(n^{-4}),\ l = \rho n - 1, \ldots, n - 1. \tag{7.24}
$$

By (7.22), (7.23), and (7.24), (7.21) reduces to

$$\prod_{k=\rho n}^{n} a_k^{m_k} b_k^{m_k} = a_{\rho n}^{m_{\rho n}} b_n^{m_n} \prod_{k=\rho n}^{n-1} a_{k+1}^{m_{k+1}} b_k^{m_k} \tag{7.25}$$
$$= a_{\rho n}^{m_{\rho n}} (1+o(n^{-4}))^{m_n} \left(\frac{1}{2}+o(n^{-4})\right)^{|\overline{m}|-m_{\rho n}-m_n+1}$$

since $\sum_{k=\rho n}^{n-1}(m_k+m_{k+1}) = |\overline{m}| - m_{\rho n} - m_n + 1$. Factoring $\frac{1}{2}$ from the main terms and combining reduces this probability to

$$a_{\rho n}^{m_{\rho n}} \left(1+o(n^{-4})\right)^{|\overline{m}|-m_{\rho n}+1} \prod_{k=\rho n}^{n-1} \left(\frac{1}{2}\right)^{m_k+m_{k+1}}.$$

Uniformly in $|\overline{m}|$, we have $\left(1+o(n^{-4})\right)^{|\overline{m}|-m_{\rho n}+1} = 1+o(1_n)$. Finally, for large enough n, uniformly in $m_{\rho n} \overset{\rho n}{\sim} v_{\rho n}$, and since $a_{\rho n}, \rho \le 1$, we can bound the term $a_{\rho n}^{m_{\rho n}}$ below:

$$a_{\rho n}^{m_{\rho n}} \ge \left(1 - \frac{1}{\rho n} + O((\rho n^2 \log n)^{-1})\right)^{3a(\rho n)^2 \log(\rho n)+\rho n}$$
$$\ge e^{-3a\rho n \log(\rho n)+O(1)} \ge e^c(\rho n)^{3\rho n(-a)}$$
$$\ge e^{n(-a+o(1_n))} n^{3\rho n(-a+o(1_n))} \ge r_{n,\rho n}^{-a+o(1_n)}.$$

All combined, this yields the exact-history s, not-skipping-$\hat{x}$-bands probability bound

$$P(h|_{|\overline{m}|} = s; \Omega_{\hat{x},n,n-1,m_n}^{\rho n-1,\ldots,n}; \mathcal{T}_{K_n}(\hat{x}) > \tau(|\overline{m}|))$$
$$\ge (1+o(1_n)) r_{n,\rho n}^{-a+o(1_n)} \prod_{k=\rho n}^{n-1} \left(\frac{1}{2}\right)^{m_k+m_{k+1}}. \tag{7.26}$$

Taking $m_n = v_n = 3an^2 \log n$ and summing over all possible maps s for each possible path ω gives us

$$P(\hat{x} \text{ is } n\text{-successful}) = (1 + o(1_n))\, q_n, \tag{7.27}$$

which, by (7.26), is (7.19) for

$$q_n \geq r_{n,\rho n}^{-a+o(1_n)} \sum_{\substack{m_{\rho n},\ldots,m_{n-1} \\ |m_k - v_k| \leq k}} \prod_{k=\rho n}^{n-1} \binom{m_{k+1} + m_k - 1}{m_k} \left(\frac{1}{2}\right)^{m_k + m_{k+1}}. \tag{7.28}$$

Note that q_n does not depend on $\hat{x}$. By (7.18), there exists $C, C' < \infty$ independent of k such that, uniformly in $m_k \overset{k}{\sim} v_k$ and $m_{k+1} \overset{k+1}{\sim} v_{k+1}$,

$$\frac{C' k^{-3a-1}}{\sqrt{\log k}} \geq \binom{m_{k+1} + m_k - 1}{m_k} \left(\frac{1}{2}\right)^{m_k + m_{k+1}} \geq \frac{C k^{-3a-1}}{\sqrt{\log k}}. \tag{7.29}$$

Since there are $2l+1$ positive terms for each l such that $m_l \overset{l}{\sim} v_l$, the sum in (7.28) is a sum of $\prod_{l=\rho n}^{n-1}(2l+1)$ terms; each of these terms is a product of $(1-\rho)n$ factors, each of the form $\binom{m_{l+1}+m_l-1}{m_l}\left(\frac{1}{2}\right)^{m_l+m_{l+1}}$. Thus, using (7.29) and some $C_1, C_1' < \infty$, we can bound the sum in (7.28) by

$$\begin{aligned}
\prod_{k=\rho n}^{n-1} \frac{C_1' k^{-3a}}{\sqrt{\log k}} &\geq \prod_{l=\rho n}^{n-1} (2l+1) \prod_{k=\rho n}^{n-1} \frac{C' k^{-3a-1}}{\sqrt{\log k}} \\
&\geq \sum_{\substack{m_{\rho n},\ldots,m_{n-1} \\ |m_l - v_l| \leq l}} \prod_{k=\rho n}^{n-1} \frac{C k^{-3a-1}}{\sqrt{\log k}} \geq \prod_{l=\rho n}^{n-1} (2l+1) \prod_{k=\rho n}^{n-1} \frac{C k^{-3a-1}}{\sqrt{\log k}} \\
&\geq \prod_{k=\rho n}^{n-1} \frac{C_1 k^{-3a}}{\sqrt{\log k}} \geq (1-\rho) n C_1^{(1-\rho)n} n^{3(1-\rho)n(-a)} \left(\prod_{k=\rho n}^{n-1} \log k \right)^{-1/2}.
\end{aligned} \tag{7.30}$$

It is obvious that a constant c is $n^{o(1_n)}$, and n^c is $(n^n)^{o(1_n)}$ for any fixed $c > 0$. Hence,

$$(1-\rho)nC_1^{(1-\rho)n} = (n^n)^{o(1_n)} = r_{n,n}^{o(1_n)}. \tag{7.31}$$

Next, $n^{3(1-\rho)n(-a)}$ combined with $r_{n,\rho n}^{-a+o(1_n)}$ yields

$$r_{n,\rho n}^{-a+o(1_n)} n^{3(1-\rho)n(-a)} = (e^n n^{3\rho n})^{-a+o(1_n)} (n^{3(1-\rho)n})^{-a} = r_{n,n}^{-a+o(1_n)}. \tag{7.32}$$

Finally,

$$\begin{aligned}\left(\prod_{k=\rho n}^{n-1} \log k\right) = n^{nx} &\implies x = \frac{\log\left(\prod_{k=\rho n}^{n-1} \log k\right)}{n \log n} \leq \frac{(1-\rho)n \log\log n}{n \log n} \to 0 \\ &\implies \left(\prod_{k=\rho n}^{n-1} \log k\right)^{-1/2} = r_{n,n}^{o(1_n)}. \end{aligned}\tag{7.33}$$

Merging (7.28)-(7.33) results in $q_n \geq r_{n,n}^{-a+o(1_n)}$. $\square$

For a given n, define

$$l(\hat{x}, \hat{y}) := \max\{m \in \{0, 1, 2, \ldots, n\} : \hat{\pi}_K(D(x, r_{n,m})) \cap \hat{\pi}_K(D(y, r_{n,m})) = \emptyset\}$$

to be the largest radius index (up to n) of discs centered at $\hat{x}$ and $\hat{y}$ that do not intersect. We now show that the covariance of $Y(n, \hat{x})$ between pairs of points depends on how far apart they are, based on this measurement.

Lemma 7.6. *Fix $\varepsilon > 0$. Then there exists $b \geq 10$ and $C = C(b, \varepsilon) < \infty$ such that* *for all* n *and* $\hat{x}, \hat{y} \in S_{K_n}$,

$$\mathbb{E}(Y(n,\hat{x})Y(n,\hat{y})) \leq \begin{cases} C^{n-l} q_n^2 n^b \left(\frac{r_{n,n}}{r_{n,l}}\right)^{a+\varepsilon} & \rho' n \leq l(\hat{x},\hat{y}) < n, \\ (1+o(1_n))q_n^2 & l(\hat{x},\hat{y}) = n. \end{cases} \tag{7.34}$$

Proof First, note that, using the index set $M_l := \{l, l+1, \ldots, n-1\}$, the same analysis at the end of the proof of Lemma 7.5 yields, for any $l \geq \rho n$, uniformly in $\hat{x} \in S_{K_n}$, $\overline{\gamma}$, and $m_k \leq 3k^2 \log k + k$,

$$P(N_{n,k}^{\hat{x}} = m_k, k \in M_l) = (1+o(1_n)) \prod_{k=l}^{n-1} \binom{m_{k+1}+m_k-1}{m_k} \left(\frac{1}{2}\right)^{m_k+m_{k+1}}. \tag{7.35}$$

Recall that $v_k = v_k(a) = 3ak^2 \log k$ and $N \overset{k}{\sim} v_k$ if $|N - v_k| \leq k$ for $\rho n \leq k < n$ and $N = 0$ if $k = 0$. We first note that, for $\rho' n \leq l(\hat{x},\hat{y}) < n$, $2r_{n,l+1} + 2 \geq d(\hat{x},\hat{y}) \geq 2r_{n,l} + 2$. Thus, there are, for some constants $C_{n,k} \approx 4\pi$,

$$|\{y : l(\hat{x},\hat{y}) = l\}| = C_{n,l+1}(r_{n,l+1}^2 - r_{n,l}^2). \tag{7.36}$$

Since $r_{n,l+2} - r_{n,l} \gg r_{n,l+1}$, it is easy to see that

$$l = l(\hat{x},\hat{y}) < n \implies \hat{\pi}_K(D(y, r'_{n,l})) \cap \hat{\pi}_K(\partial D(x, r_{n,k})_{s_k}) = \emptyset$$

for $k \neq l+1$ (the thick band at $k = n-1$ also satisfies this). Replacing hereafter l with $l \wedge n-3$, it follows that for $k \neq l+1, l+2$, the events $\{N_{n,k}^{\hat{x}} \overset{k}{\sim} v_k\}$ are measurable with respect to the σ-algebra $\mathcal{G}_{n,l\downarrow l-1}^{\hat{y}}$ (defined before Lemma 6.4), since the excursions outside these bands depend (up to error term) only on their beginning and end points.

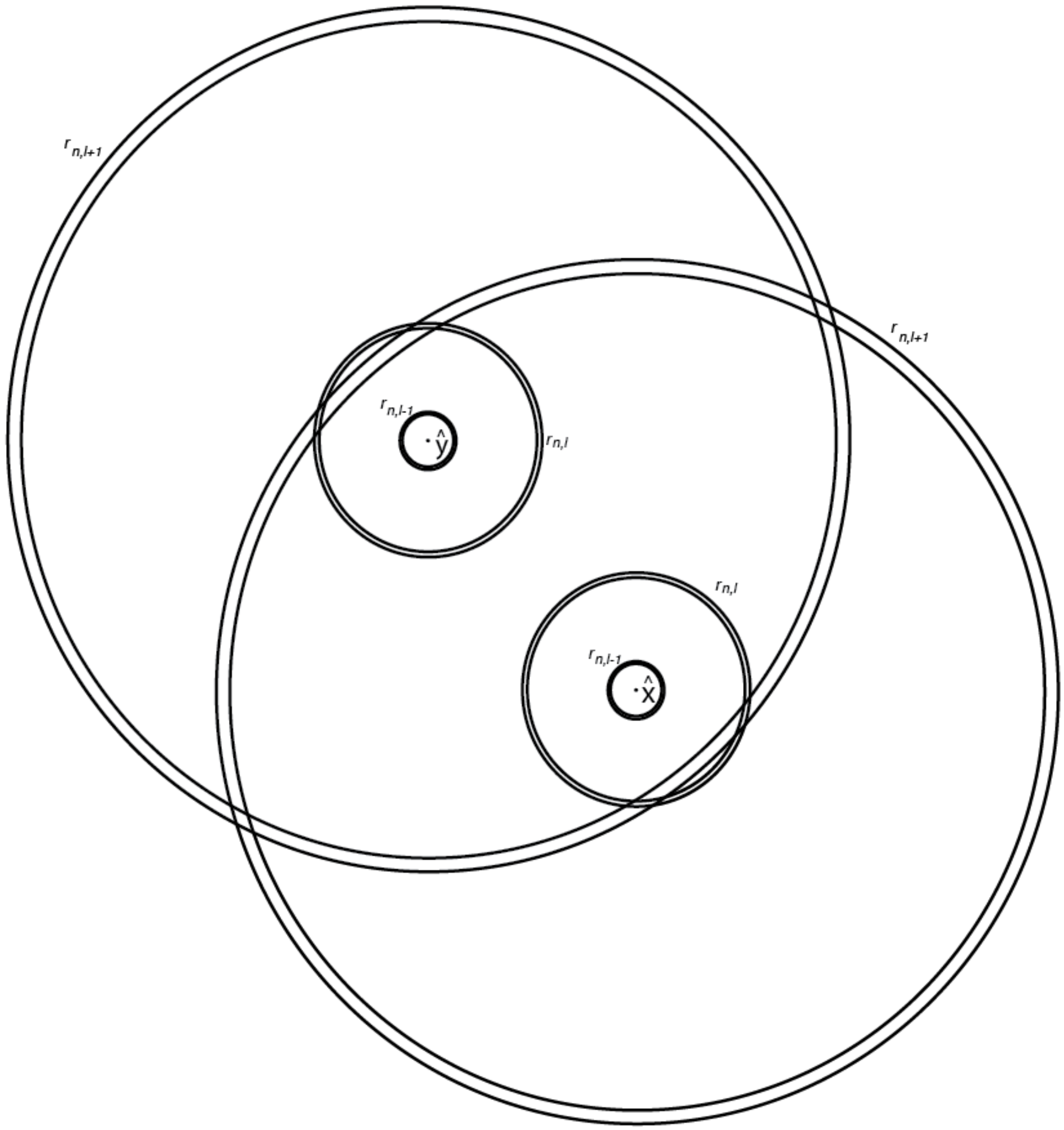

Figure 7.1: An example of $l(\hat{x}, \hat{y}) = l$ where levels l and $l+1$ have nonempty intersection

Slightly rewriting the notation of Lemma 6.5, define the set of $\hat{y}$-*faithful* paths for the set of indices A,

$$\Gamma_n^{\hat{y}}(A) := \{N_{n,i}^{\hat{y}} \overset{i}{\sim} v_i;\ i \in A\} \cap \Omega_{y,n,l,v_l+l}^{A},$$

to be the set of paths with n-successful $\hat{y}$-excursion counts on the levels of the indices of A. Using the index set $J_l = \{0, \rho n, \ldots, l-1\}$, we collect all the pertinent inner-level $\hat{y}$-based excursions, and with the index set $I_l = \{0, \rho n, \ldots, l, l+3, \ldots, n-1\}$, we combine the inner- and outer-level $\hat{x}$-faithful excursion paths, skipping the two levels where $\hat{x}$ and $\hat{y}$'s annuli cross (causing a jump in their n-success covariance).

Note that $\Gamma_n^{\hat{x}}(I_l) \in \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}$ (it skips the two levels in question). Then we have that

$$\{\hat{x} \text{ and } \hat{y} \text{ are } n\text{-successful}\} \subset \Gamma_n^{\hat{x}}(I_l) \cap \Gamma_n^{\hat{y}}(J_{l+1}).$$

Recall that, if $B \in \mathcal{G}$, $P(A \cap B|\mathcal{G}) = P(A|\mathcal{G})1_B$. Applying (6.28), and focusing on level l, for some universal constant $C_3 < \infty$,

$$\begin{aligned} P(\hat{x} \text{ and } \hat{y} \text{ are } n\text{-successful}) &\leq \sum_{m_l \overset{l}{\sim} v_l} \mathbb{E}\left(P\left(\Gamma_n^{\hat{y}}(J_l)|N^{\hat{y}}_{n,l} = m_l, \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}\right); \Gamma_n^{\hat{x}}(I_l)\right) \\ &\leq C_3 P(\Gamma_n^{\hat{x}}(I_l)) \sum_{m_l \overset{l}{\sim} v_l} P\left(\Gamma_n^{\hat{y}}(J_l)|N^{\hat{y}}_{n,l} = m_l\right). \qquad (7.37) \end{aligned}$$

Using Lemma 6.5, for some universal constant $0 < C_4 < \infty$,

$$\begin{aligned} (1+o(1_n))q_n &= P(y \text{ is } n\text{-successful}) \qquad (7.38) \\ &= \sum_{m_l \overset{l}{\sim} v_l} \mathbb{E}\left(P\left(\Gamma_n^{\hat{y}}(J_l)|N^{\hat{y}}_{n,l} = m_l, \mathcal{G}^{\hat{y}}_{n,l\downarrow l-1}\right); N^{\hat{y}}_{n,l} = m_l, \Gamma_n^{\hat{y}}(M_{l+1})\right) \\ &\geq C_4 \sum_{m_l \overset{l}{\sim} v_l} P\left(N^{\hat{y}}_{n,l} = m_l, \Gamma_n^{\hat{y}}(M_{l+1})\right) \times P(\Gamma_n^{\hat{y}}(J_l)|N^{\hat{y}}_{n,l} = m_l). \end{aligned}$$

Hence, by (7.35) and (7.29), for some universal $C_5 < \infty$,

$$\sum_{m_l \overset{l}{\sim} v_l} P\left(\Gamma_n^{\hat{y}}(J_l)|N^{\hat{y}}_{n,l} = m_l\right) \leq C_5^{n-l} q_n l \left(\prod_{k=l}^{n-1} k^{3a}\sqrt{\log k}\right). \qquad (7.39)$$

Similarly, using Lemma 6.5,

$$P(\Gamma_n^{\hat{x}}(I_l)) \leq \sum_{m_l \overset{l}{\sim} v_l} \mathbb{E}\left(P\left(\Gamma_n^{\hat{x}}(J_l)|N_{n,l}^x = m_l, \mathcal{G}_{n,l\downarrow l-1}^x\right); \Gamma_n^{\hat{x}}(M_{l+3})\right) \tag{7.40}$$
$$\leq C_6 P(\Gamma_n^{\hat{x}}(M_{l+3}) \sum_{m_l \overset{l}{\sim} v_l} P\left(\Gamma_n^{\hat{x}}(J_l)|N_{n,l}^x = m_l\right).$$

Comparing (7.40) and (7.38), and applying (7.35) and (7.29) again, we get

$$P(\Gamma_n^{\hat{x}}(I_l)) \leq C_7 l \left(\prod_{k=l}^{l+2} k^{3a}\sqrt{\log k}\right) q_n. \tag{7.41}$$

Combining (7.37), (7.39), and (7.41) proves (7.34) for $l(\hat{x}, \hat{y}) < n$.

Finally, we deal with those pairs far apart. For most pairs ($K_n^2(K_n^2 - C_{n,n}r_{n,n}^2)$ pairs for some $C_{n,n} \approx 4\pi$ of them), we have $l(\hat{x}, \hat{y}) = n$. For these, the event $\{\hat{x}$ is n-successful$\}$ is $\mathcal{G}_{n,n\downarrow n-1}^{\hat{y}}$-measurable, so by Lemma 6.5,

$$\begin{aligned} \mathbb{E}(Y(n,\hat{x})Y(n,\hat{y})) &= P(\hat{x} \text{ and } \hat{y} \text{ are } n\text{-successful}) \\ &= \mathbb{E}(P(\hat{y} \text{ is } n\text{-successful} \,|\, \mathcal{G}_{n,n\downarrow n-1}^y); \hat{x} \text{ is } n\text{-successful}) \\ &\leq (1 + O(n^{-1}(\log n)^2))(1 + o(1_n))q_n^2 = (1 + o(1_n))q_n^2. \quad \square \end{aligned} \tag{7.42}$$

We can now prove Theorem 1.1.

Let

$$V_l := \sum_{x,y \in S_{K_n}, l(\hat{x},\hat{y}) = l} \mathbb{E}(Y(n,\hat{x}), Y(n,\hat{y})), \quad l = 0, 1, \ldots, n.$$

Since, by (7.19), considering the sum $W_n := \sum_{\hat{x} \in S_{K_n}} Y(n, \hat{x})$, the number of n-successful points $\hat{x}$,

$$\mathbb{E}(W_n) = \mathbb{E}\left(\sum_{x \in S_{K_n}} Y(n,\hat{x})\right) = (1+o(1_n))K_n^2 q_n \geq K_n^{2-a+o(1_n)},$$

recall the Paley-Zygmund inequality ([20, Lemma 14.8.2]): since $W_n \in L^2(\Omega)$, for any $0 < \lambda_n < 1$, we have

$$P(W_n \geq \lambda_n \mathbb{E}(W_n)) \geq (1-\lambda_n)^2 \frac{\mathbb{E}(W_n)^2}{\mathbb{E}(W_n^2)}. \tag{7.43}$$

By (7.43), (7.17) will follow from the bottom half of (7.34) and

$$\mathbb{E}(W_n^2) = \sum_{l=0}^{n-1} V_l \leq o(1_n) K_n^4 q_n^2. \tag{7.44}$$

To obtain this bound, first note that the definition of $l(x,y)$ implies that $d(\hat{x},\hat{y}) < 2r_{n,l(x,y)+1} + 2$. Hence, on $\mathbb{Z}_{K_n}^2$ there are at most $C_0 r_{n,l+1}^2$ points $\hat{y} \in \hat{\pi}_K(D(x, r_{n,l+1}))$ (from here on, C_m are constants independent of n). Since $2\rho' < 2-a$, there exists $C_1 < \infty$ such that the covariances on the inner levels sum to

$$\begin{aligned}\sum_{l=0}^{\rho' n-1} V_l &\leq \sum_{\hat{x},\hat{y} \in \mathbb{Z}_{K_n}^2, d(\hat{x},\hat{y}) \leq 2r_{n,\rho' n}} \mathbb{E}(Y(n,\hat{x})Y(n,\hat{y})) \\ &\leq \sum_{\hat{x},\hat{y} \in \mathbb{Z}_{K_n}^2, d(\hat{x},\hat{y}) \leq 2r_{n,\rho' n}} \mathbb{E}(Y(n,\hat{x})) \leq C_1 q_n K_n^2 r_{n,\rho' n}^2 \leq o(1_n) K_n^4 q_n^2.\end{aligned} \tag{7.45}$$

Choose $\varepsilon > 0$ such that $2-a-\varepsilon > 0$ and fix $l \in [\rho' n, n)$. Then, by (7.34), the outer-level covariances are bounded by

$$V_l \leq C_2 K_n^2 r_{n,l+1}^2 q_n^2 n^b C^{n-l} \left(\frac{r_{n,n}}{r_{n,l}}\right)^{a+\varepsilon}, \tag{7.46}$$

which leads to the overall upper-level covariance bound

$$\begin{aligned}\sum_{l=\rho' n}^{n-1} V_l &\leq C_2 K_n^2 q_n^2 n^b \sum_{l=\rho' n}^{n-1} C^{n-l} r_{n,l+1}^2 \left(\frac{r_{n,n}}{r_{n,l}}\right)^{a+\varepsilon} \\ &= C_2 K_n^4 q_n^2 n^{-2\overline{\gamma}+b+6} \sum_{l=\rho' n}^{n-1} C^{n-l} \left(\frac{r_{n,l}}{r_{n,n}}\right)^{2-a-\varepsilon} \\ &\leq C_2 K_n^4 q_n^2 n^{-2} \sum_{j=1}^{n} C^j r_{n,j}^{-(2-a-\varepsilon)}.\end{aligned} \tag{7.47}$$

Combining (7.45) and (7.47) we get (7.44), which proves (7.17). Hence, we have (7.15). $\square$

8 Open Problems

We have given the asymptotic timing of a large class of infinite-range symmetric random walks on the two-dimensional torus. Some open problems to extend this work are:

- Analyze the neighborhoods and pairs of late points mentioned in [9, Theorems 1.2 and 1.3]. How is the spacing of α-late point pairs on $\mathbb{Z}_K^2$ affected by jumping walks?
- Examine the structure of the frequent points on the lattice torus, and verify that, as $K \to \infty$, their structure on $\mathbb{Z}_K^2$ looks like the planar structure.
- [9] suggests that its nearest-neighbor results may be extended to the planar Weiner sausage on the two-dimensional torus $\mathbb{T}^2$. We suggest, then, that using this class of jumping walks, this work may be extended to a larger class of "compound Poisson Weiner sausage links" on $\mathbb{T}^2$ (for example, a two-dimensional Brownian motion with exponentially-timed jumps).
- Check the ratio of late points of $\mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2}$ when limiting the coordinates at different rates and when limiting to the infinite cylinder $\mathbb{Z}^2 \times \mathbb{Z}_K$ for fixed K.
- Find tight bounds for $\hat{G}_{\hat{\pi}_K(D(0,n))}(\hat{x}, \hat{x})$, the external toral Green's function, along with annulus Green's functions on the plane and torus and expected

hitting times of these discs and annuli, and prove a full exterior toral Harnack inequality.

- Give computational rates of convergence for the number of late points, given α and p_1.

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