The Witt Ring of a Smooth Curve with Good Reduction over a Local Field

by

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Abstract

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The modern study of bilinear forms has a rich history beginning with Witt's work over fields in the 1930's, when he defined a ring structure on the set of anisotropic forms over a field. It was revived, notably by Pfister, in the 1960's. With the advent of algebraic K-Theory, much of the theory of quadratic forms over fields was generalized to a theory of quadratic spaces over rings. In the 1960's and 1970's Knebusch, among others, formulated a compatible theory for quadratic forms over schemes in which a ring analogous to Witt's ring of anisotropic forms is prominent. Calculation of such "Witt rings" is a problem of interest in modern algebraic geometry.

This thesis focuses on the calculation of the Witt ring of a smooth geometrically connected curve with good reduction over a local field. As a sub-problem, we calculate the Witt ring of a smooth gemetrically connected curve over a finite field. We present a generalization to the category of sheaves of the filtration of the Witt ring by powers of its fuundamental ideal of even rank elements. This yields a filtration by global sections which we study using étale cohomology. In the cases of interest here, this allows us to describe the Witt classes of a curve in terms of the classical invarients rank, signed discriminant, and Witt invariant.

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Contents

0.1 Introduction

The modern study of bilinear forms has a rich history beginning with Witt's work over fields in the 1930's, when he defined a ring structure on the set of anisotropic forms over a field. It was revived, notably by Pfister, in the 1960's. With the advent of algebraic K-Theory, much of the theory of quadratic forms over fields was generalized to a theory of quadratic spaces over rings. In the 1960's and 1970's Knebusch, among others, formulated a compatible theory for quadratic forms over schemes in which a ring analogous to Witt's ring of anisotropic forms is prominent. Calculation of such "Witt Rings" is a problem of interest in modern algebraic geometry.

This thesis focuses on the study of quadratic forms over varieties defined over a local field. In particular, it culminates in the calculation of the Witt ring of a smooth geometrically connected projective curve with good reduction over a local field. The Witt ring of a smooth geometrically connected projective curve over a finite field is calculated en route.

The Witt ring of a curve, C, over a finite field was initially calculated via the injection $W(C) \to W(k(C))$ of the Witt ring of a curve into the Witt ring of its function field. This allowed for the application of a number of classical results regarding quadratic forms over fields to quadratic forms defined over the curve. This technique proves generally insufficient, however, when the curve under consideration is defined over a local field, though some basic results of interest translate. The primary obstruction to the success of this technique is that the Brauer group of a curve over a local field fails to be trivial.

Over a field or ring, R, it is notable that a great deal of information has been gleaned from analysis of the filtration

$\cdots I^n \subseteq I^{n-1} \subseteq \cdots \subseteq I \subseteq W(R)$

of $W(R)$ by powers of the fundamental ideal. A generalization of this filtration to the category of sheaves has proved a useful tool in analyzing Witt rings of curves over both finite and local fields.

The primary and secondary results of this thesis are as follows:

Theorem. *Local Field Case*

Let C be a proper smooth geometrically connected projective curve with good reduction over a non-dyadic local field K with residue field $\bar{k}$ and let $\bar{C}$ be the reduction of the curve C to the residue field.

Then $W(C)$ is isomorphic to the group ring $W(\bar{C})[G]$ where G is a two element group.

Theorem. *Finite Field Case*

Let C be a proper smooth geometrically connected projective curve over a non-dyadic finite field, k.

Then $W(C)$ is isomorphic to $W(k)[{}_2Pic(C)]/\mathscr{R}$ where $W(k)[{}_2Pic(C)]$ is the group ring and $\mathscr{R}$ is the family of relations

$<1> - <s> \mathscr{L} - <t> \mathscr{M} + <st> \mathscr{L}\mathscr{M}$.

These results raise some unanswered questions that warrant future exploration.

- What is $W(X)$ if X is a variety of higher dimension over a finite field or a local field?

- What is $W(C)$ if C is a curve with bad reduction over a local field?

0.2 The Key Players

0.2.1 Bilinear Spaces and the Witt Ring

Let k be a field and let V be a k-vector space.

A bilinear form $B : V \times V \to k$ is symmetric provided $B(x, y) = B(y, x)$ for all $x, y \in V$ and non-degenerate provided it induces isomorphisms $B(x, -)$ and $B(-, y) : V \to V^\vee$ where $V^\vee = Hom_k(V, k)$ is the dual of V. A bilinear space over k is a pair (V, B) where V is a k-vector space and B is a symmetric non-degenerate bilinear form on V.

Bilinear spaces (V, B) and (V', B') are said to be isomorphic or isometric, written $(V, B) \cong (V', B')$, provided there is a vector space isomorphism $\sigma : V \to V'$ such that $B(x, y) = B'(\sigma(x), \sigma(y))$.

Equivalently, a bilinear space over k is a pair (V, φ) where $\varphi : V \to V^\vee$ is an isomorphism satisfying the following commutative diagram (this is the symmetry condition):

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V^\vee \\ \downarrow & \nearrow_{\varphi^\vee} & \\ V^{\vee\vee} & & \end{array}$$

The latter definition generalizes nicely to the categories of rings and schemes.

If X is a connected variety over a field k then a vector bundle on X is a locally free coherent $\mathcal{O}_X$-module of finite rank. The dual vector bundle of E is $E^\vee := \mathscr{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$.

A bilinear space over X is a pair $(E, \varphi)$where E is a vector bundle on X and φ a symmetric (in the diagrammatic sense) isomorphism $\varphi : E \to E^\vee$. A morphism $f : (E, \varphi) \to (F, \psi)$ of bilinear spaces is a morphism $f : E \to F$ of $\mathcal{O}_X$-modules which satisfies the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E^\vee \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^\vee} \\ F & \xrightarrow[\psi]{} & F^\vee \end{array}$$

Though it constitutes notational abuse, we may refer to a bilinear space (V, B) as either V or B and (E, φ) as either E or φ when there is no risk of confusion.

In all contexts, the set of bilinear spaces forms a semiring $Bil(-)$ under $\perp$ and $\otimes$ where $(E, \varphi) \perp (F, \xi) = (E \oplus F, \varphi \oplus \xi)$ and
$(E, \varphi) \otimes (F, \psi) = (E \otimes F, \varphi \otimes \psi)$.

If (V, B) is a bilinear space over a field k then a vector $v \in V$ is said to be isotropic if $B(v, v) = 0$. V is itself said to be isotropic if it contains an isotropic vector. A subspace U of V is totally isotropic if $B(u, v) = 0$ for all $u, v \in U$.

If E is a bilinear space on a variety X over a field k we say that a subsheaf $F \subset E$ is totally isotropic when F is contained in its orthogonal space $F^{\perp} := ker(E \to E^{\vee} \to F^{\vee})$.

A bilinear space E is metabolic provided it contains a totally isotropic subsheaf E_0 such that $rk(E_0) = \frac{1}{2} rk(E)$. Such an E_0 is called a Lagrangian of E and is a totally isotropic subsheaf of maximum possible rank. The Lagrangians are those subspaces which are precisely equal to their orthogonal spaces.

The Witt ring $W(-)$ of k or X is the semiring $Bil(-)$ of k or X modulo those spaces that are metabolic. In other words, bilinear spaces E and E'

represent the same Witt class and are called Witt equivalent provided there are metabolic spaces M and M' such that $E \perp M \cong E' \perp M'$.

The Witt ring has the structure of a ring under $\perp$ and $\otimes$. Additive inverses arise because $(E, \varphi) \perp (E, -\varphi)$ is the space whose underlying vector bundle is $E \oplus E^{\vee}$ and whose form is given by the matrix $\begin{pmatrix} \varphi & 1 \\ 1 & 0 \end{pmatrix}$. This is a metabolic space with Lagrangian $E^{\vee}$.

Note: The bundles called hyperbolic in [7] are those called metabolic here.

0.2.2 Classes Represented by Forms on Line Bundles

Let X be a proper smooth geometrically connected variety over a field k and let $Q(X)$ represent the multiplicative subgroup of $W(X)$ consisting of spaces $(\mathscr{L}, \varphi)$ whose underlying vector bundle is rank 1. Since φ may be used to define an isomorphism $\mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_X$ (and vice versa), the order of $\mathscr{L}$ is 2. Furthermore, each element of $Q(X)$ is represented by rank 1 forms on a unique line bundle of order 2 as in the following proposition and there is an element of $Q(X)$ associated to every line bundle of order 2.

Provided, $dim(X) \leq 3$, we may differentiate between classes on line bundles by viewing them in the Witt ring $W(k(X))$ of the function field via the injection $W(X) \hookrightarrow W(k(X))$ discussed in Subsection 0.2.3.

Over a field k, $< \xi >$ denotes the rank 1 form (V, φ) with $\varphi(e, e) = \xi \in k$

where e is a fixed basis of V. $<\xi_1, \ldots, \xi_n>$ denotes the sum $<\xi_1>\perp \cdots \perp<\xi_n>$.

$\mathscr{L}_{<\xi>}$ or sometimes simply $<\xi>$ will denote the Witt class represented by a form on $\mathscr{L}$ which maps to $<\xi>\in W(k(X))$. Note that $\mathscr{L}_{<\xi>} \otimes \mathscr{M}_{<\zeta>}$ maps to $<\xi\zeta>$ and $<s>\mathscr{L}_{<\xi>}$ maps to $<s\xi>$ in $W(k(C))$.

The Witt classes represented by forms on the structure sheaf $\mathcal{O}_X$ will be written $<1>$ and $<s_i>$ where the s_i are the nontrivial square classes of the base field k. Note that this notation is consistent with the above notation and that $\mathscr{L}_{<\xi>}\mathscr{L}_{<\xi>} =<1>$ and $\mathscr{L}_{<\xi>}\mathscr{L}_{<s_i\xi>} =<s_i>\mathscr{L}_{<\xi>}\mathscr{L}_{<\xi>} =<s_i>$.

Proposition 1. *If $\mathscr{L}_{<\xi>} = \mathscr{M}_{<\zeta>}$ then $\mathscr{L}$ and $\mathscr{M}$ are isomorphic as line bundles.*

Proof. *Given $\mathscr{L}, \mathscr{M} \in {}_2PicX$, if $\mathscr{L}_{<\xi>} = \mathscr{M}_{<\zeta>} \in W(X)$, $\mathscr{L}_{<\xi>} \perp M = \mathscr{M}_{<\zeta>} \perp M' \in Bil(C)$ for metabolic spaces M and M' of the same rank, $2m$.*

Taking determinants (subsection 0.2.4) on both sides, we see that

$det(\mathscr{L}_{<\xi>})det(M) = det(\mathscr{M}_{<\zeta>})det(M')$,

$\mathscr{L}_{<\xi>}(<-1>^m) = \mathscr{M}_{<\zeta>}(<-1>^m)$, *and*

$\mathscr{L}_{<\xi>} = \mathscr{M}_{<\zeta>}$.

We now answer the question of how many Witt classes are represented by forms on a given line bundle.

Proposition 2. *Fixing a line bundle $\mathscr{L} \in {}_2Pic(X)$, the Witt classes $(\mathscr{L}, \varphi)$ are in one to one correspondence with the square classes of units of the base field k.*

Proof. *First, we note that every such $\mathscr{L}$ is equipped with an isomorphism $\mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_X$, which may be used to define a nondegenerate symmmetric bilinear form on $\mathscr{L}$.*

Two forms $(\mathscr{L}, \varphi), (\mathscr{L}, \psi) \in Q$ may differ at most by a global endomorphism of $\mathscr{L}$ as in the following diagram:

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\varphi} & \mathscr{L}^\vee \\ {\scriptstyle \varphi\psi^{-1}}\downarrow & & \| \\ \mathscr{L} & \xrightarrow[\psi]{} & \mathscr{L}^\vee \end{array}$$

Given a line bundle, $\mathscr{L}$, we know that $\mathscr{E}nd(\mathscr{L}) \cong \mathcal{O}_X$. This means that the global endomorphisms of $\mathscr{L}$ are precisely the units of the base field k. Thus, φ, ψ differ by multiplication m_ℓ by some unit $\ell \in k$.

φ and ψ represent the same Witt class precisely when $\varphi \cong \psi$ which occurs if and only if $\varphi = m_\ell \circ \psi \circ m_\ell^\vee$ so that $\varphi = \ell^2\psi$.

This means that the Witt classes of X associated to $\mathscr{L}$ are in one-to-one correspondence with the square classes of the base field. Furthermore, the square classes of the base field act multiplicatively to permute the Witt *classes associated to a given line bundle.*

Note that we have obtained the following short exact sequence.

$0 \to k^\times/k^{\times 2} \to Q(X) \to {}_2Pic(X) \to 0$

Proposition 3. *The multiplicative subgroup $Q(X)$ of $W(X)$ consisting of classes represented by forms on line bundles is isomorphic to*

$W(k) \times 2Pic(X)$.

Proof. *Fixing a form on each $\mathscr{L}$, every element of $Q(X)$ may be uniquely written $< u > \mathscr{L}$ where u is a square class of units in k and $\mathscr{L}$ is an order 2 line bundle of X.*

0.2.3 The Witt Ring of a Variety as a Subring of the Witt Ring of its Function Field

Let X be a proper smooth geometrically connected variety over a field k such that $dim(X) \leq 3$.

Passing to the function field, the map $\varphi : E \to E^\vee$ yields a vector space isomorphism $E_\eta \to (E^\vee)_\eta$ where η is the generic point of X . Because $(E_\eta)^\vee$ is finitely generated, $(E^\vee)_\eta \cong (E_\eta)^\vee$.

In fact, $(E, \varphi) \to (E_\eta, \varphi_\eta)$ defines an inclusion $W(X) \hookrightarrow W(k(X))$ because $W(X) = W_{nr}(X)$ via [2]. The unramified Witt ring $W_{nr}(X)$ is the kernel of the map $W(k(X) \to \bigoplus_{x \in X^{(1)}} W(k(x))$ where $X^{(1)}$ consists of the codimension 1 points of X.

This allows us to view the Witt ring of X as a subring of the Witt ring of its function field.

There is also a map from $W(k) \to W(X)$ which is injective provided X contains a k-rational point, defining an action of $W(k)$ on $W(X)$.

Proposition 4. *If X contains a k-rational point then the map*
$i : W(k) \to W(X) :< s > \mapsto < s >$ *is injective when extended by linearity.*

Proof. *Note that i is the map obtained from the structure map $X \to Spec(k)$ via functoriality of $W(-)$. Using the k-rational point, we obtain an isomorphism* $Spec(k) \longrightarrow X \longrightarrow Spec(k)$ *. which yields an isomorphism*
$W(k) \xrightarrow{i} W(X) \longrightarrow W(k)$.

The map i defines an isomorphism from $W(k)$ onto the subring of $W(X)$ generated by rank 1 forms on the structure sheaf.

It is worth noting that the question of whether i is injective is precisely the question of whether the composition $W(k) \to W(k(X))$ is injective, which is in turn equivalent to the question of whether a non-square in the group of units $k^\times$ can become a square in $k(X)^\times$.

0.2.4 Clifford Algebras, Discriminants, The Brauer Group, and Witt Invariants

There are a number of useful invariants associated with forms, fields, and varieties. We give these definitions as if working over a variety X, but unless otherwise specified, the definitions when working over a field or ring are precisely analogous.

Determinants and Signed Determinants

To every bilinear space, E, we may associate a rank 1 space $det(E) = \bigwedge^{rk(E)} E$ called the determinant of E. In particular, the determinant of $< \xi_1, \ldots, \xi_n >$ is $< \xi_1 \cdots \xi_n >$. The determinant gives a well defined map from $Bil(X)$ to $Q(X)$. It does not, however, define a map from $W(X)$ to $Q(X)$ as the determinant of a rank $2n$ metabolic space is $< -1 >^n$, which is nontrivial if n is odd. Thus, we more frequently use the signed determinant $det_{\pm}(E) = (-1)^{\frac{n(n+1)}{2}} det(E)$.

The Witt class of $det_{\pm}(E)$ is called the signed discriminant of E and written $d_{\pm}(E)$. Furthermore, $d_{\pm}(E_\eta) = d_{\pm}(E)_\eta$.

The determinant satisfies $det(E \perp F) = det(E) \otimes det(F)$ and the signed determinant satisfies $det_{\pm}(E \perp F) = det_{\pm}(E) \otimes det_{\pm}(F)$ up to sign.

Clifford Algebras, the Brauer Group, and Witt Invariants

A map $f : E \to A$ where A is an $\mathcal{O}_X$-algebra, is said to be compatible with φ provided the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{q} & \mathcal{O}_X \\ {\scriptstyle f}\downarrow & & \downarrow \\ A & \xrightarrow[(\cdot)^2]{} & A \end{array}$$

In other words, $f : E \to A$ is compatible with φ if, locally,

$q(x) := \varphi(x, x) = f(x)^2$.

The Clifford algebra, $C(E)$, of a bilinear space, (E, φ), is the algebra which is universal with respect to compatibility with φ. $C(E)$ may be constructed on presheaves via $C(E(U)) = T(E(U))/\mathscr{R}(U)$ where $T(E(U))$ is the tensor algebra $k \oplus E(U) \oplus E(U)^{\otimes 2} \oplus \cdots$ and $\mathscr{R}(U)$ is generated by relations of the form $x \otimes x - \varphi(U)(x, x)$. It inherits the structure of a $\mathbb{Z}/2\mathbb{Z}$ graded algebra $C(E) = C_0(E) \oplus C_1(E)$ from the tensor algebra. Furthermore, if E is a space over a variety X then $C(E)_\eta = C(E_\eta)$, $C_0(E)_\eta = C_0(E_\eta)$, and $C_1(E)_\eta = C_1(E_\eta)$.

The Brauer group of a variety X consists of the Azumaya algebras over X modulo the endomorphism rings of locally free sheaves. It is a group under $\otimes$. This generalizes the concept of the Brauer group of a field, k, which consists of the central simple algebras over k modulo matrix algebras with coefficients in k.

If E has odd rank then $C_0(E)$ is an Azumaya algebra of order 2 and we define the Witt invariant $c(E)$ of E to be the class of $C_0(E)$ in the (order 2) Brauer group. If the rank of E is even then $C(E)$ is an order 2 Azumaya algebra and we define $c(E)$ to be the Brauer class of $C(E)$.

If X is proper smooth geometrically connected variety then the Brauer group of X maps into the Brauer group of $k(X)$ via $[A] \mapsto [A_\eta]$. Furthermore, $c(E)_\eta = c(E_\eta)$. This means that, in the event that the Hypotheses of [7] Proposition V.3.25 hold in the function field $k(X)$, the Witt class of a form over X is determined by parity of rank, signed discriminant, and Witt invariant.

0.2.5 A Filtration of the Witt Ring by Global Sections

If R is a ring or field, then the even rank elements of $W(R)$ form a maximal ideal $I(R)$ of $W(R)$ called the fundamental ideal and the powers $I^n(R)$ form a filtration of $W(R)$. If k is a field or local ring then $I^n(k)/I^{n+1}(k)$ is additively generated by the n-fold Pfister forms

$<< a_1, \ldots, a_n >>:=< 1, a_1 > \otimes \cdots \otimes < 1, a_n >$ (see the proof of Theorem 1).

We define presheaves

$W(-) : U \mapsto W(\mathcal{O}_X(U))$

$I(-) : U \mapsto I(\mathcal{O}_X(U))$

with associated sheaves $\tilde{W}$ and $\tilde{I}$. The sheaves $\tilde{I}^n$ form a filtration of the sheaf $\tilde{W}$. Thus, the global sections $I_n := \Gamma(X, \tilde{I}^n)$ form a filtration of $\Gamma(X, \tilde{W})$.

Assuming again that X is proper, smooth, geometrically connected, and projective and that $dim(X) \leq 3$, we obtain the following from [2].

Proposition 5. $\Gamma(X, \tilde{W}) = W(X)$.

Under these assumptions the filtration I_n by global sections is a filtration of $W(X)$.

Proposition 6. *If the rank of anisotropic forms over $k(X)$ is bounded then $E = 0 \in W(X)$ if and only if $E = 0 \in I_n/I_{n+1} \forall n$*

Proof. *Let $E = 0 \in I_n/I_{n+1}$.*

$\Gamma(X, \tilde{I}^n)/\Gamma(X, \tilde{I}^{n+1}) \hookrightarrow \Gamma(X, \tilde{I}^n/\tilde{I}^{n+1})$. *Thus, if $E = 0 \in I_n/I_{n+1}$ then $E = 0 \in \Gamma(X, \tilde{I}^n/\tilde{I}^{n+1})$.*

At the generic point, $(\tilde{I}^n/\tilde{I}^{n+1})_\eta = \tilde{I}^n_\eta/\tilde{I}^{n+1}_\eta$ and $\tilde{I}^n_\eta = I^n(k(C))$ is the n^{th} power of the fundamental ideal of $W(k(C))$.

Thus, if $E = 0 \in I_n/I_{n+1}$, then $E_\eta = 0 \in I^n(k(C))/I^{n+1}(k(C))$ and $E_\eta \in I^n(k(C))$ for all n. Since the rank of anisotropic forms is bounded, Hauptsatz X.5.1 of [7] shows that the $I^n(k(C)$ are eventually trivial.

Hence, $E_\eta = 0 \in W(k(C))$ *so that* $E = 0 \in W(C)$.

We will study the quotients I_n/I_{n+1} via the injection

$\Gamma(X, \tilde{I}^n)/\Gamma(\tilde{I}^{n+1}) \hookrightarrow \Gamma(X, \tilde{I}^n/\tilde{I}^{n+1})$.

Theorem 1. *(Generalized Theorem of Kerz, Milnor, Orlov, Vishik, Voevodsky et al)*

If X *is smooth and quasi-projective over* k *and* k *is infinite then there are isomorphisms*

$$\begin{array}{ccc} \mathscr{K}_M^n/2\mathscr{K}_M^n & \longrightarrow & \mathscr{H}^n(X_{et}, \mu_2) \\ \downarrow & \nearrow & \\ \tilde{I}^n/\tilde{I}^{n+1} & & \end{array}$$

where $\mathscr{K}_M^n$ *is the Zariski sheafification of Milnor K-theory and*

$\mathscr{H}^n(\mu_2)$ *is the Zariski sheafification of étale cohomology presheaf*

$U \mapsto H_{et}^n(U, \mu_2)$.

Proof. *These maps are defined on presheaves as follows:*

$\mathscr{K}_M^n/2\mathscr{K}_M^n \to \mathscr{H}^n(\mu_2) : \{a_1, \ldots, a_n\} \mapsto a_1 \cup \cdots \cup a_n$

$\mathscr{K}_M^n/2\mathscr{K}_M^n \to \tilde{I}^n/\tilde{I}^{n+1} : \{a_1, \ldots, a_n\} \mapsto << -a_1, \ldots, -a_n >>$

We note that this theorem holds when the variety X *is replaced by a field and étale cohomology by Galois cohomology [1]. It is also true when* $\tilde{I}^n$ *is replaced by the sheafification* $\tilde{I}_{ur,n}$ *of the presheaf*

$U \mapsto I^n(U) \cap W_{nr}(U)$ *(see [1] wherein results of [5] are presented).*

We will show that, when $X = Spec(A)$ *where* A *is a local ring of a variety,* $I^n(A)/I^{n+1}(A) = I_{ur,n}(A)/I_{ur,n+1}(A)$.

$I_{ur,n}(A)/I_{ur,n+1}(A)$ *is defined by the exact sequence*

$$0 \to I_{ur,n}(A)/I_{ur,n+1}(A) \to I^n(k(A))/I^{n+1}(k(A)) \to \coprod_{x \in Spec(A)^{(1)}} I^{n-1}(k(x)))/I^n(k(x)) \to \cdots$$

As $I_n(A)/I_{n+1}(A)$ *maps to the kernel of* $I^n(k(A))/I^{n+1}(kA) \to \coprod_{x \in Spec(A)^{(1)}} I^{n-1}(k(x)))/I^n(k(x))$*, there is a map* $I_n(A)/I_{n+1}(A) \to I_{ur,n}(A)/I_{ur,n+1}(A)$

There is also a surjection

$K^M_n(A)/2 \twoheadrightarrow I^n(A)/I^{n+1}(A) : \{a_1, \ldots, a_n\} \mapsto << -a_1, \ldots, -a_n >>$.

To see that this map is surjective, we must ensure that the Pfister forms additively generate $I^n(A)/I^{n+1}(A)$*. We proceed by induction on* n*. We know that* $W(A)$ *is additively generated by elements of rank* 1 *([7] or [6]). The induction step is trivial. We must show that* $I(A)/I^2(A)$ *is additively generated by forms* $< 1, a >$*. To see this, simply note that the quaternion form* $< 1, -a, -b, ab > = < 1, -a > \otimes < 1, -b >$ *is trivial in* $I(A)/I^2(A)$ *and so* $< a, b > = < 1, ab > \in I/I^2$.

We now have a diagram,

$$\begin{array}{ccc} K_n^M(A)/2 & \longrightarrow & I_{ur,n}(A)/I_{ur,n+1}(A) \\ & \searrow^{p} & \uparrow i \\ & & I_n(Spec(A))/I_{n+1}(Spec(A)) \end{array}$$

As $K_n^M(A)/2 \to I_{ur,n}(A)/I_{ur,n+1}(A)$ *is an isomorphism,* p *must be injective, hence an isomorphism, and so* i *is also an isomorphism.*

We can use this to obtain a description of the first three I_n/I_{n+1} (and the images of bilinear spaces therein) in terms of classical invariants and to describe representatives for these quotients in terms of classical invariants.

Proposition 7. *Suppose* 2 *is a unit on* X*.*

1. $\tilde{W}/\tilde{I} = \underline{\mathbb{Z}/2}$
2. $\tilde{I}/\tilde{I}^2 = \mathcal{O}^\times/\mathcal{O}^{\times 2}$
3. $\tilde{I}^2/\tilde{I}^3 = \underline{{}_2Br(X)}$

Proof. $\tilde{W}/\tilde{I}$ *is the sheaf defined on each open* $U \subseteq X$ *by the presheaf* $W(\mathcal{O}_X(U))/I(\mathcal{O}_X(U))$*, which is the ring* $\mathbb{Z}/2$ *for all connected* U*.*

Since $0 \to \underline{\mu_2} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ *is exact in the étale setting, there is, for each open* $U \subseteq X$*, a long exact sequence*

$$\begin{array}{ccccccc} H^0_{et}(U,\mathbb{G}_m) & \xrightarrow{\cdot 2} & H^0_{et}(U,\mathbb{G}_m) & \longrightarrow & H^1_{et}(U,\underline{\mu_2}) & \longrightarrow & H^1_{et}(U,\mathbb{G}_m) \\ & & & & & & \downarrow \\ & & & & & & H^1_{et}(U,\mathbb{G}_m) \end{array}$$

After the identifications $H^0_{et}(U,\mathbb{G}_m) = \Gamma(U,\mathbb{G}_m)$ *and* $H^1_{et}(U,\mathbb{G}_m) = Pic(U)$*, this yields a short exact sequence*

$$0 \to \Gamma(U,\mathbb{G}_m)/\Gamma(U,\mathbb{G}_m)^2 \to H^1_{et}(U,\underline{\mu_2}) \to {}_2Pic(U) \to 0$$

on each open U *so that there is an exact sequence of Zariski sheaves*

$$0 \to \mathcal{O}_X^\times/\mathcal{O}_X^{\times 2} \to \mathscr{H}^1_{et}(\underline{\mu_2}) \to \underline{{}_2Pic(X)} \to 0.$$

However, the Picard group is trivial at each stalk, so this becomes a sheaf isomorphism $\mathcal{O}^\times/\mathcal{O}^{\times 2} \cong \mathscr{H}^1_{et}(\underline{\mu_2})$.

In a similar way, we have an exact sequence

$$\begin{array}{ccccccc} H^1_{et}(U,\mathbb{G}_m) & \xrightarrow{\cdot 2} & H^1_{et}(U,\mathbb{G}_m) & \longrightarrow & H^2_{et}(U,\underline{\mu_2}) & \longrightarrow & H^2_{et}(U,\mathbb{G}_m) \\ & & & & & & \downarrow \\ & & & & & & H^2_{et}(U,\mathbb{G}_m) \end{array}$$

After the identifications $H^1_{et}(U,\mathbb{G}_m) = Pic(U)$ *and* $H^2_{et}(U,\mathbb{G}_m) = Br(U)$*, this yields a short exact sequence*

$$0 \to Pic(U))/Pic(U))^2 \to \mathscr{H}^2_{et}(U,\underline{\mu_2}) \to \underline{{}_2Br(U)} \to 0$$

Again there is no contribution from the Picard group at the level of stalks, so we have a sheaf isomorphism $\mathscr{H}^2_{et}(\underline{\mu_2}) \cong \underline{{}_2Br(X)}$

Proposition 8. *The class of* E *in* $W(X)/I$ *is determined by parity of rank.*

Proof. *At each stalk, the composition* $W(X) \to W(X)/\Gamma(X,\tilde{I}) \hookrightarrow \Gamma(X,\tilde{W}/\tilde{I})$ *is determined by parity of rank, hence the map*

$W(X) \to W(X)/\Gamma(X, \tilde{I})$ is determined by parity of rank.

Note that this means that I consists of the elements of $W(X)$ represented by symmetric spaces of even rank and that every nonzero element of $W(X)/I$ is represented by $< 1 >$ (or, indeed, any rank 1 form).

Proposition 9. *Given an even rank element of $W(X)$, The class of $E \in I/I_2$ is determined by the signed discriminant.*

Proof. *At each stalk, the composition*

$W(X) \to \Gamma(X, \tilde{I})/\Gamma(X, \tilde{I}^2) \hookrightarrow \Gamma(X, \tilde{I}/\tilde{I}^2)$ is determined by signed discriminant, hence the map $\Gamma(X, \tilde{I}) \to \Gamma(X, \tilde{I})/\Gamma(X, \tilde{I}^2)$ is determined by signed discriminant.

Note that this means that I_2 consists of those elements of $W(X)$ that are represented by even rank symmetric spaces with signed discriminant $< 1 >$. It also implies that every element of I/I_2 is represented by the form $< 1, -\delta >$ where $< \delta >$ is the signed discriminant.

Proposition 10. *Given an even rank $E \in W(X)$ whose signed discriminant is $< 1 >$, the class of E in I_2/I_3 is determined by the Witt invariant of E.*

Proof. *At each stalk, the composition*

$W(X) \to \Gamma(X, \tilde{I}^2)/\Gamma(X, \tilde{I}^3) \hookrightarrow \Gamma(X, \tilde{I}^2/\tilde{I}^3)$ is determined by its Witt invari-

ant. Hence the map

$\Gamma(X, \tilde{I}^2) \to \Gamma(X, \tilde{I})^2/\Gamma(X, \tilde{I}^3)$ *is determined by its Witt invariant.*

If $X =$ is a curve over a finite field then $k(X)$ is a C_2 field. If X is a good reduction curve over a local field then the 2-cohomological dimension of X is 3 so $H^0(X, \mathscr{H}^n(\mu_2)) = 0$ for all $n > 3$. $H^0(X, \mathscr{H}^3(\mu_2)) = H^3_{un}(X, \mu_2) = 0$ by [4] Proposition 5.2 . This gives us the following.

Proposition 11. *If X is a curve over a finite field then*

$I^n(k(X))/I^{n+1}(k(X))$ *is trivial when $n > 2$.*

If X is a curve with good reduction over a local field then $I_n(X)/I_{n+1}(X)$ is trivial when $n > 2$.

This is a very useful result. It will allow us to say that an element of the Witt ring for a curve over a finite field is completely determined by signed discriminant and rank($mod 2$). It will also allow us to say that, in the good reduction case, elements of the Witt ring of a curve over a local field are determined entirely by parity of rank, signed discriminant, and Witt invariant. In both cases, the calculation of the Witt ring is reduced to consideration of classical invariants.

Note that the good reduction assumption is essential here, as examples are known of bad reduction curves over local fields for which I_3/I_4 is nontrivial.

0.3 The Calculation over a Finite Field

0.3.1 The Witt Ring of a Finite Field

The structure of the Witt ring $W(k)$ of a non-dyadic finite field $k = \mathbb{F}_q$ (q odd) is well understood. It consists of 0, two rank one classes, $< 1 >$ and $< s >$, and a nontrivial rank 2 class which is $< 1, s >$ when $q = 1(mod 4)$ and $< 1, 1 >$ when $q = 3(mod 4)$.

$W(k)$ has a few properties which are useful for calculational purposes.

$< 1, 1 >=< s, s >$ for all nondyadic finite fields.

$< 1, 1, 1 >=< 1 >=< -1 >$ when $q = 1(mod 4)$.

$< 1, 1, 1 >=< s >=< -1 >$ when $q = 3(mod 4)$.

$< 1, 1, 1, 1 >= 0$ for all nondyadic finite fields.

0.3.2 The Witt Ring of a Curve over a Finite Field

In this section, we take C to be a smooth geometrically connected projective curve over a non-dyadic finite field k such that C contains a k-rational point.

Due to the injection $W(k) \to W(C)$ the statements of Subsection 0.3.1 hold in $W(C)$ when q is the characteristic of the base field.

We fix a form $\mathscr{L}_{<\xi>}$ on each order 2 line bundle so that the Witt classes associated to $\mathscr{L}$ are $\mathscr{L}_{<\xi>}$ and $< s > \mathscr{L}_{<\xi>}$. The fixed form on the structure sheaf will be $< 1 >$. The selection of form for all other line bundles is arbitrary. When we write $\mathscr{L}, < s > \mathscr{L} \in W(C)$, we are assuming that $\mathscr{L}$ is equipped with a fixed form.

Proposition 12. *$Br(C)$ is trivial*

Proof. *We calculate the cohomological Brauer group $H^2(C_{et}, \mathbb{G}_m)$*

Given a finite separable extension K of k, $X_K \to X$ is a finite Galois covering with Galois group $G = Gal(K/k)$.

There is a Hochschild-Serre spectral sequence ([8] Theorem III.2.20) $H^p(G, H^q(C_{K,et}, \mathbb{G}_m)) \Rightarrow H^{p+q}(C_{et}, \mathbb{G}_m)$ for each finite separable extension of the base field.

As cohomology commutes with inverse limits ([8] III.1.16), this gives a spectral sequence $H^p(\bar{G}, H^q(\bar{C}, \mathbb{G}_m)) \Rightarrow H^{p+q}(C_{et}, \mathbb{G}_m)$ where $\bar{C}$ is the extension of C to the separable closure k_{sep} of the base field. In the case where C is a curve over a finite field, $\bar{C}$ is also the extension to the algebraic closure.

This is a first quadrant spectral sequence with E^2 terms as follows:

$E^2_{0,2} = H^0(\bar{G}, H^2(\bar{C}, \mathbb{G}_m))$ *with* $H^2(\bar{C}, \mathbb{G}_m) = Br(\bar{C}) = 0$ *as* $k(\bar{C})$ *is a* C_1 *field (Tsen's theorem). Thus,* $E^2_{0,2} = 0$.

$E^2_{2,0} = H^2(\bar{G}, H^0(\bar{C}, \mathbb{G}_m)) = H^2(\bar{G}, k^\times_{sep}) = Br(k_{sep}) = 0$

$E^2_{1,1} = H^1(\bar{G}, H^1(\bar{C}, \mathbb{G}_m)) = H^1(\bar{G}, Pic(\bar{C}))$. *There is a short exact sequence* $0 \longrightarrow Pic^0(\bar{C}) \longrightarrow Pic(\bar{C}) \xrightarrow{deg} \mathbb{Z} \longrightarrow 0$

Furthermore, $H^1(\bar{G}, Pic^0(\bar{C})) = 0$ *(Lang's Theorem) and* $H^1(\bar{G}, \mathbb{Z}) = Hom_{cont}(\bar{G}, \mathbb{Z}) = 0$ *so that* $E^2_{1,1} = H^1(\bar{G}, Pic(\bar{C})) = 0$.

This shows that the cohomological Brauer group, hence the Brauer group, is trivial.

Proposition 13. *Each element of* $W(C)$ *is of the form* $< \delta >$ *or* $< 1, -\delta >$ *where* $\delta \in {}_2Pic(C)$ *is its signed discriminant. Furthermore, each of these forms is unique in* $W(C)$ *up to square classes of units.*

Proof. *By proposition 11, an arbitrary form on* C *may be equated to zero in the Witt ring after subtraction of* $< 1 >$, $< 1, -\delta >$, *or both. Thus, each element of the Witt ring has a representative of the form* $< 1 >$, $< 1, -\delta >$, *or* $< 1, 1, -\delta >$. *However, the latter is actually represented by the form* $< \delta >$ *as rank 2 forms are determined by signed discriminant and clifford invariant so that* $< 1, -\delta > = < -1, \delta >$.

Uniqueness is trivial.

The Witt ring of C gets some of its structure from the order 2 Picard group of C and some from Witt ring of the finite base field. However, there is another family of relations due to the fact that ${}_2Br(C) = 0$

Proposition 14. $< 1, -\mathscr{L}_\xi, -\mathscr{M}_\zeta, (\mathscr{L}\mathscr{M})_{\xi\zeta} >$ *is trivial in* $W(C)$.

Proof. *This form is even and has signed discriminant* $< 1 >$. *Its Witt invariant is also trivial. Thus,* $< 1, -\mathscr{L}_\xi, -\mathscr{M}_\zeta, (\mathscr{L}\mathscr{M})_{\xi\zeta} >$ *is trivial in each* I_n/I_{n+1} *and in* $W(C)$.

Corollary 1. $< \mathscr{L}_\xi, \mathscr{M}_\zeta > = < 1, \mathscr{L}\mathscr{M}_{\xi\zeta} > \in W(C)$

We can now describe the Witt ring of a curve over a finite field using generators and relations.

Proposition 15. *The Witt ring of a smooth projective curve* C *over a finite field* k *is isomorphic to the quotient* $W(k)[{}_2Pic(C)]/\mathscr{R}$ *where* $W(k)[{}_2Pic(C)]$ *is the group ring and* $\mathscr{R}$ *is the family of relations*

$< 1 > - < s > \mathscr{L} - < t > \mathscr{M} + < st > \mathscr{L}\mathscr{M} \in W(k)/{}_2Pic(C)$.

Proof. *Define a map* $W(k)[{}_2Pic(C)]/\mathscr{R} \to W(C)$ *by sending* $< s > \mathscr{L}$ *to* $< s > \mathscr{L}$ *and extending by linearity.*

This map is clearly a well defined surjection of commutative rings.

To show injectivity let $f \in W(k)[{}_2Pic(C)]/\mathscr{R}$ *map to a form* $E = < u_1\mathscr{L}_1, \ldots, u_n\mathscr{L}_n > \in W(C)$. *Using the relations* $< 1, 1 > = < -1, -1 >$ *and* $< u\mathscr{L}, v\mathscr{M} > = < 1, uv\mathscr{L}\mathscr{M} >$ *we may rewrite* E *as* $< -1, u_1 \cdots u_n\mathscr{L}_1 \cdots \mathscr{L}_n >$ *or* $< 1, u_1 \cdots u_n\mathscr{L}_1 \cdots \mathscr{L}_n >$ *in* $W(C)$. *However, using the corresponding relations* $< 1 > + < 1 > + < 1 > + < 1 >$ *and*

$<1> - <u> \mathscr{L} - <v> \mathscr{M} + <uv> \mathscr{L}\mathscr{M}$ in $W(k)[{}_2Pic(C)]/\mathscr{R}$, we may write f as either $<-1> + <u_1 \cdots u_n> \mathscr{L}_1 \cdots \mathscr{L}_n$ or $<1> + <u_1 \cdots u_n> \mathscr{L}_1 \cdots \mathscr{L}_n$. In the first case, E and f are trivial precisely when $<u_1 \cdots u_n \mathscr{L}_1 \cdots \mathscr{L}_n> = <1> \in W(K)$. In the second, both E and f are trivial precisely when $<u_1 \cdots u_n \mathscr{L}_1 \cdots \mathscr{L}_n> = <-1> \in W(K)$.

0.4 Calculation over a Local Field

In this section we consider the Witt ring of a smooth geometrically connected projective curve C with good reduction over a local field K. We assumethat C contains a K-rational point. Once again, the forms on an individual line bundle are in one to one correspondence with the rank 1 forms in the Witt ring of the base field, of which there are four. Again, equip each line bundle with a fixed form, but we must show some care when choosing the fixed form on other $\mathscr{L} \in {}_2Pic(C)$. We wish to fix a form on $\mathscr{L}$ that extends to a non-degenerate form over the extension of the curve C to the residue field. Select $<1>$ to be the fixed form on the structure sheaf.

The current stituation is shown in the following diagram

$$\begin{array}{ccccc} \bar{C} & \longrightarrow & C_\nu & \longleftarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ Spec(\bar{k}) & \longrightarrow & Spec(\mathcal{O}_\nu) & \longleftarrow & Spec(K) \end{array}$$

where $C = C_\nu \otimes K$ is the generic fiber of $C_\nu \to Spec\mathcal{O}_\nu$ and the special fiber $\bar{C} = C_\nu \otimes \bar{k}$ is smooth over $Spec(\bar{k})$.

Proposition 16. ${}_2Pic\bar{C} = {}_2PicC_\nu = {}_2PicC$

Proof. *Let T be a scheme which is also an $\mathcal{O}_{C_\nu}$-algebra.*

${}_2Pic^0_{C_\nu/\mathcal{O}_{C_\nu}} = Spec(A)$ *is a finite étale group scheme which represents the functor* ${}_2Pic^0_{C_\nu/\mathcal{O}_{C_\nu}} : T \mapsto {}_2Pic^0(C_\nu \otimes T)/{}_2Pic^0(T)$. *The elements of* ${}_2Pic^0_{C_\nu/\mathcal{O}_{C_\nu}}(T)$ *are called T-points of* ${}_2Pic^0_{C_\nu/\mathcal{O}_{C_\nu}}$

In particular, ${}_2Pic^0_{C_\nu/\mathcal{O}_{C_\nu}}(K) = {}_2Pic(C)$, ${}_2Pic^0_{C_\nu/\mathcal{O}_{C_\nu}}(\mathscr{O}_\nu) = {}_2Pic(C_\nu)$, *and* ${}_2Pic^0_{C_\nu/\mathcal{O}_{C_\nu}}(\bar{k}) = {}_2Pic(\bar{C})$.

On the other hand, the K-points of ${}_2Pic^0_{C_\nu/\mathcal{O}_{C_\nu}}$ *are K-algebra homomorphisms $A_K = A \otimes K \to K$, the $\mathscr{O}_\nu$-points are $\mathcal{O}_\nu$-algebra homomorphisms $A \to \mathscr{O}_\nu$, and the $\bar{k}$-points are $\bar{k}$-algebra homomorphisms $\bar{A} = A \otimes \bar{k} \to \bar{k}$.*

Every prime ideal of A_K is maximal, so the Chinese remainder theorem gives a decomposition $A_K = \oplus K_i$ into a product of field extensions of K. Since the idempotents generating the K_i must exist in A, we also have a decomposition $A = \oplus A_i$ into indecomposable rings. Hensel's lemma guarantees that the decomposition $\bar{A} = \oplus \bar{A}_i$ where $\bar{A}_i = A_i \otimes \bar{k}$ is also a decomposition into indecomposable rings. Furthermore, $A_i = \mathcal{O}_\nu$ precisely when $K_i = K$ and $\bar{A}_i = \bar{k}$ so that the K-points of $Pic^0_{C_\nu/\mathcal{O}_{C_\nu}}$ *are precisely the $\mathcal{O}_\nu$-points* *which are precisely the $\bar{k}$-points.*

Thus, associated to each line bundle $\mathscr{L} \in {}_2Pic(C)$, there is a unique line bundle $\mathscr{L}_\nu \in {}_2Pic(C_\nu)$. We select a Witt class represented by $(\mathscr{L}_\nu, \varphi)$ and note that $(\mathscr{L}, \varphi \otimes 1)$ represents a Witt class of C. Furthermore, $(\bar{\mathscr{L}}, \varphi \otimes 1)$ represents a Witt class of $\bar{C}$ and every rank 1 element of $W(\bar{C})$ may be obtained by base change from a form over C_ν. When we write $\mathscr{L} \in W(C)$ it is assumed that $\mathscr{L}$ is equipped with a fixed form $\varphi \otimes 1$ that was selected in this way. There are four Witt classes associated to each $\mathscr{L}$. These are $\mathscr{L}$, $< s > \mathscr{L}$, $< \pi > \mathscr{L}$, and $< s\pi > \mathscr{L}$ where $< 1 > \neq < s > \in W(\bar{k})$ and π is the uniformizing parameter of $\mathcal{O}_\nu$.

Since the forms $\mathscr{L}$ and $< s > \mathscr{L}$ arise from forms over $\bar{C}$, the forms $\mathscr{L}$ and $< s > \mathscr{L}$ behave like forms over a curve over a finite field. In particular, we have a relations $< u\mathscr{L}, v\mathscr{M} > = < 1, uv\mathscr{L}\mathscr{M} >$ and
$< \pi u\mathscr{L}, \pi v\mathscr{M} > = < \pi, \pi uv\mathscr{L}\mathscr{M} >$ in $W(C)$ where $u, v \in \bar{k}^\times / \bar{k}^{\times 2}$.

0.4.1 The Order 2 Brauer Group

Determined by Order 2 Line Bundles and Square Classes of the Base Field

Given a square class, s, of $K^\times$ and an order 2 line bundle, $\mathscr{L}$, we will construct Azumaya algebras $\tilde{Q}(< s > \mathscr{L})$ modeled after the quaternion algebras. These generalized quaternions will act as representatives for elements of the order 2 part of the Brauer group.

Let $\mathscr{L} \in {}_2Pic(C)$, $< s > \in W(\bar{k})$.

Take $f : \mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_C$ to be the isomorphism corresponding to the form on $< s > \mathscr{L}$ and let $\mathscr{L}_i := \mathscr{L}$. Let $\mathcal{O}_1 := \mathcal{O}_j := \mathcal{O}_C$. Let multiplication of graded rings on $\mathcal{O}_1 \oplus \mathscr{L}_i$ be given by f and on $\mathcal{O}_1 \oplus \mathcal{O}_j$ by the isomorphism $\mathcal{O}_j \otimes \mathcal{O}_j \to \mathcal{O}_C$ induced by the form $< \pi >$.

Define $\tilde{Q}(< s > \mathscr{L})$ to be the graded tensor product $(\mathcal{O}_1 \oplus \mathscr{L}_i)\hat{\otimes}(\mathcal{O}_1 \oplus \mathcal{O}_j)$

Note that $\tilde{Q}(< s > \mathscr{L}) = \mathcal{O}_C \oplus \mathscr{L} \oplus \mathscr{L} \oplus \mathcal{O}_C$ as an $\mathcal{O}_C$-module and that multiplication on the second copy of $\mathscr{L}$ is given by πf.

This generalized quaternion algebra will be denoted $\left(\frac{s\mathscr{L},\pi}{k(C)}\right)$. Over the function field, it is the quaternion algebra $\left(\frac{s\xi,\pi}{k(C)}\right)$ additively generated by i, j, k with $i^2 = \xi$, $j^2 = \pi$, and $ij = -ji = k$ where $< \xi >$ is the image of $\mathscr{L}$ in $W(k(C))$. The norm form of $\left(\frac{\xi,\pi}{k(C)}\right)$ is $< 1, -s\xi, -\pi, s\xi\pi >$, which is the image in $W(k(C))$ of the form $< 1, -s\mathscr{L}, -\pi, s\pi\mathscr{L} >$.

The set of $\tilde{Q}(< s > \mathscr{L})$ will comprise the order 2 Brauer group, which is calculated in the following two theorems.

Theorem 2. *The order 2 Brauer group ${}_2Br(C)$ fits into a short exact sequence*

$$0 \to \bar{k}^{\times}/\bar{k}^{\times 2} \to {}_2Br(C) \to {}_2Pic(C) \to 0.$$

Proof. *We will show that* ${}_2Br(C) \simeq H^1(\bar{C}, \mathbb{Z}/2)$*. The desired short exact sequence follows from the short exact sequence* $0 \to \mu_2 \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ *for a curve over a finite field.*

There is a localisation exact sequence

$H^2(C_\nu, \mathbb{G}_m) \to H^2(C, \mathbb{G}_m) \to H^3_{\bar{C}}(C_\nu, \mathbb{G}_m) \to H^3(C_\nu, \mathbb{G}_m) \to H^3(C, \mathbb{G}_m)$

Due to [3] 6.2 this becomes the following:

$$H^2(C_\nu, \mathbb{G}_m) \to Br(C) \to H^1(\bar{C}, \mathbb{Q}/\mathbb{Z}) \to H^3(C_\nu, \mathbb{G}_m) \to H^3(C, \mathbb{G}_m)$$

We will show that ${}_2H^2(C_\nu, \mathbb{G}_m) = 0$ *and that the map* ${}_2H^3(C_\nu, \mathbb{G}_m) \to {}_2H^3(C, \mathbb{G}_m)$ *is injective.*

We first note that $H^p(C_\nu, \mu_2) \simeq H^p(\bar{C}, \mu_2)$ *for all* p *since there are spectral sequences*

$H^p(Gal(\mathcal{O}_{\nu,sep}/\mathcal{O}_\nu), H^q(C_{\nu,sep}, \mu_2)) \Rightarrow H^n(C_\nu, \mu_2)$ *and*

$H^p(Gal(\bar{k}_{sep}/\bar{k}), H^q(\bar{C}_{sep}, \mu_2)) \Rightarrow H^n(\bar{C}, \mu_2)$

Hensel's lemma guarantees that the Galois groups are the same, so it remains to show that the cohomology groups over the separable closure are isomorphic.

We have isomorphisms of stalks $H^q(C_{\nu,sep}, \mu_2) \simeq R^q\pi_*(\mu_2)_{pt}$ *and* $H^q(\bar{C}_{sep}, \mu_2) \simeq R^q\bar{\pi}_*(\mu_2)_{pt}$ *for the following diagram*

$$\begin{array}{ccc} \bar{C}_{sep} & \xrightarrow{\ i\ } & C_{\nu,sep} \\ {\scriptstyle \bar{\pi}_{sep}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{sep}} \\ Spec(\bar{k}_{sep}) & \xrightarrow{\ i\ } & Spec(\mathcal{O}_{\nu,sep}) \end{array}$$

The proper base change theorem gives $H^p(C_\nu, \mu_2) \simeq H^p(\bar{C}, \mu_2)$.

The Kummer sequence $0 \to \mu_2 \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ *gives a ladder with short exact rows*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Pic(C_\nu)/2 & \longrightarrow & H^2(C_\nu, \mu_2) & \longrightarrow & {}_2Br(C_\nu) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\simeq & & \downarrow & & \\ 0 & \longrightarrow & Pic(\bar{C})/2 & \longrightarrow & H^2(\bar{C}\mu_2) & \longrightarrow & {}_2Br(\bar{C}) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

$Pic^0(\bar{C})$ *is finite and torsion so* ${}_2Pic^0(\bar{C})$ *and* $Pic^0(\bar{C})/2$ *must have the same finite cardinality. Similarly,* ${}_2Pic^0(C_\nu)$ *and* ${}_2Pic^0(\bar{C})$ *must have the same finite cardinality. This means that the map* $Pic(C_\nu)/2 \to Pic(\bar{C})/2$ *is an isomorphism. Since the map* $H^2(C_\nu, \mu_2) \to H^2(\bar{C}, \mu_2)$ *is also an isomorphism, so is the map* ${}_2Br(C_\nu) \to {}_2Br(\bar{C}) = 0$.

As the Brauer group is torsion, every element of $Br(C_\nu)$ *may be written as a multiple* $x = -2mx$ *of* 2. *This, together with the Kummer sequence and the triviality of the order* 2 *part of the Brauer groups gives the following* *diagram with exact rows:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_2H^3(C_\nu,\mu_2) & \longrightarrow & {}_2H^3(C_\nu,\mathbb{G}_m) & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow & & \\ 0 & \longrightarrow & {}_2H^3(\bar{C},\mu_2) & \longrightarrow & {}_2H^3(\bar{C},\mathbb{G}_m) & \longrightarrow & 0 \end{array}$$

There are spectral sequences

$E^2_{p,q} := H^p(\bar{G}, H^2(\bar{C}_{sep},\mu_2) \Rightarrow H^n(C_\nu,\mu_2)$ *and*

$E^2_{p,q} := H^p(G', H^2(C',\mu_2) \Rightarrow H^n(C,\mu_2)$

where $\bar{G} = Gal(\bar{k}_{sep}/\bar{k})$ *and* $G' = Gal(F/K)$ *where* $F = \lim\limits_{\rightarrow} F_i$ *where the* F_i *are extensions of* K *corresponding to separable extensions of* $\bar{k}$.

This means that $H^3(C_\nu,\mu_2) = H^1(\bar{G}, H^2(\bar{C}_{sep},\mu_2)) = Hom(\bar{G},\mathbb{Z}/2)$ *since the* K*-rational point determines a* $\bar{k}$*-rational point and*

$H^3(C,\mu_2) = H^1(G', H^2(C',\mu_2)) = Hom(G', H^2(C',\mu_2))$ *is contained in* $H^3(C,\mu_2)$. *In fact,* $Hom(\bar{G},\mathbb{Z}/2) = \mathbb{Z}/2$ *maps into* $Hom(G', H^2(C',\mu_2))$ *as both contain a map taking the Frobenius map to the class of the rational point.*

Theorem 3. *The order* 2 *Brauer group of a smooth curve with good reduction over a local field consists of the distinct quaternions* $\left(\frac{s\mathscr{L},\pi}{C}\right)$ *where* $s \in \bar{k}^\times/\bar{k}^{\times 2}$ *and* $\mathscr{L} \in {}_2Pic(C)$.

Proof. *It suffices to show that the quaternions are distinct. That they comprise the entire order* 2 *Brauer group follows by a cardinality argument. We* *work over the function field via the injections* ${}_2Br(C) \hookrightarrow {}_2Br(k(C))$ *and* $W(C) \hookrightarrow W(k(C))$. *We take* $< s\xi >$ *and* $< t\zeta >$ *to be the images of* $< s > \mathscr{L}$ *and* $< t > \mathscr{M}$ *in* $W(k(C))$.

If $\left(\frac{s\xi,\pi}{k(C)}\right) = \left(\frac{t\zeta,\pi}{k(C)}\right)$ *then their norm forms are equal in* $W(k(C))$ *and we have a relation*

$< 1, -\pi, -\xi, \pi\xi, -1, \pi, \zeta, -\pi\zeta >= 0 \in W(k(C))$.

But $< 1, -\pi, -\xi, \pi\xi, -1, \pi, \zeta, -\pi\zeta >=< -\xi, \zeta, \pi\xi, -\pi\zeta >=$ $< 1, -\xi\zeta, -\pi, \pi\xi\zeta >=< 1, -\pi, -\xi\zeta, \pi\xi\zeta >$ *which is the norm form of* $\left(\frac{st\xi\zeta,\pi}{k(C)}\right)$.

Thus, it suffices to show that $\left(\frac{s\xi,\pi}{K(C)}\right)$ *is nontrivial when* $< \xi > \neq < 1 >$.

If $\left(\frac{s\xi,\pi}{K(C)}\right)$ *is trivial in* ${}_2(Br(k(C))$ *then there is an identity* $a'^2 - b'^2\xi = \pi$ *in* $k(C)$ *([7] Theorem III.2.7).*

If we take $p \in \mathcal{O}_{C_\nu}$ *to be the extension of the* K*-rational point over the closed point of* $Spec(\mathcal{O}_\nu)$ *then the following are true of the local ring* $\mathcal{O}_{C_\nu,p}$:

- $\mathcal{O}_{C_\nu,p}$ *is a unique factorization domain*
- π *is not a unit of* $\mathcal{O}_{C_\nu,p}$
- π *is a prime/irreducible element of* $\mathcal{O}_{C_\nu,p}$
- *The field of fractions of* $\mathcal{O}_{C_\nu,p}$ *is* $k(C)$

This means that we can write $a' = \frac{a}{d}$ *and* $b' = \frac{b}{e}$ *where* $a, b, d, e \in \mathcal{O}_{C_\nu,p}$

and $(a,d) = (b,e) = 1$*. This gives a relation* $a^2e^2 - b^2d^2\xi = \pi d^2e^2$ *in* $\mathcal{O}_{C_\nu,p}$*. Since we are working over a unique factorization domain, we may factor all copies of* π *from both sides of this equation and reduce modulo* π *to get a relation* $\bar{a}^2 - \bar{b}^2 s\xi = 0$ *in the local ring* $\mathcal{O}_{\bar{C},p}$*. This means that* $< s\xi > = < 1 >$ *at the stalk* $\mathcal{O}_{\bar{C},p}$ *and in the function field of* $\bar{C}$*.*

We can now use the results of subsection 0.2.5 to calculate the Witt ring of the curve C. We first describe unique representatives for $W(C)$.

Theorem 4. *Each nontrivial element of the Witt ring has a representative of one of the following forms:* $< s\mathscr{L} >$, $< t\pi\mathscr{M} >$, $< 1, s\mathscr{L} >$, $< s\mathscr{L}, t\pi\mathscr{M} >$, $< \pi, t\pi\mathscr{M} >$, $< 1, s\mathscr{L}, t\pi\mathscr{M} >$, $< s\mathscr{L}, \pi, t\pi\mathscr{M} >$, *or* $< 1, s\mathscr{L}, \pi, t\pi\mathscr{M} >$ *where* $< s >, < t > \in W(\bar{k})$ *and* $\mathscr{L}, \mathscr{M} \in {}_2Pic(C)$*.*

Furthermore, these forms represent distinct Witt classes provided the fewest line bundles possible are used to represent each class and even rank forms are nontrivial except in the obvious cases.

Proof. *Let* n *be the cardinality* $|{}_2Pic(C)|$ *(this may be smaller than twice the genus).*

$W(C)/I$ *is the two element group with nontrivial representative* $< 1 >$*,* I/I_2 *has* $4n$ *elements represented by forms* $< -1, \delta >$*, and* I_2/I_3 *has* $2n$ *elements represented by quaternion norm forms* $< 1, -s\mathscr{L}, -\pi, s\pi\mathscr{L} >$*.*

This tells us two important facts.

$W(C)$ *is a ring with* $16n^2$ *elements (*$16n^2-1$ *nontrivial elements).*

Every nontrivial element of $W(C)$ *may be written*

$<1>^{d_1}\perp<-1,d>^{d_2}\perp<1,-s\mathscr{L},-\pi,s\pi\mathscr{L}>^{d_3}$, $d_i\in\{0,1\}$ *(using Proposition 11).*

We will now perform the calculations to show that

$<1>\perp<-1,\delta>\perp<1,-s\mathscr{L},-\pi,s\pi\mathscr{L}>$ *has a representative of one of the listed forms. The calculations for other values of* d_i *are analogous or obvious.*

$E=<1>\perp<-1,\delta>\perp<1,-s\mathscr{L},-\pi s\mathscr{L},\pi>=$

$<1,-1,\delta,1,-s\mathscr{L},-\pi s\mathscr{L},\pi>=<1,-s\mathscr{L},\delta,\pi,-\pi s\mathscr{L}>$.

If $\delta=<t\mathscr{M}>$ *is a form which extends to a non-degenerate form over* C_ν *then* $<1,-s\mathscr{L},\delta,\pi,-\pi s\mathscr{L}>=<st\mathscr{L}\mathscr{M},\pi,-s\pi\mathscr{L}>$

If $\delta=<t\pi\mathscr{M}>$ *is a form which does not extend to a non-degenerate form over* C_ν *then* $<1,-s\mathscr{L},\delta,\pi,-\pi s\mathscr{L}>=<1,-s\mathscr{L},st\pi\mathscr{L}\mathscr{M}>$.

It remains to show that,assuming the fewest number of line bundles possible are used to represent each form, the forms described are distinct.

We know that $W(C)$ *has* $16n^2-1$ *nontrivial elements and count the forms that cannot be written in an obvious way as a sum of fewer line bundles. There are* $2n$ *forms each* $<s\mathscr{L}>$ *and* $<t\pi\mathscr{M}>$, $2n-1$ *forms each* $<1,s\mathscr{L}>$ *and* $<\pi,t\pi\mathscr{M}>$, $(2n)^2$ *forms* $<s\mathscr{L},t\pi\mathscr{M}>$, $2n(2n-1)$ *forms* *each* $< 1, s\mathscr{L}, t\pi\mathscr{M} >$ *and* $< s\mathscr{L}, \pi, t\pi\mathscr{M} >$*, and* $(2n)^2 - 2n - 2n + 1$ *forms* $< 1, s\mathscr{L}, \pi, t\pi\mathscr{M} >$*. This is a total of* $16n^2 - 1$ *forms.*

Aternatively, considering the quotients I*,* I_1*, and* I_2*, we see that the Witt ring has* $16n^2$ *elements total and* $16n^2 - 1$ *nontrivial elements.*

A cardinality argument shows that the forms of this proposition are distinct.

Theorem 5. *$W(C)$ is isomorphic to the abelian group $W(\bar{C}) \oplus W(\bar{C})$.*

Proof. *There is an inclusion* $W(\bar{C}) \to W(C) : \bar{\mathscr{L}} \mapsto \mathscr{L}$ *which maps* $W(\bar{C})$ *onto the elements of the forms* $< s > \mathscr{L}$ *and* $< 1, s\mathscr{L}_\xi >$ *in* $W(C)$*. We identify* $W(\bar{C})$ *with its image under this inclusion. The cokernel of this map is* $< \pi > W(\bar{C})$ *and* $< \pi > W(\bar{C}) \cong W(\bar{C})$ *by factoring out* $< \pi >$*. This gives a short exact sequence.*

$0 \to W(\bar{C}) \to W(C) \to W(\bar{C}) \to 0$

We will define a left splitting map $W(C) \to W(\bar{C})$*.*

$< 1, -\pi > W(\bar{C})$ *is an ideal in* $W(C)$ *as*

$< s\mathscr{L} >< 1, -\pi > \in < 1, -\pi > W(\bar{C})$ *and*

$< s\pi\mathscr{L} >< 1, -\pi > = < s\mathscr{L} >< \pi, -1 > = < -s\mathscr{L} >< 1, -\pi >$*, which is also in* $< 1, \pi > W(\bar{C})$*. Furthermore, the composition* f *in the following diagram is an isomorphism.*

$$\begin{array}{ccc} W(\bar{C}) & \xrightarrow{i} & W(C) \\ & \searrow^{f} & \downarrow p \\ & & W(C)/ < 1, -\pi > W(\bar{C}) \end{array}$$

is an isomorphism.

The desired splitting map is $f^{-1} \circ p$.

Note that the splitting map $f^{-1} \circ p$ is a map of commutative rings.

Theorem 6. $W(C)$ *is isomorphic to the group ring* $W(\bar{C})[G]$ *where* G *is a two element group.*

Proof. *This is a corollary of theorem 5.*

Take $< 1 >$ *and* $< \pi >$ *as the representatives for* G.

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