

69-7613

TISCHLER, David Charles, 1943-
TOTALLY PARALLELIZABLE MANIFOLDS.

The City University of New York, Ph.D., 1969
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

TOTALLY PARALLELIZABLE MANIFOLDS

by

DAVID TISCHLER

A dissertation submitted to the
Graduate Faculty in Mathematics in
partial fulfillment of the requirements
for the degree of Doctor of Philosophy,
The City University of New York.

1968

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Sept. 20, 1968
date

R. Sacksteder
Chairman of Examining Committee
Professor Richard Sacksteder

September 20, 1968
date

Eldon Dyer
Executive Officer
Professor Eldon Dyer

Professor Edgar Feldman Edgar Feldman

Professor Alphonse Vasquez Alphonse Vasquez

Dr. Charles Weaver Charles Weaver
Supervisory Committee

ACKNOWLEDGEMENT

I would like to thank Professor Richard Sacksteder very much for his suggestion that I consider totally parallelizable manifolds and for his helpful advice throughout the writing of this dissertation. He has devoted much time and has said many encouraging words for which I am greatly appreciative.

TABLE OF CONTENTS

ACKNOWLEDGEMENT	iii
INTRODUCTION	v
SECTION 0.	
Preliminaries	1
SECTION 1.	
Technical Results	5
SECTION 2.	
Products of Spheres	20
SECTION 3.	
S^1 -bundles over 2-manifolds	24
SECTION 4.	
Foliations for Certain S^1 -bundles	31
REFERENCES	37
AUTOBIOGRAPHICAL STATEMENT	38

INTRODUCTION

A substantial theory has been developed to determine whether there exist sections of bundles associated to the principal bundle corresponding to the tangent bundle of a manifold. However, very little is known about finding such sections which satisfy non-linear conditions. For example, Reeb [6], p. 95, and Hirzebruch [1], Problem 9, ask: if a manifold admits a non-vanishing vector field does it admit a foliation of codimension 1. This question is still open; in fact, the answer is unknown for all odd-dimensional spheres, except for the trivial case of S^1 and for S^3 on which Reeb constructed a foliation. Another example is the problem of finding commuting vector fields on a manifold. Lima [3] and Rosenberg [7] have shown that, on S^3 and $S^1 \times S^2$ respectively, two pointwise independent vector fields cannot commute. We will be concerned here with this sort of problem.

A pair of pointwise independent vector fields is called integrable if it defines a 2-dimensional foliation. We say a manifold is totally parallelizable if it can be parallelized by vector fields each pair of which is integrable. For an n -manifold, an equivalent condition to total parallelizability is the existence of n codimension 1 oriented foliations which meet transversally. A natural question to ask is: if a manifold is parallelizable is it totally parallelizable. This question is related to the question above. We devote most of this paper to constructions which totally parallelize manifolds.

In Section 2 we show that $S^1 \times \mathbb{S}$ is totally parallelizable, where \mathbb{S} is an arbitrary product of spheres. In Section 3 we show that differentiable principal S^1 -bundles over 2-dimensional compact mani-

folds (e.g., S^3 or the lens spaces $L(p,q)$ with $q = 1$) are totally parallelizable. Since all compact orientable 3-manifolds are parallelizable, and since by Lickorish [2], they admit foliations of codimension 1, an interesting special case of the above question is: are all compact orientable 3-manifolds totally parallelizable. Aside from the S^1 -bundles mentioned above, I believe the answer is not known. In particular, it would be instructive to know the answer for the lens spaces $L(p,q)$, $q \neq 1$.

The results in Sections 2 and 3 make use of the constructions in Section 1. The main method for totally parallelizing a manifold $S^1 \times N$, where $\dim N = n - 1$ is as follows: 1) give n transversal foliations, defined by 1-forms ω^i , $1 \leq i \leq n$, on $S^1 \times C$, where $C \subset \mathbb{R}^n$; 2) for some immersion $\varphi: N \rightarrow C$, the 1-forms $(\text{id} \times \varphi)^* \omega^i$ totally parallelize $S^1 \times N$. When φ extends to an immersion $\bar{\varphi}: M \rightarrow C$ where $\partial M = N$, then we define a manifold ΣM , whose boundary is $D(M)$, the double of M ; and an immersion $\Sigma\varphi: \Sigma M \rightarrow C' \subset \mathbb{R}^{n+1}$; and $n+1$ transversal foliations on $S^1 \times C'$ such that $\text{id} \times \Sigma\varphi$ totally parallelizes $S^1 \times D(M)$ as above. Using these methods we show that for certain manifolds N , there is an immersion $\varphi: N \times S^q \rightarrow \mathbb{R}^{n+q}$ which totally parallelizes $S^1 \times N \times S^q$.

In Section 3 we represent an S^1 -bundle over a 2-manifold as the union of two product bundles whose boundaries have been identified. Then using the constructions of Section 1 we find three transversal foliations on each product bundle which agree with the identification of the boundaries.

By decomposing S^1 -bundles as in Section 3 we are able to show the existence of foliations of codimension 1 on some more S^1 -bundles.

SECTION 0. PRELIMINARIES

All manifolds and maps will be assumed to be of class C^∞ .

DEFINITION 0.1. Let M^n be an n-dimensional manifold without boundary.

A foliation of codimension 1 of M^n is a disjoint collection \mathfrak{F} of (n-1)-dimensional submanifolds L (not necessarily proper) called leaves, such that $\bigcup_{L \in \mathfrak{F}} L = M^n$. Furthermore, for each $p \in M^n$ there is a chart

(U, φ) , $\varphi: U \rightarrow R^n$, such that $p \in U$ and for all $L \in \mathfrak{F}$, $\pi \circ \varphi$ is constant on each connected component of $L \cap U$ if $L \cap U \neq \emptyset$, where π is the projection of R^n onto the first coordinate axis.

DEFINITION 0.2. A foliation \mathfrak{F} is called oriented if there is a non-vanishing vector field which meets the leaves of \mathfrak{F} transversally.

Note: From now on "foliation" will mean "oriented foliation of codimension 1".

THEOREM 0.1. (Frobenius) Let M^n be as in Definition 0.1. Let D be a field of (n-1)-planes on M^n . Suppose that locally D is spanned by

n - 1 independent vector fields X_i , $1 \leq i \leq n-1$, then $[X_i, X_j] = \sum_{k=1}^{n-1} \alpha_k X_k$, if and only if there is a foliation \mathfrak{F} of M^n such that

$D(p)$ is tangent to the leaf of \mathfrak{F} through p .

Proof. See Sternberg [10], p.132.

DEFINITION 0.3. Let M^n be an n-dimensional manifold, $\partial M^n = \emptyset$. M^n is totally parallelizable if M^n is parallelized by vector fields X_i , $1 \leq i \leq n$, such that each subset of n - 1 vector fields $X_1, \dots, \hat{X}_i, \dots, X_n$

(X_i deleted) defines a foliation. It is clear from Theorem 0.1 that total parallelizability is equivalent to the existence of parallelizing vector fields X_i and functions f_{ij}, g_{ij} , $1 \leq i, j \leq n$, such that $[X_i, X_j] = f_{ij}X_i + g_{ij}X_j$. Vector fields satisfying this condition are called pairwise integrable. Theorem 0.1 has the following dual version.

THEOREM 0.2. (Frobenius) Let ω be a non-vanishing 1-form on a manifold M^n . Then $d\omega = \theta \wedge \omega$ locally, for some 1-form θ , if and only if ω defines a foliation. In this case ω is called integrable.

Proof. See Sternberg [10], p.134.

Note that on a 2-manifold a non-vanishing 1-form ω is integrable since every 2-form can be written locally in the form $\theta \wedge \omega$.

DEFINITION 0.4. Foliations \mathcal{F}^i , $1 \leq i \leq m \leq n$, on M^n will be called transversal if there are 1-forms ω^i , $1 \leq i \leq m$, representing the \mathcal{F}^i which are independent at every point of M^n .

LEMMA 0.1. If ω^i , $1 \leq i \leq n$, on M^n are independent 1-forms, then there exist vector fields v_j , $1 \leq j \leq n$, such that $\omega_i(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Proof. Use Cramer's Rule to show that the pointwise dual basis varies smoothly.

The following proposition gives a dual formulation of total parallelizability in terms of 1-forms.

PROPOSITION 0.1. There exist 1-forms ω^i , $1 \leq i \leq n$, which define transversal foliations \mathcal{F}^i on an n -dimensional manifold M^n if and only

if M^n is totally parallelizable, in which case each leaf of each \mathfrak{F}^i is totally parallelizable.

Proof. This proposition follows in a straightforward manner from Theorem 0.2, Lemma 0.1, and the formula

$$d\omega(v_1, v_2) = v_1(\omega(v_2)) - v_2(\omega(v_1)) - \omega([v_1, v_2])$$

of Sternberg [10], p.103, where ω is a 1-form and v_1, v_2 are vector fields.

PROPOSITION 0.2. Let $\omega^i, 1 \leq i \leq n$, be 1-forms on M^n . Then $\omega^1 \wedge \dots \wedge \omega^n$ is non-vanishing if and only if the ω^i are independent at every point of M^n .

Proof. See Sternberg [10], p.15.

Let I denote $\{t \in \mathbb{R} \mid 0 \leq t \leq 1\}$.

We extend Definition 0.3 to manifolds with boundary. It is more convenient to give the definition in terms of 1-forms.

DEFINITION 0.5. An n -dimensional manifold M^n with non-null boundary N will be called totally parallelizable if and only if there exist independent 1-forms, $\omega^i, 1 \leq i \leq n$, which are integrable and such that for some collar neighborhood $\alpha: N \times I \rightarrow M$ of $N = \alpha(N \times \{0\})$,

$$i) \quad \alpha^*(\omega^i) = \pi^*(v^i), \quad 1 \leq i \leq n-1$$

where v^i is a 1-form on N , and $\pi: N \times I \rightarrow N$ is the projection map,

$$ii) \quad \alpha^*(\omega^n) = dt \quad \text{where } dt \text{ is the standard 1-form on } I.$$

DEFINITION 0.6. Let M be a compact manifold with non-null boundary N . Then $D(M)$ will denote the double of M . For details of the construction and standard facts about $D(M)$ see Munkres [5].

PROPOSITION 0.3. Let M^n be a compact n -dimensional manifold with non-null boundary N . If M^n is totally parallelizable, then $D(M)$ is totally parallelizable.

Proof. Let M_j , $j = 1, 2$, be two copies of M^n which define $D(M)$ with identifications $\varphi_j: M_j \rightarrow M^n$. Suppose ω^i , $1 \leq i \leq n$, are 1-forms which totally parallelize M^n and that $\alpha: N \times I \rightarrow M$ is as in Definition 0.5. Then we define 1-forms w^i , $1 \leq i \leq n$, on $D(M)$ as follows,

- i) $w^i = \varphi_1^*(\omega^i)$, $1 \leq i \leq n$, on M_1
- ii) $w^i = \varphi_2^*(\omega^i)$, $1 \leq i \leq n-1$, on M_2
- iii) $w^n = \varphi_2^*(-\omega^n)$, on M_2 .

Let $\beta: N \times (-1, 1) \rightarrow D(M)$ be defined by

$$\beta = \varphi_1^{-1} \circ \alpha \quad \text{on} \quad N \times [0, 1)$$

$$\beta = \varphi_2^{-1} \circ \alpha \circ \gamma^{-1} \quad \text{on} \quad N \times (-1, 0]$$

where $\gamma: [0, 1) \rightarrow (-1, 0]$ is given by $\gamma(t) = -t$. Then $\beta^*(\omega^i) = \pi^*(\omega^i)$, $1 \leq i \leq n-1$, where $\pi: N \times (-1, 1) \rightarrow N$ is the projection map, and $\beta^*(\omega^n) = dt$. Thus the two definitions of w^i , $1 \leq i \leq n$, on $\partial M_1 = \partial M_2$ agree and thus the w^i totally parallelize $D(M)$.

SECTION 1. TECHNICAL RESULTS

Let n -dimensional euclidean space R^n be given as the set of sequences $\{x_1, x_2, \dots\}$ of real numbers such that $x_j = 0$ for $j > n$. Denote by e_i the sequence $\{x_j\}$ such that $x_j = \delta_{ij}$. Denote by $R \cdot e_i$ the line of real multiples of e_i . R^{n-1} is contained in R^n as $R^{n-1} \times \{0\} \subset R^{n-1} \times R \cdot e_n$. We let \langle, \rangle denote the usual Riemannian metric on the tangent bundle of R^n . $\{\frac{\partial}{\partial x_i}\}$, $\{dx_i\}$, $1 \leq i \leq n$, will be a basis for the tangent space, respectively cotangent space, of R^n at a point.

S^1 will be the real numbers mod 2π . Let x be coordinates on the reals with $\frac{\partial}{\partial x}$, dx as above. Then the coordinates z , the tangent vector $\frac{\partial}{\partial z}$, and the cotangent vector dz are induced on S^1 from x , $\frac{\partial}{\partial x}$, dx respectively. \langle, \rangle will denote the metric on the tangent bundle of S^1 induced from the usual metric on R also denoted \langle, \rangle above.

For a topological space $A \subset B$, $\overset{\circ}{A}$ and ∂A denote the interior and boundary of A respectively. For a manifold M , ∂M denotes the boundary of M and $\overset{\circ}{M}$ is defined to be $M - \partial M$.

As in Section 0, $I = \{t \in R \mid 0 \leq t \leq 1\}$ and dt is the usual 1-form on I .

DEFINITION 1.1. An f-system (C, H) on R^n is an open set $C \subset R^n$ and a collection H of n transversal foliations on $S^1 \times C$.

DEFINITION 1.2. Given an f-system (C, H) we define a vector field

$v = v(C,H)$ on $S^1 \times C$ by the conditions:

- i) $\|v\| = 1$
- ii) $\omega^i(v) = 0$, $1 \leq i \leq n$, where the ω^i define the foliations of H .
- iii) Choosing either of the two possible orientations for v .

DEFINITION 1.3. Let $M = M^n$ be an n -dimensional manifold with non-null boundary N . M^n is said to have a transversal system if there exists an immersion $\varphi: M^n \rightarrow R^n$, and an f -system (C,H) on R^n such that $\varphi(M^n) \subset C$ and $v(C,H)$ meets $(Id \times \varphi)(S^1 \times N)$ transversally. We write $\text{sgn } v = +1$ (or -1) if v is leaving (or entering) $(Id \times \varphi)(S^1 \times M)$. $\text{Sgn } v$ is then a function on the connected components of N .

DEFINITION 1.4. Let M^n be as in Definition 1.3. M^n is said to have a 1-system if it has a transversal system for which $\text{sgn } v$ is a constant function.

DEFINITION 1.5. Let B be an $(n-1)$ -dimensional manifold without boundary. B is said to have a transversal system (1-system) if $B \times I$ has a transversal system (1-system).

Warning! This is not the obvious extension of Definition 1.3.

Remark 1.1. Suppose that B_i , $i = 1,2$, are $(n-1)$ -dimensional manifolds such that

- (i) B_1 is a relatively compact submanifold of B_2
- (ii) $\partial B_i = \emptyset$, $i = 1,2$.

If there exists an immersion $\varphi: B_2 \rightarrow R^n$ such that $\varphi(B_2) \subset C$ and

$v(C,H)$ meets $(\text{Id} \times \varphi)(S^1 \times B_2)$ transversally, then we conclude that there exists an immersion $\Phi : B_1 \times I \rightarrow \mathbb{R}^n$ such that $(\text{Id} \times \Phi)(S^1 \times B_1 \times \{t\})$ meets $v(C,H)$ transversally for all $t \in I$.

PROPOSITION 1.1. Let B_i , $i = 1, 2$, be $(n-1)$ -dimensional manifolds without boundary such that B_1 is a relatively compact submanifold of B_2 .

If B_2 has a transversal system then $S^1 \times B_1$ is totally parallelizable.

Proof. By Definition 1.5 there is an f-system (C,H) on \mathbb{R}^n , an immersion $\varphi : B_2 \times I \rightarrow \mathbb{R}^n$ such that $\varphi(B_2 \times I) \subset C$, and we can assume that $v(C,H)$ is transversal to $(\text{Id} \times \varphi)(S^1 \times B_1 \times \{t\})$ for all $t \in I$, by Remark 1.1. We claim that the 1-forms dt and $\varphi^*(\omega^i)$, $1 \leq i \leq n$, are independent on $S^1 \times B_1 \times (0,1)$. We can check this locally. For each point of $S^1 \times B_1 \times (0,1)$ choose an open set U on which φ has an inverse. Set $\eta = (\varphi^{-1})^*(dt)$. Then the assumption that v is transversal to $(\text{Id} \times \varphi)(S^1 \times B_1 \times \{t\})$ for all $t \in I$ implies that $\eta(v)$ is a non-vanishing function. Let ω^0 denote the 1-form on $S^1 \times C$ whose value on a vector b in the tangent space of $S^1 \times C$ at a point p is given by $\langle v(p), b \rangle$ where \langle , \rangle denotes the product metric. We claim that $\{\omega^i\}$, $0 \leq i \leq n$, form a collection of $n+1$ independent 1-forms. If not, then we could write $\omega^0 = \sum_{i=1}^n a_i \omega^i$ where the a_i are real valued functions. However, if we evaluate both sides of this equation on v we get a contradiction. Then by Lemma 0.1 there exist vector fields v_j , $0 \leq j \leq n$, such that $\omega^i(v_j) = \delta_{ij}$ for $0 \leq i, j \leq n$. Note that $v_0 = v$. Then $\eta \wedge \omega^1 \wedge \dots \wedge \omega^n(v_0, v_1, \dots, v_n) = \eta(v)$. Thus by Proposition 0.2 we conclude that dt and $\varphi^*(\omega^i)$, $1 \leq i \leq n$, are independent as claimed. Proposition 0.1 implies that $S^1 \times B_1 \times (0,1)$ and

$S^1 \times B_1 \times \{\frac{1}{2}\}$ are totally parallelizable. This concludes the proof of Proposition 1.1.

PROPOSITION 1.2. Let B be a compact $(n-1)$ -dimensional manifold,
 $\partial B = \emptyset$. If B has a transversal system, then $S^1 \times B$ is totally parallelizable.

Proof. Remark 1.1, taking $B_1 = B_2 = B$, and Proposition 1.1 imply this result.

PROPOSITION 1.3. Let M^n be a compact manifold with non-null boundary N .
If M^n has a transversal system then $S^1 \times N$ is totally parallelizable.

Proof. Remark 1.1, taking $B_1 = B_2 = N$, implies that N has a transversal system and so the result follows from Proposition 1.2.

DEFINITION 1.6. A k -tuple is a collection $\{M_i^n | 1 \leq i \leq k\} = \{M_i^n\}_k$ of compact n -dimensional manifolds such that

1. $\partial M_i^n = N_i$ is non-null.
2. $M_{i+1}^n \subset M_i^n$.

DEFINITION 1.7. Given two k -tuples $\{M_i^n\}_k$ and $\{\bar{M}_i^n\}_k$ we say they are equivalent and write $\{M_i^n\}_k \approx \{\bar{M}_i^n\}_k$ if and only if there exists a diffeomorphism v of M_1^n onto \bar{M}_1^n such that $v|_{M_i^n}$ is a diffeomorphism onto \bar{M}_i^n , $1 \leq i \leq k$.

DEFINITION 1.8. A k -tuple $\{M_i^n\}_k$ has a transversal system (1-system) if and only if M_1^n has a transversal system (1-system) which restricts to a transversal system (1-system) for each M_i^n , $1 \leq i \leq k$.

DEFINITION 1.9. In the case of Definition 1.8, the function $\text{sgn } v$ can be defined on the collection of connected components of $\bigcup_{i=1}^k N_i$. Let $\text{sgn}_i v$ be $\text{sgn } v$ restricted to N_i .

LEMMA 1.1. If $\{M_i^n\}_k$ has a transversal system then for any subset $i_1 < \dots < i_s < j_s < \dots < j_1$ of $\{1, \dots, k\}$, the s-tuple $\{M_{i_\alpha}^n - M_{j_\alpha}^n\}_s$ has a transversal system.

Proof. Clear.

LEMMA 1.2. If the transversal system in the hypotheses of Lemma 1.1 is a 1-system with $\text{sgn}_{i_\alpha} v = -\text{sgn}_{j_\alpha} v$, then we can replace "transversal system" by "1-system" in the conclusion of Lemma 1.1.

Proof. Clear.

Given a compact n -dimensional manifold M^n with non-null boundary N , we are going to define an $(n+1)$ -dimensional manifold denoted ΣM with boundary denoted ΣN . We need an auxiliary definition first.

DEFINITION 1.9. Call an embedding $\psi = (\psi^1, \psi^2) : [-1, 1] \rightarrow [0, 1] \times \mathbb{R}$ admissible if and only if the following conditions are satisfied:

- (1) $\psi(-1) = (1, -1)$, $\psi(1) = (1, 1)$, $\psi(0) = (0, 0)$.
- (2) There exist real numbers $0 < a < d < 1$ such that $\psi([-1, -d]) \subset [0, 1] \times \{-1\}$, $\psi([-a, a]) \subset \{0\} \times [-1, 1]$, and $\psi([d, 1]) \subset [0, 1] \times \{1\}$.
- (3) $(\psi^2)' > 0$ when $\psi^2 \neq 1, -1$.
- (4) $(\psi^1)' > 0$ on $[0, 1]$ when $\psi^1 \neq 0$.
- (5) $(\psi^1)' < 0$ on $[-1, 0]$ when $\psi^1 \neq 0$.

Here $(\psi^1)'$ and $(\psi^2)'$ are the derivatives respectively of ψ^1 and ψ^2 .

Remark. If ψ and $\bar{\psi}$ are two admissible functions, then the submanifold $\psi([-1,1])$ is diffeotopic to the submanifold $\bar{\psi}([-1,1])$ in $[0,1] \times \mathbb{R}$ via a diffeotopy leaving $\psi([-1,-c] \cup [c,1])$ fixed throughout the diffeotopy for some $0 < c < 1$.

DEFINITION 1.10. We now define ΣM and ΣN .

Choose an admissible embedding ψ as above and an embedding $\alpha : N \times [0,1] \rightarrow M$ such that $\text{im } \alpha$ is a collar neighborhood of $N = \alpha(N \times \{0\})$. Let $M' = M - \alpha(N \times [0,e])$ where $1 > e > \max(\psi_1(d), \psi_1(-d))$. $M' \times \{-1\}$ and $M' \times \{1\}$ are submanifolds of $M \times \mathbb{R}$. The image of the composition of embeddings

$$N \times [-1,1] \xrightarrow{\text{Id} \times \psi} N \times [0,1] \times \mathbb{R} \xrightarrow{\alpha \times \text{Id}} M \times \mathbb{R}$$

is a submanifold of $M \times \mathbb{R}$ denoted by M'' . Then $M'' \cup (M' \times \{-1\}) \cup (M' \times \{1\})$ forms a submanifold of $M \times \mathbb{R}$ without boundary, denoted $\Sigma_{\psi, \alpha} N$, which is diffeomorphic to $D(M)$. This is straightforward to verify.

Since there is a topological embedding of $M \times \mathbb{R}$ into itself taking $\Sigma_{\psi, \alpha} N$ onto $(\alpha(N \times \{1\}) \times [-1,1]) \cup (M \times (\{-1\} \cup \{1\}))$ it follows that $\Sigma_{\psi, \alpha} N$ bounds a submanifold of $M \times \mathbb{R}$ which we denote by $\Sigma_{\psi, \alpha} M$.

It follows from Munkres [5], lemma 6.1, p.59, that up to diffeomorphism $\Sigma_{\psi, \alpha} M$ is independent of the choice of α . Furthermore, for two admissible embeddings $\psi, \bar{\psi}$, we conclude, using the above remark, that $\Sigma_{\psi, \alpha} M$ is diffeomorphic to $\Sigma_{\bar{\psi}, \alpha} M$. Thus ΣM is well defined up to

diffeomorphism by $\Sigma M = \Sigma_{\psi, \alpha} M$, for any α and ψ as above.

DEFINITION 1.11. Given a k-tuple $\{M_i^n\}_k$ we define a k-tuple

$\Sigma(\{M_i^n\}_k) = \{\Sigma_i(M_i^n)\}_k$ as follows:

Let $u_i : M_i^n \rightarrow M_1^n$ be the inclusion map. For each i , $1 \leq i \leq k$, choose an embedding $\alpha_i : N_i \times [0, 1] \rightarrow M_i^n$ such that $\text{im } \alpha_i$ is a collar neighborhood of N_i as in Definition 1.10. Choose the α_i so that $\{u_i \circ \alpha_i(N_i \times [0, 1])\}$, $1 \leq i \leq k$, form a pairwise disjoint collection of submanifolds of M_1^n . For each i , $1 \leq i \leq k$, let $\psi_i = (\psi_i^1, \psi_i^2)$ be an admissible function. Let $\beta_i : \Sigma_{\psi_i, \alpha_i} M_i^n \rightarrow \Sigma_{\psi_1, \alpha_1} M_1^n$ be the inclusion maps.

Choose a sequence of real numbers $0 < c_k < \dots < c_1$, such that for $i \neq 1$, $\frac{c_i}{c_{i-1}} < \min(\psi_{i-1}^2(a_{i-1}), -\psi_{i-1}^2(-a_{i-1}))$, where a_i refers to ψ_i as in Definition 1.9. Define $d_i : M_1^n \times \mathbb{R} \rightarrow M_1^n \times \mathbb{R}$ by $d_i(m, t) = (m, c_i t)$, $1 \leq i \leq k$.

Then $d_i \circ \beta_i$ embeds $\Sigma_{\psi_i, \alpha_i} M_i^n$ into $\Sigma_{\psi_1, \alpha_1} M_1^n$, $1 \leq i \leq k$. We denote $d_i \circ \beta_i(\Sigma_{\psi_i, \alpha_i} M_i^n)$ by $\Sigma_{i, \psi_i, \alpha_i, c_i} M_i^n$. Then since $\Sigma_{i, \psi_i, \alpha_i, c_i} M_i^n$ is contained in the interior of $\Sigma_{i-1, \psi_{i-1}, \alpha_{i-1}, c_{i-1}} M_{i-1}^n$, we have a k-tuple $\{\Sigma_{i, \psi_i, \alpha_i, c_i} M_i^n\}_k$. However

$$\{\Sigma_{i, \psi_i, \alpha_i, c_i} M_i^n\}_k \approx \{\Sigma_{i, \psi_i', \alpha_i', c_i'} M_i^n\}_k$$

for any choice of $\psi_i, \psi_i', \alpha_i, \alpha_i', c_i, c_i'$. Thus we can drop ψ_i, α_i, c_i from the notation, and up to equivalence of k-tuples, $\Sigma(\{M_i^n\}_k) = \{\Sigma_i M_i^n\}_k$ is well defined. We denote $\partial(\Sigma_i M_i)$ by $\Sigma_i N_i$.

DEFINITION 1.12. i) An f-system (C,H) such that $dz(v(C,H)) > 0$ everywhere on $S^1 \times C$ will be called an f'-system, where dz is the 1-form induced from the standard 1-form on S^1 also denoted dz .

ii) A l'-system is a l-system which is defined by an f'-system.

LEMMA 1.3. If $\{M_i^n\}_k$ has a l'-system (C,H) , then $\Sigma\{M_i^n\}_k$ has a l'-system (C',H') . Furthermore, $\text{sgn}_i v(C',H') = \text{sgn}_i v(C,H)$, for $1 \leq i \leq k$.

Proof. By assumption there are immersions $\varphi_i : M_i \rightarrow \mathbb{R}^n$ with $\varphi_i = \varphi_1|_{M_i}$ such that $v(C,H)$ meets $(\text{Id} \times \varphi_i)(S^1 \times M_i)$ transversally.

Choose a representative of $\Sigma\{M_i^n\}_k$ with α_i such that v meets $(\text{Id} \times (\varphi_i \circ \alpha_i))(S^1 \times (N \times \{t\}))$ transversally for all $t \in I$, $1 \leq i \leq k$.

For each $1 \leq i \leq k$, we define $\Sigma\varphi_i : \Sigma_i M_i \rightarrow \mathbb{R}^{n+1}$ by the composition of the maps

$$\Sigma_i M_i \longrightarrow M_i \times \mathbb{R} \xrightarrow{\varphi_i \times \text{Id}} \mathbb{R}^n \times \mathbb{R} \cdot e_{n+1}$$

where the first map is inclusion. Note that $\Sigma\varphi_i = \Sigma_1 \varphi_1|_{\Sigma_i M_i}$. It is clear that $\Sigma\varphi_i$ is an immersion for $1 \leq i \leq k$.

We now define a subset C' of \mathbb{R}^{n+1} by $C' = C \times (r_1^-, r_1^+)$ where $r_1^+ > c_1$ and $r_1^- < -c_1$, where c_1 refers to Definition 1.11. Let w^i be the 1-forms, $1 \leq i \leq n$, which define the foliations in H . Let $\pi : \mathbb{R}^n \times \mathbb{R} \cdot e_{n+1} \rightarrow \mathbb{R}^n$ be the projection map. Choose sets of real numbers $\{r_i^+\}$ and $\{r_i^-\}$, $1 \leq i \leq k+1$, such that

$$(i) \quad r_{k+1}^+ = 0 = r_{k+1}^-$$

$$(ii) \quad r_{i+1}^+ < \psi_2(a_i) ; c_i < r_i^+ ; 1 \leq i \leq k ,$$

(iii) $r_{i+1}^- > \psi_2(-a_i)$; $r_i^- < -c_i$; $1 \leq i \leq k$, where c_i is as in Definition 1.11.

Then define H' by the 1-forms ω^i , $1 \leq i \leq n+1$, where

$$\omega^i = \pi^*(w^i) , \quad 1 \leq i \leq n$$

$$\omega^{n+1} = (K^2(x_{n+1}) + 1)^{-\frac{1}{2}}(dx_{n+1} - K(x_{n+1})dz)$$

with K a real valued function on (r_1^-, r_1^+) , satisfying the following conditions:

$$(1.1) \quad \operatorname{sgn}_i v \cdot K(p) \geq 0 \quad \text{for } p \in [r_{i+1}^+, r_i^+] , \quad 1 \leq i \leq k .$$

$$(1.2) \quad \operatorname{sgn}_i v \cdot K(p) > 0 \quad \text{for } p = \{c_i\} , \quad 1 \leq i \leq k .$$

$$(1.3) \quad \operatorname{sgn}_i v \cdot K(p) \leq 0 \quad \text{for } p \in [r_i^-, r_{i+1}^-] , \quad 1 \leq i \leq k .$$

$$(1.4) \quad \operatorname{sgn}_i v \cdot K(p) < 0 \quad \text{for } p = \{-c_i\} , \quad 1 \leq i \leq k .$$

It is clear that ω^{n+1} is integrable.

By assumption, $w^1 \wedge \dots \wedge w^n$ is a non-vanishing n -form on $S^1 \times C$. Therefore $\omega^1 \wedge \dots \wedge \omega^n$ is a non-vanishing n -form on $S^1 \times C'$. Since none of the forms ω^i , $1 \leq i \leq n$, involves dx_{n+1} , and since $(K^2 + 1)^{-\frac{1}{2}}$ is non-vanishing, we conclude that $\omega^1 \wedge \dots \wedge \omega^{n+1}$ is non-vanishing on $S^1 \times C'$. Thus (C', H') is an f -system.

Let $L_{x_{n+1}} : \mathbb{R}^n \times \mathbb{R} \cdot e_{n+1} \rightarrow \mathbb{R}^n \times \mathbb{R} \cdot e_{n+1}$ denote translation by

$x_{n+1} \in \mathbb{R} \cdot e_{n+1}$. Then let $L_*(v)$ be the vector field on $S^1 \times C \times \mathbb{R} \cdot e_{n+1}$ defined by $L_*(v)(z, x, x_{n+1}) = (L_{x_{n+1}})_* v(z, x, 0)$. Denote $v(C', H')$ by

v' . We claim that v' can be written as $v' = f_1 L_*(v) + f_2 \left(\frac{\partial}{\partial x_{n+1}} \right)$

where $f_1 : S^1 \times C' \rightarrow (0,1]$ and $f_2 : S^1 \times C' \rightarrow (-1,1)$ are functions satisfying:

$$(1.5) \quad f_2 = f_1 K dz(L_*(v)) .$$

$$(1.6) \quad f_1 = \sqrt{1 - f_2^2} .$$

It is easy to check that f_1 and f_2 are well defined by (1.5) and (1.6).

Since $\langle L_*v, \frac{\partial}{\partial x_{n+1}} \rangle = 0$ we have by (1.6) that $\langle v', v' \rangle = 1$. Furthermore,

$$\omega^i(v') = \omega^i(f_1 L_*(v) + f_2 \left(\frac{\partial}{\partial x_{n+1}} \right)) = f_1 \omega^i(v) = 0, \quad 1 \leq i \leq n ,$$

$$\omega^{n+1}(v') = (K^2 + 1)^{-\frac{1}{2}} (f_2 - f_1 K dz(L_*v)) = 0 \quad \text{by (1.5).}$$

Thus we have verified that $v' = f_1 L_*(v) + f_2 \left(\frac{\partial}{\partial x_{n+1}} \right)$.

Since $dz(v') = f_1 dz(L_*(v)) > 0$, we conclude that (C', H') is an f' -system.

Let $\psi_i = (\psi_i^1, \psi_i^2)$ be the maps used in defining $\Sigma_i M_i$. On the image of $(\text{Id} \times \text{Id} \times \psi_i) : S^1 \times N \times [-1,1] \rightarrow S^1 \times N \times I \times R \cdot e_{n+1}$ we define the 1-form

$$\delta_i = -c_i (\psi_i^2)' dt + (\psi_i^1)' dx_{n+1} .$$

$\beta = \text{Id} \times (\varphi_i \circ \alpha_i)^{-1} \times \text{Id}$ is not a well defined function but $\beta_*(L_*v)$ is a well defined vector field on $S^1 \times N \times I \times R \cdot e_{n+1}$. Then

$$\delta_i(\beta_*(L_*v)) = -c_i (\psi_i^2)' dt(\beta_*(L_*v)) .$$

By the assumption that v meets $(\text{Id} \times (\varphi_i \circ \alpha_i) \times \text{Id})(S^1 \times N \times \{t\})$ transversally for all $t \in I$, we have

$$\text{sgn}_i v \cdot dt(\beta_*(L_*v)) < 0 .$$

Let η_i be the non-vanishing 1-form defined on the submanifold $S^1 \times \Sigma_i N_i$ of $S^1 \times M_1 \times R$, $1 \leq i \leq k$;

$$(1.7) \quad \eta_i(p) = \begin{cases} (\text{Id} \times \alpha^{-1} \times \text{Id})_* \left(\frac{\delta_i}{\|\delta_i\|} \right) & \text{for} \\ p \in (\text{Id} \times \alpha \times \text{Id})(S^1 \times (N \times I) \times R \cdot e_{n+1}) , \\ dx_{n+1} & \text{for } p \in S^1 \times (M_i)' \times \{c_i\} , \\ - dx_{n+1} & \text{for } p \in S^1 \times (M_i)' \times \{-c_i\} \end{cases}$$

where $(M_i)'$ is analogous to M' in Definition 1.10.

The kernel of $\eta_i(p) : T_p \rightarrow R$ (where T_p is the tangent space of $S^1 \times M_1 \times R$ at p) is precisely the tangent space of $S^1 \times \Sigma_i N_i$ at p . This follows from the construction of $\Sigma_i N_i$. Furthermore η_i is greater than zero on the outward normal vector to $S^1 \times \Sigma_i N_i$ in $S^1 \times M_1 \times R$.

On $(\text{Id} \times \alpha \times \text{Id})(S^1 \times (N \times I) \times R \cdot e_{n+1})$ we have that

$$\begin{aligned} & \text{sgn}_i(v) \cdot \eta_i(\text{Id} \times \alpha_i^{-1} \times \text{Id})_*(v') = \\ & \text{sgn}_i(v) \frac{f_1}{\|\delta_i\|} (-c_i (\psi_i^2)') dt(\beta_*(L_*v)) + K dz(L_*v)(\psi_i^1)' \end{aligned}$$

We have that f_1 , $\|\delta_i\|$, c_i , $dz(L_*v)$, $-\text{sgn}_i(v) dt(\beta_*(L_*v))$ are all strictly greater than zero. From Definition 1.9 and the definition of K we have that

$$\text{sgn}_i v \cdot K (\psi_i^1)' \geq 0 , \quad \text{and}$$

$$\text{sgn}_i v \cdot K (\psi_i^1)' > 0 \quad \text{when } \psi_2 = c_i \quad \text{or} \quad \psi_2 = -c_i .$$

Then by Definition 1.9 we have that $(\psi_i^2)' \geq 0$, and when $(\psi_i^2) \neq c_i, -c_i$ we have that $(\psi_i^2)' > 0$.

Thus we conclude that

$$(1.8) \quad \text{sgn}_i(v) \eta_i(\beta_*(v')) > 0 \quad \text{on} \quad (\text{Id} \times \alpha \times \text{Id})(S^1 \times (N \times I) \times R \cdot e_{n+1}).$$

It is clear from the formula for v' , (1.5), (1.7), (1.1), (1.3), that

$$(1.9) \quad \text{sgn}_i(v) \eta_i(\text{Id} \circ \varphi_i^{-1} \circ \text{Id})_*(v') > 0 \quad \text{on} \quad S^1 \times (M_i)' \times (\{c_i\} \cup \{-c_i\}).$$

But (1.8) and (1.9) imply that (C', H') and $\Sigma\varphi_1$ form a $1'$ -system for $\Sigma\{M_i^n\}_k$ with $\text{sgn}_i(v') = \text{sgn}_i(v)$. This concludes the proof of Lemma 1.3.

PROPOSITION 1.4. Let M^n be a compact n-dimensional manifold with non-null boundary N . If M^n has a transversal system defined by an f' -system, then $S^1 \times M^n$ is totally parallelizable.

Proof. Let (C, H) , an f' -system on R^n , and $\varphi: M^n \rightarrow C$, an immersion, define a transversal system for M .

Choose an embedding $\alpha: N \times I \rightarrow M^n$ which gives a collar neighborhood of $N = \alpha(N \times \{0\})$, such that $v(C, H)$ is transversal to $(\text{Id} \times (\varphi \circ \alpha))(S^1 \times N \times \{t\})$ for all $t \in I$. Choose an admissible function ψ satisfying the conditions:

- (i) $\psi(x) = (0, x)$ for $x \in [-a, a]$
- (ii) $\psi(x) = (x, 1)$ for $x \in [d, 1]$
- (iii) $\psi(x) = (-x, -1)$ for $x \in [-1, -d]$

where a and d are as in Definition 1.9. Let N^+ (N^-) be that part of N on which $\text{sgn } v = +1$ (-1).

We then define a function $\bar{\varphi} : M^n \rightarrow R^{n+1}$ by the following conditions:

- (i) $\bar{\varphi}(m) = (\varphi(m), 0)$ for $m \in M - \alpha(N \times [0, d])$
- (ii) $\bar{\varphi}(m) = ((\varphi \circ \alpha) \times \text{Id})(n, \psi^1(t)) , \psi^2(t) - 1)$
if $m = \alpha(n, t)$ and $n \in N^+$,
- (iii) $\bar{\varphi}(m) = ((\varphi \circ \alpha) \times \text{Id})(n, \psi^1(-t)) , \psi^2(-t) + 1)$
if $m = \alpha(n, t)$ and $n \in N^-$.

$\bar{\varphi}$ is a well defined immersion.

Define a subset C' of R^{n+1} by $C' = C \times (-1, +1)$. Let

$K : (-1, +1) \rightarrow R$ be a function satisfying:

$$(1.10) \quad K \geq 0$$

$$(1.11) \quad K(x_{n+1}) = 0 \quad \text{for } x_{n+1} \in (-1, d] \cup [d, 1)$$

$$(1.12) \quad K(0) > 0 .$$

Let $\pi : R^n \times R \cdot e_{n+1} \rightarrow R^n$ be the projection map. We define a collection H' of oriented foliations on $S^1 \times C'$ by the 1-forms ω^i , $1 \leq i \leq n+1$; $\omega^i = \pi^*(w^i)$, for $1 \leq i \leq n$, where the w^i define the foliations in H , and

$$\omega^{n+1} = (K^2(x_{n+1}) + 1)^{-\frac{1}{2}}(dx_{n+1} - K(x_{n+1}) dz) .$$

It follows from the proof of Lemma 1.3 that (C', H') is an f-system on R^{n+1} . Let $v' = v(C', H')$. We claim that the proof of Lemma 1.3 shows that $(\text{Id} \times \bar{\varphi})(S^1 \times \overset{\circ}{M})$ meets v' transversally. Define the embedding $i : M - \alpha(N^- \times [0, d]) \rightarrow \Sigma N \cap (M \times \{x_{n+1} \in R \mid x_{n+1} \geq 0\})$ by

$$i(m) = \begin{cases} (m, +1), & \text{for } m \in M - \alpha(N \times [0, d]) \\ (\alpha \times \text{Id})((n, \psi^1(t)), \psi^2(t) - 1), & \text{for} \\ \alpha^{-1}(m) = (n, t) \in N \times [0, d] \end{cases}$$

Then $\bar{\varphi}$ restricted to $M - \alpha(N^- \times [0, d])$ can be written as

$\bar{\varphi} = L_{-1} \circ \Sigma\varphi \circ i$ where $\Sigma\varphi$ is the $\Sigma\varphi_1$ of Lemma 1.3 when we consider M as the 1-tuple $\{M_1^{n_1}\}_1$ and define $\Sigma\{M_1^{n_1}\}_1$ by the above ψ and α .

Let $L_1(C', H')$ denote the f-system given by the 1-forms $L_{-1}^*(\omega^i)$ on $S^1 \times L_{-1}(C')$, where $L_{x_{n+1}}$ is as in Lemma 1.3. Then $\Sigma\varphi \circ i$ is an immersion of $M - \alpha(N^- \times [0, d])$ into $L_1(C')$. The proof of Lemma 1.3 showed that v' (of Lemma 1.3) met $\text{Id} \times \Sigma\varphi(S^1 \times \Sigma N)$ transversally.

This implies that v' (of Lemma 1.3) meets

$\text{Id} \times (\Sigma\varphi \circ i)(S^1 \times (M - \alpha(N^- \times [0, d])))$ transversally. We have that $L_{-1}^*(\omega^i) = \omega^i$ for $1 \leq i \leq n$, and $L_{-1}^*(\omega^{n+1}) = ((K')^2 + 1)^{-\frac{1}{2}}(dx_{n+1} - K'dz)$ where $K'(x_{n+1}) = K(x_{n+1} - 1)$. Using (1.10), (1.11), (1.12) we see that

K' satisfies conditions analogous to (1.1), (1.2), and we conclude that

$v(L_{-1}(C', H'))$ meets $\text{Id} \times (\Sigma\varphi \circ i)(S^1 \times (\overset{\circ}{M} - \alpha(N^- \times [0, d])))$ transversally. But it is clear that v' meets $(\text{Id} \times \bar{\varphi})(S^1 \times (\overset{\circ}{M} - \alpha(N^- \times [0, d])))$

transversally if and only if $(\text{Id} \times (\Sigma\varphi \circ i))(S^1 \times (\overset{\circ}{M} - \alpha(N^- \times [0, d])))$

meets $v(L_{-1}(C', H'))$ transversally. A similar proof shows that v'

meets $(\text{Id} \times \bar{\varphi})(S^1 \times (\overset{\circ}{M} - \alpha(N^+ \times [0, d])))$ transversally. Hence, by

Remark 1.1 (setting $B_1 = S^1 \times (\overset{\circ}{M} - \alpha(N \times [0, \tau]))$ and $B_2 = S^1 \times \overset{\circ}{M}$) and

Proposition 1.1 we conclude that $S^1 \times (\overset{\circ}{M} - \alpha(N \times [0, \tau]))$ is totally parallelizable for $\tau \in (0, a)$.

It follows from the definition of $\bar{\varphi}$, that $\bar{\varphi} \circ \alpha$ restricted to $N^+ \times [\tau', a]$, $\tau < \tau' < a$, can be written

$$\begin{aligned}\bar{\varphi} \circ \alpha(n, t) &= L_{-1} \circ ((\varphi \circ \alpha) \times \text{Id})(n, t) \\ &= (\varphi \circ \alpha(n), t - 1) \in \mathbb{R}^n \times \mathbb{R} \cdot e_{n+1}.\end{aligned}$$

Thus $(\text{Id} \times (\bar{\varphi} \circ \alpha))^* \omega^i = (\text{Id} \times (L_{-1} \circ ((\varphi \circ \alpha) \times \text{Id})))^* \pi^* w^i$ on $S^1 \times N \times [\tau', a]$, $1 \leq i \leq n$, and

$$(1.13) \quad (\text{Id} \times (\bar{\varphi} \circ \alpha))^* \omega^{n+1} = dt \quad \text{on} \quad S^1 \times N \times [\tau', a].$$

These $(n + 1)$ 1-forms define $(n + 1)$ transversal foliations on $S^1 \times N \times [\tau', a]$. Note that $\pi \circ L_{-1} \circ \varphi \circ \alpha = \varphi \circ \alpha \circ P$ where $P : N \times I \rightarrow N$ is the projection map. Thus we have that

$$(1.14) \quad (\text{Id} \times (\bar{\varphi} \circ \alpha))^* \omega^i = (\text{Id} \times P)^* v^i, \quad \text{where}$$

$v^i = (\text{Id} \times (\varphi \circ \alpha))^* w^i$ define transversal foliations on $S^1 \times N$, $1 \leq i \leq n$.

Thus (1.13) and (1.14) imply that $S^1 \times (M - \alpha(N \times [0, \tau']))$ is totally parallelizable. Hence it is clear that $S^1 \times M$ is totally parallelizable and Proposition 1.4 is completed. /

SECTION 2. PRODUCTS OF SPHERES

Let S^{n-1} denote the points of R^n of unit length. Put polar coordinates on $R^n - \{0\}$ by identifying it with $S^{n-1} \times R^+$, $R^+ = \{r \in R \mid r > 0\}$. For $\mu > 0$, we define

$$D_\mu^n = \{x \in R^n \mid \|x\| \leq \mu\}$$

$$S_\mu^{n-1} = \{x \in R^n \mid \|x\| = \mu\}$$

Thus $S_\mu^{n-1} = S_1^{n-1} \times \{\mu\}$ in polar coordinates.

Let Σ^q denote the q^{th} iterate of Σ .

LEMMA 2.1. Let $0 < \mu_k < \mu_{k-1} < \dots < \mu_1$ be real numbers, $k \geq 1$. Then $\Sigma^q\{D_{\mu_i}^n\}_k \approx \{D_{\mu_i}^{n+q}\}_k$.

Proof. It is clear that it suffices to prove this lemma for $q = 1$.

For $q = 1$, the lemma is rather straightforward but tedious, so we omit the proof.

DEFINITION 2.1. Let $\{M_i^n\}_k$ be a k -tuple. Let N be a compact manifold without boundary. Denote by $N \times \{M_i^n\}_k$ the k -tuple $\{N \times M_i^n\}_k$, where $N \times M_i^n \subset N \times M_1^n$ in the usual way.

LEMMA 2.2. Let N , $\{M_i^n\}_k$ be as in Definition 2.1. Then

$$\Sigma(N \times \{M_i^n\}_k) \approx N \times (\Sigma\{M_i^n\}_k) .$$

Proof. This follows directly from the fact that $\partial(N \times M_1^n) = N \times \partial M_1^n$.

DEFINITION 2.2. Let Y^k , $k \geq 1$, be the subset of $[0, \frac{1}{3}]$ consisting of points y with a tertiary expansion $y = 0 \cdot a_1 a_2 a_3 \dots a_{k-1}$ such that if $a_i = 1$ then $a_j = 0$ for all $j > i$. Denote the elements of Y^k by $\{y_i^k\}$ where $0 = y_1^k < y_2^k < \dots < y_{2^{k-1}}^k = \frac{1}{3}$.

LEMMA 2.3. For $1 \leq i \leq 2^{k-2}$, $3y_i^k = y_i^{k-1}$ and $\frac{1}{3} - y_i^k = y_{j_i}^k$ where $j_i = 2^{k-1} - i + 1$.

DEFINITION 2.3. Let $I_i^k = [y_i^k, 1 - y_i^k]$, $1 \leq i \leq 2^{k-1}$. Note that $I_i^k \subset I_{i-1}^k$.

DEFINITION 2.4. Let N be a compact manifold without boundary. Then N has a k -system (k' -system) if and only if $N \times \{I_i^k\}_{2^{k-1}}$ has a 1 -system ($1'$ -system) with $\text{sgn}_i v = (-1)^{i+1}$, $k \geq 1$. Note that for $k = 1$ this definition agrees with the previous definition of a 1 -system ($1'$ -system).

LEMMA 2.4. A point has a k' -system for every $k \geq 1$.

Proof. One can easily construct the necessary k' -system so the details will be omitted.

LEMMA 2.5. If N has a k' -system then $N \times S^q$ has a $(k-1)'$ -system for $q \geq 1$.

Proof. By definition $N \times \{I_i^k\}_{2^{k-1}}$ has a $1'$ -system with $\text{sgn}_i v = (-1)^{i+1}$. By Lemma 1.3 $\Sigma^q(N \times \{I_i^k\}_{2^{k-1}})$ has a $1'$ -system with $\text{sgn}_i v = (-1)^{i+1}$. Lemma 2.2 and Lemma 2.1 imply that $N \times \{D_{\mu_i}^{q+1}\}_{2^{k-1}}$ has a $1'$ -system with $\text{sgn}_i v = (-1)^{i+1}$, $\mu_i = \frac{1}{2} - y_i^k$. For each i ,

$1 \leq i \leq 2^{k-2}$, $2^{k-1} - i + 1 > i$; let $J_i^k = D_{\mu_i}^{q+1} - D_{\mu_{j_i}}^{q+1}$, where $j_i = 2^{k-1} - i + 1$. Then by Lemmas 1.1 and 1.2, $\{N \times J_i^k\}_{2^{k-2}}$ has a $1'$ -system with $\text{sgn}_i v = (-1)^{i+1}$ because $i + (2^{k-1} - i + 1)$ is odd.

Define $\text{Id} \times \text{Id} \times \chi : N \times S^q \times I_1^{k-1} \rightarrow N \times J_1^k$ by

$$(\text{Id} \times \text{Id} \times \chi)(n, \sigma, t) = (n, \sigma, \frac{1}{2} - \frac{1}{3} t) .$$

Here J_1^k has polar coordinates induced from the inclusion of J_1^k in $D_{\mu_1}^{q+1}$.

We claim that $\text{Id} \times \text{Id} \times \chi$ restricts to a diffeomorphism of $N \times S^q \times I_i^{k-1}$ to $N \times J_i^k$ for $1 \leq i \leq 2^{k-2}$. By definition $I_i^{k-1} = [y_i^{k-1}, 1 - y_i^{k-1}]$. Thus $\chi(I_i^{k-1}) = [\frac{1}{2} - (\frac{1}{3} - \frac{1}{3} y_i^{k-1}), \frac{1}{2} - \frac{1}{3} y_i^{k-1}]$. But by Lemma 2.3,

$$\chi(I_i^{k-1}) = [\frac{1}{2} - y_{j_i}^k, \frac{1}{2} - y_i^k] ,$$

where $j_i = 2^{k-1} - i + 1$. Therefore $\chi(I_i^{k-1}) = [\mu_{j_i}, \mu_i]$ which implies that $(\text{Id} \times \text{Id} \times \chi)(N \times S^q \times I_i^{k-1}) = N \times J_i^k$. Thus $(N \times S^q) \times \{I_i^{k-1}\}_{2^{k-2}}$ has a $1'$ -system. By Definition 2.4 we have that $N \times S^q$ has a $(k-1)'$ -system.

LEMMA 2.6. If N has a k' -system then $N \times S^{q_1} \times \dots \times S^{q_{k-1}} \times D^{q_k}$ has a $1'$ -system, where $q_i \geq 1$.

Proof. This follows by induction using Lemma 2.5 and Lemma 1.3, since

$$\Sigma^{q_k}(I) = D^{q_k} .$$

THEOREM 2.1. If N is a compact manifold without boundary which has a k -system then $S^1 \times N \times S^{q_1} \times \dots \times S^{q_k}$ is totally parallelizable, where $q_i \geq 1$.

Proof. This follows from Lemma 2.6 and Proposition 1.3 or by Proposition 1.4 and Proposition 0.3.

THEOREM 2.2. $S^1 \times S^{q_1} \times \dots \times S^{q_k}$ is totally parallelizable for $q_i \geq 1$.

Proof. This follows from Theorem 2.1 and Lemma 2.4.

SECTION 3. S^1 -BUNDLES OVER 2-MANIFOLDS

In this section we consider principal circle bundles which are differentiable bundles in the sense of Steenrod [9], §6.7.

THEOREM 3.1. If (M, B_g, S^1) is a principal circle bundle with:

- (i) total space M , a compact orientable 3-manifold,
- (ii) base space B_g , a compact orientable 2-manifold, without boundary, of genus g ,
- (iii) fiber the circle S^1 , then M is totally parallelizable.

Theorem 3.1 will follow from a series of lemmas. By Proposition 0.1, it suffices to find three oriented transversal 2-foliations on M .

DEFINITION 3.1. Let Q_0 be a closed 2-disc. Let Q_j , $1 \leq j \leq g$, be disjoint closed 2-discs in the interior of Q_0 . Then we define

$$D_g = Q_0 - \bigcup_{j=1}^g Q_j .$$

Note: B_g is the double of D_g .

We denote by A_j the boundary of Q_j , $0 \leq j \leq g$. Thus $\partial D_g = \bigcup_{j=0}^g A_j$ which we denote by A . D_g^k , $k = 1, 2$, will denote the two copies of D_g which form B_g , and A^k will denote ∂D_g^k .

LEMMA 3.1. There exists an f-system (C, H) on R^3 and an embedding

$\varphi : D_g \rightarrow R^3$ satisfying the following conditions. Let the 1-forms ω^i , $1 \leq i \leq 3$, define the foliations in H on $S^1 \times C$; then

- (i) $(\text{Id} \times \varphi)^* \omega^i$, $1 \leq i \leq 3$, totally parallelize $S^1 \times D_g$

which means in particular that for some collar neighborhood of A , say given by $\alpha : A \times I \rightarrow D_g$,

$$(\text{Id} \times (\varphi \circ \alpha))^* \omega^3 = T^* dt, \quad \text{on } S^1 \times A \times I$$

$$(\text{Id} \times (\varphi \circ \alpha))^* \omega^i = P^* v^i, \quad \text{on } S^1 \times A \times I$$

where T and P , respectively, are the projections of $S^1 \times A \times I$ onto I and $S^1 \times A$; v^i are 1-forms on $S^1 \times A$, $i = 1, 2$; dt is the usual 1-form on I .

$$(ii) \quad \omega^i = (dx_i - K_i(x_i) dz) (K_i^2(x_i) + 1)^{-\frac{1}{2}} \quad \text{where } K_i : C \rightarrow R$$

depends only on x_i , $i = 1, 2$; dx_i , dz are as in the beginning of Section 1.

(iii) $K_i \circ \alpha_j$ is zero on $E_j^i \times I$ for some open set $E_j^i \subset A_j$ where α_j is the restriction of α to $A_j \times I$, for $i = 1, 2$ and $0 \leq j \leq g$.

Proof. Define the 2-tuple $\{M_i^1\}_2$ by $M_1 = [0, 1]$, $M_2 = \bigcup_{j=1}^k [a_j, b_j]$

where $0 < a_1 < b_1 < \dots < a_k < b_k < 1$. For notational convenience,

let $a_0 = 0$ and $b_0 = 1$. Let C_1 be an open neighborhood of I . Let

the function $K_1 : C_1 \rightarrow R$ be such that

$$(1) \quad K_1(a_j) < 0, \quad 0 \leq j \leq g$$

$$(2) \quad K_1(b_j) > 0, \quad 0 \leq j \leq g$$

$$(3) \quad \text{For each } j, \quad 0 \leq j \leq g, \quad \text{there is an interval } I_j \subset [a_j, b_j]$$

such that $K_1 = 0$ on I_j .

Define ω_1^1 to be a 1-form on $S^1 \times C_1$ by

$$\omega_1^1 = (dx_1 - K_1(x_1)dz)(K_1^2(x_1) + 1)^{-\frac{1}{2}}.$$

Then if we let H_1 consist of the foliation defined by ω_1^1 we have that (C_1, H_1) is a 1'-system for $\{M_1^1\}_2$. By Lemma 1.3 we have that $\Sigma\{M_1^1\}_2$ has a 1-system, defined by $\Sigma\varphi$ and the f-system (C_2, H_2) , where $C_2 = C_1 \times (r_1^-, r_1^+)$. We can assume that the function K in Lemma 1.3 is zero in a neighborhood of zero. Thus by the proof of Lemma 1.3, we can assume that H_2 consists of foliations defined by $\omega_2^1 = \pi^*(\omega_1^1)$ where π is the projection of R^2 onto $R \cdot e_1$, and

$$\omega_2^2 = (dx_2 - K_2(x_2)dz)(K_2^2(x_2) + 1)^{-\frac{1}{2}}$$

where we set $K_2 = K$. By Lemma 1.1 we have that (C_2, H_2) restricts to a transversal system for the 1-tuple $\Sigma M_1^1 - \Sigma M_2^1$. D_g is diffeomorphic to $\Sigma M_1^1 - \Sigma M_2^1$ by Lemma 2.1.

We conclude by using the transversal foliations constructed in the proof of Proposition 1.4, defined by the 1-forms ω^i , $1 \leq i \leq 3$. Using that construction we have that $\omega^1 = \pi' \circ \pi(\omega_1^1)$, $\omega^2 = \pi'(\omega_2^2)$, where π' is the projection of R^3 onto R^2 . Thus (i) and (ii) are true. One has that (iii) is true by the definition of K_1 and K_2 .

Remark 3.1. The foliations v^i , $i = 1, 2$ on $S^1 \times A$ can be calculated as follows. Let $\gamma_j = \varphi \circ \alpha_j$ and let v_j^i be v^i restricted to $S^1 \times A_j$, $i = 1, 2$; $0 \leq j \leq g$. We let $\gamma_j = (\gamma_j^1, \gamma_j^2)$ where γ_j^i , $i = 1, 2$, are real-valued functions. Then for $(z, \theta) \in S^1 \times A_j \times \{0\}$,

$$\begin{aligned} v_j^i(z, \theta) &= (\text{Id} \times \gamma_j)^* \omega^i (\text{Id} \times \gamma_j(z, \theta, 0)) \\ &= G_i(\theta) ((\gamma_j^i)')(\theta) d\theta - K_i(\gamma_j(\theta, 0)) dz \end{aligned}$$

for $i = 1, 2$; $0 \leq j \leq g$, where $(\gamma_j^i)'$ denotes the partial derivative of γ_j^i with respect to θ at $(\theta, 0)$ and $G_i(\theta) = (K_i^2(\gamma_j^i(\theta, 0)) + 1)^{-\frac{1}{2}}$.

Let (M, B_g, S^1) be the bundle in the statement of Theorem 3.1. We denote by (M^k, D_g^k, S^1) the restriction of the bundle to D_g^k , $k = 1, 2$.

LEMMA 3.2. There exists a fiber bundle diffeomorphism $\chi_k: M^k \rightarrow S^1 \times D_g^k$, $k = 1, 2$, where $S^1 \times D_g^k$ is the trivial principal S^1 -bundle.

Proof. D_g^k is contractible for $g = 0$. For $g > 0$ there is a deformation retraction of D_g^k onto the one point union of g circles. Since the only principal S^1 bundle over S^1 is the trivial bundle, we conclude that (M^k, D_g^k, S^1) has a continuous cross-section. According to Steenrod [9], §6.7, there is then a differentiable cross-section and this implies the Lemma.

The map Φ defined by $\Phi = \chi_2 \circ \chi_1^{-1} : S^1 \times A^1 \rightarrow S^1 \times A^2$ is a fiber preserving, orientation preserving diffeomorphism. Here A^1 is identified with A^2 by the identity map. Each connected component of the domain of Φ is the product bundle $S^1 \times A_j^1$. Denote Φ restricted to the j^{th} component of its domain by Φ_j , $0 \leq j \leq g$. Then

$$\Phi_j(z, \theta) = (f_j(z, \theta), \theta)$$

for some function f_j and $(z, \theta) \in S^1 \times A_j$. Consider $\mathbb{R} \times [0, 2\pi]$ as a subset of the universal covering space of $S^1 \times A_j$. The covering map is given by $\theta = \bar{\theta} \bmod 2\pi$, $z = \bar{z} \bmod 2\pi$ for $(\bar{z}, \bar{\theta}) \in \mathbb{R} \times [0, 2\pi]$.

We let $\bar{\Phi}_j$, \bar{f}_j also denote their respective liftings to $\mathbb{R} \times [0, 2\pi]$.

We let the integer $N_j = \bar{f}_j(0, 2\pi) - \bar{f}_j(0, 0)$ and note that for all

$$\bar{z} \in \mathbb{R} , \bar{f}_j(\bar{z}, 2\pi) = \bar{f}_j(\bar{z}, 0) + 2\pi N_j .$$

LEMMA 3.3. Let $\Phi_j^{N_j} : S^1 \times A_j \rightarrow S^1 \times A_j$ be given by $\Phi_j^{N_j}(z, \theta) = (z + N_j \theta, \theta)$. Then for each j , $0 \leq j \leq g$, we have that Φ_j is isotopic to $\Phi_j^{N_j}$.

Proof. Let ρ be a function from I to itself such that $\rho([0, \frac{1}{3}]) = 1$ and $\rho([\frac{2}{3}, 1]) = 0$. Then we define $F : \mathbb{R} \times [0, 2\pi] \times I \rightarrow \mathbb{R} \times [0, 2\pi]$ by

$$F(z, \theta, t) = (\rho(t)(z + N_j \theta) + (1 - \rho(t))(f_j(z, \theta)), \theta).$$

Since, for a fixed θ , $z + N_j \theta$ and $f_j(z, \theta)$ are strictly monotonic increasing functions of z , it follows that for each fixed t that $F(z, \theta, t)$ is a diffeomorphism of $\mathbb{R} \times [0, 2\pi]$. Thus F is an isotopy of the liftings of Φ_j and $\Phi_j^{N_j}$. It is clear that $F(z + 2\pi n, \theta, t) = F(z, \theta, t) + 2\pi n$ for n an integer, and that

$$F(z, 2\pi, t) = F(z, 0, t) + 2\pi N_j.$$

Thus F covers an isotopy of Φ_j and $\Phi_j^{N_j}$ on $S^1 \times A_j$. We also let $\Phi_j^{N_j}(z, \theta)$ denote $\Phi_j^{N_j}(z, \theta - \theta_0)$ for a fixed θ_0 . This does not alter the truth of the above statements.

Thus without loss of generality we assume from now on that $\Phi_j = \Phi_j^{N_j}$.

That is, we have $M = (S^1 \times D_g^1) \cup (S^1 \times D_g^2)$ with the identifications

$\Phi_j^{N_j} : S^1 \times A_j^1 \rightarrow S^1 \times A_j^2$ for $0 \leq j \leq g$. The maps $\Phi_j^{N_j}$ define the map

$\Phi : S^1 \times A^1 \rightarrow S^1 \times A^2$. Although we have three transversal foliations on

$S^1 \times D_g^1$ and on $S^1 \times D_g^2$ by Lemma 3.1, they are not compatible with the

identifications given by Φ . In the next lemma we adjust these trans-

versal foliations so that they become compatible with Φ .

Notation: In the next lemma we make the convention of dropping the subscript j for convenience (i.e., A means A_j , N means N_j , etc.). The lemma will hold whenever the same j is inserted as a subscript throughout.

Let ρ be as in the above lemma. For $i = 1, 2$, we let $\eta^i : A \rightarrow \mathbb{R}$ be a function satisfying

$$\eta^i(\theta) = 1 \quad \text{for } \theta \in A - E^i$$

$$\eta^i(\theta) = 0 \quad \text{for } \theta \in U^i \quad \text{for some open set } U^i \subset E^i, \text{ and}$$

$$\text{let } \theta_i \text{ be a fixed element of } U^i.$$

We consider $\mathbb{R} \times [\theta_i, 2\pi + \theta_i] \times I$ as a subset of the universal covering space of $S^1 \times A \times I$ with covering map π' . We can then define a diffeomorphism F_i of $\mathbb{R} \times [\theta_i, 2\pi + \theta_i] \times I$ to itself by

$$F_i(z, \theta, t) = (z + \eta^i(\pi'(\theta)) \cdot \rho(t) \cdot N \cdot (\theta - \theta_i), \theta, t)$$

where N is the integer of Lemma 3.3, $i = 1, 2$. F_i induces a diffeomorphism, also denoted F_i , of $S^1 \times A \times I$ to itself, since

$$\eta^i(\pi'(\theta)) = 0 \quad \text{for all } \theta \text{ in some neighborhood of } \theta_i \text{ and } 2\pi + \theta_i.$$

Set $\delta^i = F_i^* P^* \nu^i$ for $i = 1, 2$.

LEMMA 3.4. (i) $\delta^i = P^* \nu^i$ on $S^1 \times A \times [\frac{2}{3}, 1]$, $i = 1, 2$.

(ii) $\delta^i = P^* \Phi^{N^*} \nu^i$ on $S^1 \times A \times [0, \frac{1}{3}]$, $i = 1, 2$.

(iii) δ^1, δ^2 and dt define transversal foliations on
 $S^1 \times A \times I$.

Proof. Since $\rho = 0$ on $[\frac{2}{3}, 1]$ we have that $F_i = \text{identity}$ for $i = 1, 2$

on $S^1 \times A \times [\frac{2}{3}, 1]$. This implies (i). On $[0, \frac{1}{3}]$, $\rho = 1$. Thus $F_i = \Phi^N \times \text{Id}$ on $S^1 \times (A - E) \times [0, \frac{1}{3}]$. Since $P \circ (\Phi^N \times \text{Id}) = \Phi^N \circ P$, (ii) holds on $S^1 \times (A - E) \times [0, \frac{1}{3}]$. On $S^1 \times E$, $v^i(z, \theta) = G_i(\theta) \cdot (\gamma^1)'(\theta) d\theta$. But $F_i^* d\theta = d\theta = (\Phi^N)^* d\theta$, $G_i \circ F_i = G_i = G_i \circ \Phi^N$, and $(\gamma^1)' \circ F_i = (\gamma^1)' \circ \Phi^N$ so we conclude (ii). Using Remark 3.1 a straightforward calculation shows that on $S^1 \times A \times I$

$$\delta^i = P^* v^i - N \cdot (K_i \circ \gamma) \cdot G_i (\rho \cdot (\eta^i)' d\theta + \rho' \cdot \eta^i dt), \quad i = 1, 2$$

where $(\eta^i)'$ and ρ' are respectively the derivatives of η^i and ρ .

Another calculation yields

$$\delta^1 \wedge \delta^2 \wedge dt = P^* v^1 \wedge P^* v^2 \wedge dt +$$

$$N \cdot \rho \cdot G_1 \cdot G_2 \cdot (K_1 \cdot K_2(\gamma)) \cdot ((\eta^2)' - (\eta^1)') dz \wedge d\theta \wedge dt$$

on $S^1 \times A \times I$. If $\theta \in E^1 \cup E^2$ then $K_1(\gamma(\theta, t)) \cdot K_2(\gamma(\theta, t)) = 0$. If $\theta \notin E^1 \cup E^2$ then $(\eta^2)' = 0 = (\eta^1)'$. Thus $\delta^1 \wedge \delta^2 \wedge dt$ is non-vanishing and (iii) follows.

We complete the proof of Theorem 3.1 by defining transversal foliations h^i , $1 \leq i \leq 3$, on M as follows:

$$(1) \quad h^i \text{ is defined by } \chi_2^*(\text{Id} \times \varphi)^* \omega^i \text{ on } M^2, \quad 1 \leq i \leq 3,$$

$$(2) \quad h^i \text{ is defined by } \chi_1^*(\text{Id} \times \varphi)^* \omega^i \text{ on } M^1|_{D_g^1 - \alpha(A \times [0, 1/6])}, \\ 1 \leq i \leq 3,$$

$$(3) \quad h^i \text{ is defined by } \chi_1^*(\text{Id} \times \alpha^{-1})^* \delta^i \text{ on } M^1|\alpha(A \times [0, \frac{1}{3}]),$$

$$i = 1, 2,$$

$$(4) \quad h^3 \text{ is defined by } \chi_1^*(\text{Id} \times \alpha^{-1})^* dt \text{ on } M^1|\alpha(A \times [0, \frac{1}{3}]).$$

Lemmas 3.1, 3.3, 3.4 imply that the h^i , $1 \leq i \leq 3$, are well defined on all of M .

SECTION 4. FOLIATIONS FOR CERTAIN S^1 -BUNDLES

PROPOSITION 4.1. Let M be a compact manifold with non-null boundary N .
Then $S^1 \times M$ has an oriented foliation with $S^1 \times N$ as a leaf. Further-
more, there is a vector field v transversal to this foliation which
points outward on $S^1 \times N$.

Proof. Let $\alpha : N \times I \rightarrow M$ be a collar neighborhood of $N = \alpha(N \times \{0\})$.
 Let $\rho : I \rightarrow I$ be a function such that $\rho([0, \frac{1}{3}]) = 0$, $\rho([\frac{2}{3}, 1]) = 1$.
 We define a 1-form ω on $R \times M$ by

$$(1) \quad \omega(z, m) = dz \quad \text{for } (z, m) \in R \times (M - \alpha(N \times [0, \frac{2}{3}]))$$

$$(2) \quad \omega(z, m) = (\text{Id} \times \alpha^{-1})^* \{ \rho(t) dz + (1 - \rho(t)) dt \}$$

$$\text{for } m = (\text{Id} \times \alpha)(z, n, t) \quad \text{for } (z, n, t) \in R \times N \times [0, 1),$$

where dz is the lifting to R of the usual 1-form on S^1 also denoted dz . These two definitions of ω agree on $R \times \alpha\{N \times (\frac{2}{3}, 1)\}$. ω is clearly non-vanishing and integrable. Since $\omega(z_1, m) = \omega(z_2, m)$ for all $z_1, z_2 \in R$ we have that ω induces a non-vanishing integrable 1-form on $S^1 \times M$. It is easy to find a vector field satisfying the second part of the proposition.

PROPOSITION 4.2. Let $\eta = (M, B, p)$ be a differentiable principal S^1 -
bundle where $\partial B = \emptyset$, B is compact, p is the projection map. Let
 $H = (M', S^2, \pi)$ denote a differentiable principal S^1 -bundle on S^2 .

Suppose that there exists a commuting diagram

$$\begin{array}{ccc}
 M & \xrightarrow{F} & M' \\
 p \downarrow & & \downarrow \pi \\
 B & \xrightarrow{f} & S^2
 \end{array}$$

where F , f are continuous functions and F is a fiber bundle map.

Then M has a foliation.

Proof. Since one can approximate f arbitrarily closely by a differentiable function, we use the Riemannian structure on S^2 to show that there exists a differentiable function $g : B \rightarrow S^2$ homotopic to f . Thus by Steenrod [9], §11.5, we can assume without loss of generality that the above diagram consists completely of differentiable maps. Consider S^2 embedded as usual in R^3 . Let $h : S^2 \rightarrow [-1,1]$ be the projection onto the first coordinate axis of R^3 . Choose a regular value c in $[-1,1]$ for $h \circ f$. c lies in $(-1,1)$. Define

$$\begin{aligned}
 B^+ &= (h \circ f)^{-1}([c,1]) \\
 B^- &= (h \circ f)^{-1}([-1,c]) .
 \end{aligned}$$

B^+ and B^- are submanifolds of B with common boundary. $M|_{B^+}$ and $M|_{B^-}$ are both product bundles since $h^{-1}([c,1])$ and $h^{-1}([-1,c])$ are contractible. Thus by Proposition 4.1 we have that M has a foliation.

THEOREM 4.1. Let B be a compact 4-manifold with $\partial B = \emptyset$. Let $\eta = (M, B, \rho)$ be a differentiable S^1 -bundle. Then let $c \in H^2(B; Z)$ denote the Chern class of η . If $c^2 = 0$, then M has a foliation.

Proof. Let $E = (S^\infty, P_\infty(C), \pi)$ denote the universal S^1 -bundle. Then, by

Steenrod [9], §19.3, isomorphism classes of S^1 -bundles over a finite $c - w$ complex X are in 1-1 correspondence with homotopy classes of continuous maps from X to $P_\infty(C)$ denoted $[X, P_\infty(C)]$. If $g \in [X, P_\infty(C)]$, we denote the induced bundle over X by $g(E)$. The Chern class induces a natural equivalence of functors $ch : [\cdot, P_\infty(C)] \rightarrow H^2(\cdot ; Z)$. Hence, given a continuous map $f : B \rightarrow S^2 = P_1(C)$ there is a commutative diagram

$$\begin{array}{ccc} [B, P_\infty(C)] & \xrightarrow{\text{ch}} & H^2(B; Z) \\ \uparrow \alpha & & \uparrow f^* \\ [S^2, P_\infty(C)] & \xrightarrow{\text{ch}} & H^2(S^2; Z) \end{array}$$

where α is defined by composition with f . Let $g \in [B, P_\infty(C)]$ induce η . Then $ch(g) = c$. Since $c^2 = 0$, by Spanier [8], Theorem 8.4.11, p. 452, there exists an $f : B \rightarrow S^2 = P_1(C)$ such that $f^*(i) = c$ where i is a 2-characteristic element of $H^2(S^2; Z)$, (for the definition and existence of a 2-characteristic element in $H^2(S^2; Z)$ see Spanier [8], section 8.1.3). Let j represent $ch^{-1}(i)$. Then by commutativity of the above diagram $j \circ f$ is homotopic to g . Hence, if we set $H = j(E)$, then we have that η is isomorphic to the pullback of H by f . It is easy to see that the principal S^1 -bundle H over S^2 is bundle isomorphic to a differentiable bundle. Thus, Theorem 4.1 follows from Proposition 4.2.

Let M^n be a compact n -dimensional manifold and let $f : M \rightarrow R$ be a Morse function. We use the following notation: $f^a = f^{-1}(a)$ and $f_{[a,b]} = f^{-1}([a,b])$. The following two lemmas are immediate consequences of the proofs of Theorems 3.1 and 3.2, and Remarks 3.3 and 3.4 in Milnor [4].

LEMMA 4.1. Let $c \in [a, b]$, and let f have no critical values in $[a, b]$, except possibly c . Then f^c is a deformation retraction of $f_{[a, b]}$.

LEMMA 4.2. Let p_1, \dots, p_s be the set of critical points such that $f(p_j) = c$. Let λ_j be the index of p_j . Then for some $\epsilon > 0$ there exist embeddings $G_j : D^{\lambda_j} \rightarrow f_{[c-\epsilon, c]}$ satisfying the following properties:

$$(1) \quad G_j(\partial D^{\lambda_j}) \subset f^{c-\epsilon} \quad 1 \leq j \leq s$$

$$(2) \quad f^{c-\epsilon} \cup \left(\bigcup_{j=1}^s G_j(D^{\lambda_j}) \right) \text{ is a deformation retraction of } f_{[c-\epsilon, c]}.$$

PROPOSITION 4.3. Let E be the total space of a differentiable S^1 bundle over a compact manifold M without boundary. Let $f : M \rightarrow \mathbb{R}$ be a Morse function with critical values $c_0 < c_1 < \dots < c_N$. If E restricted to f^{c_i} is a trivial bundle for $0 \leq i \leq N$, then E has an oriented foliation.

Proof. Choose points b_i , $0 \leq i \leq N+1$ such that $b_i < c_i < b_{i+1}$. Then, by Lemma 4.1, E restricted to $f_{[b_i, b_{i+1}]}$ is a trivial bundle and hence diffeomorphic to $S^1 \times f_{[b_i, b_{i+1}]}$. Thus by Proposition 4.1 we conclude that $E|_{f_{[b_i, b_{i+1}]}}$ has an oriented foliation, $0 \leq i \leq N$. Furthermore by Proposition 4.1 we can assume that the normal vector field to the foliation is flowing outwards on $S^1 \times f_{[b_i, b_{i+1}]}$ for i even, and inwards for i odd. Thus E has an oriented foliation.

THEOREM 4.2. Let M be a compact manifold without boundary. Suppose
there exists a Morse function $f : M \rightarrow \mathbb{R}$ satisfying the following
property: for each critical point p of index 2 , with G and $f^{c-\epsilon}$
as in Lemma 4.2, $G_*(H_1(\partial D^2))$ is a direct summand of $H_1(f^{c-\epsilon})$. If
 E is the total space of a differentiable S^1 bundle over M , then E
has an oriented foliation.

Proof. Let b_i be as in the proof of Proposition 4.3. Theorem 4.2

will follow from Proposition 4.3 and the following three assertions.

(1) $E | f^{c_0}$ is a trivial bundle, (2) if $E | f^{c_i}$ is trivial then
 $E | f^{b_{i+1}}$ is a trivial bundle, and (3) if $E | f^{b_i}$ is a trivial bundle
then $E | f^{c_i}$ is a trivial bundle. Assertion (1) follows since f^{c_0}
is a finite set of points. Assertion (2) follows from Lemma 4.1. Assume
that $E | f^{b_i}$ is trivial. We assume for simplicity that p , of index
 λ , is the only critical point such that $f(p) = c_i$. (However this
assumption is not necessary.) We may assume that the ϵ of Lemma 4.2
satisfies, $0 < \epsilon$, $c_i - b_i$. Then by Lemma 4.1, $E | f^{c_i - \epsilon}$ is a trivial
bundle.

Consider the following part of the exact cohomology sequence induced
by the embedding $g : \partial D^\lambda \rightarrow f^{c_i - \epsilon}$ where g is the restriction of G to
 ∂D^λ :

$$H^2(f^{c_i - \epsilon}) \xleftarrow{j} H^2(f^{c_i - \epsilon}, \partial D^\lambda) \xleftarrow{\delta} H^1(\partial D^\lambda) \xleftarrow{g^*} H^1(f^{c_i - \epsilon}) .$$

We can identify $H^2(f^{c_i - \epsilon}, \partial D^\lambda)$ with $H^2(f^{c_i - \epsilon} \cup G(D^\lambda))$ in the usual way.

Let $c \in H^2(f^{c_i - \epsilon}, \partial D^\lambda)$ denote the Chern class of $E | f^{c_i - \epsilon} \cup G(D^\lambda)$.

Then $j(c) = 0$ since $E \mid f^{c_i^{-\epsilon}}$ is a trivial bundle. By exactness of the above sequence we conclude that there exists a $d \in H^1(\partial D^\lambda)$ such that $\delta(d) = c$. If $\lambda \neq 2$, then $H^1(\partial D^\lambda) = 0$ and thus $c = 0$. If $\lambda = 2$, we claim g^* is an epimorphism which implies that $c = 0$. By Spanier [8], Theorem 5.5.3, p. 243, there is a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \text{Hom}(H_1(f^{c_i^{-\epsilon}}; Z), Z) & \xleftarrow{\gamma} & H^1(f^{c_i^{-\epsilon}}; Z) & & \\
 & & \downarrow \alpha & & \downarrow g^* & & \\
 0 & \longleftarrow & \text{Hom}(H_1(\partial D^2; Z), Z) & \xleftarrow{\beta} & H^1(\partial D^2; Z) & \longleftarrow & \text{Ext}(H_0(\partial D^2; Z), Z) \longleftarrow 0 \\
 & & \parallel & & \parallel & & \parallel \\
 & & Z & & Z & & 0
 \end{array}$$

where α is induced by g_* . From the assumption that $g_*(H_1(\partial D^2; Z))$ is a direct summand of $H_1(f^{c_i^{-\epsilon}}; Z)$ we have that α is an epimorphism. Since $H_0(\partial D^2; Z) = Z$ we have that $\text{Ext}(H_0(\partial D^2; Z), Z) = 0$ and hence β is an isomorphism. Thus $g^* = \beta^{-1} \circ \alpha \circ \gamma$ is an epimorphism.

Thus we have shown that $E \mid f^{c_i^{-\epsilon}} \cup G(D^\lambda)$ is a trivial bundle. By Lemma 4.2 we have that $E \mid f^{c_i}$ is a trivial bundle, hence Theorem 4.2 follows.

REFERENCES

- [1] F. Hirzebruch, "Some Problems on Differentiable and Complex Manifolds", *Annals of Mathematics*, 60(1954), 213-236.
- [2] W.B.R. Lickorich, "A Foliation for 3-Manifolds", *Annals of Mathematics*, 82(1965), 414-420.
- [3] E.L. Lima, "Commuting Vector Fields on S^3 ", *Annals of Mathematics*, 81(1965), 70-81.
- [4] J.W. Milnor, Morse Theory, (Annals Studies 51), Princeton University Press, 1963.
- [5] J.R. Munkres, Elementary Differential Topology, (Annals Studies 54), Princeton University Press, 1963.
- [6] G. Reeb, Sur Certaines Propriétés Topologiques Des Variétés Feuilletées, *Actual. Sci. Industr.* 1183, 1952.
- [7] H. Rosenberg, "The rank of $S^2 \times S^1$ ", *American Journal of Mathematics*, 87(1965), 11-24.
- [8] E.H. Spanier, Algebraic Topology, McGraw-Hill, Inc., New York, N.Y., 1966.
- [9] N.E. Steenrod, The Topology of Fibre Bundles, Princeton University Press, Princeton, N.J., 1951.
- [10] S. Sternberg, Lectures on Differential Geometry, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

AUTOBIOGRAPHICAL STATEMENT

I was born in Staten Island on June 22, 1943. Soon after, my family moved to Brooklyn, New York where I went through the public school system, graduating from Midwood High School. I majored in mathematics at Columbia College and received my A.B. degree in June, 1964. I did one year of graduate work at Brandeis University before transferring to the Graduate Division of The City University of New York.

Abstract

TOTALLY PARALLELIZABLE MANIFOLDS

by

David Tischler

A substantial theory has been developed to determine whether there exist sections of bundles associated to the principal bundle corresponding to the tangent bundle of a manifold. However, very little is known about finding such sections which satisfy non-linear conditions. For example, Reeb [6], p. 95, and Hirzebruch [1], Problem 9, ask: if a manifold admits a non-vanishing vector field does it admit a foliation of codimension 1. This question is still open; in fact, the answer is unknown for all odd-dimensional spheres, except for the trivial case of S^1 and for S^3 on which Reeb constructed a foliation. Another example is the problem of finding commuting vector fields on a manifold. Lima [3] and Rosenberg [7] have shown that, on S^3 and $S^1 \times S^2$ respectively, two pointwise independent vector fields cannot commute. We will be concerned here with this sort of problem.

A pair of pointwise independent vector fields is called integrable if it defines a 2-dimensional foliation. We say a manifold is totally parallelizable if it can be parallelized by vector fields each pair of which is integrable. For an n -manifold, an equivalent condition to total parallelizability is the existence of n codimension 1 oriented foliations which meet transversally. A natural question to ask is: if a manifold is parallelizable is it totally parallelizable. This question is related to the question above. We devote most of this paper to constructions which totally parallelize manifolds.

In Section 2 we show that $S^1 \times S$ is totally parallelizable, where

S is an arbitrary product of spheres. In Section 3 we show that differentiable principal S^1 -bundles over 2-dimensional compact manifolds (e.g., S^3 or the lens spaces $L(p,q)$ with $q = 1$) are totally parallelizable. Since all compact orientable 3-manifolds are parallelizable, and since by Lickorish [2], they admit foliations of codimension 1, an interesting special case of the above question is: are all compact orientable 3-manifolds totally parallelizable. Aside from the S^1 -bundles mentioned above, I believe the answer is not known. In particular, it would be instructive to know the answer for the lens spaces $L(p,q)$, $q \neq 1$.

The results in Sections 2 and 3 make use of the constructions in Section 1. The main method for totally parallelizing a manifold $S^1 \times N$, where $\dim N = n - 1$ is as follows: 1) give n transversal foliations, defined by 1-forms ω^i , $1 \leq i \leq n$, on $S^1 \times C$, where $C \subset R^n$; 2) for some immersion $\varphi: N \rightarrow C$, the 1-forms $(\text{Id} \times \varphi)^* \omega^i$ totally parallelize $S^1 \times N$. When φ extends to an immersion $\bar{\varphi}: M \rightarrow C$ where $\partial M = N$, then we define a manifold ΣM , whose boundary is $D(M)$, the double of M ; and an immersion $\Sigma\varphi: \Sigma M \rightarrow C' \subset R^{n+1}$; and $n+1$ transversal foliations on $S^1 \times C'$ such that $\text{Id} \times \Sigma\varphi$ totally parallelizes $S^1 \times D(M)$ as above. Using these methods we show that for certain manifolds N , there is an immersion $\varphi: N \times S^q \rightarrow R^{n+q}$ which totally parallelizes $S^1 \times N \times S^q$.

In Section 3 we represent an S^1 -bundle over a 2-manifold as the union of two product bundles whose boundaries have been identified. Then using the constructions of Section 1 we find three transversal foliations on each product bundle which agree with the identification of the boundaries. By decomposing S^1 -bundles as in Section 3 we are able to show the existence of foliations of codimension 1 on some more S^1 -bundles.