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RAMSEY FUNCTIONS ON GRAPHS

by

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1. Introduction.

A graph consists of a set of vertices V and a set E (edges) of unordered distinct pairs of vertices from V. If $(v,w) \in E$ then v and w are said to be adjacent. A famous theorem of Ramsey asserts that if the number of vertices of a graph is "large", then the graph contains either a "large" set of mutually adjacent points (K_n) or a "large" set of mutually nonadjacent points ($\bar{K}_n$). In this work we examine a related problem, first examined in [8].

A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an induced subgraph of G ($\underline{H \subset G}$) if $V(H) \subseteq V(G)$ and for any $v,w \in V(H)$, $(v,w) \in E(H)$ if and only if $(v,w) \in E(G)$. We consider the set of all graphs, P, partially ordered by inclusion as an induced subgraph.

We investigate the following question, intimately related to Ramsey's Theorem: for a particular nonnegative monotonic graph invariant, f, can we find a finite number of chains (in the sense of our partial order) of graphs, with $f \to \infty$ on each chain, so that for any graph G with f(G) large there is a large graph H in one of the chains such that H is an induced subgraph of G. We call an invariant with this property a Ramsey function. This concept will be defined more precisely in Chapter 2. But we remark here that Ramsey's Theorem states that $|V(G)|$ is a Ramsey function with the two chains $0 \subset K_1 \subset K_2 \subset \ldots$ and $0 \subset \bar{K}_1 \subset \bar{K}_2 \subset \ldots$.

Here is another example of a Ramsey function. Assume that the vertices of a graph G are labeled $1,2,\ldots,n = |V(G)|$.

The adjacency matrix, $A(G)$, of a graph G has row i and column i corresponding to the vertex labeled i and is a $(0,1)$-matrix with $(i,j)=1$ iff vertex i is adjacent to vertex j. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. $A(G)$ is real and symmetric so its eigenvalues are real. The smallest eigenvalue of G is denoted $\lambda^1(G)$.

We now describe two chains of graphs. One is the chain consisting of graphs H_n. H_n has $2n+1$ points and consists of a K_{2n} and an additional point adjacent to exactly n of the points of K_{2n}. The other is the chain of graphs $K_{1,n}$. $K_{1,n}$ consists of one vertex adjacent to n mutually nonadjacent vertices. Hoffman [5] showed that $|\lambda^1(G)|$ is a Ramsey function with the two chains $0 \subset K_{1,1} \subset K_{1,2} \subset \dots$ and $0 \subset H_1 \subset H_2 \subset \dots$. What this surprising result says is that if $|\lambda^1(G)|$ is very large then G must contain either a large $K_{1,n}$ or a large H_n as an induced subgraph.

The idea of Ramsey function was stated in [8] to have a convenient language to state this theorem and other results about spectra of matrices associated with graphs. But it seems appropriate to examine other invariants of graphs, not necessarily related to eigenvalues, and to ask for each such invariant whether or not it is a Ramsey function on P. This is the theme of the thesis.

In Chapter 2 we give the precise definition of a Ramsey function and we prove that for a particular graph invariant, f, which is a Ramsey function, the corresponding chains of graphs are essentially unique. We also introduce a convenient notation which we will use for the many different graphs which occurred in chains during this investigation.

In Chapter 3 we establish that many of the well-known graph invariants are Ramsey functions. We then give a table listing invariants which we know to be Ramsey functions and the corresponding chains of graphs for each invariant.

Some results on 4 less well-known invariants are contained in Chapter 4. We first give the definition for <u>match*(G)</u> and <u>achr*(G)</u> and prove that they are Ramsey functions having (the same) six different chains of graphs corresponding. These results were first presented in [7] and are joint with Professor Hoffman. We then define the <u>Deletion number</u> of a graph and prove that it is a Ramsey function with six different chains relating to it. Finally we give the definition of the <u>Dilworth number</u> [3] of a graph and show that it is a Ramsey function by showing that if the Dilworth number of a graph G is at least a certain size then some particular large graph, H, being in one of nine different chains must appear as an induced subgraph of G.

A study of Ramsey functions for graphs might be perhaps less interesting if all invariants turn out to be Ramsey functions. In Chapter 5 we show that this is not the case and develop methods of showing that a given function is not Ramsey. We prove that certain invariants, for example, genus, chromatic number, arboricity and others are not Ramsey functions. We complete the chapter with a table of graph functions which we know are not Ramsey functions.

2. Preliminary Material.

2.1. Description of a Ramsey Function.

We begin with the definition of a Ramsey function. As the study of Ramsey functions might be applicable to structures other than graphs, we define Ramsey functions on a partially ordered set, although we will only be concerned with the set of all graphs (including the null graph) partially ordered by inclusion as an induced subgraph.

Notation: For graphs H and G, $H \subset G$ will mean that H is an induced subgraph of G.

Let P be a partially ordered set with $0 \in P$, $0 \leq a$ for all $a \in P$. Let f be a nonnegative monotonic nondecreasing function on P. So $a \leq b$ implies $f(a) \leq f(b)$.

Definition 1. For a given nonnegative monotonic nondecresing function f defined on P, a chain $0 = a_0 < a_1 < a_2 < \ldots$ for which $f(a_i) \to \infty$ is called an f-chain.

Let $F = \{\langle a_{ij} \rangle \mid i = 1, 2, \ldots, m;\ j = 0, 1, 2, \ldots\}$ be a finite set of f-chains.

Definition 2. For a given f and $F \neq \phi$ and $a \in P$ let $t_F(a) = \max \{j \mid \text{for some } i,\ a_{ij} \leq a\}$.

This is well defined since each f-chain begins with the element 0. In terms of the graphical items described in Chapter 1, these definitions may be interpreted as follows:

1) What we had described as a chain of graphs gives us an f-chain. For example, if $f = |V(G)|$ we get the f-chains $0 \subset K_1 \subset K_2 \subset K_3 \subset \ldots$ and $0 \subset \overline{K}_1 \subset \overline{K}_2 \subset \overline{K}_3 \subset \ldots$.

2) Our finite set of f-chains, F, corresponds to the finite set of chains of graphs, and

3) If we know the chains in F, and list them in matrix form $t_F(G)$ is the column index, j, of an induced subgraph of G furthest out in any of the rows of the matrix.

Definition 3. An unbounded nonnegative monotonic (nondecreasing) function f on P is a Ramsey function if there exists a finite set of f-chains, F, such that for any sequence $S = s_1, s_2, s_3, \ldots$ in P, $\sup_S f(s_i) = \infty$ implies $\sup_S t_F(s_i) = \infty$.

2.2. Essential Uniqueness of F.

In this section we will show that if f is a Ramsey function, its corresponding set of f-chains, F, is essentially unique. But first some terminology and a lemma.

Definition 4. A chain $\langle a_i \rangle \subset P$ on which $f \to \infty$ is said to be chasing an infinite sequence $\{s_i\}$ in P if there is an infinite subchain $\langle a_{i_j} \rangle$ and subsequence $\{s_{i_j}\}$ such that $a_{i_j} \leq s_{i_j}$ for each j.

Lemma 1 (Chasing Lemma). If f is a Ramsey function with finite set of f-chains, F, then any sequence $\{s_i\}$ for which $\sup_i f(s_i) = \infty$ has one of the chains of F chasing it.

Proof. Let f be a Ramsey function and F be its corresponding set of f-chains. If $S = \{s_i\}$ is any sequence such that $\sup_S f(s_i) = \infty$ then in order to establish that $\sup_S t_F(s_i) = \infty$ as required in Definition 3, the elements of one of the chains $\langle a_k \rangle$ of F are used infinitely often. This chain chases the sequence $\{s_i\}$.

A finite collection of f-chains, F, used in establishing that f is Ramsey is minimal if no one of its chains chases any other chain. Note that, if two of the f-chains used in establishing that f is Ramsey have the property that one chases the other, then the latter may be deleted from F.

Proposition 1. For a function f, the set F is essentially unique, that is, if F and F' are minimal sets of f-chains used in establishing that f is Ramsey, then each chain $\langle a_i \rangle$ in F can be paired with a chain $\langle a_i' \rangle$ in F' so that $\langle a_i \rangle$ chases $\langle a_i' \rangle$ and $\langle a_i' \rangle$ chases $\langle a_i \rangle$.

Proof. Suppose F and F' are minimal sets of f-chains used in establishing that f is a Ramsey function. Considering any chain $\langle a_i \rangle$ in F, Lemma 1 implies there is a chain $\langle a_i' \rangle$ in F' chasing it. But Lemma 1 also implies there is a chain $\langle a_j \rangle$ in F which is chasing $\langle a_i' \rangle$. Now since F is minimal and chasing is a transitive relation, $\langle a_j \rangle$ must equal $\langle a_i \rangle$. Thus $\langle a_i' \rangle$ chases $\langle a_i \rangle$ and $\langle a_i \rangle$ chases $\langle a_i' \rangle$.

2.3. Several Needed Lemmas.

We will at various times need results of the following type: if f and g are graph invariants, and f(G) has a certain lower bound, then there exists $H \subset G$ such that g(H) has a certain lower bound. The lemmas of this section are useful in making such calculations.

Lemma 2. There exists a smallest positive integer R(n;m) such that any coloring of the edges of the complete graph on R(n;m) vertices with m colors has a monochromatic complete subgraph of size n.

(This is Ramsey's Theorem).

We will write R(n) for R(n;2).

<u>Lemma 3</u> Hoffman, [4] . There exists a function $F(n)$ such that any (0,1)-matrix with at least $F(n)$ rows, all different, contains a square submatrix of size n, which after permutation of rows and columns has form I, J-I, or T. (I is the identity matrix, J is the matrix of all 1's, and $T=(t_{ij})$ has $t_{ij}=1$ iff $i \geq j$).

note: In [6] Hoffman shows that $F(n) \leq 3 \cdot 2^{R(n;4)-1}-1$.

<u>Lemma 4</u> [11, p.45]. There exists a function $Q(n)$ such that any square (0,1)-matrix of order at least $Q(n)$ with all 1's on the diagonal contains a principal submatrix of order n of form I, J, T, or T^t. (See Lemma 3 for descriptions of these). (The proof shows $Q(n) \leq R(n;4)$).

<u>Lemma 5</u>. There exists a function $M(n)$ defined for all $n \in Z^+$ such that any square (0,1)-matrix with $M(n)$ independent 1's and at most n 1's in each column has a square submatrix of size n which (after permutation of rows and columns) has form I.

<u>Proof</u>. We shall prove that $M(n) \leq Q(n+1)$. Permute the rows and columns so that we have $Q(n+1)$ independent 1's in the first $Q(n+1)$ diagonal positions and consider the principal submatrix corresponding to these positions. Applying Lemma 4 to this submatrix we find it has a principal submatrix of order n+1 of form I, J, T, or T^t. J, T, and T^t are impossible, for these would contradict the fact that every column of our matrix had at most n 1's. Therefore we find a submatrix of order n of form I. (Thus, using Lemma 4, $M(n) \leq R(n+1;4)$).

The next lemma will be used in Chapter 4 to establish that the Dilworth number is a Ramsey function.

<u>Lemma 6</u>. There exists a function $D(n)$ such that any $(0,1)$-matrix M with $D(n)$ rows and columns in which each row except the first must contain a zero and any upper row of M contains a 1 in some column where a lower row contains a 0, has a submatrix of order n which (after permutation of columns only) has form I, T^t, or $J-I$.

<u>Proof</u>. We show that $D(n) \leq 2^{2^{R(n;4)+1}+R(n;4)+1}$. Suppose M has $D(n)$ rows and the other properties described above. Either

(1) Row 1 has a 1 in a column in which $2^{2^{R(n;4)}+R(n;4)}$ of the lower rows have 0's, or

(2) For every 1 in row 1, less than $2^{2^{R(n;4)}+R(n;4)}$ of the lower rows have 0's in the same column.

In case (1) holds we get a submatrix of the form at the right where the $*$ part is yet to be determined.

$$\begin{array}{c|c} & 1 \\ \hline * & \left.\begin{matrix}0\\0\\\vdots\\0\end{matrix}\right\} 2^{2^{R(n;4)}+R(n;4)} \end{array}$$

In case (2) holds we get a submatrix of the form at the right where the $*$ part is yet to be determined.

$$\begin{array}{c|c} 111\cdots1 & 1 \\ \hline * & \left.\begin{matrix}0\\1\\1\\\vdots\\1\end{matrix}\right\} 2^{2^{R(n;4)}+R(n;4)} \end{array}$$

In either case we iterate the same procedure on the smaller submatrix $*$ and successive submatrices, seeing whether statements similar to (1) or (2) hold. At any time if the submatrix we are concerned with is of order 2^{2^A+A} we will find (by permuting columns if necessary) an additional diagonal 1 entry with a 0 entry directly below it and, if (1) holds $2^{2^{A-1}+A-1}-1$ zeros below that or if (2) holds $2^{2^{A-1}+A-1}-1$ ones below the 0 entry. After $R(n;4)+1$ iterations of this process we will have a submatrix M' of M found by permuting columns only which has all 1's on the diagonal and all 0's on the subdiagonal. M' is of order $R(n;4)+1$. Applying the matrix form of Ramsey's Theorem (see [11, p.45]) we are assured of finding a submatrix M'' of M of order n (by permuting columns only) having form I, T^t, or $J-I$.

2.4. Graph Types.

In this section we describe the notation for various graphs which have arisen in f-chains for the invariants we have considered during this investigation. We follow Harary [9] where possible.

Whenever we mention the chain G_n we mean the chain $0 \subset G_1 \subset G_2 \subset \cdots$ where we specify the n^{th} element in the chain. If G_1 and G_2 are graphs where $V(G_1) \cap V(G_2) = \phi$, the graph $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The graph $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{\text{all edges } (v_1,v_2) \mid v_1 \in V(G_1),\ v_2 \in V(G_2)\}$. nG consists of n disjoint copies of G. The complement of G, $\overline{G}$, has $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{(v_i,v_j) \mid (v_i,v_j) \notin E(G),\ i \neq j\}$.

Definition 5. A path is a sequence $v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n$ where the v_i are distinct and $e_i = (v_i, v_{i+1}) \in E(G)$. The length of a path is the number of edges in it.

Definition 6. A graph is bipartite if its vertex set can be partitioned into sets A and B so that there are no adjacencies among vertices within either set.

(a) $G_n = K_n$ consists of n mutually adjacent vertices.

(b) $G_n = \overline{K}_n$ consists of n mutually nonadjacent vertices.

(c) $G_n = K_{1,n}$ consists of a vertex adjacent to n other mutually nonadjacent vertices.

(d) $G_n = K_{n,n}$ is the bipartite graph with $|A| = |B| = n$ having each vertex of A adjacent to each vertex of B.

(e) $G_n = H_n$ has 2n+1 vertices and consists of a K_{2n} with an additional vertex adjacent to exactly n vertices of the K_{2n}.

(f) $G_n = P_n$ is a path on n vertices (therefore has length n-1).

(g) $G_n = T_n$ is a bipartite graph with $|A| = |B| = n$ and we can label the vertices of A by $1,2,\ldots,n$ and vertices of B by $1',2',\ldots,n'$ so that vertex i is adjacent to j' iff $i \geq j$.

(h) $G_n = W_n$ is a bipartite graph with $|A| = |B| = n$ and we can label the vertices of A by $1,2,\ldots,n$ and the vertices of B by $1',2',\ldots,n'$ so that vertex i is adjacent to vertex j' iff $i \neq j$.

(i) $G_n = nK_2$

(j) $G_n = nK_3$

(k) $G_n = nP_3 = nK_{1,2}$

(l) $G_n = 2K_n$

(m) $G_n = \overline{nK_2}$

(n) $G_n = K_n \cup \overline{K}_n$.

(o) $G_n = K_n + \overline{K}_n$.

(p) $G_n = 2K_n + K_1$.

(q) $G_n = K_n - \{edge\} = K_{n-2} + \overline{K}_2$.

If $G_1, G_2 \in \{K_n, \overline{K}_n\}$ have adjacency matrices $A(G_1)$ and $A(G_2)$ respectively, $V(G_1) \cap V(G_2) = \phi$ and G_3 is a bipartite graph with $|A| = |B| = n$ having adjacency matrix $A(G_3) = \begin{bmatrix} 0 & B(G_3) \\ B^t(G_3) & 0 \end{bmatrix}$ then the graph $\underline{G = (G_1 \cup G_2) \dotplus G_3}$ is the graph with adjacency matrix

$$\begin{bmatrix} A(G_1) & B(G_3) \\ B^t(G_3) & A(G_2) \end{bmatrix}.$$

(r) $G_n = (K_n \cup \overline{K}_n) \dotplus nK_2$.

(s) $G_n = (K_n \cup \overline{K}_n) \dotplus T_n$.

(t) $G_n = (K_n \cup \overline{K}_n) \dotplus W_n$.

(u) $G_n = 2K_n \dotplus nK_2 = \overline{W}_n$.

(v) $G_n = 2K_n \dotplus T_n$.

If G_1, G_2 and G_3 are bipartite graphs with disjoint vertex sets having adjacency matrices:

$$A(G_1) = \begin{bmatrix} 0 & B(G_1) \\ B^t(G_1) & 0 \end{bmatrix} \qquad A(G_2) = \begin{bmatrix} 0 & C(G_2) \\ C^t(G_2) & 0 \end{bmatrix} \qquad A(G_3) = \begin{bmatrix} 0 & D(G_3) \\ D^t(G_3) & 0 \end{bmatrix}$$

then providing the partitions of the matrices are the right size (usually nxn) the graph $\underline{G = (G_1 \cup \overline{K}_n) \dotplus_{31} G_2}$ has adjacency matrix at the right.

$$A(G) = \begin{bmatrix} 0 & B(G_1) & C^t(G_2) \\ B^t(G_1) & 0 & 0 \\ C(G_2) & 0 & 0 \end{bmatrix}$$

The adjacency matrix of the graph $\underline{G = [(G_1 \cup \bar{K}_n) \dotplus_{31} G_2] \dotplus_{23} G_3}$ is as at the right.

$$A(G) = \begin{bmatrix} 0 & B(G_1) & C^t(G_2) \\ B^t(G_1) & 0 & D(G_3) \\ C(G_2) & D^t(G_3) & 0 \end{bmatrix}.$$

With a particular correspondence between vertices of G_1 and those of G_2 or G_3 in mind, these graphs are well defined.

(w) $G_n = (K_{1,n} \cup \bar{K}_n) \dotplus_{23} nK_2$ (called an <u>n-legged spider</u>).

(x) $G_n = (T_n \cup \bar{K}_n) \dotplus_{31} T_n$.

(y1) $G_n = [(T_n \cup \bar{K}_n) \dotplus_{31} T_n] \dotplus_{23} T_n$.

(y2) $G_n = [(T_n \cup \bar{K}_n) \dotplus_{31} T_n] \dotplus_{23} K_{n,n}$.

(y3) $G_n = [(T_n \cup \bar{K}_n) \dotplus_{31} T_n] \dotplus_{23} T_n^{\neq}$ here $T_n^{\neq}$ is the graph T_n with all (i,i') edges deleted.

(y4) $G_n = [(T_n \cup \bar{K}_n) \dotplus_{31} T_n] \dotplus_{23} T_n^r$ here T_n^r has the defining relationship of T_n reversed, i.e., i adjacent to j' iff $i \geq j$ (rather than $i \leq j$).

(y5) $G_n = [(T_n \cup \bar{K}_n) \dotplus_{31} T_n] \dotplus_{23} T_n^{r\neq}$ $T_n^{r\neq}$ is the graph T_n^r with the edges (i,i') deleted.

3. The Problem for Well-known Invariants.

3.1. Monotonic Functions.

We begin with a result which follows from one application of Ramsey's Theorem.

Definition 7. The degree of a vertex is the number of edges incident with it. $\Delta(G)$ is the maximum degree of all vertices in G.

Theorem 1. $\Delta(G)$ is a Ramsey function with chains (a) and (c).

Proof. If $\Delta(G) \geq R(n)$ consider $v \in V(G)$ with degree$(v) \geq R(n)$. Let $X = \{w \mid (v,w) \in E(G)\}$. $|X| \geq R(n)$. Apply Lemma 2 to X and we find n of the vertices of X are mutually adjacent giving graph type (a), or n of the vertices of X are mutually nonadjacent. These n vertices together with v give an induced subgraph of type (c). Also $\Delta(G) \to \infty$ on the chains (a) and (c), thus $\Delta(G)$ is a Ramsey function.

The chromatic index or line chromatic number of G is the minimum number of colors needed to color the edges of G so that adjacent edges have different colors.

Corollary 1. The line chromatic number is a Ramsey function with chains (a) and (c).

Proof. This follows from Theorem 1 using Vizing's relation (see [9, p.133]): $\Delta(G) \leq$ line chromatic number$(G) \leq \Delta(G)+1$.

A matching is a set of mutually non-incident edges from E(G). The line independence number, match(G), is the number of edges in a maximum matching in G.

Theorem 2. Match(G) is a Ramsey function with chains (a), (d) and (i).

Proof. It is easy to check that each of these chains is a match-chain. We show that match(G) is a Ramsey function by showing that if match(G) $\geq$ R(R(Q(2n))) then $t_F(G) \geq n$, where F consists of the chains (a), (d), and (i). Consider a matching in G of size R(R(Q(2n))) and label the vertices so that the matching edges are (v_i, v_i'), $i=1,2,\ldots,R(R(Q(2n)))$. The v_i and v_j' are all distinct. Apply Lemma 2 to the v_i and either R(Q(2n)) of the v_i are mutually adjacent giving (a) or R(Q(2n)) are mutually nonadjacent. In the latter case assume without loss of generality the the mutually nonadjacent v_i have lowest indices. Lemma 2 applied to $v_1', v_2', \ldots, v'_{R(Q(2n)}$ implies either Q(2n) of them are mutually adjacent giving (a) or Q(2n) of them are mutually nonadjacent. In the latter case again assume lowest indices. Consider the square (0,1)incidence matrix M of size Q(2n) where row i and column j correspond to v_i and v_j' respectively with $m_{ij}=1$ iff v_i is adjacent to v_j'. M has all 1's on the diagonal. Lemma 4 applied to M implies M has a principal submatrix M' of order 2n having form I, J, T, or T^t. If M'=I, (i) appears. If M'= J, (d) appears. If M'= T consider the subgraph induced by the last n rows and first n columns of M'. This subgraph is (d). If M'= T^t, taking the first n rows and last n columns we see that (d) appears. Since each (a), (d), and (i) mentioned above is of size at least n, $t_F(G) \geq n$ and match(G) is a Ramsey function.

The point covering number of G is the minimum number of vertices from G such that every edge is incident with one of these vertices.

Corollary 2. The point covering number is a Ramsey function with chains (a), (d), and (i).

Proof. This follows from Theorem 2 since for any sequence of graphs $S=\{G_1,G_2,\ldots\}$, $\sup_{G_i\in S}$ (point covering number$(G_i)) = \infty$ iff $\sup_{G_i\in S} \text{match}(G_i)=\infty$.

Definition 8. If $X\subset V(G)$, $\underline{\Gamma(X)}=\{v \mid (v,x)\in E(G),\ x\in X\}$. If $A\subset E(G)$, $\underline{\Gamma(A)}=\{(v,w)\in E(G) \mid \ z\neq v,w \ni (v,z)\in A \text{ or } (z,w)\in A\}$. The edge absorption number of G is the minimum number of edges in a set $A\subset E(G)$ such that $A\cup\Gamma(A)=E(G)$.

Corollary 3. The edge absorption number is a Ramsey function with chains (a), (d), and (i).

The line covering number, $\underline{\alpha_1(G)}$, is the minimum number of edges from G such that every vertex is incident with one of these edges.

note: The line covering number is not defined for graphs with isolated points thus we restrict our consideration in our next theorem.

For any graph G the line graph of G, L(G), has vertex set corresponding to the edges of G and two distinct vertices of L(G) are adjacent if and only if the edges that they correspond to in G are incident.

The line graph enables us to apply Ramsey's Theorem to the edges of G in terms of them being mutually incident or mutually non-incident by applying Ramsey's Theorem to the vertices of L(G) and interpreting the result for the edges of G. We use this idea in the following:

Theorem 3. For the class of all graphs without isolated points, the line covering number, $\alpha_1(G)$, is a Ramsey function with chains (a), (c), and (i).

Proof. Each of the above chains is an α_1-chain. We show that $\alpha_1(G)$ is Ramsey by showing $\alpha_1(G) \geq R(R(R(Q(n))))$ implies $t_F(G) \geq n$, where F consists of the chains (a), (c), and (i). Consider a minimum line cover of G with at least $R(R(R(Q(n))))$ edges. Apply Lemma 2 to these edges (or more properly to the corresponding vertices of L(G)) and we find either $R(R(Q(n)))$ of these edges are mutually incident (case 1) or $R(R(Q(n)))$ of the edges are mutually disjoint (case 2).

Case 1. Call the point of mutual incidence x and call the other ends of these edges endpoints. Applying Lemma 2 to the endpoints we find either $R(Q(n)) \geq n$ of them are mutually adjacent or $R(Q(n)) \geq n$ of them are mutually nonadjacent. In the former case (a) appears. In the latter case x, together with the n mutually nonadjacent endpoints gives us (c).

Case 2. Here we have a matching of size $R(R(Q(n)))$. The proof of Theorem 2 shows that either (a) or (i) appears, each of size at least n or G contains an induced subgraph which has an adjacency matrix which can be put in the form:

$$\begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & T^t \\ T & 0 \end{bmatrix},$$

where J, T, and T^t are of order n. In either case consider the principal submatrix determined by row 1 and rows n+1 through 2n. This matrix has the form at the right and

$$\left[\begin{array}{c|c} 0 & 11\cdots 1 \\ \hline 1 & \\ 1 & 0 \\ \vdots & \\ 1 & \end{array}\right]$$

is square of order n+1. This is the adjacency matrix of (c) of size n. In all cases $t_F(G) \geq n$. Thus $\alpha_1(G)$ is a Ramsey function.

<u>Theorem 4</u>. The number of edges in the line graph of G, $|E(L(G))|$, (or equivalently, the number of adjacencies among edges of G) is a Ramsey function with chains (a), (c), (j), and (k).

<u>Proof</u>. We assume no isolated vertices or edges since these contribute nothing to $|E(L(G))|$. Each mentioned chain is a $|E(L(G))|$-chain. We show that $|E(L(G))| \geq$ $R(R(R(Q(R(R(Q(2Q(n+1)-1)))))))(R(R(R(Q(R(R(Q(2Q(n+1)-1)))))))-1)$ implies $t_F(G) \geq n$, thereby establishing the theorem. Thus suppose $|E(L(G))|$ is as large as just stated. Then $|E(G)| =$ $|V(L(G))| \geq R(R(R(Q(R(R(Q(2Q(n+1)-1)))))))$. This follows from the fact that a graph with p(p-1) edges has at least p vertices. Applying Lemma 2 to the edges of E(G) (or more properly, to the vertices of L(G)) we find either $R(R(Q(R(R(Q(2Q(n+1)-1))))))$ of the edges are mutually incident (case 1) or that same number are mutually disjoint (case 2).

<u>Case 1</u>. The point of mutual incidence is of degree at least $R(R(Q(R(R(Q(2Q(n+1)-1))))))$. Thus Theorem 1 implies (a) or (c) appears.

<u>Case 2</u>. Here we have a matching in G of size at least $R(R(Q(R(R(Q(2Q(n+1)-1))))))$. Theorem 3, part 2 implies (a) or (c) appears or G contains the graph (i) of size at least $R(R(Q(2Q(n+1)-1)))$. In this latter case we have an induced $R(R(Q(2Q(n+1)-1)))K_2$. We assumed no isolated edges so each of these K_2's is incident with another edge. Apply Lemma 2 to

these new edges (or their corresponding vertices in L(G)) and find R(Q(2Q(n+1)-1)) of them are mutually incident or R(Q(2Q(n+1)-1)) of the new edges are mutually disjoint. In the former case (c) appears. In the latter case we find R(Q(2Q(n+1)-1)) P_3's (not induced). Assume these are lined up and the vertices are labeled as in Figure 1. (v_i, w_i) are edges from the first matching and (w_i, x_i) are new edges. Apply Lemma 2 to the x_i. In case of

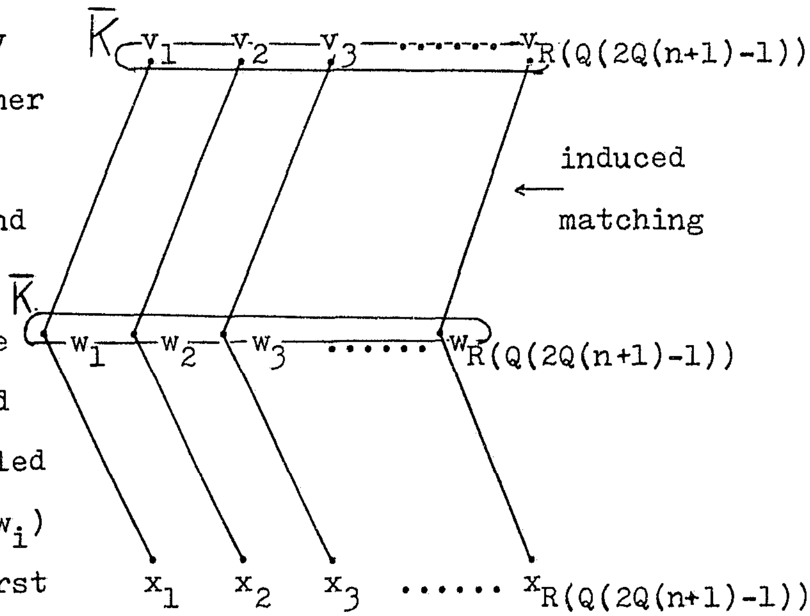

Figure 1

mutual adjacency we find (a). Otherwise, Q(2Q(n+1)-1) of the x_i are mutually nonadjacent. Assume without loss of generality that these are the x_i with lowest indices. By the proof of Theorem 3, part 2, in considering the matching edges (w_i, x_i) we find (c) or we find a graph (i) induced in G where the edges of (i) are of the form (w_j, x_j). Assume that the w_j involved have lowest indices. We thus have 2Q(n+1)-1 P_3's (not induced), each being a path (v_i, w_i, x_i). The only possible additional adjacencies are between v_i and x_j vertices, as the upper and lower matchings are induced (see Figure 2). Either Q(n+1) of the edges edges (v_i, x_i) appear (case 2a) or Q(n+1) of the edges (v_i, x_i) are missing. (case 2b). (Assume lowest indices).

Case 2a. Case 2 of Theorem 3 implies we get either (c) appearing or (i) of order n consisting of edges (v_i, x_i). The latter case gives triangles consisting of (v_i, x_i, w_i). Thus (j) appears.

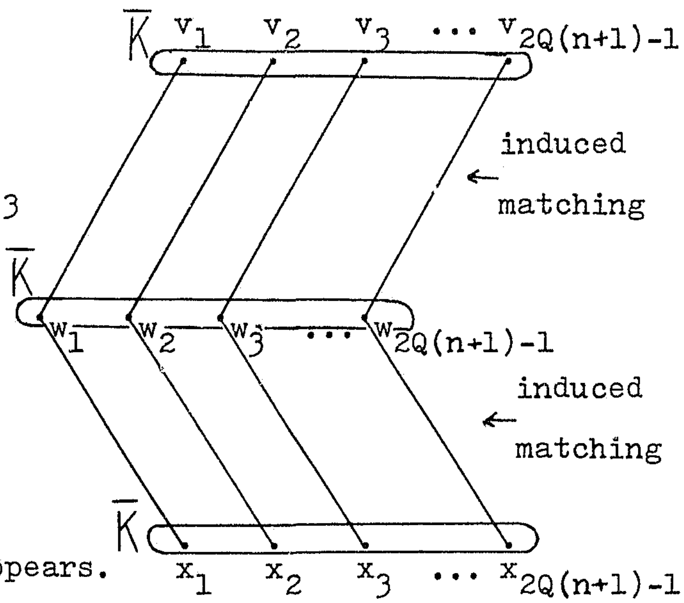

Figure 2

Case 2b. Consider the square (0,1)-incidence matrix $\overline{M}$ (for $\overline{G}$) where row i and column i correspond to v_i and x_i respectively. $\overline{m}_{ij}=1$ iff v_i is not adjacent to x_j. $\overline{M}$ is of order $Q(n+1)$ and has all 1's on the diagonal. Lemma 4 implies $\overline{M}$ has a principal submatrix $\overline{M}'$ of order n+1 having form I, J, T, or T^t. These correspond respectively to incidence matrices M' (for G) of order n+1 and form J-I, 0, T^t-I, and T-I. (We get these "complementary" matrices by interchanging the roles of 0 and 1). If M' = J-I or T^t-I, we consider the graph whose vertices correspond to the first row of M' and all columns except the first column of M'. This graph is (c) and is of size n. If M' = T-I, we consider the graph whose vertices correspond to the first column of M' and all rows except the first row of M'. This graph is (c) of size n. If M' = 0, n+1 of the paths P_3 are induced in G, thus (k) appears. Thus $|E(L(G))|$ is a Ramsey function.

3.2. Non-Monotonic Functions, f, and Corresponding Functions f'.

Ramsey functions are required to be monotonic. For certain interesting non-monotonic graph functions, f, we consider an intimately related monotonic function, f', and ask whether the function f' is a Ramsey function.

Definition 9. For a given non-monotonic function, f, its related monotonic function $f'(G) = \max_{H \subset G} f(H)$.

Odd(G) is the number of vertices of odd degree in G.

Corollary 4. Odd'(G) is a Ramsey function with chains (a), (c), and (i).

Proof. Each of the above mentioned chains is an Odd'-chain. If $\mathrm{Odd}'(G) \geq R(R(R(Q(n))))$ then $t_F(G) \geq n$. Indeed, $\mathrm{Odd}'(G) \geq R(R(R(Q(n))))$ implies $\exists\, H \subset G$ where $\mathrm{Odd}(H) \geq R(R(R(Q(n))))$ and $|E(H)| \geq R(R(R(Q(n))))$. It follows from the proof of Theorem 3 that $t_F(H) \geq n$. But $H \subset G$ implies $t_F(G) \geq n$ where F consists of the chains (a), (c), and (i).

A component of a graph is a maximal subgraph for which there is a path between every two points. $v \in V(G)$ is a cutpoint if the removal of v (and all its incident edges) increases the number of components.

The following are easy consequences of the definition of a cutpoint:

(1) For any cutpoint v_i there exist points u_i and l_i which are adjacent to v_i such that v_i is on every path from u_i to l_i (u_i-l_i path condition).

(2) Given n mutually adjacent cutpoints $v_i \in V(G)$ there exists an x_i adjacent to v_i for each i such that $(x_i, v_j) \notin E(G)$, $i \neq j$, and the x_i are mutually nonadjacent.

(3) Given n mutually nonadjacent cutpoints $v_i \in V(G)$, all adjacent to the same point z, then there exist points x_i distinct from z such that $(x_i, v_i) \in E(G)$ for each i and (x_i, z), (x_i, x_j), $(x_i, v_j) \notin E(G)$.

Let Cut(G) = the number of cutpoints in G.

Theorem 5. Cut'(G) is a Ramsey function with chains (k), (r), and (w).

Proof. The chains (k), (r), and (w) are Cut'-chains since $Cut'(G_i) \to \infty$ for each chain. We establish that Cut'(G) is a Ramsey function by showing that $Cut'(G) \geq R(nQ(R(Q(R(Q(n))))))$ implies $t_F(G) \geq n$. If Cut'(G) is as stated then there exists $H \subset G$ with $Cut(H) \geq R(nQ(R(Q(R(Q(n))))))$. Apply Lemma 2 to the cutpoints of H and find $nQ(R(Q(R(Q(n)))))$ of them are mutually adjacent of that same number of cutpoints are mutually nonadjacent. In the former case (2) implies (r) appears. In the latter case consider points u_i and l_i adjacent to each of the mutually nonadjacent cutpoints v_i so that v_i is on every u_i-l_i path. We thus have $nQ(R(Q(R(Q(n)))))$ paths of length 2, with a cutpoint in the center of each path. Either n of the paths are mutually incident (via endpoints) or $Q(R(Q(R(Q(n)))))$ of the paths are mutually disjoint. In the former case call the point of mutual incidence z. (3) implies that (w) appears. In the latter case consider the $Q(R(Q(R(Q(n)))))$ paths lined up in a row with u_i on top, v_i in the center, and l_i on the bottom. These define upper and lower matchings (see Figure 3). Recall that the v_i are mutually nonadjacent. We check for additional adjacencies. Let the remaining v_i have labels $i=1,2,\ldots,Q(R(Q(R(Q(n)))))$. Let M be the square (0,1)-incidence matrix where row i corresponds to v_i and column i corresponds to u_i. M has all 1's on the diagonal. Lemma 4 implies M has a principal submatrix, M', of order $R(Q(R(Q(n))))$ having form I, J, T, or T^t. If $M' = J$, T, or T^t then one of the u_i's is adjacent to n of the mutually nonadjacent v_i's. Thus (3) implies (w) appears. Therefore suppose $M' = I$. Apply Lemma 2 to the u_i corresponding to columns of M'. Either $Q(R(Q(n)))$ of the u_i are mutually adjacent or $Q(R(Q(n)))$ of the u_i are mutually nonadjacent. In the former case (r) appears. In the latter case we have an induced matching of order $Q(R(Q(n)))$ with edges (u_i,v_i). We may assume lowest indices. Let N be the square (0,1)-incidence matrix of order $Q(R(Q(n)))$ where row i and column i correspond to v_i and l_i respectively. N has all 1's on the diagonal. Lemma 4 implies N has a principal submatrix, N', of order $R(Q(n))$ with form I, J, T, or T^t.

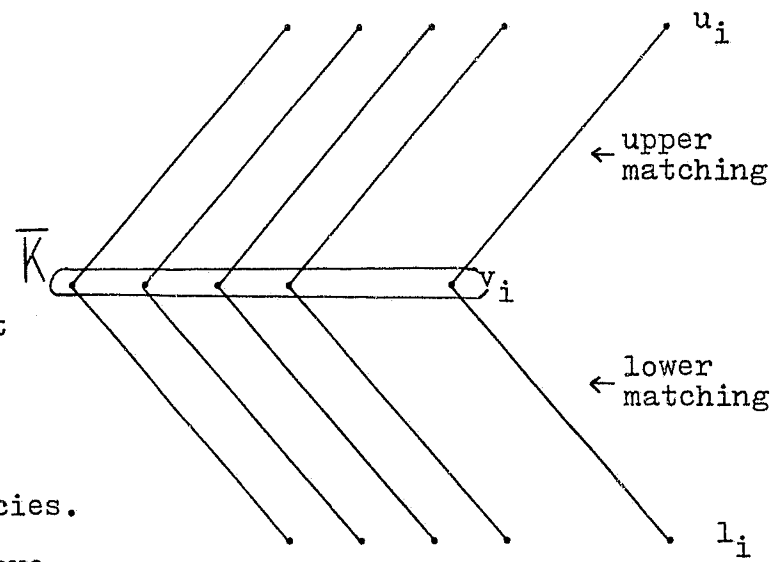

Figure 3

If $N' = J$, T, or T^t one of the l_i is adjacent to n of the mutually nonadjacent v_i. Thus (3) implies (w) appears. If $N' = I$ we apply Lemma 2 to the l_i corresponding to columns of N'. Either $Q(n)$ of the l_i are mutually adjacent or $Q(n)$ of the l_i are mutually nonadjacent. In the former case (r) appears. In the latter case we have an induced matching between the v_i and l_i. Again assume lowest indices. The only possible additional adjacencies are between u_i and l_j vertices. $(u_i, l_i) \notin E(G)$ for this would violate the u_i-l_i path condition. Let M be the square (0,1)-incidence matrix of order $Q(n)$ where row i and column i correspond to u_i and l_i respectively. M has all 0's on the diagonal. The matrix $\overline{M}$ we get from M by interchanging 0's and 1's is the corresponding incidence matrix for $\overline{G}$. $\overline{M}$ has all 1's on the diagonal. Lemma 4 applied to $\overline{M}$ implies $\overline{M}$ has a principal submatrix $\overline{M}'$ of order n with form I, J, T, or T^t (for $\overline{G}$). These correspond respectively to a matrix M' (for G) with form J-I, 0, T^t-I, and T-I. If $M' = J-I$ and $n = 1$ or $M' = T^t-I$ or T-I and $n = 1$ or 2, (w) appears. otherwise these are impossible by the u_i-l_i path condition. Thus we would have $M' = 0$, which gives (k). Thus for all cases

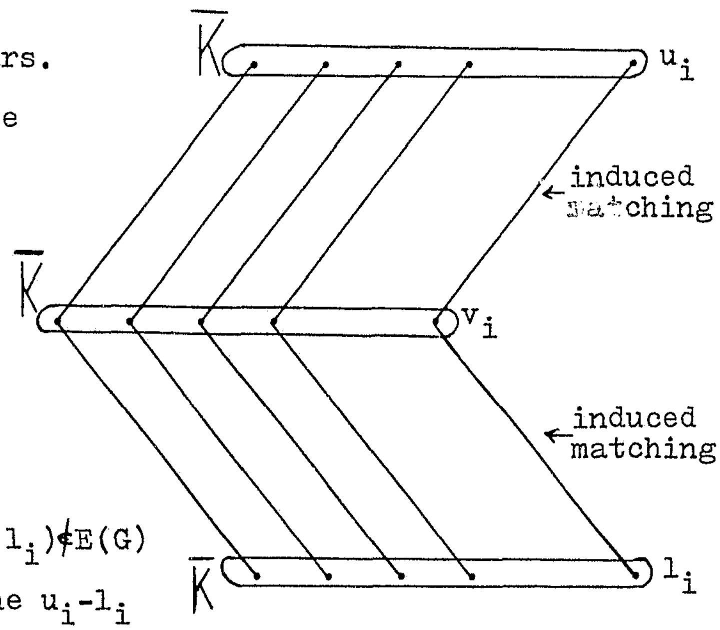

Figure 4

$t_F(G) \geq n$ and Cut'(G) is a Ramsey function.

A graph is <u>connected</u> if it has only one component. A <u>block</u> is a maximal, nontrivial connected subgraph without cutpoints. <u>Block(G)</u> is the number of blocks in G.

<u>Corollary 5</u>. Block'(G) is a Ramsey function with chains (c), (i), and (r).

<u>Proof</u>. The above chains are Block'-chains. If Block'(G) $\geq$ nR(nQ(R(Q(R(Q(n)))))) then $t_F(G) \geq n$. This follows since in some $H \subset G$ there must be either n blocks meeting at one cutpoint or Cut'(G) $\geq$ R(nQ(R(Q(R(Q(n)))))). In the former case (c) appears. In the latter case we find (k), (r) or (w) which give (i), (r), and (c) respectively.

3.3. <u>A Table of Ramsey Functions</u>.

In this section we list all the graph functions which we've found to be Ramsey functions plus several others which appear elsewhere. Any terms not defined here can be found in the cited reference or are standard and appear in [9]. When a function is not defined for all graphs, we consider the function over the largest class of graphs for which it is defined. The results below for which no reference is cited and for which the proof does not appear in this work can be gotten by very straightforward arguments. We list the f-chains in F for each of the Ramsey functions f. Dilworth number, match*, deletion number, and achr* are defined in Chapter 4, where their stated results are proved.

The <u>absorption number</u> of G is $\min_{A \subset V} |A|$ such that $A \cup \Gamma(A) = V$. (For $\Gamma(A)$ see Definition 8).

$\oplus(G)$ is the maximum number of graphs whose sum gives G.

The binding number of G is $\min_{\phi \neq X \subset V \neq \Gamma(X)} \frac{|\Gamma(X)|}{|X|}$.

Ramsey Function, f		F for f
absorption number'		(b)
achromatic number	in [7]	(a),(g),(h),(i)
achr*		(d),(i),(l),(m),(n),(o)
binding number'		(a)
bip	in [7]	(a),(g),(h),(i)
bip*	in [7]	(g),(h),(i),(m),(r),(s),(t),(u),(v)
block'		(c),(i),(r)
cut'		(k),(r),(w)
diameter'		(f)
Dilworth number		(a),(h),(i),(x),(y1),(y2),(y3),(y4),(y5)
$\Delta(G)$		(a),(c)
$\lvert E(L(G))\rvert$		(a),(c),(j),(k)
edge absorption number		(a),(d),(i)
eq	in [7]	(a),(g),(h),(i)
eq*	in [7]	(g),(h),(i),(m),(r),(s),(t),(u),(v)
$\lvert \lambda^1(G)\rvert$	in [5]	(c),(e)
line chromatic number		(a),(c)
line covering number		(a),(c),(i)
match		(a),(d),(i)
match*		(d),(i),(l),(m),(n),(o)
$\oplus'$		(a)
point covering number		(a),(d),(i)

Ramsey Function, f	F for f
point independence number	(b)
number of components'	(b)
number of bridges'	(c),(i),(r)
radius'	(f)
deletion number	(c),(k),(p),(q),(r),(u)

4. Lesser-Known Invariants.

In this chapter we give the definition of four newer graph invariants: match*, achr*, the deletion number, and the Dilworth number and establish that each is a Ramsey function.

Definition 10. The mixed achromatic number of a graph G, achr*(G), is the largest integer k such that there exists a partition $V(G)=S_1 \cup S_2 \cup \ldots \cup S_k$, where each S_i is a set of mutually nonadjacent (independent) vertices or a set of mutually adjacent vertices (a clique) but for $i \neq j$, $S_i \cup S_j$ is not an independent set and $S_i \cup S_j$ does not determine a clique.

note: This definition implies $\left|\{i \mid |S_i| = 1\}\right| \leq 1$.

We prove the following which first appeared in [7].

Theorem 6. achr*(G) is a Ramsey function with chains (d), (i), (l), (m), (n), and (o).

Proof. One may check that each of the above is an achr*-chain. We establish that achr*(G) is a Ramsey function by showing that $\text{achr}^*(G) \geq 4F(R(R(2n)))R(R(R(n)F(R(2n))))+1$ implies $t_F(G) \geq n$ where F contains the above six chains. Let $B = F(R(R(2n)))$ and $A = R(R(R(n)F(R(2n))))$. By the note following Definition 10 at most one subset in the partition of V(G) into $4BA+1$ parts has cardinality 1. Thus there are at least 2AB disjoint independent subsets S_i of V(G) each of cardinality at least 2 any pair of which are joined by an edge or there are at least 2AB disjoint cliques each of cardinality at least 2 any pair of which are joined by at least one non-edge. Assume the former. The latter case will follow directly from the argument we give for the former since by considering the complement $\overline{G}$

opposite cases hold for corresponding partitions of V(G). Note that for each achr*-chain mentioned its "complementary" chain is also an achr*-chain.

Label the independent sets $S_1, S_2, \ldots, S_{2AB}$. By hypothesis there exist vertices $v_i \in S_i$ such that $(v_1, v_2), (v_3, v_4), \ldots, (v_{2AB-1}, v_{2AB}) \in E(G)$. Consider the (0,1)-incidence matrix M with row i corresponding to v_{2i-1} and column i to v_{2i}. M has all 1's on the diagonal. Either B rows of M are all different (case 1) or A rows of M are identical (case 2).

Case 1. Consider the submatrix N of M whose B rows are all different. Applying Lemma 3, we find a square submatrix N' of N of order R(R(2n)) which (after row and column permutations were performed) has form I, J-I, or T. Applying Lemma 2 to the vertices corresponding to columns of N' we find either R(2n) are mutually nonadjacent or R(2n) of them are mutually adjacent. In either case consider the principal submatrix N" of N' corresponding to the R(2n) column vertices. Apply Lemma 2 to the row vertices of N" and find 2n of them are mutually nonadjacent or 2n of them are mutually adjacent. Consider the principal submatrix N"' corresponding to the 2n row vertices of N" . N"' has the same form as N' since we took principal submatrices. We list the possible combinations and the results of the above lemma applications:

Column vertices	Row vertices	Form of N"'	Chain appearing
nonadjacent	nonadjacent	I	(i)
"	"	T or J-I	(d)
"	adjacent	I	(n)

Column vertices	Row vertices	Form of N"'	Chain appearing
nonadjacent	adjacent	T or J-I	(o)
adjacent	nonadjacent	I	(n)
"	"	T or J-I	(o)
"	adjacent	I or T	(l)
"	"	J-I	(m)

Case 2. A rows of M are identical. Thus since there are all 1's on the diagonal, M contains a square submatrix J of order A. Apply Lemma 2 to the vertices corresponding to the columns of J and find either R(R(n)F(R(2n))) of the vertices are mutually nonadjacent or that same number are mutually adjacent. Consider the principal submatrix J' corresponding to the vertices just found. Apply Lemma 2 to the vertices corresponding to the rows of J' and find either R(n)F(R(2n)) of these vertices are mutually nonadjacent or that same number are mutually adjacent. Again consider the principal submatrix J" corresponding to the set of mutually nonadjacent (or adjacent) vertices found in applying the lemma. We consider the combination found by applying the lemma. (column vertices,row vertices) are: (mutually nonadjacent,mutually nonadjacent) then (d) appears, (mutually nonadjacent,mutually adjacent) or (mutually adjacent, mutually nonadjacent) then (o) appears. We are left to consider the case (mutually adjacent,mutually adjacent).

Recalling the note following Definition 10, each row vertex v for the matrix J" has another vertex x such that x and v are in the same independent set S_i. Consider the (0,1) incidence matrix Z where row i of Z corresponds to the row vertex corresponding to row i of J" and column i corresponds to some vertex x in the same independent set S_j as the row i vertex v. About Z we know the diagonal is all zero. Now there are R(n) rows of Z which are identical (case 2a) or F(R(2n)) rows of Z are all different (case 2b).

Case 2a. Here a submatrix Z' of Z is a 0-matrix of order R(n). Applying Lemma 2 to the vertices corresponding to columns of Z' we find either n of these vertices are mutually adjacent or n of them are mutually nonadjacent. In the former case taking these mutually adjacent x together with their corresponding v vertices from the same independent set we see (l) appears. In a similar manner for the nonadjacent x we find (n) appears.

Case 2b. Consider a submatrix of Z with F(R(2n)) rows all different. Applying Lemma 3 we find a square submatrix Z' of order R(2n) which after having permuted rows and columns has form I, J-I, or T. Apply Lemma 2 to the x vertices corresponding to columns of Z' and find 2n of the x vertices are mutually adjacent or 2n of them are mutually nonadjacent. In either case we consider the principal submatrix Z" of Z' corresponding to these vertices. Z" has form I, J-I, or T. Taking these x vertices and their corresponding mutually adjacent v-vertices from their same independent set we get the following: when the x-vertices are mutually adjacent and Z" = I or T, (l) appears. When the x are mutually adjacent and Z" = J-I, we get (m). In case of mutual nonadjacency among the x-vertices, Z" = I or T implies (n) appears, whereas Z" = J-I implies (o) appears.

<u>Definition 11</u>. Match*(G) = min $\{\text{match}(G), \text{match}(\overline{G})\}$.

The following appeared in [7].

<u>Theorem 7</u>. Match*(G) is a Ramsey function with chains (d), (i), (l), (m), (n), and (o).

<u>Proof</u>. Each of these chains is a match*-chain. We show that match*(G) $\geq F(R(R(2n)))R(nF(2n))$ implies $t_F(G) \geq n$. Let $B = F(R(R(2n)))$ and $A = R(nF(2n))$. Definition 11 implies that match(G) $\geq AB$ and match($\overline{G}$) $\geq AB$. Consider a matching labeled (v_i, w_i), $i = 1, 2, \ldots, AB$ in G. Let M be the incidence matrix where row i and column i correspond to v_i and w_i respectively. M has all 1's on the diagonal. Either B rows of M are all different (case 1) or A rows of M are identical (case 2).

<u>Case 1</u>. This is identical to case 1 of Theorem 6, thus (d), (i), (l), (m), (n), or (o) appears.

<u>Case 2</u>. M contains A identical rows. Since the diagonal of M is all 1's, we have a square submatrix J of order A. Lemma 2 applied to the vertices corresponding to the rows of J implies there is a subset V' of these vertices of cardinality nF(2n) such that V' is an independent set or V' determines a clique. Similarly there is a subset W' of the vertices corresponding to columns of J such that W' is of cardinality nF(2n) and W' is independent or W' determines a clique. If V' and W' are independent (d) appears. If V' is independent and W' determines a clique or if V' determines a clique and W' is independent then (o) appears. We are left to consider the case when both V' and W' determine a clique of order nF(2n). Recall that match($\overline{G}$) AB. This together with the fact that V' determines a clique

of order $nF(2n)$ implies that G contains a subgraph (not induced) of the form $(K_{nF(2n)} \cup \overline{K}_{nF(2n)}) \dotplus nF(2n)K_2$ (case 2a), or $(K_{nF(2n)} \cup \overline{K}_{nF(2n)}) \dotplus W_n$ (case 2b) with other possible additional adjacencies or nonadjacencies between the K and $\overline{K}$ only (i.e., the $\overline{K}$ is induced), or (l) or (m) appears.

Case 2a. Let M be the square (0,1)-incidence matrix of order $nF(2n)$ whose rows correspond to vertices of K and columns to vertices of $\overline{K}$ so that there are all 1's on the diagonal. either n rows of M are identical or $F(2n)$ rows of M are all different. In the former case M contains a square submatrix of form J and order n. This produces (o). In the latter case consider a square submatrix of order $F(2n)$ having all rows different. Lemma 3 implies that this submatrix produces a square submatrix M' of order 2n and form I, J-I, or T. If $M' = I$, (n) appears. If $M' = J-I$ or T, (o) appears.

Case 2b. This case follows directly from case 2a by considering the complement of G.

Thus in all cases $t_F(G) \geq n$ and match*(G) is a Ramsey function.

A component of a graph is complete if its points are mutually adjacent. In reading [2] we were led to formulate

Definition 12. The deletion number of a graph G, Delete(G), is the minimum number of edges which must be deleted from G so that all components become complete.

Clearly, $H \subset G$ implies $\text{Delete}(H) \leq \text{Delete}(G)$.

Lemma 7. For any $A \subseteq V(G)$ for which the graph induced by A, $\langle A \rangle$, is a clique with n of its edges in a minimum deletion there are at least n edges of G with exactly one endpoint in A.

Proof. Suppose only $m < n$ edges have only one endpoint in A. Then $\text{Delete}(\langle V-A\rangle)+n \leq \text{Delete}(G) \leq \text{Delete}(\langle V-A\rangle) + \text{Delete}(\langle A\rangle)+m = \text{Delete}(\langle V-A\rangle)+m$ which implies $n \leq m$, a contradiction. Therefore at least n edges of G have exactly one endpoint in A.

Theorem 8. Delete(G) is a Ramsey function with F consisting of the six chains: (c), (k), (p), (q), (r), and (u).

Proof. Since $\text{Delete}(G_i) \to \infty$ for each mentioned chain, the six chains are Delete(G)-chains. To prove that Delete(G) is a Ramsey function we show that Delete(G) $\geq$
R(R(R(Q(nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))))))))
implies $t_F(G) \geq n$. We assume that G has no isolated cliques since they clearly contribute nothing to Delete(G). Thus suppose Delete(G) is as large as just stated and consider a set of Delete(G) edges appearing in a minimum deletion. Applying Lemma 2 to these edges (or more properly, to the corresponding vertices of the line graph L(G)) we find either R(R(Q(nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))))))) of the edges are mutually incident (case 1) or that same number of deleted edges are mutually disjoint (case 2).

Case 1. Call the point of mutual incidence x, the edges x-edges, and call the other endpoints of the x-edges z's. Lemma 2 applied to R(2nR(R(Q(n)))) of the z's implies either 2nR(R(Q(n))) of them are mutually nonadjacent or 2nR(R(Q(n))) of the z's are mutually adjacent. In the former case (c) appears. In the latter case consider a maximal clique containing the x-edges incident with the mutually adjacent z's. This clique $\langle B\rangle$ has at least 2nR(R(Q(n))) of its edges in the deletion set. By Lemma 7 there are at least $2nR(R(Q(n)))$ edges of G having exactly one endpoint in B. If there is a point $y \notin B$ which is adjacent to n points of B then (q) appears. Thus suppose all $y \notin B$ are adjacent to at most n-1 points of B. Consider $R(R(Q(n)))$ of the edges having only one endpoint in B. Lemma 2 implies either $R(Q(n))$ of these edges are mutually incident (case 1a) or $R(Q(n))$ of these edges are mutually disjoint (case 1b).

Case 1a. The edges must be mutually incident via a point $b^* \in B$ since any $y \notin B$ is adjacent to at most n-1 points in B. Lemma 2 applied to the y points of these edges implies either Q(n) of them are mutually nonadjacent or Q(n) of them are mutually adjacent. In the former case we find (c). In the latter case the Q(n) y's can be adjacent to at most (n-1)Q(n) of the b B. Thus there are Q(n) $b \in B$ which are adjacent to none of the mutually adjacent y. n such b's, b*, and n of the y's show that (p) appears.

Case 1b. Here we have a matching between the b's and the y's. Apply Lemma 2 to the y and find either

(4) Q(n) of the y are mutually adjacent, or

(5) Q(n) of the y are mutually nonadjacent.

In either case consider the matching edges corresponding to these mutually adjacent (or nonadjacent) y. Suppose these edges are labeled (b_i, y_i), $i = 1,2,\ldots,Q(n)$. Let M be the square (0,1)-incidence matrix whose i^{th} row and i^{th} column correspond to b_i and y_i respectively. Lemma 4 implies M has a principal submatrix M' of order n having form I, J, T, or T^t.

Since each y is adjacent to at most $n-1$ b's, $M' = J$, T, and T^t are impossible. Thus $M' = I$. If (4) holds (u) appears. If (5) holds (r) appears.

Case 2. $R(R(Q(nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))))))$ of the deleted edges are mutually disjoint (giving a matching of that size). Let the deleted edges be labeled (u_i, v_i), $i = 1,2,\ldots,R(R(Q(nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))))))$. Applying Lemma 2 to the u_i we find either

(6) $R(Q(nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1))))))))))))$ of the u_i are mutually adjacent, or

(7) $R(Q(nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1))))))))))))$ of the u_i are mutually nonadjacent.

In either case we consider the v_i corresponding to the mutually adjacent (or nonadjacent) u_i. Lemma 2 implies

(8) $Q(nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))))$ of the v_i are mutually adjacent, or

(9) $Q(nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))))$ of the v_i are mutually nonadjacent.

In either case we now concentrate on the matching edges (u_i, v_i) corresponding to the mutually adjacent (or nonadjacent) v_i. We may assume the v_i have lowest possible indices. Let M be the square (0,1)-incidence matrix where row i and column i correspond to u_i and v_i respectively. M has all 1's on the diagonal since $(u_i, v_i) \in E(G)$. Applying Lemma 4 to M we see that M has a principal submatrix of order $nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1))))))))))$ having form I, J, T, or T^t. If (6) and (8) hold and $M' = I$, (u) appears.

(6), (8), and $M' = J$ we consider below as (case 2a). If (6) and (8) hold and $M' = T$ or T^t, (q) appears. If (6) and (9) hold or if (7) and(8) hold, $M' = I$ implies (r) appears. $M' = J$, T, or T^t implies (c) appears. If (7) and (9) hold $M' = J$, T, or T^t gives (c). (7), (9), and $M' = I$ gives us an induced matching of $nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))$ edges, i.e., we have that many disjoint K_2's (case 2b).

<u>Case 2a</u>. We have a complete graph on at least $4nR(R(Q(n)))$ vertices with $2nR(R(Q(n)))$ of its edges in the deletion set. By Lemma 7 and case 1 either (c), (p), (q), (r), or (u) appear.

<u>Case 2b</u>. Since there are no isolated cliques we actually have $nQ(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))$ paths of length 2. Each path is made up of a deleted edge an a "new" edge. Either n of the paths are mutually incident (via new edges) or $Q(R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))))$ of the paths are mutually disjoint. In the former case we find (c). In the latter case let the paths be labeled (u_i,v_i,l_i) where (v_i,l_i) are the new edges. The N be the square (0,1)-incidence matrix where row i and column i correspond to v_i and l_i respectively. Lemma 4 implies N has a principal submatrix N' of order $R(2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1))))))))$ with form I, J, T, or T^t. If $N' = J$, T, or T^t we find (c). Thus suppose $N' = I$. Apply Lemma 2 to the l_i corresponding to the columns of N' and we find $2Q(2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))$ of these l_i are mutually adjacent or that same number of l_i are mutually nonadjacent. In the former case (r) appears. In the latter case we consider the remaining paths (u_i,v_i,l_i). Assume lowest

indices. The only possible additional adjacencies are between u_i and l_j vertices (Figure 5). Either half of the edges (u_i, l_i) appear (case 2b1) or half of the edges (u_i, l_i) are missing (case 2b2)

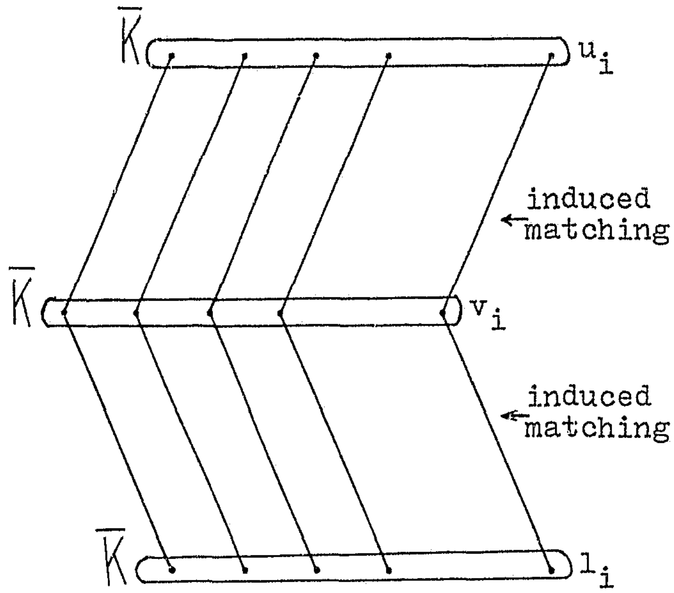

Figure 5

Case 2b1. Case 2a of Theorem 4 implies either (c) appears or we have $2nR(Q(n))R(R(R(Q(Q(R(Q(n+1)))))))$ disjoint K_3's each containing an edge from the minimum deletion. For each of these K_3's consider a maximal clique containing it. Either $nR(Q(n))$ of these cliques mutually intersect one another (case 2b1a) or $2R(R(R(Q(Q(R(Q(n+1)))))))$ of the maximal cliques are mutually disjoint (case 2b1b).

Case 2b1a. If some point is contained in n of the maximal cliques then that point together with a point from each of the disjoint K_3's in those maximal cliques shows that (c) appears. Therefore suppose any point is in at most n-1 of the cliques. Since any two of these cliques intersect and any point is in at most n-1 of the cliques, we can find $R(Q(n))$ distinct points y from different maximal cliques with all the y's adjacent to some point x in one of the cliques. Lemma 2 implies that either $Q(n)$ of the y's are mutually nonadjacent or $Q(n)$ of the y's are mutually adjacent. The former case produces (c). In the latter case suppose the y are indexed by i with y_i coming from the i^{th} maximal clique, $i=1,2,\ldots,Q(n)$. For each y_i there is a t_i in clique i distinct from y_i with t_i in the K_3 around which the i^{th} clique was built. The t_i are mutually nonadjacent. Let N be the incidence matrix between the y_i and t_i. Lemma 4 implies N has a principal submatrix N' of order n having form I, J, T, or T^t. If $N'=I$, (r) appears. If $N'=J$, T, or T^t, we find (c).

<u>Case 2b1b</u>. Since each of the mutually disjoint cliques contains an edge from the deletion set, Lemma 7 implies that the i^{th} clique C_i has a pendant edge $e_i=(p_i,a_i)$, $a_i \notin C_i$, $p_i \in C_i$. Consider $R(R(R(Q(Q(R(Q(n+1)))))))$ of the C_i so that each of their e_i is incident with no other remaining C_j. Lemma 2 implies

(10) $R(R(Q(Q(R(Q(n+1))))))$ of the e_i are mutually incident (via a_i), o

(11) $R(R(Q(Q(R(Q(n+1))))))$ of the e_i are mutually disjoint.

In either case apply Lemma 2 to the p_i in these e_i. Either

(12) $R(Q(Q(R(Q(n+1)))))$ of the p_i are mutually adjacent, or

(13) $R(Q(Q(R(Q(n+1)))))$ of the p_i are mutually nonadjacent.

When (12) holds the latter part of case 2b1a implies either (c) or (r) can be found. If (10) and (13) hold, (c) appears. Thus suppose (11) and (13) hold. Lemma 2 applied to the a_i corresponding to the mutually nonadjacent p_i implies either $Q(Q(R(Q(n+1))))$ of the a_i are mutually adjacent or $Q(Q(R(Q(n+1))))$ of the a_i are mutually nonadjacent. In the former case again the latter part of case 2b1a implies (c) or (r) appears. Thus suppose the a_i are mutually nonadjacent. We consider the incidence matrix M between these a_i and their corresponding p_i.

If on applying Lemma 4 to M we find its principal submatrix M' has form J, T, or T^t then (c) appears. If $M' = I$ we have an induced matching of Q(R(Q(n+1))) edges e_i. Since the cliques are maximal, each a_i has a b_i so that $(a_i, b_i) \notin E(G)$ where $b_i \in C_i$. For these b_i let N be the incidence matrix between b_i and p_i. If on applying Lemma 4 to N we find N' has form J, T, or T^t then (c) appears. Otherwise $N' = I$ of order R(Q(n+1)). In this latter case we apply Lemma 2 to the b_i corresponding to rows of N'. If Q(n+1) of the b_i are mutually adjacent we discover (r). If Q(n+1) of the b_i are mutually nonadjacent, we consider the paths (a_i, p_i, b_i) where the only possible additional adjacencies are between a_i and b_j, $i \neq j$. This corresponds to case 2b of Theorem 4 thus we find either (c) or (k).

Case 2b2. This also corresponds to case 2b of Theorem 4 thus (c) or (k) appears. In all cases $t_F(G) \geq n$, therefore Delete(G) is a Ramsey function.

For $v \in V(G)$, $\underline{\Gamma(v)} = \{w \mid (v,w) \in E(G)\}$. Consider the following partial order on V(G): for v, $w \in V(G)$ $v < w$ iff $\Gamma(v) \subset \Gamma(w)$. It follows that v, $w \in V(G)$ are incomparable iff $\Gamma(v) \not\subset \Gamma(w)$ and $\Gamma(w) \not\subset \Gamma(v)$.

Definition 13 [3]. The Dilworth number of a graph G, Dilworth(G), is the largest number of mutually incomparable vertices in G.

Theorem 9. Dilworth(G) is a Ramsey function with chains (a), (h), (i), (x), (y1), (y2), (y3), (y4), and (y5).

Proof. All the vertices in (a) are incomparable as are all those in (h) and (i). In any of the six other chains, all the vertices of the B-part of the first bipartite graph T_n involved are mutually incomparable. Thus Dilworth$(G_i) \to \infty$ on each of the nine chains, therefore each is a Dilworth-chain. We establish the theorem by showing that Dilworth$(G) \geq$ $R(F(R(D(R(Q(2n))))))$ implies $t_F(G) \geq n$. Let $W \subseteq V(G)$ be a set of mutually incomparable vertices such that $|W| \geq R(F(R(D(R(Q(2n))))))$. Apply Lemma 2 to these vertices and we find either $F(R(D(R(Q(2n)))))$ of them are mutually adjacent or $F(R(D(R(Q(2n)))))$ of the vertices of W are mutually nonadjacent. In the former case (a) appears. In the latter case let $W' \subset W$ be a set of $F(R(D(R(Q(2n)))))$ mutually nonadjacent, mutually incomparable vertices. Let the adjacency matrix of G, A(G), be in a form so that the first $|W'|$ rows and columns of A(G) represent vertices from W'. The upper left block of A(G) is a matrix of all 0's. Also the first $|W'|$ rows of A(G) are all different since the vertices of W' are mutually incomparable. Lemma 3 applied to the submatrix of A(G) corresponding to the first $|W'|$ rows produces a square submatrix M of order $R(D(R(Q(2n))))$ which after having permuted rows and columns has form I, J-I, or T. Applying Lemma 2 to the vertices corresponding to columns of M we find either $D(R(Q(2n)))$ of these vertices are mutually adjacent or $D(R(Q(2n)))$ of them are mutually nonadjacent. In the former case (a) appears. In the latter case consider the principal submatrix M' of M determined by the mutually nonadjacent vertices just found. M' has the same form as M. If $M' = I$ we find (i) while $M' = J-I$ gives (h). The case $M' = T$ requires further study since the information we have thus far would imply $T_n \subset G$, but Dilworth$(T_n) = 2$ for all n. We note the form of the matrix using the information we have at this point.

$$\left[\begin{array}{c|c|c} 0 & T & \\ \hline T^t & 0 & \\ \hline & & \end{array}\right]$$

Here each of the blocks we know are square of order D(R(Q(2n))). We also know that the vertices corresponding to the upper left block are mutually incomparable. Call the vertices corresponding to the upper left block W". $W'' \subset W' \subset W$. Since the $w'' \in W''$ are mutually incomparable, we know that for the upper right block

(14) Each row of the upper right block, except possibly the first, contains a zero and any upper row must contain a 1 in some column where a lower row contains a zero.

Since this condition must hold for the W" to be mutually incomparable we may now apply Lemma 6 to the upper right block. Lemma 6 implies that the upper right block has a square submatrix N of order R(Q(2n)) having form (after column permutations only) I, J-I, or T^t. In any event we apply Lemma 2 to the vertices corresponding to columns of N. If Q(2n) of these vertices are mutually adjacent we find (a). Therefore suppose Q(2n) of the column vertices are mutually nonadjacent. In this case we consider the principal submatrix N' of N corresponding to these mutually nonadjacent vertices. N' has the same form as N. If N' = I, (i) appears while N' = J-I produces (h). For the case when $N' = T^t$, we list at the right the submatrix for the information we now have. Each block is of order Q(2n) and the block in the (2,3)-position and the block in the (3,2)-position are yet to be determined.

$$\left[\begin{array}{c|c|c} 0 & T & T^t \\ \hline T^t & 0 & \\ \hline T & & 0 \end{array}\right]$$

Applying a slight variant (see [11]) of Lemma 4 we find that the (2,3)-block of the above matrix has a principal submatrix

P of order 2n having one of the forms:

$$(15)\ \begin{bmatrix} * & & & 1 \\ & * & & \\ & & \ddots & \\ 1 & & & * \end{bmatrix} \qquad (16)\ \begin{bmatrix} * & & & 1 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{bmatrix}$$

$$(17)\ \begin{bmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 1 & & & * \end{bmatrix} \quad \text{or} \quad (18)\ \begin{bmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{bmatrix} .$$

Each *entry is either a 0 or a 1. Thus n of the * are 0 or n of the * are 1. The result of these possible combinations are listed below:

Form of P	n of the *'s are	produces subgraph
(15)	0	(h)
(15)	1	(y2)
(16)	0	(y5)
(16)	1	(y4)
(17)	0	(y3)
(17)	1	(y1)
(18)	0	(x)
(18)	1	(i)

In all cases $t_F(G) \geq n$, therefore Dilworth(G) is a Ramsey function.

5. Functions which are not Ramsey Functions.

5.1. Methods of Incomparability.

In this chapter we develop methods which enable us to show that a particular function is not a Ramsey function. We then show that several particular graph invariants are not Ramsey functions. We conclude the chapter with a table of functions which we know not to be Ramsey functions.

Proposition 2. Given n and N. If for any n a_i's in P where $f(a_i) > N$ for each i, there exist $\langle b_j \rangle$ with $f(b_j) \to \infty$ and all b_j are incomparable to each a_i, then f is not a Ramsey function.

Proof. Suppose f were a Ramsey function and let the number of chains in F be n. Then given N, let a_i be the first a_{ik} in each chain with $f(a_{ik}) > N$. Suppose there exists a chain $\langle b_j \rangle$ as described above. Then by Lemma 1, one of the chains in F must be chasing the sequence $\{b_j\}$. This implies $a_i \le a_{il} \le b_l$ for some i and some b_l in the sequence $\{b_j\}$. Thus the b_j are not incomparable to all n a_i's, a contradiction. Therefore f is not a Ramsey function.

note: Proposition 2 gives a general method to establish that a function on any partially ordered set is not Ramsey. In terms of our partially ordered set of all graphs we have two "methods of incomparability" which, in conjunction with Proposition 2 enable us to show that a particular graph invariant is not a Ramsey function.

A cycle, C_n, is a sequence $v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_1$ where the v_i, $i = 1,2,\ldots,n$, are all distinct, $(v_i, v_{i+1}) = e_i \in E(G)$ $i < n$, and $(v_n, v_1) = e_n \in E(G)$. The length of a cycle is the number of edges in the cycle. The girth of a graph G, $g(G)$, is the length of the shortest cycle (if any) in G. The circumference is the length of any longest cycle. Our first incomparability method deals with the girth.

Lemma 8. If G and H are graphs, each having girth, and f is a nonnegative monotonic (nondecreasing) graph invariant, then $f(H) \geq f(G)$ and $g(H) > g(G)$ together imply that H is incomparable to G.

Proof. If H were comparable to G, f monotonic and $f(H) \geq f(G)$ would imply $G < H$. Since G is an induced subgraph of H, $g(G) \geq g(H)$, a contradiction. Thus H is incomparable to G.

The distance between two edge disjoint subgraphs of a graph is the length of the shortest path (if any) from one point in one subgraph to some point in the other. Our second method, relating to the distance between two subgraphs, is contained in our next theorem. An elementary contraction of a graph G is the identification of two adjacent points u and v. That is, we remove both u and v and insert a new point w so that w is adjacent to all points in $V(G)-\{u,v\}$ which u or v were adjacent to.

Definition 14. The contraction number of a connected graph G, Contract(G), is the minimum number of elementary contractions which must be performed on G and its resulting contracted graphs so as to form a complete graph (a clique).

Theorem 10. Contract(G) is not a Ramsey function.

Proof. Given n and $N \geq n+3$. Let G_i, $i = 1,2,\ldots,n$ be graphs with $\text{Contract}(G_i) > N$. Consider graphs consisting of $2K_m$'s separated by a single path of length p, $m \geq 3$, $p \geq 0$. We call these barbell graphs. Considering the barbell graphs for $p = 0,1,2,\ldots,n$, one of these barbell graphs is incomparable to all n of the G_i. For this B_m=($2K_m$'s separated by a path of length p), letting m be arbitrarily large makes Contract(B_m) as large as we wish while B_m remains incomparable to all the G_i. Thus by Proposition 2, Contract(G) is not a Ramsey function.

5.2. Results for Graph Invariants.

A spanning tree of a connected graph G is a minimal connected subgraph incident with every vertex of G.

note: Proposition 2 and Lemma 8, together imply that if for a given graph invariant f there exist graphs G with arbitrarily large f value and large girth, while $f(P_n)$ remains bounded then f is not a Ramsey function.

Theorem 11. Circumference(G), girth(G), and the number of spanning trees(G) are not Ramsey functions.

Proof. These follow easily from their definitions and from Proposition 2 and Lemma 8.

A trail in a graph G is a sequence $v_1,e_1,v_2,e_2,\ldots e_{n-1},v_n$ where the e_i are all distinct (the v_i need not be distinct), and $(v_i,v_{i+1})=e_i \in E(G)$.. Let Trail(G)= the length of the longest trail in G.

Theorem 12. Trail(G) is not a Ramsey function.

Proof. Given n and $N \geq 2n+2$. Consider graphs C_{mk} consisting of k cycles of length m mutually incident at a point. We call these graphs windmills. Let $\langle C_{mk} \rangle$ be a chain of windmills where m is fixed. Let G_i, $i = 1,2,\ldots,n$ be graphs with Trail(G_i) > N.

Consider the chains $\langle C_{mk}\rangle$, $3 \le m \le n+3$. Any forest G with Trail(G) large will necessarily contain an induced path of length $2n+3$. None of the windmills in the chains we are considering does, thus any forest with Trail large is incomparable to all $\langle C_{mk}\rangle$, $3 \le m \le n+3$, $k > 0$. We may assume therefore that none of the G_i are forests and will show that the tail end of one of the chains $\langle C_{mk}\rangle$ is incomparable to all the G_i. Thus suppose the G_i have girth. Then one must have girth 3 or we could use $\langle C_{3k}\rangle$ to establish Trail(G) is not Ramsey. Suppose, without loss of generality, that G_1 has girth 3. Then G_1 is incomparable to some C_{mj} for each m, $3 < m \le n+3$. For the remaining G_i, $i = 2,\ldots,n$, one must contain a cycle of length 4 or we could use the tail end of $\langle C_{4k}\rangle$ to show Trail(G) is not Ramsey. Then let G_2 have a cycle of length 4. G_2 is incomparable to some C_{mj} for each m, $4 < m \le n+3$. Continuing in this manner, by the time we reach G_n, we will either have already established that Trail is not Ramsey or we will find that G_n (and the other G_i) is incomparable to some $C_{n+3,j}$. Thus we use the tail end of $\langle C_{n+3,j}\rangle$ and show via Proposition 2 that Trail(G) is not a Ramsey function.

A graph is said to be <u>embeddable</u> on a particular surface if it can be drawn on that surface so that none of the edges intersect. The <u>genus</u> of a graph G, <u>genus(G)</u>, is the number of handles needed on a sphere so that we can embed G on the resulting surface. <u>Subdividing an edge</u> $(u,v) \in E(G)$ means deleting this edge and inserting a new point w so that (u,w) and $(v,w) \in E(G)$. Two graphs G and H are <u>homeomorphic</u> if both can be obtained from the same graph by a sequence of subdivisions of edges. Homeomorphic graphs have the same genus.

Lemma 9 (see [9, p.118]). $\text{genus}(K_n) = \left[\frac{(n-3)(n-4)}{12}\right]$.

Theorem 13. Genus(G) is not a Ramsey function.

Proof. Given n and $N > 0$, let G_i, $i = 1, 2, \dots, n$ be graphs with $\text{genus}(G_i) > N$. Let $M = \max_i \text{genus}(G_i)$. $\text{Genus}(G) > 1$ implies $g(G)$ exists. Let $L = \max_i g(G_i)$. Let K_m^k be the graph K_m with each edge subdivided k times. Consider the sequence $S = \{K_j^{L+1}\}_{j=M+3}^{\infty}$. The elements of S are incomparable to each of the G_i by Lemma 8. Also $\text{genus}(s_j) \to \infty$ by Lemma 9. Thus Proposition 2 implies that genus(G) is not a Ramsey function.

The order of a partition is the number of sets it contains. The chromatic number of a graph G, chr(G), is the smallest order of a partition of V(G) so that adjacent vertices are in different sets of the partition. For our next theorem we need:

Lemma 10 (see [9, p.129]). For all positive integers a and b there exists a graph G with $\text{chr}(G) = a$ and $g(G) > b$.

Theorem 14. The chromatic number, chr(G), is not a Ramsey function.

Proof. Given n and $N \geq 2$, let G_i, $i = 1, 2, \dots, n$ be graphs with $\text{chr}(G_i) > N$. Each G_i has girth. Let $M = \max_i \text{chr}(G_i)$. Let $L = \max_i g(G_i)$. Consider a sequence $\{H_j\}_{j=M+1}^{\infty}$ of graphs where each H_j has $\text{chr}(H_j) = j$ and $g(H_j) > L$. By Lemma 8 the H_j are incomparable to each of the G_i and clearly $\text{chr}(H_j) \to \infty$. Proposition 2 therefore implies that chr(G) is not Ramsey.

A graph is regular of degree r if all of its vertices are incident with exactly r edges. In examining whether or not a given function is a Ramsey function, several problems are rather intractable without the aid of the following powerful result of Sachs [12]:

Lemma 11. Given $r \geq 3$, $g \geq 3$, there exists a graph G which is regular of degree r and has girth g.

$\delta(G)$ = the minimum degree of the vertices of G.

Theorem 15. $\delta'(G)$ is not a Ramsey function.

Proof. Given n and $N \geq 2$, let G_i, $i = 1,2,\ldots,n$ be graphs with $\delta'(G_i) > N$. Let $r = \max_i \delta'(G_i)$. Let $M = \max_i g(G_i)$. Consider a sequence of graphs $S = \{S_j\}_{j=r}^{\infty}$ where S_j is regular of degree j and $g(S_j) = M+1$. Lemma 8 implies the S_j are incomparable to all the G_i. $\delta'(S_j) \to \infty$, thus Proposition 2 implies $\delta'(G)$ is not a Ramsey function.

A graph is a forest if it contains no cycles. The arboricity of a graph G, $\Upsilon(G)$, is the minimum number of line-disjoint spanning forests into which G can be decomposed.

Theorem 16. The arboricity, $\Upsilon(G)$, is not a Ramsey function.

Proof. Given n and $N \geq 1$. Let G_i, $i = 1,2,\ldots n$ be graphs where $\Upsilon(G_i) > N$. $g(G_i)$ exists since $\Upsilon(G_i) \geq 2$ for each i. Let $M = \max \Upsilon(G_i)$ and let $L = \max_i g(G_i)$. Let $S = \{S_j\}_{j=2M}^{\infty}$ be a sequence of graphs, regular of degree j having girth L+1. Using a formula of Nash-Williams, we find

$$\Upsilon(S_j) = \max_{A \subset S_j} \left[\frac{|E(A)|}{|V(A)| - 1}\right] \geq \left[\frac{|E(S_j)|}{|V(S_j)| - 1}\right] \geq \left[\frac{p \cdot j}{2(p-1)}\right] \geq \left[\frac{j}{2}\right] \geq M.$$

Thus by Lemma 8 all the S_j are incomparable to the G_i. Also $\Upsilon(S_j) \to \infty$, thus by Proposition 2 $\Upsilon(G)$ is not a Ramsey function.

The <u>infinite binary tree</u> is constructed as follows: begin with one vertex called the <u>root</u> said to be at level zero. The root is adjacent to two vertices at level one. Each vertex at level $k \geq 1$ is adjacent to one vertex at level $k-1$ and 2 vertices at level $k+1$ so that there are 2^k vertices at level k.

The question was raised as to whether a converse of Lemma 1 holds: if for a nonnegative monotonic function f on P, any sequence $\{S_i\}$ with $\sup_i f(S_i) = \infty$ has an f-chain chasing it, does f have to be a Ramsey function? This question was answered in the negative with the following example due to Joel Spencer and Judith Longyear: let P consist of the vertices in the infinite binary tree and $v < w$ iff v is on the path from the root to w. Let $f(v)$ be the length of the path from the root to v. For any sequence of points $\{s_i\}$ where $\sup_i f(s_i) = \infty$, we can find an f-chain chasing this sequence. However, there can be no finite set of f-chains, F, such that for any arbitrary sequence $\{S_j\}$, $\sup_j f(S_j) = \infty$ implies $\sup_j t_F(S_j) = \infty$. Thus f is not a Ramsey function.

<u>Functions, f, which are not Ramsey Functions</u>.

arboricity

chromatic number

circumference

contraction number

genus

girth

length of the longest trail

minimum degree of a vertex'

number of spanning trees.

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Autobiography

Fred Buckley was born on February 4, 1951 in Newark, New Jersey. He attended Pace University on a Full Memorial Scholarship and received a B.A., Cum Laude, in 1973. At Pace he met Wai Mui Choy, whom he married in 1975. In 1973 he began his study at the Graduate Center. During this time he taught as an adjunct lecturer in Mathematics at Lehman College for three and a half years and as an adjunct instructor in Mathematics at Pace University for a year.