Order Number 9029945

Commutation and deficiency indices: Comparing t^*t and tt^*, where t is an ordinary differential operator with smooth coefficients

Isaacs, Frank Dana, Ph.D.

City University of New York, 1990

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

A

Commutation And Deficiency Indices: Comparing t^*t And tt^*, Where t Is An Ordinary Differential Operator With Smooth Coefficients

by

Frank D. Isaacs

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1990

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

3/8/90
Date

[signature]
Executive Officer

Examining Committee:

3/8/90
Date

[signature]
Chair

3/8/90
Date

[signature]

3/8/90
Date

[signature]

Advisor: Professor Stanley Kaplan

The City University of New York

Abstract

Commutation And Deficiency Indices: Comparing t^*t And tt^*, Where t Is An Ordinary Differential Operator With Smooth Coefficients

by

Frank D. Isaacs

Advisor: Professor Stanley Kaplan

Let τ be a formal differential operator, of order n, with C^∞ coefficients, which is defined on an interval $I \subseteq \mathbf{R}$. For any interval $J \subseteq I$, let $\overline{T_0(\tau, J)}$ denote the closed minimal operator for τ operating in $L^2(J)$. Let $T_1(\tau, J)$ denote the maximal operator. If S is any symmetric operator acting in a Hilbert space, let $\mathcal{D}_\pm(S^*)$ denote the deficiency spaces of S^*, the adjoint of S. If S and T are two operators acting in a Hilbert space, and if $T \supseteq S$, let $[T : S]$ denote the dimension of the extension.

We develop information about dim $\mathcal{D}_\pm(T_1(\tau\tau^*, I))$ from information about $\overline{T_0(\tau^*\tau, I)}$, $\overline{T_0(\tau, I)}$, and $\overline{T_0(\tau^*, I)}$. Let $A = \{f \in \mathcal{D}_+(T_1(\tau^*\tau, I)) : \tau(f) \in L^2(I)\}$.

First, dim $\mathcal{D}_\pm(T_1(\tau\tau^*, I)) \geq \dim A$.

Second, $\overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}$ is a closed, symmetric, positive operator with adjoint $T_1(\tau^*, I)T_1(\tau, I)$, and

$$\begin{aligned}\dim A &= (1/2) \cdot [T_1(\tau^*, I)T_1(\tau, I) : \overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}] \\ &= [T_1(\tau, I) : \overline{T_0(\tau, I)}] \\ &= [T_1(\tau^*, I) : \overline{T_0(\tau^*, I)}].\end{aligned}$$

Third, let $I', I'' \subseteq I$ be two intervals with positive length such that $I' \cup I'' = I$, and $I' \cap I''$ is a point, c. Then $\overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}$ admits a Kodaira-type formula:

$$\begin{aligned} 4n + [T_1(\tau^*, I)T_1(\tau, I) : \overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}] &= [T_1(\tau^*, I')T_1(\tau, I') : \overline{T_0(\tau^*, I')}\,\overline{T_0(\tau, I')}] \\ &\quad + [T_1(\tau^*, I'')T_1(\tau, I'') : \overline{T_0(\tau^*, I'')}\,\overline{T_0(\tau^*, I')}], \end{aligned}$$

and $\overline{T_0(\tau^*, J)}\,\overline{T_0(\tau, J)}$, $J = I', I''$, has $4n$ boundary values at c.

Fourth, if $\sigma_e(*)$ denotes "essential spectrum," then $0 \notin \sigma_e(\tau^*\tau)$ entails $0 \notin \sigma_e(\tau)$ (whence $0 \notin \sigma_e(\tau^*\tau)$ iff $0 \notin \sigma_e(\tau)$ iff $0 \notin \sigma_e(\tau^*)$).

Fifth, we extend to arbitrary intervals a formula of Kauffman, Read, and Zettl:

$$\begin{aligned} \dim A = [T_1(\tau, I) : \overline{T_0(\tau, I)}] &\geq \dim\ker T_1(\tau^*, I) - \dim\ker \overline{T_0(\tau, I)} \\ &\quad + \dim\ker T_1(\tau, I) - \dim\ker \overline{T_0(\tau^*, I)}, \end{aligned}$$

with $0 \notin \sigma_e(\tau)$ entailing equality. In particular, when τ is formally self-adjoint,

$$\dim A = [T_1(\tau, I) : \overline{T_0(\tau, I)}] \geq 2 \cdot (\dim\ker T_1(\tau, I) - \dim\ker \overline{T_0(\tau, I)}\,).$$

Further,

$$\begin{aligned} \dim A = [T_1(\tau, I) : \overline{T_0(\tau, I)}] &\geq \dim\ker T_1(\tau^*, I') + \dim\ker T_1(\tau, I') \\ &\quad + \dim\ker T_1(\tau^*, I'') + \dim\ker T_1(\tau, I'') - 2n, \end{aligned}$$

with $0 \notin \sigma_e(\tau)$ entailing equality.

Sixth, we extend a formula of Dunford and Schwartz: if I' and I'' are as above, then

$$n + \dim\ker T_1(\tau^*, I) - \dim\ker \overline{T_0(\tau, I)} \geq \dim\ker T_1(\tau^*, I') + \dim\ker T_1(\tau^*, I''),$$

with $0 \notin \sigma_e(\tau)$ entailing equality.

Seventh, if $\dim\ker T_1(\tau^*, I) = n$, then $\dim A = \dim\ \mathcal{D}_\pm(T_1(\tau^*\tau, I))$, and $T_1(\tau^*\tau, I) = T_1(\tau^*, I)T_1(\tau, I)$.

Eighth, if $\tau\tau^*\{\ker T_1((\tau\tau^*)^2, I)\} \subseteq L^2(I)$, then $\dim A = \dim\ \mathcal{D}_\pm(T_1(\tau^*\tau, I))$, and $T_1(\tau^*\tau, I) = T_1(\tau^*, I)T_1(\tau, I)$.

Acknowledgement

Gratefully, we acknowledge the contribution of Professor Stanley Kaplan. He gave his time generously as he considered our work in its progress, and his counsel was always prompt. He allowed us the chance to search for new mathematics, and the time to turn aside from dead ends and to seek other paths. His suggestions often led to shorter proofs, and offered insight, too.

Table of Contents

Introduction

What

Let I denote an interval of the real line. Let τ denote a formal differential operator which is defined on I, and which has the form

$$\tau = a_n(t)(d/dt)^n + a_{n-1}(t)(d/dt)^{n-1} + \cdots + a_1(t)(d/dt) + a_0(t).$$

We require that $a_n(t) \neq 0$, for $t \in$ I, and that $a_0(), \ldots, a_n()$ have continuous derivatives of all orders. Let τ^* denote the formal adjoint of τ.

We consider τ, τ^*, $\tau^*\tau$, and $\tau\tau^*$ acting on various domains contained in the Hilbert space $L^2(I)$.

For any formal differential operator ζ, let $T_0(\zeta, I)$ denote the minimal operator (to be defined later) for ζ acting in $L^2(I)$. Let $\overline{T_0(\zeta, I)}$ denote the closure (to be defined later) of $T_0(\zeta, I)$, and let $T_1(\zeta, I)$ denote the maximal operator (to be defined later).

First, we develop connections between the deficiency indices of $T_0(\tau^*\tau, I)$ and those of $T_0(\tau\tau^*, I)$. These connections are related to the deficiency indices of the composition operator $\overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}$, which we show is a closed positive operator whose domain is dense in $L^2(I)$, and whose adjoint is $T_1(\tau^*, I)T_1(\tau, I)$.

Second, let I' and I'' be two sub-intervals of I, each with non-empty interior, such that $I' \cup I'' = I$, and such that $I' \cap I''$ is a single point. We relate the deficiency indices of $\overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}$ acting in $L^2(I)$, to the deficiency indices of $\overline{T_0(\tau^*, I')}\,\overline{T_0(\tau, I')}$ acting in $L^2(I')$, and to the deficiency indices of $\overline{T_0(\tau^*, I'')}\,\overline{T_0(\tau, I'')}$ acting in $L^2(I'')$.

Third, we assume that 0 does not lie in the essential spectrum (to be defined later) of τ, and examine the consequences of this for the foregoing material.

Last, we state and prove two conditions, each of which is sufficient to ensure that if g lies in the domain of $T_1(\tau^*\tau, I)$, and if it satisfies the differential equation $\tau^*\tau(v) = \pm iv$, then $\tau(g)$ (which satisfies the differential equation $\tau\tau^*(v) = \pm iv$), lies in the domain of $T_1(\tau\tau^*, I)$.

Why

All of our motivation, though none of our results, lie in the method of "commutation," which Deift [3] uses to re-prove some results of Crum [2]. For background, we sketch these.

Crum considers, in part, a regular Sturm-Liouville system

$$\begin{cases} -y'' + (q(t) - \lambda)y = 0 & 0 < t < 1 \\ y'(0) = h^0 \cdot y(0) & y'(1) = h^1 \cdot y(1) \end{cases}$$

(where $q \in C^\infty([0,1])$ and $h^0, h^1 \in \mathbf{R}$), which has eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$, and corresponding eigenfunctions $f_0, f_1, f_2, \ldots$.

From this system, he explicitly obtains a series of *associated Sturm-Liouville systems*

$$\begin{cases} -y'' + (q_n(t) - \lambda)y = 0 & 0 < t < 1 \\ \lim_{t\to 0} y(t) = 0 & \lim_{t\to 1} y(t) = 0, \end{cases}$$

where $n = 1, 2, 3, \ldots$. The potential $q_1(t) \in C^\infty([0,1])$; for $n > 1$,

$$q_n(t) \sim \begin{cases} n(n-1)t^{-2}, & \text{as } t \to 0, \\ n(n-1)(1-t)^{-2}, & \text{as } t \to 1. \end{cases}$$

Crum also obtains formulas, rational in $\{f_0, f_1, f_2, \ldots\}$, for the eigenfunctions of the associated systems, and proves that the nth system has eigenvalues $\{\lambda_n, \lambda_{n+1}, \lambda_{n+2}, \ldots\}$.

Deift re-proves the results of Crum, and relies on one aspect of "commutation:" that if T is a closed, densely defined, operator in a Hilbert space, and if T^* is its adjoint, then T^*T and TT^* are densely defined, non-negative, self-adjoint operators (see [12, p. 312], or [4, p. 1245]), and that if

$\sigma(\dots)$ denotes "spectrum of ...," then

$$\sigma(T^*T) \setminus \{0\} = \sigma(TT^*) \setminus \{0\}$$

(see [3, p. 270,273]).

That Deift uses commutation as the "pivot" (his word) in his re-proof of Crum's results, prompts us to consider the relation between deficiency indices, and commuting, for the formal differential operators $\tau^*\tau$ and $\tau\tau^*$.

Chapter 1

Preliminaries

Definition 1 1. Let $\dot{+}$ denote direct sum.

2. Let $\oplus$ denote orthogonal sum.

3. Let $\mathbf{R}$ denote the real line.

4. Let $\mathbf{C}$ denote the complex plane.

5. Very often, we will use $\mathbf{1}$ to denote the identity operator on a vector space.

6. Let I denote an interval of the real line, $\mathbf{R}$. I has end points a, b where $-\infty \leq a < b \leq \infty$. If a satisfies $-\infty < a < \infty$, and $a \in I$, we say that a is a *fixed* end point of I. Otherwise, a is a *free* end point of I.

7. Let $C^n(I)$ denote the set of complex-valued functions which are n times continuously differentiable on I, for $n = 1, 2, \ldots, \infty$. If I has a fixed end point a, then $f \in C^n(I)$ means that

$$\lim_{t \in I, t \to a} \frac{f^{(j)}(a) - f^{(j)}(t)}{a - t} = \lim_{t \in I, t \to a} f^{(j+1)}(t),$$

for all $j \leq n$, if n is finite, and for all j, if n is infinite. If $n = 0$, we omit n, and write $C(I)$.

8. Let $C_0^n(I)$ denote that subset of $C^\infty(I)$ whose functions vanish in a neighborhood of each end point of I.

9. If n is a positive integer, let $W_0^n(I)$ be the set of functions f which satisfy

   - $f \in L^2(I) \cap C_0^{(n-1)}(I)$,
   - $f^{(n-1)}$ is absolutely continuous, and
   - $f^{(n)} \in L^2(I)$.

10. If n is a positive integer, let $W^n(I)$ be the set of functions f which satisfy

    - $f \in L^2(I) \cap C^{n-1}(I)$,
    - $f^{(n-1)}$ is absolutely continuous, and
    - $f^{(n)} \in L^2(I)$.

11. If n is a positive integer, let τ denote a formal differential operator of the form

$$a_n(t)(d/dt)^n + a_{n-1}(t)(d/dt)^{n-1} + \cdots + a_1(t)(d/dt) + a_0(t),$$

where $a_n(t) \neq 0$ for $t \in I$, and $\{a_0(*), \ldots, a_n(*)\} \subseteq C^\infty(I)$. We say that τ is a *formal differential operator (of order n)*.

12. Let τ be a formal differential operator of order n defined on $I \subseteq \mathbf{R}$. If n is a positive integer, let $W_\tau^n(I)$ be the set of functions f which satisfy

    - $f \in L^2(I) \cap C^{n-1}(I)$,
    - $f^{(n-1)}$ is absolutely continuous, and
    - $\tau(f) \in L^2(I)$.

13. For $f, g \in L^2(I)$, let $\langle f, g\rangle$ denote $\int_I f(t) \cdot \overline{g(t)}\, dt$. Sometimes, when we want to emphasize the interval, we write $\langle f, g\rangle_I$.

14. Let τ be a formal differential operator of order n which is defined on an interval $I \subseteq \mathbf{R}$. Let $T_0(\tau)$ denote the operation τ with domain $W_0^n(I)$. We call $T_0(\tau)$ the *minimal operator for τ on I*. Sometimes we write $T_0(\tau, I)$ to emphasize the interval.

15. Let τ and I be as above. Let $T_1(\tau)$ denote the operation τ with domain $W_\tau^n(I)$. We call $T_1(\tau)$ the *maximal operator for τ on I*. Sometimes we write $T_1(\tau, I)$ to emphasize the interval.

Definition 2 Let T, with domain $\mathcal{D}(T)$, be an operator in a Hilbert space. T is *closed* if $\{f_n\} \subseteq \mathcal{D}(T)$, $f_n \to f$, and $T(f_n) \to g$ entail $f \in \mathcal{D}(T)$ and $T(f) = g$.

Theorem 3 From [12, p. 305]. *Let T, with domain $\mathcal{D}(T)$, be an operator in a Hilbert space. Then $\mathcal{D}(T^*)$ is dense if, and only if, T has a closed extension. $(T^*)^*$ is the smallest closed extension of T.*

Remark 4 Let $\mathcal{H}$ be a Hilbert space, with inner product $\langle f, g\rangle$. Let T, with domain $\mathcal{D}(T)$, be an operator in $\mathcal{H}$. For $f, g \in \mathcal{D}(T)$, define

$$[f, g] = \langle f, g\rangle + \langle T(f), T(g)\rangle.$$

Then T is a closed operator if, and only if, $\mathcal{D}(T)$, with inner product $[f, g]$, is a Hilbert space (see [4, p. 1186]). We sometimes call $[f, g]$ the *operator inner product for T*, and call

$$[[f]] = \{[f, f]\}^{1/2}$$

the *operator norm for T*. Sometimes we write $[f, g]_T$ and $[[f]]_T$ to emphasize the operator.

Definition 5 If T is an operator in a Hilbert space, and if T has closed extensions, let $\overline{T}$ denote the smallest closed extension of T. This is the *closure* of T.

Proposition 6 From [4, first paragraph of proof of XIII.6.2]. *Let $\mathcal{H}$ be a Hilbert space, let S be a subspace of H, and let F be a finite dimensional subspace of $\mathcal{H}$. Then $S + F$ is closed.*

Lemma 7 From [4, XIII.2.9]. *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$. Let f be a function whose square is integrable over every compact interval of*

I. Suppose that

$$\int_I f(t) \cdot \overline{\tau^*(\phi)(t)}\, dt = 0$$

for all $\phi \in W_0^n(I)$. Then (perhaps after modification on a set of measure 0) $f \in C^\infty(I)$ and $\tau(f) = 0$.

Theorem 8 From [4, XIII.2.10]. *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Let τ^* denote the formal adjoint of τ. Then*

$$T_0(\tau)^* = T_1(\tau^*).$$

Corollary 9 *Let $I \subseteq \mathbf{R}$ be an interval. Let τ be a formal differential operator defined on I. Then*

$$\overline{T_0(\tau)} = T_1(\tau^*)^*.$$

Proof: $T_0(\tau)^* = T_1(\tau^*)$, by theorem 8. $T_1(\tau^*)^*$ is well defined, since $\mathcal{D}(T_1(\tau^*))$ is dense in $L^2(I)$. Thus $T_1(\tau^*)^* = \overline{T_0(\tau)}$, by theorem 3. Q.E.D.

Corollary 10 *Let τ be a formal differential operator which is defined on an interval $J \subseteq \mathbf{R}$. Let $T_{00}(\tau)$ denote the operation τ with domain $C_0^\infty(J)$. Then $T_{00}(\tau)^* = T_1(\tau^*)$, and $\overline{T_{00}(\tau)} = \overline{T_0(\tau)}$.*

Proof: Let n denote the order of τ. In order to prove that $\overline{T_{00}(\tau)} = \overline{T_0(\tau)}$, it is enough to prove that $C_0^\infty(J)$ is dense, in the operator norm, in $\mathcal{D}(\overline{T_0(\tau)})$. In order to show *this,* it is enough to show that the elements of $W_0^n(J)$ can be approximated arbitrarily closely, in the operator norm, by elements of $C_0^\infty(J)$.

Choose $f \in W_0^n(J)$. The support of f is a compact set which lies in the interior of J. We seek $\{f_m\} \subseteq C_0^\infty(J)$ such that

$$f_m \rightarrow f, \tag{1.1}$$

$$\text{and } \tau(f_m) \rightarrow \tau(f). \tag{1.2}$$

Select a compact interval $I = [a, b]$ such that the support of f lies in the interior of I, and such that I lies in the interior of J. Evidently, τ is defined on I. Since I is compact, it is enough that $\{f_m\}$ satisfy $f_m^{(i)} \rightarrow f^{(i)}$, for i $= 0, 1, \ldots, n$.

Select $\phi \in C_0^\infty([-1,1])$ to be a positive function such that $\int_{-1}^{1} \phi(x)\,dx = 1$. We define $\phi_m(x) = m \cdot \phi(mx)$. The support of ϕ_m lies in the interior of $[-1/m, 1/m]$. We define

$$\begin{aligned} f * \phi_m(x) &= \int_{\mathbf{R}} f(s) \cdot \phi_m(x-s)\,ds \\ &= \int_{\mathbf{R}} f(x-s) \cdot \phi_m(s)\,ds. \end{aligned}$$

According to [8, p. 3, theorem 1.2.1],

- $(f * \phi_m)^{(i)}(x) = f * \phi_m^{(i)}(x)$, for $i = 0, 1, \ldots$;
- for large enough m, $f * \phi_m \in C_0^\infty(I)$; and
- $f * \phi_m \to f$ in the $L^2(I)$ norm.

Let M be large enough that $f * \phi_M \in C_0^\infty(I)$. We define $f_m(x) = f * \phi_{m+M}(x)$, for $m = 1, 2, \ldots$. For $i = 0, 1, \ldots, n$, we use integration by parts [12, p.54], and the fact that $0 = f^{(i)}(a) = f^{(i)}(b)$, to see that

$$\begin{aligned} f_m^{(i)}(x) &= f * \phi_{m+M}^{(i)}(x) \\ &= f^{(i)} * \phi_{m+M}. \end{aligned}$$

Thus, in order to show that $f_m^{(i)} \to f^{(i)}$, for $i = 0, 1, \ldots, n$, it is enough to see that if $g \in W_0^n(I)$, then $g * \phi_m \to g$. But this is the last result of [8, p.3, theorem 1.2.1] which we cited.

Now, we show that $T_{00}(\tau)^* = T_1(\tau^*)$. Let $g \in \mathcal{D}(T_{00}(\tau)^*)$. Then there is a positive constant c such that, for all $\phi \in C_0^\infty(J)$,

$$|\langle \tau(\phi), g \rangle| \leq c \cdot \|\phi\|.$$

Let $f \in W_0^n(J)$. Since we have just shown that $\overline{T_{00}(\tau)} = \overline{T_0(\tau)}$, we can find $\{f_m\} \subset C_0^\infty(J)$ such that relations 1.1 and 1.2 are true. Thus,

$$\begin{aligned} |\langle \tau(f), g \rangle| &= \lim |\langle \tau(f_m), g \rangle| \\ &\leq c \cdot \lim \|f_m\| \\ &= c \cdot \|f\|. \end{aligned}$$

Thus, $g \in \mathcal{D}(T_0(\tau)^*) = \mathcal{D}(T_1(\tau^*))$. Q.E.D.

Definition 11 Let S be a densely defined operator in a Hilbert space, such that $S \subseteq S^*$. We call S *symmetric*.

Lemma 12 From [4, XII.4.6(a)]. *Let S be a symmetric operator in a Hilbert space. Then $\overline{S}$ is symmetric.*

Definition 13 Let S be a symmetric operator in a Hilbert space. We define

$$\mathcal{D}_\pm = \{f \in \mathcal{D}(S^*) : S^*(f) = \pm i f\}.$$

Sometimes we write $\mathcal{D}_\pm(S^*)$, to emphasize the operator. Dim $\mathcal{D}_+$, and $\dim \mathcal{D}_-$, are respectively the *positive*, and *negative, deficiency indices* of S.

Lemma 14 From [4, XII.4.10]. *Let S be a symmetric operator in a Hilbert space.*

1. *$\mathcal{D}(\overline{S})$, $\mathcal{D}_+$, and $\mathcal{D}_-$ are subspaces of the Hilbert space which consists of $\mathcal{D}(S^*)$ with the inner product*

$$[f, g] = (f, g) + (S^*(f), S^*(g)).$$

2. *$\mathcal{D}(S^*) = \mathcal{D}(\overline{S}) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$, where orthogonality is with respect to the inner product $[f, g]$.*

Remark 15 From [12, p. 328]. S has self-adjoint extensions if, and only if,

$$\dim \mathcal{D}_+ = \dim \mathcal{D}_-.$$

Remark 16 If τ is a formally self-adjoint formal differential operator defined on an interval $I \subseteq \mathbf{R}$, then $T_0(\tau)$ is a symmetric operator. To see this, use integration by parts. Then

$$\mathcal{D}_\pm = \{f \in \mathcal{D}(T_1(\tau)) : \tau(f) = \pm i f\}.$$

Thus, both $\mathcal{D}_+$ and $\mathcal{D}_-$ are finite dimensional. According to lemma 14,

$$\mathcal{D}(T_1(\tau)) = \mathcal{D}(\overline{T_0(\tau)}) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-.$$

Thus,

$$\dim \, \mathcal{D}(T_1(\tau)) \, / \, \mathcal{D}(\overline{T_0(\tau)}) \; = \dim \mathcal{D}_+ + \dim \mathcal{D}_- .$$

We say that $T_1(\tau)$ is an extension of $\overline{T_0(\tau)}$ of dimension equal to $\dim \mathcal{D}_+ + \dim \mathcal{D}_-$, which is finite.

Definition 17 Let T, with domain $\mathcal{D}(T)$, and S, with domain $\mathcal{D}(S)$, be two operators in a Hilbert space, such that $T \supseteq S$. We shall use $[T : S]$ to denote $\dim \; \mathcal{D}(T) \; / \; \mathcal{D}(S)$.

Remark 18 Let τ, $\mathcal{D}_+$, and $\mathcal{D}_-$ be as in remark 16. Then

$$[T_1(\tau) : \overline{T_0(\tau)}] = \dim \mathcal{D}_+ + \dim \mathcal{D}_- < \infty.$$

Chapter 2

Formulas Related to Commuting

Remark 19 Let $I \subseteq \mathbf{R}$ be an interval. Let τ be a formal differential operator of order n, which is defined on I. Then $\tau^*\tau$ and $\tau\tau^*$ are both formally symmetric formal differential operators of order $2n$, which are defined on I.

Both $T_0(\tau^*\tau)$ and $T_0(\tau\tau^*)$ are non-negative operators. To see this, choose $\phi \in W_0^{2n}(I)$. Then

$$\begin{aligned} \langle T_0(\tau^*\tau)(\phi), \phi \rangle &= \int_I \tau^*\tau(\phi) \cdot \overline{\phi}\, dt \\ &= \int_I \tau(\phi) \cdot \overline{\tau(\phi)}\, dt \\ &\geq 0. \end{aligned}$$

Because each operator is bounded below, each has a self-adjoint extension: its Friedrichs extension [12, p.330]. Thus,

$$\begin{aligned} \dim\ \mathcal{D}_+(T_1(\tau^*\tau)) &= \dim\ \mathcal{D}_-(T_1(\tau^*\tau))\ , \text{ and} \\ \dim\ \mathcal{D}_+(T_1(\tau\tau^*)) &= \dim\ \mathcal{D}_-(T_1(\tau\tau^*))\ . \end{aligned}$$

In this chapter, we use information about $T_1(\tau^*\tau)$ to develop information about $\dim\ \mathcal{D}_+(T_1(\tau\tau^*)) = \dim\ \mathcal{D}_-(T_1(\tau\tau^*))$.

Lemma 20 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then*

1. dim $\mathcal{D}_+(T_1(\tau\tau^*)) \geq \dim\{f \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}$, *and*

2. dim $\mathcal{D}_-(T_1(\tau\tau^*)) \geq \dim\{f \in \mathcal{D}_-(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}$.

Proof: We prove the first formula; the second is proved similarly.

We know that $f \in \mathcal{D}_+(T_1(\tau^*\tau))$ if, and only if, $f \in \mathcal{D}(T_1(\tau^*\tau))$ and f satisfies the differential equation $\tau^*\tau(v) - iv = 0$. Now, f satisfying $\tau^*\tau(v) - iv = 0$ entails $f \in C^\infty(I)$ [4, XIII.1.4]. Thus, $f \in \mathcal{D}_+(T_1(\tau^*\tau))$ if, and only if, f satisfies $\tau^*\tau(v) - iv = 0$ and $f \in L^2(I)$.

Similarly, $g \in \mathcal{D}_+(T_1(\tau\tau^*))$ if, and only if, g satisfies $\tau\tau^*(v) - iv = 0$, and $g \in L^2(I)$.

Suppose $f \in \mathcal{D}_+(T_1(\tau^*\tau))$. Then f satisfies $\tau^*\tau(v) - iv = 0$. Thus, $f \in C^\infty(I)$. Thus, by definition of τ (see definition 1), $\tau(f) \in C^\infty(I)$; this shows that $\tau\tau^*(\tau(f))$ is well-defined. We have

$$\begin{aligned}\tau\tau^*(\tau(f)) - i\tau(f) &= \tau(\tau^*\tau(f)) - i\tau(f) \\ &= \tau(if) - i\tau(f) \\ &= 0.\end{aligned}$$

Thus, $\tau(f)$ satisfies $\tau\tau^*(v) - iv = 0$.

We summarize: if $f \in \mathcal{D}_+(T_1(\tau^*\tau))$, then $\tau(f) \in \mathcal{D}_+(T_1(\tau\tau^*))$ if, and only if, $\tau(f) \in L^2(I)$.

Thus,

$$\dim \mathcal{D}_+(T_1(\tau\tau^*)) \geq \dim\{\tau(f) : f \in \mathcal{D}_+(T_1(\tau^*\tau)) \text{ and } \tau(f) \in L^2(I)\}.$$

But $f \in \mathcal{D}_+(T_1(\tau^*\tau))$ and $f \neq 0$ entail $\tau(f) \neq 0$; this is because $\tau(f) = 0$ and $\tau^*\tau(f) = if$ entail $0 = if$, whence $0 = f$.

Thus, dim $\mathcal{D}_+(T_1(\tau\tau^*)) \geq \dim\{f \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}$. Q.E.D.

Definition 21 Let S, with domain $\mathcal{D}(S)$, and T, with domain $\mathcal{D}(T)$, be two operators in a Hilbert space. We define

$$\mathcal{D}(TS) = \{f \in \mathcal{D}(S) : S(f) \in \mathcal{D}(T)\}.$$

If $f \in \mathcal{D}(TS)$, we define $TS(f) = T(S(f))$.

Remark 22 Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Suppose that $f \in \mathcal{D}_{\pm}(T_1(\tau^*\tau))$, and that $\tau(f) \in L^2(I)$. Then $f \in C^{\infty}(I)$ (see comments in the proof of lemma 20), and $\tau(f) \in L^2(I)$. Thus, $f \in \mathcal{D}(T_1(\tau))$. Further, $\tau(f) \in C^{\infty}(I)$, because the coefficients of τ belong to $C^{\infty}(I)$ (see definition 1); and

$$\tau^*(\tau(f)) = \pm i f \in L^2(I).$$

Thus, $\tau(f) \in \mathcal{D}(T_1(\tau^*))$.

Thus, $f \in \mathcal{D}(T_1(\tau^*)T_1(\tau))$, and f satisfies $T_1(\tau^*)T_1(\tau)(v) = \pm iv$.

Thus,

$$\{f \in \mathcal{D}_{\pm}(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\} \subseteq \mathcal{D}_{\pm}(T_1(\tau^*)T_1(\tau)) .$$

We use the next several lemmas to show that

$$\dim\{f \in \mathcal{D}_{\pm}(T_1(\tau^*\tau)) \ : \tau(f) \in L^2(I)\}$$

equals the positive, or the negative, deficiency index of $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$.

Lemma 23 *Let $I \subseteq \mathbf{R}$ be an interval. Let τ be a formal differential operator which is defined on I. Then*

1. $\overline{T_0(\tau^*\tau)} \subseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$,

2. $[\overline{T_0(\tau^*)}\,\overline{T_0(\tau)} : \overline{T_0(\tau^*\tau)}] < \infty$, *and*

3. $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ *is closed, symmetric, and non-negative.*

Proof:

1. We show that $\mathcal{D}(\overline{T_0(\tau^*\tau)}) \subseteq \mathcal{D}(\overline{T_0(\tau)})$. Let $f \in \mathcal{D}(\overline{T_0(\tau^*\tau)})$. Then there is $\{\phi_m\} \subseteq C_0^{\infty}(I)$ such that $\phi_m \to f$ and $T_0(\tau^*\tau)(\phi_m) \to \overline{T_0(\tau^*\tau)}(f)$: see corollary 10. We will show that $f \in \mathcal{D}(\overline{T_0(\tau)})$. Given $\varepsilon > 0$, there exists a natural number M such that $l, m \geq M$ entails $\| \phi_m - \phi_l \| < \varepsilon$ and $\| T_0(\tau^*\tau)(\phi_m) - T_0(\tau^*\tau)(\phi_l) \| < \varepsilon$. Thus,

$$|\langle T_0(\tau^*\tau)(\phi_m - \phi_l), \phi_m - \phi_l\rangle| \leq \| T_0(\tau^*\tau)(\phi_m - \phi l) \| \cdot \| (\phi_m - \phi_l) \| < \varepsilon^2.$$

We restate this:

$$\begin{aligned}
|\int_I \tau^*\tau(\phi_m-\phi_l)\cdot\overline{(\phi_m-\phi_l)}\,dt| &< \varepsilon^2 \\
\int_I |\tau(\phi_m-\phi_l)|^2\,dt &< \varepsilon^2 \text{ (integration by parts)} \\
\| T_0(\tau)(\phi_m-\phi_l)\|^2 &< \varepsilon^2.
\end{aligned}$$

Thus, $\{T_0(\tau)(\phi_m)\}$ is a cauchy sequence in $L^2(I)$. Let $F = \lim T_0(\tau)(\phi_m)$. Then $f \in \mathcal{D}(\overline{T_0(\tau)})$, and $\overline{T_0(\tau)}(f) = F$.

2. We show that $\overline{T_0(\tau)}(f) \in \mathcal{D}(\overline{T_0(\tau^*)})$.

   $\{T_0(\tau)(\phi_m)\} \subseteq C_0^\infty(I)$, because $\{\phi_m\} \subseteq C_0^\infty(I)$, and because all the coefficients of τ belong to $C^\infty(I)$. From paragraph 1, we know that $T_0(\tau)(\phi_m) \to \overline{T_0(\tau)}(f)$.

   Also, since $T_0(\tau^*\tau)(\phi_m) \to \overline{T_0(\tau^*\tau)}(f)$, we see that

$$T_0(\tau^*)\,T_0(\tau)(\phi_m) = T_0(\tau^*\tau)(\phi_m) \to \overline{T_0(\tau^*\tau)}(f).$$

   Thus, $\overline{T_0(\tau)}(f) \in \mathcal{D}(\overline{T_0(\tau^*)})$, and $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}(f) = \overline{T_0(\tau^*\tau)}(f)$.

3. We summarize the preceding: $\overline{T_0(\tau^*\tau)} \subseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$.

4. To show that $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is a closed operator.

   From theorem 8, $\overline{T_0(\tau)}^* = T_1(\tau^*)$. Thus $T_1(\tau^*)\,\overline{T_0(\tau)}$ is self-adjoint [12, p. 312].

   From paragraph 3,

$$\overline{T_0(\tau^*\tau)} \subseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}. \tag{2.1}$$

   From definition 21, and the fact that $\mathcal{D}(\overline{T_0(\tau^*)}) \subseteq \mathcal{D}(T_1(\tau^*))$, we know that

$$\overline{T_0(\tau^*)}\,\overline{T_0(\tau)} \subseteq T_1(\tau^*)\,\overline{T_0(\tau)}. \tag{2.2}$$

   From lines 2.1 and 2.2, $\overline{T_0(\tau^*\tau)} \subseteq T_1(\tau^*)\,\overline{T_0(\tau)}$. Taking adjoints yields

$$\overline{T_0(\tau^*\tau)}^* \supseteq (T_1(\tau^*)\,\overline{T_0(\tau)})^*. \tag{2.3}$$

But $T_1(\tau^*)\overline{T_0(\tau)}$ is self-adjoint, and theorem 8 says that $\overline{T_0(\tau^*\tau)}^* = T_1(\tau^*\tau)$. This, and line 2.3, show that

$$T_1(\tau^*)\,\overline{T_0(\tau)} \subseteq T_1(\tau^*\tau). \tag{2.4}$$

Lines 2.1, 2.2,and 2.4 yield

$$\overline{T_0(\tau^*\tau)} \subseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)} \subseteq T_1(\tau^*\tau).$$

This shows that $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is a finite dimensional extension of $\overline{T_0(\tau^*\tau)}$.

Equivalently, $\mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$ is a finite dimensional extension of $\mathcal{D}(\overline{T_0(\tau^*\tau)})$, which is a closed subspace of the Hilbert space consisting of the set $\mathcal{D}(T_1(\tau^*\tau))$, with the inner product

$$[f,g] = \langle f,g\rangle + \langle T_1(\tau^*\tau)(f), T_1(\tau^*\tau)(g)\rangle.$$

From proposition 6, we conclude that $\mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$ is a closed subspace of $\mathcal{D}(T_1(\tau^*\tau))$. From this, and from remark 4, we conclude that $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is a closed operator.

5. We show that $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is symmetric. Let $f,g \in \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$. In particular,

$$g \in \mathcal{D}(\overline{T_0(\tau)}) \subseteq \mathcal{D}(T_1(\tau)) = \mathcal{D}(\overline{T_0(\tau^*)}^*)\ .$$

Thus,

$$\begin{aligned}
\langle \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}(f), g\rangle &= \langle \overline{T_0(\tau)}(f), \overline{T_0(\tau^*)}^*(g)\rangle \\
&= \langle \overline{T_0(\tau)}(f), T_1(\tau)(g)\rangle \\
&= \langle \overline{T_0(\tau)}(f), \overline{T_0(\tau)}(g)\rangle.
\end{aligned}$$

Also,

$$\overline{T_0(\tau)}(g) \in \mathcal{D}(\overline{T_0(\tau^*)}) \subseteq \mathcal{D}(T_1(\tau^*)) = \mathcal{D}(\overline{T_0(\tau)}^*)\ .$$

Thus,

$$\begin{aligned}
\langle \overline{T_0(\tau)}(f), \overline{T_0(\tau)}(g)\rangle &= \langle f, \overline{T_0(\tau)}^*\overline{T_0(\tau)}(g)\rangle \\
&= \langle f, T_1(\tau^*)\,\overline{T_0(\tau)}(g)\rangle \\
&= \langle f, \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}(g)\rangle.
\end{aligned}$$

6. We show that $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is non-negative.

For $g \in \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$,

$$\begin{aligned} \langle \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}(g), g\rangle &= \langle \overline{T_0(\tau)}(g), \overline{T_0(\tau^*)}^*(g)\rangle \\ v &= \langle \overline{T_0(\tau)}(g), T_1(\tau)(g)\rangle \\ &= \langle \overline{T_0(\tau)}(g), \overline{T_0(\tau)}(g)\rangle \\ &\geq 0. \end{aligned}$$

Q.E.D.

Remark 24 In this remark, *orthogonal* will be with respect to an operator inner product of the form

$$[f,g] = \langle f,g\rangle + \langle T(f), T(g)\rangle,$$

for some operator T.

As before, τ is a formal differential operator defined on an interval $I \subseteq \mathbf{R}$.

By the general theory of Hilbert spaces,

$$[T_1(\tau) : \overline{T_0(\tau)}] = \dim(\ \mathcal{D}(T_1(\tau)) \ \ominus\ \mathcal{D}(\overline{T_0(\tau)})\ ).$$

According to [4, the second paragraph of the proof of XIII.2.27], $\dim(\ \mathcal{D}(T_1(\tau)) \ominus \mathcal{D}(\overline{T_0(\tau)})\ )$ is equal to the number of linearly independent solutions of the equation

$$T_1(\tau^*)T_1(\tau)(v) = -v. \tag{2.5}$$

Thus, $[T_1(\tau) : \overline{T_0(\tau)}] < \infty$. (Compare this with remarks 16 and 18, where τ is assumed to be self-adjoint.)

Lemma 25 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then*

$$[T_1(\tau) : \overline{T_0(\tau)}] = [T_1(\tau^*) : \overline{T_0(\tau^*)}].$$

Proof:

1. Let $f \neq 0$, and let it satisfy

$$T_1(\tau^*)T_1(\tau)(v) = -v. \tag{2.6}$$

   Then $T_1(\tau)(f) \neq 0$. This is because $T_1(\tau)(f) = 0$ and $T_1(\tau^*)T_1(\tau)(f) = -f$ entail $f = 0$, which contradicts $f \neq 0$. This shows that if f satisfies equation 2.6, then the map $f \mapsto T_1(\tau)(f)$ is one-one.

2. Similarly, let $g \neq 0$, and let it satisfy

$$T_1(\tau)T_1(\tau^*)(v) = -v. \tag{2.7}$$

   Then $T_1(\tau^*)(g) \neq 0$,and the map $g \mapsto T_1(\tau^*)(g)$ is one-one.

3. Let f be as above. We show that $T_1(\tau)(f)$ satisfies equation 2.7.

   Suppose that $f \in \mathcal{D}(T_1(\tau))$. Then f, $T_1(\tau)(f) \in L^2(I)$. Moreover,

$$\begin{aligned} T_1(\tau)T_1(\tau^*)(T_1(\tau)(f)) &= T_1(\tau)(T_1(\tau^*)T_1(\tau)(f)) \\ &= T_1(\tau)(-f) = -T_1(\tau)(f). \end{aligned}$$

   We also know that the map $f \mapsto T_1(\tau)(f)$ is one-one. Thus, the number of linearly independent solutions to equation 2.7 must be greater than, or equal to, the number of linearly independent solutions to equation 2.6.

   The above, together with remark 24, entails $[T_1(\tau^*) : \overline{T_0(\tau^*)}] \geq [T_1(\tau) : \overline{T_0(\tau)}]$.

4. Recall that $(\tau^*)^* = \tau$. If we reverse the roles of τ and τ^*, we may conclude that

$$[T_1(\tau) : \overline{T_0(\tau)}] \geq [T_1(\tau^*) : \overline{T_0(\tau^*)}].$$

5. Thus, $[T_1(\tau) : \overline{T_0(\tau)}] = [T_1(\tau^*) : \overline{T_0(\tau^*)}]$. Q.E.D.

In the following theorem and lemma, ***orthogonality*** refers to the operator inner product $[f,g] = \langle f,g\rangle + \langle S^*(f), S^*(g)\rangle$. For notation, see definitions 13 and 17.

Theorem 26 From [4, XII.4.12(b)]. *Let S, with domain $\mathcal{D}(S)$, be a closed symmetric operator in a Hilbert space. Let G be a closed subspace of $\mathcal{D}_+(S) \oplus \mathcal{D}_-(S)$. Then $S^*|_{\mathcal{D}(S)\oplus G}$ is self-adjoint if, and only if, G is the graph of an isometry mapping all of $\mathcal{D}_+(S)$ onto all of $\mathcal{D}_-(S)$.*

Lemma 27 *Let S, with domain $\mathcal{D}(S)$, be a closed symmetric operator in a Hilbert space with inner product (f,g). Let A be a self-adjoint extension of S. Then*

$$1/2 \cdot [S^* : S] = [A : S] = \dim\ \mathcal{D}_+(S)\ .$$

Proof: Because S has a self-adjoint extension, $\dim\ \mathcal{D}_+(S) = \dim\ \mathcal{D}_-(S)$. According to remarks 16 and 18,

$$[S^* : S] = 2 \cdot \dim\ \mathcal{D}_+(S)\ . \tag{2.8}$$

By the previous theorem, $\mathcal{D}(A) = \mathcal{D}(S) \oplus G$, where G is the graph of an isometry from $\mathcal{D}_+(S)$ to $\mathcal{D}_-(S)$. Thus

$$[A : S] = \dim G. \tag{2.9}$$

Let $V : \mathcal{D}_+(S) \to \mathcal{D}_-(S)$ denote the isometry. If $g \in G$, there is $d \in \mathcal{D}_+(S)$ such that $g = d + V(d) = (1 + V)(d)$.

The map $(1 + V) : \mathcal{D}_+(S) \to G$ is onto.

Let $d \in \ker(1 + V)$. Then $d + V(d) = 0$, yielding $d = -V(d)$. But $-V(d) \in \mathcal{D}_-(S)$, which entails $[d, -V(d)] = 0$. Thus $[d, d] = [d, -V(d)] = 0$. Thus $d = 0$. So $\ker(1 + V) = (0)$.

Since $1 + V$ is onto and one-one, $\dim G = \dim\ \mathcal{D}_+(S)$. The lemma follows from this, and from equations 2.8 and 2.9. Q.E.D.

Lemma 28 *Let τ be a formal differential operator, which is defined on an interval $I \subset \mathbf{R}$. Then*

$$[T_1(\tau^*)T_1(\tau) : T_1(\tau^*)\overline{T_0(\tau)}] = [T_1(\tau) : \overline{T_0(\tau)}].$$

Proof:

1. As in remark 24, $[T_1(\tau) : \overline{T_0(\tau)}]$ equals the number of linearly independent solutions of $T_1(\tau^*)T_1(\tau)(v) = -v$. This number does not exceed $2n$, the number of linearly independent

solutions to $\tau^* \tau(v) = -v$. Let $m = [T_1(\tau) : \overline{T_0(\tau)}]$. Then there are m linearly independent functions $r_1, \ldots, r_m$ which satisfy $T_1(\tau^*)T_1(\tau)(v) = -v$. Span $\{r_1, \ldots, r_m\} = \mathcal{D}(T_1(\tau)) \ominus \mathcal{D}(\overline{T_0(\tau)})$, where orthogonality refers to the operator inner product for $T_1(\tau)$. Thus, $\{r_1, \ldots, r_m\}$ are linearly independent, modulo $\mathcal{D}(\overline{T_0(\tau)})$. Because $\mathcal{D}(T_1(\tau^*)\overline{T_0(\tau)}) \subseteq \mathcal{D}(\overline{T_0(\tau)})$, $\{r_1, \ldots, r_m\}$ are linearly independent, modulo $\mathcal{D}(T_1(\tau^*)\overline{T_0(\tau)})$.

Also, $\{r_1, \ldots, r_m\} \subseteq \mathcal{D}(T_1(\tau^*)T_1(\tau))$.

Thus, $[T_1(\tau) : \overline{T_0(\tau)}] \leq [T_1(\tau^*)T_1(\tau) : T_1(\tau^*)\overline{T_0(\tau)}]$.

2. Clearly,

$$T_1(\tau^*\tau) \supseteq T_1(\tau^*)T_1(\tau) \supseteq T_1(\tau^*)\overline{T_0(\tau)}. \tag{2.10}$$

From relation 2.2 in the proof of lemma 23, we have

$$T_1(\tau^*)\overline{T_0(\tau)} \supseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}. \tag{2.11}$$

From paragraph 3 in the proof of lemma 23, we have

$$\overline{T_0(\tau^*)}\,\overline{T_0(\tau)} \supseteq \overline{T_0(\tau^*\tau)}. \tag{2.12}$$

Relations 2.10, 2.11, and 2.12 yield

$$T_1(\tau^*\tau) \supseteq T_1(\tau^*)T_1(\tau) \supseteq T_1(\tau^*)\overline{T_0(\tau)} \supseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)} \supseteq \overline{T_0(\tau^*\tau)}. \tag{2.13}$$

From this relation, and the fact that $[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] < \infty$, we conclude that

$$l = [T_1(\tau^*)T_1(\tau) : T_1(\tau^*)\overline{T_0(\tau)}] < \infty. \tag{2.14}$$

Let $\{s_1, \ldots, s_l\}$ be a minimal, complete set of coset representitives for

$$\mathcal{D}(T_1(\tau^*)T_1(\tau)) \,/\, \mathcal{D}(T_1(\tau^*)\overline{T_0(\tau)}).$$

If $\{c_1, \ldots, c_l\} \subseteq \mathbf{C}$ is such that $\sum_{i=1}^{l} c_i \cdot s_i \in \mathcal{D}(T_1(\tau^*)\overline{T_0(\tau)})$, then $0 = c_1 = \cdots = c_l$.

We have $s_i \in \mathcal{D}(T_1(\tau))$, and $T_1(\tau)(s_i) \in \mathcal{D}(T_1(\tau^*))$, for $i = 1, \ldots, l$.

Let $\{c_1, \ldots, c_l\} \subseteq \mathbf{C}$ be such that $\sum_{i=1}^{l} c_i \cdot s_i \in \mathcal{D}(\overline{T_0(\tau)})$. Then

$$\overline{T_0(\tau)}(\sum_{i=1}^{l} c_i \cdot s_i) = T_1(\tau)(\sum_{i=1}^{l} c_i \cdot s_i) = \sum_{i=1}^{l} c_i \cdot T_1(\tau)(s_i) \subseteq \mathcal{D}(T_1(\tau^*).)$$

Thus, $\sum_{i=1}^{l} c_i \cdot s_i \in \mathcal{D}(T_1(\tau^*)\overline{T_0(\tau)})$, from which we conclude that $0 = c_1 = \ldots = c_l$.

We restate this: $\{s_1, \ldots, s_l\}$ are linearly independent,modulo $\mathcal{D}(\overline{T_0(\tau)})$.

Since $\{s_1, \ldots, s_l\} \subseteq \mathcal{D}(T_1(\tau))$, then

$$[T_1(\tau) : \overline{T_0(\tau)}] \geq l = [T_1(\tau^*)T_1(\tau) : T_1(\tau^*)\overline{T_0(\tau)}].$$

Q.E.D.

Lemma 29 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then*

$$[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)] = [T_1(\tau^*)T_1(\tau) : T_1(\tau^*)\overline{T_0(\tau)}].$$

Proof: We have two chains of subspaces:

$$\mathcal{D}(T_1(\tau^*)T_1(\tau)) \supseteq \mathcal{D}(T_1(\tau^*)\overline{T_0(\tau)}) \supseteq \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) \text{ , and} \tag{2.15}$$

$$\mathcal{D}(T_1(\tau^*)T_1(\tau)) \supseteq \mathcal{D}(\overline{T_0(\tau^*)}T_1(\tau)) \supseteq \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) \text{ .} \tag{2.16}$$

$T_1(\tau^*)\overline{T_0(\tau)}$ and $\overline{T_0(\tau^*)}T_1(\tau)$ are both self-adjoint extensions of $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$, which is a closed, positive, symmetric, and densely defined operator (see lemma 23). Let the deficiency indices of $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ be (d,d). We use lemma 27, with $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ as the symmetric operator.

Lemma 27, with the self-adjoint $T_1(\tau^*)\overline{T_0(\tau)}$, yields

$$\frac{1}{2}[(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = [T_1(\tau^*)\overline{T_0(\tau)} : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = d. \tag{2.17}$$

Lemma 27, with the self-adjoint $T_1(\tau^*)\overline{T_0(\tau)}$, yields

$$\frac{1}{2}[(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = [\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = d. \tag{2.18}$$

Equations 2.17 and 2.18 yield

$$d = [T_1(\tau^*)\overline{T_0(\tau)} : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = [\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}]. \tag{2.19}$$

From the second isomorphism theorem for vector spaces, and chain 2.15, we have

$$\frac{\mathcal{D}(T_1(\tau^*)T_1(\tau)) \;/\; \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})}{\mathcal{D}(T_1(\tau^*)\overline{T_0(\tau)}) \;/\; \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})} \approx \frac{\mathcal{D}(T_1(\tau^*)T_1(\tau))}{\mathcal{D}(T_1(\tau^*)\overline{T_0(\tau)})}.$$

Thus,

$$\begin{aligned}[][T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] \quad &- \quad [T_1(\tau^*)\overline{T_0(\tau)} : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] \\ &= \quad [T_1(\tau^*)T_1(\tau) : T_1(\tau^*)\overline{T_0(\tau)}].\end{aligned}$$

This, and equation 2.19 yield

$$[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] - d = [T_1(\tau^*)T_1(\tau) : T_1(\tau^*)\overline{T_0(\tau)}]. \tag{2.20}$$

From chain 2.16, equation 2.19, and the same procedure, we get

$$[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] - d = [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)]. \tag{2.21}$$

Q.E.D.

Lemma 30 *Let τ be a formal differential operator defined on an interval $I \subseteq \mathbf{R}$. Then*

$$[T_1(\tau) : \overline{T_0(\tau)}] \geq [\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}].$$

Proof: Clearly, $T_1(\tau^*)T_1(\tau) \supseteq \overline{T_0(\tau^*)}T_1(\tau) \supseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$. From relation 2.13, and the fact that $[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] < \infty$, we have
$[\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] < \infty$.

Let $\{t_1, \ldots, t_q\}$ be a minimal, complete set of coset representatives for the vector space

$$\mathcal{D}(\overline{T_0(\tau^*)}T_1(\tau)) \;/\; \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) \;.$$

If $\{c_1, \ldots, c_q\} \subseteq \mathbf{C}$ are such that

$$\sum_{i=1}^{q} c_i \cdot t_i \in \; \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) \;,$$

then $0 = c_1 = \cdots = c_q$.

We know that $t_i \in \; \mathcal{D}(T_1(\tau))$, and that $T_1(\tau)(t_i) \in \; \mathcal{D}(\overline{T_0(\tau^*)})$, for $1 \leq i \leq q$.

Suppose that $\{c_1, \ldots, c_q\} \subseteq \mathbf{C}$ are such that $\sum_{i=1}^{q} c_i \cdot t_i \in \mathcal{D}(\overline{T_0(\tau)})$. Then

$$\overline{T_0(\tau)}(\sum_{i=1}^{q} c_i \cdot t_i) = T_1(\tau)(\sum_{i=1}^{q} c_i \cdot t_i) = \sum_{i=1}^{q} c_i \cdot T_1(\tau)(t_i) \in \mathcal{D}(\overline{T_0(\tau^*)}) .$$

Thus, $\sum_{i=1}^{q} c_i \cdot t_i \in \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$, from which we conclude that $0 = c_1 = \cdots = c_q$.

We restate this: $\{t_1, \ldots, t_q\}$ are linearly independent, modulo $\mathcal{D}(\overline{T_0(\tau)})$. Since $\{t_1, \ldots, t_q\} \subseteq \mathcal{D}(T_1(\tau))$, then

$$[T_1(\tau) : \overline{T_0(\tau)}] \geq q = [\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}].$$

Q.E.D.

Proposition 31 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then*

1. $(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* = T_1(\tau^*)T_1(\tau)$, *and*

2. $(T_1(\tau^*)T_1(\tau))^* = \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$.

Proof: From lemma 23, $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is a closed, symmetric, non-negative, and densely defined operator. Thus, it has deficiency indices (d, d), where d is a non-negative integer.

$\overline{T_0(\tau^*)}T_1(\tau)$ is a self-adjoint extension of $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$. Thus,

$$(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* \supseteq (\overline{T_0(\tau^*)}T_1(\tau))^* = \overline{T_0(\tau^*)}T_1(\tau) \supseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}.$$

Thus,

$$\mathcal{D}((\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^*) \supseteq \mathcal{D}(\overline{T_0(\tau^*)}T_1(\tau)) \supseteq \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) .$$

From this chain of subspaces, and from the second isomorphism theorem for vector spaces, we get

$$\frac{\mathcal{D}((\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^*) \;/\; \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})}{\mathcal{D}(\overline{T_0(\tau^*)}T_1(\tau)) \;/\; \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})} \approx \frac{\mathcal{D}((\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^*)}{\mathcal{D}(\overline{T_0(\tau^*)}T_1(\tau))}.$$

This yields

$$\begin{aligned}[(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] &- [\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] \\ &= [(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}T_1(\tau)]. \qquad (2.22)\end{aligned}$$

From lemma 27, the left side equals $d = \dim\ \mathcal{D}_+((\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^*)$. Thus,

$$d = [(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}T_1(\tau)]. \tag{2.23}$$

Also, from lemma 27

$$d = [\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}]. \tag{2.24}$$

From equations 2.23 and 2.24, and from lemma 30, we conclude that

$$[T_1(\tau) : \overline{T_0(\tau)}] \geq [(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}T_1(\tau)]. \tag{2.25}$$

Let A and B be two densely defined operators in a Hilbert space. Recall, from [12, p.300], that

$$(AB)^* \supseteq B^*A^*.$$

We use this relation to assert

$$(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* \supseteq T_1(\tau^*)T_1(\tau). \tag{2.26}$$

Further, definition 21, and the fact that $\mathcal{D}(T_1(\tau^*)) \supseteq \mathcal{D}(\overline{T_0(\tau^*)})$ entail

$$T_1(\tau^*)T_1(\tau) \supseteq \overline{T_0(\tau^*)}T_1(\tau). \tag{2.27}$$

From relations 2.26 and 2.27, we get

$$(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* \supseteq T_1(\tau^*)T_1(\tau) \supseteq \overline{T_0(\tau^*)}T_1(\tau). \tag{2.28}$$

Thus,

$$[(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}T_1(\tau)] \geq [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)]. \tag{2.29}$$

Relations 2.25 and 2.29 yield

$$\begin{aligned} [T_1(\tau) : \overline{T_0(\tau)}] &\geq [(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}T_1(\tau)] \\ &\geq [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)]. \end{aligned} \tag{2.30}$$

But lemmas 28 and 29 give

$$[T_1(\tau) : \overline{T_0(\tau)}] = [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)]. \tag{2.31}$$

Relations 2.30 and 2.31 entail

$$[(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* : \overline{T_0(\tau^*)}T_1(\tau)] = [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)]. \tag{2.32}$$

From relations 2.32 and 2.28, we conclude that

$$\mathcal{D}((\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^*) \;=\; \mathcal{D}(T_1(\tau^*)T_1(\tau)) \;,$$

whence

$$(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* = T_1(\tau^*)T_1(\tau), \tag{2.33}$$

which proves the first assertion of this proposition.

Now, after taking the adjoint of equation 2.33, we get

$$(T_1(\tau^*)T_1(\tau))^* = ((\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^*)^*. \tag{2.34}$$

According to theorem 3, $((\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^*)^*$ is the smallest closed operator which contains $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$. But, by the results of lemma 23, $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is a closed operator. Thus

$$((\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^*)^* = \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}.$$

From this, and from equation 2.34, follows the second assertion of this proposition. Q.E.D.

Corollary 32 *$T_1(\tau^*)T_1(\tau)$ is a closed operator, under the hypotheses of the preceding proposition.*

Proof: This is because adjoint operators are closed. Q.E.D.

Corollary 33 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then the following are equal:*

1. $[T_1(\tau) : \overline{T_0(\tau)}]$,
2. $[T_1(\tau^*)T_1(\tau) : T_1(\tau^*)\overline{T_0(\tau)}]$,
3. $[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)]$,
4. $[T_1(\tau^*)\overline{T_0(\tau)} : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}]$,

5. $[\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}]$,

6. $[T_1(\tau^*) : \overline{T_0(\tau^*)}]$,

7. $[T_1(\tau)T_1(\tau^*) : T_1(\tau)\overline{T_0(\tau^*)}]$,

8. $[T_1(\tau)T_1(\tau^*) : \overline{T_0(\tau)}T_1(\tau^*)]$,

9. $[T_1(\tau)\overline{T_0(\tau^*)} : \overline{T_0(\tau)}\,\overline{T_0(\tau^*)}]$,

10. $[\overline{T_0(\tau)}T_1(\tau^*) : \overline{T_0(\tau)}\,\overline{T_0(\tau^*)}]$.

Proof:

1. *1* = *2* follows from lemma 28.

2. *2* = *3* follows from lemma 29.

3. *4* = *5* follows from equation 2.19 in the proof of lemma 29.

4. *2* = *5* follows from equations 2.23 and 2.24 in the proof of proposition 31, and the result that

$$(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* = T_1(\tau^*)T_1(\tau).$$

5. *1* = *6* follows from lemma 25.

6. *6* = *7* = *8* = *9* = *10* follows from the immediately preceding arguments, but where we substitute τ^* for τ, and use the fact that $(\tau^*)^* = \tau$. Q.E.D.

Corollary 34 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then*

$$\begin{aligned}
[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] &= [T_1(\tau^*) : \overline{T_0(\tau^*)}] + [T_1(\tau) : \overline{T_0(\tau)}], \\
&= 2[T_1(\tau^*) : \overline{T_0(\tau^*)}], \\
&= 2[T_1(\tau) : \overline{T_0(\tau)}].
\end{aligned}$$

Proof: From equation 2.22 in the proof of proposition 31, and from the first assertion of that proposition, we get

$$[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = [\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] + [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)]. \quad (2.35)$$

From corollary 33, we know that

$$[\overline{T_0(\tau^*)}T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = [T_1(\tau^*) : \overline{T_0(\tau^*)}],$$

and that

$$[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)] = [T_1(\tau) : \overline{T_0(\tau)}].$$

Making these substitutions into equation 2.35 gives the first result.

From lemma 25, we know that

$$[T_1(\tau^*) : \overline{T_0(\tau^*)}] = [T_1(\tau) : \overline{T_0(\tau)}].$$

The rest of the corollary follows from this. Q.E.D.

Corollary 35 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then the following four statements are equivalent:*

1. $T_1(\tau^*\tau) = T_1(\tau^*)T_1(\tau)$,
2. $\overline{T_0(\tau^*\tau)} = \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$,
3. $[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] = 2 \cdot [T_1(\tau) : \overline{T_0(\tau)}]$,
4. $[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] = 2 \cdot [T_1(\tau^*) : \overline{T_0(\tau^*)}]$.

Proof: Here, we prove that number 1 implies number 2. We assume

$$T_1(\tau^*\tau) = T_1(\tau^*)T_1(\tau).$$

Taking adjoints, we get $(T_1(\tau^*\tau))^* = (T_1(\tau^*)T_1(\tau))^*$. Using corollary 9 and proposition 31, we get $\overline{T_0(\tau^*\tau)} = \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$.

To prove that number 2 implies number 1, reverse the above argument, but using theorem 8 instead of corollary 9.

Here, we prove that number 1 implies number 3. If number 1 is true, then number 2 is also true. Explicitly, we have

$$T_1(\tau^*\tau) = T_1(\tau^*)T_1(\tau), \quad \text{and}$$
$$\overline{T_0(\tau^*\tau)} = \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}.$$

To get the desired result, we make these substitutions into the left side of the conclusion of corollary 34.

Here, we prove that number 3 implies number 2. From relation 2.13, we have

$$\overline{T_0(\tau^*\tau)} \subseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)} \subseteq T_1(\tau^*)T_1(\tau) \subseteq T_1(\tau^*\tau). \tag{2.36}$$

From remark 18, we know that

$$[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] < \infty. \tag{2.37}$$

From corollary 34, we have

$$[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = 2[T_1(\tau) : \overline{T_0(\tau)}].$$

Assuming that number 3 is true, the preceding equation yields

$$[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = [T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}].$$

This last equation, together with relations 2.36 and 2.37, leads to $T_1(\tau^*\tau) = T_1(\tau^*)T_1(\tau)$, and $\overline{T_0(\tau^*\tau)} = \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$.

That number 3 is equivalent to number 4 follows from lemma 25. Q.E.D.

Corollary 36 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then*

$$\begin{aligned} [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] &= 2 \cdot \dim\{f \in \mathcal{D}_+(T_1(\tau^*\tau)) \ : \tau(f) \in L^2(I)\} \\ &= 2 \cdot \dim\{f \in \mathcal{D}_-(T_1(\tau^*\tau)) \ : \tau(f) \in L^2(I)\}. \end{aligned}$$

Proof: From lemma 23, $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is a symmetric operator. From proposition 31,

$$(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* = T_1(\tau^*)T_1(\tau).$$

In the notation of definition 13,

$$\mathcal{D}_\pm(T_1(\tau^*)T_1(\tau)) = \{f \in \mathcal{D}(T_1(\tau^*)T_1(\tau)) : T_1(\tau^*)T_1(\tau)(f) = \pm if\}.$$

According to lemma 14,

$$\mathcal{D}(T_1(\tau^*)T_1(\tau)) = \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) \oplus \mathcal{D}_+(T_1(\tau^*)T_1(\tau)) \oplus \mathcal{D}_-(T_1(\tau^*)T_1(\tau)),$$

where the orthogonality refers to the operator inner product for $T_1(\tau^*)T_1(\tau)$. According to lemma 23, $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is non-negative. Thus, it has a self-adjoint extension: its Friedrichs extension (see [12, p. 330]). Therefore,

$$\dim \mathcal{D}_+(T_1(\tau^*)T_1(\tau)) = \dim \mathcal{D}_-(T_1(\tau^*)T_1(\tau)).$$

According to remark 22, if $g \in \{f \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}$, then g must satisfy $T_1(\tau^*)T_1(\tau)(v) = iv$. Thus,

$$\{f \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\} \subseteq \mathcal{D}_+(T_1(\tau^*)T_1(\tau)).$$

On the other hand, let $f \in \mathcal{D}_+(T_1(\tau^*)T_1(\tau))$. Then

$$T_1(\tau^*)T_1(\tau)(f) = if.$$

Since $T_1(\tau^*)T_1(\tau) \subseteq T_1(\tau^*\tau)$, then f satisfies

$$T_1(\tau^*\tau)(v) = iv.$$

Also, according to definition 21, $T_1(\tau)(f) \in \mathcal{D}(T_1(\tau^*))$. In particular, $T_1(\tau)(f) \in L^2(I)$. Thus,

$$\mathcal{D}_+(T_1(\tau^*)T_1(\tau)) \subseteq \{f \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}.$$

Thus, $\mathcal{D}_+(T_1(\tau^*)T_1(\tau)) = \{f \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}$, whence

$$\dim \mathcal{D}_+(T_1(\tau^*)T_1(\tau)) = \dim\{f \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}.$$

Similarly,

$$\dim \mathcal{D}_-(T_1(\tau^*)T_1(\tau)) = \dim\{f \in \mathcal{D}_-(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}.$$

To sum up, the following quantities are all equal:

1. $[T_1(\tau^*)T_1(\tau) : \overline{\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}}]$,

2. $\dim \mathcal{D}_+(T_1(\tau^*)T_1(\tau)) + \dim \mathcal{D}_-(T_1(\tau^*)T_1(\tau))$,

3. $2 \cdot \dim \mathcal{D}_+(T_1(\tau^*)T_1(\tau))$,

4. $2 \cdot \dim \mathcal{D}_-(T_1(\tau^*)T_1(\tau))$,

5. $2 \cdot \dim\{f \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}$,

6. $2 \cdot \dim\{f \in \mathcal{D}_-(T_1(\tau^*\tau)) : \tau(f) \in L^2(I)\}$.

Q.E.D.

Corollary 37 *Let τ be a formal differential operator defined on an interval $I \subseteq \mathbf{R}$. Then*

$$\frac{1}{2} \cdot [T_1(\tau^*)T_1(\tau) : \overline{\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}}]$$

equals each of the quantities from corollary 33.

Proof: Start with equation 2.22 in the proof of lemma 31, and use the fact that $(\overline{\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}})^* = T_1(\tau^*)T_1(\tau)$:

$$\begin{aligned}[T_1(\tau^*)T_1(\tau) : \overline{\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}}] &- [\overline{T_0(\tau^*)}T_1(\tau) : \overline{\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}}] \\ &= [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)].\end{aligned}$$

Thus,

$$\begin{aligned}[T_1(\tau^*)T_1(\tau) : \overline{\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}}] &= [\overline{T_0(\tau^*)}T_1(\tau) : \overline{\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}}] \\ &\quad + [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}T_1(\tau)] \\ &= 2 \cdot [\overline{T_0(\tau^*)}T_1(\tau) : \overline{\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}}],\end{aligned}$$

by corollary 33. The desired conclusion now follows from an application of corollary 33 to this last equation. Q.E.D.

Corollary 38 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then* dim $\mathcal{D}_{\pm}(T_1(\tau\tau^*))$ *is greater than, or equal to,*

1. $(1/2)[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}]$;

2. *any item from corollary 33.*

Proof:

1. Number 1 follows from lemma 20 and corollary 36.

2. Number 2 follows from the preceding, and corollary 37.

Q.E.D.

Chapter 3

Boundary Values at Fixed End Points

Remark 39 The last corollary gives a lower bound for the dimensions of the deficiency spaces $\mathcal{D}_\pm(T_1(\tau^*\tau))$, in terms of information about the two operators $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$, and $(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* = T_1(\tau^*)T_1(\tau)$ (both of which are restrictions of $T_1(\tau^*\tau)$).

Up until now, we have placed no restriction on the interval I. In particular, I might have two free end points. This could happen, for example, when the formal differential operator τ is singular at both ends of I.

It can happen that information about dimensions of extensions is more easily obtained when τ is singular at only one end point, than when τ is singular at both end points. With this in mind, we continue with τ defined on $I \subseteq \mathbf{R}$ with end points $a < b$. Then we choose $c \in I$ such that $a < c < b$. Let $I' = [a, c] \cap I$. Let $I'' = [c, b] \cap I$. I' and I'' each has at least one fixed end point. Due to the presence of a fixed end point, quantities like

$$[T_1(\tau^*, I')T_1(\tau, I') : \overline{T_0(\tau^*, I')}\,\overline{T_0(\tau, I')}]$$

might be more easily calculated than quantities like

$$[T_1(\tau^*,I)T_1(\tau,I):\overline{T_0(\tau^*,I)}\,\overline{T_0(\tau,I)}].$$

We will derive relations between

$$[T_1(\tau^*,I)T_1(\tau,I):\overline{T_0(\tau^*,I)}\,\overline{T_0(\tau,I)}],$$

and quantities which are defined on the two subintervals, I' and I''.

Then we will show that, at a fixed end point c, $\overline{T_0(\tau^*)}\ \overline{T_0(\tau)}$ is like $\overline{T_0(\tau^*\tau)}$, in a way which we will describe below. This result is not used in subsequent material.

This material is directly motivated by the development in [4, XIII.2].

Definition 40 From [4, XIII.2.17]. Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. $\mathcal{D}(T_1(\tau))$, with the operator inner product, is a Hilbert space. A *boundary value for* τ is a continuous linear functional, A, on $\mathcal{D}(T_1(\tau))$, which vanishes on $\mathcal{D}(T_0(\tau))$. If $A(f) = 0$ for each $f \in \mathcal{D}(T_1(\tau))$ which vanishes in a neighborhood of a, then A will be called a *boundary value at a*.

Boundary value at b is defined similarly.

Theorem 41 From [4, XIII.2.19]. *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. The space of boundary values for τ is the direct sum of the spaces of boundary values at a, and of boundary values at b.*

Theorem 42 From [4, XIII.2.20]. *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. Choose $c \in I$ such that $a < c < b$, and let τ' be the restriction of τ to $I' = I \cap [a, c]$. Then there exists a one-one linear mapping of the space of boundary values for τ' at a onto the space of boundary values for τ at a.*

Corollary 43 From [4, XIII.2.21]. *Under the hypotheses of the preceding theorem, τ and τ' have the same number of independent boundary conditions at a.*

Corollary 44 From [4, XIII.2.22]. *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$. Then τ has at most n linearly independent boundary values at either end point of I.*

Definition 45 From [4, XIII.2.17]. A *complete set of bounday values at* a is a maximal linearly independent set of boundary values at a.

Corollary 46 From [4, XIII.2.23]. *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points a, b, and suppose that a is a fixed end point. Then the functionals $A_i(f) = f^{(i)}(a), i = 0, 1, \ldots, n-1$, form a complete set of boundary values for τ at a.*

The following proposition is a slight adaptation of [4, XIII.2.25].

Proposition 47 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. Let $c \in I$ such that $a < c < b$. Let $I' = I \cap [a, c]$. Let $I'' = I \cap [c, b]$. Then*

$$[T_1(\tau, I) : \overline{T_0(\tau, I)}] = [T_1(\tau, I') : \overline{T_0(\tau, I')}] + [T_1(\tau, I'') : \overline{T_0(\tau, I'')}] - 2n.$$

Proof: By theorem 41, $[T_1(\tau, I) : \overline{T_0(\tau, I)}]$ equals the sum of the number of independent boundary values for τ at a, and of those at b. Since c is a fixed end point, it follows from corollary 46, and theorems 41 and 42 that $[T_1(\tau, I') : \overline{T_0(\tau, I')}]$ and $[T_1(\tau, I'') : \overline{T_0(\tau, I'')}]$ each exceed, by n, the number of independent boundary values at a, and at b, respectively. The desired conclusion is thus evident. Q.E.D.

Proposition 48 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. Let $c \in I$ such that $a < c < b$. Let $I' = I \cap [a, c]$. Let $I'' = I \cap [c, b]$. Then $[T_1(\tau^*, I) T_1(\tau, I) : \overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}]$ is equal to each of*

1. $2 \cdot [T_1(\tau, I') : \overline{T_0(\tau, I')}] + 2 \cdot [T_1(\tau, I'') : \overline{T_0(\tau, I'')}] - 4n$;

2. $2 \cdot [T_1(\tau^*, I') : \overline{T_0(\tau^*, I')}] + 2 \cdot [T_1(\tau^*, I'') : \overline{T_0(\tau^*, I'')}] - 4n$; *and*

3.

$$[T_1(\tau^*, I')\, T_1(\tau, I') : \overline{T_0(\tau^*, I')}\, \overline{T_0(\tau, I')}]$$

$$+ [T_1(\tau^*, I'')\, T_1(\tau, I'') : \overline{T_0(\tau^*, I'')}\, \overline{T_0(\tau, I'')}] - 4n.$$

Proof: To prove part 1: From the preceding proposition,

$$2 \cdot [T_1(\tau, I') : \overline{T_0(\tau, I')}] + 2 \cdot [T_1(\tau, I'') : \overline{T_0(\tau, I'')}] - 4n = 2 \cdot [T_1(\tau, I) : \overline{T_0(\tau, I)}]. \tag{3.1}$$

Then, from corollary 34,

$$[T_1(\tau^*, I)\, T_1(\tau, I) : \overline{T_0(\tau^*, I)}\, \overline{T_0(\tau, I)}] = 2 \cdot [T_1(\tau, I) : \overline{T_0(\tau, I)}]. \tag{3.2}$$

Equating the left sides of equations 3.1 and 3.2 gives result 1.

Part 2 follows from part 1 and from corollary 33, parts 1 and 6.

Part 3 follows from part 1, and from the fact that equation 3.2, in the proof of part 1, applies to I' and I'', too. Q.E.D.

Remark 49 The preceding proposition expresses relationships between

$$[T_1(\tau^*, I)\, T_1(\tau, I) : \overline{T_0(\tau^*, I)}\, \overline{T_0(\tau, I)}],$$

and quantities which are defined with respect to I', and to I'', each of which has at least one fixed end point.

Corollary 50 *Under the hypotheses of the preceding proposition,*

$$\dim\ \mathcal{D}_\pm(T_1(\tau\tau^*, I))$$

is greater than, or equal to, each of

1. $[T_1(\tau, I') : \overline{T_0(\tau, I')}] + [T_1(\tau, I'') : \overline{T_0(\tau, I'')}] - 2n$;

2. $[T_1(\tau^*, I') : \overline{T_0(\tau^*, I')}] + [T_1(\tau^*, I'') : \overline{T_0(\tau^*, I'')}] - 2n$; *and*

3.

$$(1/2)\cdot[T_1(\tau^*,I')T_1(\tau,I'):\overline{T_0(\tau^*,I')}\,\overline{T_0(\tau,I')}]$$

$$+\,(1/2)\cdot[T_1(\tau^*,I'')T_1(\tau,I''):\overline{T_0(\tau^*,I'')}\,\overline{T_0(\tau,I'')}]-2n.$$

Proof: From the preceding proposition, and corollary 38. Q.E.D.

Remark 51 In the following material, we define *boundary value for* $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$, and then show that at fixed end points, $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ has as many boundary values as has $\tau^*\tau$ (i.e., as many as has $\overline{T_0(\tau^*\tau)}$).

Definition 52 Let τ be a formal differential operator which is defined on an interval $I\subseteq\mathbf{R}$ with end points $a<b$. $\mathcal{D}(T_1(\tau^*)\,T_1(\tau))$, with the operator inner product, is a Hilbert space. A *boundary value for* $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is a continuous linear functional, A, on $\mathcal{D}(T_1(\tau^*)\,T_1(\tau))$ which vanishes on $\mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$. If $A(f)=0$ for each $f\in\mathcal{D}(T_1(\tau^*)\,T_1(\tau))$ which vanishes in a neighborhood of a, then A will be called a *boundary value for* $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ *at* a.

Boundary value for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ *at* b is defined similarly.

Lemma 53 From [4, XIII.2.16]. *Let τ be a formal differential operator, of order n, which is defined on an interval $I\subseteq\mathbf{R}$, and let J be a compact subinterval of I. For $f\in\mathcal{D}(T_1(\tau))$, define a norm*

$$|f|_J=\sum_{i=0}^{n-1}\max_{t\in J}\{|f^{(i)}(t)|\}+\left(\int_J|f^{(n)}(t)|^2\,dt\right)^{1/2}.$$

Then

1. *The space $W^n(J)$ (see definition 1) is complete in the norm $|f|_J$, and*

2. *If $\{f_n\}$ is a sequence in $\mathcal{D}(T_1(\tau))$ such that $\{f_n\}$ and $\{\tau(f_n)\}$ converge in $L^2(I)$, then the sequence $\{f_n\}$ converges in the topology of $W^n(I)$ defined by the norm $|f|_J$.*

Remark 54 Under the hypotheses of the preceding lemma, let R denote the map

$$\begin{aligned} R:\ \mathcal{D}(T_1(\tau,I)) &\rightarrow W^n(J)\\ f &\mapsto f|_J \end{aligned}$$

which maps $f \in \mathcal{D}(T_1(\tau, I))$ to its restriction to the interval J. R is clearly onto, because J is compact.

$T_1(\tau, I)$ is an adjoint operator (see theorem 8), whence it is a closed operator (see [12, p. 300], which entails that $\mathcal{D}(T_1(\tau, I))$, with the operator inner product, is a (complete) Hilbert space (see remark 4). Also, according to part 1 of the preceding lemma, the space $W^n(J)$, with the metric induced by the norm $|f|_J$, is a complete space.

Since R is defined everywhere in $\mathcal{D}(T_1(\tau, I))$, since $W^n(J)$, with the metric induced by the norm $|f|_J$, is complete, and since R is onto, the closed graph theorem tells us that the operator R is bounded. Thus, there exists a constant $C > 0$, depending on τ, I, and J, such that, for all $f \in \mathcal{D}(T_1(\tau, I))$,

$$|f|_J \leq C \cdot [[f]]_{T_1(\tau, I)}. \tag{3.3}$$

The following lemma is implicit in [4, XIII.2.19].

Lemma 55 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. Let $c, d \in I$ such that $a < c < d < b$. Choose $f_a \in C^\infty(I)$ such that $f_a(t) \equiv 1$ on $[a, c)$, and $f_a(t) \equiv 0$ on $(d, b]$. Choose $f_b \in C^\infty(I)$ such that $f_b(t) \equiv 0$ on $[a, c)$, and $f_b(t) \equiv 1$ on $(d, b]$. Then*

1.

$$\begin{aligned} F_a : \mathcal{D}(T_1(\tau)) &\rightarrow \mathcal{D}(T_1(\tau)) \\ h &\mapsto f_a h, \text{ and} \end{aligned} \tag{3.4}$$

$$\begin{aligned} \tilde{F}_a : \mathcal{D}(\overline{T_0(\tau)}) &\rightarrow \mathcal{D}(\overline{T_0(\tau)}) \\ h &\mapsto f_a h \end{aligned} \tag{3.5}$$

are continuous in the operator norm, and

2.

$$F_b : \mathcal{D}(T_1(\tau)) \rightarrow \mathcal{D}(T_1(\tau)) \qquad (3.6)$$
$$h \mapsto f_b h, \text{ and}$$

$$\tilde{F}_b : \mathcal{D}(\overline{T_0(\tau)}) \rightarrow \mathcal{D}(\overline{T_0(\tau)}) \qquad (3.7)$$
$$h \mapsto f_b h$$

are continuous in the operator norm.

Proof: We prove part 1; part 2 is proved similarly.

First, we prove that the mapping 3.4 is defined everywhere in its domain. Let $h \in \mathcal{D}(T_1(\tau))$. Notice that f_a is a bounded function. Let $k > 0$ be a bound for f_a. Since $h \in L^2(I)$, and since

$$\|f_a h\|_I \leq k \cdot \|h\|_I, \qquad (3.8)$$

then $f_a h \in L^2(I)$. Also

$$\begin{aligned} \int_I |\tau(f_a h)|^2 dt &= \int_a^c + \int_c^d + \int_d^b |\tau(f_a h)|^2 dt \\ &= \int_a^c |\tau(h)|^2 dt + \int_c^d |\tau(f_a h)|^2 dt. \end{aligned} \qquad (3.9)$$

The first integral on the right side of equation 3.9 is finite because $h \in \mathcal{D}(T_1(\tau))$.

Now we show that the second integral on the right side of equation 3.9 is finite. Recall, from definition 1, that the coefficient functions of τ have continuous derivatives of all orders. By hypothesis, this is also true of f_a. Recall further, from definition 1, that $\mathcal{D}(T_1(\tau)) = W_\tau^n(I)$. Thus, if $h \in \mathcal{D}(T_1(\tau))$, then $h, h', \ldots, h^{(n-1)}$ are all continuous functions. Thus,

$$\tau(f_a h) = f_a \cdot \tau(h) + \tau_a(h), \qquad (3.10)$$

where τ_a is a differential operator of order n-1, with $C^\infty(I)$ coefficients. Because $f_a(t) \equiv 0$ on $(d, b]$, the coefficients of τ_a vanish on $(d, b]$. The coefficient functions of τ_a vanish on $[a, c)$, because $f_a(t) \equiv 1$ on $[a, c)$. Thus, the coefficients of τ_a vanish outside of $[c, d]$, a compact interval.

The function f_a is bounded, with bound k. Also, since $h \in \mathcal{D}(T_1(\tau))$, then $\tau(h) \in L^2(I)$. Thus,

$$\|f_a \tau(h)\|_I \leq k \cdot \|\tau(h)\|_I \leq k \cdot [[h]]_{T_1(\tau)}. \tag{3.11}$$

Because the coefficients of τ_a vanish outside of $[c, d]$, then

$$\|\tau_a(h)\|_I = \|\tau_a(h)\|_{[c,d]}. \tag{3.12}$$

Further, because all the coefficients of τ_a, as well as $h, h', \ldots, h^{(n-1)}$, are continuous on $[c, d]$, then

$$\begin{aligned} \|\tau_a(h)\|_{[c,d]} &\leq (d-c)^{1/2} \cdot \sup_{t \in [c,d]} \{|\tau_a(h)(t)|\} \\ &\leq k' \cdot |h|_{[c,d]}, \end{aligned} \tag{3.13}$$

where $k' \geq 0$ is a constant, and $|h|_{[c,d]}$ refers to the special norm of lemma 53, for $J = [c, d]$. Then by remark 54, and lines 3.12 and 3.13,

$$\|\tau_a(h)\|_I \leq k'' \cdot [[h]]_{T_1(\tau)}, \tag{3.14}$$

for some constant $k'' \geq 0$. Equation 3.10, and relations 3.11 and 3.14 show that

$$\|\tau(f_a h)\|_I \leq k''' \cdot [[h]]_{T_1(\tau)}, \tag{3.15}$$

for some constant $k''' \geq 0$. Since $[[h]]_{T_1(\tau)} < \infty$, lines 3.10–3.15 show that the second integral on the right side of equation 3.9 is finite. Thus, the map F_a is defined everywhere in its domain.

More is true. Using relation 3.8, we see that

$$\|f_a h\|_I \leq k \cdot [[h]]_{T_1(\tau)}. \tag{3.16}$$

This, and relation 3.15, yield

$$\{\|f_a h\|_I^2 + \|\tau(f_a h)\|_I^2\}^{1/2} \leq k'''' \cdot [[h]]_{T_1(\tau)},$$

for some $k'''' > 0$. Thus, the map

$$F_a : \mathcal{D}(T_1(\tau)) \rightarrow \mathcal{D}(T_1(\tau))$$

is continuous.

As for the mapping $\tilde{F}_a$, we must first show that $h \in \mathcal{D}(\overline{T_0(\tau)})$ implies $f_a h \in \mathcal{D}(\overline{T_0(\tau)})$. By theorem 41, it is enough to show that $B_a(f_a h) = 0$, and $B_b(f_a h) = 0$ for every B_a, a boundary value at a, and every B_b, a boundary value at b. Since $f_a(t) \equiv 0$ on $(d, b]$, then $f_a(t)h(t) \equiv 0$ on $(d, b]$. Hence, $B_b(f_a h) = 0$. On $[a, c)$, $f_a(t)h(t) = h(t)$. Hence, $f_a(t)h(t) - h(t) \equiv 0$ on $[a, c)$. Thus, $B_a(f_a h - h) = 0$. But $B_a(h) = 0$, since $h \in \mathcal{D}(\overline{T_0(\tau)})$. Thus, $B_a(f_a h) = 0$.

Now, to see that $\tilde{F}_a$ is a continuous map, it is enough to notice that $\tilde{F}_a = F_a|_{\mathcal{D}(\overline{T_0(\tau)})}$, and that the metric induced on $\mathcal{D}(\overline{T_0(\tau)})$ by the $\overline{T_0(\tau)}$ operator norm, is equal to the restriction, to $\mathcal{D}(\overline{T_0(\tau)})$, of the metric induced on $\mathcal{D}(T_1(\tau))$ by the $T_1(\tau)$ operator norm. Q.E.D.

The next theorem is a slight adaptation of [4, XIII.2.19].

Theorem 56 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. The space of boundary values for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ is the direct sum of the space of boundary values for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ at a, and the space of those at b.*

Proof: Choose $c, d \in I$ such that $a < c < d < b$. Choose $f_a, f_b \in C^\infty(I)$ such that

1. $f_a(t) + f_b(t) = 1$,
2. $f_a(t) \equiv 1$ on $[a, c)$,
3. $f_a(t) \equiv 0$ on $(d, b]$,
4. $f_b(t) \equiv 0$ on $[a, c)$, and
5. $f_b(t) \equiv 1$ on $(d, b]$.

Let A be a boundary value for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$. We show that the linear functional on $\mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$, A_a, which is defined by

$$\begin{aligned} A_a :\ \mathcal{D}(T_1(\tau^*)T_1(\tau)) &\rightarrow \mathbf{C} \\ h &\mapsto A(f_a h), \end{aligned}$$

is a boundary value for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ at a.

First, we show that

$$\hat{F}_a : \mathcal{D}(T_1(\tau^*)\,T_1(\tau)) \;\rightarrow\; \mathcal{D}(T_1(\tau^*)\,T_1(\tau)) \qquad (3.17)$$
$$h \;\mapsto\; f_a h$$

is a continuous map. One application of lemma 55, to $\mathcal{D}(T_1(\tau^*\tau))$, shows that

$$F_a : \mathcal{D}(T_1(\tau^*\tau)) \;\rightarrow\; \mathcal{D}(T_1(\tau^*\tau)) \qquad (3.18)$$
$$h \;\mapsto\; f_a h$$

is continuous. Thus, the restricted map

$$F_a|_{\mathcal{D}(T_1(\tau^*)\,T_1(\tau))} : \mathcal{D}(T_1(\tau^*)\,T_1(\tau)) \;\rightarrow\; \mathcal{D}(T_1(\tau^*\tau)) \qquad (3.19)$$

is continuous.

A second application of lemma 55, to $\mathcal{D}(T_1(\tau))$, shows that

$$F_a : \mathcal{D}(T_1(\tau)) \;\rightarrow\; \mathcal{D}(T_1(\tau)) \qquad (3.20)$$
$$h \;\mapsto\; f_a h$$

is continuous. To see that the map of line 3.17 is continuous, it is enough to show that the image of the map of line 3.19 lies in $\mathcal{D}(T_1(\tau^*)\,T_1(\tau))$. This we do now. Let $h \in \mathcal{D}(T_1(\tau^*)\,T_1(\tau))$. From line 3.19, we see that

$$\tau^*(\tau(f_a h)) = \tau^*\tau(f_a h) \in L^2(I). \qquad (3.21)$$

From line 3.20, we see that $\tau(f_a h) \in L^2(I)$. This, together with line 3.21, shows that $\tau(f_a h) \in \mathcal{D}(T_1(\tau^*))$. Thus, the image of the map in line 3.19 lies in $\mathcal{D}(T_1(\tau^*)\,T_1(\tau))$. Thus, the map of line 3.17 is continuous.

Thus, the linear functional $A_a = A \circ \hat{F}_a$ is a continuous linear functional on $\mathcal{D}(T_1(\tau^*)\,T_1(\tau))$.

Now, let $h \in \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$. We must show that $A_a(h) = 0$. It is enough to show that $f_a h \in \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$. An application of lemma 55, with respect to the formal operator τ, shows that $f_a h \in \mathcal{D}(\overline{T_0(\tau)})$. Further, since

$$h \in \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) \subseteq \mathcal{D}(T_1(\tau^*)\,T_1(\tau)) \,,$$

an application of line 3.17 shows that $f_a h \in \mathcal{D}(T_1(\tau^*)\,T_1(\tau))$. Thus, $f_a h \in \mathcal{D}(T_1(\tau^*)\,\overline{T_0(\tau)})$. Next, we show that $\tau(f_a h) \in \mathcal{D}(\overline{T_0(\tau^*)})$. Since $\tau(f_a h) \in \mathcal{D}(T_1(\tau^*))$, it is enough to show that $B_a(\tau(f_a h)) = 0$, and that $B_b(\tau(f_a h)) = 0$, for every B_a, a boundary value for τ^* at a, and every B_b, a boundary value for τ^* at b. Since $f_a(t) \equiv 0$ on $(d, b]$, then $[\tau(f_a h)](t) \equiv 0$ on $(d, b]$. Thus, $B_b(\tau(f_a h)) = 0$. Since $f_a(t) \equiv 1$ on $[a, c)$, then $[\tau(f_a h)](t) = [\tau(h)](t)$ on $[a, c)$, whence

$$[\tau(f_a h) - \tau(h)](t) \equiv 0$$

on $[a, c)$. Thus

$$B_a(\tau(f_a h) - \tau(h)) = 0.$$

But $B_a(\tau(h)) = 0$, since $\tau(h) \in \mathcal{D}(\overline{T_0(\tau^*)})$. Thus, $B_a(\tau(f_a h)) = 0$.

The preceding shows that A_a is a boundary value for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$. Now we show that A_a is a boundary value at a. Let $h \in \mathcal{D}(T_1(\tau^*)\,T_1(\tau))$ such that $h(t) \equiv 0$ near a. Then $(f_a h)(t) \equiv 0$ near a, and near b. Thus, using definition 1 and lemma 23, we see that

$$f_a h \in \mathcal{D}(T_0(\tau^*\tau)) \subseteq \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) .$$

Thus, $A(f_a h) = 0$. Thus, $A_a(h) = 0$.

Similar arguments show that the linear functional A_b, defined by

$$\begin{aligned} A_b : \mathcal{D}(T_1(\tau^*)T_1(\tau)) &\rightarrow \mathbf{C} \\ h &\mapsto A(f_b h) \end{aligned}$$

is a boundary value for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ at b.

Since $f_a + f_b \equiv 1$, the $A_a + A_b = A$. This shows that every boundary value is the sum of a boundary value at a, and one at b.

Now, let A be a boundary value both at a, and at b. Let $h \in \mathcal{D}(T_1(\tau^*)\,T_1(\tau))$. Then $(f_a h)(t) \equiv 0$ near b. Thus $A(f_a h) = 0$. Similarly, $(f_b h)(t) \equiv 0$ near a. Thus $A(f_b h) = 0$. Thus

$$A(h) = A(f_a h) + A(f_b h) = 0.$$

Thus $A \equiv 0$. This shows that the space of boundary values is the direct sum of those at a, and those at b. Q.E.D.

The next proposition is a slight adaptation of theorem 42.

Proposition 57 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. Choose $e \in I$ such that $a < e < b$. Let $I' = I \cap [a, e]$. Then there exists a one-one linear mapping of the space of boundary values for $\overline{T_0(\tau^*, I')}\,\overline{T_0(\tau, I')}$ at a, onto the space of boundary values for $\overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}$ at a.*

Proof: Choose $c, d \in I'$ such that $a < c < d < e$. Choose $f \in C^\infty(I)$ such that $f(t) \equiv 1$ on $[a, c)$. and such that $f(t) \equiv 0$ on $(d, b]$. Let $\tilde{F}_1$ denote the map

$$\begin{aligned} \tilde{F}_1 : \mathcal{D}(T_1(\tau^*, I)\,T_1(\tau, I)) &\rightarrow \mathcal{D}(T_1(\tau^*, I)\,T_1(\tau, I)) \\ g &\mapsto fg. \end{aligned}$$

According to an argument in the proof of theorem 56, $\tilde{F}_1$ is continuous. Also, $(fg)(t) \equiv 0$ for $t \in (d, b]$, since the same holds for f. Thus, the restriction map,

$$\begin{aligned} R : \text{Image } (\tilde{F}_1) &\rightarrow \mathcal{D}(T_1(\tau^*, I')\,T_1(\tau, I')) \\ fg &\mapsto fg|_{I'} \end{aligned}$$

is an isometry. Thus,

$$F_1 = R \circ \tilde{F}_1 : \mathcal{D}(T_1(\tau^*, I)\,T_1(\tau, I)) \rightarrow \mathcal{D}(T_1(\tau^*, I')\,T_1(\tau, I'))$$

is continuous.

Let M, and M', denote the spaces of boundary values at a for $\overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)}$, and for $\overline{T_0(\tau^*, I')}\,\overline{T_0(\tau, I')}$, respectively. We define a linear map,

$$\begin{aligned} \Phi_1 : M' &\rightarrow M \\ A' &\mapsto A' \circ F_1. \end{aligned}$$

Now we show that Φ_1 is one-one, and onto.

Let $\tilde{F}_2$ denote the map

$$\begin{aligned}\tilde{F}_2 : \mathcal{D}(T_1(\tau^*, I')\, T_1(\tau, I')) &\rightarrow \mathcal{D}(T_1(\tau^*, I')\, T_1(\tau, I')) \\ g &\mapsto fg.\end{aligned}$$

Using the same argument as that in the proof of theorem 56, $\tilde{F}_2$ is continuous. Also, $(fg)(t) \equiv 0$ on $(d, e]$. Then fg may be extended to an element of $\mathcal{D}(T_1(\tau^*, I)\, T_1(\tau, I))$ by defining $(fg)(t) \equiv 0$ on $[e, b]$. This extension defines an isometric inclusion

$$E : \text{Image}\,(\tilde{F}_2) \rightarrow \mathcal{D}(T_1(\tau^*, I)\, T_1(\tau, I))\ .$$

Thus, the map

$$F_2 = E \circ \tilde{F}_2 : \mathcal{D}(T_1(\tau^*, I')\, T_1(\tau, I')) \rightarrow \mathcal{D}(T_1(\tau^*, I)\, T_1(\tau, I))$$

is continuous.

Let Φ_2 denote the map

$$\begin{aligned}\Phi_2 : M &\rightarrow M' \\ A &\mapsto A \circ F_2.\end{aligned}$$

If $g \in \mathcal{D}(T_1(\tau^*, I)\, T_1(\tau, I))$, then g and $F_2F_1(g)$ agree in $[a, c)$. Therefore, for every $A \in M$, and every $g \in \mathcal{D}(T_1(\tau^*, I)\, T_1(\tau, I))$,

$$(\Phi_1\Phi_2 A)(g) = A((F_2F_1)g) = A(f).$$

Thus $\Phi_1\Phi_2$ is the identity map, which shows that $\Phi_1(M') = M$. In the same way, $\Phi_2\Phi_1$ may be seen to be the identity map on M', which shows that Φ_1 is one-one. Q.E.D.

Corollary 58 *Under the hypotheses of the preceding proposition,*

$$\overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)} \text{ and } \overline{T_0(\tau^*, I')}\,\overline{T_0(\tau, I')}$$

have the same number of linearly independent boundary values at a.

Proposition 59 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$, and suppose that a is a fixed end point of I. Then $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ has 2n independent boundary conditions at a.*

Proof: By relying on the preceding corollary, we may suppose that b is also a fixed end point. Thus, $I = [a, b]$ is compact.

We show that

$$T_1(\tau^*)\,T_1(\tau) = T_1(\tau^*\tau). \tag{3.22}$$

From lemma 23, we know that $\overline{T_0(\tau^*\tau)} \subseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$. Taking adjoints, and then using theorem 8 and proposition 31, yields

$$T_1(\tau^*\tau) = (\overline{T_0(\tau^*\tau)})^* \supseteq (\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})^* = T_1(\tau^*)\,T_1(\tau). \tag{3.23}$$

Thus, to prove equation 3.22, it enough to prove

$$T_1(\tau^*\tau) \subseteq T_1(\tau^*)\,T_1(\tau).$$

To prove *this*, it is enough to prove that $f \in \mathcal{D}(T_1(\tau^*\tau))$ entails $\tau(f) \in \mathcal{D}(T_1(\tau^*))$. From definition 1, $\mathcal{D}(T_1(\tau^*\tau)) = W^{2n}_{\tau^*\tau}(I)$. Thus,

$$f, f', \ldots, f^{(n-1)}, f^{(n)} \in C(I).$$

Since the coefficient functions of τ are all continuous, then $\tau(f) \in C(I)$. Since I is compact, then $\tau(f) \in L^2(I)$. Finally, since

$$\tau^*(\tau(f)) = T_1(\tau^*\tau)(f) \in L^2(I),$$

we conclude that $\tau(f) \in \mathcal{D}(T_1(\tau^*))$, and equation 3.22 is proven.

Thus, $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)} = (T_1(\tau^*)\,T_1(\tau))^* = T_1(\tau^*\tau)^* = \overline{T_0(\tau^*\tau)}$. Thus, on I, $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ and $\overline{T_0(\tau^*\tau)}$ have the same number of independent boundary values at a. It follows from corollary 46 that $\overline{T_0(\tau^*\tau)}$ has 2n independent boundary values at a. The same must be true of $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$.
Q.E.D.

Corollary 60 *Under the hypotheses of the preceding proposition, the functionals $A_i(g) = g^{(i)}(a)$, $i = 0, 1, \ldots, 2n-1$, form a complete set of boundary values for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ at a.*

Proof: For $f \in \mathcal{D}(T_1(\tau^*\tau))$, define $\tilde{A}_i(f) = f^{(i)}(a)$, $i = 0, 1, \ldots, 2n-1$. According to corollary 46, these functionals are continuous, with respect to the operator inner product. Thus, their restrictions to $\mathcal{D}(T_1(\tau^*)\,T_1(\tau))$ are also continuous. Let $A_i(f)$, $i = 0, 1, \ldots, 2n-1$, denote the restrictions.

Here, we show that the functionals $\{A_0, A_1, \ldots, A_{2n-1}\}$ are independent on $\mathcal{D}(T_1(\tau^*)T_1(\tau))$. According to corollary 46, $\{\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_{2n-1}\}$ are independent on $\mathcal{D}(T_1(\tau^*\tau))$. Thus, we may choose $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{2n-1} \in \mathcal{D}(T_1(\tau^*\tau))$ such that $\tilde{A}_i(\tilde{h}_j) = \delta_{ij}$ (the Kronecker delta), for $0 \le i, j \le 2n-1$. Now, we choose $f_a \in C^\infty(I)$ such that $f_a(t) \equiv 1$ for t near, or equal to, a, and such that f_a has compact support. Let $h_i = \tilde{h}_i f_a$. Each h_i has compact support. Because $f_a \in C^\infty(I)$, and because f_a has compact support, $\{h_0, h_1, \ldots, h_{2n-1}\} \subseteq \mathcal{D}(T_1(\tau^*\tau))$. Because $f_a \equiv 1$ near a, then

$$\tilde{A}_i(h_j) = \tilde{A}_i(\tilde{h}_j) = \delta_{ij},$$

for $0 \le i, j \le 2n-1$. Moreover, by relying on the fact that each h_i has compact support, and using an argument from the proof of the preceding proposition,

$$\{h_0, h_1, \ldots, h_{2n-1}\} \subseteq \mathcal{D}(T_1(\tau^*)\,T_1(\tau)) .$$

Thus, $A_i(h_j) = \tilde{A}_i(h_j) = \delta_{ij}$, for $0 \le i, j \le 2n-1$. Thus, $\{A_0, A_1, \ldots, A_{2n-1}\}$ are independent functionals on $\mathcal{D}(T_1(\tau^*)\,T_1(\tau))$.

If $f \in \mathcal{D}(T_1(\tau^*)\,T_1(\tau))$ and f vanishes near a, then $A^{(i)}(f) = f^{(i)}(a) = 0$, for $i = 0, 1, \ldots, 2n-1$.

To show that $\{A_i\}$ are independent boundary values for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ at a, it only remains to show that these functionals vanish on $\mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$. First, we show that if $f \in \mathcal{D}(\overline{T_0(\tau)})$, then $f^{(i)}(a) = 0$, for $i = 0, 1, \ldots, n-1$. Suppose that $\{\phi_m\} \subseteq W_0^n(I)$ such that $[[\phi_m - f]]_{\overline{T_0(\tau)}} \to 0$, where $[[f]]_{\overline{T_0(\tau)}}$ denotes the operator norm. We have ${\phi_m}^{(i)}(a) = 0$, for $i = 0, 1, \ldots, n-1$ and for all m. Thus

$$|f^{(i)}(a)| = \lim_{m\to\infty} |f^{(i)}(a) - {\phi_m}^{(i)}(a)| \le K \cdot [[f - \phi_m]]_{\overline{T_0(\tau)}} \to 0,$$

for some $K > 0$. Thus $f^{(i)}(a) = 0$, for $i = 0, 1, \ldots, n-1$. Second, we note that the same is true for any $g \in \mathcal{D}(\overline{T_0(\tau^*)})$: $g^{(i)}(a) = 0$, for $i = 0, 1, \ldots, n-1$.Now, choose $f \in \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$. Then $\tau(f) \in \mathcal{D}(\overline{T_0(\tau^*)})$. Thus, according to the material immediatly preceding, $[\tau(f)]^{(i)}(a) = 0$, for $i = 0, 1, \ldots, n-1$.

In particular, for $i = 0$, we have

$$0 = \tau(f)(a) = c_n(a)f^{(n)}(a) + \cdots + c(a)f(a).$$

Since $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$, because $f \in \mathcal{D}(\overline{T_0(\tau)})$, and since $c_n(a) \neq 0$, then $f^{(n)}(a) = 0$.

To prove that $f^{(n+1)}(a) = \cdots = f^{(2n-1)}(a) = 0$, we use induction on i, for $1 \leq i \leq n-1$. From material above, $[\tau(f)]^{(i)}(a) = 0$. Then, algebra shows that

$$[\tau(f)]^{(i)}(a) - c_n(a)f^{(n+i)}(a)$$

is a linear combination of $\{f(a), f'(a), \ldots, f^{(n+i-1)}(a)\}$. But previous induction has shown that $0 = f(a) = \cdots = f^{(n+i-1)}(a)$. Thus $c_n(a)f^{(n+i)}(a) = 0$. But $c_n(a) \neq 0$. Hence, $f^{(n+i)}(a) = 0$. Completing the induction shows that

$$\begin{aligned} 0 &= f(a) = \cdots = f^{(2n-1)}(a) \\ &= A_0(f) = \cdots = A_{2n-1}(f). \end{aligned}$$

We conclude that $\{A_0, \ldots, A_{2n-1}\}$ are 2n independent boundary values for $\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$ at a.

By the previous proposition, this space of boundary values at a has dimension 2n. So $\{A_0, \ldots, A_{2n-1}\}$ forms a complete set of boundary values. Q.E.D.

The proof of the following corollary is contained in the previous proof.

Corollary 61 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with a fixed endpoint a. If $f \in \mathcal{D}(\overline{T_0(\tau)})$, then $f^{(i)}(a) = 0$, for $i = 0, 1, \ldots, n-1$.*

Remark 62 Under the hypotheses of the preceding corollary, $\ker \overline{T_0(\tau)} = (0)$. This is because the initial value problem

$$\begin{aligned} \tau(v) &= 0; \\ v^{(i)}(a) &= 0, 0 \le i \le n-1; \end{aligned}$$

has a unique solution (see [4, XIII.1.3]), and $v(t) \equiv 0$ is it.

Chapter 4

When $0 \notin \sigma_e(\tau)$

Remark 63 Now we derive a lower bound for

$$[T_1(\tau^*)\,T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}]$$

in terms of $\dim\ker\overline{T_0(\tau)}$, $\dim\ker\overline{T_0(\tau^*)}$, $\dim\ker T_1(\tau)$, and $\dim\ker T_1(\tau^*)$. This lower bound is exact when

1. Range $\overline{T_0(\tau)}$ is closed, or
2. Range $\overline{T_0(\tau^*)}$ is closed, or
3. Range $\overline{T_0(\tau^*\tau)}$ is closed, or
4. Range $\overline{T_0(\tau\tau^*)}$ is closed.

We show that these four conditions are equivalent. These matters lead us to an extension of a formula of [4, XIII.3.2], which relates $\dim\ker T_1(\tau, I)$ to $\dim\ker T_1(\tau, I')$ and $\dim\ker T_1(\tau, I'')$, where $I \subseteq \mathbf{R}$ is an interval with end points $a < b$, where $a < c < b$, and where $I' = I \cap [a, c]$, and $I'' = I \cap [c, b]$.

Lemma 64 *Let A and B be two linear manifolds contained in a Hilbert space H. Let $\{f_1, \ldots, f_n\}$ be vectors such that*

$$A + span\{f_1, \ldots, f_n\} = B.$$

Then $[B:A] \geq [\overline{B}:\overline{A}]$. *If* A *is closed, then* $[B:A] = [\overline{B}:\overline{A}]$.

Proof: Let $C = \text{span }\{f_1, \ldots, f_n\}$. We have $[B:A] \leq \dim C$. In particular, if $A \cap C = (0)$, then

$$[B:A] = \dim C. \tag{4.1}$$

Because $B = A + C$, we have

$$\begin{aligned} A &\subseteq B &\subseteq A + C, \text{ whence} \\ \overline{A} &\subseteq \overline{B} &\subseteq \overline{A+C}. \end{aligned} \tag{4.2}$$

Moreover,

$$\overline{A+C} \subseteq \overline{A} + C, \tag{4.3}$$

because $\overline{A} + C$ is closed (see proposition 6), and because $A + C \subseteq \overline{A} + C$. From relations 4.2 and 4.3, we see that

$$\overline{A} \subseteq \overline{B} \subseteq \overline{A} + C.$$

This last relation entails

$$[\overline{B}:\overline{A}] \leq [\overline{A} + C : \overline{A}] \leq \dim C. \tag{4.4}$$

Now choose $C' \subseteq C$ such that $A + C' = A + C$, and such that $C' \cap A = (0)$. From relation 4.4, we see that $[\overline{B}:\overline{A}] \leq \dim C'$. From equation 4.1, we have $\dim C' = [B:A]$. Thus,

$$[\overline{B}:\overline{A}] \leq [B:A].$$

If A is closed, then $B = A + C$ is also closed, by proposition 6. In this case, $A = \overline{A}$ and $B = \overline{B}$, so that

$$[B:A] = [\overline{B}:\overline{A}].$$

Q.E.D.

Definition 65 From [4, XIII.6.1]. Let T be a closed operator in a Hilbert space. The set of complex numbers such that the range of $\lambda \mathbf{1} - T$ is not closed is called the ***essential spectrum*** of T, and is denoted by $\sigma_e(T)$.

If τ is a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$, then the essential spectrum of the closed operator $T_1(\tau)$ is called the *essential spectrum* $\sigma_e(\tau)$ of τ.

Lemma 66 From [4, XIII.6.2]. *Let $\mathcal{H}$ be a Hilbert space, and suppose that $\mathcal{H} = S + F$, where S is a subspace and F is finite dimensional. Let T be a bounded operator from $\mathcal{H}$ to another Hilbert space $\mathcal{H}_1$. Then $T(S)$ is closed if, and only if, $T(\mathcal{H})$ is closed.*

Corollary 67 From [4, XIII.6.3 or XIII.7.3]. *Let τ be a formal differential operator, which is defined on an interval $I \subseteq \mathbf{R}$, and let T be any closed extension of $T_0(\tau)$. Then $\sigma_e(\tau) = \sigma_e(T)$.*

Lemma 68 *Let T, with domain $\mathcal{D}(T)$, be a linear operator in a Hilbert space $\mathcal{H}$. Let $\tilde{T}$ be a linear operator such that $\tilde{T} \subseteq T$, and such that Range $(\tilde{T})$ is closed and has finite codimension in $\mathcal{H}$. Then Range (T) is closed.*

Proof:

$$\text{Range } (T)/\text{Range } (\tilde{T}) \subseteq \mathcal{H}/\text{Range } (\tilde{T}).$$

Thus,

$$[\text{Range } (T) : \text{Range } (\tilde{T})] \leq [\mathcal{H} : \text{Range } (\tilde{T})] < \infty.$$

So there exists a finite dimensional space $F \subseteq$ Range (T) such that

$$\text{Range } (T) = \text{Range } (\tilde{T}) + F.$$

Then proposition 6 shows that Range (T) is closed. Q.E.D.

Theorem 69 From [6, theorem IV.1.2, p. 95; and remark II.7.2(ii), p. 75]. *Let T be a densely defined, closed operator in a Hilbert space. Then Range (T) is closed if, and only if, Range (T^*) is closed.*

Definition 70 From [5, p. 195,193].

1. A *Fredholm operator* in a Hilbert space is a closed operator with closed range such that $\dim\ker T < \infty$, and $\dim\ker T^* < \infty$.

2. When T is Fredholm, the *index* of T, denoted ind (T), is defined to be $\dim\ker T^* - \dim\ker T$.

Theorem 71 From [5, p. 195]. *Let S and T be two Fredholm operators acting in a Hilbert space, and let $\mathcal{D}(S)$ be dense. Then ST is Fredholm, and*

$$ind\ (ST) = ind\ (S) + ind\ (T).$$

Remark 72 Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$. Then $\ker T_1(\tau)$, $\ker \overline{T_0(\tau)}$, $\ker T_1(\tau^*)$, and $\ker \overline{T_0(\tau^*)}$ all have dimension less than, or equal to, n.

Proposition 73 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$. The following are equivalent:*

1. $0 \notin \sigma_e(\tau)$;

2. $0 \notin \sigma_e(\tau^*)$;

3. $0 \notin \sigma_e(\tau^*\tau)$;

4. $0 \notin \sigma_e(\tau\tau^*)$.

Proof:

1. Part *1* implies part *2* : $0 \notin \sigma_e(\tau)$ implies Range $(T_1(\tau))$ is closed. So, according to theorem 69,

$$\text{Range } (\overline{T_0(\tau^*)}) = \text{Range } (T_1(\tau)^*)$$

is closed. So, by corollary 67, Range $(T_1(\tau^*))$ is closed.

2. Part *2* implies part *1* : By the method immediately preceding, and the fact that $(\tau^*)^* = \tau$.

3. Part *1* implies part *3* : If $0 \notin \sigma_e(\tau)$, then $0 \notin \sigma_e(\tau^*)$. So Range $(T_1(\tau))$ is closed, and Range $(T_1(\tau^*))$ is closed. By definition 70 and remark 72, $T_1(\tau)$ and $T_1(\tau^*)$ are Fredholm operators. By theorem 71, $T_1(\tau^*)T_1(\tau)$ is a Fredholm operator. Thus, Range $(T_1(\tau^*)\,T_1(\tau))$ is

closed. So $0 \notin \sigma_e(T_1(\tau^*)\, T_1(\tau))$. By proposition 31, $(T_1(\tau^*)T_1(\tau))^* = \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$. Thus, by theorem 69, $0 \notin \sigma_e(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)})$. But lemma 23 shows that $\overline{T_0(\tau^*)}\overline{T_0(\tau)}$ is a closed extension of $\overline{T_0(\tau^*\tau)}$. So, by corollary 67, $0 \notin \sigma_e(\tau^*\tau)$.

4. Part *1* implies part *4* : By the method immediately preceding, but considering $T_1(\tau)\, T_1(\tau^*)$ instead of $T_1(\tau^*)\, T_1(\tau)$.

5. Part *3* implies part *2* : $0 \notin \sigma_e(\tau^*\tau)$ entails $0 \notin \sigma_e(\overline{T_0(\tau^*)}\, T_1(\tau))$. Thus, Range $(\overline{T_0(\tau^*)}\, T_1(\tau))$ is closed. Recall, from [12, p. 312], that $\overline{T_0(\tau^*)}\, T_1(\tau)$ is self-adjoint. Thus,

$$\text{Range } (\overline{T_0(\tau^*)}\, T_1(\tau))^{\perp} = \ker \overline{T_0(\tau^*)}\, T_1(\tau) = \ker T_1(\tau).$$

By remark 72, $\dim\ker T_1(\tau) \leq n$. Thus, $\dim(\text{Range } (\overline{T_0(\tau^*)}\, T_1(\tau)))^{\perp} < \infty$.

Let T of lemma 68 correspond to $\overline{T_0(\tau^*)}$. Let $\tilde{T}$ of lemma 68 correspond to

$$\overline{T_0(\tau^*)}|_{T_1(\tau)[\ \mathcal{D}(\overline{T_0(\tau^*)}\, T_1(\tau))\ ]}.$$

We notice that Range $(\tilde{T})$ = Range $(\overline{T_0(\tau^*)}\, T_1(\tau))$, which is closed and has finite codimension. Also, $\tilde{T} \subseteq T$. Thus Range (T) = Range $(\overline{T_0(\tau^*)})$ is closed, by lemma 68. Thus, $0 \notin \sigma_e(\tau^*)$, by corollary 67.

6. Part *4* implies part *1* : By the method immediately preceding, but using τ^* instead of τ, and using the fact that $(\tau^*)^* = \tau$.

Q.E.D.

The next proposition extends, to arbitrary intervals, proposition 4.4 of [10, p. 17].

Proposition 74 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then*

$$\begin{aligned}[T_1(\tau) : \overline{T_0(\tau)}] \quad \geq \quad & \dim\ker T_1(\tau) - \dim\ker \overline{T_0(\tau^*)} \\ & + \dim\ker T_1(\tau^*) - \dim\ker \overline{T_0(\tau)}. \end{aligned} \tag{4.5}$$

Equality holds if $0 \notin \sigma_e(\tau)$ (equivalently, if $0 \notin \sigma_e(\tau^*)$).

Proof: By theorem 8, $\overline{T_0(\tau)}^* = T_1(\tau^*)$; by corollary 9, $T_1(\tau)^* = \overline{T_0(\tau^*)}$. Thus,

$$\begin{aligned} L^2(I) &= \overline{\text{Range } (\overline{T_0(\tau)})} \oplus \ker T_1(\tau^*) \\ &= \overline{\text{Range } (T_1(\tau))} \oplus \ker \overline{T_0(\tau^*)}. \end{aligned}$$

By remark 24, $[T_1(\tau) : \overline{T_0(\tau)}] < \infty$. Thus, $[\text{Range } (T_1(\tau)) : \text{Range } (\overline{T_0(\tau)})] < \infty$. Let Π denote the natural projection

$$\begin{aligned} \Pi : \mathcal{D}(T_1(\tau)) &\rightarrow \mathcal{D}(T_1(\tau)) \,/\, \mathcal{D}(\overline{T_0(\tau)}) \\ f &\mapsto f + \mathcal{D}(\overline{T_0(\tau)}) \,. \end{aligned}$$

Let T denote the induced linear operator,

$$\begin{aligned} T : \mathcal{D}(T_1(\tau)) \,/\, \mathcal{D}(\overline{T_0(\tau)}) &\rightarrow \text{Range } (T_1(\tau))/\text{Range } (\overline{T_0(\tau)}), \\ f + \mathcal{D}(\overline{T_0(\tau)}) &\mapsto T_1(\tau)(f) + \text{Range } (\overline{T_0(\tau)}). \end{aligned}$$

Notice that T is onto. Also, $\ker T = \Pi(\ker T_1(\tau))$; thus, $\dim\ker T = \dim\ker T_1(\tau) - \dim\ker \overline{T_0(\tau)}$. Using a result of finite dimensional linear algebra, we have

$$\begin{aligned} \dim\ \mathcal{D}(T_1(\tau)) \,/\, \mathcal{D}(\overline{T_0(\tau)}) &= \dim\ker T + \dim\{\text{Range } (T_1(\tau))/\text{Range } (\overline{T_0(\tau)})\} \\ &= \dim\ker T_1(\tau) - \dim\ker \overline{T_0(\tau)} \\ &\quad + \dim\{\text{Range } (T_1(\tau))/\text{Range } (\overline{T_0(\tau)})\}. \end{aligned}$$

Thus,

$$\begin{aligned} [T_1(\tau) : \overline{T_0(\tau)}] &= \dim\ker T_1(\tau) - \dim\ker \overline{T_0(\tau)} \\ &\quad + \dim\{\text{Range } (T_1(\tau))/\text{Range } (\overline{T_0(\tau)})\}, \\ &\geq \dim\ker T_1(\tau) - \dim\ker \overline{T_0(\tau)} \\ &\quad + \dim\{\overline{\text{Range } (T_1(\tau))}/\overline{\text{Range } (\overline{T_0(\tau)})}\}, \end{aligned} \tag{4.6}$$

where the last inequality is a consequence of lemma 64. Thus

$$\begin{aligned} [T_1(\tau) : \overline{T_0(\tau)}] &\geq \dim\ker T_1(\tau) - \dim\ker \overline{T_0(\tau)} \\ &\quad + \dim\{\overline{\text{Range } (T_1(\tau))} \ominus \overline{\text{Range } (\overline{T_0(\tau)})}\}. \end{aligned} \tag{4.7}$$

But

$$\begin{aligned}
&\overline{\text{Range }(T_1(\tau))} \ominus \overline{\text{Range }(\overline{T_0(\tau)})} \\
&= \{f | f \in \overline{\text{Range }(T_1(\tau))} \text{ and } f \perp \overline{\text{Range }(\overline{T_0(\tau)})}\}, \\
&= \{f | f \perp \ker \overline{T_0(\tau^*)} \text{ and } f \in \ker T_1(\tau^*)\}, \\
&= \ker T_1(\tau^*) \ominus \ker \overline{T_0(\tau^*)}.
\end{aligned}$$

Thus

$$\dim \overline{\text{Range }(T_1(\tau))} \ominus \overline{\text{Range }(\overline{T_0(\tau)})} = \dim \ker T_1(\tau^*) - \dim \ker \overline{T_0(\tau^*)}.$$

Substituting this last equation into relation 4.7 yields relation 4.5, as desired.

Now we address the question of equality. $T_1(\tau)$ is a closed operator. By remark 4, $\mathcal{D}(T_1(\tau))$, with the operator inner product, is a Hilbert space. We notice that $[T_1(\tau) : \overline{T_0(\tau)}] < \infty$, and that $\mathcal{D}(\overline{T_0(\tau)})$ is a subspace of $\mathcal{D}(T_1(\tau))$. Applying lemma 66, we see that Range $(T_1(\tau))$ is closed in $L^2(I)$ if, and only if, Range $(\overline{T_0(\tau)})$ is closed in $L^2(I)$. That these ranges are closed in $L^2(I)$ is equivalent to $0 \notin \sigma_e(\tau)$, (and to $0 \notin \sigma_e(\tau^*)$, by proposition 73). In this situation,

$$[\text{Range }(T_1(\tau)) : \text{Range }(\overline{T_0(\tau)})] = \overline{[\overline{\text{Range }(T_1(\tau))} : \text{Range }(\overline{T_0(\tau)})]},$$

and then relations 4.6, 4.7, and 4.5 are equalities. Q.E.D.

Example 75 Here is an example of the preceding proposition. We consider $\tau = d/dx + x$ on the interval $[0, \infty)$. $\tau^* = -d/dx + x$.

If f satisfies the equation $\tau(v) = 0$, then $f \in \text{span } \{e^{-x^2/2}\}$. Thus, $\dim \ker T_1(\tau) = 1$.

If f satisfies the equation $\tau^*(v) = 0$, then $f \in \text{span } \{e^{x^2/2}\}$. But $e^{x^2/2} \notin L^2([0, \infty))$. Thus $\ker T_1(\tau^*) = (0)$.

Since $[0, \infty)$ has a fixed end point, then, by remark 62,

$$\ker \overline{T_0(\tau)} = \ker \overline{T_0(\tau^*)} = (0).$$

Thus, by proposition 74,

$$
\begin{aligned}
[T_1(\tau):\overline{T_0(\tau)}] &\geq \dim\ker T_1(\tau) - \dim\ker \overline{T_0(\tau^*)} \\
&\quad + \dim\ker T_1(\tau^*) - \dim\ker \overline{T_0(\tau^*)}, \\
&\geq 1-0+0-0=1.
\end{aligned}
$$

We may conclude that $[T_1(\tau):\overline{T_0(\tau)}]=1$, once we show that $0\notin \sigma_e(\tau)$. To do this, according to corollary 67 and proposition 73, it is enough to show that $0\notin \sigma_e(\overline{T_0(\tau^*)})$. To show that $0\notin \sigma_e(\overline{T_0(\tau^*)})$, it is enough to show that

$$
\|T_0(\tau^*)(\phi)\|^2 \geq \|\phi\|^2,
$$

for all $\phi\in W_0^2([0,\infty))$. This we do:

$$
\begin{aligned}
\|T_0(\tau^*)(\phi)\|^2 &= \left|\int_0^\infty (-d/dx+x)\phi\cdot\overline{(-d/dx+x)\phi}\,dx\right|, \\
&= \left|\int_0^\infty (d/dx+x)(-d/dx+x)\phi\cdot\overline{\phi}\,dx\right|, \\
&= \left|\int_0^\infty [(-d^2/dx^2)(\phi)\cdot\overline{\phi}+(x^2+1)|\phi|^2]\,dx\right| \\
&= \left|\int_0^\infty |\phi'|^2dx+\int_0^\infty (x^2+1)|\phi|^2dx\right|, \\
&= \int_0^\infty |\phi'|^2dx+\int_0^\infty (x^2+1)|\phi|^2dx, \\
&\geq \int_0^\infty |\phi|^2dx=\|\phi\|^2.
\end{aligned}
$$

Thus, $[T_1(\tau):\overline{T_0(\tau)}]=1$.

Corollary 76 *Let τ be a formally self-adjoint formal differential operator which is defined on an interval $I\subseteq\mathbf{R}$. Then*

$$
[T_1(\tau):\overline{T_0(\tau)}]\geq 2\cdot(\dim\ker T_1(\tau)-\dim\ker\overline{T_0(\tau)}).
$$

Equality holds when $0\notin\sigma_e(\tau)$.

Proof: From proposition 74. Q.E.D.

Corollary 77 *Let τ be a formal differential operator which is defined on an interval $I\subseteq\mathbf{R}$. Then*

$$
\begin{aligned}
[T_1(\tau^*)\,T_1(\tau):\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] &\geq 2\cdot(\dim\ker T_1(\tau)-\dim\ker\overline{T_0(\tau^*)}) \\
&\quad +2\cdot(\dim\ker T_1(\tau^*)-\dim\ker\overline{T_0(\tau)}).
\end{aligned}
$$

Equality holds when $0 \notin \sigma_e(\tau)$ (equivalently, when $0 \notin \sigma_e(\tau^*)$, or when $0 \notin \sigma_e(\tau^*\tau)$).

Proof: From corollary 34,

$$[T_1(\tau)\,T_1(\tau^*) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = 2 \cdot [T_1(\tau) : \overline{T_0(\tau)}].$$

To the right side of this equation, we apply the result of proposition 74.

The remarks about equality follow from the same remarks in proposition 74, and also, from the fact that $0 \notin \sigma_e(\tau^*\tau)$ if, and only if, $0 \notin \sigma_e(\tau)$: see proposition 73. Q.E.D.

Remark 78 Equality may hold in proposition 74, or in corollary 76, even though $0 \in \sigma_e(\tau)$. An example is $\tau = -(d^2/dx^2)$, operating in $L^2(\mathbf{R})$.

1. $\ker T_1(\tau) = \ker \overline{T_0(\tau)} = (0)$.

2. Recalling remarks 16 and 18, we notice that $[T_1(\tau) : \overline{T_0(\tau)}] = 0$. This is because the only solutions to $\tau(v) = \pm iv$ are

   $$e^{\mp\sqrt{i}x} = e^{\mp(1+i)x/\sqrt{2}}.$$

   Now, $|e^{\mp\sqrt{i}x}| = e^{\mp x/\sqrt{2}}$; thus, $e^{\mp\sqrt{i}x} \notin L^2(\mathbf{R})$.

3. Thus,

   $$0 = [T_1(\tau) : \overline{T_0(\tau)}] \geq 2 \cdot (\dim\ker T_1(\tau) - \dim\ker \overline{T_0(\tau)}) = 0.$$

4. Finally, we show that $0 \in \sigma_e(\tau)$. $\overline{T_0(\tau)}$ is a symmetric operator. From the results of paragraph 2, above, we see that

   $$\dim\, \mathcal{D}_+(T_1(\tau)) = \dim\, \mathcal{D}_-(T_1(\tau)) = 0.$$

   Thus,according to comment 15, $\overline{T_0(\tau)}$ has a self-adjoint extension. Since $[T_1(\tau) : \overline{T_0(\tau)}] = 0$, this extension must be trivial. Thus $\overline{T_0(\tau)}$ is self-adjoint. By Plancherel's theorem for $L^2(\mathbf{R})$, $\overline{T_0(\tau)}$ is unitarily equivalent to

   $$M_{x^2} : f(x) \mapsto x^2 f(x).$$

   It is clear that $0 \in \sigma_e(M_{x^2})$. Thus, $0 \in \sigma_e(\overline{T_0(\tau)})$. Hence, by corollary 67, $0 \in \sigma_e(\tau)$.

In the material which follows, we apply proposition 74, and corollaries 76 and 77, to the material of propositions 47 and 48.

Theorem 79 From [4, XIII.7.2]. *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Then $\lambda \in \sigma_e(\tau)$ if, and only if, there exists a sequence $\{f_n\} \subseteq \mathcal{D}(T_0(\tau))$, which is bounded in the $L^2(I)$ norm, and such that $\{(\tau-\lambda)f_n\}$ converges, but $\{f_n\}$ has no strongly convergent subsequence.*

Corollary 80 *Let τ be as above. Let $J \subseteq I$ be an interval. If $0 \notin \sigma_e(\tau, I)$, then $0 \notin \sigma_e(\tau, J)$.*

Proof: If $0 \in \sigma_e(\tau, J)$, there would exist $\{f_n\} \subseteq \mathcal{D}(T_0(\tau, J))$ with $\{\tau(f_n)\}$ convergent, but $\{f_n\}$ having no convergent subsequence.

Since $\mathcal{D}(T_0(\tau, J)) \subseteq \mathcal{D}(T_0(\tau, I))$ (see definition 1), we would have $\{f_n\} \subseteq \mathcal{D}(T_0(\tau, I))$, with $\{\tau(f_n)\}$ convergent, and $\{f_n\}$ having no convergent subsequence. This would entail $0 \in \sigma_e(\tau, I)$, a contradiction. Q.E.D.

Proposition 81 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. Choose $c \in I$ such that $a < c < b$. Let $I' = I \cap [a, c]$. Let $I'' = I \cap [c, b]$. Then*

$$\begin{aligned}[T_1(\tau, I) : \overline{T_0(\tau, I)}] \quad &\geq \quad \dim\ker T_1(\tau, I') + \dim\ker T_1(\tau^*, I') \\ &\quad + \dim\ker T_1(\tau, I'') + \dim\ker T_1(\tau^*, I'') - 2n.\end{aligned}$$

Equality holds when $0 \notin \sigma_e(\tau, I)$ (equivalently, when $0 \notin \sigma_e(\tau^*, I)$).

Proof: From proposition 47,

$$\begin{aligned}[T_1(\tau, I) : \overline{T_0(\tau, I)}] \quad &= \quad [T_1(\tau, I') : \overline{T_0(\tau, I')}] \\ &\quad + [T_1(\tau, I'') : \overline{T_0(\tau, I'')}] - 2n. \end{aligned} \tag{4.8}$$

From proposition 74, for $J = I'$ or for $J = I''$,

$$\begin{aligned}[T_1(\tau, J) : \overline{T_0(\tau, J)}] \quad &\geq \quad \dim\ker T_1(\tau, J) - \dim\ker \overline{T_0(\tau^*, J)} \\ &\quad + \dim\ker T_1(\tau^*, J) - \dim\ker \overline{T_0(\tau, J)}. \end{aligned} \tag{4.9}$$

But J has a fixed end point at c, so that, by remark 62,

$$\ker \overline{T_0(\tau, J)} = \ker \overline{T_0(\tau^*, J)} = (0).$$

Thus,

$$[T_1(\tau, J) : \overline{T_0(\tau, J)}] \geq \dim\ker T_1(\tau, J) + \dim\ker T_1(\tau^*, J). \tag{4.10}$$

Substituting this result into relation 4.8, for $J = I'$ and for $J = I''$, yields the desired result.

As to equality, relation 4.10 is an equality if $0 \notin \sigma_e(\tau, J)$ or if $0 \notin \sigma_e(\tau^*, J)$.

Now, $0 \notin \sigma_e(\tau, I)$ if, and only if, $0 \notin \sigma_e(\tau^*, I)$ (proposition 73), and these conditions imply that $0 \notin \sigma_e(\tau, J)$ and that $0 \notin \sigma_e(\tau^*, J)$, for $J = I'$ and $J = I''$ (corollary 80).

Thus, $0 \notin \sigma_e(\tau, I)$, or $0 \notin \sigma_e(\tau^*, I)$, implies that relation 4.10 is an equality, whence the conclusion of this proposition is an equality. Q.E.D.

Corollary 82 *In the previous proposition, let τ be formally self-adjoint. Then*

$$[T_1(\tau, I) : \overline{T_0(\tau, I)}] \geq 2 \cdot \dim\ker T_1(\tau, I') + 2 \cdot \dim\ker T_1(\tau, I'') - 2n.$$

Equality holds if $0 \notin \sigma_e(\tau)$.

Proof: This follows from proposition 81, since $\tau = \tau^*$. Q.E.D.

Example 83 Here is an example of proposition 81. We consider $\tau = (d/dx) + x$, on $I = \mathbf{R}$. Let c be any real number.

If f satisfies $\tau(v) = 0$, then $f \in \text{span}\{e^{-x^2/2}\}$. Thus,

$$\dim\ker T_1(\tau, (-\infty, c]) = \dim\ker T_1(\tau, [c, \infty)) = 1.$$

If f satisfies $\tau^*(v) = 0$, then $f \in \text{span}\{e^{x^2/2}\}$. Thus,

$$\dim\ker T_1(\tau^*, (-\infty, c]) = \dim\ker T_1(\tau^*, [c, \infty)) = 0.$$

τ is of order 1.

By an argument similar to the one in example 75, $0 \notin \sigma_e(\tau, \mathbf{R})$. Thus,

$$[T_1(\tau, \mathbf{R}) : \overline{T_0(\tau, \mathbf{R})}] = 1 + 0 + 1 + 0 - 2 \cdot 1 = 0.$$

Thus, $T_1(\tau,\mathbf{R}) = \overline{T_0(\tau,\mathbf{R})}$.

Proposition 84 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. Let $c \in I$ such that $a < c < b$. Let $I' = I \cap [a,c]$. Let $I'' = I \cap [c,b]$. Then*

$$[T_1(\tau^*,I)\,T_1(\tau,I) : \overline{T_0(\tau^*,I)}\,\overline{T_0(\tau,I)}] \geq$$
$$2\cdot(\dim\ker T_1(\tau,I') + \dim\ker T_1(\tau^*,I'))$$
$$+\,2\cdot(\dim\ker T_1(\tau,I'') + \dim\ker T_1(\tau^*,I'')) - 4n.$$

Equality holds when $0 \notin \sigma_e(\tau)$ (equivalently, $0 \notin \sigma_e(\tau^*)$) (equivalently, $0 \notin \sigma_e(\tau^*\tau)$).

Proof: We start with the third assertion of proposition 48, and then apply the conclusion of corollary 77, to the expression

$$[T_1(\tau^*,I')\,T_1(\tau,I') : \overline{T_0(\tau^*,I')}\,\overline{T_0(\tau,I')}],$$

keeping in mind that, by remark 62,

$$\dim\ker\overline{T_0(\tau^*,I')} = \dim\ker\overline{T_0(\tau,I')} = 0,$$

since I' has the fixed end point c. The same argument applies to the interval I''.

As for equality, the three hypotheses about essential spectrum are equivalent, according to proposition 73. If any one holds, then $0 \notin \sigma_e(\tau,I)$, whence $0 \notin \sigma_e(\tau,I')$ by corollary 80. Thus,

$$[T_1(\tau^*,I')\,T_1(\tau,I') : \overline{T_0(\tau^*,I')}\,\overline{T_0(\tau,I')}] = 2\cdot(\dim\ker T_1(\tau,I') + \dim\ker T_1(\tau^*,I')).$$

A similar statement is true for I''. The equality for the relation which is stated in this proposition follows from this. Q.E.D.

Corollary 85 *Under the hypotheses of the preceding proposition,*

$$\begin{aligned}\dim\ \mathcal{D}_\pm(T_1(\tau\tau^*,I)) \quad &\geq \quad \dim\ker T_1(\tau,I') + \dim\ker T_1(\tau^*,I') \\ &\quad + \dim\ker T_1(\tau,I'') + \dim\ker T_1(\tau^*,I'') - 2n.\end{aligned}$$

Proof: This follows from the first assertion of corollary 38, and proposition 84. Q.E.D.

The following lemma is an extension of [4, XIII.3.2].

Lemma 86 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$ with end points $a < b$. Choose $c \in I$ such that $a < c < b$. Let $I' = I \cap [a, c]$. Let $I'' = I \cap [c, b]$. Then*

$$\dim\ker T_1(\tau^*, I') + \dim\ker T_1(\tau^*, I'') \leq \dim\ker T_1(\tau^*, I) - \dim\ker \overline{T_0(\tau, I)} + n.$$

Equality holds when $0 \notin \sigma_e(\tau, I)$ (equivalently $0 \notin \sigma_e(\tau^*, I)$).

Proof: Let $T_0(\tau, I') \oplus T_0(\tau, I'')$ denote the operation τ with the domain $W_0^n(I') \oplus W_0^n(I'')$. $T_0(\tau, I') \oplus T_0(\tau, I'')$ is densely defined in $L^2(I)$; thus, it has an adjoint.

It is clear that

$$\{T_0(\tau, I') \oplus T_0(\tau, I'')\}^* = T_1(\tau^*, I') \oplus T_1(\tau^*, I'').$$

$T_1(\tau^*, I') \oplus T_1(\tau^*, I'')$ is densely defined; thus, $T_0(\tau, I') \oplus T_0(\tau, I'')$ has a closure, $\overline{T_0(\tau, I') \oplus T_0(\tau, I'')}$ (see theorem 3). It is easy to see that

$$\overline{T_0(\tau, I') \oplus T_0(\tau, I'')} = \overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}.$$

We point out that $\ker\{\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}\} = (0)$, because the same is true of $\overline{T_0(\tau, I')}$, and of $\overline{T_0(\tau, I'')}$. Now we show that

$$[\overline{T_0(\tau)} : \overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}] = n. \tag{4.11}$$

We choose $\{\psi_0, \psi_1, \ldots, \psi_{n-1}\} \subset W_0^n(I)$ such that $\psi_i^{(j)}(c) = 1$ if $i = j$, and $= 0$ if $i \neq j$, for $0 \leq i, j \leq n - 1$. Clearly,

$$\mathcal{D}(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}) \ + \ \text{span}\ \{\psi_0, \ldots, \psi_{n-1}\} \subseteq \ \mathcal{D}(\overline{T_0(\tau, I)})\ . \tag{4.12}$$

Now, let $\phi \in W_0^n(I)$. We define

$$f(x) = \phi(x) - \sum_{i=1}^{n-1} \phi^{(i)}(c) \cdot \psi_i(x).$$

Then $f^{(i)}(c) = 0$, for $0 \le i \le n-1$. Integration by parts shows that $f|_J \in \mathcal{D}(\{T_1(\tau^*, J)\}^*)$, for $J = I'$ or $J = I''$. Hence, $f \in \mathcal{D}(\{T_0(\tau, I') \oplus T_0(\tau, I'')\}^{**})$. That is, $f \in \mathcal{D}(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')})$. Thus,

$$\mathcal{D}(T_0(\tau, I)) \subseteq \mathcal{D}(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}) + \text{span}\,\{\psi_0, \ldots, \psi_{n-1}\}.$$

The right side of this relation must be closed in the $T_1(\tau, I)$–operator norm, because it is a finite dimensional extension of $\mathcal{D}(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')})$, which is closed in the $T_1(\tau, I)$–operator norm (see proposition 6). Thus,

$$\mathcal{D}(\overline{T_0(\tau, I)}) \subseteq \mathcal{D}(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}) + \text{span}\,\{\psi_0, \ldots, \psi_{n-1}\}. \tag{4.13}$$

Lines 4.12 and 4.13 entail line 4.11.

Let Π denote the natural projection

$$\begin{aligned} \Pi : \mathcal{D}(\overline{T_0(\tau, I)}) \quad &\rightarrow \quad \mathcal{D}(\overline{T_0(\tau, I)}) \,/(\, \mathcal{D}(\overline{T_0(\tau, I')}) \,\oplus\, \mathcal{D}(\overline{T_0(\tau, I'')})\,), \\ f \quad &\mapsto \quad f + (\, \mathcal{D}(\overline{T_0(\tau, I')}) \,\oplus\, \mathcal{D}(\overline{T_0(\tau, I'')})\,). \end{aligned}$$

Let T denote the induced linear operator,

$$\begin{aligned} T : \mathcal{D}(\overline{T_0(\tau, I)}) \,/\, \mathcal{D}(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}) \quad &\rightarrow \quad \text{Range}\,(\overline{T_0(\tau, I)})/\text{Range}\,(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}), \\ f + \mathcal{D}(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}) \quad &\mapsto \quad \overline{T_0(\tau, I)}(f) + \text{Range}\,(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}). \end{aligned}$$

Notice that T is onto. Also, $\ker T = \Pi(\ker \overline{T_0(\tau, I)})$. Thus,

$$\dim\ker T = \dim\ker \overline{T_0(\tau, I)} - \dim\ker(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}) = \dim\ker \overline{T_0(\tau, I)}.$$

Using a result of finite dimensional algebra, we have

$$\begin{aligned} &\dim\ \mathcal{D}(\overline{T_0(\tau, I)}) \,/\, \mathcal{D}(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')}) \\ &= \dim\ker T + \dim\{\text{Range}\,(\overline{T_0(\tau, I)})/\text{Range}\,(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')})\}. \\ &= \dim\ker \overline{T_0(\tau, I)} + \dim\{\text{Range}\,(\overline{T_0(\tau, I)})/\text{Range}\,(\overline{T_0(\tau, I')} \oplus \overline{T_0(\tau, I'')})\}. \end{aligned}$$

Thus,

$$[\overline{T_0(\tau,I)} : \overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')}] =$$

$$\dim\ker \overline{T_0(\tau,I)} + \dim\{\text{Range } (\overline{T_0(\tau,I)})/\text{Range } (\overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')})\}. \tag{4.14}$$

Lines 4.11 and 4.14 entail

$$n = \dim\ker \overline{T_0(\tau,I)} + \dim\{\text{Range } (\overline{T_0(\tau,I)})/\text{Range } (\overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')})\}.$$

Now, using lemma 64, we have

$$n \geq \dim\ker \overline{T_0(\tau,I)} + \dim\{\overline{\text{Range } (\overline{T_0(\tau,I)})}/\overline{\text{Range } (\overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')})}\}. \tag{4.15}$$

Thus,

$$n \geq \dim\ker \overline{T_0(\tau,I)} + \dim\{\overline{\text{Range } (\overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')})}^{\perp} \ominus \overline{\text{Range } (\overline{T_0(\tau,I)})}^{\perp}\}.$$

Thus,

$$n \geq \dim\ker \overline{T_0(\tau,I)} + \dim\ker\{T_1(\tau^*,I) \oplus T_1(\tau^*,I'')\} - \dim\ker T_1(\tau^*,I).$$

Thus,

$$n \geq \dim\ker \overline{T_0(\tau,I)} + \dim\ker T_1(\tau^*,I') + \dim\ker T_1(\tau^*,I'') - \dim\ker T_1(\tau^*,I), \tag{4.16}$$

which is what we sought to prove.

This relation is an equality when

$$\dim\{\overline{\text{Range } (\overline{T_0(\tau,I)})}/\overline{\text{Range } (\overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')})}\}$$

$$= \dim\{\text{Range } (\overline{T_0(\tau,I)})/\text{Range } (\overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')})\}. \tag{4.17}$$

From line 4.11, we deduce that

$$\dim\{\text{Range } (\overline{T_0(\tau,I)})/\text{Range } (\overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')})\} \leq n.$$

Thus, Range $(\overline{T_0(\tau,I)})$ is closed if, and only if, Range $(\overline{T_0(\tau,I')} \oplus \overline{T_0(\tau,I'')})$ is closed if, and only if, $0 \notin \sigma_e(\tau,I)$ (equivalently, $0 \notin \sigma_e(\tau^*,I)$). In this case, lines 4.17, 4.15, and 4.16 are equalities. Q.E.D.

Chapter 5

When $T_1(\tau^*)T_1(\tau) = T_1(\tau^*\tau)$

Remark 87 In this chapter, we state and prove two conditions such that

$$T_1(\tau^*\tau) = T_1(\tau^*)T_1(\tau). \tag{5.1}$$

This equation is equivalent to

$$g \in \mathcal{D}_\pm(T_1(\tau^*\tau)) \text{ entails } \tau(g) \in L^2(I). \tag{5.2}$$

We demonstrate this. According to relation 2.13,

$$\overline{T_0(\tau^*\tau)} \subseteq \overline{T_0(\tau^*)}\,\overline{T_0(\tau)} \subseteq T_1(\tau^*)T_1(\tau) \subseteq T_1(\tau^*\tau). \tag{5.3}$$

Thus, $[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] \geq [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}]$. According to remarks 18 and 19,

$$[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] = 2 \cdot \dim\ \mathcal{D}_+(T_1(\tau^*\tau))\ . \tag{5.4}$$

According to corollary 36,

$$[T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] = 2 \cdot \dim\{g \in \mathcal{D}_+(T_1(\tau^*\tau))\ : \tau(g) \in L^2(I)\}. \tag{5.5}$$

If equation 5.1 is true, then $\overline{T_0(\tau^*\tau)} = \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}$, according to corollary 35. Hence,

$$\begin{aligned} 2 \cdot \dim\ \mathcal{D}_+(T_1(\tau^*\tau)) &= [T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] \\ &= [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}] \\ &= 2 \cdot \dim\{g \in \mathcal{D}_+(T_1(\tau^*\tau))\ : \tau(g) \in L^2(I)\}. \end{aligned} \tag{5.6}$$

Certainly, $\mathcal{D}_+(T_1(\tau^*\tau)) \supseteq \{g \in \mathcal{D}_+(T_1(\tau^*\tau)) : \tau(g) \in L^2(I)\}$. This, and equation 5.6, require that statement 5.2 be true.

On the other hand, if statement 5.2 is true, then the right sides of equations 5.4 and 5.5 are equal. Thus,

$$[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] = [T_1(\tau^*)T_1(\tau) : \overline{T_0(\tau^*)}\,\overline{T_0(\tau)}].$$

Then, using relation 5.3, we would have equation 5.1.

Lemma 88 From [9, prob. 5.8, p. 233] *Let T, with domain $\mathcal{D}(T)$, and $\hat{T}$, with domain $\mathcal{D}(\hat{T})$, be closed, densely defined operators in a Hilbert space, such that* $\ker(T)$, $\ker(T^*)$, $\ker(\hat{T})$, *and* $\ker(\hat{T}^*)$ *are all finite dimensional, and such that $0 \notin \sigma_e(T)$, and $0 \notin \sigma_e(\hat{T}^*)$.*

If $T \subseteq \hat{T}$, and ind $(T) =$ ind $(\hat{T})$, then $T = \hat{T}$.

Proof:

1. $T \subseteq \hat{T}$ entails $\ker T \subseteq \ker \hat{T}$. Thus,

$$\dim\ker T - \dim\ker \hat{T} \leq 0.$$

2. $T \subseteq \hat{T}$ entails Range $(T) \subseteq$ Range $(\hat{T})$, which entails $\ker T^* \supseteq \ker \hat{T}^*$. Thus,

$$\dim\ker T^* - \dim\ker \hat{T}^* \geq 0.$$

3. ind $(T) =$ ind $(\hat{T})$ entails

$$\dim\ker T^* - \dim\ker T = \dim\ker \hat{T}^* - \dim\ker \hat{T}.$$

Thus,

$$\dim\ker T^* - \dim\ker \hat{T}^* = \dim\ker T - \dim\ker \hat{T}.$$

From paragraph 2, the left side of this last equation is non-negative, and from paragraph 1, the right side is non-positive. Thus,

$$0 = \dim\ker T^* - \dim\ker \hat{T}^* = \dim\ker T - \dim\ker \hat{T}.$$

From this, we see that $\ker T = \ker \hat{T}$, and that $\ker T^* = \ker \hat{T}^*$. This last fact, together with the facts that Range (T) is closed, and that Range $(\hat{T})$ is closed, leads us to conclude that Range $(T) =$ Range $(\hat{T})$.

4. Let $g \in \mathcal{D}(\hat{T})$. If $g \in \ker \hat{T}$, then $g \in \ker T$, whence $g \in \mathcal{D}(T)$. If $g \notin \ker \hat{T}$, then

$$\hat{T}(g) \in \text{Range } (\hat{T}) = \text{Range } (T).$$

Thus, there is $h \in \mathcal{D}(T)$ such that $T(h) = \hat{T}(g)$. Since $h \in \mathcal{D}(\hat{T})$, then $\hat{T}(h - g) = 0$, which leads to

$$h - g \in \ker \hat{T} = \ker T \subseteq \mathcal{D}(T).$$

From this, and the fact that $h \in \mathcal{D}(T)$, we conclude that $g \in \mathcal{D}(T)$. Q.E.D.

Proposition 89 *Let τ be a formal differential operator of order n, which is defined on an interval $I \subseteq \mathbf{R}$. Suppose that $0 \notin \sigma_e(\tau^*\tau, I)$, and that $\dim \ker T_1(\tau^*, I) = n$.*

Then $T_1(\tau^*\tau, I) = T_1(\tau^*, I)T_1(\tau, I)$.

Proof: Let a and b, with $a < b$, denote the end points of I. Pick $c \in I$ such that $a < c < b$. Let $I' = I \cap [a, c]$. Let $I'' = I \cap [c, b]$.

Relying on relation 2.13, we have

$$\overline{T_0(\tau^*\tau, I)} \subseteq \overline{T_0(\tau^*, I)}\,\overline{T_0(\tau, I)} \subseteq T_1(\tau^*, I)T_1(\tau, I) \subseteq T_1(\tau^*\tau, I).$$

By remark 87, it is enough to show that $\tau(g) \in L^2(I)$, for all $g \in \mathcal{D}_\pm(T_1(\tau^*\tau, I))$. To do this, we will show that $\tau(g) \in L^2(I')$, and $\tau(g) \in L^2(I'')$. To show that $\tau(g) \in L^2(I')$, it is enough to show that

$$T_1(\tau^*\tau, I') = T_1(\tau^*, I')\,T_1(\tau, I').$$

We will use lemma 88 to do this.

According to corollary 80, $0 \notin \sigma_e(\tau^*\tau, I')$.

We rely on relation 2.13 to see that $T_1(\tau^*\tau, I') \supseteq T_1(\tau^*, I')\,T_1(\tau, I')$, whence

$$\ker T_1(\tau^*\tau, I') \supseteq \ker T_1(\tau^*, I')T_1(\tau, I'). \tag{5.7}$$

Now we show that

$$\text{ind }(\overline{T_0(\tau^*\tau, I')}) = \text{ind }(\overline{T_0(\tau^*, I')}\,\overline{T_0(\tau, I')}).$$

Since I' has a fixed end point,

$$\ker \overline{T_0(\tau^*\tau, I')} = \ker \overline{T_0(\tau^*, I')}\,\overline{T_0(\tau, I')} = (0), \tag{5.8}$$

by remark 62. Now, choose $h \in \ker T_1(\tau^*\tau, I')$. We have $\tau^*\tau(h) = 0$. Thus, $\tau^*(\tau(h)) = 0$. Since $\dim\ker T_1(\tau^*, I) = n$, then $\dim\ker T_1(\tau^*, I') = n$. From $\tau^*(\tau(h)) = 0$, and from $\dim\ker T_1(\tau^*, I') = n$, we conclude that $\tau(h) \in L^2(I')$. Thus,

$$h \in \ker T_1(\tau^*, I')\, T_1(\tau, I').$$

Thus,

$$\ker T_1(\tau^*\tau, I') \subseteq \ker T_1(\tau^*, I')\, T_1(\tau, I'). \tag{5.9}$$

From lines 5.7 and 5.9, we conclude that

$$\ker T_1(\tau^*\tau, I') = \ker T_1(\tau^*, I')\, T_1(\tau, I').$$

From this, and relation 5.8, we conclude that

$$\text{ind }(T_1(\tau^*\tau, I')) = \text{ind }(T_1(\tau^*, I')\, T_1(\tau, I')).$$

Now we can use lemma 88 to conclude that $T_1(\tau^*\tau, I') = T_1(\tau^*, I')\, T_1(\tau, I')$. Similarly, $T_1(\tau^*\tau, I'') = T_1(\tau^*, I'')\, T_1(\tau, I'')$. Q.E.D.

Remark 90 Under the hypotheses of the preceding proposition, we have

$$\dim\ \mathcal{D}_{\pm}(T_1(\tau^*\tau, I)) = \dim\{g \in \mathcal{D}_{\pm}(T_1(\tau^*\tau, I)) : \tau(g) \in L^2(I)\},$$

according to an argument in remark 87. Now, if $g \in \mathcal{D}_{\pm}(T_1(\tau^*\tau, I))$, and if $g \neq 0$, then $\tau(g) \neq 0$. If this were *not* true, we would have

$$\begin{aligned} \tau(g) &= 0 \\ \tau^*\tau(g) &= \tau^*(0) \end{aligned}$$

$$\begin{aligned} \pm ig &= 0 \\ g &= 0, \end{aligned}$$

which is a contradiction. Thus, the map $g \mapsto \tau(g)$ is 1-1. Thus,

$$\dim \mathcal{D}_{\pm}(T_1(\tau^*\tau, I)) = \dim\{\tau(g) \in L^2(I) : g \in \mathcal{D}_{\pm}(T_1(\tau^*\tau, I))\}. \tag{5.10}$$

Thus, by lemma 20, we have

$$\dim \mathcal{D}_{\pm}(T_1(\tau^*\tau, I)) \leq \dim \mathcal{D}_{\pm}(T_1(\tau\tau^*, I)) .$$

Another way to state this is

$$[T_1(\tau^*\tau, I) : \overline{T_0(\tau^*\tau, I)}] \leq [T_1(\tau\tau^*, I) : \overline{T_0(\tau\tau^*, I)}].$$

Example 91 Here is a theorem, from [4, XIII.6.23], which we will use in this example:

Let

$$\tau = -(d^2/dx^2) + q(x)$$

be a real self adjoint second order formal differential operator defined on an interval $I = (0, b]$, *where* $b > 0$. *Then*

1. *if* $\liminf_{x \to 0} x^2 q(x) > 3/4$, *then* τ *has no boundary values at zero; and*

2. *if* $\limsup_{x \to 0} |x^2 q(x)| < 3/4$, *then* τ *has two boundary values at zero.*

Let $p > 0$. On the interval $I = (0, 1]$, we consider $\tau = d/dx + p/x$. Clearly, $\tau^* = -d/dx + p/x$. Notice that $f(x) = x^p \in L^2(I)$, and that $\tau^*(x^p) = 0$. Thus,

$$\dim \ker T_1(\tau^*) = \text{ order of } \tau^* = 1. \tag{5.11}$$

Notice further that

$$\begin{aligned} \tau^*\tau &= -(d^2/dx^2) + (p^2 + p)/x^2, \text{and} \\ \tau\tau^* &= -(d^2/dx^2) + (p^2 - p)/x^2. \end{aligned}$$

1. For both $\tau^*\tau$ and $\tau\tau^*$, the end point one is a fixed end point. Thus, both $\tau^*\tau$ and $\tau\tau^*$ have two boundary values at one (see corollary 46).

2. Let $p = 1$. We apply the above theorem to

$$\tau^*\tau = -(d^2/dx^2) + 2/x^2$$

to see that $\tau^*\tau$ has no boundary values at zero. If we combine this with the result of paragraph 1, we see that

$$[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] = 2. \tag{5.12}$$

Next, we apply the above theorem to

$$\tau\tau^* = -(d^2/dx^2)$$

to see that $\tau\tau^*$ has two boundary values at zero. If we combine this with the result of paragraph 1, we see that

$$[T_1(\tau\tau^*) : \overline{T_0(\tau\tau^*)}] = 4. \tag{5.13}$$

Equations 5.11, 5.12, and 5.13 show that remark 90 is verified, that

$$4 = [T_1(\tau\tau^*) : \overline{T_0(\tau\tau^*)}] \geq [T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] = 2.$$

3. Let $p = 1/4$. We apply the above theorem to

$$\tau^*\tau = -(d^2/dx^2) + 5/(16x^2)$$

to see that $\tau^*\tau$ has two boundary values at zero. If we combine this with the result of paragraph 1, we see that

$$[T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] = 4. \tag{5.14}$$

Next, we apply the above theorem to

$$\tau\tau^* = -(d^2/dx^2) - 3/(16x^2)$$

to see that $\tau\tau^*$ has two boundary values at zero. If we combine this with the result of paragraph 1, we see that

$$[T_1(\tau\tau^*) : \overline{T_0(\tau\tau^*)}] = 4. \tag{5.15}$$

Equations 5.11, 5.14, and 5.15 show that remark 90 is verified, that

$$4 = [T_1(\tau\tau^*) : \overline{T_0(\tau\tau^*)}] \geq [T_1(\tau^*\tau) : \overline{T_0(\tau^*\tau)}] = 4.$$

Remark 92 The second condition is that

$$\begin{aligned} \ker T_1((\tau\tau^*)^2) &\subseteq \mathcal{D}(T_1(\tau\tau^*)) \text{ entails} \\ T_1(\tau^*)T_1(\tau) &= T_1(\tau^*\tau). \end{aligned}$$

First, we need to state several lemmas. The first, due to I. M. Glazman, is a specialization of [5, p. 193, lemma 2.1].

Lemma 93 *Let a Hilbert space $\mathcal{H}$ be decomposed into a direct sum of a subspace S and a finite dimensional subspace F : $\mathcal{H} = S \dot{+} F$. Let $\{r_1, r_2, \ldots, r_k\}$ be a basis for F, and let M be a linear manifold which is dense in $\mathcal{H}$. Then*

1. *$M \cap S$ is dense in S;*
2. *the space $\mathcal{H}$ can be represented as a direct sum of subspaces S and F', where $F' \subseteq M$;*
3. $\dim F = \dim F'$ *(this follows from a remark in the last part of the proof of [5, lemma 2.1]; and*
4. *there exists $\{m_1, \ldots, m_k\} \subseteq F'$ such that $\det(\langle r_i, m_j \rangle, 1 \leq i, j \leq k) \neq 0$ (this follows from remarks in the proof of [5, lemma 2.1]).*

Lemma 94 *Let T, with domain $\mathcal{D}(T)$, be a closed operator in a Hilbert space, such that $0 \notin \sigma_e(T)$, and such that $\dim \ker T < \infty$. Then there is a positive number p such that*

$$p \cdot \|f\| \leq \|T(f)\|,$$

for all $f \in \mathcal{D}(T) \ominus \ker T$.

Proof: $\mathcal{D}(T)$, with the operator inner product, is a Hilbert space, because T is a closed operator (see remark 4). Let $[f,g]$ denote the operator inner product.

In this Hilbert space, the orthogonal complement of $\ker T$ is a subspace, and equals

$$\begin{aligned} & \{f \in \mathcal{D}(T) : \forall k \in \ker T, [f,k] = 0\} \\ = & \{f \in \mathcal{D}(T) : \forall k \in \ker T, \langle f,k \rangle = 0\} \\ = & \mathcal{D}(T) \ominus \ker T. \end{aligned}$$

Let T_r denote the operator with domain $\mathcal{D}(T) \ominus \ker T$, and such that

$$T_r(f) = T(f), \text{for } f \in \mathcal{D}(T) \ominus \ker T.$$

T_r is a closed operator because it is a restriction of T to $\mathcal{D}(T) \ominus \ker T$, which is a subspace of $\mathcal{D}(T)$. Clearly, Range (T) = Range (T_r). Also, $\ker T_r = (0)$. This shows that T_r defines a 1-1 map

$$\begin{aligned} T_r : \mathcal{D}(T) \ominus \ker T & \rightarrow \text{Range } (T) \\ f & \mapsto T(f). \end{aligned}$$

The inverse map T_r^{-1} is well defined. It is closed because T_r is closed (see [4, XII.1.2]). We conclude that T_r^{-1} is a bounded map (see [12, p. 306]). Thus, there exists a positive constant c, such that, for all $g \in$ Range (T_r) = Range (T),

$$||T_r^{-1}(g)|| \leq c \cdot ||g||.$$

But $T_r^{-1}(g) \in \mathcal{D}(T) \ominus \ker T$. As g assumes every value in Range (T_r), $T_r^{-1}(g)$ asumes every value in $\mathcal{D}(T) \ominus \ker T$. Thus, for all $f \in \mathcal{D}(T) \ominus \ker T$, there exists $g \in$ Range (T_r) such that $f = T_r^{-1}(g)$, and

$$||f|| = ||T_r^{-1}(g)|| \leq c \cdot ||g|| = c \cdot ||T(f)||.$$

If we let $p = 1/c$, the lemma is proven. Q.E.D.

Lemma 95 *Let τ be a formal differential operator, of order n, which is defined on an interval $I \subseteq \mathbf{R}$, such that $0 \notin \sigma_e(\tau^*\tau)$. Suppose $g \in \mathcal{D}_\pm(T_1(\tau^*\tau))$. Then there is a positive number c such* *that, for all $\phi \in W_0^{2n}(I)$,*

$$|\langle T_0(\tau\tau^*)(\phi), \tau(g)\rangle| \leq c \cdot \|g\| \cdot \|T_0(\tau\tau^*)(\phi)\|.$$

Proof: If $\phi = 0$, the assertion is clearly true. Now we assume that $\phi \neq 0$.

$$\begin{aligned}
|\langle T_0(\tau\tau^*)(\phi), \tau(g)\rangle| &= |\int_I \tau\tau^*(\phi) \cdot \overline{\tau(g)}\, dt| \\
&= |\int_I \tau^*(\phi) \cdot \overline{\tau^*\tau(g)}\, dt| \\
&= |\int_I \tau^*(\phi) \cdot \overline{\pm ig}\, dt| \\
&\leq \|T_0(\tau^*)(\phi)\| \cdot \|g\| \\
&= \frac{\|T_0(\tau^*)(\phi)\|}{\|T_0(\tau\tau^*)(\phi)\|} \cdot \|T_0(\tau\tau^*)(\phi)\| \cdot \|g\|. \qquad (5.16)
\end{aligned}$$

Now, $\tau\tau^*$ may be represented as

$$a_{2n}(t)(d/dt)^{2n} + a_{2n-1}(t)(d/dt)^{2n-1} + \cdots + a_1(t)(d/dt) + a_0(t),$$

where $a_{2n}(t) \neq 0$ for $t \in I$, and $\{a_0(), a_1(), \ldots, a_{2n}()\} \subseteq C^\infty(I)$; thus, no solution of the equation $\tau\tau^*(v) = 0$ has compact support in the interior of I. Thus $T_0(\tau\tau^*)(\phi) \neq 0$.

Further, we have

$$\begin{aligned}
T_0(\tau\tau^*) \subseteq \overline{T_0(\tau\tau^*)} &\subseteq \overline{T_0(\tau)}\,\overline{T_0(\tau^*)} \text{ (lemma 23)} \\
&\subseteq T_1(\tau)\,\overline{T_0(\tau^*)}.
\end{aligned}$$

Thus,

$$|\langle T_0(\tau\tau^*)(\phi), \tau(g)\rangle| \leq \frac{\|\overline{T_0(\tau^*)}(\phi)\|}{\|T_1(\tau)\,\overline{T_0(\tau^*)}(\phi)\|} \cdot \|T_0(\tau\tau^*)(\phi)\| \cdot \|g\|.$$

Because we assume that $0 \notin \sigma_e(\tau^*\tau)$, then $0 \notin \sigma_e(\tau)$ (proposition 73). Thus, $0 \notin \sigma_e(T_1(\tau))$. Thus, by the previous lemma, we know that there is a positive number p such that, for all $f \in \mathcal{D}(T_1(\tau)) \ominus \ker T_1(\tau)$,

$$p \cdot \|f\| \leq \|T_1(\tau)(f)\|$$

But $\overline{T_0(\tau^*)}(\phi) \in$ Range $(\overline{T_0(\tau^*)})$, so that $\overline{T_0(\tau^*)}(\phi) \perp \ker T_1(\tau)$. Thus,

$$0 < p \cdot \|\overline{T_0(\tau^*)}(\phi)\| \leq \|T_1(\tau)\,\overline{T_0(\tau^*)}(\phi)\|.$$

From this, we see that

$$\frac{||T_0(\tau^*)(\phi)||}{||T_0(\tau\tau^*)(\phi)||} = \frac{||T_0(\tau^*)(\phi)||}{||T_1(\tau)\,T_0(\tau^*)(\phi)||} \leq \frac{1}{p}.$$

Substituting this result into relation 5.16 yields the desired result. Q.E.D.

Lemma 96 *Let T, with domain $\mathcal{D}(T)$, be an operator in a Hilbert space. Suppose that T has a closure, $\overline{T}$. Then*

$$\overline{Range\ (T)} = \overline{Range\ (\overline{T})}.$$

Proof: Range (T) is dense in Range $(\overline{T})$, and Range $(\overline{T})$ is dense in $\overline{\text{Range } (\overline{T})}$. Thus, Range (T) is dense in $\overline{\text{Range } (\overline{T})}$. Q.E.D.

Proposition 97 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$, such that $0 \notin \sigma_e(\tau^*\tau)$. If*

$$\ker T_1((\tau\tau^*)^2) \subseteq \mathcal{D}(T_1(\tau\tau^*)) ,$$

then $T_1(\tau^*)T_1(\tau) = T_1(\tau^*\tau)$.

Proof: Let n denote the order of τ. Relying on relation 2.13, we have $T_1(\tau^*)T_1(\tau) \subseteq T_1(\tau^*\tau)$. It remains to show that $T_1(\tau^*\tau) \subseteq T_1(\tau^*)T_1(\tau)$, or equivalently, that $\mathcal{D}(T_1(\tau^*\tau)) \subseteq \mathcal{D}(T_1(\tau^*)T_1(\tau))$. According to lemma 14,

$$\mathcal{D}(T_1(\tau^*\tau)) \;=\; \mathcal{D}(\overline{T_0(\tau^*\tau)}) \;\oplus\; \mathcal{D}_+(T_1(\tau^*\tau)) \;\oplus\; \mathcal{D}_-(T_1(\tau^*\tau)) \,.$$

Using relation 2.13, we have

$$\mathcal{D}(\overline{T_0(\tau^*\tau)}) \;\subseteq\; \mathcal{D}(\overline{T_0(\tau^*)}\,\overline{T_0(\tau)}) \;\subseteq\; \mathcal{D}(T_1(\tau^*)T_1(\tau)) \,.$$

It only remains now to show that

$$\mathcal{D}_+(T_1(\tau^*\tau)) \;\oplus\; \mathcal{D}_-(T_1(\tau^*\tau)) \;\subseteq\; \mathcal{D}(T_1(\tau^*)T_1(\tau)) \,.$$

To do this, it is enough to show that if $g \in \mathcal{D}_\pm(T_1(\tau^*\tau))$, then $\tau(g) \in L^2(I)$. We show that $\tau(g) \in \mathcal{D}(T_1(\tau^*\tau)) \subseteq L^2(I)$. The rest of the proof is divided into 5 steps.

1. Here, we start with $\tau(g)$, and then construct a bounded linear functional on Range $(\overline{T_0(\tau\tau^*)})$, which is induced by $\tilde{q} \in$ Range $(\overline{T_0(\tau\tau^*)})$.

By proposition 73, $0 \notin \sigma_e(\tau\tau^*)$. Thus, Range $(\overline{T_0(\tau\tau^*)}) = \overline{\text{Range } (\overline{T_0(\tau\tau^*)})}$.

According to lemma 95, there is a positive number c such that, for all $\tau\tau^*(\phi) \in \tau\tau^*(W_0^{2n}(I))$,

$$|\int_I \tau\tau^*(\phi) \cdot \overline{\tau(g)}\, dx| \leq c \cdot \|g\| \cdot \|\tau\tau^*(\phi)\|.$$

This says that $\tau(g)$ induces a bounded linear functional on $\tau\tau^*(W_0^{2n}(I)) =$ Range $(T_0(\tau\tau^*))$.

By continuity, we can extend this bounded linear functional to one defined on Range $(\overline{T_0(\tau\tau^*)})$. because Range $(T_0(\tau\tau^*))$ is dense in Range $(\overline{T_0(\tau\tau^*)}) = \overline{\text{Range } (\overline{T_0(\tau\tau^*)})}$: see lemma 96. Since Range $(\overline{T_0(\tau\tau^*)})$ is a subspace of $L^2(I)$, the Riesz-Frechet theorem tells us that there is a unique $\tilde{q} \in$ Range $(\overline{T_0(\tau\tau^*)})$ such that, for all $\tau\tau^*(\phi) \in$ Range $(T_0(\tau\tau^*))$,

$$\int_I \tau\tau^*(\phi) \cdot \overline{\tilde{q}}\, dx = \int_I \tau\tau^*(\phi) \cdot \overline{\tau(g)}\, dx.$$

Restating this: for all $\phi \in W_0^{2n}(I)$,

$$\int_I \tau\tau^*(\phi) \cdot (\overline{\tilde{q}} - \overline{\tau(g)})\, dx = 0.$$

From this, and from lemma 7, we conclude that $\tilde{q} - \tau(g)$ is a solution of the differential equation $\tau\tau^*(v) = 0$.

Also, we want to show that $g \neq 0$ entails $\tilde{q} \neq 0$. Suppose there exists g such that $\tilde{q} = 0$. Since $\tau(g) - \tilde{q}$ satisfies the differential equation $\tau\tau^*(v) = 0$, $\tilde{q} = 0$ entails that $\tau(g)$ satisfy the equation $\tau\tau^*(v) = 0$. Thus:

$$\begin{aligned}
\tau\tau^*(\tau(g)) &= 0, \\
\pm i\tau(g) &= 0, \\
\tau(g) &= 0, \\
\tau^*\tau(g) &= 0, \\
\pm ig &= 0, \\
g &= 0.
\end{aligned}$$

2. (a) Here, we construct a second bounded linear functional, an extension of the first one, which is defined on Range $(T_1(\tau\tau^*))$.

As noted above, Range $(\overline{T_0(\tau\tau^*)})$ is a subspace of $L^2(I)$; its orthocomplement is $\ker T_1(\tau\tau^*)$, which is finite dimensional. Let $m = \dim\ker T_1(\tau\tau^*)$.

$W_0^{2n}(I)$ is dense in $L^2(I)$. Because $0 \notin \sigma_e(\tau\tau^*)$, Range $(T_1(\tau\tau^*))$ is a subspace of $L^2(I)$; its orthocomplement is $\ker \overline{T_0(\tau\tau^*)}$, which is finite demensional. Thus, by lemma 93, $W_0^{2n}(I) \cap$ Range $(T_1(\tau\tau^*))$ is dense in Range $(T_1(\tau\tau^*))$.

Range $(\overline{T_0(\tau\tau^*)})$ is a subspace of Range $(T_1(\tau\tau^*))$. We have

$$\text{Range } (T_1(\tau\tau^*)) = \text{Range } (\overline{T_0(\tau\tau^*)}) \oplus [\text{Range } (T_1(\tau\tau^*)) \ominus \text{Range } (\overline{T_0(\tau\tau^*)})].$$

But

$$\begin{aligned} &\text{Range } (T_1(\tau\tau^*)) \ominus \text{Range } (\overline{T_0(\tau\tau^*)}) \\ &= \{f : f \in \text{Range } (T_1(\tau\tau^*)) \text{ and } f \perp \text{Range } (\overline{T_0(\tau\tau^*)})\} \\ &= \{f : f \perp \ker \overline{T_0(\tau\tau^*)} \text{ and } f \in \ker T_1(\tau\tau^*)\} \\ &= \ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}. \end{aligned}$$

Moreover,

$$\dim[\ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}] \le \dim\ker T_1(\tau\tau^*) < \infty.$$

This shows that the orthogonal complement of Range $(\overline{T_0(\tau\tau^*)})$ in Range $(T_1(\tau\tau^*))$ is finite dimensional. Let $l = \dim[\ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}]$. Let $\{r_1, \ldots, r_l\}$ be a basis for $\ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}$.

Because $\dim[\text{Range } (T_1(\tau\tau^*)) \ominus \text{Range } (\overline{T_0(\tau\tau^*)})] < \infty$, and because

$$W_0^{2n}(I) \cap \text{Range } (T_1(\tau\tau^*))$$

is dense in Range $(T_1(\tau\tau^*))$, lemma 93 leads us to conclude that there is an l-dimensional space

$$F' \subseteq W_0^{2n}(I) \cap \text{Range } (T_1(\tau\tau^*)),$$

with basis $\{\phi_1, \ldots, \phi_l\}$ such that

- Range $(\overline{T_0(\tau\tau^*)}) \cap F' = (0)$;
- Range $(T_1(\tau\tau^*)) =$ Range $(\overline{T_0(\tau\tau^*)}) + F'$;
- $1 = \|\phi_1\| = \cdots = \|\phi_l\|$; and
- $\det(\langle \phi_i, r_j\rangle, 1 \leq i, j \leq l) \neq 0$.

Because $\tau\tau^*(W_0^{2n}(I))$ is dense in Range $(\overline{T_0(\tau\tau^*)})$, then $\tau\tau^*(W_0^{2n}(I)) + F'$ is dense in Range $(T_1(\tau\tau^*))$. Let M denote $\tau\tau^*(W_0^{2n}(I)) + F'$.

Let P denote the orthogonal projection onto $\ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}$. If $\Phi \in M$, then

$$\Phi = \tau\tau^*(\phi) + \sum_{i=1}^{l} c_i \cdot \phi_i, \tag{5.17}$$

where $\{c_1, \ldots, c_l\} \subseteq \mathbf{C}$, and $\tau\tau^*(\phi) \in \tau\tau^*(W_0^{2n}(I))$. In 2b and 2c, below, we will show that $c_1, \ldots, c_l$, and $\tau\tau^*(\phi)$ depend continuously on Φ.

(b) From equation 5.17, we get

$$P(\Phi) = \sum_{i=1}^{l} c_i \cdot P(\phi). \tag{5.18}$$

From this equation, we derive a system of l equations, in the unknowns $c_1, \ldots, c_l$:

$$\begin{cases} \langle P(\Phi), \phi_1\rangle & = & \sum_{j=1}^{l} c_j \cdot \langle P(\phi_j), \phi_1\rangle \\ \vdots & & \vdots \\ \langle P(\Phi), \phi_l\rangle & = & \sum_{j=1}^{l} c_j \cdot \langle P(\phi_j), \phi_l\rangle. \end{cases} \tag{5.19}$$

The system 5.19 has at least one solution, since we assumed that $\Phi \in M$. To show that the system has exactly one solution, and that we may use Cramer's rule to find it, it is enough to show that

$$\det(\langle P(\phi_j), \phi_i\rangle, 1 \leq j, i \leq l) \neq 0.$$

Since P is an orthogonal projection, it is enough to show that

$$\det(\langle P(\phi_j), P(\phi_i)\rangle, 1 \leq j, i \leq l) \neq 0.$$

Since $\langle *, * \rangle$ is the bilinear form corresponding to the positive definite quadratic form $\|*\|^2$, it is enough to show that $\{P(\phi_1), \ldots, P(\phi_l)\}$ is a linearly independent set. This we do now:

Suppose that $\{a_1, \ldots, a_l\} \subseteq \mathbf{C}$ is such that $\sum_{i=1}^{l} a_i \cdot P(\phi_i) = 0$. Then

$$\begin{aligned} \sum_{i=1}^{l} a_i \cdot \phi_i &= \sum_{i=1}^{l} a_i \cdot P(\phi_i) + \sum_{i=1}^{l} a_i \cdot (1 - P)(\phi_i), \\ &= \sum_{i=1}^{l} a_i \cdot (1 - P)(\phi_i). \end{aligned} \tag{5.20}$$

We claim that the right side of this last equation lies in Range $(\overline{T_0(\tau\tau^*)})$.

First, let $k' \in \ker \overline{T_0(\tau^*\tau)}$. Then, for $1 \leq i \leq l$, we have $\langle \phi_i, k' \rangle = 0$, because

$$\text{span } \{\phi_1, \ldots, \phi_l\} \subseteq \text{Range } (T_1(\tau^*\tau)) = \{\ker \overline{T_0(\tau^*\tau)}\}^{\perp}.$$

Also, $\langle P(\phi_i), k' \rangle = \langle \phi_i, P(k') \rangle = \langle \phi_i, 0 \rangle = 0$, since P is the orthogonal projection onto $\ker T_1(\tau^*\tau) \ominus \ker \overline{T_0(\tau^*\tau)}$. Thus, we have $\langle (1 - P)(\phi_l), k' \rangle = 0$.

Second, let $k'' \in \ker T_1(\tau^*\tau) \ominus \ker \overline{T_0(\tau^*\tau)}$. Then

$$\langle (1 - P)(\phi_i), k'' \rangle = \langle \phi_i, (1 - P)(k'') \rangle = \langle \phi_i, k'' - k'' \rangle = 0.$$

Thus, for

$$k \in \ker T_1(\tau^*\tau) = \ker \overline{T_0(\tau^*\tau)} \oplus (\ker T_1(\tau^*\tau) \ominus \ker \overline{T_0(\tau^*\tau)}),$$

$\langle (1 - P)(\phi_i), k \rangle = 0$. Thus, $(1 - P)(\phi_i) \in \ker T_1(\tau^*\tau)^{\perp} = \text{Range } (\overline{T_0(\tau^*\tau)})$. Thus, the left side of equation 5.20 lies in Range $(\overline{T_0(\tau^*\tau)})$. Now, in sub-paragraph 2a, above,

$$F' = \text{span}\{\phi_1, \ldots, \phi_l\}$$

is such that

$$\text{span}\{\phi_1, \ldots, \phi_l\} \cap \text{Range } (\overline{T_0(\tau\tau^*)}) = (0).$$

Thus $\sum_{i=1}^{l} a_i \cdot \phi_i = 0$. Since $\{\phi_1, \ldots, \phi_l\}$ is a linearly independent set, we must have $0 = a_1 = \cdots = a_l$. Thus, $\{P(\phi_1), \ldots, P(\phi_l)\}$ is a linearly independent set.

So, we may use Cramer's rule to solve system 5.19. For all $i, 1 \le i \le l$,

$$c_i = \frac{\begin{vmatrix} \langle P(\phi_1), \phi_1 \rangle & \cdots & \langle P(\phi_1), \phi_l \rangle \\ \vdots & & \vdots \\ \langle P(\phi_{i-1}), \phi_1 \rangle & \cdots & \langle P(\phi_{i-1}), \phi_l \rangle \\ \langle P(\Phi), \phi_1 \rangle & \cdots & \langle P(\Phi), \phi_l \rangle \\ \langle P(\phi_{i+1}), \phi_1 \rangle & \cdots & \langle P(\phi_{i+1}), \phi_l \rangle \\ \vdots & & \vdots \\ \langle P(\phi_l), \phi_1 \rangle & \cdots & \langle P(\phi_l), \phi_l \rangle \end{vmatrix}}{\det(\langle P(\phi_j), \phi_i \rangle, 1 \le j, i \le l)}.$$

We showed above that $\{P(\phi_1), \ldots, P(\phi_l)\}$ is a linearly independent set. Thus, the denominator above is non-zero. From the form of the solution, it is clear that c_i depends linearly on Φ. We abbreviate the above formula:

$$c_i = \sum_{j=1}^{l} e_{ij} \cdot \langle P(\Phi), \phi_j \rangle,$$

where $\{e_{ij}\} \subseteq \mathbf{C}$. Thus,

$$\begin{aligned} |c_i| &\le \sum_{j=1}^{l} |e_{ij}| \cdot \|P(\Phi)\| \cdot \|\phi_j\|, \\ &\le \sum_{j=1}^{l} |e_{ij}| \cdot \|\Phi\| \cdot \|\phi_j\|, \end{aligned}$$

because P is an orthogonal projection. Thus,

$$|c_i| \le \sum_{j=1}^{l} |e_{ij}| \cdot \|\Phi\|,$$

because we chose $\{\phi_i\}$ so that $1 = \|\phi_1\| = \cdots = \|\phi_l\|$.

This shows that the rule for calculating c_i yields a bounded linear mapping from M to $\mathbf{C}$, which we denote by $c_i(*)$. Let $\|c_i\|$ denote the bound of $c_i(*)$.

(c) Here, we show that $\tau\tau^*(\phi)$, from formula 5.17, depends continuously on Φ. We re-arrange formula 5.17:

$$\tau\tau^*(\phi) = \Phi - \sum_{i=1}^{l} c_i(\Phi) \cdot \phi_i.$$

Thus,

$$\begin{aligned}
\|\tau\tau^*(\phi)\| &\leq \|\Phi\| + \sum_{i=1}^{l} |c_i(\Phi)| \cdot \|\phi_i\| \\
&\leq \|\Phi\| + \sum_{i=1}^{l} \|c_i\| \cdot \|\Phi\| \cdot \|\phi_i\| \\
&= (1 + \sum_{i=1}^{l} \|c_i\|) \cdot \|\Phi\|,
\end{aligned}$$

since $1 = \|\phi_1\| = \ldots = \|\phi_l\|$. Denote by Z the correspondence between Φ and $\tau\tau^*(\phi)$. Then

$$\|Z(\Phi)\| \leq (1 + \sum_{i=1}^{l} \|c_i\|) \cdot \|\Phi\|.$$

So Z is a bounded linear map $Z : M \rightarrow \tau\tau^*(W_0^{2n}(I))$.

(d) Now, we look at the linear functional on M which is induced by $\tau(g)$, and show that it is bounded. Choose $\Phi \in M$. Then

$$\begin{aligned}
|\int_I \Phi \cdot \overline{\tau(g)}\, dx| &= |\int_I \{Z(\Phi) + \sum_{i=1}^{l} c_i(\Phi) \cdot \phi_i\} \cdot \overline{\tau(g)}\, dx| \\
&\leq |\int_I Z(\Phi) \cdot \overline{\tau(g)}\, dx| \\
&\quad + \sum_{i=1}^{l} |\int_I c_i(\Phi) \cdot \phi_i \cdot \overline{\tau(g)}\, dx|. \qquad (5.21)
\end{aligned}$$

Now, $Z(\Phi) \in \tau\tau^*(W_0^{2n}(I))$, so that the results of paragraph 1 show that

$$\begin{aligned}
|\int_I Z(\Phi) \cdot \overline{\tau(g)}\, dx| &\leq c \cdot \|Z(\Phi)\| \cdot \|g\| \\
&\leq c \cdot \|Z\| \cdot \|\Phi\| \cdot \|g\|, \qquad (5.22)
\end{aligned}$$

where $c > 0$.

Now, we look at the second addend on the right side of line 5.21. Let J be the convex hull of $\bigcup_{i=1}^{l}$ support(ϕ_i). J is contained in the interior of I, since support(ϕ_i) is contained in the interior of I, for $1 \leq i \leq l$. Let $\|\tau(g)\|_J$ denote

$$\{\int_J |\tau(g)|^2 dx\}^{1/2}.$$

Then

$$\begin{aligned}
\sum_{i=1}^{l} |\int_I c_i(\Phi)\cdot\phi_i\cdot\overline{\tau(g)}\,dx| &= \sum_{i=1}^{l} |\int_J c_i(\Phi)\cdot\phi_i\cdot\overline{\tau(g)}\,dx| \\
&= \sum_{i=1}^{l} |c_i(\Phi)|\cdot|\int_J \phi_i\cdot\overline{\tau(g)}\,dx| \\
&\leq \sum_{i=1}^{l} |c_i(\Phi)|\cdot\|\phi_i\|\cdot\|\tau(g)\|_J \\
&= \sum_{i=1}^{l} |c_i(\Phi)|\cdot\|\tau(g)\|_J \text{ (since} \\
&\quad 1=\|\phi_1\|=\cdots=\|\phi_l\|) \\
&\leq \sum_{i=1}^{l} \|c_i\|\cdot\|\Phi\|\cdot\|\tau(g)\|_J \\
&= \|\Phi\|\cdot\sum_{i=1}^{l}\|c_i\|\cdot\|\tau(g)\|_J. \qquad (5.23)
\end{aligned}$$

By referring back to lines 5.21, 5.22, and 5.23, we get

$$|\int_I \Phi\cdot\overline{\tau(g)}\,dx| \leq \|\Phi\|\cdot[c\cdot\|Z\|\cdot\|g\|+\sum_{i=1}^{l}\|c_i\|\cdot\|\tau(g)\|_J].$$

This shows that $\tau(g)$ induces a bounded linear functional on M. By continuity, we can extend this to a bounded linear functional on the closure of M, which is Range $(T_1(\tau\tau^*))$. By using the Riesz- Frechet theorem, we infer a unique $q \in$ Range $(T_1(\tau\tau^*))$ such that, for all $\Phi \in M$,

$$\int_I \Phi\cdot\overline{\tau(g)}\,dx = \int_I \Phi\cdot\overline{q}\,dx.$$

In particular, for all $\tau\tau^*(\phi) \in \tau\tau^*(W_0^{2n}(I)) \subsetneq M$,

$$\int_I \tau\tau^*(\phi)\cdot\overline{\tau(g)}\,dx = \int_I \tau\tau^*(\phi)\cdot\overline{q}\,dx. \qquad (5.24)$$

We compare the previous equation with the linear functional defined in paragraph 1. Paragraph 1 showed that there is $\tilde{q} \in$ Range $(\overline{T_0(\tau\tau^*)})$ such that, for all $\tau\tau^*(\phi) \in \tau\tau^*(W_0^{2n}(I))$,

$$\int_I \tau\tau^*(\phi)\cdot\overline{\tau(g)}\,dx = \int_I \tau\tau^*(\phi)\cdot\overline{\tilde{q}}\,dx. \qquad (5.25)$$

We subtract line 5.24 from line 5.25, and see that, for all $\phi \in W_0^{2n}(I)$,

$$0 = \int_I \tau\tau^*(\phi)\cdot\overline{q-\tilde{q}}\,dx.$$

Thus, $q - \tilde{q}$ is a solution of the differential equation $\tau\tau^*(v) = 0$ (lemma 7). Since

$$\tilde{q}, q \in \text{Range } (T_1(\tau\tau^*)) \subseteq L^2(I),$$

we see that

$$q - \tilde{q} \in \ker T_1(\tau\tau^*) \cap \text{Range } (T_1(\tau\tau^*)) = \ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}.$$

(e) Let g satisfy the equation $T_1(\tau^*\tau)(v) = \pm iv$. We show that $q = 0$ entails $g = 0$. Notice that $\tilde{q}$ is the projection of q onto Range $(\overline{T_0(\tau\tau^*)})$; this is true since

$$q - \tilde{q} \in \ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}.$$

If $q = 0$, then $\tilde{q} = 0$. To have $g \neq 0$ and $q = 0$ would entail $g \neq 0$ and $\tilde{q} = 0$, which contradicts paragraph 1, above. Thus, $q = 0$ entails $g = 0$.

3. Let m denote $\dim \ker T_1(\tau\tau^*)$. Recall, from paragraph 2, that

$$l = \dim[\ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}],$$

and that $\{r_1, \ldots, r_l\}$ is a basis for $\ker T_1(\tau\tau^*) \ominus \ker \overline{T_0(\tau\tau^*)}$. Then $m - l = \dim \ker \overline{T_0(\tau\tau^*)}$. If $m = l$, then $\{r_1, \ldots, r_m\} = \{r_1, \ldots, r_l\}$ is a basis for $\ker T_1(\tau\tau^*)$.

If $m > l$, we choose $\{r_{l+1}, \ldots, r_m\}$ to be a basis for $\ker \overline{T_0(\tau\tau^*)}$; then $\{r_1, \ldots, r_m\}$ is a basis for $\ker T_1(\tau\tau^*)$.

$W_0^{2n}(I)$ is dense in $L^2(I)$. Range $(\overline{T_0(\tau\tau^*)})$ is closed, according to corollary 67; moreover, its orthocomplement in $L^2(I)$, $\ker T_1(\tau\tau^*)$, has dimension $m < \infty$. Thus, according to lemma 93, there exists $\{\phi_1, \ldots, \phi_m\} \subseteq W_0^{2n}(I)$ such that

- $\text{span}\{\phi_1, \ldots, \phi_m\} \cap \text{Range } (\overline{T_0(\tau\tau^*)}) = (0)$;
- $\text{span}\{\phi_1, \ldots, \phi_m\} + \text{Range } (\overline{T_0(\tau\tau^*)}) = L^2(I)$;
- $\det((\phi_i, r_j) 1 \leq i, j \leq m) \neq 0$; and
- $1 = \|\phi_1\| = \cdots = \|\phi_m\|$.

(Note: the set $\{\phi_1, \ldots, \phi_m\}$ could contain the set $\{\phi_1, \ldots, \phi_l\}$ of paragraph 2, but this is not necessary.)

Now we define a linear operator, Π, on $L^2(I)$. For $f \in L^2(I)$, we define

$$\Pi(f) = \frac{\begin{vmatrix} f & \langle f, r_1\rangle & \cdots & \langle f, r_m\rangle \\ \phi_1 & \langle \phi_1, r_1\rangle & \cdots & \langle \phi_1, r_m\rangle \\ \vdots & \vdots & & \vdots \\ \phi_m & \langle \phi_m, r_1\rangle & \cdots & \langle \phi_m, r_m\rangle \end{vmatrix}}{\det(\langle \phi_i, r_j\rangle, 1 \leq i, j \leq m)}. \tag{5.26}$$

This operator is well-defined, since the denominator is not zero. Here are some of its properties:

(a) $\langle \Pi(f), r_i\rangle = 0$, for $1 \leq i \leq m$. Thus,

$$\Pi(f) \in (\text{span } \{r_1, \ldots, r_m\})^{\perp} = \text{Range } (\overline{T_0(\tau\tau^*)}).$$

(b) If $f \in$ Range $(\overline{T_0(\tau\tau^*)})$, then $\langle f, r_i\rangle = 0$, for $0 \leq i \leq m$. From this, we see that $\Pi(f) = f$. Thus,

$$\text{Range } (\Pi) \supseteq \text{Range } (\overline{T_0(\tau\tau^*)}).$$

(c) Sub-paragraphs 3a and 3b entail Range $(\Pi) =$ Range $(\overline{T_0(\tau\tau^*)})$.

(d) We expand line 5.26:

$$\Pi(f) = f + \sum_{i=1}^{m} (-1)^{i-1} \cdot \phi_i \cdot \frac{\begin{vmatrix} \langle f, r_1\rangle & \cdots & \langle f, r_m\rangle \\ \langle \phi_1, r_1\rangle & \cdots & \langle \phi_1, r_m\rangle \\ \vdots & & \vdots \\ \langle \phi_{i-1}, r_1\rangle & \cdots & \langle \phi_{i-1}, r_m\rangle \\ \langle \phi_{i+1}, r_1\rangle & \cdots & \langle \phi_{i+1}, r_m\rangle \\ \vdots & & \vdots \\ \langle \phi_m, r_1\rangle & \cdots & \langle \phi_m, r_m\rangle \end{vmatrix}}{\det(\langle \phi_i, r_j\rangle, 1 \leq i, j \leq m)}.$$

We re-express this:

$$\Pi(f) = f + \sum_{i=1}^{m} \phi_i \cdot C_i(f),$$

where $C_i : L^2(I) \to \mathbf{C}$ is a linear functional, for $1 \le i \le m$. Each $C_i(f)$ may be rewritten

$$C_i(f) = \sum_{i=1}^{m} e_{ij} \cdot \langle f, r_j \rangle,$$

where $\{e_{ij}\} \subseteq \mathbf{C}$. This shows that each $C_i(*)$ is a bounded linear functional; let $||C_i||$ denote its bound, for $1 \le i \le m$.

(e) For any $f \in L^2(I)$,

$$\Pi(f) = f + \sum_{i=1}^{m} C_i(f) \cdot \phi.$$

Thus,

$$\begin{aligned} ||\Pi(f)|| &\le ||f|| + \sum_{i=1}^{m} ||C_i|| \cdot ||f|| \cdot ||\phi|| \\ &= (1 + \sum_{i=1}^{m} ||C_i||) \cdot ||f||, \end{aligned}$$

since $1 = ||\phi_1|| = \cdots = ||\phi_m||$. This shows that Π is a bounded operator. Let $||\Pi||$ denote its bound.

(f) If $f \in W_0^{2n}(I)$, then

$$\Pi(f) \in \text{span } \{f, \phi_1, \ldots, \phi_m\} \subseteq W_0^{2n}(I).$$

4. (a) Now we show that $q \in \mathcal{D}(T_1(\tau\tau^*))$. In sub-paragraph 2d, above, we showed that $q - \tilde{q} \in \ker T_1(\tau\tau^*)$. Thus, to show that $q \in \mathcal{D}(T_1(\tau\tau^*))$, it is enough to show that $\tilde{q} \in \mathcal{D}(T_1(\tau\tau^*))$. We do this by finding a positive constant α such that, for all $\phi \in W_0^{2n}(I)$,

$$|\langle T_0(\tau\tau^*)(\phi), \tilde{q}\rangle| \le \alpha \cdot ||\phi||.$$

(b) Let $\phi \in W_0^{2n}(I)$. As in paragraph 3,

$$\phi = \Pi(\phi) - \sum_{i=1}^{m} C_i(\phi) \cdot \phi_i.$$

We noted in sub-paragraph 3e, above, that if $\phi \in W_0^{2n}(I)$, then $\Pi(\phi) \in W_0^{2n}(I)$. Also, $\sum_{i=1}^m C_i(\phi)\cdot\phi_i \in W_0^{2n}(I)$. Thus,

$$\begin{aligned} |\langle T_0(\tau\tau^*)(\phi)\,,\tilde{q}\rangle| &\le |\langle T_0(\tau\tau^*)(\Pi(\phi))\,,\tilde{q}\rangle| \\ &\quad + |\langle T_0(\tau\tau^*)[\sum_{i=1}^m C_i(\phi)\cdot\phi_i]\,,\tilde{q}\rangle|. \end{aligned}$$

First, we look at the second addend:

$$\begin{aligned} &|\langle T_0(\tau\tau^*)[\sum_{i=1}^m C_i(\phi)\cdot\phi_i]\,,\tilde{q}\rangle| \\ &\le \sum_{i=1}^m |C_i(\phi)|\cdot|\langle T_0(\tau\tau^*)(\phi_i)\,,\tilde{q}\rangle| \\ &\le \sum_{i=1}^m ||C_i||\cdot||\phi||\cdot||T_0(\tau\tau^*)(\phi_i)||\cdot||\tilde{q}|| \\ &= [\,||\tilde{q}||\cdot\sum_{i=1}^m ||C_i||\cdot||T_0(\tau\tau^*)\phi||\,]\cdot||\phi||. \end{aligned}$$

This shows that the second addend represents a bounded linear functional on $W_0^{2n}(I)$.

(c) Now, we look at the first addend. In paragraph 3c, above, we noted that Range (Π) = Range $(\overline{T_0(\tau\tau^*)})$. Hence, there exists $f \in \mathcal{D}(\overline{T_0(\tau\tau^*)})$ such that

$$\overline{T_0(\tau\tau^*)}(f) = \Pi(\phi).$$

We want to show that $T_0(\tau\tau^*)(\Pi(\phi)) \in$ Range $(\overline{T_0((\tau\tau^*)^2)})$. Note that $0 \notin \sigma_e((\tau\tau^*)^2)$ (see proposition 73). Hence, Range $(\overline{T_0((\tau\tau^*)^2)})$ is a subspace of $L^2(I)$. Thus, we will be able to conclude that

$$T_0(\tau\tau^*)(\Pi(\phi)) \in \text{Range } (\overline{T_0((\tau\tau^*)^2)})$$

as soon as we show that

$$T_0(\tau\tau^*)(\Pi(\phi)) \perp \ker T_1((\tau\tau^*)^2).$$

This we do next.

Let $k \in \ker T_1((\tau\tau^*)^2)$. Since we assume that

$$\ker T_1((\tau\tau^*)^2) \subseteq \mathcal{D}(T_1(\tau\tau^*))\text{ , then } T_1(\tau\tau^*)(k) \in L^2(I).$$

Also, $T_1(\tau\tau^*)(k) \in \mathcal{D}(T_1(\tau\tau^*))$, because

$$T_1(\tau\tau^*)\,T_1(\tau\tau^*)(k) = (\tau\tau^*)^2(k) = 0,$$

and because $T_1(\tau\tau^*)(k) \in C^\infty(I)$(this last is true because we assume that the coefficients of τ, and hence those of $\tau\tau^*$ and of $(\tau\tau^*)^2$, belong to $C^\infty(I)$). We calculate:

$$\begin{aligned}
\langle T_0(\tau\tau^*)(\Pi(\phi)), k\rangle &= \langle \Pi(\phi), T_1(\tau\tau^*)(k)\rangle \\
&= \langle \overline{T_0(\tau\tau^*)}(f), T_1(\tau\tau^*)(k)\rangle \\
&= \langle f, T_1(\tau\tau^*)T_1(\tau\tau^*)(k)\rangle \\
&= \langle f, 0\rangle = 0.
\end{aligned}$$

(d) Thus, there is $\tilde{f} \in \mathcal{D}(\overline{T_0((\tau\tau^*)^2)})$ such that

$$\overline{T_0((\tau\tau^*)^2)}(\tilde{f}) = T_0(\tau\tau^*)(\Pi(\phi)).$$

We know that $\tilde{f} \in \mathcal{D}(\overline{T_0(\tau\tau^*)})$ because lemma 23 entails $\overline{T_0((\tau\tau^*)^2)} \subseteq \overline{T_0(\tau\tau^*)}\,\overline{T_0(\tau\tau^*)}$.

We now show that $\overline{T_0(\tau\tau^*)}(\tilde{f}) = \Pi(\phi)$. We have

$$\overline{T_0(\tau\tau^*)}\,\overline{T_0(\tau\tau^*)}(\tilde{f}) = \overline{T_0((\tau\tau^*)^2)}(\tilde{f}) = \overline{T_0(\tau\tau^*)}(\Pi(\phi)),$$

which shows that $\overline{T_0(\tau\tau^*)}(\tilde{f}) - \Pi(\phi) \in \ker T_1(\tau\tau^*)$.

On the other hand, both $\overline{T_0(\tau\tau^*)}(\tilde{f})$ and $\Pi(\phi)$ belong to

$$\text{Range }(\overline{T_0(\tau\tau^*)}) = \ker T_1(\tau\tau^*)^\perp,$$

which entails $\overline{T_0(\tau\tau^*)}(\tilde{f}) - \Pi(\phi) \in \ker T_1(\tau\tau^*)^\perp$.

Taken together,these conditions entail $\overline{T_0(\tau\tau^*)}(\tilde{f}) - \Pi(\phi) = 0$.

To summarize, there exists $\{\psi_n\} \subseteq W_0^{4n}(I)$ such that

- $\psi_n \to \tilde{f}$,
- $\tau\tau^*(\psi_n) \to \overline{T_0(\tau\tau^*)}(\tilde{f}) = \Pi(\phi)$, and
- $(\tau\tau^*)^2(\psi_n) \to \overline{T_0((\tau\tau^*)^2)}(\tilde{f}) = T_0(\tau\tau^*)(\Pi(\phi))$.

(e) Now we can show that $\langle T_0(\tau\tau^*)(\Pi(*)), \tilde{q}\rangle$ represents a bounded linear functional on $W_0^{2n}(I)$.

For all $\phi \in W_0^{2n}(I)$,

$$
\begin{aligned}
|\langle T_0(\tau\tau^*)(\Pi(\phi)), \tilde{q}\rangle| &= \lim_{n\to\infty} |\langle(\tau\tau^*)^2(\psi), \tilde{q}\rangle| \text{ (see last sub-paragraph)} \\
&= \lim_{n\to\infty} \left|\int_I (\tau\tau^*)^2(\psi_n)\cdot \overline{\tilde{q}}\, dx\right| \\
&= \lim_{n\to\infty} \left|\int_I (\tau\tau^*)^2(\psi_n)\cdot \overline{\tau(g)}\, dx\right| \text{ (see paragraph 1)} \\
&= \lim_{n\to\infty} \left|\int_I \tau\tau^*(\psi_n)\cdot \overline{\pm i\tau(g)}\, dx\right| \text{ (integration by parts)} \\
&= \lim_{n\to\infty} \left|\int_I \tau\tau^*(\psi_n)\cdot \overline{\tilde{q}}\, dx\right| \text{ (see paragraph 1)} \\
&= \left|\int_I \Pi(\phi)\cdot \overline{\tilde{q}}\, dx\right| \text{ (see last sub-paragraph)} \\
&\le ||\Pi(\phi)||\cdot||\tilde{q}|| \\
&\le ||\Pi||\cdot||\phi||\cdot||\tilde{q}||.
\end{aligned}
$$

(f) We put together sub-paragraphs 4b and 4e to conclude that, for all $\phi \in W_0^{2n}(I)$,

$$|\langle T_0(\tau\tau^*)(\phi), \tilde{q}\rangle| \le \{\, ||\tilde{q}|| \cdot \sum_{i=1}^{m} ||C_i||\cdot||T_0(\tau\tau^*)(\phi_i)|| + ||\tilde{q}||\cdot||\Pi||\,\}\cdot||\phi||.$$

Thus, $\tilde{q} \in \mathcal{D}(T_1(\tau\tau^*))$. Further,

$$
\begin{aligned}
\langle T_0(\tau\tau^*)(\phi), q\rangle &= \langle T_0(\tau\tau^*)(\phi), \tilde{q}\rangle + \langle T_0(\tau\tau^*)(\phi), q-\tilde{q}\rangle \\
&= \langle T_0(\tau\tau^*)(\phi), \tilde{q}\rangle,
\end{aligned}
$$

because $q - \tilde{q} \in \ker T_1(\tau\tau^*) = \text{Range}\,(\overline{T_0(\tau\tau^*)})^{\perp}$. Thus,

$$|\langle T_0(\tau\tau^*)(\phi), q\rangle| = |\langle T_0(\tau\tau^*)(\phi), \tilde{q}\rangle|,$$

and we see that $q \in \mathcal{D}(T_1(\tau\tau^*))$.

5. Here, we show that $\tau(g) = q$.

In paragraph 2, we found $q \in \text{Range}\,(T_1(\tau\tau^*))$ which has the property that, for all $\phi \in W_0^{2n}(I)$,

$$\int_I \tau\tau^*(\phi)\cdot\overline{\tau(g)}\, dx = \int_I \tau\tau^*(\phi)\cdot\overline{q}\, dx.$$

Thus, for all $\phi \in W_0^{2n}(I)$,

$$\int_I \tau\tau^*(\phi) \cdot \overline{\tau(g) - q}\, dx = 0.$$

From this, and from lemma 7, we conclude that $\tau(g) - q$ satisfies the differential equation $\tau\tau^*(v) = 0$, which entails that $\tau(g) - q \in C^\infty(I)$. Since $\tau(g)$ satisfies the equation $\tau\tau^*(v) = \pm iv$, then $\tau(g) \in C^\infty(I)$. Thus,

$$q = \tau(g) - (\tau(g) - q) \in C^\infty(I).$$

Thus,

$$\begin{aligned} \tau\tau^*(q) &= \tau\tau^*(\tau(g)) - \tau\tau^*(\tau(g) - q) \\ &= \tau\tau^*(\tau(g)) \\ &= \pm i\tau(g). \end{aligned}$$

We conclude that $\tau(g) \in L^2(I)$, because $q \in \mathcal{D}(T_1(\tau\tau^*))$ (see paragraph 4).

Because $\tau(g) \in L^2(I)$, because $\tau(g) \in C^\infty(I)$, and because $\tau\tau^*(\tau(g)) = \pm i\tau(g)$, we see that $\tau(g) \in \mathcal{D}(T_1(\tau\tau^*))$, and that $\tau(g) \in$ Range $(T_1(\tau\tau^*))$.

In paragraph 2, we chose $q \in$ Range $(T_1(\tau\tau^*))$ so that, for all $\Phi \in M \cap$ Range $(T_1(\tau\tau^*))$,

$$\int_I \Phi \cdot \overline{\tau(g)}\, dx = \int_I \Phi \cdot \overline{q}\, dx.$$

Since $M \cap$ Range $(T_1(\tau\tau^*))$ is dense in Range $(T_1(\tau\tau^*))$, we conclude that $\tau(g) = q$. Q.E.D.

Remark 98 Under the hypotheses of the preceding proposition, we argue thus: according to lemma 31,

$$\begin{aligned} \overline{T_0(\tau^*\tau)} &= (T_1(\tau^*\tau))^* \\ &= (T_1(\tau^*)\, T_1(\tau))^* \\ &= \overline{T_0(\tau^*)}\, \overline{T_0(\tau)}. \end{aligned}$$

Corollary 99 *Let τ be a formal differential operator which is defined on an interval $I \subseteq \mathbf{R}$. Suppose that $0 \notin \sigma_e(\tau\tau^*)$ (equivalently, $0 \notin \sigma_e(\tau^*\tau)$), and that ind $(\{T_1(\tau\tau^*)\}^2) =$ ind $(T_1((\tau\tau^*)^2))$.*

Then $T_1(\tau^*\tau) = T_1(\tau^*)T_1(\tau)$.

Proof: The two suppositions about essential spectra are equivalent by proposition 73.

We show that $\ker T_1((\tau\tau^*)^2) \subseteq \mathcal{D}(T_1(\tau\tau^*))$; this lets us rely on the conclusion of the previous proposition to reach the desired result.

Lemma 23 entails $\overline{T_0((\tau\tau^*)^2)} \subseteq \{\overline{T_0(\tau\tau^*)}\}^2$. Taking adjoints, and using theorem 8 and proposition 31, yields $T_1((\tau\tau^*)^2) \supseteq \{T_1(\tau\tau^*)\}^2$. According to lemma 88, this last relation, together with the supposition about indices, entails $T_1((\tau\tau^*)^2) = \{T_1(\tau\tau^*)\}^2$. Now, clearly, $\ker T_1((\tau\tau^*)^2) \subseteq \mathcal{D}(T_1(\tau\tau^*))$. Q.E.D.

Example 100 The book [10, p. 70-73] presents an example of T.T. Read, of which the following is part.

Let n be a positive integer. For $j \in \{1, \ldots, n\}$, let i_j be a non-negative integer. Now consider the formal operator

$$M_j = d/dx + i_j + 1/(j+2),$$

defined on $[0,\infty)$. Consider, too, the formal operator

$$\tau(f) = M_1^* \cdots M_n^*(e^{x/2}f),$$

defined on $[0,\infty)$), and consider the formally self-adjoint operator $\tau\tau^*$, defined on $[0,\infty)$. Then the following are true:

1. *$T_0(\tau\tau^*)$ is a positive operator, and thus has deficiency indices (d_1, d_1);*

2. *$0 \notin \sigma_e(\tau\tau^*)$;*

3. *$T_0((\tau\tau^*)^2)$ is a positive operator, and thus has deficiency indices (d_2, d_2);*

4. *$0 \notin \sigma_e((\tau\tau^*)^2)$;*

5. *$d_2 \geq d_1$;*

6. *d_1 equals the number of i_j satisfying $0 = i_j$; and*

7. *$d_2 - d_1$ equals the number of i_j satisfying $0 = i_j$, or $1 = i_j$.*

Now, we choose any positive integer l. Then we choose another positive integer $n \geq l$. Then we choose $\{i_1, \ldots, i_n\}$ such that card $\{i_j : i_j = 0\} = l$, and card $\{i_j : i_j = 1\} = 0$. Then, for the operators $\tau\tau^*$ and $(\tau\tau^*)^2$ of the above paragraph, $d_1 = l$, and $d_2 - d_1 = l$, whence $d_2 = 2 \cdot l$. Since $0 \notin \sigma_e(\tau\tau^*)$, and $0 \notin \sigma_e((\tau\tau^*)^2)$, we have, by corollary 76,

$$\begin{aligned} l &= d_1 = \text{ ind } (\overline{T_0(\tau\tau^*)}), \text{ and} \\ 2 \cdot l &= d_2 = \text{ ind } (\overline{T_0((\tau\tau^*)^2)}). \end{aligned}$$

Further, ind $(\overline{T_0(\tau\tau^*)}^{\,2}) = 2 \cdot \text{ind } (\overline{T_0(\tau\tau^*)}) = 2 \cdot l$. This yields

$$\text{ind } (\overline{T_0(\tau\tau^*)}^{\,2}) = 2 \cdot l = \text{ind } (\overline{T_0((\tau\tau^*)^2)}).$$

From this, corollary 99 lets us conclude that

$$T_1(\tau^*\tau) = T_1(\tau^*)T_1(\tau).$$

Example 101 Let $\zeta = -(d^2/dx^2) + q(x)$ be a formal differential operator defined on the interval $[-\delta, \infty)$, where $\delta > 0$. If q is real-valued, then ζ is formally self-adjoint. This is the only case we consider.

We seek to express ζ in the form

$$\zeta = -(d^2/dx^2) + q(x) = (d/dx + f(x))(-d/dx + f(x)) = \tau\tau^*$$

on the interval $I = [0, \infty)$, where $f \in C^\infty(I)$. An elementary calculation shows that this is possible if, and only if, f satisfies the Riccati equation

$$v(x)^2 + v'(x) = q(x), \tag{5.27}$$

for $x \in [0, \infty)$. The elementary theory of Riccati equations ([7, p. 273-277], for example) shows that f will satisfy equation 5.27 if, and only if, f has the form s'/s, where s is a solution of the differential equation

$$-v'' + qv = 0, \tag{5.28}$$

such that $s(x) \neq 0$, for $x \in [0, \infty)$.

Since $q \in C^\infty([0,\infty)$, every solution of equation 5.28 belongs to $C^\infty([0,\infty))$ ([4, XIII.1.4]).

In order to find solutions of equation 5.28 which have no zeros in $[0,\infty)$, we consider solutions to equation 5.28 on the interval $[-\delta,\infty)$, and further restrict attention to $q \in C^\infty([-\delta,\infty))$ such that $q > 0$. Then any solution of equation 5.28 has at most one zero in $[-\delta,\infty)$ ([1, problem 8, p. 262]). Choose $x_0 \in [-\delta, 0)$, and $r \in \mathbf{R}, r \neq 0$. Then, according to what we have just said, the solution s to the initial value problem

$$-v'' + qv = 0;$$

$$v(x_0) = 0; \qquad v'(x_0) = r;$$

has no zero in $[0,\infty)$. For such an s, we have

$$\begin{aligned}\zeta &= -(d^2/dx^2) + q(x) \\ &= (d/dx + s'(x)/s(x))(-d/dx + s'(x)/s(x)) = \tau\tau^*, \end{aligned} \tag{5.29}$$

for $x \in [0,\infty)$.

In order to offer an application of proposition 97, we impose one last restriction on ζ : that $0 \notin \sigma_e(\zeta)$. This will be true, for example, if $q(x) \geq \varepsilon > 0$, for $x \in [0,\infty)$, which implies that the $T_0(\zeta)$, and hence $\overline{T_0(\zeta)}$, is bounded below by ϵ. To see this, notice that

$$T_0(\zeta) = T_0(-d^2/dx^2) + T_0(\,q(x)\,)$$

(where $T_0(\,q(x)$ denotes the operation of multiplying by $q(x)$ on the domain $W_0^2([0,\infty)\,)$, and that $T_0(-d^2/dx^2)$ is bounded below by zero, and that $T_0(\,q(x)\,)$ is bounded below by ϵ.

Now, here is a specialization of a theorem of Read ([11, p. 357-366]):

Let $\zeta = -(d^2/dx^2) + q(x)$ be a formal differential operator, which is defined on $[0,\infty)$.and which is such that q is real-valued and bounded below.

Then neither $T_1(\zeta)$, nor $T_1(\zeta^2)$, has a boundary value at ∞.

We know that $T_1(\zeta)$ has two boundary values at 0, and $T_1(\zeta^2)$ has four boundary values at 0 (corollary 46). Thus, by the theorem of Read, ζ has two boundary values, and ζ^2 has four. From definition 40, we see that

1. $2 = \dim(\mathcal{D}(T_1(\zeta)) \ominus \mathcal{D}(\overline{T_0(\zeta)}))$, and

2. $4 = \dim(\mathcal{D}(T_1(\zeta^2)) \ominus \mathcal{D}(\overline{T_0(\zeta^2)}))$,

where othogonality refers to the operator inner product.

From another point of view, remark 18 shows that

1. $\dim(\mathcal{D}(T_1(\zeta)) \ominus \mathcal{D}(\overline{T_0(\zeta)})) = [T_1(\zeta) : \overline{T_0(\zeta)}]$, and

2. $\dim(\mathcal{D}(T_1(\zeta^2)) \ominus \mathcal{D}(\overline{T_0(\zeta^2)})) = [T_1(\zeta^2) : \overline{T_0(\zeta^2)}]$.

Thus,

1. $[T_1(\zeta) : \overline{T_0(\zeta)}] = 2$, and

2. $[T_1(\zeta^2) : \overline{T_0(\zeta^2)}] = 4$.

Recall that $0 \notin \sigma_e(\zeta)$. Thus, $0 \notin \sigma_e(\zeta^2)$ (proposition 73). Recall *index* from definition 70. Remembering that ζ and ζ^2 are formally self-adjoint, and using corollary 76, we see that

1. ind $(T_1(\zeta)) = -1$, and

2. ind $(T_1(\zeta^2)) = -2$.

From this, and from theorem 71, we have

$$\text{ind } (T_1(\zeta)^2) = \text{ind } (T_1(\zeta^2)).$$

Using equation 5.29, we re-state this:

$$\text{ind } (T_1(\tau\tau^*)^2) = \text{ind } (T_1((\tau\tau^*)^2)).$$

Now, using corollary 99, we have

$$T_1(\tau^*\tau) = T_1(\tau^*)T_1(\tau).$$

Bibliography

[1] G. Birkhoff and G.-C. Rota,*Ordinary Differential Equations,* Ginn and Co., Boston, 1962

[2] M. M. Crum, "Associated Sturm-Liouville Systems," *Quart. J. Math.*, Oxford(2), vol. 6(1955), p. 121-127

[3] P.A. Deift, "Applications of a Commutation Formula," *Duke J. Math.*, vol. 45(1978), p. 267-310

[4] N. Dunford and J.T. Schwartz, *Linear Operators Part II: Spectral Theory*, Interscience Publishers, New York, 1963

[5] I.C. Gohberg and M.S. Krein, "The Basic Propositions on Defect Numbers, Root Numbers, and Indices of Linear Operators", *Amer. Math. Soc. Transl., Ser. 2*, vol. 13(1960), p. 185-264

[6] S. Goldberg, *Unbounded Linear Operators*, McGraw- Hill Book Co., New York, 1966

[7] E. Hille, *Lectures on Ordinary Differential Equations,* Addison-Wesley Publishing Co., Reading, Mass., 1969

[8] L. Hörmander, *Linear Partial Differential Operators*, second revised printing, Springer-Verlag, Berlin, 1964

[9] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976

[10] R.M. Kauffman, T.T. Read, and A. Zettl, *The Deficiency Index Problem for Powers of Ordinary Differential Expressions*, Lecture Notes in Mathematics, vol. 621, Springer-Verlag, Berlin, 1977

[11] T.T. Read, "On the Limit Point Condition for Polynomials in a Second Order Differential Expression ", *J. London Math. Soc. (2)*, vol. 10(1975), p. 357-366

[12] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar Publishing Co., New York, 1955