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Novel features of planar fermions: Anyon superconductivity and the quantum Hall effect

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NOVEL FEATURES OF PLANAR FERMIONS : ANYON SUPERCONDUCTIVITY AND THE QUANTUM HALL EFFECT

by

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ABSTRACT

NOVEL FEATURES OF PLANAR FERMIONS : ANYON SUPERCONDUCTIVITY AND THE QUANTUM HALL EFFECT

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In this work we have discussed some remarkable features associated with systems where the particles are permitted to move only along a plane, the third degree of freedom having been effectively frozen out. From a theoretical point of view, the restriction of the dimensionality of space to two is interesting in the sense that anyons, particles with any spin and any statistics may exist in planar systems. Anyons, as may be expected, have some rather novel features which are absent in higher dimensions. In this work, we have dwelt on one such feature : the superfluidity of a gas of anyonic particles. We have further demonstrated that if the anyons are coupled minimally to electromagnetism, the magnetic field is expelled from within the system, resulting in superconductivity. This is the so-called anyon superconductivity, which is one of the candidates for explaining the properties of planar superconductors. Further, planar fermionic systems subjected to strong magnetic fields normal to the plane support the various quantum Hall effects. The single-particle spectrum for such a system consists of the equi-spaced Landau levels, the gap between two successive levels being the cyclotron frequency. If the applied magnetic field is strong enough, the system is projected to the lowest Landau level and hence the development of

a field theory of fermions in the lowest Landau levels becomes of more than cursory interest. In this work, we have developed the field theory of such fermions, coupled to external perturbative electromagnetic fields, when the system is projected onto the lowest Landau level. If we further confine the fermions to a finite portion of the plane through some suitably chosen confining potential, the field theory that we develop enables us to discuss the excitations associated with the boundary of the system. These excitations, dubbed as the edge-excitations in the literature, are shown to be important in the maintenance of the original electromagnetic gauge invariance of the system. In summary, the main thrust of this work is in the discussion of anyon superconductivity and the development of a field theory for fermions in the lowest Landau level, the unifying theme being the physics of planar fermions.

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Contents

Chapter 1

Introduction

A generic feature of planar (2+1 dimensional) fermions relativistic or otherwise, coupled to electromagnetism is that, on integrating the fermions out, a special type of term is generated in the effective action (Redlich 1984), (Abouelsaood 1985) which is not of the familiar Maxwell type. This term, bilinear in the gauge potentials, is the Chern-Simon (CS) term (Deser et al 1982), (Hagen, 1984). The CS term recurs in physically diverse situations and is the underlying theme that runs through the present work. Here, we focus our attention on two of these, anyonic superconductivity and the quantum Hall effect.

With the recent experimental realization of planar systems, a whole set of new and exciting possibilities has opened up in theoretical physics. Some remarkable experimental discoveries (Von Klitzing et al 1980), (Tsui et al 1982) in the past decade have fuelled an extensive exploration of possibilities in two spatial dimensions.

One of the most interesting phenomena peculiar to two dimensions is the existence of anyons (Leinaas & Myrrheim 1977). As the name suggests, anyons are particles with"any" spin obeying "any" statistics as opposed to just fermions and bosons. For a long time, fermions and bosons were the only two possibilities as we live in a world that is inherently three dimensional. In this context one should also mention the interesting development of parastatistics, which results from higher dimensional representations of the permutation group. Unfortunately, no particles obeying parastatistics have been observed in nature and consequently, parastatistics has been of little more than epistemological interest. However, as has been discovered of late, there do exist situations where the third dimension is essentially frozen, rendering it superfluous. Here the tantalizing possibility exists that anyons might produce new phenomenology.

Apart from the possible existence of anyons, there is another planar phenomenon that merits mention in this context. This is the quantum Hall effect, which basically involves planar electrons moving in a strong magnetic field normal to the plane.

What is common to both phenomena, however is the emergence of a CS term for the applied electromagnetic field, when some or all of the fermionic degrees of freedom have been integrated out. This CS term, which bears different significance in the two cases, is nonetheless the unifying motif for these apparently unrelated topics.

In the first part of the thesis, we shall describe an effort to understand superconductivity in planar materials in terms of anyon physics. We prove that in a system of anyons coupled minimally to electromagnetism, the free energy is minimized when B, the external magnetic field, is 0. This is the Meissner effect. Further, when this magnetic field is zero, we demonstrate that the spectrum of excitations of the system contains a massless excitation at low momenta. This signals superfluidity. These two observations, in conjunction, indicate superconductivity.

The planar electrons are coupled to suitably chosen slowly varying electromagnetic potentials. The expectation values for the fermionic density and current are obtained within the derivative expansion scheme (Sakita & Su 1988), which is justified in the light of the slowly varying nature of the fields. The computation involves the extensive use of the concept of Landau levels, which are the single particle states for electrons in a magnetic field, and of the idea of guiding centre coordinates which expose the enormous degeneracy that is inherent in such problems. These ideas occur again in the second part of the thesis, reinforcing the notion of the basic similarity of the two problems at a very pragmatic level. Once the average currents are obtained, we integrate them functionally with respect to the gauge potentials to obtain the desired effective action. This effective action is seen to contain a CS-like term which is the most important term for our purposes. As discussed later in the introduction, a system of anyons may be expressed as a system of fermions coupled to a "statistical" gauge field governed by a CS term. It is shown that this tree-level CS term cancels against the CS term generated at the one loop level mentioned above in order to minimize the free energy. This cancellation, in turn, causes the real magnetic field to be expelled from the sample. Thus, the generation of a CS-like term in the effective action is of vital importance for the existence of the Meissner effect. Further, once this cancellation goes through, we show that the density fluctuation of the anyons, described by the higher derivative terms in the effective action, contains a massless mode. This establishes the superfluidity of the anyon fluid. It would be of some pedagogical value to address the issue of the connection between the existence of a massless mode in the spectrum and the phenomenon of superfluidity(Chang 1990). Let us imagine a system of bosons at the temperature $T = 0$, with some suitably weak two-body interaction between the particles. The non-interacting ground state of the system at $T = 0$ exhibits Bose-condensation. A straightforward perturbative analysis of the inverse of the full propagator, taking the condensed ground state into account, yields the excitation spectrum to be

$$\epsilon(k) \equiv |k^0| = \sqrt{(\vec{k}^2/2m)^2 + 2n_0 V(k)\vec{k}^2/2m},$$

where $V(k)$ is the fourier transform of the two-body potential. From this, one notes that as $k \rightarrow 0$, $\epsilon(k)$ depends linearly on k. This dependence is given by

$$\epsilon(k) = ck,$$

where $c \equiv \sqrt{n_0 V(0)/m}$. The effective mass of this excitation is given by

$$m^* = \lim_{k \rightarrow 0} k/(\partial \epsilon(k)/\partial k) = \lim_{k \rightarrow 0} k/c = 0.$$

Thus in the presence of the Bose-condensate, the excitation behaves as a massless particle travelling with the sound velocity c (the phonon mode). This linearity of the spectrum, arising out of the massless mode, is crucial to the understanding of superfluidity.

A superfluid is characterized by the absence of viscosity. We shall first establish that if the spectrum is linear, the fluid exhibits no viscosity for an object moving in it with a velocity $v < c$. Consider a classical object of mass

M moving in a superfluid with a velocity $\vec{V}$. The momentum and energy of this moving object are

$$\vec{P} = M\vec{V},$$
$$E = \frac{\vec{P}^2}{2M}.$$

We assume that there are no phonons initially. The only way this object can lose energy in the fluid is through the emission of one or more phonons. Let us first consider the situation where one phonon is emitted. Energy and momentum conservation imply that

$$\frac{\vec{P}^2}{2M} = \frac{\vec{P}'^2}{2M} + cp,$$

where

$$\vec{P} = \vec{P}' + \vec{p}.$$

Here, $\vec{p}$ is the momentum associated with the phonon and $\vec{P}'$ is the momentum of the classical object after the phonon has been emitted. This implies, on eliminating $\vec{P}'$,

$$\vec{V} \cdot \vec{p} = cp + \frac{p^2}{2M} \geq cp.$$

This indicates that the emission of a phonon is forbidden if $|\vec{V}| < c$. Thus if the object is moving slower than sound in the fluid, it cannot lose its energy through the production of a phonon, which is tantamount to losing energy to the fluid through viscosity. This conclusion, as may be checked trivially, holds even for multiphonon emissions. The importance of the linearity of the spectrum is demonstrated quite easily. Imagine that the dispersion curve, instead of being linear, is quadratic. Namely,

$$\epsilon(p) = \frac{p^2}{2m},$$

with $m \neq 0$. Then an analysis similar to the one carried out above would indicate that excitations in the fluid may be produced as soon as

$$\vec{V} \cdot \vec{p} \geq \frac{p^2}{2m}$$

or

$$V \geq \frac{p}{2m}.$$

Thus if the momentum of the excitation goes to zero, the object can lose energy through viscosity even if it moves arbitrarily slowly through the fluid. Thus, the fluid cannot be a superfluid. So, in conclusion, a massless excitation in the spectrum, which is the same as a linear dispersion relation, leads to superfluidity in a bosonic fluid at $T = 0$.

In the second part of the thesis, we address the problem of the quantum Hall effect. The system at hand comprises of planar electrons under the influence of a strong magnetic field orthogonal to the plane and coupled in addition to slowly varying perturbative, electromagnetic potentials imposed externally on the system. The exactly soluble single particle problem is the celebrated Landau problem, where the single particle eigenstates are the Landau levels. These Landau levels are highly degenerate, $\frac{eB}{2\pi}$ being the number of degenerate states per unit area. The guiding centre coordinate X, to be suitably defined later, commutes with the single particle Hamiltonian and measures the degeneracy of a given Landau level. The gap between successive Landau levels is given by $\omega_{\mathrm{cyclotron}} = \frac{B}{m}$. This means that in the event of a high cyclotron frequency, a considerable amount of energy is required to excite an electron to a higher Landau level. So, to describe low energy phenomena, it is most appropriate to discuss the physics of the lowest Landau level (L.L.L.). There are two kinds of excitations of a quantum Hall system (Dahm et al 1985). One, the so called bulk magnetoplasmon arises from the excitation to a higher Landau level and is thus beyond the scope of the low energy regime that we have discussed. The other, the chiral surface magnetoplasmons, are gapless excitations along the edge of the sample. These low energy excitations are produced by the perturbations we have applied on the system. The action governing these excitations on the edge is a 1+1 dimensional action (the edge of a 2+1 dimensional system being 1+1 dimensional). We have extracted this lower dimensional action from the underlying action described above. The interior of the sample, on the other hand, is seen to be described by an effective gauge field action, which contains the by now ubiquitous CS term. It is however well-known that a CS term defined on a compact manifold is not gauge invariant, the non-invariance residing on the boundary of the manifold. So the effective description of the interior of the sample is gauge non-invariant, which is unacceptable. We have, however shown that the action describing the edge of the sample is also gauge noninvariant at the quantum level (Jackiw & Rajaraman1985), (Faddeev & Shatashvili 1986). Further, we have demonstrated that the two gauge dependences cancel against each other, rendering the complete description gauge-invariant. Thus, the edge action, in addition to governing the edge excitations of the system, ensures, through its anomalous nature, that the overall gauge invariance of the system is restored. The effective action for the lowest Landau level that we have obtained by integrating the higher Landau levels out, is the starting point for all these discussions and should be a suitable point of departure for further investigations in this direction.

This introduction, by now, should have served its purpose by making it amply clear to the reader that the entire thesis is in effect the evaluation of the role of the same CS term in different areas of nonrelativistic planar physics.

Chapter 2

Anyons and Anyon Superconductivity

2.1 An introduction to Anyons

As discussed in the introduction to the thesis, the world of two dimensional physics offers fascinating possibilities in that it permits exotic specimens to be added to an already crowded menagerie of particles. These exotic particles, peculiar to planar (two spatial dimensions) systems, can theoretically have any spin and any statistics, consistent with a generalized spin-statistics theorem and have consequently been dubbed "anyons"(Wilczek 1982a,b).

In 3+1 dimensions, the unitary representations of the rotation group SO(3), restrict the angular momenta to integral and half-integral values. In the 2+1 dimensional case, however, the rotation group SO(2) is abelian and admits of unitary representations with continuous values of the angular momentum. So, in principle, particles can have any spin in planar physics.

One must however clearly specify what one means by "statistics" before claiming that anyons can have any statistics. There is a well established set of ideas in quantum physics which is embodied in the word "statistics." In quantum statistical mechanics, the rules of counting states are drastically modified from the classical rules, in one of two possible ways, depending on the nature of the quantum mechanical interference arising among identical particles. If the quantum particles are identical bosons, the amplitudes for configurations which differ only by a permutation of two particles interfere constructively - that is, with a relative phase which is +1. In contrast, for identical fermions, the relative phase is destructive, with a relative phase of -1. The effect of this difference is quite tremendous: any number of bosons may occupy a single quantum state, while the occupation number for fermions is restricted to the values 0 or 1 (the Pauli exclusion principle). Because of the tight relationship between the counting rules for the occupation of states (in statistical mechanics) and the relative exchange phase associated with exchanging two identical particles (in quantum mechanics), the word "statistics" has come to refer to both. The "amplitude for a configuration" (for example, N particles at $\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N\}$) is of course the wavefunction $\psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$. Let us call the amplitude which differs only by a single permutation $\psi' \equiv \psi(\vec{r}_2, \vec{r}_1, \ldots, \vec{r}_N)$. The identity of the particles means that the probability density is unaltered by the exchange:

$$|\psi'|^2 = |\psi|^2$$

or

$$\psi' = p\psi$$

where p is a simple phase $e^{i\theta}$. A second exchange gives $\psi'' \equiv p^2\psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$. Requiring that the wavefunction be single valued, we obtain $p^2 = 1$. That is the permutation operator has only two eigenvalues

$$p = \pm 1.$$

Thus only two kinds of wavefunctions are possible, symmetric and anti-symmetric. However this property of wavefunctions does not itself determine the possible phases which may arise upon physical exchange of identical particles. In fact, it transpires that while in three spatial dimensions, fermions and bosons are the only possibilities, a continuum of possibilities (fractional statistics) exists in two spatial dimensions.

We are used to thinking that the permutation eigenvalue p controls the main physical properties of a system. It turns out, however, that the quantity which is crucial to the physics is not p but rather the phase which arises from an adiabatic transport of two identical particles along a path which gives an actual physical exchange. This latter phase (which we call η) depends on both the wavefunction and the Hamiltonian. Since η does not necessarily equal p, one must specify as to which one is meant by the term"statistics." We shall choose it to mean η. It may be shown that in three spatial dimensions, two exchanges are topologically equivalent to no exchange. Thus the phase assigned to two exchanges must be that assigned to the case of no exchange. This means in turn that $\eta^2 = 1$ or $\eta = \pm 1$. So, p can be identified with η and bosons and fermions are the only two possibilities.

In two spatial dimensions, on the other hand, two exchanges are not topologically equivalent to no exchange, nor are n exchanges, for $n \neq 0$. This is because the trajectory traced by the exchange of two identical particles in configuration space cannot be continuously shrunk to a point, or in other words, the configuration space is not simply-connected. Thus it is possible to consistently assign any value to η. If we denote as C^2 the configuration space of identical particles in two dimensions, the group structure of the paths in C^2 is the so called "braid" group rather than the permutation group which is the corresponding group structure in C^3. This is the formal reason for the vast wealth of possibilities in planar physics. The acquired phase depends generically on the path followed during the exchange. In fact even if only two particles are exchanged, the acquired phase depends in an essential manner on the positions of the other particles of the system.

From the above it is clear that the relative winding of particle world lines is of the utmost significance in two spatial dimensions. A nice way to keep track of this relative winding is through a "fictitious" Aharonov-Bohm (Aharonov & Bohm 1959) effect.

The interchange of two particles, say a and b, defines two paths C_1 and C_2 along which the particles a and b are transported to the original locations of b and a respectively. C_1 and C_2 together form a closed oriented loop. Let us pick the paths so that no other particles are within the loop. Depending on whether the loop is clockwise or anticlockwise, the operation is denoted by $P_-(a,b)$ or $P_+(a,b)$. In two spatial dimensions, in accordance with what we have said earlier,

$$P_\pm(a,b)\psi(\vec{r}_1,\ \ldots,\ \vec{r}_N) = -e^{\pm i\eta}\psi(\vec{r}_1,\ \ldots,\ \vec{r}_N)$$

where η need not be 0 or π. This is the boundary condition that has to be satisfied by an N-anyon wave function.

The Schrödinger equation for a system of "free" anyons is

$$i\partial_t \psi(\vec{r}_1, \ldots, \vec{r}_N) = -\frac{\hbar^2}{2m} \sum_{a=1}^{N} \nabla_a^2 \psi(\vec{r}_1, \ldots, \vec{r}_N),$$

where the boundary condition has been specified above. This nontrivial boundary condition hides an interaction between the anyons. Therefore unless $\eta = 0, \pi$ the anyon gas is not really free. The energy of a many-anyon system is not the sum of single particle energies. To expose the interaction it is convenient to make a singular gauge transformation and go over to a new gauge (Arovas *et al.* 1985). Let

$$\chi(\vec{r}_1, \ldots, \vec{r}_N) \equiv e^{i\omega(\vec{r}_1, \ldots, \vec{r}_N)} \psi(\vec{r}_1, \ldots, \vec{r}_N)$$

where $\omega(\vec{r}_1, \ldots, \vec{r}_N) = \frac{\eta}{\pi} \sum_{a<b} \tan^{-1} \frac{y_a - y_b}{x_a - x_b}$. This gauge transformation becomes singular when $\vec{r}_a = \vec{r}_b$. The Schrödinger equation satisfied by χ can be easily shown to be

$$i\partial_t \chi(\vec{r}_1, \ldots, \vec{r}_N) = \sum_{a=1}^{N} (-\frac{\hbar^2}{2m})[\vec{\nabla}_a - i\vec{A}_a(\vec{r}_1, \ldots, \vec{r}_N)]^2 \chi(\vec{r}_1, \ldots, \vec{r}_N)$$

where $\vec{A}_a(\vec{r}_1, \ldots, \vec{r}_N) \equiv \vec{\nabla}_a \omega(\vec{r}_1, \ldots, \vec{r}_N)$. So,

$$\vec{A}_a^j(\vec{r}_1, \ldots, \vec{r}_N) = -\frac{\eta}{\pi} \sum_{b \neq a} \epsilon^{jk} \partial_k^a \ln |\vec{r}_a - \vec{r}_b|.$$

Further, $P_\pm(a,b)\chi(\vec{r}_1, \ldots, \vec{r}_N) = -\chi(\vec{r}_1, \ldots, \vec{r}_N)$. So the new wavefunction defines a N-fermion system with a specific long range interaction given by $\vec{A}$. The wavefunction is a regular single-vallued function of $\{\vec{r}_1, \ldots, \vec{r}_N\}$, as opposed to the original wavefunction. So a "free" anyon system with a multivalued wavefunction may be looked upon as a system of fermions with a specified interaction and a single-valued wavefunction.

For definiteness let us look briefly into the two-anyon problem. On making the singular gauge transformation, the Hamiltonian becomes

$$H = \frac{1}{2m}[\vec{p}_1 - \vec{A}_1]^2 + \frac{1}{2m}[\vec{p}_2 - \vec{A}_2]^2.$$

Here,

$$\vec{A}_1 = \frac{\eta}{\pi}\frac{\hat{z}\times(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^2};\ \vec{A}_2 = \frac{\eta}{\pi}\frac{\hat{z}\times(\vec{r}_2-\vec{r}_1)}{|\vec{r}_1-\vec{r}_2|^2}.$$

Let us go over to the C.M. and relative coordinates. So,

$$\vec{R} \equiv \frac{1}{2}(\vec{r}_1 + \vec{r}_2);\ \vec{r} \equiv \vec{r}_1 - \vec{r}_2.$$

The corresponding canonical momenta are

$$\vec{P} = \vec{p}_1 + \vec{p}_2;\ \vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2).$$

Thus the Hamiltonian may be rewritten as

$$H = \frac{\vec{P}^2}{4m} + \frac{1}{m}(\vec{p} - \frac{\eta}{\pi}\frac{\hat{z}\times\vec{r}}{\vec{r}^2})^2.$$

The C.M. motion is independent of the statistics as expected. The relative coordinate describes a particle of mass $\frac{m}{2}$ minimally coupled to a vector potential $\vec{A} = \frac{\eta}{\pi}\frac{\hat{z}\times\vec{r}}{\vec{r}^2}$ which leads to a magnetic field $B = 2\eta\delta(\vec{r})$. This is a flux tube of strength 2η with the particle at a distance $|\vec{r}|$ from the tube. Further, because of the fermionic nature of the particles subsequent to the singular gauge transformation,

$$\psi(r,\theta) = -\psi(r,\theta+\pi).$$

Going over to cylindrical coordinates for $\vec{r}$, the Schrödinger equation for the relative coordinate is solved and the solution is of the form

$$\psi_{\text{rel}}(r,\theta) = e^{i(2l+1)\theta} J_{|(2l+1)+\frac{\eta}{\pi}|}(\sqrt{mE_{\text{rel}}}\,r),$$

where l is an integer. This satisfies the above boundary condition on the wavefunction. It is easily seen that

$$\chi_{\rm rel}(r,\theta) \equiv e^{i\frac{\eta}{\pi}\theta}\psi_{\rm rel}(r,\theta)$$

is an anyonic wavefunction as

$$\chi_{\rm rel}(r,\theta+\pi) = -e^{i\eta}\chi_{\rm rel}(r,\theta).$$

If we also use cylindrical coordinates $\{R,\Theta\}$ for $\vec{R}$, the C.M. wavefunction may be written as $e^{iL\Theta}J_{\rm L}(\sqrt{4mE_{\rm C.M.}}R)$. So, the full wavefunction is written as

$$\psi(\vec{R},\vec{r}) = e^{iL\Theta+\{(2l+1)++\frac{\eta}{\pi}\}\theta}J_{\rm L}(\sqrt{4mE_{\rm C.M.}}R)J_{|(2l+1)+\frac{\eta}{\pi}|}(\sqrt{mE_{\rm rel}}r).$$

This, as expected, cannot be factorized into a product of two suitable one-anyon wavefunctions.

Clearly, an N-anyon problem with or without interactions gets to be quite intractable. An alternative and equivalent approach to the N-body problem with identical particles is through second-quantization, where the Schrödinger problem is replaced by a non-relativistic field theory.

Consider a Lagrangian

$$\mathcal{L} = i\psi^\dagger\partial_t\psi - \frac{1}{2m}[(\partial_k - ia^k)\psi]^\dagger[(\partial_k - ia^k)\psi],$$

where ψ is a fermionic Schrödinger field and

$$a^j(\vec{r}) = -\frac{\eta}{\pi}\int d\vec{x}'\ \epsilon^{jk}\frac{(r-r')_k}{|\vec{r}-\vec{r}'|^2}\psi^\dagger(\vec{r}')\psi(\vec{r}').$$

Here the gauge-fixing condition $\vec{\nabla} \cdot \vec{a} = 0$ has been adopted. Since particle-number is conserved by the Hamiltonian, we can work in a fixed particle-number sector of the Fock space. The Schrödinger wavefunction is given by

$$\psi(\vec{r}_1, \ldots, \vec{r}_N) \equiv \langle 0|\hat{\psi}(\vec{r}_1)\ldots\hat{\psi}(\vec{r}_N)|N\rangle,$$

where $\langle 0|$ and $|N\rangle$ are the vacuum and N-particle states respectively. It can be shown that starting from the given Lagrangian, the familiar N-particle Schrödinger equation for $\psi(\vec{r}_1, \ldots, \vec{r}_N)$ is recovered.

This Lagrangian, in turn is obtained from

$$\mathcal{L}_0 = -\frac{1}{4\eta}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + i\psi^\dagger D_0\psi - \frac{1}{2m}|D_k\psi|^2.$$

Here, $D_0 \equiv \partial_0 + ia_0; D_k \equiv \partial_k - ia^k$. This is a theory of non-relativistic fermions coupled minimally to a gauge field whose motion is governed by a Chern-Simon (CS) term. The corresponding equations of motion are

$$-\frac{1}{4\eta}\epsilon^{\mu\nu\rho}f_{\nu\rho} = j^\mu.$$

$$i\partial_t\psi = [-\frac{1}{2m}D_k^2 + a_0]\psi.$$

Here,$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu$; $j^0 \equiv \psi^\dagger\psi$; $j^k \equiv -\frac{i}{2m}[\psi^\dagger D_k\psi - (D_k\psi)^\dagger\psi]$. Since $f_{\mu\nu}$ is determined by the fermionic current according to the equation of motion above, a^μ does not represent any additional degree of freedom and can be eliminated in favour of the matter fields. In the Coulomb gauge, $\vec{\nabla} \cdot \vec{a} = 0$, we obtain

$$a_0(\vec{r}) = -\frac{\eta}{\pi}\int d\vec{r'}\ \epsilon^{jk}\frac{(r-r')_j}{|\vec{r}-\vec{r'}|^2}j^k(\vec{r'})$$

and

$$a^j(\vec{r}) = -\frac{\eta}{\pi}\int d\vec{r'}\ \epsilon^{jk}\frac{(r-r')_k}{|\vec{r}-\vec{r'}|^2}j^0(\vec{r'}).$$

When these expressions are substituted back into $\mathcal{L}_0$, and a definite ordering is adopted for the fermion operators, we get back $\mathcal{L}$. But $\mathcal{L}$ as stated above, in turn leads to the familiar N-anyon Schrödinger equation. So, a theory of non-relativistic fermions, coupled to a CS gauge field, leads to a theory of anyons.

Now in the recently discovered high T_c superconducting materials, a striking feature is the planarity of the samples. The third dimension effectively decouples from the system. This effective planarity makes it extremely tempting to hypothesize as to the role of anyonic excitations in the physics of such superconductors.

The Lagrangian $\mathcal{L}_0$ described above is a very convenient point of departure for discussing anyon superconductivity. Coupling to real electromagnetism is effected by adding the real electromagnetic potentials to the "statistical" potentials in the covariant derivatives in $\mathcal{L}_0$.

Starting from this Lagrangian, we go on to show that an anyon gas placed in an external electromagnetic field, exhibits superfluidity and the Meissner effect. This is taken to imply superconductivity.

2.2 The heuristics of anyon superconductivity

Before invoking the formal machinery of quantum field theory, it will be instructive to investigate informally into why a gas of anyons coupled to electromagnetism is expected to superconduct. A mean field (Fetter *et al.* 1989), (Chen *et al.* 1989) approach is the most useful for this discussion. In the context of anyons, this involves replacing the flux tubes carried by the charges by a uniform magnetic field with the same flux density. It is obvious that this approximation is valid when the density of flux tubes (or, equivalently particles) is high and fluctuations in the number density are small, i.e., in the high density, low temperature regime.

With the given statistical potential, the magnetic field at the site of the i^{th} charge is given by

$$b_i = \nabla \times \vec{a}_i = 2\eta \sum_{i \neq j} \delta(\vec{r}_i - \vec{r}_j).$$

Let us assume that the density of anyons per unit area is given by $\bar{\rho}$. Then the uniform magnetic field to be used in the mean-field approach is given by

$$b = 2\eta\rho.$$

The flux per unit area due to this magnetic field is the same, on the average as that obtained from the expression for the magnetic field given earlier. From a pragmatic point of view, this means that we now have reduced the problem to one describing the motion of fermions in a uniform magnetic field, the magnetic field being proportional to the mean particle density. The single particle problem is easily solved, the energy eigenstates being the Landau levels. The gap between successive levels is given by $\frac{b}{2m} \equiv \omega_{\mathbf{cyclotron}}$. Each Landau level is highly degenerate, the number of such states per unit area being $\frac{b}{2\pi}$. But here, $b = 2\eta\rho$. So, the degeneracy is given by

$$deg = \frac{\eta\bar{\rho}}{\pi}.$$

If η takes on the special values of $\eta = \frac{\pi}{n}$, where n is any integer, the degeneracy is simply,

$$deg = \frac{\bar{\rho}}{n}.$$

Since $\bar{\rho}$ is the density of particles and each level can contain $\frac{\bar{\rho}}{n}$ particles per unit area, clearly n Landau levels will be completely filled. The next available single particle state is the next Landau level which is separated by a gap ω. Thus there is a gap for single particle excitations. If however n is not an integer, the last Landau level is not completely filled and there is no gap in the single particle excitations. Hence the parameter fractions $\eta = \frac{\pi}{n}$ appear to be special and the states formed at these fractions should be particularly stable.

To prove that the states at these special fractions are superconducting, we have to study the effect of adding a real magnetic field B to the fictitious field b. First let us consider the case where B is parallel to b. Here the degeneracy, deg, of the Landau levels increases to

$$deg = \frac{b + B}{2\pi}.$$

But the particle number per unit area remains unchanged at

$$\bar{\rho} = \frac{nb}{2\pi}.$$

Here, n is the number of Landau levels filled before the real magnetic field is turned on. So, the highest Landau level is now only partially filled. Let us denote its filling fraction by $(1 - x)$. From the conservation of the density of particles, it follows that

$$(n - 1)\frac{b + B}{2\pi} + (1 - x)\frac{b + B}{2\pi} = \bar{\rho} = \frac{nb}{2\pi}$$

from which we see that

$$(b+B)x = Bn.$$

The total energy of $\bar{\rho}$ particles is given by

$$\begin{aligned} E &= \frac{b+B}{2\pi}\omega[\sum_{j=0}^{n-2}(j+\frac{1}{2}) + (n-\frac{1}{2})(1-x)], \\ &= \frac{(b+B)^2}{2\pi m}[\frac{n^2}{2} - (n-\frac{1}{2})x] \\ &= \frac{n^2}{4\pi m}[b^2 + \frac{bB}{n} - B^2(1-\frac{1}{n})]. \end{aligned}$$

For small external magnetic fields B, the energy is seen to grow linearly with B. Thus, to minimize the energy, the anyon gas behaves as a diamagnet and expels external flux.

If the external magnetic field is aligned antiparallel to b, the degeneracy of the Landau levels decreases- i.e.,

$$deg = \frac{b-B}{2\pi}.$$

So, some of the particles have to occupy the $(n+1)^{\text{th}}$ Landau level. Let us denote the filling fraction of the highest level by x. From the conservation of particles, we obtain

$$x = \frac{B}{b-B}n.$$

Correspondingly, the total energy of the system is given by

$$E = \frac{n^2}{4\pi m}[b^2 + \frac{bB}{n} - B^2(1+\frac{1}{n})].$$

Once again it is quite obvious that the anyon gas behaves diamagnetically to reduce its energy. This establishes the Meissner effect in the anyon gas.

To establish superconductivity, one has to further demonstrate that the spectrum of the fluctuations about the mean density contains a massless mode. This would indicate that the collective excitation in the system is massless, thereby establishing superfluidity. A very heuristic argument may be made as follows: consider a very long wavelength density fluctuation (a collective excitation). Then $\bar{\rho}$, although varying, is approximately constant over macroscopic lengths. Within each such macroscopic area, b remains a constant. Hence, locally the system always has n filled Landau levels. Thus such a collective excitation does not require any particle to be excited to a higher Landau level and consequently requires no energy to be produced. Such a density wave is therefore massless. This, qualitatively is the origin of superfluidity in the anyon gas.

With these qualitative arguments, we are now ready for a formal demonstration of anyon superconductivity (Hosotani & Chakravarty 1990), (Randjbar-Daemi *et al.* 1990), (Panigrahi *et al.* 1990), (Banks & Lykken 1990), (Lykken *et al.* 1991).

2.3 Computing the effective action for an electron gas coupled to electromagnetism

We consider a system of planar electrons subject to an external electromagnetic potential whose frequency is much smaller than the natural frequency of the fermionic system. The Hamiltonian for this system is

$$H = \int d\vec{x}\ [\frac{1}{2m}|\{\vec{\nabla} - ie\vec{A}(\vec{x},t)\}\psi(\vec{x},t)|^2 + e\psi^\dagger(\vec{x},t)\psi(\vec{x},t)A_0(\vec{x},t)] \quad (2.1)$$

The Partition function of the system is, in the standard path integral representation (e.g. Sakita 1985),

$$Z[A] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\Big[-\int d\vec{x}\ \int_0^\beta d\tau\ \bar{\psi}(\vec{x},t)(\partial_\tau + h - \mu)\psi(\vec{x},t)\Big] \qquad (2.2)$$

where $h \equiv \frac{1}{2m}(\vec{p} - e\vec{A})^2 + eA_0$ and μ is the chemical potential of the system.

Let $A_0 \equiv -iA_\tau$; $\langle\tau|\hat{p}_\tau = -i\partial_\tau\langle\tau|$; $\hat{\pi}_\tau = \hat{p}_\tau - eA_\tau(\hat{x})$; $\hat{\pi}_i = \hat{p}_i - eA_i(\hat{x})$. From equation 2.2,

$$Z[A] = \det[i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu] \qquad (2.3)$$

Let $W[A] \equiv -\ln Z[A]$. This is the effective action. So,

$$W[A] = -\mathrm{tr}\ln\ [i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu]. \qquad (2.4)$$

The average fermionic currents are defined as

$$\langle j_\tau(x)\rangle \equiv \frac{\delta W[A]}{\delta A_\tau(x)}; \qquad \langle j_k(x)\rangle \equiv \frac{\delta W[A]}{\delta A_k(x)} \qquad (2.5)$$

We wish to obtain expressions for $\langle j_\tau\rangle$ and $\langle\vec{j}\rangle$ in terms of A_μ and functionally integrate them with respect to A_τ and $\vec{A}$, respectively, to recover an expression for $W[A]$. So

$$\langle j_\tau(x)\rangle = -ie\langle\vec{x},\tau|\frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu}|\vec{x},\tau\rangle \qquad (2.6)$$

and

$$\langle j_k(x)\rangle = \frac{e}{2m}\langle\vec{x},\tau|\hat{\pi}_k\frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} + \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu}\hat{\pi}_k|\vec{x},\tau\rangle \qquad (2.7)$$

To regulate the formal expressions, we use the Pauli-Villar regulator (Pauli & Villars 1949), even though for notational simplicity we do not mention it explicitly.

Formally, it is straightforward to demonstrate that the currents in equations 2.6 and 2.7 are conserved. The effective action in equation 2.4 is therefore formally gauge invariant. Further the effective action is also invariant under space reflection. Namely, $W[A] = W[A']$ where $A'_\mu = \pm A_\mu(\tau, x, y)$; $+$ for $\mu = \tau, y$ and $-$ for $\mu = x$.

Since $A_\mu(\vec{x}, \tau)$ is slowly varying in $\vec{x}$ and τ, we use the inhomogeneity expansion technique (Sakita & Su 1988) to evaluate $\langle j_\tau \rangle$ and $\langle j_k \rangle$.

We note that $|\vec{x}, \tau\rangle = e^{-ix.\hat{p}}|0\rangle \equiv \hat{U}_0|0\rangle$. So

$$\langle \vec{x}, \tau | \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} | \vec{x}, \tau \rangle = \langle 0 | \hat{U}_0^\dagger \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} \hat{U}_0 | 0 \rangle$$

and

$$\langle \vec{x}, \tau | \hat{\pi}_k \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} | \vec{x}, \tau \rangle = \langle 0 | \hat{U}_0^\dagger \hat{\pi}_k \hat{U}_0 \hat{U}_0^\dagger \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} \hat{U}_0 | 0 \rangle$$

Further, if

$$\hat{U}_1 \equiv e^{ie[\hat{x}_i . A_i + \frac{1}{2}\hat{x}_i . \hat{x}_j \partial_i A_j + \frac{1}{3!}\hat{x}_i . \hat{x}_j \hat{x}_k \partial_i \partial_j A_k \ldots]},$$

$\hat{U}_1|0\rangle = |0\rangle$. Thus

$$\langle \vec{x}, \tau | \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} | \vec{x}, \tau \rangle = \langle 0 | \hat{U}_1^\dagger \hat{U}_0^\dagger \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} \hat{U}_0 \hat{U}_1 | 0 \rangle$$

and

$$\langle \vec{x}, \tau | \hat{\pi}_k \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} | \vec{x}, \tau \rangle = \langle 0 | [\hat{U}_1^\dagger \hat{U}_0^\dagger \hat{\pi}_k \hat{U}_0 \hat{U}_1] \hat{U}_1^\dagger \hat{U}_0^\dagger \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} \hat{U}_0 \hat{U}_1 | 0 \rangle$$

Again consider

$$\hat{U}_2 \equiv e^{-i\frac{e}{2}B\hat{x}\hat{y}}$$

where $B \equiv \partial_x A_y - \partial_y A_x$. So $\langle 0|\hat{U}_2^\dagger = \langle 0|$. Thus

$$\langle \vec{x}, \tau| \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} |\vec{x}, \tau\rangle = \langle 0|\hat{V}^\dagger \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} \hat{V}|0\rangle$$

$$\langle \vec{x}, \tau|\hat{\pi}_k \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} |\vec{x}, \tau\rangle = \langle 0|\hat{V}^\dagger \hat{\pi}_k \hat{V}\hat{V}^\dagger \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} \hat{V}|0\rangle$$

Now $\hat{\pi}_x \equiv \hat{p}_x - eA_x(\hat{x})$. So,

$$\langle 0|\hat{U}_0^\dagger \hat{\pi}_x \hat{U}_0 = \langle 0|[\hat{p}_x - eA_x(x)]$$

This implies $\langle 0|\hat{V}^\dagger \hat{\pi}_x \hat{V} = \langle 0|\hat{p}_x$ and $\langle 0|\hat{V}^\dagger \hat{\pi}_y \hat{V} = \langle 0|\hat{p}_y$. Also

$$\hat{V}^\dagger \frac{1}{i\hat{\pi}_\tau + \frac{1}{2m}\hat{\vec{\pi}}^2 - \mu} \hat{V} = \frac{1}{\hat{P}_0 + i\hat{\Delta}_\tau + \frac{1}{2m}[\vec{\Pi}.\vec{\Delta} + \vec{\Delta}.\vec{\Pi} + \vec{\Delta}^2]}$$

where $\hat{P}_0 \equiv i\hat{p}_\tau + \hat{H}_0 - \mu'$; $\hat{\Pi}_x = \hat{p}_y$; $\hat{\Pi}_y \equiv \hat{p}_y - e\hat{x}B(x)$; $\hat{H}_0 \equiv \frac{1}{2m}[\hat{\Pi}_x^2 + \hat{\Pi}_y^2$; $\mu' \equiv \mu + ieA_\tau(x)$ and

$$\hat{\Delta}_\tau \equiv -e[\hat{x}_i \partial_i A_\tau + \frac{1}{2}\hat{x}_i \hat{x}_j \partial_i \partial_j A_\tau + \hat{\tau}\partial_\tau A_\tau + \frac{1}{2}\hat{\tau}^2 \partial_\tau^2 A_\tau + \hat{x}_i \tau^2 \partial_i \partial_t A_\tau + \ldots]$$

$$\hat{\Delta}_x \equiv -e\hat{\tau}\partial_\tau A_x - \frac{e}{2}\hat{\tau}^2 \partial_\tau^2 A_x - e\hat{\tau}\hat{x}\partial_\tau \partial_x A_x - e\hat{\tau}\hat{y}\partial_\tau \partial_y A_x + \frac{e}{3}\hat{x}\hat{y}\partial_x B + \frac{e}{3}\hat{y}^2 \partial_y B \ldots$$

$$\hat{\Delta}_y \equiv -e\hat{\tau}\partial_\tau A_y - \frac{e}{2}\hat{\tau}^2 \partial_\tau^2 A_y - e\hat{\tau}\hat{x}\partial_\tau \partial_x A_y - e\hat{\tau}\hat{y}\partial_\tau \partial_y A_y + \frac{e}{3}\hat{x}\hat{y}\partial_y B + \frac{e}{3}\hat{x}^2 \partial_x B \ldots$$

So,

$$\langle j_\tau(x)\rangle = ie\langle 0| \frac{1}{\hat{P}_0 + i\hat{\Delta}_\tau + \frac{1}{2m}[\vec{\Pi}.\vec{\Delta} + \vec{\Delta}.\vec{\Pi} + \vec{\Delta}^2]} |0\rangle \tag{2.8}$$

$$\begin{aligned}\langle j_k(x)\rangle &= \frac{e}{2m}\langle 0|\Pi_k \frac{1}{\hat{P}_0 + i\hat{\Delta}_\tau + \frac{1}{2m}[\vec{\Pi}.\vec{\Delta} + \vec{\Delta}.\vec{\Pi} + \vec{\Delta}^2]} \\ &+ \frac{1}{\hat{P}_0 + i\hat{\Delta}_\tau + \frac{1}{2m}[\vec{\Pi}.\vec{\Delta} + \vec{\Delta}.\vec{\Pi} + \vec{\Delta}^2]}\Pi_k|0\rangle \end{aligned} \tag{2.9}$$

We use

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A}\dots$$

to expand

$$\frac{1}{\hat{P}_0 + i\hat{\Delta}_\tau + \frac{1}{2m}[\vec{\Pi}.\vec{\Delta} + \vec{\Delta}.\vec{\Pi} + \vec{\Delta}^2]}.$$

The higher iterations are dropped as they involve higher spatial and temporal derivatives. Also, $[\hat{\Pi}_x, \hat{\Pi}_y] = ieB(x)$. Let us take $eB(x) < 0$ for definiteness. So

$$[\hat{\Pi}_x, \hat{\Pi}_y] = -i|eB(x)| \equiv \frac{-i}{l^2} \qquad (2.10)$$

So $\hat{\Pi}_y$ is like a coordinate with $\hat{\Pi}_x$ as the corresponding canonical momentum. For $eB > 0$, the roles of Π_x and Π_y are reversed. Let

$$\hat{a} \equiv \frac{l}{\sqrt{2}}(\hat{\Pi}_x - i\hat{\Pi}_y); \quad \hat{a}^\dagger \equiv \frac{l}{\sqrt{2}}(\hat{\Pi}_x + i\hat{\Pi}_y); \qquad (2.11)$$

So

$$[\hat{a}, \hat{a}^\dagger] = 1$$

We introduce guiding center coordinates (Kubo *et al.* 1965) as

$$\hat{X} \equiv \hat{x} - l^2\hat{\Pi}_y; \quad \hat{Y} \equiv \hat{y} + l^2\hat{\Pi}_x \qquad (2.12)$$

So,

$$[\hat{X}, \hat{\Pi}_i] = [\hat{Y}, \hat{\Pi}_i] = [\hat{X}, \hat{H}_0] = [\hat{Y}, \hat{H}_0] = 0; \ [\hat{X}, \hat{Y}] = il^2 \qquad (2.13)$$

Also,

$$\hat{\Pi}_x = \frac{1}{\sqrt{2}l}(\hat{a} + \hat{a}^\dagger); \quad \hat{\Pi}_y = \frac{i}{\sqrt{2}l}(\hat{a} - \hat{a}^\dagger) \qquad (2.14)$$

We choose a basis $\{|n\rangle \otimes |X\rangle\}$ where $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$; $n = 0\dots\infty$ and $\hat{X}|X\rangle = X|X\rangle$. We note that

$$\int_{-\infty}^{\infty} dX \ |\langle\vec{x}|n, X\rangle|^2 = \frac{1}{2\pi l^2} \qquad (2.15)$$

Further,

$$[\hat{r},\hat{p}_r]=i;\quad [\hat{r},\frac{1}{\hat{p}_0}]=\frac{1}{\hat{p}_0^2}$$
$$[\hat{\Pi}_x,\frac{1}{\hat{p}_0}]=\frac{i}{ml^2}\frac{1}{\hat{p}_0}\hat{\Pi}_y\frac{1}{\hat{p}_0}$$
$$[\hat{\Pi}_y,\frac{1}{\hat{p}_0}]=-\frac{i}{ml^2}\frac{1}{\hat{p}_0}\hat{\Pi}_x\frac{1}{\hat{p}_0} \tag{2.16}$$

Again, let $\hat{O}$ and $\hat{O}'$ consist of $\hat{P}_0$ and $\hat{\Pi}_i$. Then

$$\langle 0|\hat{O}\hat{x}\hat{O}'|0\rangle = \langle 0|\hat{O}(\hat{X}+l^2\hat{\Pi}_y)\hat{O}'|0\rangle = l^2\langle 0|[\hat{O},\hat{\Pi}_y]\hat{O}'|0\rangle = l^2\langle 0|\hat{O}[\hat{\Pi}_y,\hat{O}']|0\rangle \tag{2.17}$$

$$\langle 0|\hat{O}\hat{y}\hat{O}'|0\rangle = \langle 0|\hat{O}(\hat{Y}+l^2\hat{\Pi}_x)\hat{O}'|0\rangle = l^2\langle 0|[\hat{\Pi}_x,\hat{O}]\hat{O}'|0\rangle = l^2\langle 0|\hat{O}[\hat{O}',\hat{\Pi}_x]|0\rangle \tag{2.18}$$

We can use these to remove all reference to the guiding center coordinates from the matrix elements. In that case we can use equation 2.15 to do the integral over X, leaving only the sum over n to be performed.

Armed with these relations, let us look at $\langle j_r(x)\rangle$.

$$\langle j_r(x)\rangle = \langle j_r(x)\rangle^{(0)} + \langle j_r(x)\rangle^{(1)} + \dots$$

where $\langle j_r(x)\rangle^{(0)} = ie\langle 0|\frac{1}{\hat{P}_0}|0\rangle$

$$\langle j_r(x)\rangle^{(0)} = -ie\langle 0|\frac{1}{\hat{P}_0}[i\hat{\Delta}+\frac{1}{2m}\{\vec{\Pi}.\vec{\Delta}+\vec{\Delta}.\vec{\Pi}+\vec{\Delta}^2\}]\frac{1}{\hat{P}_0}|0\rangle \tag{2.19}$$

$$\begin{aligned}\langle j_r(x)\rangle^{(0)} &= ie\langle 0|\frac{1}{\hat{P}_0}|0\rangle \\ &= ie\int dX \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{\langle 0|n,X,m\rangle\langle n,X,m|0\rangle}{i\xi_m+E_n-\mu'}\end{aligned}$$

where $\xi_m \equiv (m+\frac{1}{2})\frac{2\pi}{\beta}$ are the Matsubara frequencies . and $E_n = (n+\frac{1}{2})\omega$ where $\omega \equiv \frac{|eB|}{m} = \frac{1}{ml^2}$. Also, $\langle \tau = 0|m\rangle = \frac{1}{\sqrt{\beta}}$. Let $\Gamma_n \equiv \xi_m + E_n - \mu'$. So,

$$\begin{aligned}
\langle j_\tau(x)\rangle^{(0)} &= \frac{ie}{\beta}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\Gamma_n}\int dX \ |\langle 0|n,X\rangle|^2 \\
&= \frac{ie}{\beta}.\frac{1}{2\pi l^2}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\Gamma_n} \\
&= \frac{ie}{2\pi l^2}\sum_{n=0}^{\infty}\frac{1}{e^{\beta(E_n-\mu')+1}} \\
&\equiv \frac{ie}{2\pi l^2}\gamma(|eB|, A_\tau)
\end{aligned} \tag{2.20}$$

Similarly, we compute $\langle j_\tau(x)\rangle^{(1)}$ and obtain

$$\langle j_\tau(x)\rangle^{(1)} = -\frac{e^2}{2\pi\omega}[\gamma + \omega\frac{\partial\gamma}{\partial\omega}]\vec{\nabla}\cdot\vec{E} \tag{2.21}$$

where $E_x \equiv \partial_\tau A_x - \partial_x A_\tau$ and $E_y \equiv \partial_\tau A_y - \partial_y A_\tau$. Again,

$$\begin{aligned}
\langle j_x(x)\rangle &= \frac{e}{2m}\langle 0|\hat{\Pi}_x\frac{1}{ip_\tau + \hat{H}_0 - \mu' + i\hat{\Delta}_\tau + \frac{1}{2m}(\vec{\Pi}\cdot\vec{\Delta} + \vec{\Delta}\cdot\vec{\Pi} + \vec{\Delta}^2)}|0\rangle \\
&+ \frac{e}{2m}\langle 0|\frac{1}{ip_\tau + \hat{H}_0 - \mu' + i\hat{\Delta}_\tau + \frac{1}{2m}(\vec{\Pi}\cdot\vec{\Delta} + \vec{\Delta}\cdot\vec{\Pi} + \vec{\Delta}^2)}\hat{\Pi}_x|0\rangle
\end{aligned} \tag{2.22}$$

This may be computed in a similar fashion to yield

$$\langle j_x(x)\rangle^{(0)} = 0 \tag{2.23}$$

$$\langle j_x(x)\rangle^{(1)} = \frac{ie^2}{2\pi}\gamma F_{\tau y} - \frac{e^2}{2\pi m}\lambda(|eB(x)|, A_\tau)\partial_y B(x) \tag{2.24}$$

where

$$\lambda = \sum_{n=0}^{\infty}(n+\frac{1}{2})\frac{1}{1+exp[\beta(E_n-\mu')]}.$$

Similarly,

$$\langle j_x(y)\rangle^{(0)} = 0 \qquad (2.25)$$

$$\langle j_x(y)\rangle^{(1)} = -\frac{ie^2}{2\pi}\gamma F_{\tau x} + \frac{e^2}{2\pi m}\lambda(|eB(x)|, A_\tau)\partial_x B(x) \qquad (2.26)$$

At T=0, $\gamma(|eB|, A_\tau)$ is independent of the real part of A_τ and possesses the property of being a "staircase" function of $(\mu + ieA_\tau)/eB$. Accordingly, $\lambda = \gamma^2/2 + \gamma$. At finite T, the staircase function is smoothed out and there is no simple relation between λ and γ. Upto now we have assumed that $eB < 0$, but the same expression holds for $eB > 0$. This is to be expected from parity considerations.

In the approach taken here, the higher order corrections are given by the higher order derivatives of the fields. Upto the given order, the current, as it stands, is not conserved at $T \neq 0$. The origin of the problem is that the inhomogeneity expansion does not respect the boundary condition for finite temperature. However, the current is conserved for zero temperature and in the case of finite temperature but static electromagnetic fields. We therefore restrict our attention to these two particular cases.

The effective action at T=0 is now straightforwardly obtained. It is given by

$$\begin{aligned} W[A] &= \frac{ie^2}{2\pi}\int d\vec{x}d\tau\ \epsilon(eB)\gamma A_\tau(x)B(x) \\ &+ \frac{e^2}{8\pi m}\int d\vec{x}d\tau\ \gamma^2 B^2(x) \\ &+ \frac{e^2 m}{4\pi}\int d\vec{x}d\tau\ \frac{\gamma}{|eB|}\vec{E}^2(x) \end{aligned} \qquad (2.27)$$

where

$$\epsilon(eB) = +1, \; eB > 0, \\ = -1, \; eB < 0.$$

Thus we see that a CS-like term has been generated in the effective action when the fermions are integrated out. This effective action can now be adapted for use in the case of anyon superconductivity.

2.4 Anyonic superconductivity and the existence of massless excitations

In this section, we use the effective Lagrangian computed in section 2.3 to discuss anyonic superconductivity. Anyons are quasi-particle excitations of fractional statistics. So we are looking at a system of charged particles of fractional statistics whose Hamiltonian is given by

$$H = \int \vec{dx} \, [\frac{1}{2m} |(\nabla - ie(\vec{a} + \vec{A}))\psi(\vec{x})|^2 + e\bar{\psi}(\vec{x})\psi(\vec{x})A_0(\vec{x})] \qquad (2.28)$$

where $\vec{a}$ is the solution of

$$b \equiv \partial_x a_y - \partial_y a_x = \frac{e}{\mu_0}\bar{\psi}\psi \qquad (2.29)$$

For simplicity, we hereafter set $e = 1$ which can be done by rescaling the gauge fields. According to this Hamiltonian, the charged particles (anyons) are in the statistical magnetic field b which is proportional to the anyon density. As a consequence the anyon acquires an extra phase when it goes around other anyons. The phase depends on the parameter μ_0 and the value of μ_0 determines the statistics of the particles (Arovas *et al.* 1985). Solving equation 2.29 in the Coulomb gauge we may write the partition function for the system as

$$Z = \int D\psi D\bar{\psi} Da_i \ \delta(\partial_i a_i) \ \delta(b - \frac{1}{\mu_0}\bar{\psi}\psi) \exp[-S] \qquad (2.30)$$

where

$$S = \int d\vec{x} d\tau \ [\bar{\psi}(\partial_\tau - iA_\tau + \frac{1}{2m}(\vec{p} - (\vec{A} + \vec{a}))^2 - \mu)\psi]$$

Let

$$\delta(b - \frac{1}{\mu_0}\bar{\psi}\psi) = \int Da_\tau exp[i \int d\tau d\vec{x} \ (\bar{\psi} a_\tau \psi - \mu_0 b a_\tau) \qquad (2.31)$$

So the partition function can be written as

$$Z = \int D\psi D\bar{\psi} Da_i Da_\tau \ \delta(\partial_i a_i) \exp[-S'] \qquad (2.32)$$

where

$$S' = \int d\vec{x} d\tau \ [i\mu_0 b a_\tau \ + \ \bar{\psi}[i(p\tau - (a_\tau + A_\tau)) \ + \ \frac{1}{2m}(\vec{p} - (\vec{A} + \vec{a}))^2 \ - \ \mu]\psi]$$

We do the fermionic integration to get

$$Z = \int Da_i Da_\tau \ \delta(\partial_i a_i) \exp(-[i\mu_0 \int d\vec{x} d\tau \ b a_\tau \ + \ W_{\text{eff}}[a + A]) \qquad (2.33)$$

$W_{\text{eff}}[a + A]$ has already been computed in section 2.3 with A of section 2.3 replaced by $(a + A)$ here.

Thus we have an expression for the partition function of the system of anyons. It is interesting to note that

$$\int d\vec{x} d\tau \ b \ a_\tau$$

is the full C-S term in the Coulomb gauge. The Coulomb gauge condition yields

$$a_i(x) \;=\; \epsilon_{ij}\partial_j\phi(x)$$

and

$$b(x) \;=\; -\nabla^2\phi(x),$$

whence, formally

$$a_i(x) \;=\; -\epsilon_{ij}\frac{\partial_j}{\nabla^2}b(x) \tag{2.34}$$

Using equation 2.34, the functional integral is trivially converted from one over a_i to one over b. It is also worth noting that the signature of $b+B$ is crucial in defining the creation and destruction operators in $\hat{H}_0$. The signature manifests itself in the expression for $< j_i(x) >$.

Now, after carrying out the fermionic integration, we get

$$Z \;=\; \int DbDa_\tau\; \exp[T_1+T_2+T_3+T_4] \tag{2.35}$$

where

$$\begin{aligned}
T_1 &= -i\int d\vec{x}d\tau\; \mu_0 a_\tau b \\
T_2 &= -\frac{i}{2\pi}\int d\vec{x}d\tau\; \gamma|(b+B)|(a_\tau+A_\tau) \\
T_3 &= -\frac{m}{4\pi}\int d\vec{x}d\tau\; \frac{\gamma}{|(b+B)|}\;(\vec{e}+\vec{E})^2 \\
T_4 &= -\frac{1}{8\pi m}\int d\vec{x}d\tau\; \gamma^2\;(b+B)^2
\end{aligned}$$

where γ is given by 1.20 with $B \rightarrow (b+B)$ and $A_\tau \rightarrow (a_\tau+A_\tau)$. It is to be understood that the potential $a_i(x)$ appearing in the statistical electric field has been expressed in terms of $b(x)$. A more explicit expression will be given later when we study the spectrum of the collective excitation. At this point we intend to study two aspects of the system. First, we want to show that the cancellation of the tree level and the induced C-S term minimizes the free energy of the system. Secondly, we want to demonstrate the existence of a massless mode once the cancellation of the C-S term is achieved. The exact analysis being quite complicated, we, in what follows use the fact that the higher derivative terms in the Lagrangian are much smaller compared to the C-S term and that they are negligible in the first approximation.

In accordance with the above arguments we write

$$\begin{aligned} Z &\simeq \int Da_\tau Db \, \exp[-i \int d\vec{x} d\tau \, [\mu_0 b + \frac{\gamma}{2\pi}|(b+B)|]a_\tau] \\ &= \int_{b+B<0} Da_\tau Db \, \exp[-i \int d\vec{x} d\tau \, [\mu_0 b - \frac{\gamma}{2\pi}|(b+B)|]a_\tau] \\ &+ \int_{b+B>0} Da_\tau Db \, \exp[-i \int d\vec{x} d\tau \, [\mu_0 b + \frac{\gamma}{2\pi}|(b+B)|]a_\tau] \end{aligned} \tag{2.36}$$

$B(x)$ being the external field, we first choose it to be positive: $B(x) \geq 0$. Now since μ_0 is a parameter in the theory we fix it to be $\mu_0 = -\frac{N}{2\pi}$. The case $N = 2$ corresponds to that of semions. So,

$$Z = \int_{b+B<0} Db \, \delta[\frac{N+\gamma}{2\pi} b + \frac{\gamma}{2\pi} B] + \int_{b+B>0} Db \, \delta[\frac{\gamma}{2\pi} B - (N-\gamma)\frac{b}{2\pi}] \tag{2.37}$$

Since $\gamma > 0$ and $N > 0$;

$$g(b) = \gamma(b+B) + Nb|_{b=b_0} = 0$$

does not have a consistent solution for $b + B < 0, \; B > 0$.

Thus the first term in the partition function is zero. In the second term let

$$f(b) = (N-\gamma)b - \gamma B \tag{2.38}$$

If $f(b_0) = 0$,

$$(N-\gamma)b_0 = \gamma B. \tag{2.39}$$

Defining $\xi = \frac{\mu}{b+B}$ we get

$$b = \frac{\mu}{\xi} - B$$

and hence

$$(N-\gamma)(\frac{\mu}{\xi_0} - B) = \gamma B$$

or

$$(N-\gamma)\frac{\mu}{\xi_0} = NB \tag{2.40}$$

In our calculations, at a point x, $B(x) > 0$ and μ is fixed. ξ_0 changes due to changes in B.

Also, at $T = 0$,

$$\gamma = \sum_{n=0}^{\infty} \theta[(n+\frac{1}{2})\frac{1}{m} - \xi] \tag{2.41}$$

For semions $N = 2$, so

$$[2 - \sum_{n=0}^{\infty} \theta[(n+\frac{1}{2})\frac{1}{m} - \xi_0]\frac{\mu}{\xi}] = 2B \tag{2.42}$$

We solve this equation graphically.

$$Z \simeq \int Db\ \delta(f(b))$$

or

$$Z = \sum_{b_0} \Pi_{x,\tau} |\frac{\partial f}{\partial b}|^{-1}_{b=b_0} \tag{2.43}$$

where b_0 is obtained from equation 2.42. Also

$$\frac{\partial f}{\partial b} = (2-\gamma) - \frac{\partial \gamma}{\partial b}(b+B) = (2-\gamma) - \sum_{n=0}^{\infty} \delta(\ b + B - \frac{m\mu}{(n+1/2)}). \tag{2.44}$$

Now, from the graph, we see that if

$$\frac{2\mu m}{5} < b_0 + B < \frac{2\mu m}{3},$$

$\gamma = 2$. Equation 2.42 has a solution provided $B = 0$. In this case, $|\frac{\partial f}{\partial b}|^{-1} = \infty$ and gives a large contribution to Z. If, however, b_0 lies somewhere else, a solution to equation 2.42 exists even if $B \neq 0$. In this case $\gamma \neq 2$ so that $|\frac{\partial f}{\partial b}|_{b=b_0}$ is finite.

A similar analysis may be carried out for $B(x) \leq 0$. As in equation 2.36 we write

$$\begin{aligned} Z &\simeq \int Da_\tau Db \ \exp[-i\int d\vec{x}d\tau \ (\mu_0 b + \frac{\gamma}{2\pi}|b+B|)a_\tau] \\ &= \int_{b+B<0, B\leq 0} Da_\tau Db \ \exp[-i\int d\vec{x}d\tau \ (\mu_0 b + \frac{\gamma}{2\pi}|b+B|)a_\tau] \\ &+ \int_{b+B>0, B\leq 0} Da_\tau Db \ \exp[-i\int d\vec{x}d\tau \ (\mu_0 b + \frac{\gamma}{2\pi}|b+B|)a_\tau] \quad (2.45) \end{aligned}$$

Let $\mu_0 = -\frac{N}{2\pi}$. Thus,

$$Z = \int_{b+B<0, B\leq 0} Db\delta[Nb + \gamma(b+B)] + \int_{b+B>0, B\leq 0} Db \ \delta[-Nb + \gamma(b+B)] \quad (2.46)$$

Let

$$\rho(b) \equiv Nb + \gamma(b+B) \quad (2.47)$$

$$\sigma(b) \equiv -Nb + \gamma(b+B) \quad (2.48)$$

Now we require the zeros of $\rho(b)$ and $\sigma(b)$. From equation 2.47

$$Nb + \gamma(b_0 + B) = 0 \quad (2.49)$$

Let $\xi \equiv \frac{\mu}{b+B}$. If $b + B < 0, \xi < 0$. If $b + B > 0, \xi > 0$. So from equation 2.49, $(N+\gamma)(\frac{\mu}{\xi} - B) = -\gamma B$. Or

$$(N+\gamma)\frac{\mu}{\xi} = NB \quad (2.50)$$

Here $\xi < 0$. Again from equation 2.48,

$$(N-\gamma)\frac{\mu}{\xi} = NB \tag{2.51}$$

Here $\xi > 0$. In equation 2.50 let $B < 0$ be such that the solution exists when $\gamma = 1$. A solution for equation 2.51 also exists simultaneously. For these solutions,

$$Z_1 \simeq \sum_{b_0} \int Db\ \delta(b-b_0)|\frac{\partial \rho}{\partial b}|_{b=b_0}^{-1} + \sum_{b_0} \int Db\delta(b-b_0)|\frac{\partial \sigma}{\partial b}|_{b=b_0}^{-1} \tag{2.52}$$

If, however, B=0, a solution for equation 2.50 does not exist, but a solution for equation 2.51 exists with $N = \gamma$. Now,

$$\frac{\partial \sigma}{\partial b} = (\gamma - N) \tag{2.53}$$

So,

$$Z_2 \simeq \sum_{b_0} \int Db\ \delta(b-b_0)|\gamma - N|^{-1} \tag{2.54}$$

So the free energy $F_1 \equiv -lnZ_1 \gg F_2 \equiv -lnZ_2$. Thus, an absolute minimum of F is attained if B=0 even if we start from $B \leq 0$. We conclude, therefore, that the free energy has a strong minimum at $B(x) = 0$.

With this heuristic argument in mind let us look at the partition function more carefully. Z gets its maximum contribution from the region of the b(x) integration that yields $N = \gamma$. So for $N = 2$,

$$\frac{2\mu m}{5} < b_0 + B < \frac{2\mu m}{3} \tag{2.55}$$

and

$$Z = \iint Da_\tau\, Db\ \exp[A_1 + A_2 + A_3 + A_4 + A_5 + A_6] \tag{2.56}$$

where

$$
\begin{aligned}
A_1 &= -\frac{i}{\pi}\int dx\ [(b+B)A_\tau + Ba_\tau] \\
A_2 &= -\frac{1}{2\pi m}\int dx\ (b+B)^2 \\
A_3 &= -\frac{m}{2\pi}\int dx\ \frac{1}{|(b+B)|}\,[\ (D_i b)^2 + (\partial_i a_\tau)^2 + 2\partial_i a_\tau\ \epsilon_{ij}\ D_j b] \\
A_4 &= \int dx\ \vec{E}^2 \\
A_5 &= -2\int dx\ \epsilon_{ij}\ D_j b(x)\ E_i \\
A_6 &= -2\int dx\ \partial_i a_\tau\ E_i \\
D_i &= \frac{\partial_\tau \partial_i}{\nabla^2}
\end{aligned}
$$

But from an order of magnitude estimate, we note that $A_1 \gg A_2, A_3, A_4, A_5, A_6$. Also, in the low momentum regime, $(\partial_i a_\tau)^2 \ll (D_i b)^2$. So even though we may drop $(\partial_i a_\tau)^2$ when doing the a_τ integration, we may not drop $(D_i b)^2$ in the b integration. So,

$$
Z \;=\; \delta(B)\int Db\ \exp[P+Q+R], \tag{2.57}
$$

where

$$
\begin{aligned}
P &= -\frac{i}{\pi}\int dx\ bA_\tau \\
Q &= -\frac{1}{2\pi m}\int dx\ b^2 \\
R &= \frac{m}{2\pi}\int dx\ \frac{1}{|b|}\,[\ (D_i b)^2 + \vec{E}^2 - 2\epsilon_{ij}\ D_j b(x)\ E_i]
\end{aligned}
$$

This indicates that $B(x) = 0$ which is the Meissner effect. Further in doing the $b(x)$ integration we would like to do a saddle point calculation. Let

$$
h[\ b\ ] \;=\; P+Q+R. \tag{2.58}
$$

For the extremum,

$$0 = -\frac{i}{\pi}\ \nabla^2 A_\tau - \frac{1}{\pi m}\ \nabla^2 b_0 - \frac{m}{\pi|b_0|}\ \partial_\tau{}^2 b_0 + \frac{m}{\pi|b_0|}\ \epsilon_{ij}\partial_\tau\partial_j E_i \tag{2.59}$$

Since A_τ is the external scalar potential, $\nabla^2 A_\tau = 0$. $\vec{E}$, the external electric field is time-independent. So $b_0 =$ constant is a solution of equation 2.59 b_0 lies in the range

$$\frac{2\mu m}{5} \le b_0 \le \frac{2\mu m}{3}$$

Further,

$$\frac{\delta^2 h(b)}{\delta b(x)\delta b(y)} = -\frac{1}{\pi}\ [\ \frac{1}{m} + \frac{m}{|b_0|}\partial_\tau{}^2\nabla^{-2}\ ]\ \delta(x-y) \tag{2.60}$$

So,

$$h[\ b\ ] \simeq h[\ b_0\ ] - \int dx\ \eta(x)[\ \frac{1}{\pi m} + \frac{m}{\pi|b_0|}\ \partial_\tau{}^2\nabla^{-2}\ ]\eta(x) \tag{2.61}$$

So

$$Z \simeq \delta(B)e^{h[b_0]}\int D\eta\ e^{S[\eta]}$$

where

$$S[\ \eta\ ] \equiv -\frac{1}{\pi m}\ \int dx\ \eta[1 + \frac{m^2}{|b_0|}\partial_\tau{}^2\nabla^{-2}]\eta \tag{2.62}$$

Thus if we rescale as

$$\frac{1}{(\pi m)^{1/2}}\ \frac{1}{|p|}\ \tilde{\eta}(\vec{p}, p_0) \equiv \tilde{\xi}(\vec{p}, p_0),$$

where as $\beta \to \infty$, $\frac{1}{\beta}\sum_n \to \frac{1}{2\pi}\int dp_0$, we see that the propagator of the $\xi(x,\tau)$ field is

$$(p^2 + [\frac{m}{(b_0)^{1/2}}]^2\ p_0^2)^{-1} \tag{2.63}$$

which is the propagator for a massless excitation. So the fluctuation around a constant background is a propagating massless mode.

With the above arguments, we have shown, at least at $T = 0$, that the anyon gas is a superfluid which expels the external applied magnetic field.

2.5 Summary

In this work, we have computed the effective Lagrangian using the inhomogeneity expansion method. The perturbations have been carried out about a local vacuum consisting of filled Landau levels, because of which a CS-like term appears in the nonrelativistic Schrödinger field theory. This effective Lagrangian has then been used to study the low energy behaviour of an anyon gas coupled to electromagnetism. Free-energy considerations have revealed that the cancellation of the induced and the tree-level CS term is favoured. This in turn has indicated that the external magnetic field is screened and that the collective mode in the density fluctuation of the anyons is massless. These two features, taken in conjunction, is taken to indicate superconductivity.

A caveat that ought to be mentioned is that the current computed in the text is not conserved at $T \neq 0$. This is because the inhomogeneity expansion does not respect the boundary condition for finite temperature. Thus the effective Lagrangian obtained is strictly gauge-invariant only if $T = 0$ or if the electromagnetic fields are static. Further analysis is indicated for looking at anyon superconductivity in more general situations.

Chapter 3

The electromagnetic interactions of electrons in the lowest landau level

3.1 Introduction

In this chapter, we wish to discuss the field theory of planar fermions in a strong magnetic field normal to the plane and coupled minimally to other electromagnetic potentials. the fields associated with these other potentials are taken to be small in comparison with the magnetic field. Thus, their effect may be treated as a perturbation on the basic problem of a planar electron gas in a strong magnetic field.

The solution to the one-body problem, which is the celebrated Landau problem, is already known. The energy eigenstates are the uniformly spaced Landau levels (L.L.) with a gap of $\omega \equiv \frac{B}{m}$ between successive levels. Each L.L. has a huge degeneracy equal to $\frac{B}{2\pi}$ per unit area. In the case of a strong magnetic field, the gap in the single-particle spectrum is large and consequently, one may rightly imagine the lowest Landau level (L.L.L.) to play a paramount role in the corresponding many-body problem.

It is thus useful to develop an effective field theory of the L.L.L. which in particular would be relevant to the theory of the fractional quantum Hall effect, whose essential physics results from the rearrangement of the degenerate states in a single L.L.. Some of the machinery for the first quantized approach to the L.L.L. states has already been developed by Girvin and Jach (Girvin & Jach 1983). In what follows, we have emphasized the second-quantized field theoretic approach to the problem.

The additional electromagnetic potentials in the problem can, in principle, excite electrons from the L.L.L. to higher L.L.. If we are interested in the dynamics of the L.L.L., however, these transitions, which appear as intermediate steps in a process, may be encapsuled into an effective Lagrangian involving only the L.L.L. degrees of freedom. In practise we consider, as stated above, only weak perturbative fields whose associated energy scale is much smaller than the energy scales associated with the unperturbed problem. This enables us to compute this effective Lagrangian through a perturbative technique. The resulting effective Lagrangian shows a rather complicated coupling of the perturbing potential to the L.L.L. field. Further, it does not evince the manifest gauge invariance of the microscopic Lagrangian that one starts out with. However, interestingly enough, gauge covariance can be seen to be effected through a non-linear realization of W_∞ gauge transformations (Ray & Sakita 1993).

There are other related points of interest regarding a field theory of the

L.L.L.. The fermion field for the L.L.L. may be written as

$$\hat{\psi}(x,y,t) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{n!}} \hat{c}_n(t), \tag{3.1}$$

where,

$$z \equiv \sqrt{\frac{B}{2}}(x+iy), \bar{z} \equiv z^{\dagger}.$$

The modes $\hat{c}_n$ satisfy $\{\hat{c}_n, \hat{c}_m^{\dagger}\} = \delta_{n,m}$. However the anticommutator for the fields is more complicated since the L.L.L. single-particle states do not form a complete set of states. As a result,

$$\begin{aligned} \{\psi(z_1,\bar{z}_1),\ \bar{\psi}(z_2,\bar{z}_2)\} &= \{z_1|z_2\} \\ &\equiv \frac{B}{2\pi} e^{-\frac{1}{2}|z_1-z_2|^2 + \frac{1}{2}(\bar{z}_1 z_2 - \bar{z}_2 z_1)}. \end{aligned} \tag{3.2}$$

The presence of this bi-local kernel instead of a local delta-function leads to certain peculiarities and renders some results extrapolated from theories with a local anticommutator, null and void (Martinez and Stone 1993). In particular, the current operator for the L.L.L. is seen to deviate from the naive expectation.

Further, as discussed below, the effective Lagrangian for the L.L.L. can be fruitfully used to discuss the so called edge-states in a finite sample of the quantum Hall fluid. The 1+1 dimensional action for the fermions on the boundary of such a sample can be extracted from the 2+1 dimensional effective Lagrangian for the L.L.L.. The excitations governed by this boundary action are precisely the edge excitations of the sample, corresponding to the experimentally observed surface magnetoplasmons (Dahm *et al.* 1985).

The importance of the edge states for the observability of the Quantum Hall effect was first pointed out by Halperin (Halperin 1982). More recently, Wen (Wen 1990), Stone (Stone 1991a,b) and Fröhlich and Kerler (Fröhlich 1991) have delineated the role of these edge waves in maintaining the electromagnetic $U(1)$ invariance of the system of planar electrons confined to a droplet and placed in a strong magnetic field orthogonal to the plane. They noted that when perturbative electromagnetic potentials are coupled to the electrons in the droplet, the fermionic degrees of freedom may be integrated out to obtain an effective action defined over the domain of the droplet. This effective action generically contains an Abelian Chern-Simons term with its coefficients given by the quantized Hall conductance. Further it is well known that a Chern-Simons term defined on a compact space is gauge non-invariant, the non-invariance manifesting itself through a surface term.

To restore gauge invariance, Wen (Wen 1990) postulated a boundary action, expressed as a bilinear in the perturbing potential, the coefficient of the bilinear being the current-current correlator of the boundary current. If one requires that the gauge-variation of this boundary action should cancel against that of the Chern-Simons term, one gets conditions on the correlator. This condition enables one to show that the low energy excitations of the boundary Hamiltonian are massless. Further, the components of the currents in the momentum space are the creation operators for these massless excitations from the vacuum state.

The light-cone components of these currents satisfy the chiral $U(1)$ Kac-Moody algebra. Wen gives a field theory whose currents satisfy this algebra. It is a theory of chiral fermions living on the boundary and interacting with the perturbative electromagnetic fields on the boundary in a fashion dictated by minimal coupling. But since chiral fermions in $1+1$ dimensions are equivalent to chiral bosons, a bosonic realization can also be given. Such a bosonic construction has been given by Stone (Stone 1991a).

Iso, Karabali and Sakita (Iso *et al.* 1992a) have studied the 2+1 dimensional non-relativistic fermionic theory and have bosonized the theory from the onset. In the fermionic theory, the strong magnetic field projects the electrons to the lowest Landau level (L.L.L.). In the bosonic language this L.L.L. condition leads to a classical configuration in the form of a droplet whose shape is determined by the confining potential. Namely, from $A_0(x,y) = \mu$ we obtain a curve $y = y(x)$ which is the boundary of the droplet (Iso *et al.* 1992a). The bosonic fluctuations around this classical configuration are precisely the chiral bosonic edge waves. The vibration of the surface just means the excitation of electrons from just below the Fermi surface to just above it. Since the energy levels here are quasi-continuous, the excitations are gapless. The magnetic field causes the electrons to rotate in a given direction and thus the edge waves are chiral. This present work can be taken to be an analysis of the same problem in the presence of additional perturbative electromagnetic fields.

In previous works on the edge waves of the Quantum Hall droplet, the boundary degree of freedom appears as a postulated construct designed to maintain gauge invariance. Our purpose here is to demystify the origin of the edge waves. We start from the original fermionic degrees of freedom and extract those that describe the edge waves. On doing this we see that the bulk of the droplet is naturally described by an effective Lagrangian involving the perturbing potential. The electromagnetic interaction of the edge waves is also obtained and we comment on the $U(1)$ gauge-invariance of this effective description.

Let us consider a system of planar (2+1 dimensional) electrons in a magnetic field B set up orthogonal to the plane. This magnetic field creates Landau levels on the plane. Each Landau level has a degeneracy given by $B/2\pi$ per unit area. The gap between Landau levels is given by $\omega = B/m$ where m is the mass of an electron. Working with the single-particle Hamiltonian

$$h_0 = \frac{1}{2m}(\vec{p} - \vec{A})^2, \tag{3.3}$$

where $\vec{A} = B(0, x)$ in the Landau gauge, we see that the wave function for the nth Landau level is given by

$$\langle \vec{x} | n, X \rangle = \sqrt{B/2\pi}(B/\pi)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{iBXy} e^{-\frac{B}{2}(x-X)^2} H_n(\sqrt{B}(x - X)) \tag{3.4}$$

where X, the centre of the classical cyclotron motion is arbitrary and is a measure of the degeneracy. The corresponding eigen-energy is $E_n = (n + \frac{1}{2})\omega$ which contains no reference to X. So given n, the index of the Landau level, the electron can be anywhere on the plane, depending on the value of X chosen. However, if an uniform electric field is also turned on, say in the x-direction, where $\vec{E} = (E, 0)$, the energy eigenvalues are

$$E_n(X) = (n + \frac{1}{2})\omega + EX - E^2/2m\omega^2. \tag{3.5}$$

The corresponding eigenfunction is

$$\langle \vec{x} | n, X \rangle = \sqrt{B/2\pi}(B/\pi)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{iBXy} e^{-\frac{B}{2}(x-X-X_0)^2} H_n(\sqrt{B}(x - X - X_0)), \tag{3.6}$$

where $X_0 \equiv -E/m\omega^2$. So the degeneracy in X is lifted and the electron will drift to large negative values of X to minimize the energy. We can use this external electrostatic potential $A_0 = Ex$ to define a droplet of electrons. We fill all the single-particle states with $n = 0$ and $X \leq 0$. Physically, these electrons will exist in the negative half-plane. We call this a droplet and identify the y-axis as the edge of the droplet.

In a second-quantized description of this system, we choose as the ground state, the state in the Fock-space with all the $X \leq 0$ for $n = 0$ filled. So, the Fermi-surface in this case coincides with the physical edge of the droplet. The droplet is incompressible as all the electronic states in it are filled. The only way to excite it perturbatively is to excite electrons with X small and negative to small and positive values of X. With reference to the ground state defined earlier, this corresponds to producing neutral electron-hole pairs. This leads to a deformation of the linear profile of the edge . These are the edge excitations described in the literature. Due to the magnetic field, they are chiral in nature, as has been mentioned earlier.

In this work, we wish to extract the effective Lagrangian for these excitations and the nature of their electromagnetic interactions from the original microscopic Lagrangian of the planar fermions. Further, we shall investigate the role of the edge excitations in preserving the original electromagnetic $U(1)$ gauge invariance of the system. For this purpose we derive the effective Lagrangian for the electrons in the L.L.L. , interacting with perturbative external electromagnetic potentials, whose typical frequency is much less than the cyclotron energy and the inverse of the magnetic length. Thus we keep terms only up to the quadratic in the electromagnetic potentials and consistently drop the higher spatial and time derivatives of these potentials.

3.2 Computing the effective action for the lowest Landau level

The kinematics of electrons in the L.L.L. has already been discussed by Girvin and Jach (Girvin & Jach 1983). However, to make the discussion reasonably self-contained, we introduce our own notation in the following.

The Lagrangian for planar electrons in a magnetic field normal to the plane is

$$\int d\vec{x}\ \psi^\dagger(\vec{x},t)\Big(i\partial_t - h_0\Big)\psi(\vec{x},t) \tag{3.7}$$

The corresponding action is given by

$$-S_0 = \langle\psi|\hat{p}_t + h_0|\psi\rangle, \tag{3.8}$$

where $\langle t|\hat{p}_t = -i\partial_t\langle t|$ and $\psi(\vec{x},t) \equiv \langle\vec{x},t|\psi\rangle$ is the Schrödinger wave field for electrons. The single-particle Hamiltonian, h_0, is

$$h_0 = \frac{1}{2m}\vec{\Pi}^2.$$

$\vec{\Pi} \equiv \vec{p} - \vec{A}$, and $\vec{A} \equiv \frac{B}{2}(y,-x)$ is the vector potential for the magnetic field in the symmetric gauge. B is of dimension $(\mathrm{mass})^2$. So there are two parameters with mass dimension, $\omega = B/m$ and $\sqrt{B}$, both of which are much greater than the typical frequency of the external electromagnetic fields. Define $\hat{\pi} \equiv \frac{1}{\sqrt{2B}}(\hat{\Pi}^x - i\hat{\Pi}^y)$ and $\hat{\pi}^\dagger \equiv \frac{1}{\sqrt{2B}}(\hat{\Pi}^x + i\hat{\Pi}^y)$. So, $\hat{\pi}$ and $\hat{\pi}^\dagger$ are dimensionless. Their commutator is

$$[\hat{\pi},\hat{\pi}^\dagger] = 1$$

Dropping the zero-point energy, the Hamiltonian is written as

$$h_0 = \omega\hat{\pi}^\dagger\hat{\pi} \tag{3.9}$$

Further, we define the guiding centre coordinate (Kubo *et al.* 1965) operators

$$\hat{X} \equiv \hat{x} - \frac{1}{B}\hat{\Pi}^y \qquad \text{and} \qquad \hat{Y} \equiv \hat{y} + \frac{1}{B}\hat{\Pi}^x$$

and their holomorphic form

$$\hat{a} \equiv \sqrt{B/2}(\hat{X} + i\hat{Y}) \qquad \text{and} \qquad \hat{a}^\dagger \equiv \sqrt{B/2}(\hat{X} - i\hat{Y})$$

The commutators are

$$[\hat{X}, \hat{Y}] = \frac{i}{B} \qquad \text{and} \qquad [\hat{a}, \hat{a}^\dagger] = 1 \tag{3.10}$$

$\hat{a}$ and $\hat{a}^\dagger$ are dimensionless. Further,

$$[\hat{a}, \hat{\pi}] = [\hat{a}, \hat{\pi}^\dagger] = [\hat{a}^\dagger, \hat{\pi}] = [\hat{a}^\dagger, \hat{\pi}^\dagger] = 0 \tag{3.11}$$

Thus we have two sets of independent raising and lowering operators. We choose to expand $|\psi\rangle$ in the $\{|n, \zeta\rangle\}$ basis, where $|n, \zeta\rangle \equiv |n\rangle \otimes |\zeta\rangle$; i.e

$$\hat{\pi}^\dagger \hat{\pi}|n\rangle = n|n\rangle \qquad \text{and} \qquad \hat{a}|\zeta\rangle = \zeta|\zeta\rangle \tag{3.12}$$

namely the coherent state. Let

$$|\psi\rangle = \sum_{n=0}^{\infty} |\psi_n\rangle \qquad \text{such that} \qquad \langle m, \bar{\zeta}, t|\psi\rangle = \delta_{m,n}\hat{\psi}_n(\bar{\zeta}, t) \tag{3.13}$$

Thus, the action is written as

$$-S_0 = \sum_{n=0}^{\infty} \langle \psi_n|\hat{p}_t + \omega n|\psi_n\rangle \tag{3.14}$$

In addition to the coherent state representation, let us introduce the x-representation and the y-representation given by

$$(\text{x} - \text{representation}) : \hat{X}|x\rangle = x|x\rangle, \langle x|x'\rangle = \delta(x - x')$$

$$(\text{y}-\text{representation}) : \hat{Y}|y\rangle = y|y\rangle, \langle y|y'\rangle = \delta(y-y')$$

The inner product between these representations and the coherent state representation may be computed to be

$$\langle x|\zeta\rangle = (\frac{B}{\pi})^{\frac{1}{4}} e^{-\frac{1}{2}(x^2B+z^2)+\sqrt{2B}x\zeta}$$

$$\langle y|\zeta\rangle = (\frac{B}{\pi})^{\frac{1}{4}} e^{-\frac{1}{2}(x^2B-z^2)-\sqrt{2B}iy\zeta}$$

Further,

$$\langle y|x\rangle = \sqrt{\frac{B}{2\pi}} e^{-ixyB}$$

In the symmetric gauge that we have adopted,

$$\langle\vec{x}|\hat{\pi} = -i(\partial_z + \frac{1}{2}\bar{z})\langle\vec{x}|$$

This means that the L.L.L. Schrödinger wave field has to satisfy the following condition

$$\langle\vec{x}|\hat{\pi}|\psi_0(t)\rangle = 0 \longrightarrow (\partial_z + \frac{1}{2}\bar{z})\psi(\vec{x},t) = 0$$

The L.L.L. field which satisfies this condition is given by

$$\psi(\vec{x},t) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \sum_l \frac{\bar{z}^l}{\sqrt{l!}} C_l(t)$$

where

$$\{C_m, C_n\} = \delta_{m,n}.$$

In the absence of interactions, it is easily seen that the time independent unitary transformation

$$C_n \rightarrow C_n' = u_{n,m}C_m$$

where

$$u = e^{-i\xi},$$

ξ being a hermitean operator, is a symmetry of the aciton. This is a global W_∞ symmetry.

We now introduce electromagnetic perturbations through minimal coupling. The action is

$$-S = \langle\psi|\hat{p}_t + h_0 + h_{int}|\psi\rangle \tag{3.15}$$

where

$$h_{int} = \frac{1}{2m}[-\vec{\Pi}.\vec{\mathcal{A}} - \vec{\mathcal{A}}.\vec{\Pi} + \vec{\mathcal{A}}^2] + A_0.$$

Let

$$A \equiv \frac{1}{\sqrt{2B}}(\mathcal{A}^x + i\mathcal{A}^y) \qquad \text{and} \qquad A^\dagger \equiv \frac{1}{\sqrt{2B}}(\mathcal{A}^x - i\mathcal{A}^y)$$

which are functions of

$$\hat{z} \equiv \sqrt{\frac{B}{2}}(\hat{x} + i\hat{y}) = \hat{a} - i\hat{\pi}^\dagger \qquad \text{and} \qquad \hat{\bar{z}} \equiv \sqrt{\frac{B}{2}}(\hat{x} - i\hat{y}) = \hat{a}^\dagger + i\hat{\pi}$$

So, A, $A^\dagger$, z, $\bar{z}$ are all dimensionless. We note,

$$[\hat{\pi}, A(\hat{z}, \hat{\bar{z}})] = -i\partial_z A(\hat{z}, \hat{\bar{z}}) \qquad \text{and} \qquad [\bar{A}(\hat{z}, \hat{\bar{z}}), \hat{\pi}^\dagger] = i\partial_{\bar{z}} \bar{A}(\hat{z}, \hat{\bar{z}})$$

So, the interaction hamiltonian is rewritten as

$$h_{\text{int}} = - - \frac{\omega}{2}[A\hat{\pi} + \hat{\pi}^\dagger \bar{A} + \hat{\pi}A + \bar{A}\hat{\pi}^\dagger] + \omega A\bar{A} + A_0 \tag{3.16}$$

Now, an arbitrary function f of $\hat{z}$, $\hat{\bar{z}}$ may be written as

$$f(\hat{z}, \hat{\bar{z}}) = \sum_{n,m} \frac{1}{m!n!}(-i)^n(i)^m(\hat{\pi}^\dagger)^n(\hat{\pi})^m \ddagger \partial_z{}^n \partial_{\bar{z}}{}^m f(z, \bar{z})\big|_{z=\hat{a}^\dagger, \bar{z}=\hat{a}} \ddagger$$

where ‡ ‡ means antinormal ordering of $\hat{a}$ and $\hat{a}^\dagger$. That is, $\hat{a}$ is always placed to the left of $\hat{a}^\dagger$. This ordering arises naturally from the normal ordering adopted for $\hat{\pi}$ and $\hat{\pi}^\dagger$. Utilizing this, we write

$$h_{\rm int} = m_{00} + m_{10}\hat{\pi}^\dagger + m_{01}\hat{\pi} + m_{11}\hat{\pi}^\dagger\hat{\pi} + m_{20}\hat{\pi}^{\dagger 2} + m_{02}\hat{\pi}^2 + \ldots \qquad (3.17)$$

$$\begin{aligned}
m_{00} &= \frac{i\omega}{2}(\partial_z A - \partial_{\bar{z}}\bar{A}) + \omega A\bar{A} + A_0 + \ldots \\
m_{10} &= -\omega\bar{A} + \frac{\omega}{2}\partial_z(\partial_z A - \partial_{\bar{z}}\bar{A}) - i\omega\partial_z(A\bar{A}) - i\partial_z A_0 + \ldots \\
m_{01} &= -\omega A + \frac{\omega}{2}\partial_{\bar{z}}(\partial_z A - \partial_{\bar{z}}\bar{A}) + i\omega\partial_{\bar{z}}(A\bar{A}) + i\partial_{\bar{z}} A_0 + \ldots \\
m_{11} &= i\omega(\partial_z A - \partial_{\bar{z}}\bar{A}) + \ldots \\
m_{20} &= i\omega\partial_z\bar{A} + \ldots \\
m_{02} &= -i\omega\partial_{\bar{z}} A + \ldots
\end{aligned}$$

The antinormal ordering of $\hat{a}$ and $\hat{a}^\dagger$ is tacit in the above. From equation 3.15,

$$\begin{aligned}
-S &= \langle\psi_0|\hat{p}_t|\psi_0\rangle + \langle\psi_0|h_{\rm int}|\psi_0\rangle \\
&+ \sum_{n\neq 0}[\langle\psi_0|h_{\rm int}|\psi_n\rangle + \langle\psi_n|h_{\rm int}|\psi_0\rangle] + \sum_{n,n'\neq 0}\langle\psi_n|\hat{p}_t + h_{\rm int} + h_0|\psi_{n'}\rangle
\end{aligned}$$

Integrating the modes with $n \neq 0$ out, we get

$$-S_{\rm eff} = \langle\psi_0|\hat{p}_t|\psi_0\rangle + \langle\psi_0|h_{\rm int}|\psi_0\rangle - \langle\psi_0|h_{\rm int}\text{``}\frac{1}{\hat{p}_t + h_0 + h_{\rm int}}\text{''}h_{\rm int}|\psi_0\rangle \quad (3.18)$$

"$\frac{1}{\hat{p}_t + h_0 + h_{\rm int}}$" is just a notation. It means that all the intermediate states exclude $n = 0$. So,

$$-S_{\rm eff} = \langle\psi_0|\hat{p}_t|\psi_0\rangle + H_{\rm eff}{}^{(0)} + H_{\rm eff}{}^{(1)} + H_{\rm eff}{}^{(2)} + \langle\psi|h_{\rm int}\frac{1}{h_0}\hat{p}_t\frac{1}{h_0}h_{\rm int}|\psi\rangle + \ldots \qquad (3.19)$$

In this expansion and in all subsequent discussions, the occurrence of $\frac{1}{h_0}$ is automatically taken to signify that $n = 0$ is omitted from the intermediate states. We have dropped the subscript "0" from $|\psi\rangle$.

$$
\begin{aligned}
H^{(0)}_{\text{eff}} &\equiv \langle\psi|h_{\text{int}}|\psi\rangle \\
H^{(1)}_{\text{eff}} &\equiv -\langle\psi|h_{\text{int}}\frac{1}{h_0}h_{\text{int}}|\psi\rangle \\
H^{(2)}_{\text{eff}} &\equiv \langle\psi|h_{\text{int}}\frac{1}{h_0}h_{\text{int}}\frac{1}{h_0}h_{\text{int}}|\psi\rangle
\end{aligned}
$$

We choose the external perturbing fields to be slowly varying in space and time. Thus, we drop all derivatives higher than the leading order. This means effectively that we may truncate the sum over all $n > 0$ to a sum over the first few terms. Further, since $|\frac{A^\mu}{\sqrt{B}}| \ll 1$, we drop terms arising out of the higher iterations of "$\frac{1}{\bar{p}_t + h_0 + h_{\text{int}}}$." In computing H_{eff} we come across expressions like

$$\langle\psi| \ddagger A(\hat{a}, \hat{a}^\dagger) \ddagger |\psi\rangle$$

$$\langle\psi| \ddagger A(\hat{a}, \hat{a}^\dagger) \ddagger \ \ddagger B(\hat{a}, \hat{a}^\dagger) \ddagger |\psi\rangle$$

$$\langle\psi| \ddagger A(\hat{a}, \hat{a}^\dagger) \ddagger \ \ddagger B(\hat{a}, \hat{a}^\dagger) \ddagger \ \ddagger C(\hat{a}, \hat{a}^\dagger) \ddagger |\psi\rangle.$$

These are written as

$$\langle\psi| \ddagger A(\hat{a}, \hat{a}^\dagger) \ddagger |\psi\rangle = \int d^2z\ e^{-|z|^2}\bar{\psi}(z,t)\ A(z,\bar{z},t)\psi(\bar{z},t)$$

$$
\begin{aligned}
\langle\psi| \ddagger A(\hat{a}, \hat{a}^\dagger) \ddagger \ \ddagger B(\hat{a}, \hat{a}^\dagger) \ddagger |\psi\rangle &= \int d^2z\ ve^{-|z|^2}\bar{\psi}(z,t)\Big[A(z,\bar{z},t)B(z,\bar{z},t) \\
&\quad -\partial_{\bar{z}}A(z,\bar{z},t)\partial_z B(z,\bar{z},t)\ldots\Big]\psi(\bar{z},t)
\end{aligned}
$$

$$\begin{aligned}\langle\psi|\ddagger A(\hat{a},\hat{a}^{\dagger})\ddagger\ \ddagger B(\hat{a},\hat{a}^{\dagger})\ddagger\ \ddagger C(\hat{a},\hat{a}^{\dagger})\ddagger|\psi\rangle &= \int d^2z\ e^{-|z|^2}\bar{\psi}(z,t)\Big[ABC \\ &- A(\partial_{\bar{z}}B)(\partial_z C) \\ &- (\partial_{\bar{z}}A)\partial_z(BC)+\ldots\Big]\psi(\bar{z},t)\end{aligned}$$

where $d^2z \equiv d(\text{Re}z)d(\text{Im}z)/\pi$. We write

$$A_0(\vec{x},t) = Ex + \mathcal{A}_0(\vec{x},t) \tag{3.20}$$

where Ex is a fixed background. So

$$\begin{aligned}-\langle\psi|h_{\text{int}}|\psi\rangle &= -\langle\psi|m_{00}|\psi\rangle \\ &= -\int d^2z\ \rho(z,\bar{z},t)m_{00}(z,\bar{z},t) \\ &= -\int d^2z\ \rho(z,\bar{z},t)[\frac{i\omega}{2}\Omega + \omega A\bar{A} + \mathcal{A}_0]\end{aligned} \tag{3.21}$$

Again,

$$\begin{aligned}\langle\psi|h_{\text{int}}|\psi\rangle &= \frac{1}{\omega}\langle\psi|m_{01}.m_{10} + m_{02}.m_{20}|\psi\rangle \\ &= \int d^2z\ \rho(z,\bar{z},t)[\omega A\bar{A} + i(A\partial_z a_0 - \bar{A}\partial_{\bar{z}}a_0) \\ &= \frac{E}{\sqrt{2B}}\{i(A-\bar{A}) + (\partial_z + \partial_{\bar{z}})A\bar{A}\}]\end{aligned} \tag{3.22}$$

Further,

$$\begin{aligned}\langle\psi|h_{\text{int}}\frac{1}{h_0}h_{\text{int}}\frac{1}{h_0}h_{\text{int}}|\psi\rangle &= -\frac{1}{\omega^2}\langle\psi|m_{01}.m_{00}.m_{10} + m_{01}.m_{11}.m_{10} + m_{01}.m_{01}.m_{20} \\ &+ m_{02}.m_{10}.m_{10} + \frac{1}{2}m_{02}.m_{00}.m_{20} + m_{02}.m_{11}.m_{20}|\psi\rangle \\ &= -\frac{E}{\sqrt{2B}}\int d^2z\ \rho(z,\bar{z},t)[(z+\bar{z})A\bar{A} + \frac{3}{2}(\bar{A}-A)\Omega\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}(z+\bar{z})(\bar{A}\partial_{\bar{z}} - A\partial_z)\Omega + A\partial_z\bar{A} \\
&+ \quad \bar{A}\partial_{\bar{z}}A - \frac{1}{2}(z+\bar{z})(\partial_{\bar{z}}A)(\partial_z\bar{A})]
\end{aligned} \tag{3.23}
$$

Also,

$$
\begin{aligned}
i\langle\psi|h_{\rm int}\frac{1}{{h_0}^2}\partial_t h_{\rm int}|\psi\rangle &= \frac{i}{\omega^2}\langle\psi|m_{01}.\partial_t m_{10} + \frac{1}{2}m_{02}.\partial_t m_{20}|\psi\rangle \\
&= \frac{i}{\omega^2}\int d^2z\ \rho(z,\bar{z},t)[m_{01}.\partial_t m_{10} - \partial_{\bar{z}}m_{01}\partial_z\partial_t m_{10} \\
&+ \frac{1}{2}m_{02}\partial_t m_{20} - \frac{1}{2}\partial_{\bar{z}}m_{02}\partial_z\partial_t m_{20} + \ldots] \\
&= \int d^2z\ \rho(z,\bar{z},t)(iA\partial_t\bar{A} + \ldots)
\end{aligned} \tag{3.24}
$$

Again,

$$
\begin{aligned}
-\langle\psi|h_{\rm int}\frac{1}{{h_0}^2}h_{\rm int}p_t|\psi\rangle &\rightarrow i\langle\psi|h_{\rm int}\frac{1}{{h_0}^2}h_{\rm int}\partial_t|\psi\rangle \\
&= \frac{i}{\omega^2}\langle\psi|[m_{01}.m_{10} + \frac{1}{2}m_{02}.m_{20}]\partial_t|\psi\rangle \\
&= \int d^2z\ e^{-|z|^2}\bar{\psi}(z,t)[A\bar{A} - \frac{1}{2}(\partial_{\bar{z}}A)(\partial_z\bar{A}) - \frac{1}{2}A\partial_z\Omega \\
&+ \frac{1}{2}\bar{A}\partial_{\bar{z}}\Omega](i\partial_t)\psi(\bar{z},t)
\end{aligned} \tag{3.25}
$$

Putting everything together, we get,

$$
\begin{aligned}
L_{\rm eff} &= \int d^2z\ e^{-|z|^2}\bar{\psi}(z,t)[1 + A\bar{A} - \frac{1}{2}(\partial_{\bar{z}}A(\partial_z\bar{A}) - \frac{1}{2}A\partial_z\Omega + \frac{1}{2}\bar{A}\partial_{\bar{z}}\Omega](i\partial_t)\psi(\bar{z},t) \\
&= -\int d^2z\ \rho(z,\bar{z},t)[a_0 + \frac{i\omega}{2}\Omega - i(A\partial_z a_0 - \bar{A}\partial_{\bar{z}}a_0) - iA\partial_t\bar{A}] \\
&+ \frac{E}{\sqrt{2B}}\int d^2z\ \rho(z,\bar{z},t)\{(z+\bar{z}) - i(A-\bar{A}) - (\partial_z + \partial_{\bar{z}})A\bar{A} + (z+\bar{z})A\bar{A} \\
&- \frac{1}{2}(z+\bar{z})(\partial_{\bar{z}}A)(\partial_z\bar{A}) + \frac{3}{2}(\bar{A}-A)\Omega \\
&+ \frac{1}{2}(z+\bar{z})(\bar{A}\partial_{\bar{z}} - A\partial_z)\Omega + A\partial_z\bar{A} + \bar{A}\partial_{\bar{z}}A\}
\end{aligned} \tag{3.26}
$$

This effective Lagrangian is not in the canonical fermionic form. To reduce it to a standard form, we transform ψ and $\bar{\psi}$ appropriately. Let

$$\begin{aligned}\psi(\bar{z},t) &\rightarrow [1-\frac{1}{2}\ddagger(A\bar{A}-\frac{1}{2}(\partial_{\bar{z}}A)(\partial_z\bar{A})-\frac{1}{2}(A\partial_z-\bar{A}\partial_{\bar{z}})\Omega \\ &+ \frac{iE}{\omega\sqrt{2B}}(A-\bar{A}))(\partial_{\bar{z}},\bar{z},t)\ddagger]\psi(\bar{z},t) \end{aligned} \tag{3.27}$$

$$\begin{aligned}\bar{\psi}(z,t) &\rightarrow [1-\frac{1}{2}\ddagger(A\bar{A}-\frac{1}{2}(\partial_{\bar{z}}A)(\partial_z\bar{A})-\frac{1}{2}(A\partial_z-\bar{A}\partial_{\bar{z}})\Omega \\ &+ \frac{iE}{\omega\sqrt{2B}}(A-\bar{A}))(\partial_z,z,t)\ddagger]\bar{\psi}(z,t) \end{aligned} \tag{3.28}$$

The Jacobian of this transformation, which is obviously not unity, depends on the perturbative gauge fields and is gauge non-invariant. This non-invariance, however cancels against the gauge non-invariance of the determinant that arises from integrating the higher L.L. out, as may be seen explicitly. One can express the Jacobian of this transformation as the partition function of a fermionic degree of freedom. This, when combined with the partition function for the higher L.L., yields a gauge invariant quantity. Thus we do not have to worry about the gauge dependence arising out of the Jacobian as well as the higher L.L. in subsequent discussions of gauge invariance. Under these transformations, the effective Lagrangian reduces to

$$\begin{aligned}L_{\text{eff}} &= \int d^2z\; e^{-|z|^2}\bar{\psi}(z,t)i\partial_t\psi(\bar{z},t) \\ &- \int d^2z\;\rho(z,\bar{z},t)[a_0+\frac{i\omega}{2}\Omega-i(A\partial_z a_0-\bar{A}\partial_{\bar{z}}a_0)-\frac{i}{2}(A\partial_t\bar{A}-\partial_t A\bar{A})] \\ &- \frac{E}{\sqrt{2B}}\int d^2z\;\rho(z,\bar{z},t)[(z+\bar{z})-i(A-\bar{A})-\frac{1}{2}(\partial_z+\partial_{\bar{z}})A\bar{A}+(\bar{A}-A)\Omega \\ &+ A\partial_z\bar{A}+\bar{A}\partial_{\bar{z}}A]\end{aligned} \tag{3.29}$$

This effective action indicates how the fermionic field representing the L.L.L. couples to the external electromagnetic potentials.

The microscopic action for the system of planar electrons in external electromagnetic fields is invariant under local gauge transformations. A gauge transformation of the electromagnetic potentials can be compensated for by suitable local redefinitions of the phase of the fermionic fields. In the effective action, however, the only remaining fermionic field is that describing the L.L.L.. Thus we may expect that the transformation of this fermionic field required to negate the effect of gauge transformation is somewhat more complicated than mere local redefinition of phase. The required transformation, which is unitary, may be explicitly obtained. The effective action is given by,

$$\begin{aligned} -S_{\text{eff}} &= \langle\psi|p_t + h_{\text{int}}|\psi\rangle - \langle\psi|h_{\text{int}}\text{“}\frac{1}{h_0}\text{”}h_{\text{int}}|\psi\rangle \\ &+ \langle\psi|h_{\text{int}}\text{“}\frac{1}{h_0}\text{”}(p_t + h_{\text{int}})\text{“}\frac{1}{h_0}\text{”}h_{\text{int}}|\psi\rangle \end{aligned} \tag{3.30}$$

Under a gauge transformation,

$$\delta h_{\text{int}} = i[p_t + h_0 + h_{\text{int}}, \Lambda]$$

where $i[p_t, \Lambda] = \partial_t \Lambda$. So, the effective action transforms by,

$$\begin{aligned} -\delta S_{\text{eff}} &= \langle\psi|\delta v|\psi\rangle - \langle\psi|\delta v\frac{1}{h_0}v|\psi\rangle - \langle\psi|v\frac{1}{h_0}\delta v|\psi\rangle \\ &+ \langle\psi|\delta v\frac{1}{h_0}(p_t + v)\frac{1}{h_0}v|\psi\rangle + \langle\psi|v\frac{1}{h_0}(p_t + v)\frac{1}{h_0}\delta v|\psi\rangle \\ &+ \langle\psi|v\frac{1}{h_0}\delta v\frac{1}{h_0}v|\psi\rangle \end{aligned} \tag{3.31}$$

Here $v \equiv h_{\text{int}}$. A straightforward set of manipulations yields

$$-\delta S_{\text{eff}} = i\langle\psi|(p_t + v)|0\rangle\langle 0|\Lambda|\psi\rangle - i\langle\psi|\Lambda|0\rangle\langle 0|(p_t + v)|\psi\rangle$$

$$
\begin{aligned}
&+\quad i\langle\psi|\Lambda|0\rangle\langle 0|v\frac{1}{h_0}v|\psi\rangle - i\langle\psi|v\frac{1}{h_0}v|0\rangle\langle 0|\Lambda|\psi\rangle \\
&+\quad i\langle\psi|v\frac{1}{h_0}\Lambda|0\rangle\langle 0|(p_t+v)|\psi\rangle - i\langle\psi|(p_t+v)|0\rangle\langle 0|\Lambda\frac{1}{h_0}v|\psi\rangle \\
&+\quad \ldots \qquad (3.32)
\end{aligned}
$$

To cancel out the Λ dependence in the above, it is quite easy to see that we require

$$
\delta|\psi\rangle \quad = \quad -i|0\rangle\langle 0|\Lambda|\psi\rangle + i|0\rangle\langle 0|\Lambda\frac{1}{h_0}v|\psi\rangle \qquad (3.33)
$$

$$
\delta\langle\psi| \quad = \quad i\langle\psi|\Lambda|0\rangle\langle 0| - i\langle\psi|v\frac{1}{h_0}\Lambda|0\rangle\langle 0| \qquad (3.34)
$$

To write these transformations out explicitly in the chosen representation, we use

$$
\Lambda \quad = \quad l_{00} + l_{01}\pi + l_{10}\pi^\dagger + l_{11}\pi^\dagger\pi + l_{02}\pi^2 + l_{20}{\pi^\dagger}^2 + \ldots
$$

where

$$
\begin{aligned}
l_{00} &= \Lambda \\
l_{01} &= i\partial_{\bar{z}}\Lambda \\
l_{10} &= -i\partial_z\Lambda \\
l_{11} &= \partial_z\partial_{\bar{z}}\Lambda \\
l_{02} &= -\frac{1}{2}{\partial_{\bar{z}}}^2\Lambda \\
l_{20} &= -\frac{1}{2}{\partial_z}^2\Lambda
\end{aligned}
$$

where antinormal ordering of a and $a^\dagger$ is implicit. Thus, after a straightforward calculation using the same set of techniques as we did for the calculation of the effective Lagrangian, we obtain

$$\delta\psi(\bar{z},t) = \ddagger[-i\Lambda + (\partial_{\bar{z}}\Lambda)\bar{A} + \ldots](\partial_{\bar{z}},\bar{z},t)\ddagger\psi(\bar{z},t) \quad (3.35)$$

$$\delta\bar{\psi}(z,t) = \ddagger[i\Lambda + (\partial_{z}\Lambda)A + \ldots](\partial_{z},z,t)\ddagger\bar{\psi}(z,t) \quad (3.36)$$

However, to obtain a conventional fermionic Lagrangian, we have transformed the fermionic fields as in equations 3.27 and 3.28. For these redefined fields, the corresponding transformation that is required is directly obtained from equations 3.35 and 3.36. This is,

$$\delta\psi(\bar{z},t) = \ddagger[-i\Lambda + \frac{1}{2}(\partial_{\bar{z}}\Lambda)\bar{A} - \frac{1}{2}(\partial_{z}\Lambda)A\ldots](\partial_{\bar{z}},\bar{z},t)\ddagger\psi(\bar{z},t) \quad (3.37)$$

$$\delta\bar{\psi}(z,t) = \ddagger[i\Lambda - \frac{1}{2}(\partial_{\bar{z}}\Lambda)\bar{A} + \frac{1}{2}(\partial_{z}\Lambda)A\ldots](\partial_{z},z,t)\ddagger\bar{\psi}(z,t) \quad (3.38)$$

This transformation, which is unitary, is a particular class of W_∞ transformations. As discussed further on, W_∞ transformations are the most general unitary transformations that preserve the particle number and the L.L.L. condition (Iso *et al.* 1992b). It is easily verified the the above transformation of the fermionic field is adequate for cancelling out the Λ dependence arising out of the gauge transformation of equation 3.29.

3.3 Constructing the droplet

In the previous sections, we have obtained the effective Lagrangian for the fermion field representing the L.L.L. and have demonstrated that this Lagrangian is invariant under gauge transformations.

We can use this effective Lagrangian to discuss the electromagnetic interaction of a quantum Hall fluid when it is confined to a droplet. In what follows, we have used a constant electric field in the x-direction to provide a boundary for the quantum Hall fluid by confining the electrons to the negative half plane. With this droplet that we construct, we can address questions regarding electromagnetic gauge invariance in the droplet. It will be seen that in the context of the droplet, the effective Lagrangian splits naturally into two pieces, one describing the electronic states in the interior of the droplet and the other describing the dynamics of the electrons on the 1+1 dimensional boundary of the droplet. We shall see that individually, neither of these two pieces is gauge invariant but that, when considered together lead to a gauge invariant description of the droplet.

When bosonized, the action governing the dynamics of the fermions on the boundary of the droplet describes chiral bosons which are the fluctuations of the boundary of the incompressible droplet under electromagnetic perturbations.

If we imagine, momentarily that the fluctuating potentials have been switched off, the effective Lagrangian from equation 3.29 yields an effective action

$$S_{\text{eff}}^{(0)} = \langle\psi|(i\partial_t - \frac{E}{\sqrt{2B}}(\hat{a} + \hat{a}^\dagger))|\psi\rangle \tag{3.39}$$

where $|\psi\rangle$ is a second quantized operator. But $(\hat{a} + \hat{a}^\dagger)/\sqrt{2B} = \hat{X}$. This implies that the dominant part of the effective Hamiltonian is

$$E\langle\psi|\hat{X}|\psi\rangle = E\int_{-\infty}^{\infty} dX\ \psi^\dagger(X)X\psi(X) \tag{3.40}$$

where $\psi(X) = \langle X|\psi\rangle$ and $\{|X\rangle\}$ is the basis of the coordinate representation.

We define the droplet to be such that all the single particle states up to the zero energy state are filled. So $X = 0$ is the Fermi surface. But, for a large magnetic field, $X \simeq x$, the real spatial coordinate. So, in physical space, the edge of the droplet is at $x = 0$. The ground state as defined above is

$$|G\rangle \equiv \prod_{X \leq 0} \psi^{\dagger}(X)|0\rangle \tag{3.41}$$

where $|0\rangle$ is the Fock vacuum. With respect to the ground state we redefine the fermion operators appropriately as particle and antiparticle (hole) operators.

Excitation of this droplet by means of the fluctuating potentials means the destruction of an electron within the Fermi sea and the creation of an electron outside of the Fermi sea. In terms of the state $|G\rangle$, this translates into the creation of a neutral particle-antiparticle excitation from the ground state.

Given that the perturbing potential is small and slowly varying in space-time, (this is the justification for the derivative expansion we have performed) we would expect only those electrons within some distance $\lambda \ll 1/\sqrt{B}$ to participate in the neutral excitations. In fact, an expansion in X about $X = 0$ yields, to leading order, an action for fermions interacting with A, $\bar{A}$, $\mathcal{A}_0$ on the boundary of the droplet ($X = 0$). The neutral particle-antiparticle excitations mentioned earlier are actually the neutral excitations around the filled Fermi sea for this boundary action.

To extract this boundary action from S_{eff}, we write the density operator as

$$\begin{aligned}\hat{\rho}(z,\bar{z},t) &= \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dX' \ \langle\psi|X,t\rangle\langle X',t|\psi\rangle\langle X|z\rangle\langle\bar{z}|X'\rangle e^{-|z|^2} \\ &= \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dX' \ \psi^{\dagger}(X,t)\psi(X',t)\langle X|z\rangle\langle\bar{z}|X'\rangle e^{-|z|^2}\end{aligned}$$

where $\{\psi^{\dagger}(X,t),\psi(X',t)\} = \delta(X-X')$. We note that if $X, X' \leq 0$, $\psi^{\dagger}, \psi$ exchange their roles with respect to $|G\rangle$. This implies that

$$\begin{aligned}\rho(z,\bar{z},t) &= \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dX' \ [: \psi^{\dagger}(X,t)\psi(X',t) : \\ &+ \Theta(-X)\delta(X-X')]\langle X|z\rangle\langle\bar{z}|X\rangle e^{-|z|^2}\end{aligned}$$

where : : indicates normal ordering with respect to $|G\rangle$. The term independent of the fermion fields is

$$\int_{-\infty}^{0} dX \ \langle X|z\rangle\langle\bar{z}|X\rangle e^{-|z|^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\sqrt{B}x} d\gamma \ \exp(-\gamma^2) \sim \Theta(-x) \qquad (3.42)$$

The last expression is valid in the large B limit. When this is inserted in equation 3.29, we get an effective action involving only $\mathcal{A}_{\mu}$, called the bulk action.

$$\begin{aligned}S_{\text{eff}}^{\text{bulk}} &\simeq -\frac{B}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \ \int_{-\infty}^{\infty} dy \ \Big[Ex + \mathcal{A}_0 - \frac{1}{2m}b + \frac{E}{B}\mathcal{A}^y \\ &+ \frac{1}{2B}\epsilon^{\mu\nu\rho}\mathcal{A}_{\mu}\partial_{\nu}\mathcal{A}_{\rho} - \frac{2E}{(2B)^2}\Big\{2\mathcal{A}^x\partial_y\mathcal{A}^y + x(\mathcal{A}^x\partial_y b - \mathcal{A}^y\partial_x b) - 3\mathcal{A}^y b\Big\}\Big] \\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dy \ \mathcal{A}^0(y,t)\mathcal{A}^y(y,t)\end{aligned}$$

where $b \equiv \partial_x \mathcal{A}^y - \partial_y \mathcal{A}^x$, $\mathcal{A}^\mu(y,t) \equiv \mathcal{A}^\mu(x=0,y,t)$. We see that $S_{\text{eff}}^{\text{bulk}}$ contains a Chern-Simons term. It exists only in the bulk of the droplet. Under gauge transformation, $\mathcal{A}^i \to \mathcal{A}^i - \partial_i\Lambda$ and $\mathcal{A}^0 \to \mathcal{A}^0 + \partial_t\Lambda$,

$$\delta S_{\text{eff}}^{\text{bulk}} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dy\ \Lambda(y,t)[\partial_y \mathcal{A}_0(y,t) + \frac{E}{B}\partial_y \mathcal{A}^y(y,t)] \tag{3.43}$$

So the gauge dependence is through a boundary term. The fermionic part of S_{eff} is obtained by inserting $:\rho(z,\bar{z},t):$ in place of $\rho(z,\bar{z},t)$ in equation 3.29. We expand the fermionic part of S_{eff} and keep only the low momentum ($X \sim 0$) fermions. Thus we get

$$\begin{aligned} S_{\text{bdry}} &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dY\ \psi^\dagger(Y,t)\Big[\{i\partial_t - \mathcal{A}_0(Y,t)\} \\ &+ \frac{E}{B}\{-i\partial_Y - \mathcal{A}^y(Y,t)\}\Big]\psi(Y,t). \end{aligned} \tag{3.44}$$

Here normal ordering with respect to the ground state is implicit. $\psi(Y,t) \equiv \langle Y|\psi\rangle$ is actually the Fourier transform of $\langle X|\psi\rangle$ used in equation 3.40, since $[\hat{X},\hat{Y}] = \frac{i}{B}$. We however continue to use the same symbol ψ. Thus

$$\int_{-\infty}^{\infty} dt\ \int d^2\zeta e^{-|\zeta|^2}\psi^\dagger(\zeta,t)i\partial_t\psi(\bar{\zeta},t) = \int_{-\infty}^{\infty} dt\ \int_{-\infty}^{\infty} dY\ \psi^\dagger(Y,t)i\partial_t\psi(Y,t). \tag{3.45}$$

Also,

$$\langle\psi|E\hat{X}|\psi\rangle = i\frac{E}{B}\psi^\dagger(Y,t)\partial_Y\psi(Y,t). \tag{3.46}$$

which we have used in deriving equation 3.45 The neutral excitations of electron–hole pairs are the neutral excitations of this boundary action since we have expressed the low momentum ($X \simeq 0$) part of the original normal ordered fermion action as this boundary action. The electrons in the high momentum eigen states are unaffected as long as the momenta of the perturbing potentials are $\ll \sqrt{B}$. Since we are interested in the low energy perturbations of the droplet, $S_{\rm eff}^{\rm bulk}$ is the net effect of the fermions inside of the droplet. $S_{\rm bdry}$ is classically gauge invariant. But quantum mechanically this theory is an anomalous gauge theory (Jackiw & Rajaraman 1985 ; Faddeev & Shatashvili 1986). This means that the quantized theory will not be gauge invariant. The gauge parameter dependence of the theory can be best extracted by writing it in the bosonic language.

By studying the bosonization of the L.L.L. fermions, the following result was obtained by Iso, Karabali and Sakita (Iso *et al.* 1992a). The Lagrangian is written in the form of (2.19) in terms of the L.L.L. fermion fields, namely

$$L_{\rm eff} = \int d\vec{x}\ \bar{\psi}(\vec{x},t) i\partial_t \psi(\vec{x},t) - \int d\vec{x}\ \bar{\psi}(\vec{x},t)\psi(\vec{x},t) f(\vec{x},t) \qquad (3.47)$$

where f is a function of $\vec{\mathcal{A}}$ and $\mathcal{A}_0$, as given in (2.19) and $\psi(\vec{x},t)$ satisfies the L.L.L. condition, $(\partial_z + \frac{1}{2}\bar{z})\psi(\vec{x},t) = 0$. In the bosonized form,these fields are replaced by bosonic Schrödinger fields. In the bosonized form, the Lagrangian is precisely given by equation 3.47, where ψ is replaced by a bosonic Schrödinger wave field which obeys the nonlinear L.L.L. condition, $(\partial_z + \frac{1}{2}\bar{z} - \int \frac{\bar{\psi}\psi(\vec{x}')}{z-z'})\psi(\vec{x},t) = 0$. This L.L.L. condition can be approximately solved in the droplet approximation. This solution is that the density $\bar{\psi}\psi(\vec{x})$ is equal to $B/2\pi$ inside of a certain region of space (droplet) and zero outside. The dynamical variable is then the boundary fluctuations of the uniform density around this "classical" droplet configuration. We denote these fluctuations in the following by $\phi(Y,t)$. The first term in (3.12) gives the first term in the free chiral bosonic Lagrangian (Iso *et al.* 1992a), (Floreanini &

Jackiw 1987), (Sonnenschein 1988), in terms of $\phi(Y,t)$. Working within this droplet approximation, we obtain

$$\begin{aligned} L_{\text{eff}} &= \frac{B^2}{8\pi}\int dY dY'\ \phi(Y,t)\ \epsilon(Y-Y')\dot{\phi}(Y',t) - \frac{B}{2\pi}\int dY[-\mathcal{A}_0 - \frac{E}{B}\mathcal{A}^y]\phi(Y) \\ &- \frac{BE}{2\pi}\int dY\ \frac{1}{2}\phi^2(Y) - \frac{B}{2\pi}\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy\ \Big[\mathcal{A}_0 - \frac{1}{2m}b + Ex + \frac{E}{B}\mathcal{A}^y \\ &+ \frac{1}{2B}\epsilon^{\mu\nu\rho}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\rho - \frac{2E}{(2B)^2}\Big\{2\mathcal{A}^x\partial_y\mathcal{A}^y + x(\mathcal{A}^x\partial_y b - \mathcal{A}^y\partial_x b) - 3\mathcal{A}^y b\Big\}\Big] \\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dy\ \mathcal{A}^0(y,t)\mathcal{A}^y(y,t) \end{aligned} \tag{3.48}$$

where $\epsilon(Y-Y')$ is equal to 1 for $Y > Y'$ and -1 for $Y < Y'$. The equation of motion for $\phi(Y,t)$ is

$$\frac{B^2}{4\pi}\int dY'\ \epsilon(Y-Y')\ \dot{\phi}(Y',t) - \frac{BE}{2\pi}\phi(Y,t) + \frac{B}{2\pi}(\mathcal{A}_0 + \frac{E}{B}\mathcal{A}^y) = 0 \tag{3.49}$$

This means quantum mechanically

$$[\partial_t - \frac{E}{B}\partial_Y]\langle\phi(Y,t)\rangle = -\frac{1}{B}\partial_Y(\mathcal{A}_0 + \frac{E}{B}\mathcal{A}^y) \tag{3.50}$$

where $\langle\ldots\rangle$ denotes the quantum mechanical average over ϕ. Now under gauge transformation, the change in the partition function due to the change in the action for ϕ is

$$\begin{aligned} \langle\delta S^{\text{bdry}}_{\text{eff}}\rangle &= \frac{B}{2\pi}\int dt \int dY\ [-\partial_t\Lambda + \frac{E}{B}\partial_Y\Lambda]\langle\phi(Y,t)\rangle \\ &= -\frac{B}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dY\ \Lambda(Y,t)(\partial_t - \frac{E}{B}\partial_Y)\langle\phi(Y,t)\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dY\ \Lambda(Y,t)[\partial_Y\mathcal{A}_0 + \frac{E}{B}\partial_Y\mathcal{A}^y] \end{aligned} \tag{3.51}$$

by equation 3.50. So comparing equations 3.43 and 3.3 we see that all the Λ-dependence precisely cancels out. We have therefore demonstrated explicitly that all the gauge non-invariance of the bulk, which appears as a boundary term is precisely removed by the gauge non-invariance of the chiral bosonic action governing the surface oscillations of the droplet.

This emphasizes the importance of the edge states in maintaining gauge invariance in the system.

3.4 An alternative derivation of the effective action

As we have noted, the effective Lagrangian for the fermionic field in the lowest Landau level displays some rather interesting features. For one thing, it does not display the manifest gauge invariance of the original microscopic action. Actually, as has been observed recently, (Sakita 1993) gauge covariance is seen to be realized through a nonlinear realization of W_∞ gauge transformations. This observation has been utilized to derive the effective Lagrangian (albeit not to the full extent) in a rather interesting fashion. In this section we wish to append a discussion of this extremely elegant computation, following the above reference very closely.

The single component, spin-polarized fermion field is given by

$$\hat{\psi}(x,y,t) = \sqrt{\frac{B}{2\pi}}e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\langle z|n\rangle\hat{c}_n(t), \tag{3.52}$$

where,

$$\langle z|n\rangle \equiv \frac{\bar{z}^n}{\sqrt{n!}}, z \equiv \sqrt{\frac{B}{2}}(x+iy), \bar{z} \equiv z^{\dagger}.$$

$|z\rangle$ is the coherent state basis discussed earlier. The modes $\hat{c}_n$ satisfy $\{\hat{c}_n, \hat{c}^\dagger_m\} = \delta_{n,m}$. Let us consider a time-dependent unitary transformation in the space of the $\{\hat{c}_n\}$. Namely,

$$\hat{c'}_n = u_{n,m}\hat{c}_m \equiv \langle n|\hat{u}|m\rangle\hat{c}_m. \tag{3.53}$$

So, the transformed field is given by

$$\begin{aligned}\hat{\psi}'(z,\bar{z},t) &= \sqrt{\frac{B}{2\pi}}\sum_{\mathrm{m,n=0}}^{\infty}\langle z|n\rangle e^{-\frac{1}{2}|z|^2}\langle n|\hat{u}|m\rangle\hat{c}_m(t). \\ &= \sqrt{\frac{B}{2\pi}}\sum_{\mathrm{n=0}}^{\infty}e^{-\frac{1}{2}|z|^2}\langle z|\hat{u}|n\rangle\hat{c}_n(t).\end{aligned} \tag{3.54}$$

Let us further specialize to infitesimal unitary transformations. So, we consider $\hat{u} \equiv I + i \ddagger\hat{\xi}\ddagger$, where ξ is a real function of $\hat{a}$ and $\hat{a}^\dagger$ and an anti-normal ordering prescription has been adopted for $\hat{a}$, $\hat{a}^\dagger$ which is denoted by $\ddagger\ \ddagger$. Thus,

$$\begin{aligned}\delta\hat{\psi}(z,\bar{z},t) &= i\sqrt{\frac{B}{2\pi}}\sum_{\mathrm{n=0}}^{\infty}e^{-\frac{1}{2}|z|^2}\langle z|\ddagger\hat{\xi}\ddagger|n\rangle\hat{c}_n(t) \\ &= i\ddagger\xi(\partial_{\bar{z}}+\frac{z}{2},\bar{z},t)\ddagger\hat{\psi}(z,\bar{z},t).\end{aligned} \tag{3.55}$$

Two points are immediately obvious from the above, namely, the total fermion number in the L.L.L. is conserved under this transformation and the L.L.L. condition is also preserved since $\partial_{\bar{z}} + \frac{z}{2}$ commutes with $\partial_z + \frac{\bar{z}}{2}$. The most general unitary transformation with the above two properties is the W_∞ transformation.

Let us now demand that the effective action for the L.L.L. be W_∞ gauge-invariant. This requires the introduction of a W_∞ gauge potential. The

Lagrangian is given by

$$L = \int dxdy\ \bar{\psi}(x,y,t)(i\partial_t - W(x,y,t))\psi(x,y,t), \tag{3.56}$$

where W is the W_∞ gauge potential. This gauge potential is split up into two pieces for convenience.Namely,

$$W(x,y,t) = V(x,y) + \mathcal{A}(x,y,t). \tag{3.57}$$

The function $V(x,y)$ is a given function.

$$V(x,y) = Ex. \tag{3.58}$$

If we perform the W_∞ transformation on the fermion fields, gauge covariance is restored if

$$\delta\mathcal{A} = \partial_t\xi + \frac{1}{B}\{\{\xi,\ V\}\} + \frac{1}{B}\{\{\xi,\ \mathcal{A}\}\}, \tag{3.59}$$

where $\{\{\ ,\ \}\}$ is the Moyal bracket defined by

$$\{\{\xi_1,\ \xi_2\}\} = iB\sum_{\mathrm{n}=1}^{\infty}\frac{(-1)^n}{n!}(\partial_z^n\xi_1\partial_{\bar{z}}^n\xi_2 - \partial_z^n\xi_2\partial_{\bar{z}}^n\xi_1). \tag{3.60}$$

In the limit of large B, this goes over to the Poisson bracket:

$$\longrightarrow_{\mathrm{B}\to\infty} \{\xi_1,\ \xi_2\}_{\mathrm{P.B.}} = \epsilon_{0\mathrm{ij}}\partial^i\xi_1\partial^j\xi_2.$$

Thus in the limit of a large magnetic field,

$$\delta\mathcal{A}(x,y,t) \approx \partial_t\xi - v\partial_y\xi + \frac{1}{B}\epsilon_{0\mathrm{ij}}\partial^i\xi\partial^j\mathcal{A}, \tag{3.61}$$

where $v \equiv \frac{E}{B}$. In addition to the background electric field applied to the system, let us imagine that further weak space -time dependent fields are also coupled to the system. Let A^μ be the corresponding gauge potential. The Lagrangian for the L.L.L. should be invariant against gauge transformations of this potential. This in turn implies that $\mathcal{A}$ and the W_∞ gauge transformations should be realized in terms of A^μ and its gauge transforms. Since the W_∞ gauge transformations discussed above are nonabelian while the gauge transformations of A^μ are abelian, the W_∞ transformations must be realized nonlinearly in terms of the abelian gauge transformations. We further assume that the energy scale associated with A^μ is much lower than that associated with E,B. thus, the higher derivatives of A^μ may be neglected. So $\delta\mathcal{A}$ should be realized in terms of $\partial^\mu\Lambda$, where $A^\mu \rightarrow A^\mu + \partial^\mu\Lambda$.

Since ξ and Λ are both infinitesimal, ξ should be linear in Λ. We expand $\mathcal{A}, \xi$ in powers of A^μ and its derivatives. By looking at the structure of the W_∞ gauge transformation, we can decide upon the form of this expansion to be:

$$\begin{aligned} \mathcal{A} &= a_\mu A^\mu + \frac{1}{B} A^\mu a_{\mu\nu\rho} \partial^\nu A^\rho + \ldots \\ \xi &= \Lambda + \frac{1}{B} \partial^\mu \Lambda \xi_{\mu\nu} A^\nu + \ldots . \end{aligned} \tag{3.62}$$

Here the a's and $\xi_{\mu\nu}$ are dimensionless quantities to be determined. We take them to be products of constants and the ∂^μ's.

If $\mathcal{A}$ is given as above, we can compute $\delta\mathcal{A}$ since δA^μ is given. We may then compare this with equation 3.61 to determine the a's and $\xi_{\mu\nu}$. If we are interested only in the minimum realization up to, the quadratic in A^μ, we obtain:

$$\mathcal{A} = A_0 - vA^y + \frac{1}{B}\epsilon_{0ij}A^i\partial^j A^0 - \frac{1}{2B}\epsilon_{0ij}A^i\partial^0 A^j + \frac{v}{2B}[\partial^y(A^x A^y) - \partial^x(A^y)^2] + \ldots . \tag{3.63}$$

This is precisely the expression for the effective potential that has been obtained earlier in the chapter. Thus, demanding that the effective action be invariant under W_∞ gauge transformations which are to be nonlinearly realized in terms of abelian U(1) gauge transformations, we get explicit forms for the W_∞ potential and also for the parameter ξ involved in the transformation of the L.L.L. fermion fields.

In the aforementioned article, the W_∞ gauge field theory that has been developed for the electrons in the L.L.L. has been further utilised to derive the effective Lagrangians for circular droplets and for the $\nu = 1$ quantum Hall system.

3.5 Summary

In this chapter we have studied the electromagnetic interactions of a quantum Hall droplet.More generally, we have investigated the field theory of fermions in the lowest Landau level when the system has been coupled to external electromagnetic perturbations. In practice, we have integrated the higher Landau levels out and within the framework of a derivative-expansion scheme have obtained a gauge invariant effective action for the electrons in the L.L.L. We know that $2+1$ dimensional electrons in the L.L.L. are equivalent to $1+1$ dimensional electrons, the configuration space of the $2+1$ dimensional system being the phase space of the $1+1$ dimensional system, (Iso *et al.* 1992a). In the Thomas-Fermi picture (e.g. Kittel 1986), the $1+1$ dimensional electron gas occupies a region of constant density in phase space. This region in the $1+1$ deimensional phase space coincides precisely with the physical droplet of electrons in the L.L.L. created by the background electrostatic potential. The bulk of the droplet which corresponds to a filled fermi sea contributes an effective action, called the bulk action, in terms of the perturbative electromagnetic potentials. The bulk action is not gauge invariant. This non-invariance, however, is spurious as it is cancelled by the gauge non-invariance of the $1+1$ dimensional edge system. Thus the basic mechanism for the preservation of gauge invariance as suggested in (Wen 1990), (Stone 1991a,b) and (Fröhlich & Kerler 1991) is seen to be valid. Moreover, we have a well-defined and systematic procedure for isolating the $1+1$ dimensional edge from the original $2+1$ dimensional system.

The case of the fractional Hall droplet can be similarly handled, at least phenomenologically, by writing the bosonic density $\bar{\psi}\psi(\vec{x})$ in the discussion following 3.47, as $\nu B/2\pi$ inside of the droplet, where ν is the appropriate filling factor.

Chapter 4

Conclusion

The physics of planar systems, comprising anyons or electrons has been the main thrust of this thesis. The planarity of the system has given rise to many new phenomena not encountered in higher or lower dimensions. Of these, we have addressed two which have enjoyed immense popularity in the current literature, anyon superconductivity and the quantum Hall effect.

In this work, we have envisioned and subsequently realized an unified treatment of both these problems. Since anyons may be regarded as fermions with an additional statistical interaction of a specified nature, we may treat both planar anyons and planar electrons on a similar footing. Both of these are coupled to electromagnetism, the dominant field being a magnetic field normal to the plane. The residual fields are incorporated perturbatively. The unperturbed single-particle spectrum is thus the set of Landau levels. The corresponding many-particle states are filled Landau levels, with the remaining fields producing perturbations about these states. The basic technique adopted here is to integrate out all or most of the fermionic modes associated with the Landau levels to arrive at an effective Lagrangian containing a few fermionic modes coupled to the perturbing potentials in a non-trivial fashion.

In the case of anyon superconductors, we integrate out all the fermionic modes while in the case of the Hall effect, we concentrate on the effective theory of the L.L.L.. The residual electromagnetic fields are further considered to be slowly varying in space-time. This has enabled us to adopt a derivative expansion scheme to evaluate the functional determinant arising out of the fermionic Grassmann integrals.

In both cases, a Chern-Simons (CS) term involving the gauge potentials has emerged from the fermionic integrations. For anyons in an electromagnetic field, this CS term (composed of the statistical potentials) generated at the one-loop level, cancels against the CS term put in at the tree-level to transmute fermions to anyons. This cancellation, which is seen to be energetically favourable, leads in turn to the expulsion of the real magnetic field from the sample (Meissner effect) and to the existence of a massless density fluctuation (Superfluidity). Thus the generation of a CS term in the effective Lagrangian is seen to be of vital importance to anyon superconductivity.

The CS term that is generated in the case of Hall effect by integrating out all but the L.L.L. modes is again seen to be of paramount importance in the discussion of the effective field theory of the L.L.L., especially when the sample considered is of finite size. In such a situation, the CS term is part of the action describing the bulk of the sample. This leads to the gauge non-invariance of the bulk action, since the CS term defined on a finite manifold is not gauge-invariant, the gauge dependent term residing on the boundary of the manifold. This in turn leads us to suspect the existence of degrees of freedom on the boundary of the sample, which are governed by an anomalous gauge theory, so that the non-invariance from this anomaly can cancel against that from the bulk. Indeed, a careful analysis of the effective Lagrangian for the L.L.L., which we have undertaken, reveals these edge modes governed by the action for the chiral Schwinger model which is almost the paradigm of an anomalous gauge theory. This lower dimensional theory, appropriately bosonized, is seen to have the well known edge excitations in its spectrum. Further, it has been confirmed that the gauge-dependence from the edge cancels precisely against the gauge-dependence from the bulk. This is an actual physical instance of the well-known Callan-Harvey mechanism, discovered in the context of string theories. Thus the presence of a CS term has not only necessitated the existence of the edge excitations, but has also enabled us to study concepts normally reserved for more esoteric disciplines in physics.

Thus in conclusion, it would be fair to claim that the CS term, which exists as part of the action only in the physics of odd-dimensional systems, is of the primary importance in the physics of these systems. We have exposed its significance in only two situations. Further, the issue of stability of the rigid incompressible quantum Hall droplet against perturbing electromagnetic fields has been addressed, we believe for the first time in the literature. This work, in our opinion, offers a perspective on an important pedagogical and physical problem, the projection of the dynamics of planar electrons in a strong magnetic field to the highly degenerate L.L.L..

An interesting further development that may be pursued would be to consider turning on the Coulomb repulsion between the electrons, which would bring in the scenario typical of the fractional Hall effect, a system that is yet to be explained from a purely microscopic point of view.

Chapter 5

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