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Invariants of 1-Relator Groups
and
Residual Properties of
Amalgamated Products

by

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Introduction

This work is concerned with properties of the lower central series of finitely generated 1-relator groups and amalgamated products. It was motivated by the results and techniques of W. Magnus and A. I. Mal'cev, specifically by Magnus's proof that free groups are residually torsion-free nilpotent and Mal'cev's proof that the free product of two residually torsion-free nilpotent groups is residually torsion-free nilpotent. Mal'cev's theorem naturally leads to the question of under what circumstances an amalgamated product of two residually torsion-free nilpotent groups will again be residually torsion-free nilpotent. In light of the result of Magnus, a special case of this is the question of when the amalgamated product of two free groups will be residually torsion-free nilpotent. In particular, it was conjectured by G. Baumslag in the 1960s that the amalgamated product of two free groups over an isolated cyclic subgroup will be residually torsion-free nilpotent.

After an introduction to various basic definitions and background material in Chapter I, these questions about amalgamated products are considered in Chapters II and III. We use the notation $\mathcal{N}$ to denote the class of nilpotent groups, $\mathcal{T}$ to denote the class of finitely generated torsion-free nilpotent groups, and $\mathrm{R}\mathcal{N}$ and $\mathrm{R}\mathcal{T}$ for residually nilpotent groups and residually finitely generated torsion-free nilpotent groups respectively. In addition to these classes of groups, we consider groups in which extraction of roots is unique whenever possible and residually finite p-groups. We denote these classes by $\mathcal{U}$ and $\mathrm{R}\mathcal{F}_p$ respectively. These classes are related as follows: $\mathrm{R}\mathcal{T} \subset \mathcal{U}$, $\mathrm{R}\mathcal{F}_p \subset \mathrm{R}\mathcal{N}$, and $\mathrm{R}\mathcal{T} \subset \mathrm{R}\mathcal{F}_p$ for all primes p.

In Chapter II we give several sets of necessary conditions for the amalgamated product of two $\mathcal{U}$-groups to be a $\mathcal{U}$-group. In Chapter III we begin with amalgamated products of finitely generated torsion-free nilpotent groups over cyclic and isolated cyclic subgroups. We prove that the latter will be $\mathrm{R}\mathcal{F}_p$ for almost all primes p. We then consider amalgamated products of two free groups over isolated cyclic subgroups. Using a theorem by J. P. Labute on 1-relator groups, we affirm the validity of Baumslag's conjecture in a large class of special cases.

The above result of Labute's concerns the Lazard Lie ring of a finitely generated 1-relator group G.

Considering the relator r as an element of the free group F on the generators of G, the theorem states, in part, that if r is not a proper power modulo $\gamma_{wt(r)+1}F$, then the lower central factors of G are free abelian and there is a formula for their ranks which depends only on the number of generators and the weight of the relator.

The main result of this work is a generalization of Labute's theorem. We prove that the torsion-free ranks of the lower central factors of any finitely generated 1-relator group depend only on the number of generators and the weight of the relator. This is the subject of Chapter IV. The proof relies on the theory of nilpotent $\mathcal{D}$-groups and Lie algebras.

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Notation

$H \cong G$	H is isomorphic to G
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
G/H	the quotient group of G by the normal subgroup H
$G\backslash H$	the set of elements of G which are not contained in H
$gp(X)$	the smallest subgroup of G containing the subset X
$gp_G(X)$	the smallest normal subgroup of G containing the subset X
x^y	$y^{-1}xy$
$[x,\ y]$	$x^{-1}y^{-1}xy$
$[H,\ K]$	$gp(\{\,[h,\ k] \mid h \in H,\ k \in K\})$
$\gamma_2 G$	$[G,\ G]$
$\gamma_n G$	$[\gamma_{n-1}G,\ G]$, the n^{th} term of the lower central series of G
$\langle X\,;\,R\rangle$	the presentation for a group with generators X and relators R
$\{A * B\,;\, H \overset{\varphi}{=} K\}$	the amalgamated product of A and B identifying H with K
$A \underset{H}{*} B$	the amalgamated product of A and B over the subgroup $A \cap B = H$
$\mathcal{F}_p$	the class of finite p-groups
$\mathcal{N}$	the class of nilpotent groups
$\mathcal{T}$	the class of finitely generated torsion-free nilpotent groups
$\mathrm{R}\mathcal{M}$	the class of groups which are residually $\mathcal{M}$-groups for some class $\mathcal{M}$
$\tau(G)$	the torsion subgroup of the (nilpotent) group G
$\mathcal{U}$	the class of groups in which n^{th}-roots are unique whenever they exist
$\mathcal{D}$	the class of groups in which n^{th}-roots are unique and always exist
$id_D(X)$	the smallest $\mathcal{D}$-group ideal of D containing the subset X
$id_L(J)$	the smallest ideal of the Lie algebra L containing the subset J
$\mathbf{N}$	the natural numbers $\{0, 1, 2, \ldots\}$
$\mathbf{Z}$	the integers
$\mathbf{Z}^+$	the positive integers
$\mathbf{Q}$	the rational numbers
$\mathbf{Q}G$	the rational group ring of G
IG	augmentation ideal of the rational group ring of G
$gr(A)$	$\bigoplus_{n=1}^{\infty} \frac{\gamma_n A}{\gamma_{n+1} A}$, the graded algebra associated to the lower central series of A
$A^{[\,\cdot\,]}$	the commutation Lie algebra of the associative algebra A

Chapter I

Preliminaries

1 Presentations

We define free groups and presentations of groups.

Let G be a group and suppose X is a subset of G. Let $gp(X)$ denote the smallest subgroup of G containing X. Then

$$gp(X) = \{x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_n}^{\epsilon_n} \mid n \geq 0,\ x_{i_j} \in X,\ \epsilon_j = \pm 1,\ j \in \mathbf{N}\}.$$

If this is in fact all of G, then we say that the set X generates G. Every group has generating sets since we may always take all of G as the generating set of G.

Let w be an element of $gp(X)$. Then w possesses an expression of the form

$$w = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_n}^{\epsilon_n}. \tag{1}$$

When we wish to refer to the expression (1) for w as opposed to the element of the group G represented by this expression, we will speak of the *word* w. If the expression (1) has the property that whenever $x_{i_j} = x_{i_{j+1}}$ then $\epsilon_j + \epsilon_{j+1} \neq 0$, then we say that this word is freely reduced. Suppose G is generated by $X \subset G$ and every non-empty, freely-reduced word is different from the identity. Then we say G is a free group, freely generated by X, or that G is free on X. In this case, we may also refer to X as a basis for the free group G.

Free groups possess the following *universal mapping property:*

Theorem 1.1 *Let F be a free group on X. For any group G and any set map σ of X into G there exists a unique homomorphism of groups $\varphi : F \longrightarrow G$ such that $\varphi(x) = \sigma(x)$ for all x in X.*

We note that this theorem serves as an alternate definition for free groups. In fact, we can replace the word *group* in all its occurrences to define free objects in other categories. For example, a free *associative algebra* A on a set X is an associative algebra A together with a subset X of A such that X generates A as an algebra and any set map σ from X to an *associative algebra* B extends uniquely to a homomorphism of *associative algebras* from A to B.

Given any set X, there always exists a free group with basis X and as we remarked earlier, every group G possesses a generating set. Thus, by the universal mapping property for free groups, for any group G there exists a free group F and a homomorphism φ mapping F onto G. The First Isomorphism Theorem states that

Theorem 1.2 *If θ is a homomorphism mapping a group H onto a group G, then the kernel of θ is a normal subgroup of H and $H/ker\theta \cong G$.*

Consequently, every group G is isomorphic to a quotient of a free group. A *presentation* is a means of defining a specific group as the quotient of a free group. Let F be the free group on X and let $R \subset F$. We write

$$G = \langle X \; ; \; R \rangle \tag{2}$$

to express that G is isomorphic to the quotient group $F/gp_F(R)$ where $gp_F(R)$, the normal closure of R in F, is the smallest normal subgroup of F containing R. The elements of R are called *defining relators.*

Some classifications of groups depend on how they can be presented. If G possesses a presentation (2) in which the set X is finite we say G is *finitely generated.* If both X and R are finite, then G is *finitely presented.* If R consists of a single element, G is termed a *1-relator group.*

2 Amalgamated Products

We define amalgamated products of groups and state two classic results on normal forms and length.

Let $A = \langle X \ ; \ R \rangle$ and $B = \langle Y \ ; \ S \rangle$ be presentations of two groups. Suppose A and B contain subgroups H and K respectively and φ is an isomorphism mapping H onto K. Then the group given by the presentation

$$G = \langle X \cup Y \ ; \ R \cup S \cup \{h\varphi(h)^{-1} \mid h \in H\} \rangle$$

is called the *amalgamated product* of A and B amalgamating H with K according to φ. It is denoted

$$G = \{ A * B ; H \overset{\varphi}{=} K \} \quad \text{or} \quad A \underset{H}{*} B.$$

If H and K are trivial, then $G = A * B$, the ordinary free product of A and B.

It turns out that A and B inject into G. For convenience, we can identify A with the subgroup of G generated by X, and B with the subgroup of G:

$$\langle Y \cup H \ ; \ S \cup \{h\varphi(h)^{-1} \mid h \in H\} \rangle$$

and take the point of view that $A \cap B = H$.

As in the case of free groups, amalgamated products possess a universal mapping property. If C is any group with α and β a pair of homomorphisms mapping A to C and B to C respectively such that $\alpha(h) = \beta(\varphi(h))$ for all h in H, then there exists a unique homomorphism mapping G into C which simultaneously extends α and β. In the terminology of Category Theory, the amalgamated product is a push-out in the category of groups.

We will employ the following two results on amalgamated products. For their proofs and further discussion of this material we refer the reader to [22].

Proposition 2.1 (Schreier, 1927) *Suppose G is an amalgamated product,*

$$G = A \underset{H}{*} B.$$

Let S_α be a set of left coset representatives of A modulo H in which the identity is the representative of the coset H. Similarly, let S_β be a set of left coset representatives of B modulo H which contains 1.

Then every $g \in G$ *can be expressed uniquely in the form*

$$g = s_1 s_2 \cdots s_k h \tag{3}$$

where

1. *each* $s_i \in S_\alpha \cup S_\beta - \{1\}$,
2. *no two adjacent components* s_i, s_{i+1} *are contained in the same factor of* G, *and*
3. $h \in H$.

We will say that the right-hand side of equation (3) is the *normal form* of g, and that k is the *length* of the normal form of g, which we denote $l(g)$.

Proposition 2.2 *Let* G *be an amalgamated product as above. If* $g \in G$ *and*

$$g = g_1 g_2 \cdots g_k \tag{4}$$

where $k > 1$ *and adjacent terms* g_i, g_{i+1} *are never contained in the same factor, then*

$$l(g) = k.$$

Note that it is implicit in the hypothesis of Proposition 2.2 that if $G = A \underset{H}{*} B$, then each g_i in the product (4) is an element of either $A \backslash H$ or $B \backslash H$. Products with this property are said to be *strictly alternating.*

Corollary 2.3 *In an amalgamated product* $G = A \underset{H}{*} B$, *every strictly alternating product*

$$g = g_1 g_2 \cdots g_k, \qquad k \geq 1$$

is different from the identity.

3 Residual Properties and Nilpotent Groups

Given a class of groups C, it is natural to ask when C will be closed under formation of amalgamated products. Our investigations focus on groups which are residually nilpotent, a property which may be defined using the lower central series, and the closely related class of residually finite p-groups. We remind the reader of the following basic definitions (see [1]).

A property of groups will be called ***abstract*** if it is closed under isomorphism. Let $\mathcal{M}$ be an abstract property or class of groups. Then we say that a group G is ***residually*** $\mathcal{M}$ if for every non-trivial element g of G there is a homomorphic image H_g of G such that H_g is an $\mathcal{M}$-group and the image of g in H_g is not the identity. Thus, we can think of G as approximated by the collection of $\mathcal{M}$-groups $\{H_g\}_{g \in G}$. Equivalently, G is residually $\mathcal{M}$ if for each non-trivial element g there exists a normal subgroup N_g of G such that g is not contained in N_g and G/N_g has the property $\mathcal{M}$. For example, G is residually a finite p-group for some prime p, and we write $G \in \mathrm{R}\mathcal{F}_p$, if for any non-trivial g in G there is a normal subgroup N_g of G such that $g \notin N_g$ and G/N_g is finite with order a power of p.

Let G be a group. Given two elements x and y, the *commutator* of x and y is

$$[x, y] = x^{-1}y^{-1}xy.$$

Given two subgroups H and K of G, let $[H, K]$ denote the subgroup of G generated by the set $\{\, [h, k] \mid h \in H, k \in K \,\}$. The terms of the *lower central series* of G are defined recursively as follows:

$$\begin{aligned} \gamma_1 G &= G \\ \gamma_{n+1} G &= [\gamma_n G, G] \qquad (n \geq 1). \end{aligned}$$

In consequence of this definition, $\gamma_i G$ is a normal subgroup of G for all i and

$$G = \gamma_1 G \geq \gamma_2 G \geq \cdots \geq \gamma_i G \geq \cdots .$$

A group is said to be ***nilpotent*** of class c if $\gamma_{c+1} G = \{1\}$ and c is the smallest natural number with this property. Nilpotent groups may be viewed as a generalization of the non-identity abelian groups, which are nilpotent of class 1. In some sense they are also a generalization of finite p-groups, which are always nilpotent. Consequently, the class of residually finite p-groups, $\mathrm{R}\mathcal{F}_p$, is contained in the class of residually nilpotent groups, which we denote $\mathrm{R}\mathcal{N}$.

If G is nilpotent, then for any prime p, the elements of order a power of p constitute a normal subgroup of G called the Sylow p-subgroup of G. The elements of finite order form a normal subgroup which is a direct sum of the Sylow p-subgroups of G. This subgroup is termed the *torsion subgroup* of G and is denoted $\tau(G)$. Finitely generated nilpotent groups satisfy the *maximal condition*; that is, they can have no infinite properly ascending chain of subgroups. This is equivalent to the condition that all subgroups are finitely generated. Consequently, if G is a finitely generated nilpotent group, then $\tau(G)$ is finite. Moreover, if $\{g_1, g_2, \ldots, g_n\}$ is any finite set of elements of finite order in a nilpotent group, then $gp(g_1, g_2, \ldots, g_n)$ is finite (see [1, page 1]). Clearly, for any nilpotent group G, $G/\tau(G)$ is a torsion-free nilpotent group. We shall let $\mathcal{T}$ denote the class of all finitely generated torsion-free nilpotent groups. If G can be approximated by groups which are not merely nilpotent, but torsion-free nilpotent, then we say G is *residually torsion-free nilpotent.*

It is easy to see that

Lemma 3.1 *If $H \trianglelefteq G$, then*

$$\gamma_i(G/H) = ((\gamma_i G)H)/H.$$

Thus, as a corollary to Lemma 3.1, the quotient group $G/\gamma_i G$ is always nilpotent. More generally, we have:

Corollary 3.2 *Let N be a normal subgroup of G. Then G/N is nilpotent if and only if $\gamma_i(G) \subseteq N$ for some $i < \infty$.*

Theorem 3.3 *A group G is residually nilpotent if and only if the terms of the lower central series of G have trivial intersection.*

Proof: Beginning with the forward direction, let S be the set of all normal subgroups N of G such that G/N is nilpotent. By Corollary 3.2, for any N in S, there exists $n < \infty$, such that $\gamma_n G$ is contained in N. Since the intersection of the terms of the lower central series of G will be contained in $\gamma_n G$ for any n,

$$\bigcap_{i=1}^{\infty} \gamma_i G \subseteq \bigcap_{N \in S} N.$$

Since G is residually nilpotent, for any nontrivial element g, there is a subgroup N in S with $g \notin N$. Hence, the subgroups contained in the collection S have trivial intersection and therefore so does the lower central series of G.

Conversely, if $\bigcap_{i=1}^{\infty} \gamma_i G = \{1\}$, then, given any non-trivial element g of G, there must be some term of the lower central series which does not contain g. Since the quotient of G by any term of its lower central series will be nilpotent, G is residually nilpotent. □

Thus, if G is residually nilpotent, then for any non-trivial element g of G there is a unique positive integer e such that

$$g \in \gamma_e G, \quad g \notin \gamma_{e+1} G.$$

We term such an e the *weight* of g and we write $wt_G(g) = e$, or simply $wt(g) = e$, when there is no ambiguity.

4 Commutator Identities and Basic Sequences

Commutation defines a non-associative binary operation on a group. We note the following identities for commutators which were first introduced by P. Hall. We use the notation x^y for $y^{-1}xy$.

$$[a, bc] = [a, c]\,[a, b]\,[[a, b], c] \tag{5}$$

$$[ab, c] = [a, c]\,[[a, c], b]\,[b, c] \tag{6}$$

$$1 = [[a, b], c^a]\,[[c, a], b^c]\,[[b, c], a^b] \tag{7}$$

Any matched arrangement of brackets can be interpreted using commutation. We define a *bracket arrangement* of weight n recursively as follows. The unique bracket arrangement of weight 1 is $[*]$. A bracket arrangement of weight n is obtained as the commutator of two bracket arrangements of weight l and m respectively where $l + m = n$. If β is a bracket arrangement of weight n and $a_1, a_2, \ldots, a_n$ are any elements of G, not necessarily unique, then $\beta(a_1, a_2, \ldots, a_n)$ is said to be a commutator of weight n in $a_1, a_2, \ldots, a_n$.

There are particular sequences of commutators in the generators of a group which prove to be very useful. In order to define these *basic commutators*, we must begin with a simpler structure than a group. A groupoid is a set with a binary operation. Unlike a semigroup, the product in a groupoid need not be associative. If Z is the finite set $\{z_1, z_2, \ldots, z_q\}$, then the elements of the free groupoid $\mathcal{G}$ on Z consist of all properly bracketed strings of z's. Two such elements will be equal if and only if they are identical. Thus, for any element g of $\mathcal{G}$ there is a well defined length $l(g)$, namely, the number of z's which appear in g. We further observe that if $l(g) > 1$, then there exist a unique pair of elements u and v of $\mathcal{G}$ such that $g = uv$.

A sequence $b_1, b_2, \ldots$ of elements of $\mathcal{G}$ is termed a *basic sequence* in Z if it meets the following three conditions:

1. $z_1, z_2, \ldots, z_q$ are contained in the sequence.
2. If $l(b_i) < l(b_j)$, then $i < j$.
3. If $u \in \mathcal{G}$ and $l(u) > 1$, then u is contained in $\{b_i\}_{i=1}^{\infty}$ if and only if $u = b_i b_j$ where $j < i$ and, if the $l(b_i)$ is greater than 1, $b_i = b_l b_k$, where $k < l$ and $j \geq k$.

If Γ is any groupoid with finite generating set $X = \{x_1, x_2, \ldots, x_q\}$, then the map $z_i \mapsto x_i$ defines a homomorphism of groupoids θ from $\mathcal{G}$ to Γ. A sequence of elements of Γ is called a basic sequence in X if it is the image under θ of a basic sequence in Z. Suppose G is any group with finite generating set $X = \{x_1, x_2, \ldots, x_q\}$. We observe that G is a groupoid under the operation $f \cdot g = [f, g]$. We term the elements of a basic sequence in X *basic commutators.*

As a result of the commutator identities (5), (6), and (7), the commutators of weight n in any basic sequence $\{\beta_i\}_{i=1}^{\infty}$ in X generate $\gamma_n G$ modulo $\gamma_{n+1} G$. Moreover, if G is free on X then $\gamma_n G / \gamma_{n+1} G$ is a free abelian group freely generated modulo $\gamma_{n+1} G$ by the basic commutators of weight n in any basic sequence in X (see [1, page 37]).

5 Group Rings and the Augmentation Ideal

The group ring of G over the rational numbers, $\mathbf{Q}G$, is a vector space over $\mathbf{Q}$ with the elements of the group G as basis. A typical element is of the form

$$\sum_{g\in G} r_g g$$

where all but finitely many of the coefficients $r_g \in \mathbf{Q}$ are zero. Addition is coordinatewise. Multiplication is defined by

$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{g\in G} s_g g\right) = \sum_{g\in G}\sum_{xy=g} r_x s_y g$$

which is known as Cauchy multiplication. There is a homomorphism ε from $\mathbf{Q}G$ to $\mathbf{Q}$ defined by

$$\varepsilon\left(\sum_{g\in G} r_g g\right) = \sum_{g\in G} r_g.$$

The kernel of this *augmentation map* is called the *augmentation ideal*, and we denote it by IG.

6 Lie Algebras and Gradings

We define Lie algebras, giving the example of the commutation Lie algebra of an associative algebra, and discuss the category of graded Lie algebras.

Let K be a field. An algebra A over K is a K-vector space with a bilinear multiplication $*$; that is, for u, v, and w in A and k in K,

$$\begin{aligned} u*(v+w) &= u*v+u*w \\ (v+w)*u &= v*u+w*u \\ k(u*v) &= (ku)*v = u*(kv). \end{aligned}$$

If this multiplication is associative, then we say that A is an *associative algebra.* If the multiplication satisfies the conditions that $u*u=0$ for all u in A and the Jacobi identity

$$(u*v)*w+(v*w)*u+(w*u)*v=0,$$

then A is termed a *Lie algebra.*

Given any associative algebra A, we can define a Lie algebra $A^{[\,,\,]}$ called the *commutation Lie algebra*. Addition and scalar multiplication are unchanged. The new Lie algebra bracket multiplication is defined by

$$[u, v] = u * v - v * u.$$

If X is a subset of an algebra A over a field K, then we say that X *generates A as an algebra* if every element of A may be expressed as a K-linear combination of products of elements of X. Free objects exist in the categories of associative and Lie algebras. We term X a basis of a free Lie algebra L over K if X is a subset of L which generates L as an algebra and if any set map from X to a Lie algebra A over K extends to a unique Lie algebra homomorphism from L to A.

Whether L is a free Lie algebra over K or a free nilpotent Lie algebra over K or a free object in some other category of Lie algebras, a free basis for L should not be confused with a vector space basis for L. In the case of a free basis X, not only do we have the appropriate universal mapping property, but we are not limited to linear combinations of elements of X for the generation of L.

An algebra A is said to be *graded* if it possesses a direct sum decomposition $\bigoplus_{i=1}^{\infty} A_i$ such that $A_i * A_j \subseteq A_{i+j}$. The elements of A_i are termed *homogeneous of degree* i and every element u of A has a unique expression of the form $a = a_1 + a_2 + \cdots$ where $a_i \in A_i$ and all but a finite number of the a_i are equal to zero. We term the a_i *homogeneous components* of a. If $A = \bigoplus_{i=1}^{\infty} A_i$ and $B = \bigoplus_{i=1}^{\infty} B_i$ are graded algebras over K, then φ mapping A to B is a homomorphism of graded algebras if it is an algebra homomorphism with $\varphi(A_i) \subseteq B_i$ for all i. An ideal J of A is said to be *homogeneous* if, in the decomposition of any element j of J

$$j = j_1 + j_2 + \cdots \qquad j_i \in A_i,$$

we have $j_i \in J$ for all i. Equivalently, $J = \bigoplus_{i=1}^{\infty} J \cap A_i$.

For clarity, we now restrict ourselves to Lie algebras. However, all statements apply equally to associative algebras after suitable change of notation. For further details, we refer the reader to [10]. Given an arbitrary Lie algebra A, if there exist subspaces

$$A = J_1 \supseteq J_2 \supseteq \cdots$$

such that $[J_i, J_j] \subseteq J_{i+j}$, then we say that $\{J_i\}_{i=1}^{\infty}$ is a filtration in A. We can use such a filtration to construct a graded Lie algebra by taking the vector space $\bigoplus_{i=1}^{\infty} J_i/J_{i+1}$ and imposing the multiplication

$$[a_i + J_{i+1}, a_j + J_{j+1}] = [a_i, a_j] + J_{i+j+1}.$$

As an example of a filtration, we introduce the lower central series of a Lie algebra A. If A_1 and A_2 are subspaces of the Lie algebra A, we define $[A_1, A_2]$ to be the subspace spanned by all products $[a_1, a_2]$ with $a_i \in A_i$. We define the lower central series of a Lie algebra using the same notation as for a group:

$$\begin{aligned} \gamma_1 A &= A \\ \gamma_2 A &= [A, A] \\ \gamma_n A &= [\gamma_{n-1} A,\ A] \qquad n > 1. \end{aligned}$$

A is termed nilpotent if $\gamma_n A = 0$ for some finite n, and residually nilpotent if $\bigcap_{n=1}^{\infty} \gamma_n A = 0$. The lower central series is a filtration, and we denote the graded algebra associated to this filtration by

$$gr(A) = \bigoplus_{n=1}^{\infty} \frac{\gamma_n A}{\gamma_{n+1} A}\ .$$

We will refer to $gr(A)$ as the *associated graded algebra* or the *associated Lie algebra* of A.

7 Groups with Unique Roots

We state the definitions of $\mathcal{U}$-groups, $\mathcal{D}$-groups, and $\mathcal{D}$-group ideals, and introduce the Mal'cev completion of a torsion-free nilpotent group [1, 5].

In a group G, if there exists a solution to the equation

$$x^n = g$$

where n is a positive integer and $g \in G$, then we term such a solution an n^{th}-root of g. A group G is termed a *$\mathcal{U}$-group* if, for any $n \in \mathbf{Z}^+$, n^{th}-roots are unique whenever they exist. That is, if $f, g \in G$, and

$$f^n = g^n,$$

then $f = g$. A group G is a $\mathcal{D}$-group if it contains unique n^{th}-roots for all of its elements for all values of n. We note that all residually torsion-free nilpotent groups are $\mathcal{U}$-groups [1, page 10]. Thus, if an amalgamated product is to be residually torsion-free nilpotent, it must certainly be a $\mathcal{U}$-group.

Given a subset S of a $\mathcal{D}$-group D, let $id_D(S)$ denote the smallest normal subgroup of D containing S such that the quotient group $D/id_D(S)$ is a $\mathcal{D}$-group. If N is a normal subgroup of D such that D/N is a $\mathcal{D}$-group, then we term N a $\mathcal{D}$-group ideal. Thus, $id_D(S)$ is the smallest $\mathcal{D}$-group ideal of D containing S.

Any torsion-free nilpotent group B can be embedded in a nilpotent $\mathcal{D}$-group mB. This is the Mal'cev completion of B and it is unique up to isomorphism. We will generally identify B with its embedded image in mB. The Mal'cev completion possesses the following properties (see [21] and [1, pages 18,49]):

Lemma 7.1 *Let A and B be torsion-free nilpotent groups. Then*

1. *mB is a nilpotent $\mathcal{D}$-group of the same nilpotency class as B.*
2. *Any homomorphism θ mapping A into B can be extended uniquely to a homomorphism $m\theta$ mapping mA into mB.*
3. *Any element of mB has a non-trivial power which lies in B.*

8 Completions and the Lower Central Series

In this section we establish several results regarding completions of nilpotent groups and the lower central series.

Unlike ordinary $\mathcal{D}$-groups, the terms of the lower central series of a nilpotent $\mathcal{D}$-group are $\mathcal{D}$-group ideals [21]. We use this property to prove the following:

Lemma 8.1 *If B is a finitely generated torsion-free nilpotent group of class c, then*

$$id_{mB}(\gamma_k B) = \gamma_k mB.$$

Proof: Since $mB/\gamma_k mB$ is a $\mathcal{D}$-group and $\gamma_k B \subset \gamma_k mB$, we know that $id_{mB}(\gamma_k B)$ is contained in $\gamma_k mB$. For the other containment, we use descending induction on k. For any group G and any positive integer n, $\gamma_n G$ is generated modulo $\gamma_{n+1} G$ by the set of all commutators of weight n in the elements of G. By Lemma 7.1, mB is nilpotent of class c and every element of mB has a non-trivial power lying in B. Therefore, $\gamma_c mB$ is generated by commutators of the form $w = \beta(a_1, \ldots, a_c)$ where β is any bracket arrangement of weight c and $a_i^{\alpha_i} \in B$ for some $\alpha_i \in \mathbf{Z}^+$. Since mB is nilpotent of class c,

$$w^{\alpha_1 \cdots \alpha_c} = \beta(a_1^{\alpha_1}, \ldots, a_c^{\alpha_c}) \in \gamma_c B.$$

Thus, $w \in id_{mB}(\gamma_c B)$ and consequently $\gamma_c mB \subset id_{mB}(\gamma_c B)$.

Next we assume that the proposition holds for all terms of the lower central series beyond $k < c$. $\gamma_k mB$ is generated modulo $\gamma_{k+1} mB$ by commutators of the form $w = \beta(a_1, \ldots, a_k)$ where β is any bracket arrangement of weight k and $a_i^{\alpha_i} \in B$ for some $\alpha_i \in \mathbf{Z}^+$. In this case,

$$w^{\alpha_1 \cdots \alpha_k} = \beta(a_1^{\alpha_1}, \ldots, a_k^{\alpha_k}) \text{ modulo } \gamma_{k+1} mB.$$

By the inductive hypothesis, $\gamma_{k+1} mB = id_{mB}(\gamma_{k+1} B)$, which is contained in $id_{mB}(\gamma_k B)$. Hence, $w^{\alpha_1 \cdots \alpha_k} \in id_{mB}(\gamma_k B)$ and therefore $w \in id_{mB}(\gamma_k B)$. □

Since we are identifying B with its image in mB, we consider that, for any subgroup H of B, the completion mH is a subgroup of mB. In fact, mH can be defined as the collection of elements of mB which have a non-trivial power lying in H (see [21]). It follows from a result of Plotkin (see [12, page 254]), that if H is a normal subgroup of B, then mH is a normal subgroup of mB. In particular, $m\gamma_k B$ is a normal subgroup of mB and therefore $m\gamma_k B = id_{mB}(\gamma_k B)$. Lemma 8.1 then has the following corollaries:

Corollary 8.2 *With B as above, $m\gamma_k B = \gamma_k mB$.*

Corollary 8.3 *If F is a free nilpotent group of class c, then $\gamma_k mF \cap F = \gamma_k F$.*

Proof: By Corollary 8.2, $\gamma_k mF \cap F = m\gamma_k F \cap F$. Since $F/\gamma_k F$ is torsion-free, $m\gamma_k F \cap F = \gamma_k F$. □

9 Nilpotent $\mathcal{D}$-Groups and Lie Algebras

We discuss the correspondence between nilpotent $\mathcal{D}$-groups and rational nilpotent Lie algebras. In particular, we examine the behavior of free nilpotent $\mathcal{D}$-groups and the lower central series under this correspondence.

The importance of nilpotent $\mathcal{D}$-groups can be seen in the following theorem. The finitely generated case was proved by Mal'cev [21]. The infinitely generated case follows from a generalization by Lazard of Jennings' Theorem (see [1, pages 47-52]).

Theorem 9.1 *There is an equivalence between the categories of nilpotent $\mathcal{D}$-groups and rational nilpotent Lie algebras.*

One method of realizing this equivalence is due to Jennings. What follows is a brief summary of some of his results [11]. He proved that if G is a finitely generated torsion-free nilpotent group, then the rational group ring of G, $\mathbb{Q}G$, is residually nilpotent as an augmented algebra. In other words, if IG is the augmentation ideal of $\mathbb{Q}G$, then

$$\bigcap_{n=1}^{\infty} IG^n = 0.$$

In consequence, we can form the IG-adic completion $\widehat{\mathbb{Q}G}$ of $\mathbb{Q}G$. $\widehat{\mathbb{Q}G}$ may be identified with a ring of "power series"

$$r = r_0 + \sum_{i=1}^{\infty} r_i \qquad r_0 \in \mathbf{Q},\ r_i \in IG^i \quad (i \geq 1).$$

We can then use the ordinary power series expansion of the log and exp functions

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$
$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

to define 1–1 mappings from $1+\widehat{IG}$ to $\widehat{IG}$ and from $\widehat{IG}$ to $1+\widehat{IG}$ respectively. As usual, exp and log are inverses. We restrict our attention to G as a subset of $1+\widehat{IG}$. Let $LogG$ denote the $\mathbf{Q}$-subspace of $\widehat{\mathbb{Q}G}$ spanned by $\{\log g \mid g \in G\}$. Jennings proves [11, page 186]:

Theorem 9.2 *$LogG$ is a nilpotent Lie subalgebra of the commutation Lie ring $\widehat{QG}^{[\,,\,]}$.*

We will refer to $LogG$ as *the Lie algebra* of G. We can impose a group structure on $LogG$ by the Baker-Campbell-Hausdorff formula [1, 10]. This formula expresses $z = \log(\exp x \cdot \exp y)$ in terms of x and y. According to it,

$$z = \sum_{i=0}^{\infty} h_i(x, y)$$

where $h_i(x, y)$ is a rational linear combination of Lie brackets of weight i in x and y. The first several terms are

$$z = x + y + \frac{1}{2}[x, y] + \cdots .$$

The product $\log a * \log b$ is defined as $\sum_{i=0}^{\infty} h_i(\log a, \log b)$. It follows that

$$\exp(\log a * \log b) = a \cdot b.$$

If G is a finitely generated torsion-free nilpotent group, then $\exp(LogG)$ is the Mal'cev completion of G [1, page 50]. If G is a nilpotent $\mathcal{D}$-group, then $(LogG, *) \cong G$. Thus, log and exp move us between the categories of torsion-free nilpotent groups and rational nilpotent Lie algebras. We assume the following results concerning this correspondence [23, page 305]. In each case, D is a nilpotent $\mathcal{D}$-group and L is a rational nilpotent Lie algebra.

Lemma 9.3 $\exp(LogD) = D$ *and* $Log(\exp L) = L$

Lemma 9.4 *If H is a $\mathcal{D}$-group ideal of D, then $LogH$ is an ideal of $LogD$.*

Lemma 9.5 *If M is an ideal of L, then* $\exp M$ *is a $\mathcal{D}$-group ideal of* $\exp L$.

Lemma 9.6 *Given nilpotent $\mathcal{D}$-groups A and B, any homomorphism φ mapping A into B induces a homomorphism of rational Lie algebras φ^* from $LogA$ to $LogB$, and the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\log"'] & B \arrow[d, "\log"] \\
LogA \arrow[r, "\varphi^*"] & LogB
\end{tikzcd}$$

Lemma 9.7 *If H is a $\mathcal{D}$-group ideal of D, then*

$$Log\left(\frac{D}{H}\right) \cong \frac{Log D}{Log H} .$$

Next, we examine the Lie algebra of a free nilpotent $\mathcal{D}$-group.

Lemma 9.8 *Suppose E is a free nilpotent $\mathcal{D}$-group of class c with basis $\{x_1, x_2, \ldots, x_q\}$. Then $LogE$ is a free rational nilpotent Lie algebra of class c with basis $\{\log x_1, \ldots, \log x_q\}$.*

Proof: First we verify that $\log x_1, \ldots, \log x_q$ generate $LogE$ as a rational Lie algebra. $LogE$ is the rational linear span of the set of all $\log g$ such that g is an element of the group E. Any non-trivial $g \in E$ may be expressed as a product in $\{x_1, x_2, \ldots, x_q\}$, $g = x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}$, with each α_j a non-zero integer. Since $\log g = \alpha_1 \log x_{i_1} * \cdots * \alpha_n \log x_{i_n}$, by the Campbell-Baker-Hausdorff formula, $\log g$ is a rational linear combination of brackets in $\log x_1, \ldots, \log x_q$.

Next we demonstrate that this set of generators possesses the correct mapping property. Suppose L is a rational nilpotent Lie algebra of class less than or equal to c. Then any set map

$$x_i \mapsto \exp l_i \qquad l_i \in L$$

extends to a unique $\mathcal{D}$-group homomorphism φ mapping E to $\exp L$. By Lemmas 9.3 and 9.6, φ induces a unique homomorphism of rational Lie algebras

$$\varphi^* : \; LogE \longrightarrow L$$

so that $\varphi^*(\log x_i) = \log(\varphi x_i) = \log(\exp l_i) = l_i$. Thus, any set map

$$\log x_i \mapsto l_i \; \in \; L$$

extends to a unique homomorphism φ^* mapping $LogE$ into L. □

Lemma 9.9 *If D is a nilpotent $\mathcal{D}$-group, then $\gamma_k LogD = Log\gamma_k D$ for any positive integer k.*

Proof: Since the proposition is trivial for $k = 1$, let k be a positive integer greater than 1. Suppose β is a bracket arrangement of weight k, and $a_1, \ldots, a_k$ are any, not necessarily distinct, elements of D. It follows from the Baker-Campbell-Hausdorff formula that

$$\log\beta(a_1,\ a_2,\ldots,\ a_k) = \beta(\log a_1,\ \log a_2,\ldots,\ \log a_k) + \sum_j q_j P_j \tag{8}$$

where the last term is a rational linear combination of Lie brackets of weight greater than k, each containing $\log a_1, \log a_2, \ldots, \log a_k$. Hence,

$$\log\beta(a_1,\ a_2,\ldots,\ a_k) \in \gamma_k LogD.$$

Any element d of $\gamma_k D$ is a product $\beta_1\beta_2\cdots\beta_n$ where each β_i is a commutator of weight k in elements of D. Then

$$\log d = \log(\beta_1\beta_2\cdots\beta_n) = \log\beta_1 * \log\beta_2 * \cdots * \log\beta_n$$

where $*$ denotes the Baker-Campbell-Hausdorff multiplication. Consequently, $\log d$ is a rational linear combination of Lie products in $\log\beta_1, \log\beta_2, \ldots, \log\beta_n$. Since each $\log\beta_i$ is contained in the Lie subalgebra $\gamma_k LogD$, $\log d$ is contained in $\gamma_k LogD$. Thus, $Log\gamma_k D$ is contained in $\gamma_k LogD$.

Conversely, we observe that equation (8) can be written

$$\beta(\log a_1,\ \log a_2,\ldots,\ \log a_k) = \log\beta(a_1,\ a_2,\ldots,\ a_k) - \sum_j q_j P_j.$$

We can rewrite each term of $\sum_j q_j P_j$ in this manner and then iterate this procedure for higher weight bracket terms that arise. Since D is nilpotent, this process terminates. It then follows that $\beta(\log a_1,\ \log a_2,\ldots,\ \log a_k)$ is contained in $Log\gamma_k D$. Any element of $\gamma_k LogD$ is a rational linear combination of terms of this form. Since $Log\gamma_k D$ is a Lie subalgebra of $LogD$, it contains $\gamma_k LogD$. Thus, they are equal. □

10 The Lazard Lie Algebra of a Nilpotent $\mathcal{D}$-Group

We give the definition of the Lazard Lie ring of a group and compare the Lazard Lie algebra of a nilpotent $\mathcal{D}$-group D to the associated graded algebra of $LogD$.

Let G be a group. The quotients of consecutive terms of the lower central series can be used to form a Lie ring

$$gr(G) = \bigoplus_{n=1}^{\infty} \frac{\gamma_n G}{\gamma_{n+1} G}\ .$$

Addition is coordinatewise and induced by the group operation in G, so that if $a_n, g_n \in \gamma_n G$, then

$$a_n\gamma_{n+1}G + g_n\gamma_{n+1}G = a_n g_n \gamma_{n+1}G.$$

Let $b_m \in \gamma_m G$. We impose a bracket multiplication on $gr(G)$ by defining

$$[a_n\gamma_{n+1}G,\ b_m\gamma_{m+1}G] = [a_n,\ b_m]\gamma_{n+m+1}G \tag{9}$$

where the bracket on the right-hand side of equation (9) indicates group commutation in G. As a consequence of the commutator identities given by equations (5), (6), and (7), under this multiplication $gr(G)$ is a Lie ring over the integers. This construction was introduced by Lazard [15], and $gr(G)$ is known as the Lazard Lie ring of G.

If D is a nilpotent $\mathcal{D}$-group, then $\gamma_n D/\gamma_{n+1}D$ is an abelian $\mathcal{D}$-group. We can define an action of the field of rational numbers on $gr(D)$ as follows: If d is an element of $\gamma_n D$, a an integer, and b a positive integer, then

$$\frac{a}{b}(d\gamma_{n+1}D) = u^a\gamma_{n+1}D$$

where $u\gamma_{n+1}D$ is the unique b^{th}-root of $d\gamma_{n+1}D$. Under this action, $gr(D)$ is a rational Lie algebra.

Given a nilpotent $\mathcal{D}$-group D, we now have two means of associating a graded rational Lie algebra to D. We can form $gr(D)$ or, as we saw in Section 6, we can use the lower central series of the rational Lie algebra $Log D$ to form the associated graded algebra $gr(Log D)$.

Proposition 10.1 *If D is a nilpotent $\mathcal{D}$-group, then $gr(D)$ and $gr(LogD)$ are isomorphic as graded Lie algebras.*

Proof:

$$gr(Log D) = \bigoplus_{i=1}^{\infty} \frac{\gamma_i(Log D)}{\gamma_{i+1}(Log D)}.$$

By Lemma 9.9, $\gamma_k Log D = Log \gamma_k D$. Thus,

$$\frac{\gamma_i(Log D)}{\gamma_{i+1}(Log D)} = \frac{Log(\gamma_i D)}{Log(\gamma_{i+1}D)}.$$

By Lemma 9.7,

$$\frac{Log(\gamma_i D)}{Log(\gamma_{i+1}D)} \cong Log\left(\frac{\gamma_i D}{\gamma_{i+1}D}\right).$$

$\gamma_i D/\gamma_{i+1} D$ is an abelian group; therefore,

$$Log\left(\frac{\gamma_i D}{\gamma_{i+1} D}\right) \cong \frac{\gamma_i D}{\gamma_{i+1} D}\,.$$

Thus, the additive structures of $gr(D)$ and $gr(\log D)$ are isomorphic. If a_n and b_m are arbitrary elements of $\gamma_n D$ and $\gamma_m D$ respectively, then the mapping

$$\log a_n + \gamma_{n+1} LogD \mapsto a_n + \gamma_{n+1} D$$

defines an isomorphism of Lie algebras since, by the Campbell-Baker-Hausdorff formula,

$$[\log a_n, \log b_m] - \log[a_n, b_m] \in \gamma_{n+m+1} LogD.$$

Consequently, $gr(D)$ is isomorphic to $gr(LogD)$. □

11 The Associated Lie Algebra of a Quotient

Suppose A is a Lie algebra. We see here that the associated graded algebra of a quotient of A is a quotient of the associated graded algebra of A.

Lemma 11.1 *Let A be a Lie algebra and suppose J is an ideal of A. Put*

$$gr_A(J) = \bigoplus_{n=1}^{\infty} \frac{J \cap \gamma_n A + \gamma_{n+1} A}{\gamma_{n+1} A}\,.$$

Then

$$gr\left(\frac{A}{J}\right) \cong \frac{gr(A)}{gr_A(J)}\,.$$

Proof: Let

$$J_n = \frac{J \cap \gamma_n A + \gamma_{n+1} A}{\gamma_{n+1} A} \qquad \text{and} \qquad A_n = \frac{\gamma_n A}{\gamma_{n+1} A}\,.$$

Note that J_n is contained in A_n. First we will establish that $gr_A(J) = \bigoplus_{n=1}^{\infty} J_n$ is an ideal of $gr(A) = \bigoplus_{n=1}^{\infty} A_n$. To that end, let $j_n \in J \cap \gamma_n A$ and $a_m \in \gamma_m A$. Observe that

$$[a_m + \gamma_{m+1} A,\ j_n + \gamma_{n+1} A] = [a_m,\ j_n] + \gamma_{m+n+1} A.$$

Since J is an ideal of the Lie algebra A, $[a_m,\ j_n]$ is an element of J, as well as of $\gamma_{m+n} A$. It follows that $[A_m,\ J_n]$ is contained in J_{m+n}, and $gr_A(J)$ is an ideal of $gr(A)$.

Each homogeneous term of $gr(A/J)$ of degree n is canonically isomorphic to

$$\frac{\gamma_n A}{\gamma_{n+1}A} \Big/ \left(\frac{\gamma_n A}{\gamma_{n+1}A} \bigcap \frac{J + \gamma_{n+1}A}{\gamma_{n+1}A} \right). \tag{10}$$

It is clear that

$$J_n \subset \frac{\gamma_n A}{\gamma_{n+1}A} \bigcap \frac{J + \gamma_{n+1}A}{\gamma_{n+1}A}.$$

Conversely, if $t + \gamma_{n+1}A$ is an element of the intersection of $\gamma_n A/\gamma_{n+1}A$ with $(J + \gamma_{n+1}A)/\gamma_{n+1}A$, then there must exist elements a and j of $\gamma_n A$ and J respectively which represent the same coset of $\gamma_{n+1}A$ as t. Since $j + \gamma_{n+1}A = a + \gamma_{n+1}A$, it follows that j is an element of $J \cap \gamma_n A$. Thus, $t + \gamma_{n+1}A = j + \gamma_{n+1}A$ is an element of J_n, and the intersection of $\gamma_n A/\gamma_{n+1}A$ and $J + \gamma_{n+1}A/\gamma_{n+1}A$ is equal to J_n. Hence, the quotient (10) is identical to A_n/J_n, and the vector space structures of $gr(A/J)$ and $gr(A)/gr_A(J)$ are isomorphic via the map φ defined by

$$(a_n + J) + \gamma_{n+1}(A/J) \mapsto (a_n + \gamma_{n+1}A) + J_n.$$

Consider the following two homogeneous elements of $gr(A/J)$:

$$\tilde{a} = (a_n + J) + \gamma_{n+1}(A/J) \qquad \text{and} \qquad \tilde{b} = (b_m + J) + \gamma_{m+1}(A/J).$$

Then

$$\begin{aligned} [\tilde{a}, \tilde{b}] &= [a_n + J,\ b_m + J] + \gamma_{n+m+1}(A/J) \\ &= ([a_n,\ b_m] + J) + \gamma_{n+m+1}(A/J). \end{aligned}$$

Since $[A_i, J_j] \subseteq J_{i+j}$,

$$\begin{aligned} [\varphi(\tilde{a}), \varphi(\tilde{b})] &= [(a_n + \gamma_{n+1}A) + J_n,\ (b_m + \gamma_{m+1}A) + J_m] \\ &= [a_n + \gamma_{n+1}A,\ b_m + \gamma_{m+1}A] + J_{n+m} \\ &= ([a_n,\ b_m] + \gamma_{n+m+1}A) + J_{n+m} \\ &= \varphi[\tilde{a}, \tilde{b}]. \end{aligned}$$

Thus, φ is an isomorphism of graded Lie algebras. □

12 The Free Rational Lie Algebra

We demonstrate a grading of the free Lie algebra L and establish that there is a canonical isomorphism between L and the associated graded algebra $gr(L)$.

Suppose F is the free group on $\{x_1, x_2, \ldots, x_q\}$. Then $gr(F) \otimes \mathbf{Q}$ is a free rational Lie algebra on $\{x_1\gamma_2 F, x_2\gamma_2 F, \ldots, x_q\gamma_2 F\}$. This provides us with a realization of the free rational Lie algebra L as a graded Lie algebra, one given exactly by the Lazard Lie algebra of the free group F. Put

$$L_i = \frac{\gamma_i F}{\gamma_{i+1} F} \otimes \mathbf{Q} \quad \text{and} \quad L = \bigoplus_{i=1}^{\infty} L_i.$$

For any Lie algebra A, and any positive integer n, $\gamma_n A$ consists of linear combinations of all bracket arrangement of weight n in the elements of A. Thus,

$$\gamma_n L = \bigoplus_{k=1}^{\infty} \sum_{\substack{i_1+\cdots+i_n=k \\ \beta \text{ of weight } n}} \beta(L_{i_1}, L_{i_2}, \ldots, L_{i_n}). \tag{11}$$

Moreover, the grading of the free Lie algebra L has the property that

$$L_n = \sum_{\beta \text{ of weight } n} \beta(L_1, L_1, \ldots, L_1).$$

Let $\gamma_n L[k]$ denote the k^{th} homogeneous component of $\gamma_n L$.

Lemma 12.1 *Let n be a positive integer. For all $k \geq n$, $\gamma_n L[k] = L_k$.*

Proof: By equation (11),

$$\gamma_k L[k] = \sum_{\beta \text{ of weight } k} \beta(L_1, L_1, \ldots, L_1).$$

Thus, $\gamma_k L[k] = L_k$. If $k \geq n$, then $\gamma_k L$ is contained in $\gamma_n L$. Since $\gamma_m L = \bigoplus_i \gamma_m L[i]$ for any positive integer m, we have

$$L_k = \gamma_k L[k] \subseteq \gamma_n L[k] \subseteq L_k.$$

□

Corollary 12.2 *$gr(L) \cong L$.*

Proof: In consequence of Lemma 12.1,

$$\frac{\gamma_n L}{\gamma_{n+1} L} = \frac{0 \oplus \cdots \oplus 0 \oplus L_n \oplus L_{n+1} \oplus \cdots}{0 \oplus \cdots \oplus 0 \oplus 0 \oplus L_{n+1} \oplus \cdots} \cong L_n.$$

So for each coset $a_n + \gamma_{n+1} L$ in $\gamma_n L / \gamma_{n+1} L$, there exists a unique $\hat{a}_n \in L_n$ such that

$$\hat{a}_n + \gamma_{n+1} L = a_n + \gamma_{n+1} L.$$

The assignment $a_n + \gamma_{n+1} L \mapsto \hat{a}_n$ defines a vector space isomorphism from $gr(L)$ to L. By definition,

$$[a_n + \gamma_{n+1} L,\ b_m + \gamma_{m+1} L] = [a_n,\ b_m] + \gamma_{n+m+1} L.$$

If $a_n \in L_n$ and $b_m \in L_m$, then $[a_n, b_m] \in L_{n+m}$. Thus, our map is a Lie algebra homomorphism and $gr(L)$ is once again the free rational Lie algebra of rank q. □

13 Principal Ideals of Graded Lie Algebras

Given an arbitrary graded Lie algebra A, we wish to characterize the ideal of A generated by a single element.

Lemma 13.1 *Suppose A is a graded Lie Algebra over a field K*

$$A = \bigoplus_{i=i}^{\infty} A_i$$

and let u be an element of A. Then any bracket arrangement

$$\beta(a_{i_1}, a_{i_2}, \ldots, a_{i_j}, u, a_{i_{j+1}}, \ldots, a_{i_n}) \tag{12}$$

with $a_{i_t} \in A_{i_t}$ is equal to a right normed commutator

$$[b_{l_p}[b_{l_{p-1}} \cdots [b_{l_1}, u] \cdots]]$$

where $b_{l_t} \in A_{l_t}$ and

$$i_1 + i_2 + \cdots + i_n = l_1 + l_2 + \cdots + l_p.$$

Proof: The proof is by induction on $v = i_1 + i_2 + \cdots + i_n$. The statement clearly holds when this sum is equal to 1. Let $v > 1$, and assume that the claim holds for all bracket arrangements (12) for which the corresponding sum is less than v. Without loss of generality,

$$\beta(a_{i_1}, \ldots, u, \ldots, a_{i_n}) = \left[\beta_1(a_{i_1}, \ldots, u, \ldots, a_{i_q}),\ \beta_2(a_{i_{q+1}}, \ldots, a_{i_n})\right].$$

By the inductive hypothesis,

$$\beta_1(a_{i_1}, \ldots, u, \ldots, a_{i_q}) = [b_{l_p}[b_{l_{p-1}} \cdots [b_{l_1}, u] \cdots]]$$

with $i_1 + i_2 + \cdots + i_q = l_1 + l_2 + \cdots + l_p$. Thus,

$$\beta(a_{i_1}, \ldots, u, \ldots, a_{i_n}) = \left[[b_{l_p} \cdots [b_{l_1}, u] \cdots],\ \beta_2(a_{i_{q+1}}, \ldots, a_{i_n})\right].$$

Put $l_{p+1} = i_{q+1} + \cdots + i_n$ and $b_{l_{p+1}} = -\beta_2(a_{i_{q+1}}, \ldots, a_{i_n})$. Then

$$\beta(a_{i_1}, \ldots, u, \ldots, a_{i_n}) = [b_{l_{p+1}}[b_{l_p} \cdots [b_{l_1}, u] \cdots]]$$

with $b_{l_t} \in A_{l_t}$ for $t = 1, \ldots, p+1$ and

$$l_1 + l_2 + \cdots + l_{p+1} = i_1 + i_2 + \cdots + i_n.$$

□

Corollary 13.2

$$id_A(u) = \sum_{\substack{\forall (i_1, \ldots, i_n) \\ \forall n}} [A_{i_n}[A_{i_{n-1}} \cdots [A_{i_1}, u] \cdots]] \tag{13}$$

Proof: Let J denote the right-hand side of equation (13). Clearly $J \subseteq id_A(u)$. Moreover, by Lemma 13.1, $id_A(u) \subseteq J$. □

14 The Adjoint Action

We introduce the universal enveloping algebra $\mathcal{U}(L)$ of a Lie algebra L and show that the adjoint representation extends to an action of $\mathcal{U}(L)$ on L. For further details, we refer the reader to [10].

Given any Lie algebra L over a field K, there exists an associative algebra $\mathcal{U}(L)$ over K such that L is a Lie subalgebra of the commutation Lie algebra $\mathcal{U}(L)^{[\,,\,]}$. Moreover, the *Universal Enveloping Algebra* $\mathcal{U}(L)$ of L has the property that any Lie algebra homomorphism from L to a commutation Lie algebra $A^{[\,,\,]}$ of an associative algebra A over K can be uniquely extended to a homomorphism of associative algebras from $\mathcal{U}(L)$ to A.

If $l \in L$, then the mapping $ad(l)$ which is defined by left multiplication

$$ad(l) : x \mapsto [l, x]$$

is a linear transformation. Moreover, since multiplication is bilinear, the adjoint mapping which sends

$$l \mapsto ad(l)$$

is a vector space homomorphism from L into the algebra $\hom_K(L, L)$ of linear transformations of the vector space L. As a consequence of the Jacobi identity and anti-symmetry,

$$\begin{aligned} ad([l_1, l_2])(x) &= [[l_1, l_2], x] \\ &= [l_1, [l_2, x]] - [l_2, [l_1, x]] \\ &= ad(l_1) \circ ad(l_2)(x) - ad(l_2) \circ ad(l_1)(x) \\ &= \big(ad(l_1) \circ ad(l_2) - ad(l_2) \circ ad(l_1)\big)(x)\,. \end{aligned}$$

Hence, the adjoint mapping is a Lie algebra homomorphism from L to the commutation Lie algebra $\hom_K(L, L)^{[\,,\,]}$. Consequently, ad extends to a unique homomorphism θ of associative algebras

$$\theta : \mathcal{U}(L) \longrightarrow \hom_K(L, L).$$

Chapter II

Amalgamated Products of $\mathcal{U}$-Groups

15 Powers in Amalgamated Products

We define isolated and malnormal subgroups and demonstrate a relationship between the length of an element and the lengths of its powers in an amalgamated product over an isolated subgroup.

Definition: Let A be a group, $H \leq A$. H is said to be an *isolated subgroup* of A if for $a \in A$ and any positive integer $n \in \mathbf{Z}^+$, $a^n \in H$ implies that $a \in H$.

Lemma 15.1 *Suppose that $G = A \underset{H}{*} B$, where H is an isolated subgroup of both A and B. Let $g \in G$ and $n \in \mathbf{Z}^+$. If $l(g)$ is even, then $l(g^n) = n \cdot l(g)$.*

Proof: If $l(g) = k$, then, by Proposition 2.1, after choosing sets of coset representatives S_α and S_β, g has a unique normal form

$$g =_. s_1 s_2 \cdots s_k h.$$

Since k is even, s_k and s_1 are not elements of the same factor. Therefore,

$$g^n = s_1 s_2 \cdots s_{k-1}(s_k h) s_1 s_2 \cdots s_{k-1}(s_k h) \cdots s_1 s_2 \cdots s_{k-1}(s_k h)$$

is a strictly alternating product. By Proposition 2.2, $l(g^n) = n \cdot k$. □

The following comments apply to any amalgamated product of two groups. Suppose that $G = A \underset{H}{*} B$, where $H = A \cap B$. Let S_α and S_β be sets of coset representatives of A and B respectively, each containing 1. Suppose g is an element of G of odd length, $l(g) = k = 2r + 1$. Then we can *cyclically reduce* g by conjugating it to an element $\hat{g}$ of G such that $l(\hat{g})$ is either even or equal to 1. We see this as follows:

Suppose g has normal form

$$g = s_1 s_2 \cdots s_k h.$$

Observe that

$$s_1^{-1} g s_1 = s_2 \cdots s_{k-1} \left(s_k h s_1 \right). \tag{14}$$

Since k is odd, s_k and s_1 are elements of the same factor of G. Without loss of generality, we assume the factor to be A. Hence, the subproduct $(s_k h s_1)$ in equation (14) is interpreted as a product in A. It can be uniquely expressed as

$$(s_k h s_1) = s_1' h_1$$

where $s_1' \in S_\alpha$ and $h_1 \in H$. If $s_1' \neq 1$, then $s_1^{-1} g s_1$ has normal form

$$s_1^{-1} g s_1 = s_2 \cdots s_{k-1} s_1' h_1.$$

Hence, $l(s_1^{-1} g s_1) = k - 1$, which is even. Otherwise, if $s_1' = 1$, then

$$s_1^{-1} g s_1 = s_2 \cdots s_{k-1} h_1.$$

Since s_2 and s_{k-1} are both elements of B, the process repeats.

In general, there are two possibilities. Suppose there exists $1 \leq j \leq r$ such that

$$\begin{aligned}
s_k h s_1 &= h_1 \in H \\
s_{k-1} h_1 s_2 &= h_2 \in H \\
&\vdots \\
s_{k-j+2} h_{j-2} s_{j-1} &= h_{j-1} \in H \\
s_{k-j+1} h_{j-1} s_j &= s_j' h_j
\end{aligned}$$

with $s'_j \in S_\alpha \cup S_\beta$ and $s'_j \neq 1$. Then $\hat{g} = s_j^{-1} \cdots s_1^{-1} g s_1 \cdots s_j$ has normal form

$$\hat{g} = s_{j+1} \cdots s_{k-j} s'_j h_j$$

and $l(\hat{g}) = k - 2j + 1$. Otherwise,

$$s_{k-r+1} \cdots s_k h s_1 s_2 \cdots s_r = h_r \in H.$$

In this case, $\hat{g} = s_r^{-1} \cdots s_1^{-1} g s_1 \cdots s_r$ has normal form $s_{r+1} h_r$ and $\hat{g}$ has length 1.

We are now ready to prove the following lemma:

Lemma 15.2 *Suppose that $G = A \ast_H B$, where H is an isolated subgroup of both A and B. Let $g \in G$ and $n \in \mathbf{Z}^+$. If $l(g) \leq 1$, then $l(g) = l(g^n)$. Otherwise, $l(g^n) \geq l(g)$ and they have the same parity.*

Proof: Suppose that $l(g) \leq 1$. We may assume without loss of generality that $g \in A$. Since H is an isolated subgroup of A, $g^n \in H$ if and only if $g \in H$. Consequently, $l(g^n) = 0$ if and only if $l(g) = 0$. Thus, our first claim is proved.

If $l(g) = k$ is even, then the proposition follows from Lemma 15.1. Suppose then that k is odd; $k = 2r + 1$. Let g have normal form

$$g = s_1 s_2 \cdots s_k h$$

and let $\hat{g}$ be the cyclic reduction of g.

Case 1 $\hat{g} = s_{j+1} \cdots s_{k-j} s'_j h_j$ for some $j \leq r$.

Since $g = (s_1 \cdots s_j)\hat{g}(s_1 \cdots s_j)^{-1}$,

$$\begin{aligned} g^n &= g^{n-1} g \\ &= (s_1 \cdots s_j)\hat{g}^{n-1}(s_1 \cdots s_j)^{-1} g \\ &= s_1 \cdots s_j \left(s_{j+1} \cdots s_{k-j}(s'_j h_j)\right)^{n-1} s_{j+1} \cdots s_{k-1} \left(s_k h\right). \end{aligned} \tag{15}$$

The right-hand side of equation (15) is a strictly alternating product. Thus, Proposition 2.2 implies $l(g^n) = (n-1)(k - 2j + 1) + k$, which is greater than $l(g)$ and odd.

Case 2 $\hat{g} = s_{r+1}h_r$.

In this case,

$$\begin{aligned} g^n &= s_1 \cdots s_r \hat{g}^n s_r^{-1} \cdots s_1^{-1} \\ &= s_1 s_2 \cdots s_r \left(s_{r+1}h_r\right)^n s_r^{-1} \cdots s_2^{-1} s_1^{-1}. \qquad (16) \end{aligned}$$

$s_{r+1}h_r$ is an element of $A \backslash H$ or $B \backslash H$. Since H is isolated in both A and B, $(s_{r+1}h_r)^n$ is not contained in H. Consequently, the right-hand side of equation (16) is a strictly alternating product and $l(g^n) = 2r + 1 = k$. □

Definition: H is said to be a *malnormal subgroup* of A if for any $a \in A \backslash H$,

$$a^{-1}Ha \cap H = \{1\}.$$

Equivalently, given any $a \in A$, if there exists a non-trivial element h of H such that

$$a^{-1}ha \in H,$$

then $a \in H$.

Observe that malnormality implies isolation as follows: Suppose H is a malnormal subgroup of A and a is an element of A such that $a^n \in H$ for some positive integer n. Then $a^{-1}a^n a \in H$ and, by malnormality, $a \in H$.

16 Amalgamated Products with Unique Roots

We state several sets of conditions under which an amalgamated product of two $\mathcal{U}$-groups will be a $\mathcal{U}$-group. Theorems 16.2 and 16.5 are stated solely in terms of restrictions placed on the amalgamated subgroup. Theorem 16.3 makes some of the same assumptions about the amalgamated subgroup, but it is more general and is stated in terms of a property of the amalgamated product.

All of the theorems in this section require the following lemma:

Lemma 16.1 *Suppose A and B are $\mathcal{U}$-groups, and that $G = A \underset{H}{*} B$ where H is an isolated subgroup of both A and B. Suppose g and f are elements of G, and n is a positive integer such that*

$$g^n = f^n.$$

If $l(f) \leq 1$, then $f = g$. If $l(f)$ is even, then $f = gh$ for some $h \in H$.

Proof: By Proposition 2.1, having chosen appropriate sets of coset representatives, g and f have unique normal forms,

$$g = s_1 s_2 \cdots s_k h_g \qquad \text{and} \qquad f = t_1 t_2 \cdots t_l h_f.$$

Case 1 $l(f) = l \leq 1$.

Without loss of generality, we may assume that $f \in A$. We will demonstrate that g must be an element of A as well. The length of f^n is less than or equal to 1. Since $g^n = f^n$, the length of g^n is less than or equal to 1, and by Lemma 15.2, $l(g) \leq 1$. Hence, g is an element of A or B. If $g \in B$, then $g^n \in A \cap B = H$ and by isolation, $g \in H$. Thus, f and g are elements of the $\mathcal{U}$-group A, and it follows that $f = g$.

Case 2 $l(f) = l$ is even.

By Lemma 15.2, k is even, and by Lemma 15.1,

$$n \cdot l = l(f^n) = l(g^n) = n \cdot k.$$

Hence, $l = k$. The normal form of f^n begins with $t_1 t_2 \cdots t_k$ and the normal form of g^n begins with $s_1 s_2 \cdots s_k$. Thus, by the uniqueness of normal forms,

$$t_1 = s_1,\ t_2 = s_2, \ldots,\ t_k = s_k$$

and $g^{-1} f = h_g^{-1} h_f \in H$. □

Theorem 16.2 *Suppose A and B are $\mathcal{U}$-groups, and that $G = A \underset{H}{*} B$ where H is an isolated subgroup of both A and B. If H is a malnormal subgroup of B, then G is a $\mathcal{U}$-group.*

Proof: Let $n \in \mathbf{Z}^+$, and suppose that g and f are elements of G such that

$$g^n = f^n.$$

Let $l(f) = k$. By Lemma 16.1, if $k \leq 1$, then $g = f$. Suppose $k > 1$. Let

$$f = s_1 s_2 \cdots s_k h_f$$

be the normal form of f. If k is even, by Lemma 16.1, $f = gh$ for some $h \in H$ and g has normal form

$$g = s_1 s_2 \cdots s_k h_g.$$

Since $g^{-1} f^n = g^{n-1}$, these two must have the same normal form. Observe that

$$g^{-1} f^n = h s_1 s_2 \cdots s_k h_f s_1 s_2 \cdots s_k h_f \cdots s_1 s_2 \cdots s_k h_f,$$

$$g^{n-1} = s_1 s_2 \cdots s_k h_g s_1 s_2 \cdots s_k h_g \cdots s_1 s_2 \cdots s_k h_g.$$

In particular, this implies that there exist $h', h'' \in H$ such that $h s_1 = s_1 h'$ and $h' s_2 = s_2 h''$. Thus,

$$s_1^{-1} h s_1 \in H \qquad \text{and} \qquad s_2^{-1} h' s_2 \in H.$$

Since either s_1 or s_2 is in $B \backslash H$, and H is malnormal in B, it follows that $h = 1$ and therefore $f = g$.

If k is odd, then let $\hat{f} = u^{-1} f u$ be the cyclic reduction of f. Put $\hat{g} = u^{-1} g u$. Then

$$\hat{f}^n = \hat{g}^n.$$

Since $l(\hat{f})$ is either even or less than or equal to 1, $\hat{f} = \hat{g}$ and consequently $f = g$. □

Theorem 16.3 *Suppose A and B are $\mathcal{U}$-groups, and that $G = A \underset{H}{*} B$ where H is an isolated subgroup of both A and B. If there exists a $\mathcal{U}$-group U and a homomorphism*

$$\psi : G \longrightarrow U$$

such that ψ is injective on H, then G is a $\mathcal{U}$-group.

Proof: Suppose $f, g \in G$ with $f^n = g^n$ for some $n \in \mathbf{Z}^+$. By Lemma 16.1, we need only consider the case that $l(f) > 1$. Moreover, if $l(f)$ is odd, we can replace f by its cyclically reduced conjugate

$\hat{f} = u^{-1}fu$ and g by $\hat{g} = u^{-1}gu$ and apply the even case or the length 1 case. Thus, we may assume $l(f)$ is even. By Lemma 16.1, $f = gh$ for some $h \in H$. By hypothesis, $g^n = (gh)^n$; hence,

$$\psi(g)^n = (\psi(g)\psi(h))^n.$$

U is a $\mathcal{U}$-group; therefore, $\psi(g) = \psi(g)\psi(h)$ and $\psi(h) = 1$. Since ψ is injective on H, $h = 1$, and thus, $f = g$. □

Corollary 16.4 *Let A be a $\mathcal{U}$-group. Suppose φ is an isomorphism*

$$\varphi : A \longrightarrow \varphi(A).$$

If H is an isolated subgroup of A and $G = A \mathop{*}_{H} \varphi(A)$ where H is amalgamated with its image under φ, then G is a $\mathcal{U}$-group.

In particular, this applies to a free group F amalgamated with an isomorphic copy of itself $\varphi(F)$ over a cyclic subgroup $gp(a) = \varphi(gp(a))$ where a is not a proper power in F. The fact that such groups are $\mathcal{U}$-groups is already known since it was proved by G. Baumslag that they are residually free [4].

Theorem 16.5 *Suppose A and B are $\mathcal{U}$-groups, and that $G = A \mathop{*}_{H} B$ where H is an isolated subgroup of both A and B. If H is central in A and B, then G is a $\mathcal{U}$-group.*

Proof: Once again, we may confine ourselves to the case that $l(f)$ is even. By Lemma 16.1, $f = gh$ for some $h \in H$. We have

$$g^n = f^n = (gh)^n = g^n h^n;$$

therefore, h^n is trivial. Since H is torsion-free, $h = 1$ and $f = g$. □

Corollary 16.6 *The amalgamated product of two torsion-free abelian groups over an isolated subgroup is a $\mathcal{U}$-group.*

Chapter III

Amalgamated Products Over Cyclic Subgroups

17 Torsion-Free Nilpotent Groups

We define compatible-$\mathrm{R}\mathcal{F}_p$-filtrations, and employ them to prove several results on amalgamated products of finitely generated torsion-free nilpotent groups.

Let $\mathcal{M}$ be a class of groups, and suppose A is a $\mathrm{R}\mathcal{M}$-group. Suppose $\{A_i\}_{i=1}^{\infty}$ is a descending chain of normal subgroups of A

$$A = A_1 \geq A_2 \geq \cdots \geq A_i \geq \cdots$$

with the following properties:

1. A/A_i is an $\mathcal{M}$-group for all $i = 1, 2, \ldots,$
2. $\cap_{i=1}^{\infty} A_i = 1.$

Then we term $\{A_i\}$ a $\mathrm{R}\mathcal{M}$-*filtration* for A. Let G be an amalgamated product

$$G = \{\, A * B\,;\; H \overset{\varphi}{=} K \,\}. \tag{17}$$

If $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ are $\mathrm{R}\mathcal{M}$-filtrations for A and B respectively such that φ induces an isomorphism between HA_i/A_i and KB_i/B_i for all $i = 1, 2, \ldots$, then we say that $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ are *compatible-*$\mathrm{R}\mathcal{M}$*-filtrations*, with respect to the amalgam (17).

Let $\mathcal{T}$ denote the class of finitely generated torsion-free nilpotent groups. In [3], G. Baumslag proves that

Theorem 17.1 *If* A, $B \in \mathcal{T}$ *and* G *is the amalgamated product*

$$G = \{\, A * B\,;\; H \overset{\varphi}{=} K \,\},$$

then $\{A^{p^i}\}_{i=1}^{\infty}$ *and* $\{B^{p^i}\}_{i=1}^{\infty}$ *are compatible-*$\mathrm{R}\mathcal{F}$*-filtrations for almost all primes* p.

Since A/A^{p^i} is a finitely generated nilpotent group in which every element has order a power of p, A/A^{p^i} is a finite p-group for each i. Thus, $\{A^{p^i}\}_{i=1}^{\infty}$ and $\{B^{p^i}\}_{i=1}^{\infty}$ are compatible-$\mathrm{R}\mathcal{F}_p$-filtrations for almost all primes p. We also require the following result of G. Higman [9]:

Theorem 17.2 *The amalgamated product of two finite p-groups over a cyclic subgroup is* $\mathrm{R}\mathcal{F}_p$.

We are now ready to prove

Theorem 17.3 *The amalgamated product of two finitely generated torsion-free nilpotent groups over a cyclic subgroup is a free extension of a* $\mathrm{R}\mathcal{F}_p$*-group for almost all primes* p.

Proof: Let $A, B \in \mathcal{T}$, $a \in A$, $b \in B$. Put $H = gp(a)$, $K = gp(b)$, and suppose that φ is an isomorphism from H onto K with $\varphi(a) = b$. Put

$$G = \{\, A * B\,;\; H \overset{\varphi}{=} K \,\}.$$

As we know from Theorem 17.1, $\{A^{p^i}\}_{i=1}^{\infty}$ and $\{B^{p^i}\}_{i=1}^{\infty}$ are compatible-$\mathrm{R}\mathcal{F}_p$-filtrations for almost all primes p. Fix such a prime p. For $i = 1, 2, \ldots$, put

$$G_i = \{\, A/A^{p^i} * B/B^{p^i}\,;\; HA^{p^i}/A^{p^i} \overset{\varphi}{=} KB^{p^i}/B^{p^i} \,\}.$$

Let θ_i denote the homomorphism from G onto G_i which extends the canonical homomorphisms from A onto A/A^{p^i} and from B onto B/B^{p^i}. By Theorem 17.2, G_i is $\mathrm{R}\mathcal{F}_p$ for all i.

Suppose u is a non-trivial element of A. Since $\cap_{i=1}^{\infty} A^{p^i} = 1$, there exists j such that $u \notin A^{p^j}$. Therefore, $\theta_j(u) = u_j$ is not trivial. Since G_j is $\mathrm{R}\mathcal{F}_p$, there is a normal subgroup N_{u_j} of G_j, which does not contain u_j, with G_j/N_{u_j} a finite p-group. Let $K = ker\theta_j$. Under the isomorphism between G_j and G/K, N_{u_j} corresponds to a subgroup N_u/K, where N_u is a normal subgroup of G containing K. Since u_j is not contained in N_{u_j}, it follows that u is not contained in N_u. Moreover, G/N_u is isomorphic to G_j/N_{u_j}. Hence, G/N_u is a finite p-group.

Let N be the intersection of all normal subgroups of G of index a power of p. Then N is a normal subgroup of G and, as we see from the last paragraph, $N \cap A = 1 = N \cap B$. By a theorem of H. Neumann (see for example [16, p. 212]), it follows that N is a free group. Next consider G/N. We claim that G/N is $\mathrm{R}\mathcal{F}_p$. To see this, let gN be an element of G/N, and suppose gN is contained in every normal subgroup $\tilde{M}$ of G/N of index a power of p. Every such subgroup $\tilde{M}$ is of the form M/N, where M is a normal subgroup of G. Note that $[G : M] = [G/N : \tilde{M}]$. Consequently, modulo N, g is contained in every normal subgroup M of G of index a power of p. Hence, g is contained in N, and gN is trivial. Therefore, G/N is $\mathrm{R}\mathcal{F}_p$. □

If we add the condition that the amalgamated cycle is isolated, we get a stronger result.

Theorem 17.4 *The amalgamated product of two finitely generated torsion-free nilpotent groups over an isolated cyclic subgroup is* $\mathrm{R}\mathcal{F}_p$ *for almost all primes* p.

The proof of this theorem will be built-up from the following results. In a variation on a theorem of G. Baumslag [3] we have

Theorem 17.5 *Let* $\mathcal{M}$ *and* $\mathcal{C}$ *be classes of groups. Suppose that* $\{A_i\}$ *and* $\{B_i\}$ *are compatible-*$\mathrm{R}\mathcal{M}$*-filtrations for the amalgamated product*

$$G = \{\, A * B \,;\, H \overset{\varphi}{=} K \,\}.$$

In addition, suppose that $G_i = \{\, A/A_i * B/B_i \,;\, HA_i/A_i \overset{\varphi}{=} KB_i/B_i \,\}$ *is a* $\mathcal{C}$*-group for all* i*. If*

$$\bigcap_{i=1}^{\infty} A_i H = H \quad \text{and} \quad \bigcap_{i=1}^{\infty} B_i K = K, \tag{18}$$

then G *is* $\mathrm{R}\mathcal{C}$*. If* $\mathcal{C}$ *is a residual property, then* G *is a* $\mathcal{C}$*-group.*

Proof: Let θ_i be the homomorphism from G onto G_i which extends the canonical homomorphisms mapping A onto A/A_i and B onto B/B_i. Since G_i is a $\mathcal{C}$-group for all i, it is sufficient to show that for any non-trivial element g in G, there exists j such that $\theta_j(g)$ is a non-trivial element of G_j.

If g is contained in A or B, then the filtrations $\{A_i\}$ and $\{B_i\}$ insure that this is the case. Let us assume then that g is not in A or B. By Propositions 2.1 and 2.2, there is a unique positive integer $n > 1$ such that g has an expression as a strictly alternating product in A and B

$$g = u_1 u_2 \cdots u_n.$$

Each u_i is an element of either $A \backslash H$ or $B \backslash K$. Therefore, by condition (18), we can choose j sufficiently large that

$$\theta_j(g) = \theta_j(u_1)\theta_j(u_2)\cdots\theta_j(u_n)$$

is a strictly alternating product in A/A_j and B/B_j. By Corollary 2.3, $\theta_j(g)$ is a non-trivial element of G_j, and we are done. □

As an application of this theorem, we have the following:

Corollary 17.6 *Let A, $B \in \mathcal{T}$, $a \in A$, $b \in B$, and let G be the amalgamated product*

$$G = \{\, A * B\,;\ gp(a) \overset{\varphi}{=} gp(b)\,\}.$$

If $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ are compatible-$\mathrm{R}\mathcal{F}_p$-filtrations such that

$$\bigcap_{i=1}^{\infty} A_i gp(a) = gp(a) \quad \text{and} \quad \bigcap_{i=1}^{\infty} B_i gp(b) = gp(b)\,, \tag{19}$$

then G is $\mathrm{R}\mathcal{F}_p$.

Proof: By Theorem 17.2, G_i is $\mathrm{R}\mathcal{F}_p$ for all i. □

It only remains to state the following result of G. Baumslag [3].

Theorem 17.7 *Let $A \in \mathcal{T}$ and let H be an isolated subgroup of A. Then*

$$\bigcap_{i=1}^{\infty} HA^{p^i} = H$$

for almost all primes p.

The proof of our theorem is now immediate.

Proof of Theorem 17.4: Let $A, B \in \mathcal{T}$, and $a \in A$, $b \in B$ where $gp(a)$ and $gp(b)$ are isomorphic isolated subgroups. Let G be the amalgamated product

$$G = \{\, A * B\,;\; gp(a) \overset{\varphi}{=} gp(b)\,\}.$$

As we know from Theorem 17.1, for almost all primes p, $\{A^{p^i}\}_{i=1}^{\infty}$ and $\{B^{p^i}\}_{i=1}^{\infty}$ are compatible-$\mathrm{R}\mathcal{F}_p$-filtrations for the amalgam. Moreover, by Theorem 17.7, for a possibly smaller collection, but still for all but finitely many primes p, these filtrations satisfy condition (19). Thus, Corollary 17.6 applies, and we obtain the desired result. □

18 Free Groups

We use Theorem 17.4 to give a proof of the already known result (see [3, page 195]) that the amalgamated product of two free groups over a maximal cycle is residually nilpotent. We also state certain conditions under which such an amalgamated product will be residually *torsion-free* nilpotent.

It follows directly from a theorem of Baumslag [2, page 277] that

Lemma 18.1 *If F is a free group and f is an element of F which is not a proper power, then there exists a positive integer n such that f is not an element of $\gamma_{n+1}F$, and f is not a power modulo $\gamma_{n+1}F$.*

Corollary 18.2 *For such n, $gp(f\gamma_{n+1}F)$ is an isolated subgroup of $F/\gamma_{n+1}F$.*

All we require now is the following:

Lemma 18.3 *If A is a free group and a is an element of A which is not a proper power, then*

$$\bigcap_{i=1}^{\infty} gp(a)\,\gamma_i A = gp(a)\,. \tag{20}$$

Proof: Choose n so that a is not contained in $\gamma_{n+1}A$. For any element g contained in the left-hand side of equation (20), there is a sequence of integers $\{\alpha_i\}$, and a sequence of $d_i \in \gamma_{n+i}A$ such that

$$g = a^{\alpha_i} d_i \qquad (i \geq 1). \tag{21}$$

It follows that $a^{\alpha_i-\alpha_j}$ is contained in $\gamma_{n+i}A$ for all $j > i$. Since $A/\gamma_{n+i}A$ is torsion-free, and a is not contained in $\gamma_{n+i}A$, it must be that $\alpha_i = \alpha_j$ for all $j > i$. But then equation (21) implies that $d_j = d_i$ for all $j > i$. Hence, $d_i = d_1 \in \cap_{k=n+1}^{\infty}\gamma_k A = \cap_{k=1}^{\infty}\gamma_k A = 1$, and $g \in gp(a)$. □

Theorem 18.4 *The amalgamated product of two free groups over maximal cycles is residually nilpotent.*

Proof: Let A and B be free groups with elements a and b respectively where a and b are not proper powers. To simplify notation, let $A_n = \gamma_n A$ and $B_n = \gamma_n B$ for all n. By Lemma 18.1, we can choose n sufficiently large that a is not a proper power modulo A_{n+1} and b is not a proper power modulo B_{n+1}. By Corollary 18.2 $gp(aA_{n+i}) = gp(a)\,A_{n+i}/A_{n+i}$ and $gp(bB_{n+i}) = gp(b)\,B_{n+i}/B_{n+i}$ are isolated subgroups of A/A_{n+i} and B/B_{n+i} respectively for all $i = 1, 2, \ldots$. Moreover, since the latter are torsion-free, these cyclic subgroups are isomorphic via $\varphi : a \mapsto b$. Thus, for all $i \geq 1$ we can form the amalgamated products

$$G_i = \{\, A/A_{n+i} * B/B_{n+i}\,;\; gp(a)\,A_{n+i}/A_{n+i} \overset{\varphi}{=} gp(b)\,B_{n+i}/B_{n+i}\,\}.$$

By Theorem 17.4, for each i, G_i is $\mathrm{R}\mathcal{F}_p$ for almost all primes p and is therefore $\mathrm{R}\mathcal{N}$. We now apply Lemma 18.3 and Theorem 17.5. □

It is a long-standing conjecture that the amalgamated product of two free groups over a maximal cycle is residually torsion-free nilpotent. In the remainder of this section, we observe certain conditions under which this will be the case. We will require the following definitions and results. Recall that in a $\mathrm{R}\mathcal{N}$-group G, the *weight* of a non-trivial element g is the largest natural number n such that $g \in \gamma_n G$.

Definition: An element g of a $\mathrm{R}\mathcal{N}$-group G will be termed *Labute primitive* if $g\gamma_{e+1}G$ is not a proper power in $\gamma_e G/\gamma_{e+1}G$ where e is the weight of g.

Theorem 18.5 (Labute [13]) *Let r be a Labute primitive element of weigth e in the free group F on $x_1, x_2, \ldots, x_q$. If G is a 1-relator group with presentation*

$$G = \langle x_1,\, x_2, \ldots, x_q\,;\; r\rangle,$$

then the Lazard Lie ring gr(G) *is isomorphic to* gr$(F)/id(r\gamma_{e+1}F)$. *Moreover, it is additively free abelian, and the rank of the* n^{th} *homogeneous term is given by the formula*

$$l(n,e,q) = \frac{1}{n}\sum_{d|n}\left[\sum_{0\le i\le[\frac{d}{e}]} (-1)^i \frac{d}{d+i-ei}\binom{d+i-ie}{i} q^{d-ei}\right]. \tag{22}$$

Definition: Suppose g is a non-trivial element of weight e in a free group G. Since $\gamma_e G/\gamma_{e+1}G$ is a free abelian group, there exists a largest natural number m such that $g\gamma_{e+1}G$ is an m^{th}-power in $\gamma_e G/\gamma_{e+1}G$. We term this the *maximum exponent* of g.

Being Labute primitive is equivalent to having maximum exponent 1. Observe that if $\beta_1, \beta_2, \ldots, \beta_p$ are distinct basic commutators of weight e in any basic sequence in the free generators of G, and $g = \beta_1^{m_1}\beta_2^{m_2}\cdots\beta_p^{m_p}$, then the maximum exponent of g is equal to the greatest common divisor of the integers $m_1, m_2, \ldots, m_p$, which we denote by $gcd(m_1, m_2, \ldots, m_p)$.

Lemma 18.6 *Let* $a \in A$, $b \in B$ *where* A *and* B *are free groups. Then* ab^{-1} *is Labute primitive as an element of* $F = A * B$ *if and only if* a *and* b *meet one of the following conditions:*

1. *the one of lower weight is Labute primitive, or*
2. *the weight of* a *is equal to the weight of* b*, but their maximum exponents are relatively prime.*

Proof: Let A have free basis $X = \{x_1, x_2, \ldots\}$ and B free basis $Y = \{y_1, y_2, \ldots\}$. Put $F = A * B$. Then F is free on $X \cup Y$. We select a basic sequence $\{\beta_i\}$ in $X \cup Y$ with

$$x_1 < x_2 < \cdots < y_1 < y_2 < \cdots.$$

We observe that $\{\beta_i\}$ contains basic sequences $\{c_j\}$ in X and $\{d_l\}$ in Y as disjoint subsequences. Moreover, we may assume $\{\beta_i\}$ is ordered so that if $\beta_s = c_j$ and $\beta_t = d_l$ are both of weight n, then $s < t$.

Let $wt_A(a) = e$, $wt_B(b) = f$, and assume without loss of generality that $wt_A(a) \le wt_B(b)$. Let $r = ab^{-1}$ and put $\rho = wt_F(r)$. We can uniquely express a, b and r as follows:

$$a \equiv c_{j_1}^{\gamma_1} c_{j_2}^{\gamma_2} \cdots c_{j_n}^{\gamma_n} \quad \text{modulo } \gamma_{e+1}A \text{ or } \gamma_{e+1}F$$

$$\begin{aligned} b^{-1} &\equiv d_{l_1}^{\delta_1} d_{l_2}^{\delta_2} \cdots d_{l_k}^{\delta_k} \quad \text{modulo } \gamma_{f+1}B \text{ or } \gamma_{f+1}F \\ r &\equiv \beta_{i_1}^{m_1} \beta_{i_2}^{m_2} \cdots \beta_{i_p}^{m_p} \quad \text{modulo } \gamma_{\rho+1}F \end{aligned}$$

where c_{j_t}, d_{l_t}, and β_{i_t} are basic commutators of weight e, f, and ρ respectively, all contained in the basic sequence $\{\beta_i\}$, with $c_{j_1} < \cdots < c_{j_n} < d_{l_1} < \cdots < d_{l_k}$ and $\beta_{i_1} < \beta_{i_2} < \cdots < \beta_{i_p}$.

Case 1 $wt_A(a) < wt_B(b)$.

In this case, $r \equiv c_{j_1}^{\gamma_1} c_{j_2}^{\gamma_2} \cdots c_{j_n}^{\gamma_n}$ modulo $\gamma_{e+1}F$. The basic commutators of weight e in the basic sequence $\{\beta_i\}$ in $X \cup Y$ are linearly independent modulo $\gamma_{e+1}F$. Consequently, $c_{j_1}^{\gamma_1} c_{j_2}^{\gamma_2} \cdots c_{j_n}^{\gamma_n}$ is not contained in $\gamma_{e+1}F$, and $wt_F(r) = e$. Thus,

$$\beta_{i_1}^{m_1} \beta_{i_2}^{m_2} \cdots \beta_{i_p}^{m_p} \equiv c_{j_1}^{\gamma_1} c_{j_2}^{\gamma_2} \cdots c_{j_n}^{\gamma_n} \quad \text{modulo } \gamma_{e+1}F,$$

where both sides of this congruence consist of ascending products of basic commutators of weight e in the basic sequence $\{\beta_i\}$. By linear independence, these products are identical; that is, $n = p$, $c_{j_t} = \beta_{i_t}$, and $\gamma_t = m_t$. The maximum exponent of r and the maximum exponent of a are both $gcd(m_1, \ldots, m_p)$. Hence, r is Labute primitive in F if and only if a is Labute primitive in A.

Case 2 $wt_A(a) = wt_B(b)$.

Then $r \equiv c_{j_1}^{\gamma_1} \cdots c_{j_n}^{\gamma_n} d_{l_1}^{\delta_1} \cdots d_{l_n}^{\delta_k}$ modulo $\gamma_{e+1}F$. As before, by linear independence, $wt_F(r) = e$. We have

$$\beta_{i_1}^{m_1} \beta_{i_2}^{m_2} \cdots \beta_{i_p}^{m_p} \equiv c_{j_1}^{\gamma_1} \cdots c_{j_n}^{\gamma_n} d_{l_1}^{\delta_1} \cdots d_{l_k}^{\delta_k} \quad \text{modulo } \gamma_{e+1}F, \tag{23}$$

where both sides of this congruence consist of ascending products of basic commutator of weight e in the basic sequence $\{\beta_i\}$. By linear independence, the right-hand side and left-hand side of congruence (23) must be identical. The maximum exponents of a and b are $gcd(m_1, m_2, \ldots, m_n)$ and $gcd(m_{n+1}, m_{n+2}, \ldots, m_p)$ respectively. Since

$$gcd(m_1, \ldots, m_p) = gcd(gcd(m_1, \ldots, m_n), gcd(m_{n+1}, \ldots, m_p)),$$

the maximum exponents of a and b are relatively prime if and only if r is Labute primitive. □

Corollary 18.7 *The amalgamated product of two free groups over maximal cycles will be residually torsion-free nilpotent if the generators of the amalgamated cycles satisfy either condition of Lemma 18.6.*

Proof: Let $a \in A$, $b \in B$ where A and B are free groups with a and b not proper powers. If either the one of lower weight is Labute primitive, or their weights are equal but their maximum exponents are relatively prime, then by Lemma 18.6 ab^{-1} is Labute primitive as an element of the free group $F = A * B$. By Theorem 18.4, the amalgamated product $G = \{A * B\,;\ gp(a) = gp(b)\}$ is residually nilpotent. By Theorem 18.5 (Labute), $G/\gamma_n G$ is torsion-free for all n. Thus, the lower central series of G is a residually torsion-free nilpotent-filtration of G. □

Chapter IV

Invariants of 1-Relator Groups

In the following chapter we prove a generalization of Theorem 18.5 (Labute) which demonstrates an interesting invariant of 1-relator groups. The proof relies on the theory of nilpotent $\mathcal{D}$-groups and Lie algebras.

Let G be a finitely generated 1-relator group

$$G = \langle x_1,\ x_2, \ldots,\ x_q\ ;\ r \rangle$$

and let $F(X)$ denote the free group on $X = \{x_1,\ x_2, \ldots,\ x_q\}$. As Magnus showed, free groups are residually nilpotent [18]. Hence, there exists a unique positive integer e such that

$$r \in \gamma_e F(X), \quad r \notin \gamma_{e+1} F(X).$$

As defined earlier, e is termed the weight of r and we write $wt(r) = e$. In the following, we will prove that for any positive integer n, the torsion-free rank of the n^{th} lower central factor of G,

$$\gamma_n G / \gamma_{n+1} G,$$

is given by formula (22) and thus depends only on n, q, and e.

19 Quotients and Completions

In Section 7, we stated that any torsion-free nilpotent group can be completed to a nilpotent $\mathcal{D}$-group. The first thing we shall require for our proof of the invariance of torsion-free rank is a more detailed understanding of completions of nilpotent groups. We begin with a result relating completions of quotients and quotients of completions.

Lemma 19.1 *If B is a finitely generated torsion-free nilpotent group and Y is any subset of B, then there exists a homomorphism*

$$\varphi : B/gp_B(Y) \longrightarrow mB/id_{mB}(Y)$$

with $ker\varphi = \tau(B/gp_B(Y))$.

Proof: Let H denote the quotient of $B/gp_B(Y)$ by its torsion subgroup and let D denote the nilpotent $\mathcal{D}$-group $mB/id_{mB}(Y)$. We have the following commutative diagram:

$$\begin{tikzcd}
B \arrow[r, "\pi_1"] \arrow[d, "i"'] & B/gp_B(Y) \arrow[r, "\pi_2"] \arrow[d, "\varphi"'] & H \arrow[d, "j"] \\
mB \arrow[r, "q"] & D \arrow[r, "\pi"] & mH
\end{tikzcd}$$

By Lemma 7.1, the composition $\pi_2 \circ \pi_1$ induces a homomorphism $m(\pi_2 \circ \pi_1)$ mapping mB to mH. Since Y is contained in the kernel of $m(\pi_2 \circ \pi_1)$, by definition $id_{mB}(Y)$ is contained in the kernel of $m(\pi_2 \circ \pi_1)$ as well. Consequently, this homomorphism factors through D as $m(\pi_2 \circ \pi_1) = \pi \circ q$. Similarly, since Y is contained in the kernel of $q \circ i$, this composition factors through $B/gp_B(Y)$. Thus, we obtain a homomorphism φ mapping $B/gp_B(Y)$ to D and, since π_1 is surjective,

$$\pi \circ \varphi = j \circ \pi_2 \tag{24}$$

We have only to identify $ker\varphi$. Since D is torsion-free, $\tau(B/gp_B(Y))$ is contained in the kernel of φ. Hence, φ factors through H.

$$\begin{tikzcd}
B/gp_B(Y) \arrow[r, "\pi_2"] \arrow[d, "\varphi"'] & H \arrow[d, "j"] \arrow[dl, "\phi"'] \\
D \arrow[r, shift left, "\pi"] & mH \arrow[l, shift left, "m\phi"]
\end{tikzcd}$$

Thus, we have $\phi : H \longrightarrow D$ with

$$\varphi = \phi \circ \pi_2. \tag{25}$$

By Lemma 7.1, ϕ extends to a homomorphism $m\phi : mH \longrightarrow D$ and

$$m\phi \circ j = \phi. \tag{26}$$

It is clear from equation (26) that if $m\phi$ is injective, then so is ϕ, in which case $ker\varphi = \tau(B/gp_B(Y))$ as desired. In fact, we will show that $m\phi$ is an isomorphism. By equations (25) and (24) $\pi \circ \phi \circ \pi_2 = j \circ \pi_2$. Since π_2 is surjective, this implies that

$$\pi \circ \phi = j. \tag{27}$$

We are now in a position to see that π and $m\phi$ are mutual inverses. As a consequence of equations (26) and (27), $\pi \circ m\phi \circ j = j$ and $m\phi \circ \pi \circ \phi = \phi$. The images of j and ϕ generate mH and D respectively as $\mathcal{D}$-groups. Hence, $\pi \circ m\phi$ and $m\phi \circ \pi$ are the identity mappings on these respective groups. □

Corollary 19.2

$$m\left(\frac{B}{gp_B(Y)} \Big/ \tau\left(\frac{B}{gp_B(Y)}\right)\right) \cong \frac{mB}{id_{mB}(Y)}.$$

Corollary 19.3 *If $B/gp_B(Y)$ is torsion-free, then*

$$m\left(\frac{B}{gp_B(Y)}\right) \cong \frac{mB}{id_{mB}(Y)}.$$

20 Lower Central Factors

For any group G, the factors of consecutive terms of the lower central series are abelian groups. Given an abelian group A, $tfr(A)$ will denote the torsion-free rank of A, that is the number of infinite cycles in the direct sum decomposition. If A is torsion-free, then $rank(A) = tfr(A)$. In this section, we establish the equality of the torsion-free ranks of the lower central factors of a torsion-free nilpotent group and the corresponding ranks of the lower central factors of its $\mathcal{D}$-completion.

Lemma 20.1 *Let G be a group. Suppose H is a normal subgroup of G such that all elements of H have finite order. If $B = G/H$, then*

$$tfr\left(\frac{\gamma_n B}{\gamma_{n+1} B}\right) = tfr\left(\frac{\gamma_n G}{\gamma_{n+1} G}\right)$$

for all positive integers n.

Proof: Since $B = G/H$,

$$\frac{\gamma_n B}{\gamma_{n+1} B} \cong \frac{\gamma_n G}{\gamma_{n+1} G} \Big/ \left(\frac{\gamma_n G}{\gamma_{n+1} G} \bigcap \frac{H\,\gamma_{n+1} G}{\gamma_{n+1} G}\right). \tag{28}$$

Let $\mathcal{T}$ denote the intersection of $\gamma_n G/\gamma_{n+1} G$ with $H\gamma_{n+1} G/\gamma_{n+1} G$. Since $\gamma_n G/\gamma_{n+1} G$ is an abelian group, it is a $\mathbf{Z}$-module and possesses a direct sum decomposition

$$\frac{\gamma_n G}{\gamma_{n+1} G} = U \oplus V$$

where U is a free $\mathbf{Z}$-module and V is a torsion module. All elements of H are of finite order in G, therefore $\mathcal{T} \subseteq V$. Hence, equation (28) implies

$$\frac{\gamma_n B}{\gamma_{n+1} B} \cong U \oplus \frac{V}{\mathcal{T}}.$$

Thus, we see that

$$tfr\left(\frac{\gamma_n B}{\gamma_{n+1} B}\right) = rank(U) = tfr\left(\frac{\gamma_n G}{\gamma_{n+1} G}\right).$$

□

Corollary 20.2 *Let G be a group, c a positive integer, and let B denote the quotient of $G/\gamma_{c+1} G$ by its torsion subgroup. Then*

$$tfr\left(\frac{\gamma_n B}{\gamma_{n+1} B}\right) = tfr\left(\frac{\gamma_n G}{\gamma_{n+1} G}\right)$$

for all n less than c.

Proof: Put $G_c = G/\gamma_{c+1}G$. If n is less than c, then $\gamma_n G_c/\gamma_{n+1}G_c$ is isomorphic to $\gamma_n G/\gamma_{n+1}G$. By Lemma 20.1

$$tfr\left(\frac{\gamma_n B}{\gamma_{n+1}B}\right) = tfr\left(\frac{\gamma_n G_c}{\gamma_{n+1}G_c}\right) = tfr\left(\frac{\gamma_n G}{\gamma_{n+1}G}\right).$$

□

Lemma 20.3 *If B is a finitely generated torsion-free nilpotent group, then*

$$tfr\left(\frac{\gamma_n B}{\gamma_{n+1}B}\right) = rank\left(\frac{\gamma_n mB}{\gamma_{n+1}mB}\right). \tag{29}$$

Proof: The proof is by induction on the nilpotency class c. If $c = 1$, then B is abelian. By Lemma 7.1, mB is abelian as well. Thus, we need only verify that $tfr(B)$ is equal to $rank(mB)$. Since B is torsion-free, it is a finitely generated free abelian group and is isomorphic to $\mathbf{Z}^{\mathrm{N}}$ where N is the torsion-free rank of B. Then mB is isomorphic to $m(\mathbf{Z}^{\mathrm{N}})$, which is just $\mathbf{Q}^{\mathrm{N}}$.

Next, let $c > 1$ and assume the proposition is true for nilpotent groups of class less than c. We will begin by showing that equation (29) holds for any n strictly less than c. Let H denote the quotient of $B/\gamma_c B$ by its torsion subgroup. By Corollary 20.2

$$tfr\left(\frac{\gamma_n H}{\gamma_{n+1}H}\right) = tfr\left(\frac{\gamma_n B}{\gamma_{n+1}B}\right). \tag{30}$$

By Lemmas 19.2 and 8.1, mH is isomorphic to $mB/\gamma_c mB$. Consequently,

$$\frac{\gamma_n mH}{\gamma_{n+1}mH} \cong \frac{\gamma_n mB}{\gamma_{n+1}mB}. \tag{31}$$

Since H is a torsion-free nilpotent group of class $c-1$, the inductive hypothesis applies and

$$tfr\left(\frac{\gamma_n H}{\gamma_{n+1}H}\right) = rank\left(\frac{\gamma_n mH}{\gamma_{n+1}mH}\right).$$

Combining this with equations (30) and (31), we obtain the desired result for $n < c$. It only remains to verify that $tfr(\gamma_c B) = rank(\gamma_c mB)$. Since $\gamma_c B$ is a finitely generated torsion-free abelian group $tfr(\gamma_c B) = rank(m\gamma_c B)$. Moreover, by Corollary 8.2 $m\gamma_c B = \gamma_c mB$. □

21 First Reduction

We recall that G is a finitely generated 1-relator group

$$G = \langle x_1,\ x_2, \ldots,\ x_q\ ;\ r\rangle$$

with $wt(r) = e$ relative to the free group $F(X)$. We are interested in the torsion-free ranks of the lower central factors of G. However, rather than working with G, we will obtain a nilpotent $\mathcal{D}$-group H, of arbitrary class c, with the property that for all $n < c$

$$tfr\left(\frac{\gamma_n G}{\gamma_{n+1} G}\right) = rank\left(\frac{\gamma_n H}{\gamma_{n+1} H}\right).$$

Given any positive integer $c > e$, let $F_c = F(X)/\gamma_{c+1}F(X)$, the free nilpotent group of class c on $X = \{x_1, x_2, \ldots, x_q\}$. Put $\rho = r\gamma_{c+1}F(X)$. Then

$$\frac{G}{\gamma_{c+1}G} \cong \frac{F_c}{gp_{F_c}(\rho)}$$

and $wt_{F_c}(\rho) = e$. Since F_c is a torsion-free nilpotent group we can identify it with its image in the $\mathcal{D}$-completion mF_c, the free nilpotent $\mathcal{D}$-group of class c on $\{x_1, x_2, \ldots, x_q\}$. To simplify our notation, we will fix c. Let F denote F_c and let E denote the completion mF.

Let B denote the quotient of the nilpotent group $G/\gamma_{c+1}G$ by its torsion subgroup. By Corollary 20.2, for all $n < c$

$$tfr\left(\frac{\gamma_n G}{\gamma_{n+1} G}\right) = tfr\left(\frac{\gamma_n B}{\gamma_{n+1} B}\right).$$

Moreover, since B is a finitely generated torsion-free nilpotent group we can apply Lemma 20.3 to obtain

$$tfr\left(\frac{\gamma_n G}{\gamma_{n+1} G}\right) = rank\left(\frac{\gamma_n mB}{\gamma_{n+1} mB}\right).$$

Let H denote $E/id_E(\rho)$. By Corollary 19.2, $mB \cong H$. Thus, we have proved:

Proposition 21.1 *For all $n < c$*

$$tfr\left(\frac{\gamma_n G}{\gamma_{n+1} G}\right) = rank\left(\frac{\gamma_n H}{\gamma_{n+1} H}\right).$$

Since c was arbitrary, we conclude that the torsion-free rank of any lower central factor of the one-relator group G can be obtained as the dimension of the corresponding homogeneous term of the graded rational Lie algebra

$$gr(H) = \bigoplus_{i=1}^{\infty} \frac{\gamma_i H}{\gamma_{i+1} H}$$

for appropriately chosen c.

22 The Lie Algebra of a 1-Relator Nilpotent $\mathcal{D}$-Group

We have determined that the desired torsion-free ranks of our 1-relator group can be obtained from $gr(H)$ where $H = E/id_E(\rho)$, E the free nilpotent $\mathcal{D}$-group of class c on $\{x_1, x_2, \ldots, x_q\}$. By Proposition 10.1, $gr(H) \cong gr(LogH)$. In light of this, we turn our attention to the Lie algebra of H. In this section, we demonstrate that the Lie algebra of a 1-relator nilpotent $\mathcal{D}$-group is a 1-relator rational nilpotent Lie algebra; that is, a free rational nilpotent Lie algebra quotiented by a principal ideal.

By Lemmas 9.4 and 9.7, since H is the quotient $E/id_E(\rho)$, the Lie algebra $LogH$ is isomorphic to $LogE/Log\left(id_E(\rho)\right)$.

Proposition 22.1 $Log\left(id_E(\rho)\right) = id_{LogE}(\log\rho)$.

Proof: Put $K = id_E(\rho)$ and $J = id_{LogE}(\log\rho)$. By Lemma 9.4 $LogK$ is an ideal of $LogE$. Since it contains $\log\rho$, by definition we have $J \subseteq LogK$. Conversely, by Lemmas 9.5 and 9.3, $\exp J$ is a $\mathcal{D}$-group ideal of $\exp(LogE) = E$. It contains $\exp(\log\rho) = \rho$, therefore $K \subseteq \exp(J)$. Hence, $LogK \subseteq J$, and so they are equal. □

Thus,

$$LogH \cong \frac{LogE}{id_{LogE}(\log\rho)}. \tag{32}$$

Let L denote the free rational Lie algebra of rank q and let L_c denote the quotient $L/\gamma_{c+1}L$. By Lemma 9.8, $LogE \cong L_c$. We now choose a lift of $\log\rho$ to L. Let φ be the isomorphism mapping L_c

onto *LogE*. Then we select $w \in L$ such that $\varphi(w + \gamma_{c+1}L) = \log\rho$ and φ induces an isomorphism

$$\frac{L_c}{id_{L_c}(w + \gamma_{c+1}L)} \cong \frac{LogE}{id_{LogE}(\log\rho)}\,. \tag{33}$$

Thus, *LogH* is isomorphic to the 1-relator nilpotent Lie algebra $L_c/id_{L_c}(w + \gamma_{c+1}L)$. We make one further observation. Put $Y = L/id_L(w)$. Then, using equations (32) and (33), we see that

$$LogH \cong \frac{Y}{\gamma_{c+1}Y}\,. \tag{34}$$

23 Invariance of Weight

We will need to know that the weight of w as an element of L is the same as the weight of r in $F(X)$. To that end we prove

Lemma 23.1 *For* $c > wt_F(\rho)$

$$wt_F(\rho) = wt_E(\rho) = wt_{LogE}(\log\rho) = wt_L(w)\,.$$

Proof: We will work from left to right. Suppose $k \leq c$. Recall that E is the $\mathcal{D}$-completion of F. By Corollary 8.3, $\gamma_kE \cap F = \gamma_kF$. Since ρ is an element of F, ρ will be contained in γ_kE if and only if ρ is an element of γ_kF. Thus, $wt_F(\rho) = wt_E(\rho)$.

The next step is handled by Lemma 9.9. Since $\gamma_kLogE = Log\gamma_kE$, if $\rho \in \gamma_kE$, then $\log\rho \in \gamma_kLogE$. Conversely, if $\log\rho \in \gamma_kLogE$, then ρ is contained in $\exp(Log\gamma_kE)$. Since γ_kE is a nilpotent $\mathcal{D}$-group, $\exp(Log\gamma_kE) = \gamma_kE$. Therefore, $wt_E(\rho) = wt_{LogE}(\log\rho)$.

Since the isomorphism from L_c onto *LogE* takes $w + \gamma_{c+1}L$ to $\log\rho$,

$$wt_{LogE}(\log\rho) = wt_{L_c}(w + \gamma_{c+1}L)\,.$$

When we take a quotient of a group, weights can only increase. Thus, $wt_L(w) \leq wt_{L_c}(w + \gamma_{c+1}L)$. On the other hand,

$$\gamma_kL_c = \frac{\gamma_kL}{\gamma_{c+1}L}$$

Consequently, if $w + \gamma_{c+1}L$ is contained in γ_kL_c, then $w \in \gamma_kL$. Hence, $wt_{L_c}(w + \gamma_{c+1}L)$ is equal to $wt_L(w)$. □

24 The Induced Lie Algebra of a 1-Relator Lie Algebra

In consequence of Propositions 10.1 and 21.1, equation (34), and Lemma 23.1, the torsion-free rank of the n^{th} lower central factor of our 1-relator group G will be equal to the dimension of the n^{th} homogeneous term of the graded Lie algebra

$$gr(H) \cong gr\left(\frac{Y}{\gamma_{c+1}Y}\right)$$

where $Y = L/id_L(w)$, L is the free rational Lie algebra of rank q, and $wt_L(w) = e$. Observe that as long as we choose $c > n$, these dimensions will be the same as for

$$gr(Y) = gr\left(\frac{L}{id_L(w)}\right).$$

Therefore, we now consider this Lie algebra. We will eventually see that although w is not a homogeneous element of L, $gr(Y)$ is isomorphic to a quotient of L by the ideal generated by a homogeneous element. This isomorphism will be revealed through the identification of two ideals of $gr(L)$.

By Lemma 11.1

$$gr\left(\frac{L}{id_L(w)}\right) \cong \frac{gr(L)}{gr_L(id_L(w))}.$$

We consider two ideals of $gr(L)$:

$$gr_L(id_L(w)) = \bigoplus_{i=1}^{\infty} \frac{id_L(w) \cap \gamma_i L + \gamma_{i+1} L}{\gamma_{i+1} L} \tag{35}$$

and the ideal of $gr(L)$ generated by the homogeneous element $w + \gamma_{e+1}L$

$$id_{gr(L)}(w + \gamma_{e+1}L). \tag{36}$$

Since $gr_L(id_L(w))$ is an ideal of $gr(L)$ and contains $w + \gamma_{e+1}L$,

$$id_{gr(L)}(w + \gamma_{e+1}L) \subseteq gr_L(id_L(w)).$$

We will prove that these two ideals, (35) and (36), are equal in order to apply the following theorem, which is a special case of a result of Labute's on free Lie rings over integral domains [13].

Theorem 24.1 [Labute] *If $\mathcal{L}$ is a free Lie algebra of finite rank q and ω is a homogeneous element of $\mathcal{L}$ of weight e, then the rank of the n^{th} homogeneous term of $\mathcal{L}/id_{\mathcal{L}}(\omega)$ is given by formula (22) and hence depends only on n, q, and e.*

25 The Action of the Universal Enveloping Algebra

As we saw in Section 14, the adjoint representation of a Lie algebra extends to an action of the Universal Enveloping Algebra. We explore this action in the case of the free Lie algebra L. The results we obtain will allow us to characterize the ideal $id_L(w)$ and to prove the equality of our two ideals of $gr(L)$.

Consider the free rational Lie algebra L on $\{x_1, x_2, \ldots, x_q\}$. As we saw in Section 12, L possesses a natural grading which we denote $L = \bigoplus_{i=1}^{\infty} L_i$. The universal enveloping algebra U of L is the free associative algebra on $\{x_1, x_2, \ldots, x_q\}$. Thus, U is naturally graded, $U = \bigoplus_{i=1}^{\infty} U_i$ where U_n consists of all $\mathbf{Q}$-linear combinations of monomials in $\{x_1, x_2, \ldots, x_q\}$ of homogeneous degree n. Also put $U_0^+ = U$ and

$$U_n^+ \quad = \quad \{u = u_0 + u_1 + \cdots \in U \mid 0 = u_0 = u_1 = \cdots = u_{n-1}\} \quad (n \geq 1).$$

In U, the weight of an element is the degree of its first non-zero homogeneous component. Thus,

$$U_n^+ = \{u \in U \mid wt_U(u) \geq n\}.$$

As in Section 14, let θ denote the homomorphism of associative algebras mapping U into $\hom(L, L)$ which extends the adjoint map on L. For convenience of notation, we will sometimes write $u * y$ to denote $\theta(u)(y)$.

Lemma 25.1 *$L = \bigoplus_{i=1}^{\infty} L_i$ is a graded U-module under the action defined by θ.*

Proof: U_n has a basis consisting of all formally distinct monomials in $x_1, x_2, \ldots, x_q$ of homogeneous degree n. Let u be a typical such monomial

$$u = x_{i_1} x_{i_2} \cdots x_{i_n}.$$

L_m has a basis of all basic commutators of weight m in a basic sequence $\{\beta_i\}$ in $\{x_1, x_2, \ldots, x_q\}$. Let β be one of these. Observe that

$$\begin{aligned} u * \beta &= \theta(x_{i_1} x_{i_2} \cdots x_{i_n})(\beta) \\ &= \big(\theta(x_{i_1}) \circ \theta(x_{i_2}) \circ \cdots \circ \theta(x_{i_n})\big)(\beta) \\ &= \big(ad(x_{i_1}) \circ ad(x_{i_2}) \circ \cdots \circ ad(x_{i_n})\big)(\beta) \\ &= [x_{i_1}[x_{i_2}[\cdots[x_{i_n}, \beta]\cdots]]]. \end{aligned}$$

Thus, $u * \beta \in L_{n+m}$. □

Corollary 25.2 *If u is a homogeneous element of U and y is a homogeneous element of L, then either $u * y = 0$ or else*

$$wt_L(u * y) = wt_U(u) + wt_L(y).$$

It was proved by Labute [14] that

Theorem 25.3 [Labute] *If y is a non-zero homogeneous element of L and $u \in U$, then $u * y = 0$ if and only if*

$$u = \sum_v b_v \cdot \theta(v) y \cdot v \qquad (b_v \in U), \tag{37}$$

for some finite set of $v \in U$.

For some fixed, but arbitrary choice of elements v and b_v of U, we consider the mapping from L to U which takes an arbitrary element l of L to $\sum_v b_v \cdot \theta(v) l \cdot v$. We will refer to such mappings as *ψ-forms.* The next two lemmas demonstrate several important properties of ψ-forms.

Lemma 25.4 *Let z and y be elements of L and let Ψ be a ψ-form. Then*

1. $\Psi(z) * z = 0,$
2. $\Psi(z) * y = -\Psi(y) * z.$

Proof: For some finite set of pairs v and b_v, Ψ takes an arbitrary element l of L to $\sum_v b_v \cdot \theta(v)l \cdot v$.

$$\begin{aligned}
\Psi(z) * y &= \theta\left(\sum_{v\in U} b_v \cdot \theta(v)z \cdot v\right)(y) \\
&= \sum_{v\in U} \theta\big(b_v \cdot \theta(v)z \cdot v\big)(y) \\
&= \sum_{v\in U} \theta(b_v) \circ \theta\big(\theta(v)z\big) \circ \theta(v)(y) .
\end{aligned}$$

Since $\theta(v)z \in L$,

$$\begin{aligned}
\theta\big(\theta(v)z\big)\big(\theta(v)y\big) &= ad\big(\theta(v)z\big)\big(\theta(v)y\big) \\
&= [\theta(v)z,\ \theta(v)y] .
\end{aligned}$$

Since $[\theta(v)z, \theta(v)z] = 0$, $\Psi(z) * z = 0$. More generally,

$$\begin{aligned}
\Psi(z) * y &= \sum_{v\in U} \theta(b_v)\big([\theta(v)z, \theta(v)y]\big) \\
&= -\sum_{v\in U} \theta(b_v)\big([\theta(v)y, \theta(v)z]\big) \\
&= -\theta\left(\sum_{v\in U} b_v \cdot \theta(v)y \cdot v\right)(z) \\
&= -\Psi(y) * z .
\end{aligned}$$

□

A ψ-form Ψ will be termed a *homogeneous ψ-form of weight n* if $\Psi(z)$ is homogeneous of weight $wt(z) + n$ for all homogeneous z in L.

Lemma 25.5 *Suppose Ψ is a ψ-form and y is a homogeneous element of L for which $\Psi(y)$ is a homogeneous element of U of weight $wt(y)+n$. Then there exists a homogeneous ψ-form $\tilde{\Psi}$ of weight n such that $\tilde{\Psi}(y) = \Psi(y)$.*

Proof: For some finite set of pairs v and b_v elements of U,

$$\Psi(y) = \sum_v b_v \cdot \theta(v)y \cdot v .$$

Without loss of generality, we may assume that the b_v's are homogeneous elements of U. Each v is of the form

$$v = v_1 + v_2 + \cdots \qquad v_i \in U_i$$

with all but a finite number of v_i equal to zero. Since $\Psi(y)$ is homogeneous of degree $wt(y)+n$, by Corollary 25.2

$$\Psi(y) = \sum_{v} \sum_{i+j=n-wt(b_v)} b_v \cdot \theta(v_i)y \cdot v_j .$$

Let Φ be the map from L to U taking an arbitrary element z to

$$\Phi(z) = \sum_{v} \sum_{i+j=n-wt(b_v)} b_v \cdot \theta(v_i)z \cdot v_j . \tag{38}$$

Then $\Phi(y) = \Psi(y)$. Moreover, if z is a homogeneous element of L, then $\Phi(z)$ is a sum of terms of homogeneous degree $wt(z)+n$. Thus, $\Phi(z)$ is homogeneous of degree $wt(z)+n$. For each v consider the pairs i and j such that $i+j = n - wt(b_v)$. If $i = j$, then the term

$$b_v \cdot \theta(v_i)z \cdot v_j \tag{39}$$

is a ψ-form. All other terms of the inner sum of equation (38) can be grouped in pairs

$$b_v \cdot \theta(v_i)z \cdot v_j + b_v \cdot \theta(v_j)z \cdot v_i. \tag{40}$$

where $i < j$. Since expression (39) is a ψ-form and expression (40) is equal to the ψ-form

$$b_v \cdot \theta(v_i + v_j)z \cdot (v_i + v_j) - b_v \cdot \theta(v_i)z \cdot v_i - b_v \cdot \theta(v_j)z \cdot v_j,$$

equation (38) can be rewritten as a sum of ψ-forms. Hence, there exists a ψ-form $\tilde{\Psi}$ such that $\Phi(z) = \tilde{\Psi}(z)$ for all $z \in L$. □

Given an element $y \in L$, let $y[j]$ denote the homogeneous component of y of degree j.

Lemma 25.6 *If $u \in U$ and $y \in L$ of weights μ and e respectively such that $u * y[\mu + e] = 0$, then there exists $v \in U$ of weight strictly greater than μ such that $v * y = u * y$.*

Proof: Put $u_\mu = u[\mu]$ and $y_e = y[e]$. We can express u and y as

$$\begin{aligned} u &= u_\mu + \tilde{u}, \qquad \tilde{u} \in U^+_{\mu+1}; \\ y &= y_e + \tilde{y}, \qquad wt(\tilde{y}) \geq e . \end{aligned}$$

Then $u * y = u_\mu * y_e + u_\mu * \tilde{y} + \tilde{u} * y$. By Corollary 25.2, $u_\mu * y_e = u * y[\mu + e] = 0$. Thus, by Theorem 25.3, u_μ is equal to a ψ-form evaluated at y_e. Moreover, since u_μ and y_e are homogeneous, by Lemma 25.5 there exists a homogeneous ψ-form Ψ of weight $\mu - e$ such that $\Psi(y_e) = u_\mu$. Using Lemma 25.4

$$\begin{aligned} u * y &= \Psi(y_e) * \tilde{y} + \tilde{u} * y \\ &= \Psi(y_e) * \tilde{y} - \Psi(\tilde{y}) * \tilde{y} + \tilde{u} * y \\ &= -\Psi(\tilde{y}) * y_e - \Psi(\tilde{y}) * \tilde{y} + \tilde{u} * y \\ &= (\tilde{u} - \Psi(\tilde{y})) * y. \end{aligned}$$

Put $v = (\tilde{u} - \Psi(\tilde{y}))$. Then $v * y = u * y$ and, since $wt(\tilde{y}) > e$, $wt(v) > wt(u)$. □

Corollary 25.7 *Let u and y be elements of U and L respectively. If the first $wt(y) + wt(u) + n$ homogeneous terms of $u * y$ are zero, then there exists an element v in U such that $wt(v) > wt(u) + n$ and $v * y = u * y$.*

Proof: The proof is by induction on n. If we set $n = 0$, the proposition is identical to Lemma 25.6. Hence, we suppose $n > 0$ and assume the proposition holds for all positive integers less than n. By inductive hypothesis, there exists $u' \in U$ such that $wt(u') \geq wt(u) + n$ and $u' * y = u * y$. If $wt(u') > wt(u) + n$, the proposition holds. Therefore, let us assume that $wt(u') = wt(u) + n$. By hypothesis, $u' * y[wt(u') + wt(y)] = 0$. Consequently, by Lemma 25.6 there exists $v \in U$ such that $wt(v) > wt(u')$ and $v * y = u' * y$. Hence, $wt(v) > wt(u) + n$ and $v * y = u * y$. □

26 The Identification of Ideals

We view the ideal $id_L(w)$ in terms of the action of the universal enveloping algebra U on the free rational Lie algebra L. This will enable us to prove the equality of the ideals $gr_L(id_L(w))$ and $id_{gr(L)}(w + \gamma_{e+1}L)$.

For any element l in L, let $\theta(U)(l)$ denote the set of all $\theta(u)(l)$ for u in U. As a consequence of Lemma 13.1, a typical element z of $id_L(w)$ of weight greater than e can be expressed in the form

$$z = \sum_j r_j [l_{j,n_j}[\cdots[l_{j,1},\ w]\cdots]] \qquad r_j \in \mathbf{Q},\ l_{j,i} \in L. \tag{41}$$

Put

$$u = \sum_j r_j l_{j,n_j} \cdots l_{j,1}.$$

Then, since θ is a homomorphism of associative algebras and extends ad_L, $\theta(u)(w) = z$. Moreover,

$$\theta(x_{i_1} x_{i_2} \cdots x_{i_n})(w) = [x_{i_1}[x_{i_2}[\cdots [x_{i_n}, w] \cdots]]]$$

for any monomial $x_{i_1} x_{i_2} \cdots x_{i_n}$ in $\{x_1, x_2, \ldots, x_q\}$. Thus, $id_L(w) = \theta(U)(w)$, so that the action of the universal enveloping algebra U defines the principal ideals of L. However, there is more we can say about the relationship between $id_L(w)$ and the action of U. The key to the identification of our two ideals of $gr(L)$ is contained in the following result:

Theorem 26.1 $id_L(w) \cap \gamma_{e+n} L = \theta(U_n^+)(w)$.

Proof: Since $id_L(w) \cap \gamma_e L = id_L(w)$ and $U_0^+ = U$, by the above comments we need only consider the case that $n > 0$. By Lemma 25.1, $\theta(U_n^+)(w) \subseteq id_L(w) \cap \gamma_{e+n} L$. For the converse, suppose y is an arbitrary element of $id_L(w)$ of weight $e + m$ for some $m > 0$. We claim that there exists an element $v \in U_m^+$ such that $y = v * w$. The proof of this claim is as follows:

Since $id_L(w) = \theta(U)(w)$, we can find $u \in U$ such that $y = u * w$. Put $k = m - wt(u)$. If $k = 0$, then $u \in U_m^+$ and our claim is proved. Therefore, assume $k > 0$. The first $wt(w) + wt(u) + (k - 1)$ terms of $u * w$ are equal to zero. Hence, by Corollary 25.7 there exists an element $v \in U$ with $v * w = u * w = y$ and

$$wt(v) \geq wt(u) + k = m.$$

Thus, $v \in U_m^+$ and our claim is proved.

Suppose z is contained in $id_L(w) \cap \gamma_{e+n} L$ for some positive integer n. Then $wt(z) = e + m \geq e + n$. As we have just shown, this implies $z \in \theta(U_m^+)(w) \subseteq \theta(U_n^+)(w)$. □

Corollary 26.2 $gr_L(id_L(w)) = id_{gr(L)}(w + \gamma_{e+1} L)$.

Proof: Consider an arbitrary non-trivial element of

$$\frac{id_L(w) \cap \gamma_{e+n} L + \gamma_{e+n+1} L}{\gamma_{e+n+1} L}.$$

It is a coset of $\gamma_{e+n+1}L$

$$y + \gamma_{e+n+1}L$$

with representative $y \in id_L(w) \cap \gamma_{e+n}L$. Since $id_L(w) \cap \gamma_{e+n}L = \theta(U_n^+)(w)$, there exists $v \in U_n^+$ such that $y = v * w$. Decomposing w and v into homogeneous terms, we have

$$\begin{aligned} w &= w_e + w_{e+1} + \cdots \qquad & w_i \in L_i \\ v &= v_n + v_{n+1} + \cdots \qquad & v_i \in U_i\,. \end{aligned}$$

Then $y[n+e] = v_n * w_e$ and

$$\begin{aligned} y + \gamma_{e+n+1}L &= y[n+e] + \gamma_{e+n+1}L \\ &= v_n * w_e + \gamma_{e+n+1}L\,. \end{aligned}$$

Since $v_n \in U_n$, it is of the form

$$v_n = \sum_{\alpha \in A} s_\alpha x_{\alpha_n} \cdots x_{\alpha_1} \qquad s_\alpha \in \mathbf{Q},\ x_{\alpha_i} \in \{x_1, x_2, \ldots, x_q\}$$

for some finite set of multi-indices A. Then the element of $id_{gr(L)}(w_e + \gamma_{e+1}L)$

$$\sum_{\alpha \in A} s_\alpha [x_{\alpha_n} + \gamma_2 L[\cdots [x_{\alpha_1} + \gamma_2 L, w_e + \gamma_{e+1}L] \cdots]]$$

is equal to

$$\begin{aligned} \sum_{\alpha \in A} s_\alpha [x_{\alpha_n}[\cdots [x_{\alpha_1}, w_e] \cdots]] + \gamma_{e+n+1}L &= v_n * w_e + \gamma_{e+n+1}L \\ &= y + \gamma_{e+n+1}L \end{aligned}$$

Thus, $y + \gamma_{n+1}L \in id_{gr(L)}(w_e + \gamma_{e+1}L) = id_{gr(L)}(w + \gamma_{e+1}L)$. $\square$

27 Proof of the Invariance of Torsion-Free Ranks

Theorem 27.1 *Let G be a finitely generated 1-relator group, $G = \langle x_1,\ x_2, \ldots,\ x_q\ ;\ r \rangle$, and let $wt(r) = e$ relative to the free group $F(X)$ on $\{x_1, x_2, \ldots, x_q\}$. Then, for any positive integer n, the torsion-free rank of the n^{th} lower central factor of G depends only on n, q, and e.*

Proof: For some fixed but arbitrary positive integer $c > e$, let $H = E/id_E(\rho)$ where E is the free nilpotent $\mathcal{D}$-group of class c, $m(F(X)/\gamma_{c+1}F(X))$, and $\rho = r\gamma_{c+1}F(X)$. Let n be a positive integer less than c. In Proposition 21.1, we saw that $tfr(\gamma_n G/\gamma_{n+1}G)$ is equal to the dimension of the n^{th} homogeneous component of $gr(H)$. Since H is a nilpotent $\mathcal{D}$-group, by Proposition 10.1 $gr(H) \cong gr(Log H)$. The Lie algebra of a quotient is isomorphic to the quotient of the Lie algebras. Thus,

$$Log H \cong \frac{Log E}{Log\,(id_E(\rho))}.$$

By Lemma 22.1, $Log\,(id_E(\rho))$ is equal to the ideal of $Log E$ generated by $\log \rho$. Therefore

$$Log H \cong \frac{Log E}{id_{Log\,E}(\log \rho)}.$$

$Log E$ is a free rational nilpotent Lie algebra of class c and rank q. Hence, it is isomorphic to $L/\gamma_{c+1}L$ where L is the free rational Lie algebra of rank q. Thus,

$$Log H \cong \frac{L}{id_L(w)} \Big/ \gamma_{c+1}\left(\frac{L}{id_L(w)}\right)$$

where w is a lift of $\log \rho$. Consequently, since n is less than c, $tfr(\gamma_n G/\gamma_{n+1}G)$ will be the dimension of the n^{th} homogeneous component of the graded Lie algebra

$$gr\left(\frac{L}{id_L(w)}\right).$$

As a consequence of Corollary 26.2

$$gr\left(\frac{L}{id_L(w)}\right) \cong \frac{gr(L)}{id_{gr(L)}(w + \gamma_{e+1}L)}.$$

By Corollary 12.2, $gr(L)$ is a free rational Lie algebra of rank q, and by Lemma 23.1, $w + \gamma_{e+1}L$ is a homogeneous element of $gr(L)$ of weight e. By Theorem 24.1 (Labute), the rank of the n^{th} homogeneous component of $gr(L)/id_{gr(L)}(w + \gamma_{e+1}L)$ is given by formula (22). Hence,

$$tfr\left(\frac{\gamma_n G}{\gamma_{n+1}G}\right) = l(n, q, e)$$

and thus depends only on n, q, and e. □

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