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**RESIDUAL SOLVABILITY OF GENERALIZED FREE
PRODUCTS**

by

DELARAM KAHROBAEI

A dissertation submitted to the Graduate Faculty in Mathematics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy, The City University of New York

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Abstract

RESIDUAL SOLVABILITY of GENERALIZED FREE
PRODUCTS

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The notion of residual properties was first introduced by Philip Hall [22] in 1954. In this work we show that in general $G = \{A * B; C\}$, the generalized product of two finitely generated nilpotent groups A and B , amalgamating C , is not perfect. As a consequence, when the factors A and B are torsion-free, G is guaranteed to be residually solvable by any of the following conditions: 1. the amalgamated subgroup C is **central** in both factors; 2. the amalgamated subgroup C is of **finite index** in at least one of the factors; 3. the factors A and B are isomorphic, via an isomorphism that agrees with the isomorphism that identifies the amalgamated subgroups C_A and C_B (this type of generalized free product is called a **double**); 4. the amalgamated subgroup C is **cyclic**; 5. one of the factors is **abelian**. The generalized free products of finitely generated torsion-free nilpotent groups are not necessarily residually solvable. We demonstrate that using an example due to Gilbert Baumslag [11].

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New York
August 28, 2003

D.K.

**To the memory of my aunt, Alam Taj Hayati,
my first physics and mathematics teacher.**

Preface

The study of residual solvability of groups was initiated by Philip Hall in 1954 [22]. Residual finiteness of generalized free products of finitely generated nilpotent groups was studied by Gilbert Baumslag [3] primarily. This work is concerned with residually solvability of this kind of structures.

To start, in **Chapter 1**, we give alternative definitions of nilpotent groups, based on upper and lower central series and their properties, we then recall some properties of finitely generated torsion nilpotent groups and their center which reflect in the entire group; the Frattini subgroup is the topic of another subsection in this chapter. Next we bring in polycyclic groups and their relation to nilpotent groups; definition of polycyclic-by-finite groups and Hirsch number is one of the subsections. Definition, theorems and important properties of Mal'cev completion, such as, the fundamental theorem of torsion-free nilpotent groups.

In **Chapter 2**, we define generalized central products for an arbitrary number of groups. In the literature this was introduced

mostly for two groups. We collect related results concerning injective mappings of the factors of these kinds of structures. This is the context in which the classical definition of generalized free products of arbitrary number of groups [7] and the related theorems. We then prove some lemmas about removing torsion from amalgamated products of nilpotent groups. Afterwards, we define the tree products of groups [36] and we list some related theorems on subgroups of generalized free products.

Then in **Chapter 3**, we define residual properties of classes of groups and we give different terms which are used to describe residual properties of groups, such as separation and discrimination. Next we concentrate on residual solvability of groups: we give alternative definitions (one in terms of solvable filtration) of this property, we prove their equivalence. After that, we study the facts about solvability and residual solvability.

In **Chapter 4**, we give a complete formulation and related results of the structure of the abelianization of the generalized free products of groups. Then we study this abelianization in the case where the factors are finitely generated nilpotent groups. After that we study the effect of the order of the abelianization of amalgamated products of nilpotent groups.

The results of **Chapter 5**, are involved in residual solvability of generalized free products of finitely generated nilpotent groups.

We first prove that this generalized free products is not perfect in general, regardless of finitely generated nilpotent groups being torsion-free. Then we study the different case and put conditions of the amalgamated subgroups and observe that these kinds of structures are almost residually solvable. But that is not the case in general. We show by an example by Baumslag that they are not residually solvable [11], this example actually shows that these structures are not poly-residually-solvable in general.

In **Chapter 6**, we deal with another class of groups, namely free groups and residually solvable groups. We give a very simple and short proof of a theorem due to Karrass and Solitar which gives an alternative answer to a question of Peter Neumann, listed under item *F22* in the “Open Problems” on “Free Groups” section [12] of <http://www.grouptheory.org>.

Then we discuss residual solvability for the case where the factors are residually solvable. We particularly study the case of doubles.

In **Chapter 7**, we discuss residual solvability of one-relator groups. Gilbert Baumslag’s result [5] shows that positive one-relator groups are residually solvable. There examples of other one-relator groups which residually solvable. One of the most important one’s are Baumslag-Solitar groups, as well as the class of groups introduced by Baumslag in [6]. In this chapter bring new different cases where these kinds of groups in both cases.

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Introduction

The notion of residual properties was first introduced by Philip Hall in 1954 [22]. Let X be a class of groups. G is residually- X if and only if, for every non-trivial element g in G there is an epimorph of G in X such that the element corresponding to g is not the identity. If Y is another class of group, then we denote by $X \cdot Y$ the class of those groups G which possess a normal subgroup N in X such that G/N is in class Y .

A group G is nilpotent if it has at least one central series. A group G is called solvable, if there exists a natural number n , so that the n -th terms of its derived group is the trivial group. In particular we like to remind that nilpotent groups satisfy the maximal condition, where solvable groups fail to satisfy this condition. Another lemma which is worth recalling is that, a solvable group satisfies the maximal condition if and only if it is polycyclic. Quotient, subgroups, homomorphic images, cartesian product of

solvable(Nilpotent) groups are again solvable(Nilpotent).

In this thesis, we consider the finitely generated groups.

Abelian \rightarrow Nilpotent \rightarrow polycyclic \rightarrow solvable + maximal condition \rightarrow residually solvable \rightarrow meta-residually solvable \rightarrow poly-residually-solvable.

In 1963 Gilbert Baumslag studied residual finiteness of generalized free products of finitely generated torsion-free nilpotent groups, [3]. He showed first that if both factors are non-abelian then this amalgamation is not residually-finite, he found conditions where G is residually finite. He actually proved in general that this kind of structure is free-by-residually-finite, or meta-residually-finite, and if the amalgamating subgroup is closed in both factors then it is residually finite. For when the amalgamation subgroup is cyclic, he showed that these kinds of groups are residually finite. A few years after, in 1968 one of his students, Joan Landman Dyer, continued his work [19] on residual finiteness of this kind of structures. She showed in particular that if the factors are not necessarily torsion-free, these kind of groups are not free-by-residually-finite (by taking isomorphic factors of class three), but rather residually-finite-by-free-by-residually-finite.

In this work we consider the generalized free product of finitely generated nilpotent groups, and discuss how close this structure of groups is to residual solvability. We show that how the condition of amalgamating subgroup affects on the residual solvability of this kind of groups. In a more specific way we study the algebraic extension of these kind of groups.

Effect of abelianization on amalgamated product of nilpotent groups

We give a complete explanation of abelianization of the generalized free products. In particular, we can see how the abelianization of this kind of groups with nilpotent factors is not trivial. Using Frattini theorem, which shows that the commutator subgroup of a finitely generated nilpotent group is a non-generator set. As a result, we find out that the generalized free products of two finitely generated nilpotent groups are not perfect.

Theorem 0.0.1. *The generalized free products of two finitely generated nilpotent groups amalgamated a subgroup of them is not perfect.*

This property of finitely generated nilpotent groups plays a critical role to approach our question. For example finitely generated

polycyclic factors that satisfy the maximal condition, but fail to satisfy the condition of Frattini theorem, turns out to be perfect in several examples.

Cyclic amalgamated subgroup and residual solvability

Another case that we consider, is when the amalgamating subgroup is cyclic. It turns out that choosing an appropriate solvable filtration [6] of each factor so that the generator of amalgamating subgroup doesn't lie in the n -th term of upper central series and we prove that it is residually-solvable-by-solvable.

Theorem 0.0.2. *The generalized free products of two finitely generated torsion-free nilpotent groups amalgamating a cyclic subgroup is an extension of a residually solvable group by a solvable group.*

The amalgamated subgroup is central in both factors

We then define the generalized central products of arbitrary number of groups, and show that each factor inject to this kind of product. We then look at the case where the amalgamating subgroup is central in both factors, and we show that using the generalized central products that these kind of groups turn out to be free-by-nilpotent. We note that a nilpotent extension of a free group is not residually nilpotent but rather residually solvable.

Theorem 0.0.3. *The generalized free products of two finitely generated nilpotent groups amalgamating central subgroups of them is a nilpotent extension of a free group.*

As a corollary, the generalized free products of finitely generated abelian groups are free-by-abelian, i.e. residually solvable. We can actually generalize this theorem to arbitrary number of factors.

Corollary 0.0.4. *G is residually solvable.*

Theorem 0.0.5. *The generalized free products of arbitrary number of finitely generated nilpotent groups of bounded class, amalgamating central subgroups of them, is an extension of a free group by a nilpotent group.*

Theorem 0.0.6. *The generalized free products of arbitrary number of abelian groups is an extension of a free group by an abelian group.*

The case where one of the factors is abelian

We then consider the case where just one of the factors is abelian, it turns out that these groups with an abelian factor is again residually solvable. In fact, free-by-nilpotent.

Theorem 0.0.7. *The generalized free products of a finitely generated torsion-free abelian and a finitely generated torsion-free nilpotent group is an abelian extension of a residually solvable.*

The case when the amalgamated subgroup is of finite index in at least one of the factors

Theorem 0.0.8. *The generalized free product of two finitely generated torsion-free nilpotent groups amalgamated subgroup is of finite index in at least one of the factors, is an extension of a free group by a torsion-free nilpotent group.*

Doubles of nilpotent groups and residual solvability

The case of two factors

Theorem 0.0.9. *Let A be a finitely generated nilpotent group, C be a subgroup of A , and $\bar{}$ be an isomorphic mapping of A onto A . Then the generalized free product of A and \bar{A} amalgamated C with \bar{C} ,*

$$G = \{A * \bar{A}; C = \bar{C}\},$$

is an extension of a free group by a finitely generated nilpotent group. Furthermore, it is residually solvable.

The arbitrary number of factors

The theorem (5.8.1) can be generalized to the arbitrary number of isomorphic nilpotent groups.

Theorem 0.0.10. *Let $\{A_i | i \in I\}$ be a arbitrary indexed family of isomorphic torsion-free nilpotent groups, such that $\bigcap_{i \in I} A_i = C$, and let G be the generalized free product of A_i s amalgamated C :*

$$G = \left\{ \prod_{i \in I}^* A_i; C \right\}.$$

Then G is residually solvable.

A negative theorem, generalized free products of nilpotent groups sometimes may fail to be residually solvable.

Gilbert Baumslag remarked in [11], by a counter example that not every subgroup of generalised free product of two finitely generated torsion-free nilpotent groups is indicable. We use this example to prove the following theorem:

Theorem 0.0.11. *There exist two finitely generated torsion-free non-abelian free nilpotent group A and B , such that the amalgamated product of them with abelian amalgamation,*

$$G = \{A * B; C_A = C_B\},$$

is not residually solvable.

Abelian amalgamating subgroup does not suffice for residual solvability.

The second part of this work is on residual solvability of generalized free products of residual solvable groups (free groups). This part has been motivated by a question of Peter Neumann, [12] which is one of the open problems in Magnus webpage, *Is it possible that the free product $\{A * B; C_A = C_B\}$ with amalgamation, where A, B are free groups of finite ranks, C_A, C_B are finitely generated subgroups of A, B , respectively, is a simple group?* [F22]

Ruth Camm in 1953 [15] constructed an example that if the amalgamating subgroup is of infinite rank, in which this amalgamated product is simple. Burger and Mozes in 1997 [14] constructed a simple amalgamated products where all factors and amalgamated subgroups are finitely generated torsion-free free groups. Karrass and Solitar proved that

Theorem 0.0.12. *Let H be a finitely generated subgroup of a free group F and the index $[F : H]$ infinite. Then there exists a non-trivial normal subgroup N of F such that $N \cap H = \{1\}$.*

In 1998 Ivanov and Schupp gave an alternative proof of this theorem using small cancellation theory as well as brought up two

corollaries which give an affirmative answer to the question of Peter Neumann. In 2000 I gave a very simple and short proof in [26], [25], which I use Kurosh subgroup theorem as well as a theorem of Marshall Hall. In this work we show the complete procedure of this proof as well.

In this work we proved for the case of doubles of residually solvable groups in general (free groups), that this kind of groups are meta-residually-solvable in general.

Theorem 0.0.13. *Let A be a residually solvable group, C be a subgroup of A , and $\bar{}$ be an isomorphic mapping of A onto A . Then the generalized free product of A and \bar{A} amalgamated C with \bar{C} ,*

$$G = \{A * \bar{A}; C = \bar{C}\}$$

is an extension of a free group by a residually solvable group. Furthermore, G is meta-residually-solvable.

Effect of solubly separability on the amalgamated subgroup and residual solvability

We look at the case where the amalgamated subgroup is solvably

separable and we show that in this case these kinds of amalgamation is residually solvable.

Theorem 0.0.14. *Let θ be an isomorphism from a group A onto itself, C be a subgroup of A , and G be the amalgamated product of A and \bar{A} with C amalgamated:*

$$G = \{A * \bar{A}; C = \bar{C}\}.$$

If A is residually solvable then G is also residually solvable, provided that C be solubly separable in A .

Solvable separability is a sufficient condition for residual solvability. A negative theorem

Note that the condition of solvable separability of the amalgamated subgroup in the factors, in the case of doubles is necessary. The following negative theorem shows that the amalgamated product of doubles is not residually solvable where the factors are residually solvable groups.

Theorem 0.0.15. *Let A be a finitely generated residually solvable group, and C be a normal subgroup of A , such that A/C be perfect. Let θ be an isomorphic mapping of A onto itself. Then*

$$D = \{A * \bar{A}; C = \bar{C}\}$$

*is meta-residually solvable, but **not** residually solvable.*

Chapter 1

Basic Notations and results

1.1 Nilpotent groups

Definition 1.1.1. ([4]) A series is called normal if each term in the series is a normal subgroup of the succeeding term.

Definition 1.1.2. Let X be a property (or class) of groups. A finite normal series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_l = G \quad (1)$$

of G is termed a poly- X series for G if

$$G_{i+1}/G_i \in X \text{ for } i = 0, \dots, l-1.$$

Definition 1.1.3. A group G is termed poly- X if it has at least

one poly- X series. The length of the series (1) is l .

Lemma 1.1.1. (*[35]*) *Suppose X is a property (or class) of groups such that every subgroup of a group with X also has X . Then every subgroup of a poly- X group is again poly- X .*

Definition 1.1.4. G is called an extension of A by Q if there exist A and Q and a short exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1.$$

Definition 1.1.5. Let X be a property (or class) of groups. G is meta- X if there exist A and Q of property (or class) X and a short exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1.$$

Note 1.1.2. *An extension of a finitely presented group by another finitely presented group is finitely presented. In particular polycyclic groups are finitely presented.*

Definition 1.1.6. The center of G is denoted by ξG ; so

$$\xi G = \{x \in G \mid [x, y] = 1 \text{ for all } y \in G\}.$$

Definition 1.1.7.

$$1 = G_0 \leq G_1 \leq \cdots \leq G_l = G$$

is an invariant series, if

1) $G_i \triangleleft G_{i+1}$;

2) G_i be invariant in G , i.e. $\forall \text{Aut}\phi : G \rightarrow G$, then $\phi(G_i) = G_i$.

In other words, if G_i is a normal subgroup of G for $i = 1, \dots, l$.

Example 1.1.3. *Derived series is an example of an invariant series.*

Definition 1.1.8. An invariant series (1) is termed a central series

if

$$G_{i+1}/G_i \leq \xi(G/G_i) \text{ for } i = 0, 1, \dots, l-1.$$

Definition 1.1.9. G is nilpotent if it has at least one central series.

The class of a nilpotent group is the least of the lengths of its central series.

Definition 1.1.10. ([37]) Let F be the free group on a finite set X , such that $|X| = n$, and c be a positive integer. Any group isomorphic to $G = F/\gamma_{c+1}(F)$ is called a free nilpotent group of rank n and class c .

Definition 1.1.11. The upper central series of G is defined to be the series

$$1 = \xi_0 G \leq \xi_1 G \leq \xi_2 G \leq \cdots \leq \xi_n G \leq \cdots$$

where inductively

$$\xi_{i+1} G / \xi_i G = \xi(G / \xi_i G) \quad (i \geq 0).$$

So $\xi_1 G = \xi G$.

Definition 1.1.12. The lower central series of G is defined to be the series

$$G = \gamma_1 G \geq \gamma_2 G \geq \cdots \geq \gamma_n G \geq \cdots$$

where inductively

$$\gamma_{n+1}G = [\gamma_n G, G] \quad (n \geq 1).$$

Proposition 1.1.4. *G is nilpotent if $\gamma_{c+1}G = 1$ for some c . We call the least such c , the class of G .*

Note 1.1.5. *$\xi_c G = G$ if and only if $\gamma_{c+1}G = 1$.*

Definition 1.1.13. A non-empty class ν of groups is termed a variety (of groups) if it is closed under homomorphic images, subgroups and cartesian products.

Lemma 1.1.6. *Subgroups, homomorphic images and cartesian products of nilpotent groups of class at most c are again nilpotent of class at most c . In other words the class \mathcal{N}_c of all nilpotent groups of class at most c is a variety of groups.*

Lemma 1.1.7. *Every subgroup of a poly-nilpotent group is again poly-nilpotent.*

Proof. This will can be proved by applying lemma 1.1.6 and 1.1.1.

□

Definition 1.1.14. A group G satisfies the maximal condition if it has no infinite strictly increasing chain of subgroups. This is equivalent to the condition that all subgroups of G are finitely generated.

Theorem 1.1.8. *Subgroups of finitely generated nilpotent groups are finitely generated.*

Proof. Let C be a subgroup of a finitely generated nilpotent group G of class c .

$$G = \gamma_1 G > \gamma_2 G > \cdots > \gamma_{c+1} G = 1.$$

There exists i such that $C \leq \gamma_i G$ and $C \not\leq \gamma_{i+1} G$.

$$\text{For } i = c \quad C \leq \gamma_c G = \gamma_c G / \gamma_{c+1} G$$

which is finitely generated abelian, and therefore C is finitely generated.

Let ϕ be a homomorphism

$$\phi : C \rightarrow \gamma_i G / \gamma_{i+1} G.$$

Note that $\gamma_i G / \gamma_{i+1} G$ is finitely generated abelian.

$\exists a_1, \dots, a_n \in C$ such that $\gamma_i G / \gamma_{i+1} G = gp(a_1\phi, \dots, a_n\phi)$.

$$\forall a \in C \quad a\phi = (a_1\phi)^{\alpha_1} \dots (a_n\phi)^{\alpha_n} = (a_1^{\alpha_1} \dots a_n^{\alpha_n})\phi.$$

Therefore

$$(a_1^{\alpha_1} \dots a_n^{\alpha_n} a^{-1})\phi = 1,$$

$$\text{i.e. } a_1^{\alpha_1} \dots a_n^{\alpha_n} a^{-1} \in \ker \phi.$$

Hence,

$$a = a_1^{\alpha_1} \dots a_n^{\alpha_n} k \quad (k \in \ker \phi).$$

Put

$$D = C \cap \gamma_{i+1} G.$$

Note that

$$k \in D \leq \gamma_{i+1}G.$$

Therefore D is finitely generated by the induction hypothesis. Note that

$$C = gp(a_1, \dots, a_n, D)$$

which is again finitely generated. □

Note 1.1.9. *Subgroups of finitely generated solvable groups are **not** necessarily finitely generated.*

1.2 The torsion subgroups of nilpotent groups

Note 1.2.1. τA is the set of elements of finite order in A .

Lemma 1.2.2. τA is the torsion subgroup of A .

Proof. Let $a, b \in A$ be two elements of finite order in A . Hence $gp(x, y)$ has finite order. So the order of $x \cdot y$ is finite. Hence τA is a subgroup of A . □

Lemma 1.2.3. Let A be a nilpotent group. Then the torsion subgroup of A , τA is normal in A , i.e. $\tau A \triangleleft A$.

Proof. Since the order of an element is invariant under conjugation, we may prove that $\tau A \triangleleft A$. □

Lemma 1.2.4. A finitely generated nilpotent torsion group is finite.

Proof. Finitely generated nilpotent groups satisfy the maximal condition 1.1.8, so τA is finitely generated. Hence a finitely generated

nilpotent torsion group is finite since its generators have finite order. \square

Lemma 1.2.5. *Let A be a finitely generated nilpotent group. Then the quotient $A/\tau A$ is torsion-free.*

Proof. We want to show that the only element of finite order in $A/\tau A$ is the identity. If $(a\tau A)^n$ is the identity in $A/\tau A$ for some coset $a\tau A$, then $a^n \in \tau A$ and a is in τA as well. So $a\tau A = \tau A$. \square

Lemma 1.2.6. *Finite nilpotent groups are simply the direct products of their Sylow p -subgroups.*

1.3 Properties of the center of a nilpotent group reflected in the entire group

The properties of the center of a nilpotent group are often reflected in the entire group. One such result is the following:

Lemma 1.3.1. *(Philip Hall, [4]) A finitely generated nilpotent group G is finite if and only if its center is finite.*

Proof. By induction on c , the class of G . If $c = 1$ there is nothing to prove.

Suppose $c > 1$. If $\xi_1 G$ has order n , $a \in \xi_2 G$ and $x \in G$ then

$$[x, a^n] = [x, a]^n = 1.$$

So $a^n \in \xi_1 G$ and $\xi_2 G / \xi_1 G$ is finite since $G / \xi_1 G$ satisfies the maximal condition.

Inductively $G / \xi_1 G$ is finite, and so is G . □

Theorem 1.3.2. *Let G be a nilpotent group and $\{1\} \neq H \triangleleft G$.*

Then

$$H \cap \xi G \neq \{1\}.$$

Proof. Proof by induction c , the nilpotency class of G . Note that if $H \leq \xi G = \xi_1$ then there is nothing to prove. Obviously, $H \cap \xi_c \neq \{1\}$. We want to show that if $H \cap \xi_i \neq \{1\}$, then $H \cap \xi_{i-1} \neq \{1\}$ for $i > 1$. Let $1 \neq h \in H \cap \xi_i$; if $h \notin \xi G$, since $h \in \xi G$ the proof is complete. So

$$\exists g \in G \text{ such that } h_1 = [h, g] \neq 1.$$

Observe that $h_1 \in H$, since $H \triangleleft G$. Also

$$h_1 \in [\xi_i, G] \leq \xi_{i-1}.$$

□

Corollary 1.3.3. *Let $\phi : G \rightarrow K$ be a homomorphism of nilpotent groups. Then,*

$$\phi \text{ is monic} \Rightarrow \phi|_{\xi G} \text{ is monic}.$$

Corollary 1.3.4. *Let G be a nilpotent group. G is torsion-free if and only if ξG is torsion-free.*

Lemma 1.3.5. *If G is torsion-free nilpotent group, then*

$$\forall x, y \in G \ x^n = y^n \Leftrightarrow x = y.$$

Lemma 1.3.6. *Let G be a torsion-free nilpotent group. Then $G/\xi G$ is also torsion-free.*

Proof. Let $x \in G$ such that $x^m \in \xi G$.

$$\forall g \in G \ [x^m, g] = 1$$

$$(g^{-1}xg)^m = x^m \Leftrightarrow g^{-1}xg = x.$$

Therefore $[x, g] = 1$, so $x \in \xi G$. □

Corollary 1.3.7. *If G is a torsion-free nilpotent group, then $\xi_{n+1}G/\xi_nG$ is torsion-free abelian for any n .*

Note that this corollary doesn't hold for the lower central series.

Lemma 1.3.8.

$$[x, yz] = [x, z][x, y]^z$$

$$[xy, z] = [x, z]^y[y, z].$$

Proof. In any group,

$$x^y = x[x, y]$$

and so

$$x^{yz} = x[x, z][x, y]^z = x[x, yz].$$

Similarly,

$$(xy)^z = xy[xy, z] = x^z y^z = x[x, z]y[y, z] = xy[x, z]^y[y, z].$$

□

Note 1.3.9. $[y, x] = [x, y]^{-1}$.

Proof.

$$[y, x] = [x, y^{-1}]^y = [x^{-1}, y]^x = [y^{-1}, x^{-1}]^{xy} = [x, y]^{-1}.$$

□

1.4 The Frattini subgroup

Definition 1.4.1. ΦG , a Frattini subgroup of G , is the intersection of all maximal subgroups of G . In particular, ΦG is characteristic.

Definition 1.4.2. A maximal subgroup of a group is a proper subgroup which is not contained in any larger proper group.

Definition 1.4.3. Let $H \leq G$. H is characteristic if $H\alpha \leq H \forall \alpha \in \text{Aut } G$.

Definition 1.4.4. $g \in G$ is called to be a non-generator element of G , if $G = gp(X, g)$ then $G = gp(X)$.

Theorem 1.4.1. (*Frattini*) ΦG is the set of all non-generators in G .

Theorem 1.4.2. Let G be a finitely generated nilpotent group.

Then

$$G' \leq \Phi G.$$

Proof. By induction on c , the nilpotency class of G . We must show that if $gp(X, G') = G$, then $gp(X) = G$.

If $c = 1$, G is abelian and hence $G' = 1$ and $gp(X, G') = gp(X) = G$.

Now, let $c > 1$. We assume the theorem holds for all finitely generated nilpotent groups of class less than c .

$$\gamma_c G = gp([u, v] | u \in \gamma_c G, v \in G)$$

$$G/\gamma_c G = gp(X\gamma_c G/\gamma_c G, G'\gamma_c G/\gamma_c G).$$

So, if $g \in G$, then

$$g = w(x) \cdot z \quad (z \in \gamma_c G)$$

$$u = w_1(x) \cdot z_1 \quad (z_1 \in \gamma_c G \leq \xi G),$$

$$v = w_2(x) \cdot z_2 \quad (z_2 \in \gamma_c G \leq \xi G).$$

Hence,

$$\begin{aligned} [u, v] &= [w_1(x)z_1, w_2(x)z_2] \\ &= [w_1(x), w_2(x)] \in gp(X). \end{aligned}$$

That is to say $\gamma_c G \leq gp(X)$, hence $G = gp(X)$. \square

Corollary 1.4.3. *Let G be a nilpotent group and $X \subset G$. Then X generates G if and only if $X\phi$ generates G_{ab} , where $\phi : G \rightarrow G/[G, G]$ is a canonical homomorphism.*

Theorem 1.4.4. *Suppose G is a finitely generated nilpotent group. Then G is finite if and only if G_{ab} is finite.*

Proof. By induction on c , the nilpotency class of G . \square

1.5 Polycyclic-by-finite groups. Hirsch number

Lemma 1.5.1. *Finitely generated nilpotent groups are polycyclic.*

Proof. This can be proved by observing that if G is a finitely generated nilpotent group of class c then, $\gamma_c G$ is also finitely generated.

Let G be a finitely generated nilpotent group of class c . If $c \leq 1$, then G is abelian, and hence G is polycyclic. Now assume that $c > 1$. The last non-trivial term in the lower central series of G is $\gamma_c G$. $\gamma_c G$ is abelian and finitely generated. Therefore $\gamma_c G$ is polycyclic. The quotient $G/\gamma_c G$ is nilpotent of class $c - 1$. By induction on c , $G/\gamma_c G$ is polycyclic. Thus G is polycyclic. \square

Note 1.5.2. *There exists finitely generated solvable groups that are not polycyclic.*

Lemma 1.5.3. *A solvable group satisfies maximal condition if and only if it is polycyclic.*

Corollary 1.5.4. *Polycyclic groups are solvable.*

Lemma 1.5.5. *Finitely generated abelian groups are polycyclic.*

Lemma 1.5.6. *Quotient groups of polycyclic groups are polycyclic.*

Lemma 1.5.7. *Subgroups of polycyclic groups are polycyclic.*

Lemma 1.5.8. *Polycyclic groups satisfies the maximal condition.*

In other words, subgroups of finitely generated polycyclic groups are finitely generated.

Definition 1.5.1. The group G is said to be polycyclic-by-finite if G has a subnormal series

$$1 \triangleleft G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G,$$

where G_i/G_{i-1} is either infinite cyclic or finite. If G has a normal series of this type, then G is said to be strongly polycyclic-by-finite.

Lemma 1.5.9. *A finitely generated torsion-free nilpotent group is polycyclic-by-finite.*

Proof. Let G be a finitely generated torsion-free nilpotent group.

G is polycyclic, therefore it is also polycyclic-by-finite. \square

Lemma 1.5.10. (*[35]*) *Let G be a polycyclic-by-finite group, the Hirsch number of G is denoted by $h(G)$. $h(G)$ is an invariant of G , i.e. it is independent of the series used to define it. Suppose $H \leq G$ and $N \triangleleft G$; then the followings are true:*

1. $h(H) \leq h(G)$.
2. $h(H) = h(G) \iff [G : H] < \infty$.
3. $h(G) = h(N) + h(G/N)$.

1.6 Effect of indicability on the order of finitely generated nilpotent groups

Definition 1.6.1. A group A is termed indicable if there exists a homomorphism of A onto the infinite cyclic group.

Lemma 1.6.1. *A finitely generated group G is indicable if and only if G_{ab} is infinite.*

Theorem 1.6.2. *(Higman, [23]) Every finitely generated torsion-free nilpotent group is indicable.*

Corollary 1.6.3. *The abelianization of a finitely generated torsion-free nilpotent group is infinite.*

Proof. Let A be a finitely generated torsion-free nilpotent group. Then, by theorem 1.6.2, A is indicable. Hence by lemma 1.6.1 A is infinite. □

Lemma 1.6.4. *The abelianization of an infinite finitely generated nilpotent group is again infinite.*

Proof. Let A be an infinite finitely generated nilpotent group. Then there is a canonical homomorphism from A onto $A/\tau A$, where τA is the torsion group of A . Since $A/\tau A$ is a finitely generated torsion-free nilpotent group, by corollary 1.6.3, $A/\tau A$ is infinite. Hence A is infinite. \square

The alternate proof of lemma 1.6.4, can be found in theorem 1.4.4.

Theorem 1.6.5. (*K. A. Hirsch, [4]*) *Let G be a finitely generated nilpotent group. Then G can be embedded as a subgroup of finite index in a direct product $D = A \times B$, where A is finite and B is torsion-free.*

Proof. Let $T = \tau G$. Then T is finite. So, using the fact that G is residually finite, there is a normal subgroup N of G of finite index such that

$$N \cap T = 1.$$

Put

$$A = G/N,$$

$$B = G/T.$$

Then A is finite and B is torsion-free. Since $N \cap T = 1$, the mapping

$$\phi : g \rightarrow (gN, gT) \quad (g \in G)$$

of G into $D = A \times B$,

is a monomorphism. Moreover, $N\phi$ is a subgroup of finite index in B , and hence $G\phi$ is of finite index in D . This completes the proof. □

1.7 The Mal'cev completion. Fundamental theorem of torsion-free nilpotent groups and related results

Definition 1.7.1. ([27]) A group is called complete if, for any element a of G and an arbitrary natural number n , the equation $x^n = a$ has at least one solution in G , or, in other words, if every root of every element of G belongs to G .

Theorem 1.7.1. *A finitely generated nilpotent group is complete if and only if it contains no proper subgroup of finite index.*

Corollary 1.7.2. *Every group is contained in a complete group.*

Note 1.7.3. ([4]) *Let \mathcal{D} denote the set of the class of groups in which extraction of roots is always uniquely possible.*

Definition 1.7.2. ([4]) Let us term a minimal torsion-free nilpotent \mathcal{D} -group, $m(G)$, containing a given torsion-free nilpotent group G , a Mal'cev completion of G .

Lemma 1.7.4. ([4]) *Let G be a finitely generated torsion-free nilpotent group of class c , let g be an element of G and let n be a positive integer. Then G can be embedded in a finitely generated torsion-free nilpotent group of class c in which g has an n -th root.*

Corollary 1.7.5. ([4]) *Every finitely generated torsion-free nilpotent group can be embedded in a nilpotent \mathcal{D} -group.*

Theorem 1.7.6. (Mal'cev, [4]) *Let A be a torsion-free nilpotent group. Then A can be embedded in a torsion-free nilpotent \mathcal{D} -group.*

Corollary 1.7.7. ([4]) *Let G, H be torsion-free nilpotent groups, and let ϕ be a homomorphism of G into H . If $m(G)$ and $m(H)$ are any Mal'cev completions of G and H , then ϕ can be extended uniquely to a homomorphism $m(\phi)$ of $m(G)$ onto $m(H)$.*

Corollary 1.7.8. ([4]) *Let G be a torsion-free nilpotent group. If $m(G)$ and $m'(G)$ are Mal'cev completions of G then they are isomorphic.*

Theorem 1.7.9. (*Mal'cev fundamental theorem of torsion-free nilpotent groups*) *If G_1^* and G_2^* are two completions of G , then there exists an isomorphism between them that extends the identity automorphism of G , and this isomorphism is unique.*

1.8 Perfect groups

Definition 1.8.1. ([18]) A group H is called perfect if $H = H'$ (i.e. H is equal to its own commutator subgroup).

Lemma 1.8.1. *Every non-abelian simple group is perfect.*

Lemma 1.8.2. *If H and K are perfect subgroups of a group G then $gp(H, K)$ is also perfect.*

Lemma 1.8.3. *The subgroup of a group G generated by any collection of perfect subgroups is perfect.*

Lemma 1.8.4. *Any conjugate of a perfect subgroup is perfect.*

Lemma 1.8.5. *Any group G has a unique maximal perfect subgroup and this subgroup is normal.*

Chapter 2

Products

2.1 Generalized central product and some related results

Definition 2.1.1. ([34]) A group G is said to be the central product of its normal subgroups G_1, \dots, G_n , if

$$G = G_1 G_2 \cdots G_n,$$

$$[G_i, G_j] = 1, \text{ for } i \neq j, \text{ and}$$

$$G_i \cap \prod_{i \neq j} G_j = \xi G, \text{ for all } i.$$

Note 2.1.1. Since $\xi G_i \leq \xi G$, it follows that

$$\xi G_i = \xi G.$$

Lemma 2.1.2. $\mu_i : G_i \rightarrow G$ is monic.

Proof. Map $G_i \xrightarrow{\mu_i} G$, in such a way that $g_i \mapsto g_i$. We want to show that the kernel of this map is trivial. If

$$g_i \in \ker \mu_i,$$

$$\text{then } \mu_i(g_i) = 1.$$

But on the other hand

$$\mu_i(g_i) = g_i = 1.$$

This completes the proof of lemma (2.1.2), that μ_i is an injective homomorphism. □

Another definition of the generalized central product is:

Definition 2.1.2. Suppose that

$$\{A_i = \langle X_i; R_i \rangle \mid i \in I\}$$

is an indexed family of presentations

$$A_i = \langle X_i; R_i \rangle$$

of the groups A_i , and suppose C is another group equipped with monomorphisms

$$\phi_i : C \rightarrow A_i \ (i \in I) \text{ and } C \leq \xi A_i.$$

We term the group A defined by the presentation

$$A = \langle \cup X_i; \cup R_i \cup \{c\phi_i c^{-1} \phi_j \mid c \in C, i, j \in I\} \cup \{[x_i, x_j] = 1 \mid i, j \in I\} \rangle$$

where we assume that the X_i are disjoint, the generalized central product of A_i amalgamating the central subgroup C .

$$A = \prod_{i \in I}^{\times} \{A_i; C\}.$$

If $C = 1$, then A is termed the direct product of A_i s.

According to Dyck's theorem, there are canonical homomorphisms μ_i of each A_i to A . It can be proved that μ_i are monomorphisms, and, if we identify A_i with $A_i \mu_i$, then

$$c\phi_i = c\phi_j \ \forall c \in C, i, j \in I.$$

So, we can identify C with any of its images $C\phi_i$ which have already identified with $C\phi_j \mu_i$. $A_i \cap A_j = C$ ($i, j \in I, i \neq j$) and $A =$

$gp(\cup_{i \in I} A_i)$. A can be also shown as the following:

$$A = \left(\prod_{i \in I}^{\times} A_i \right) / gp(c\phi_i c^{-1} \phi_j | i, j \in I, c \in C).$$

Lemma 2.1.3. μ_i is a monomorphism.

Proof. Recall

$$\mu_i : A_i \rightarrow \left(\prod_{i \in I} A_i \right) / gp(c\phi_i c^{-1} \phi_j | i, j \in I, c \in C).$$

Put

$$S = gp(c\phi_i c^{-1} \phi_j | i, j \in I, c \in C), \text{ where } C \xrightarrow{\phi_i} A_i.$$

We want to show that μ_i is a monomorphism. In order to do this

we must show that

$$\ker \mu_i = 1.$$

Let $a \in A_i$ be an element in the kernel of μ_i , $a \in \ker \mu_i$, so that

$\mu_i(a) = 1$. We want to show that $a = 1$. A generic element in S

has the following form:

$$s = (c_1 \phi_{i_1} c_1^{-1} \phi_{j_1}) \cdots (c_n \phi_{i_n} c_n^{-1} \phi_{j_n}).$$

Since $\mu_i(a) = 1$, then a is an element in S , and takes the following form:

$$a = (c_1 \phi_{i_1} c_1^{-1} \phi_{j_1}) \cdots (c_n \phi_{i_n} c_n^{-1} \phi_{j_n}).$$

Let's consider two cases: the case where none of i_r and j_r are equal to i , and the case where some of them are equal to i .

Case 1: Since none of the subscripts are equal to i , this implies that

$$a \in gp(A_k | k \neq i).$$

But on the other hand, a is an element in A_i as well. Therefore

$$a \in A_i \cap gp(A_k | i \neq k) = 1 \quad (\text{by a property of direct product}).$$

This implies that $a = 1$.

Case 2: Now suppose that some of the indexes are equal to i , ($i_l = i$). Note that $c\phi_i$, for $c \in C$, is central in A_i . We thus have:

$$a = \left(\prod_i c_l \phi_{i_l} c_l^{-1} \phi_{j_l} \right) \left(\prod_{k_m \neq i} c_k \phi_{k_1} c_k^{-1} \phi_{k_1} \cdots c_n \phi_{k_n} c_n^{-1} \phi_{k_n} \right).$$

Note that the following is equal to one, since we consider this in the i -th place, and follows similar to the previous case

$$\left(\prod_{k_m \neq i} c_k \phi_{k_1} c_k^{-1} \phi_{k_1} \cdots c_n \phi_{k_n} c_n^{-1} \phi_{k_n} \right) = 1.$$

Now

$$\prod_i c_i \phi_i = \left(\prod c_i \right) \phi_i = c \phi_i.$$

Hence

$$a = (c \phi_i) \left(\prod_{j_i \neq i} c_l^{-1} \phi_{j_i} \right).$$

We have

$$ac^{-1} \phi_i = \left(\prod_{j_i \neq i} c_l^{-1} \phi_{j_i} \right) \in A_i \cap gp(A_k | i \neq i) = 1.$$

So

$$a = c \phi_i.$$

This implies that $a = 1$, since

$$a = c \phi_i = \left(\prod_i c_l \phi_i c_l^{-1} \phi_{j_i} \right).$$

The right hand side has an even number of factors, so this is possible only if it is equal to one. This completes the proof that each factor injects in A . \square

Let's look at the case where there are only two factors:

Lemma 2.1.4. *Define*

$$\mu : A \rightarrow (A \times B) / gp(c\phi(c)^{-1} | c \in C_A) \text{ where } C_A \stackrel{\phi}{\cong} C_B.$$

μ is an injective homomorphism.

Proof. Put

$$S = gp(c(c^{-1}\phi) | c \in C_A) \text{ where } C_A \stackrel{\phi}{\cong} C_B.$$

Let

$$\mu : A \rightarrow S$$

be defined by

$$\mu : a \mapsto aS.$$

We must show that the kernel of this map is trivial, i.e show for an element $a \in A$ such that a is in the kernel of μ , $a = 1$. Now for an arbitrary element of $a \in A$,

$$a\mu = aS$$

$$a\mu = ac(c^{-1}\phi)S.$$

Now let

$$a \in \ker \mu.$$

Thus

$$ac(c^{-1}\phi) = 1$$

$$ac = c^{-1}\phi$$

$$a = 1.$$

Therefore μ is a monomorphism. □

Lemma 2.1.5. (*[35]*) *A direct product of finitely many nilpotent groups is nilpotent.*

Note 2.1.6. *A direct product of infinitely many nilpotent groups is not nilpotent.*

Lemma 2.1.7. *The generalized central product of finitely many nilpotent groups is nilpotent.*

Proof. The generalized central product of nilpotent groups is the quotient of the direct product of finitely many nilpotent groups, (which is again nilpotent, pg. 6. [35]) and another nilpotent group. The quotient of nilpotent groups is nilpotent, as a corollary of lemma (1.1.6). Therefore the generalized central product of finitely many nilpotent group is nilpotent. \square

Lemma 2.1.8. *The generalized central product of finitely many solvable groups is solvable.*

Corollary 2.1.9. *The generalized central product for an arbitrary number of abelian groups is abelian.*

Proof. The generalized central product of abelian groups is the quotient of the direct product of an arbitrary number of abelian

groups, which is again abelian [35], and another abelian group.

The quotient of abelian groups is abelian, as a corollary of lemma

(3.3.5).

□

2.2 Generalized free product

Suppose that ([7])

$$\{G_i = \langle X_i; R_i \rangle \mid i \in I\}$$

is an indexed family of presentations $G_i = \langle X_i; R_i \rangle$ of the groups G_i , and suppose that H is another group equipped with monomorphisms

$$\phi_i : H \longrightarrow G_i \quad (i \in I).$$

Then we term the group G defined by the presentation

$$G = \langle \bigcup_{i \in I} X_i; \bigcup_{i \in I} R_i \cup \{h\phi_i h^{-1} \mid h \in H, i \in I\} \rangle \quad (1)$$

where we assume that the X_i are disjoint, the generalized free product of the G_i amalgamating H . We express the fact that G is given by such a presentation by writing

$$G = \left\{ \prod_{i \in I}^* G_i; H \right\}.$$

We assume that in (1) each $h\phi_i$ is expressed as an X_i product so that (1) actually looks like a presentation. According to Dyck's

theorem, there are canonical homomorphisms μ_i of which each G_i to G . It may be proved that the μ_i are monomorphisms, and if we identify G_i with $G_i\mu_i$, then $h\phi_i = h\phi_j$ for all $h \in H$, $i, j \in I$. So we can identify H with any one of its images $H\phi_i$ (which we have already identified with $H\phi_i\mu_i$) and it then follows that

$$G_i \cap G_j = H \quad (i, j \in I, i \neq j),$$

i.e., the G_i intersects precisely in H . Note that

$$G = gp\left(\bigcup_{i \in I} G_i\right).$$

It follows that every element $g \in G$ can be expressed in the form

$$g = y_1 \cdots y_n h \quad (n \geq 0) \quad (2)$$

where

$$(i) \quad y_i \in G_{i_j} - H \quad \text{and} \quad i_j \neq i_{j+1} \quad \text{for} \quad j = 1, \dots, n-1;$$

$$(ii) \quad h \in H.$$

In particular if $g \notin H$, then (2) can be rewritten as

$$g = z_1 \cdots z_n \quad (n \geq 1) \quad (3)$$

where

(iii) $z_j \in G_{i_j} - H$ and $i_j \neq i_{j+1}$ for $j = 1, \dots, n-1$.

Theorem 2.2.1. (*O. Schreier*) Suppose that $G = \{\prod^* G_i; H\}$.

Then

(a) the G_i embed into G ;

(b) $G_i \cap G_j = H$ ($i, j \in I, i \neq j$);

(c) every product of the form (3) satisfying (iii) is not equal to 1 in G .

Conversely, suppose that G is a group and that H is a subgroup of G . Furthermore, suppose that $\{G_i | i \in I\}$ is an indexed family of subgroups of G , that G is generated by these subgroups G_i , and that $G_i \cap G_j = H$ ($i, j \in I, i \neq j$). Then

$$G = \{\prod_{i \in I}^* G_i; H\}$$

if and only if every product of the form (3) satisfying (iii) is not equal to 1 in G .

Lemma 2.2.2. *Let*

$$G = \left\{ \prod_{i \in I}^* G_i; H \right\}.$$

Suppose that for each $i \in I$, F_i is a subgroup of G_i and that

$$F_i \cap H = K = F_j \cap H \quad (i, j \in I).$$

Then

$$gp\left(\bigcup_{i \in I} F_i\right) = \left\{ \prod_{i \in I}^* F_i; K \right\}.$$

Proof. Every strictly alternating $\bigcup_{i \in I} F_i$ -product, relative to K , is non-trivial because it is strictly alternating $\bigcup_{i \in I} F_i$ -product, relative to H . □

The following is the special case of the above theorem, where there are just two factors.

Lemma 2.2.3. *Let G be the free product of its subgroups A and B with H amalgamated. If A_1 is a subgroup of A , and B_1 is a subgroup of B , and if $A_1 \cap H = H_1 = B_1 \cap H$, then $G_1 = gp(A_1, B_1)$ is the amalgamated product of A_1 and B_1 with H_1 amalgamated.*

Proof. Note that $A_1 \cap B_1 = H_1$, by a simple computation. By definition $gp(A_1, B_1) = A_1 B_1$. We need to show that $gp_G(A_1, B_1) = \{A_1 * B_1; H_1\}^{\text{set}} \cong A_1 B_1$.

$$\forall w \in gp(A_1, B_1) \quad w = a_1 b_1 a_2 b_2 \cdots a_n b_n.$$

We want to show that if $n > 0$, then $w \neq 1$. If $w = 1$, then $a_1 b_1 a_2 b_2 \cdots a_n b_n$ is alternating in A_1, B_1 , and so in A, B . Thus $n = 0$, because G is a free product of A and B . Now using the universal property of amalgamated product, every element can be uniquely expressed as alternating product of them, and that it is again an element in the group G_1 . Take an element $x \in G_1 = A_1 B_1$. Transform it to the unique construction in G then we go back again in G_1 and all are doable since we always remain in the group G_1 . □

Proposition 2.2.4. ([29]) *Let H be a normal subgroup of A , and K be a normal subgroup of B . If N is the normal subgroup of $A * B$*

generated by H and K , then

$$(A * B)/N \simeq (A/H) * (B/K).$$

Proof. Observe that $(A * B)/N$ can be defined by adding relators, each of which is in A or in B , to the relators of $(A * B)$. \square

Proposition 2.2.5. ([29]) *Let G be the free product of A and B with the subgroups H and K amalgamated. Suppose that M and N are normal subgroups of A and B , respectively, such that $H \cap M = 1 = K \cap N$. If P is the normal subgroup of G generated by M and N , then G/P is the free product of A/M and B/N with the isomorphic subgroups HM/M and KN/N amalgamated.*

Proof. Using the presentation of G and adding on as relators words that generate M and N ; since $H \cap M = 1 = K \cap N$, $HM/M \simeq H$ and $KN/N \simeq K$. If $U_i(a_\nu)$ generates H in G , then $U_i(a_\nu)$ generates HM/M in G/P . Similarly, if $V_j(b_\mu)$ generates K in G , $V_j(b_\mu)$ generates KN/N in G/P . Moreover, if $U_i(a_\nu)$ into $V_j(b_\mu)$

induces an isomorphism between H and K , it also induces an isomorphism between HM/M and KN/N . \square

Proposition 2.2.6. (*[7]*) *Let $G = \{A * B; H = K\}$. Then any homomorphism α of A into a group L , and β of B into a group L , such that $\alpha = \beta\phi$ on H , can be extended to a homomorphism γ of G into L .*

2.3 Removing torsion from amalgamated product of nilpotent groups

Lemma 2.3.1. *Let A and B be two finitely generated nilpotent groups, C be a proper subgroup of A and B , and $G = \{A * B; C\}$, the amalgamated product of A and B , amalgamating C . Then*

$$gp_G(\tau A, \tau B) = \{\tau A * \tau B; \tau C\}.$$

Proof. Recall that τA and τB are torsion subgroups of A and B respectively. Since

$$\tau A \cap C = \tau C = \tau B \cap C.$$

Then by applying lemma (2.2.3):

$$gp_G(\tau A, \tau B) = \{\tau A * \tau B; \tau C\}.$$

□

Lemma 2.3.2. *Let A and B be two finitely generated nilpotent groups, C be a proper subgroup of A and B , and $G = \{A * B; C\}$, the amalgamated product of A and B , amalgamating C . Then there is an epimorphism from G onto a torsion-free group G^* , amalgamated product of finitely generated torsion-free nilpotent groups $A/\tau A$ and $B/\tau B$ with amalgamating subgroup $C/\tau C$,*

$$G^* = \{A/\tau A * B/\tau B; C/\tau C\}.$$

Proof. Let

$$\mu : G \rightarrow G^*$$

$$\mu : \{A * B; C\} \rightarrow \{A/\tau A * B/\tau B; C/\tau C\}$$

be defined by

$$\mu : a \mapsto a \tau A, \quad b \mapsto b \tau B.$$

Note that

$$c \mapsto c \tau A,$$

$$c \mapsto c \tau B,$$

$$c \mapsto c \tau A \cap c \tau B = c \tau C.$$

So μ agrees on the amalgamating subgroup C , by proposition (2.2.6), μ can be extended to a homomorphism from G to G^* , and by Von Dyck's theorem (4.2.2), μ is an epimorphism. \square

Lemma 2.3.3. *Let A and B be two finitely generated nilpotent groups, C be a proper subgroup of A and B , and $G = \{A * B; C\}$, the generalized free product of A and B . Then G is an extension of an amalgamated product of finitely generated nilpotent torsion groups by a torsion-free group, (amalgamated product of finitely generated torsion-free nilpotent groups).*

Proof. We claim that there is short exact sequence:

$$1 \rightarrow T \rightarrow G \rightarrow G^* \rightarrow 1$$

$$1 \rightarrow \{\tau A * \tau B; \tau C\} \rightarrow \{A * B; C\} \rightarrow \{A/\tau A * B/\tau B; C/\tau C\} \rightarrow 1$$

By lemma (2.3.1),

$$T = gp_G(\tau A, \tau B) = \{\tau A * \tau B; \tau C\}.$$

By lemma (2.3.2), there is a homomorphism from G onto G^* , and

we can calculate the kernel of this map,

$$\ker \mu = gp_G(\tau A, \tau B) = T.$$

□

Corollary 2.3.4.

$$G/gp_G(\tau A, \tau B) \simeq G^* = \{A/\tau A * B/\tau B; C/\tau C\},$$

$$G/\{\tau A * \tau B; \tau C\} \simeq G^* = \{A/\tau A * B/\tau B; C/\tau C\}.$$

2.4 Tree product

2.4.1 Historical overview

With regard to abstract groups, Hanna Neumann showed in the 1950s that, in general, subgroups of amalgamated products are not amalgamated products anymore, but “generalized free products”.

At the end of 1960s A. Karrass and D. Solitar described such subgroups by means of tree products and HNN-constructions. The descriptions was given by the Bass-Serre theory, in which groups acting on graphs in order to employ geometric intuition: the fundamental group of a graph of groups generalizes both amalgamated products, HNN-extensions and tree products.

Definition 2.4.1. Suppose Γ is a tree, and (G, γ) is a graph of groups. We term $\pi_1(G, \Gamma, \Gamma)$ a tree product.

Lemma 2.4.1. *Suppose that (G, Γ) is a graph of groups, and let T be a maximal tree in Γ . Then if we restrict G to T , we get*

an associated graph of groups (G, T) . Then $\pi_1(G, T, T)$ is a subgroup of $\pi_1(G, T, T)$. Moreover, $\pi_1(G, \Gamma, \Gamma)$ is an HNN-extension of $\pi_1(G, T, T)$ with one stable letter for each $y \in A - T$, where A is an orientation of Γ .

2.4.2 Subgroups of amalgamated free products

([10]) Let $G = \{A * B; C\}$. The tree Γ of G :

$V(\Gamma) = (\text{Set of all the cosets of } gA \text{ of } A \text{ in } G) \cup (\text{set of all cosets } gB \text{ of } B \text{ in } G).$

$E(\Gamma) = (\text{Set of all the cosets } gC \text{ of } C \text{ in } G).$

Now let $H \leq G$. We want to see what can be said about H .

Note that H acts on Γ . So $H \simeq \pi_1(H, H \setminus \Gamma, U)$. So H is an HNN-extension of a tree product of stabilizer of vertices and edges in Γ (under H).

Let's compute some stabilizers of some vertices.

Suppose $h \in H$ stabilizes gA :

$$hgA = gA$$

$$g^{-1}hgA = A$$

$$g^{-1}hg \in A$$

$$h \in H \cap gHg^{-1}$$

$$H_{gA} = H \cap gAg^{-1}$$

$$H_{gB} = H \cap gBg^{-1}$$

$$H_{gC} = H \cap gCg^{-1}.$$

Corollary 2.4.2. *If $H \triangleleft G$, and*

$$H \cap A = \{1\} = H \cap B,$$

then H is free.

Corollary 2.4.3. *If $H \triangleleft G$, and*

$$H \cap C = \{1\},$$

then

$$H = \prod_{i \in I}^* X_i * F,$$

where the X_i are conjugates of subgroups of A and B and F is free.

Theorem 2.4.4. (*H. Neumann, Karrass, Solitar*) Suppose $G = \{A * B; C\}$ is an amalgamated product. Then every subgroup H of G is an HNN-extension of a tree product in which the vertex groups are conjugates of subgroups of either A or B and the edge groups are conjugates of subgroups of C . The associated subgroups involved in the HNN-extension are also conjugates of subgroups of C .

Lemma 2.4.5. Let K be the HNN-extension of a tree product T , stable letters F . Then $K/gp_K(T) \simeq F$. Furthermore F is free on the set of stable letters.

Theorem 2.4.6. (*A.G. Kurosh*) Let $G = A * B$ be the free product of its subgroups A and B , and let $H < G$. Then H is a tree product of conjugates of subgroups of A and B , and a free group.

Chapter 3

Residual properties

3.1 Preliminaries

Definition 3.1.1. Let X be a class of groups. Then G is residually- X if and only if, for each $1 \neq x \in G$, there is an epimorph of G in X such that the element corresponding to x is not the identity. In other words, G is residually- X if for each $g \in G$, ($g \neq 1$) there exists a normal subgroup N of G such that $g \notin N$ and $G/N \in X$.

Another way of describing residual properties is:

Definition 3.1.2. Let X be a class of groups, and G be a group. Then X separates G if and only if G is residually- X .

Definition 3.1.3. X discriminates G if $\forall n \geq 1$ and for all $\{g_1, \dots, g_n\} \subset$

G , ($g_i \neq 1$) there exist $N \triangleleft G$ such that $g_i \notin N$ for ($i = 1, \dots, n$) and $G/N \in X$.

Lemma 3.1.1. *If X is a class of groups, then G is residually- X . Furthermore, G is residually- X if and only if G is residually-residually- X .*

3.2 Equivalent definitions for residual solvability

Definition 3.2.1. A group G is said to be *residually solvable* if, for every $1 \neq a \in G$, there exists a normal subgroup $N \triangleleft G$ such that $a \notin N$ and G/N is solvable.

Another definition of residually solvable is:

Definition 3.2.2. A group G is said to be *residually solvable* if, for every $a \in G$ $a \neq 1$, there exists a solvable group S and a homomorphism $\phi : G \rightarrow S$ such that $a\phi \neq_S 1$.

However, these two definitions are equivalent, as the next lemma shows.

Lemma 3.2.1. *Definition (3.2.1) is equivalent to definition (3.2.2).*

Proof. (\Rightarrow): Suppose G satisfies definition (3.2.1). Put

$$S = G/N,$$

and let

$$\phi : G \rightarrow S$$

be the canonical projection. Then S is a solvable group, and since

$$a \notin N,$$

$$a\phi \neq_S 1.$$

So G satisfies definition (3.2.2).

(\Leftarrow): Suppose G satisfies definition (3.2.2). Then let

$$S' = \mathfrak{S}(\phi) \leq S.$$

If S is solvable of length i , commutators of weight i are equal to 1 in S . Then commutators of weight i in S' are also equal to 1, since $S' \leq S$. So S' is also solvable, of length $\leq i$. Since

$$G/\ker(\phi) \simeq S'$$

is a solvable group of length $\leq i$, we know that

$$\ker(\phi) \geq \delta_i(G).$$

Since

$$a\phi \neq_S 1,$$

we know that

$$a \notin \ker(\phi).$$

But

$$\ker(\phi) \geq \delta_i(G),$$

so

$$a \notin \delta_i(G).$$

Taking

$$N = \delta_i(G) \triangleleft G,$$

we get that

$$a \notin N \text{ and } G/N \text{ is solvable (of length } \leq i).$$

Thus, G satisfies definition (3.2.1). □

Another definition of residually solvable is:

Definition 3.2.3. A group G is said to be *residually solvable* if, for every $a \in G$ $a \neq 1$, there exists an integer $i \in \mathbb{N}$ such that $a \notin \delta_i G$.

This definition is equivalent to the previous definitions (3.2.1) and (3.2.2), as the next lemma shows.

Lemma 3.2.2. *Definitions (3.2.1) and (3.2.2) are equivalent to definition (3.2.3).*

Proof. (\Rightarrow): Suppose G satisfies definition (3.2.1). Then for every non-trivial $a \in G$ there exists an $N \triangleleft G$ such that $a \notin N$ and G/N is a solvable group of length $i = i(a)$. So products of weight i commutators in G/N equal 1. Also products of weight i commutators in G must be in N . In other words,

$$\delta_i G \leq N.$$

Since $a \notin N$, it follows that

$$a \notin \delta_i G.$$

Thus, G satisfies definition (3.2.3).

(\Leftarrow): Suppose G satisfies definition (3.2.3). Then for every non-trivial $a \in G$ there exists an $i = i(a) \in \mathbb{N}$ such that $a \notin \delta_i G$.

Put

$$N = \delta_i G \triangleleft G.$$

Then

G/N is solvable (of length $\leq i$)

and $a \notin N$.

Thus, G satisfies definition (3.2.1). □

Another definition of residually solvable is:

Definition 3.2.4. A group G is said to be *residually solvable* if

the set

$$\Delta(G) = \{\delta_\lambda G\}_{\lambda \in \mathbb{N}}$$

is a solvable filtration of G .

This definition is equivalent to the previous definition (3.2.3), as the next lemma shows.

Lemma 3.2.3. *Definition (3.2.4) is equivalent to definition (3.2.3).*

Proof. To show that $\Delta(G)$ is a solvable filtration of G we must, by definition, show that

1. $\forall \lambda \in \mathbb{N}, \delta_\lambda G \triangleleft G$.
2. $\forall \lambda \in \mathbb{N}, G/\delta_\lambda G$ is solvable.
3. $\bigcap_{\lambda \in \mathbb{N}} \delta_\lambda G = \{1\}$.

Now property (1) and (2) are true for any group G , so we only need to consider property (3).

(\Rightarrow): Suppose G satisfies definition (3.2.4). Then $\Delta(G)$ is a solvable filtration, and, in particular, $\Delta(G)$ satisfies property (3).

Then, given any non-trivial $a \in G$, since

$$a \notin \bigcap_{\lambda \in \mathbb{N}} \delta_\lambda G,$$

there must be some least $i = i(a)$ for which $a \notin \delta_i G$. Thus we have shown that G satisfies the conditions of definition (3.2.3).

(\Leftarrow): Suppose G satisfies definition (3.2.3). Then given any non-trivial $a \in G$, there is some $i = i(a)$ for which $a \notin \delta_i G$. So for any non-trivial $a \in G$, a cannot be in $\bigcap_{\lambda \in \mathbb{N}} \delta_\lambda G$. Thus

$$\bigcap_{\lambda \in \mathbb{N}} \delta_\lambda G = \{1\},$$

so $\Delta(G)$ satisfies (3) and is a solvable filtration of G . Thus, G satisfies the conditions of definition (3.2.4). \square

3.3 Some facts about solvability and residual solvability

Lemma 3.3.1. *Subgroups of solvable groups are solvable.*

Proof. Let G be a solvable group of length i , and let $H \leq G$.

Then every weight i commutator of elements in H is also a weight i commutator of elements in G , and so is equal to 1. Thus

$$\delta_i H = 1,$$

and H is solvable of length $\leq i$. □

Lemma 3.3.2. *Every subgroup of a poly-solvable group is again poly-solvable.*

Proof. This can be proved by applying lemma (3.3.1) and (1.1.1).

□

Lemma 3.3.3. *Subgroups of residually solvable groups are residually solvable.*

Proof. Let G be a residually solvable group, and let $H \leq G$. To show that H is residually solvable (by using definition (3.2.2)), let $a \in H$ be a non-trivial element. Then $a \in G$, so

$\exists \quad S$ a solvable group,

and $\phi : G \rightarrow S$,

such that $a\phi \neq_S 1$.

Let

$$S' = H\phi \geq S.$$

Then S' is solvable, by lemma (3.3.1), and

$\phi|_H : H \rightarrow S'$ satisfies $a\phi|_H \neq 1$.

□

Lemma 3.3.4. *Every subgroup of a poly-residually-solvable group is again poly-residually-solvable.*

Proof. This can be proved by applying lemma (3.3.3) and (1.1.1).

□

Lemma 3.3.5. *Quotients of solvable groups are solvable.*

Proof. Let G be a solvable group of length i , and $N \triangleleft G$. Then $\delta_i(G/N) = \delta_i GN = 1$, so G/N is solvable. \square

Lemma 3.3.6. *Let P be any group, and S be any solvable group of length i , and $\phi : P \rightarrow S$, then $\ker \phi \geq \delta_i P$.*

Proof. A commutator of weight i in P is sent by ϕ to a commutator of weight i in S , which is equal to one since S is solvable of length i . That shows that

$$\delta_i P \geq \ker \phi.$$

\square

Lemma 3.3.7. *Let P be any group, $K \trianglelefteq P$ and $K \geq \delta_i P$. Then P/K is solvable of length i .*

Proof. We want to show that P/K is solvable of length $\leq i$, i.e.

we want to show that

$$\delta_i(P/K) = \delta_i P/K = 1,$$

i.e. $\delta_i(P) \leq K$.

Now

$$P/K = P/\delta_i P/K/\delta_i P.$$

We know that $P/\delta_i P$ is solvable. Now using the lemma (3.3.5),

P/K is solvable. □

Lemma 3.3.8. *Let P be any group, and $K \triangleleft P$ satisfying: P/K is solvable and K be solvable. Then P is solvable. In other words, a solvable extension of a solvable group is again solvable.*

Lemma 3.3.9. *Let P be any group, and $K \triangleleft P$ satisfying: P/K is polycyclic, and K be poly-cyclic then P is poly-cyclic. In other words, a polycyclic extension of a polycyclic group is again polycyclic.*

Note 3.3.10. *Let P be any group, and $K \triangleleft P$ satisfying: P/K is nilpotent and K be nilpotent, then P is **not** nilpotent. In other*

words, a nilpotent extension of a nilpotent group is **not** nilpotent.

Proof. As a counter example, Σ_3 is metabelian, but not nilpotent.

□

Lemma 3.3.11. *Let P be any group, and $K \triangleleft P$ satisfying: P/K is solvable and K be residually solvable, then P is residually solvable.*

Proof. We want to show that P is residually solvable. So, for any $1 \neq w \in P$, we must find a solvable group S and $\phi : P \rightarrow S$ such that $w\phi \neq 1$.

Consider two cases, where $w \in P \setminus K$, and $w \in K$.

Case 1: $w \in P \setminus K$.

Let

$$S = P/K,$$

(note that S is solvable), and let

$$\phi : P \rightarrow P/K.$$

Now since $w \notin K$,

$$w\phi \neq 1.$$

Case 2: $w \in K$.

Since K is residually solvable, then there exists a solvable group

S and

$$\psi : K \rightarrow S$$

$$\text{where } \psi(w) \neq 1.$$

Suppose P/K is generated by

$$\{a_1K, a_2K, \dots, a_lK\}.$$

Now let

$$\phi : P \rightarrow S$$

$$\text{such that } \phi|_K \equiv \psi$$

$$\text{and } a_i\phi = 1.$$

Since

$$w \in K,$$

$$w\phi = w\psi \neq 1.$$

We claim that ϕ is a homomorphism. We know $\text{gen}(P) = \text{gen}(K) \cup \{a_1, \dots, a_l\}$.

(i) ψ is a homomorphism of $K \rightarrow S$, so every relator of K is sent to 1 in S .

(ii) Suppose r is a relator of P . If one replaces all a_i 's in r by 1, one will get a relator of K .

To show that ϕ is a homomorphism, we must show that every relator of P is sent to 1 in S . Let's take a relator r of P

$$\begin{aligned} r &= a_i K_i \cdots a_j K_j \\ &= 1 \cdot \psi(K_i) \cdots 1 \cdot \psi(K_j) \\ &= \psi(K_i \cdots K_j) \\ &=_{S} 1, \text{ by (ii) and (i).} \end{aligned}$$

Therefore, by Von Dyck's theorem, ϕ is a homomorphism. \square

Corollary 3.3.12. *A solvable extension of a residually solvable group is residually solvable.*

Note 3.3.13. *Let P be any group, and $K \triangleleft P$ satisfying: P/K is nilpotent and K be residually nilpotent then P is **not** residually nilpotent. In other words, a nilpotent extension of a residually nilpotent group is **not** residual nilpotent.*

Proof. As a counter example, one can consider that Σ_3 is metabelian, but not nilpotent. Subgroups of index 3 in Σ_3 are not normal in Σ_3 . Now if one considers the elements of order 3 in Σ_3 , there is no normal subgroup which does not contain that element. \square

Lemma 3.3.14. *Nilpotent groups are residually solvable.*

3.4 The filtration approach to residual solvability

Definition 3.4.1. A family $(A_\lambda | \lambda \in \Lambda)$ of normal subgroups of A is termed a *solvable filtration* of A if A/A_λ is solvable for every $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} A_\lambda = \{1\}$.

Definition 3.4.2. Let $(A_\lambda | \lambda \in \Lambda)$ of normal subgroups of A be a *solvable filtration* of A . We shall say that H is *solubly separable* in A if

$$\bigcap_{\lambda=1}^{\infty} HA_\lambda = H,$$

i.e. if $a \in A$, $a \notin H$, then $\exists \lambda$ such that $a \notin HA_\lambda$.

Example 3.4.1. $\{1\}$ is solubly separable in A , is equivalent to A is residually solvable.

Definition 3.4.3. Let $H \leq A$, then $(A_\lambda | \lambda \in \Lambda)$ is called an H -filtration of H , if

$$\bigcap_{\lambda \in \Lambda} HA_\lambda = H.$$

Note 3.4.2. *If A posses an H -filtration of $(A_\lambda)_{\lambda \in \Lambda}$, then H is solvably separable in A .*

Definition 3.4.4. Two equally indexed filtration $(A_\lambda | \lambda \in \Lambda)$ and $(B_\lambda | \lambda \in \Lambda)$ of A and B respectively are termed (H, K, ϕ) -compatible if

$$(A_\lambda \cap H)\phi = B_\lambda \cap K \quad (\forall \lambda \in \Lambda).$$

Note 3.4.3. *Under these circumstances ϕ induces for each $\lambda \in \Lambda$ an isomorphism ϕ_λ from HA_λ/A_λ to KB_λ/B_λ .*

Proposition 3.4.4. ([3]) *Let $(A_\lambda | \lambda \in \Lambda), (B_\lambda | \lambda \in \Lambda)$ be solvable (H, K, ϕ) -compatible filtration of the residually solvable groups A and B respectively. Suppose $(A_\lambda | \lambda \in \Lambda)$ is an H -filtration of A and $(B_\lambda | \lambda \in \Lambda)$ is a K -filtration of B . If, for every $\lambda \in \Lambda$,*

$$\{A/A_\lambda * B/B_\lambda; HA_\lambda/A_\lambda = KB_\lambda/B_\lambda\} \text{ is residually solvable,}$$

then so is

$$G = \{A * B; H = K\}.$$

Observation 3.4.5. *Let $\{A_\lambda\}_{\lambda \in \Lambda}$, $\{B_\lambda\}_{\lambda \in \Lambda}$ be families of normal subgroups of A and B respectively, satisfying the conditions:*

- (i) A/A_λ and B/B_λ are solvable,
- (ii) $HA_\lambda/A_\lambda \stackrel{\phi_\lambda}{\cong} KB_\lambda/B_\lambda$,
- (iii) $\bigcap_{\lambda \in \Lambda} HA_\lambda = H$ and $\bigcap_{\lambda \in \Lambda} KB_\lambda = K$,
- (iv) $\bigcap_{\lambda \in \Lambda} A_\lambda = \{1\} = \bigcap_{\lambda \in \Lambda} B_\lambda$,
- (v) $G_\lambda = \{A/A_\lambda * B/B_\lambda; HA_\lambda/A_\lambda \stackrel{\phi_\lambda}{\cong} KB_\lambda/B_\lambda\}$.

Suppose that the families $\{A_\lambda\}_{\lambda \in \Lambda}$, $\{B_\lambda\}_{\lambda \in \Lambda}$ are (H, K, ϕ) -compatible, i.e. for each $\lambda \in \Lambda$ the mapping

$$\phi_\lambda : hA_\lambda \rightarrow (h\phi)B_\lambda \text{ is an isomorphism.}$$

*Then $G = \{A * B; H = K\}$ is residually solvable.*

Now we want to explore the above proposition in reverse:

Problem 3.4.6. *We assume that $G = \{A * B; H = K\}$ is residually solvable, then*

$\exists N_i \trianglelefteq G$ s. t. G/N_i is solvable and

$\bigcap_{i \in I} N_i = 1$ where I is an indexed set.

Put

$$A_i = A \cap N_i \ (\forall i \in I),$$

$$B_i = B \cap N_i \ (\forall i \in I).$$

Hence

$$\bigcap_{i \in I} A_i = \{1\} = \bigcap_{i \in I} B_i.$$

Now there exist normal subgroups A_i of A and B_i of B such that

A/A_i and B/B_i are solvable,

and

$$\bigcap_{i \in I} A_i = \{1\} = \bigcap_{i \in I} B_i.$$

Note that

$$HN_i/N_i \leq G/N_i$$

and

$$KN_i/N_i \leq G/N_i.$$

Thus

$$HN_i/N_i \stackrel{\phi_i}{=} KN_i/N_i,$$

$$H/H \cap N_i \stackrel{\phi_i}{\simeq} K/K \cap N_i.$$

This allows us to form

$$G_i = \{A/A_i * B/B_i; HA_i/A_i = KB_i/B_i\}.$$

We are interested in answering two questions:

Q1) Does G_i ($\forall i \in I$) have to be residually solvable?

Q2) Are H, K solubly separable in A, B respectively? i.e. Does

$\bigcap_{i \in I} HA_i = H$ and $\bigcap_{i \in I} KB_i = K$? (i.e. If $a \in A, a \notin H$ then does there exist $i \in I$ such that $a \notin HA_i$.)

3.5 Residual solvability of free groups and free products

Theorem 3.5.1. (*Levi*) *Free groups are residually solvable.*

Proof. Let $F = gp(X)$ be free on the set X . For every word in $f \in F$, f can be expressed as follows:

$$f = x_1^{\epsilon_1} \cdot x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} \quad (x_i \in X)$$

We distinguish two cases:

Case 1: The exponent sum of one of the generators in f is not equal to zero.

Hence $f \notin \delta_2 F$. Now there is a canonical homomorphism from F onto $F/\delta_2 F$ (an abelian group), in which the image of f in $F/\delta_2 F$ is not 1.

Case 2: The exponent sum of all of the generators is zero in f .

We proceed by induction on the length of f . Let $t = x$, and let N

be the normal closure of X in F ,

$$N = gp_F(X) = gp(t^{-n}xt^n | n \in \mathbb{Z}, x \in X).$$

We observe that N is free on x^{t^n} , and the length of f is shorter in N than in F . So F is residually solvable. \square

Theorem 3.5.2. (*Magnus*) *Free product of residually solvable groups are residually solvable.*

Chapter 4

Abelianization and generalized free products

4.1 Some results on the structure of the abelianization

Lemma 4.1.1. *Let G be the amalgamated product of two groups A and B amalgamating C ,*

$$G = \{A * B; C\}.$$

Then the abelianization of G , G_{ab} takes the following form:

$$G_{ab} = (A_{ab} \times B_{ab}) / gp(\bar{c}\alpha\bar{c}^{-1}\beta | c \in C),$$

where α is the monomorphism from C into A , and β is the monomorphism from C into B .

Proof. Let σ be a homomorphism from $A * B$ into $A_{ab} \times B_{ab}$,

$$\sigma : A * B \rightarrow A_{ab} \times B_{ab}.$$

Put $N = gp(\bar{c}\alpha\bar{c}^{-1}\beta|c \in C)$.

Now $\bar{\sigma}$ induces to a homomorphism of G onto $(A_{ab} \times B_{ab})/N$.

$$\bar{\sigma} : G \rightarrow (A_{ab} \times B_{ab})/N.$$

We claim that the kernel of this map is $\delta_2 G = [G, G]$. In order to prove this, it suffices to show that $\delta_2 G \leq \ker \bar{\sigma}$. Suppose $g \in G$, then $g =_{\delta_2 G} ab$. If $g \in \ker \bar{\sigma}$, i.e. $\bar{\sigma}(g) = 1$, then

$$\bar{\sigma}(g) = \bar{a}N\bar{b}N = (\bar{a}\bar{b})N.$$

Therefore $\bar{a}\bar{b} \in N$. That is to say that we can express $\bar{a}\bar{b}$ as the

product of elements in N ,

$$\bar{a}\bar{b} = \bar{c}_1\alpha \bar{c}_1^{-1}\beta \cdots \bar{c}_k\alpha \bar{c}_k^{-1}\beta,$$

$$\bar{a} = (\bar{c}_1 \cdots \bar{c}_k)\alpha,$$

$$\bar{b} = (\bar{c}_1^{-1} \cdots \bar{c}_k^{-1})\beta,$$

$$\bar{a}\bar{b} = (\bar{c}_1 \cdots \bar{c}_k)\alpha (\bar{c}_1^{-1} \cdots \bar{c}_k^{-1})\beta,$$

i.e. $ab \in_{\delta_2 G} N$.

So $\ker \bar{\sigma} = \delta_2 G$.

$$G_{ab} \simeq G/\delta_2 G$$

$$G_{ab} \simeq G/\ker \bar{\sigma}$$

$$G_{ab} \simeq (A_{ab} \times B_{ab})/N.$$

That completes the proof of lemma (4.1.1). \square

Lemma 4.1.2. *Let G be the amalgamated product of A and B identifying C_A with C_B ,*

$$G = \{A * B; C_A = C_B\}.$$

Then the abelianization of G , G_{ab} maps onto

$$D = A_{ab}/gp(\bar{c}_a | c_a \in C_A) \times B_{ab}/gp(\bar{c}_b^{-1} | c_b \in C_B).$$

Proof. By lemma (4.1.1), the abelianization of G can be expressed as below:

$$G_{ab} = (A_{ab} \times B_{ab})/gp(\bar{c}_a \bar{c}_b^{-1}).$$

Put

$$N = gp(\bar{c}_a \bar{c}_b^{-1})$$

$$N_1 = gp(\bar{c}_a)$$

$$N_2 = gp(\bar{c}_b^{-1}).$$

Let θ to be a map from G_{ab} into D ,

$$\theta : G_{ab} \rightarrow D$$

$$\theta : (A_{ab} \times B_{ab})/N \rightarrow A_{ab}/N_1 \times B_{ab}/N_2$$

be defined by:

$$\bar{a}N \mapsto \bar{a}N_1$$

$$\text{and } \bar{b}N \mapsto \bar{b}N_2.$$

We want to show that θ is an epimorphism.

A typical element w in N takes the form $w = \bar{c}_a \bar{c}_b^{-1}$.

$$w = \bar{c}_a \bar{c}_b^{-1} \mapsto (\bar{c}_a N_1, \bar{c}_b^{-1} N_2) = (N_1, N_2).$$

So, by Von Dyck's theorem, (4.2.2), θ is an epimorphism. This completes the proof of lemma (4.1.2). \square

The following is another proof of lemma (4.1.2), using the presentation of groups.

Proof. Let A and B have the following presentation:

$$A = \langle X; R \rangle \quad \text{and}$$

$$B = \langle Y; S \rangle .$$

So the abelianization of A and B have the following presentation:

$$A_{ab} = \langle X; R \cup \{[x_i, x_j] \mid \forall x_i, x_j \in X\} \rangle$$

$$B_{ab} = \langle Y; S \cup \{[y_i, y_j] \mid \forall y_i, y_j \in Y\} \rangle .$$

The direct product of A_{ab} and B_{ab} has the following presentation,

since by definition of the direct product, all element of A_{ab} commute with all elements of B_{ab} :

$$A_{ab} \times B_{ab} = \langle X \cup Y; \quad R \cup S \cup \{[x_i, x_j] \mid \forall x_i, x_j \in X\} \\ \cup \{[y_i, y_j] \mid \forall y_i, y_j \in Y\} \\ \cup \{[x, y] \mid \forall x \in X, \forall y \in Y\} \rangle .$$

Recall that α is the monomorphism from C into A , and β is the monomorphism from C into B , and put

$$N = gp(c\alpha \ c\beta^{-1} \mid c \in C)$$

$$N_1 = gp(c\alpha \mid c \in C)$$

$$N_2 = gp(c\beta^{-1} \mid c \in C).$$

Now, by lemma (4.1.1), G_{ab} , the abelianization of G , has the following presentation:

$$\begin{aligned}
 G_{ab} = (A_{ab} \times B_{ab})/N = & \langle X \cup Y; R \cup S \cup \{[x_i, x_j] | \forall x_i, x_j \in X\} \\
 & \cup \{[y_i, y_j] | \forall y_i, y_j \in Y\} \\
 & \cup \{[x, y] | \forall x \in X, \forall y \in Y\} \\
 & \cup \{c\alpha c\beta^{-1} | c \in C\} \rangle .
 \end{aligned}$$

Let's find the presentation of each factor of D ,

$$A_{ab}/N_1 = \langle X; R \cup \{[x_i, x_j] | \forall x_i, x_j \in X\} \cup \{c\alpha | c \in C\} \rangle$$

$$B_{ab}/N_2 = \langle Y; S \cup \{[y_i, y_j] | \forall y_i, y_j \in Y\} \cup \{c\beta^{-1} | c \in C\} \rangle .$$

The direct product of them, D , has the following presentation:

$$\begin{aligned}
 D = A_{ab}/N_1 \times B_{ab}/N_2 = & \langle X \cup Y; R \cup S \cup \{[x_i, x_j] | \forall x_i, x_j \in X\} \\
 & \cup \{[y_i, y_j] | \forall y_i, y_j \in Y\} \\
 & \cup \{c\alpha | c \in C\} \\
 & \cup \{c\beta^{-1} | c \in C\} \\
 & \cup \{[x(c\alpha), y(c\beta^{-1})] | \forall x \in X, \forall y \in Y, c \in C\} \rangle .
 \end{aligned}$$

Now define θ to be a map from G_{ab} into D , by sending $x_i \mapsto x_i$ and $y_i \mapsto y_i$. Then $N \mapsto 1$. So, by using Von Dyck's theorem, (4.2.2), θ defines a homomorphism from G_{ab} onto D , and that completes the second proof of lemma (4.1.2). \square

4.2 The case of nilpotent factors

Theorem 4.2.1. *The generalized free product of two finitely generated nilpotent groups amalgamating a proper subgroup of them is not perfect.*

Proof. Let A and B be two finitely generated nilpotent groups, C_A and C_B be proper subgroups of A and B respectively. We want to show that the generalized free product of them,

$$G = \{A * B; C_A = C_B\} \text{ is not perfect.}$$

That is, we want to show that abelianization of G , G_{ab} is not trivial.

By lemma (4.1.2), G_{ab} maps onto

$$D = A_{ab}/gp(\bar{c}_a | c_a \in C_A) \times B_{ab}/gp(\bar{c}_b^{-1} | c_b \in C_B).$$

where \bar{c}_a and \bar{c}_b are the images of $c_a \in C_A$ and $c_b \in C_B$ in A_{ab} and B_{ab} respectively.

We claim that D is not trivial under the condition of theorem

(5.1.1). In order to prove the claim, we will prove,

$$A_{ab}/gp(\bar{c}_a) \neq \{1\} \quad \text{and}$$

$$B_{ab}/gp(\bar{c}_b^{-1}) \neq \{1\}.$$

Recall that C_A is a proper subgroup of A . Now, by using the Frattini subgroup theorem, (1.4.2), this implies that $gp(C_A, \delta_2 A)$ is a proper subgroup of A , where $\delta_2 A = [A, A]$. So

$$C_A < A$$

$$gp(C_A, \delta_2 A) < A$$

$$A/gp(C_A, \delta_2 A) \neq \{1\}.$$

Now by using the third isomorphism theorem and the above relation, we have:

$$\begin{aligned} \frac{A_{ab}}{gp(\bar{c}_a)} &\simeq \frac{\frac{A}{\delta_2 A}}{\frac{gp(C_A, \delta_2 A)}{\delta_2 A}} \\ &\simeq \frac{A}{gp(C_A, \delta_2 A)} \\ &\neq \{1\}. \end{aligned}$$

Similarly

$$B_{ab}/gp(\bar{c}_b^{-1}) \neq \{1\}.$$

This proves our claim that under the condition of the theorem (5.1.1),

$$D \neq \{1\} \quad \text{and so}$$

$$G_{ab} \neq \{1\}.$$

Hence G is not perfect. This completes the proof of the theorem (5.1.1). □

Theorem 4.2.2. (*Dyck*) Suppose $G = \langle X; R \rangle$ and $D = \langle X; R \cup S \rangle$. If γ and ν are the respective presentation maps, then $x\gamma \mapsto x\nu$ ($x \in X$) defines a homomorphism of G onto D .

4.3 Effects of the order of the abelianization of amalgamated products of nilpotent groups

Theorem 4.3.1. (*G. Baumslag, [11]*) *The amalgamated product of two finitely generated torsion-free nilpotent groups is indicable.*

Lemma 4.3.2. *The abelianization of amalgamated product of two finitely generated torsion-free nilpotent groups is infinite.*

Proof. Let A and B be two finitely generated torsion-free nilpotent groups, C be a proper subgroup of them, and G be the amalgamated product of them, amalgamating C ,

$$G = \{A * B; C\}.$$

G is indicable by a theorem of G. Baumslag (4.3.1), [11]. Therefore G_{ab} , the abelianization of G , is infinite, by lemma (1.6.1). This completes the proof of the theorem. \square

Theorem 4.3.3. *The abelianization of amalgamated product of two finitely generated nilpotent groups, is infinite only if one of the*

factors is infinite.

Proof. There exists an epimorphism from G onto a finitely generated torsion-free nilpotent group, by lemma (2.3.2),

$$G^* = \{A/\tau A * B/\tau B; C/\tau C\}$$

the amalgamated product of finitely generated torsion-free nilpotent groups. Therefore G^*_{ab} is infinite, by (4.3.2). This implies that G_{ab} is infinite. □

4.3.1 An example

Example 4.3.4. *Let*

$$A = \langle a, b, c; [a, b] = c, [a, c] = 1, [b, c] = 1 \rangle$$

be a nilpotent group of class 2, and

$$B = \langle e, f, g; [e, f] = g^2, [e, g] = 1, [f, g] = 1 \rangle$$

be an abelian group, where $h(A) = 3$ and $h(B) = 2$. Let $C_A = \text{gp}(b, c)$ and $C_B = \text{gp}(e, f)$.

Define an isomorphism ϕ which maps

$$b \mapsto f \quad \text{and} \quad c \mapsto e.$$

Form the free product of A and B , identifying C_A with C_B :

$$G = \{A * B; C_A = C_B\}.$$

$$G_{ab} = \langle a, b, c, e, f, g; b = f, c = e, [a, b] = c = 1, [a, c] = 1, [b, c] = 1,$$

$$[e, f] = g^2 = 1, [e, g] = 1, [f, g] = 1 \rangle$$

$$G_{ab} = \langle a, b, g; [a, b] = 1, g^2 = 1, [b, g] = 1 \rangle$$

We thus observe:

1. $gp(g) \triangleleft B$, and $B/gp(g)$ is a free abelian group of rank 2 (infinite).
2. $C_B \triangleleft B$, $h(C_B) = 2$, and B/C_B is abelian of order 2.
3. $[egp(g), fgp(g)] = gp(g)$.
4. B_{ab} is infinite, since the exponent sum of generator e is zero in every relator.
5. B is abelian.
6. The image of C_B in B_{ab} has index 2.

Chapter 5

Residual solvability of generalized free products of finitely generated nilpotent groups

5.1 Effect of abelianization on amalgamated product of nilpotent groups

Theorem 5.1.1. *The generalized free product of two finitely generated nilpotent groups amalgamated a proper subgroup of them is not perfect.*

Proof. Let A and B be two finitely generated nilpotent groups, C_A and C_B be proper subgroups of A and B respectively. We want to

show that the generalized free product of them,

$$G = \{A * B; C_A = C_B\} \text{ is not perfect.}$$

That is to show that the abelianization of G , G_{ab} is not trivial.

By lemma (4.1.2), G_{ab} maps onto

$$D = A_{ab}/gp(\bar{c}_a | c_a \in C_A) \times B_{ab}/gp(\bar{c}_b^{-1} | c_b \in C_B).$$

where \bar{c}_a and \bar{c}_b are the images of $c_a \in C_A$ and $c_b \in C_B$ in A_{ab} and B_{ab} respectively.

We claim that D is not trivial under the condition of theorem (5.1.1). In order to prove the claim, we will prove,

$$A_{ab}/gp(\bar{c}_a) \neq \{1\} \text{ and}$$

$$B_{ab}/gp(\bar{c}_b^{-1}) \neq \{1\}.$$

Recall that C_A is a proper subgroup of A . Now by using the Frattini subgroup theorem, (1.4.2), this implies that $gp(C_A, \delta_2 A)$

is a proper subgroup of A , where $\delta_2 A = [A, A]$. So

$$C_A < A$$

$$gp(C_A, \delta_2 A) < A$$

$$A/gp(C_A, \delta_2 A) \neq \{1\}.$$

Now, by using the third isomorphism theorem and the above relation, we have:

$$\begin{aligned} \frac{A_{ab}}{gp(\bar{c}_a)} &\simeq \frac{\frac{A}{\delta_2 A}}{\frac{gp(C_A, \delta_2 A)}{\delta_2 A}} \\ &\simeq \frac{A}{gp(C_A, \delta_2 A)} \\ &\neq \{1\}. \end{aligned}$$

Similarly

$$B_{ab}/gp(\bar{c}_b^{-1}) \neq \{1\}.$$

This proves our claim that, under the condition of the theorem (5.1.1)),

$$D \neq \{1\} \quad \text{and so}$$

$$G_{ab} \neq \{1\}.$$

Hence G is not perfect. This completes the proof the theorem
(5.1.1). \square

5.2 The amalgamated subgroup is central in both factors

Theorem 5.2.1. *The generalized free product of two finitely generated torsion-free nilpotent groups amalgamated central subgroups of them, is a nilpotent extension of a free group.*

Proof. Suppose that A and B are finitely generated nilpotent groups and G is an amalgamated product of A and B , with amalgamated subgroup C , where C is central in both A and B :

$$G = \{A * B; C\}.$$

Let S be the central product of A and B , (2.1.2):

$$S = (A \times B) / gp(c(c\phi^{-1}) | c \in C) \text{ where } C_A \stackrel{\phi}{\cong} C_B.$$

S contains a canonical homomorphism of A and B . Since these homomorphisms coincide on C , by proposition (2.2.6), they can be extended to a homomorphism μ . Let K be the kernel of μ . Since S is nilpotent by lemma (2.1.7), G/K is nilpotent. Now, μ restricted

to A or B is one-to-one by lemma (2.1.4). So, by a theorem of Hanna Neumann, K is free. Therefore G is a nilpotent extension of free group. \square

Corollary 5.2.2. *G is residually solvable.*

Proof. Note that free groups are residually solvable by (3.5.1). So G is an extension of a residually solvable group by a nilpotent group. By a corollary of lemma (3.3.11), G is residually solvable. Note that G is not necessarily residually nilpotent by note (3.3.10).

\square

We can generalize the above theorem to generalized free product of finite number of finitely generated torsion-free nilpotent groups.

Theorem 5.2.3. *The generalized free product of finite number of finitely generated torsion-free nilpotent groups, amalgamating central products of them, is an extension of a free group by a nilpotent group.*

Proof. Suppose that $\{A_i | i \in I\}$, is a finite indexed family of finitely generated nilpotent groups and G be the generalized free product of A_i , with amalgamated subgroup C :

$$G = \left\{ \prod_{i \in I}^* A_i; C \right\}.$$

Let S be the generalized central product of A_i (2.1.2):

$$S = \prod_{i \in I}^{\times} A_i / gp(c\phi_i c^{-1}\phi_j | i, j \in I, c \in C).$$

S contains a canonical homomorphisms of A_i for $i \in I$. Since these homomorphisms coincide on C , they can be extended to a homomorphism μ from G into S . Let K be the kernel of μ . Since S is nilpotent by lemma (??), it follows that G/K is nilpotent. By lemma (2.1.3), μ is one-to-one restricted to each factor, i.e.

$$K \cap A_i = 1 \text{ for all } i \in I.$$

So, by a theorem of Hanna Neumann (2.4.2), K is free. Hence G is a nilpotent extension of a free group. \square

5.2.1 The case of abelian factors

Corollary 5.2.4. *The generalized free product of two abelian groups is an extension of a free group by an abelian group.*

Proof. We can proceed the proof in a similar way as in theorem (5.2.1). □

Corollary 5.2.5. *The amalgamated product of two abelian groups, is residually solvable.*

Proof. An extension of a residually solvable group by an abelian group is again residually solvable by lemma (3.3.11). □

We can generalize the above theorem to generalized free product of arbitrary number of abelian groups.

Theorem 5.2.6. *The generalized free product of arbitrary number of abelian groups is an extension of a free group by an abelian group.*

Proof. Suppose that $\{A_i | i \in I\}$, is an indexed family of abelian groups and G be the generalized free product of A_i , with amalgamated subgroup C :

$$G = \left\{ \prod_{i \in I}^* A_i; C \right\}.$$

Let S be the generalized central product of A_i (2.1.2):

$$S = \prod_{i \in I}^{\times} A_i / gp(c\phi_i c^{-1}\phi_j | i, j \in I, c \in C).$$

S contains a canonical homomorphisms of A_i for $i \in I$. Since these homomorphisms coincide on C , they can be extended to a homomorphism μ from G into S . Let K be the kernel of μ . Since S is abelian by lemma (2.1.9), it follows that G/K is abelian. By lemma (2.1.3), μ is one-to-one restricted to each factor, i.e.

$$K \cap A_i = 1 \text{ for all } i \in I.$$

So, by a theorem of Hanna Neumann (2.4.2), K is free. Hence G is an abelian extension of a free group. \square

Corollary 5.2.7. *G is residually solvable.*

5.3 Homomorphic mappings and their kernels

Lemma 5.3.1. *There is an epimorphism from*

$$K = \left\{ \prod_{i \in I}^* B_i; C \right\}$$

onto

$$M = \prod_{i \in I}^* B_i / gp_{B_i}(C).$$

Proof. Suppose that $\{B_i = \langle Y_i; S_i \rangle \mid i \in I\}$ is an indexed family of presentations $B_i = \langle Y_i; S_i \rangle$ of the groups B_i and suppose that C is another group equipped with monomorphisms: $\phi_i : C \rightarrow B_i$ ($i \in I$).

Then we form the group K defined by the presentation

$$K = \left\langle \bigcup_{i \in I} Y_i; \bigcup_{i \in I} S_i \cup \{c\phi_i c^{-1} \phi_j \mid c \in C, i, j \in I\} \right\rangle$$

where we assume that the Y_i are disjoint, and K is the generalized free product of the B_i amalgamated C . Put

$$M = \prod_{i \in I}^* B_i / gp_{B_i}(C)$$

defined by the following presentation:

$$M = \langle \bigcup_{i \in I} Y_i; \bigcup_{i \in I} S_i \bigcup \{c\phi_i \mid i \in I, c \in C\} \rangle .$$

Define $\theta : y_i \mapsto y_i$. Since the relators of K is identity in M .

Therefore θ is an epimorphism from K onto M . \square

Lemma 5.3.2. *Suppose that $\{B_i \mid i \in I\}$ be an indexed family of finitely generated torsion-free nilpotent groups, and suppose C is another group equipped with monomorphisms $\phi_i : C \rightarrow B_i$ ($i \in I$). Let $B_{i,1} \triangleleft B_i$ containing such that $h(B_{i,1}) = h(B_i) - 1$. There is an epimorphism from*

$$\begin{aligned} K &= \left\{ \prod_{i \in I}^* B_i; C \right\} \\ \text{onto } M &= \prod_{i \in I}^* B_i / B_{i,1} \simeq \prod_{i \in I}^* \mathbb{Z}. \end{aligned}$$

Proof. The lemma can be verified using Von Dyck's theorem, in the similar manner of proving lemma (5.3.1). \square

The following lemma is an interpretation of a lemma by G. Baumslag, [11], when he proved that G is indicable.

Lemma 5.3.3. *Let A and B be finitely generated torsion-free nilpotent groups, C be the common proper subgroup of them, and $G = \{A * B; C\}$ be the generalized free product of A and B amalgamating with C . There exists a homomorphism μ from G into A/D , where D is a normal subgroup of A containing C . Furthermore, the kernel of this homomorphism contains C in its kernel.*

Proof. There is a homomorphism α of A onto A/D that contains C in its kernel by a property of finitely generated nilpotent groups. Let β be the map from B onto the identity subgroup of A/D . By proposition (2.2.6), α and β can be extended to a homomorphism from G onto A/D . □

Lemma 5.3.4. *Let A, B, C, G , and ϕ be as theorem (5.3.3). Furthermore, suppose that C is normal in one of the factors. Then the kernel of the defined homomorphism can be expressed as follows:*

$$K = \left\{ \prod_{a \in A-C}^* B^a; C \right\}.$$

Proof. We define ϕ as follows: $a \mapsto aC$ for $a \in A$, and $b \mapsto C$ for $b \in B$. Note that ϕ agrees on C , and one can verify by Von Dyck's theorem that ϕ is a homomorphism. Hence

$$K = \ker \phi = gp_G(B, C) = gp_A(B) = gp\left(\bigcup_{a \in A} B^a\right) \quad (1)$$

We can also verify:

$$\bigcap_{a \in A-C} B^a = C \quad (2)$$

with the following argument:

It suffices to show that

$$B \cap B^a = C \quad \text{for all } a \in A - C.$$

We verify this by using the normality of C in A and the universal property of amalgamated product. The typical elements of B and B^a take the following form:

$$b_1 = a^{-1}b_2a \quad (b_1, b_2 \in B, a \in A - C).$$

The above equality holds only if $b_2 \in C$. Now since C is a normal subgroup of A , $a^{-1}b_2a \in C$. Which implies that $b_1 \in C$.

In particular, if $g \notin C$, then it can be expressed in the form

$$g = b_{i_1}^{a_{i_1}} b_{i_2}^{a_{i_2}} \cdots b_{i_n}^{a_{i_n}} \quad (n \geq 1)$$

where $b_{i_j} \in B - C$, $a_{i_j} \in A - C$ and $a_{i_j} \neq a_{i_{j+1}}$. Since $g \neq_G 1$, we can conclude that g is not equal to one in K as well.

$$g = b_{i_1}^{a_{i_1}} b_{i_2}^{a_{i_2}} \cdots b_{i_n}^{a_{i_n}} \neq_K 1. \quad (3)$$

Now by using (1), (2) and (3) a theorem of O. Schreier (2.2.1), we can express K as follows:

$$K = \left\{ \prod_{a \in A-C}^* B^a; C \right\}.$$

□

5.4 The amalgamated subgroup is central in one of the factors

5.4.1 An example with one of the factors is abelian

Example 5.4.1. *Let $A = gp(a, b, c)$ be a free abelian group, and $B = gp(x, y, z)$ be a free nilpotent group of class 2. $C_A = gp(a^2, b)$ and $C_B = gp(y, z)$ are free abelian group of rank 2. If we form*

$$G = \{A * B; a^2 = x, b = z\},$$

then G is residually solvable. Note that C_B is normal but not central in B .

Proof.

$$A = \langle a, b, c; [a, b], [b, c], [a, c] \rangle \text{ free of rank 3,}$$

$$C_A = gp(a^2, b) = \langle a^2, b; [a^2, b] \rangle \text{ free abelian of rank 2,}$$

$$B = \langle x, y, z; [x, y] = z, [x, z], [y, z] \rangle \text{ free nilpotent of class 2 and of rank 3}$$

$$C_B = gp(y, z) = \langle y, z; [y, z] \rangle \text{ free abelian of rank 2.}$$

Define an isomorphism ϕ which maps

$$a^2 \mapsto y \text{ and } b \mapsto z.$$

Form the generalized free product of A and B identifying C_A with

C_B :

$$G = \{A * B; C_A = C_B\}.$$

Note that $C_B \triangleleft B$ but not central in B , but observe that $\xi B \cap C_B \neq \{1\}$.

C_A is central and therefore normal in A .

Note that the quotient group B/C_B is an infinite cyclic group generated by xC_B ,

$$B/C_B = gp(xC_B).$$

Let K be the normal closure of A in G ,

$$K = gp_G(A) = gp_B(A) = gp(A^{x^i} | i \in \mathbb{Z}).$$

Note that G/K is an infinite cyclic group.

$$G/K = gp(xK).$$

Note that

$$a^{x^i} = b^i a^2 \quad \text{and} \quad b^{x^i} = b \quad \forall i \in \mathbb{Z}.$$

Hence

$$A \cap A^{x^i} = C_A \quad \forall i \in \mathbb{Z}$$

$$A_i := gp(A, A^{x^i}) = \{A * A^{x^i}; C_A\}$$

$$\forall w_i \in A_i \exists w_j \in A_j \text{ s.t. } w_i = w_j^{x^{(i-j)}}$$

$$K = \langle A_i, x^i (i \in \mathbb{Z}); H_i = H_j^{x^{(i-j)}} (i, j \in \mathbb{Z}) \rangle.$$

K is an HNN-extension, where the base groups are the union of A_i ($i \in \mathbb{Z}$), the stable letters are $x_i = x^i$ ($i \in \mathbb{Z}$), and associated subgroups are H_i , where H_i is a subgroup of A_i (for $i \in \mathbb{Z}$). By using the subgroup theorem, in particular lemma (5.3.3), we can express K as generalized free product of conjugates of free abelian groups.

$$K = \left\{ \prod_{i \in \mathbb{Z}}^* A^{x^i}; C_A \right\}.$$

Hence, by the theorem (5.2.6), K is an extension of a free group by an abelian group. Consider the following short exact sequence:

$$1 \rightarrow K \rightarrow G \rightarrow B/C_B \simeq \mathbb{Z} \rightarrow 1$$

G/K is an infinite cyclic group, therefore G is an extension of a residually solvable group by an infinite cyclic group, and therefore G is residually solvable by lemma (3.3.11). \square

5.4.2 A theorem on residual solvability

Theorem 5.4.2. *The generalized free product of two finitely generated torsion-free abelian and nilpotent groups is an abelian extension of a residually solvable.*

Proof. Let A and B be two finitely generated torsion-free abelian and nilpotent group respectively, and G be the amalgamated product of them with amalgamated subgroup C .

$$G = \{A * B; C_A = C_B\}.$$

It can be observed that C is a direct factor of a subgroup A_1 of A of finite index, i.e.

$$A_1 = C \times H \quad \text{where } [A : A_1] = n < \infty.$$

There is an epimorphism from G onto $A/A_1 = \bigcup_{i=1}^n a_i A_1$, where a_i are distinct set of coset representatives of A_1 in A . The kernel of this map, K is $gp_G(B, A_1)$.

$$\begin{aligned} K &= gp_G(B, A_1) \\ &= \left\{ \prod_{i=1}^n B^{a_i} * A_1; C \right\}. \end{aligned}$$

Now let D be the normal closure of finitely many copies of B in K .

$$\begin{aligned} D &= gp_K\left(\bigcup_{i=1}^n B^{a_i}\right) \\ &= \left\{ \prod_{i=1}^n B^{a_i}; C \right\}. \end{aligned}$$

Now since $[a_i, C] = 1$, then D is the generalized free product of finite number of doubles, and hence D is residually solvable. So K is an extension of a residually solvable group by an abelian group,

that is to say that K is residually solvable by lemma (3.3.11). Therefore G is an extension of a residually solvable group by a finite abelian group. And this completes the proof of the theorem that G is residually solvable. \square

Corollary 5.4.3. *G is residually solvable.*

5.5 The case when the amalgamated subgroup is of finite index in at least one of the fac- tors

Theorem 5.5.1. *The generalized free product of two finitely generated torsion-free nilpotent groups amalgamated a proper subgroup of them in which this subgroup is of finite index in one of the factors, is an extension of a free group by a torsion-free nilpotent group.*

Proof. Let A and B be two finitely generated torsion-free nilpotent groups, and let C be a proper subgroup of A and B , such that the index of C in one of the factors, say, A be finite. We want to show that the generalized free product of A and B with C amalgamated,

$$G = \{A * B; C\}$$

is a finitely generated torsion-free nilpotent extension of a free group. Let $m(A)$ and $m(B)$ be Mal'cev completion of A and B

respectively, by (1.7.2). Since the index of C in A is finite, then $m(A)$ is also a Mal'cev completion of C . There exists a monomorphism μ from C into B . Then μ can be extended uniquely to a homomorphism $m(\mu)$ of $m(A)$ onto $m(B)$, by corollary (1.7.7,[4]). Now, by proposition (2.2.6), there is a homomorphism from G into $m(B)$, which agrees on C , and injective restricted to A and B . Let K be the kernel of this homomorphism, we have

$$K \cap A = \{1\} = K \cap B.$$

Therefore K is free by a well-known theorem of Hanna Neumann.

Hence G is a torsion-free nilpotent extension of a free group. \square

Corollary 5.5.2. *G is residually solvable.*

Proof. Since free groups are residually solvable by a theorem of Levi, (3.5.1), and G is free-by-nilpotent, hence G is residually-solvable-by-solvable. So G is residually solvable. \square

Observation 5.5.3. on theorem (5.5.1) *Suppose that the index of C in A is finite; $[A : C] = k < \infty$, $A = \bigcup_{i=1}^k x_k C$.*

It follows that $h(A) = h(C) = h(gp_A(C))$, and $h(A/gp_A(C)) = 0$.

Also $A/gp_A(C)$ is a finite group. Consider the following short exact sequence:

$$1 \rightarrow K \rightarrow G \xrightarrow{\theta} B/gp_B(C) \text{ (a torsion-free nilpotent group)} \rightarrow 1.$$

Where K is the kernel of θ . One can show that

$$K = gp_G(A, C) = gp_G(A) = gp_B(A) = \left\{ \prod_{b \in B}^* A^b; C_A \right\}.$$

K is also residually solvable, since a subgroup of a residually solvable group is residually solvable.

5.6 Cyclic amalgamated subgroup and residual solvability

Theorem 5.6.1. *The generalized free product of two finitely generated torsion-free nilpotent groups amalgamated a cyclic subgroup is an extension of a residually solvable group by a solvable group.*

Proof. Let A and B be two finitely generated torsion-free nilpotent groups. Let a be a nontrivial element of A , and b be a non-trivial element of B . We can find $m \geq 1$ and $n \geq 1$ such that $1 \neq a \in \xi_{m+1}A$, $1 \neq a \notin \xi_m A$, and $1 \neq b \in \xi_{n+1}B$ and $1 \neq b \notin \xi_n B$, (where $\xi_i A$ is the i -th term of upper central series of A , and $\xi_j B$ is the j -th term of upper central series of B). Let $G = \{A * B; a = b\}$ be the generalized free product of A and B amalgamated a with b . Let D be the central product of $A/\xi_m A$ and $B/\xi_n B$ amalgamated $a\xi_m A$ with $b\xi_n B$.

$$D = \{A/\xi_m A \times B/\xi_n B; a\xi_m A = b\xi_n B\}.$$

Note that

$$C\xi_n B/\xi_n B \simeq C/C \cap \xi_n B,$$

is an infinite cyclic group, similarly

$$C\xi_m A/\xi_m A \simeq C/C \cap \xi_m A$$

is an infinite cyclic group. That confirms the fact that

$$a\xi_m A = b\xi_n B.$$

Map G into D . Let K be the kernel of this map. $K \cap C = \{1\}$, where $C = gp(a) = gp(b)$. Therefore K is a free product of conjugates of subgroups of A and B , and a free group by the corollary (2.4.2,[33]). So K is residually solvable. Hence G is an extension of a residually solvable group by a solvable group. \square

Corollary 5.6.2. *G is residually solvable.*

Proof. In theorem (5.6.1), we have proved that G is an extension of a residually solvable group by a solvable group. Now by lemma (3.3.11), we conclude that G is residually solvable. \square

5.7 Homomorphic mappings from doubles, significant properties of their kernels

Lemma 5.7.1. *If A is a group, C is a subgroup of A , ϕ is an isomorphic mapping of A onto a group B , and D is the amalgamated product of A and B amalgamated C with $C\phi$,*

$$D = \{A * B; C = C\phi\},$$

then there is a homomorphism from D either one of the factors, and the kernel of this homomorphism, K , is:

$$gp(a(a\phi)^{-1} | a \in A).$$

Furthermore this map is injective on each of the factor.

Proof. Let α be the homomorphism from A onto itself, and β be the homomorphism B onto the inverse of the isomorphic copy of A , i.e. $\beta = \phi^{-1}$. These homomorphism can be extended to a homomorphism from D onto A , by proposition (2.2.6). By the

way that α and β are defined, we realize that this homomorphism is one-to-one restricted to either A or B . \square

Lemma 5.7.2. *Let A, B, C, D, K and ϕ be as above. Then K is not central in D . In other words, $[K, D] \neq \{1\}$.*

Proof. First, note that the center of D is $\xi D = \xi A \cap \xi B \cap C$. By lemma (6.4.1), $K \cap A = \{1\}$, and hence $\xi A \cap K = \{1\}$. This implies that $[K, D] \neq \{1\}$. \square

Lemma 5.7.3. *Let A, B, C, D, K and ϕ be as above. Furthermore, let C be normal in A . Then K commutes with C , i.e. $[C, K] = 1$.*

Proof. Let $c, c' \in C$, and $a \in A$. Since $C \triangleleft A$ and $C\phi \triangleleft B$, and ϕ is an isomorphism, we can do the following computation:

$$a^{-1}ca = c' = c'\phi = (a^{-1}\phi)(c\phi)(a\phi) = (a^{-1}\phi)c(a\phi).$$

Note that $a(a\phi)^{-1} \in K$, so

$$ca(a\phi)^{-1} = a(a\phi)^{-1}c$$

$$[C, K] = \{1\}.$$

This completes the proof of lemma (6.4.3). \square

Corollary 5.7.4. *Let A, B, C, D, K and ϕ be as above. Then K is free*

Proof. Since

$$K \cap A = \{1\} = K \cap A\phi.$$

So, by a theorem of Hanna Neumann (2.4.2), K is free. \square

5.8 Doubles of nilpotent groups and residual solvability

5.8.1 The case of two factors

Theorem 5.8.1. *Let A be a finitely generated nilpotent group, C be a subgroup of A , and $'\bar{\cdot}'$ be an isomorphic mapping of A onto A . Then the generalized free product of A and \bar{A} , amalgamating C with \bar{C} ,*

$$G = \{A * \bar{A}; C = \bar{C}\},$$

is an extension of a free group by a finitely generated nilpotent group.

Proof. Let ϕ be a homomorphism from G onto A , $\phi : G \rightarrow A$. Then ϕ restricted to A and \bar{A} are an injective homomorphism. That is to say that the kernel of this homomorphism K does not meet any conjugate of A or \bar{A} . Then K is free by theorem, (2.4.2).

The kernel of this homomorphism is $K = \ker \phi = gp(a\bar{a}^{-1} | a \in A)$.

$$A \cap K = \{1\} = \bar{A} \cap \ker \phi.$$

Therefore G is an extension of a free group by a finitely generated nilpotent group. Since free groups are residually solvable by (3.5.1), so G is residually solvable by lemma (3.3.11). \square

Corollary 5.8.2. *G is residually solvable.*

5.8.2 Arbitrary number of factors

Theorem (5.8.1) can be generalized to the arbitrary number of isomorphic nilpotent groups.

Theorem 5.8.3. *Let $\{A_i | i \in I\}$ be a arbitrary indexed family of isomorphic torsion-free nilpotent groups such that $\bigcap_{i \in I} A_i = C$, and let G be the generalized free product of the A_i s amalgamating C :*

$$G = \left\{ \prod_{i \in I}^* A_i; C \right\}.$$

Then G is residually solvable.

Proof. Let $\phi : G \rightarrow A_i$ (for $i \in I$) be an epimorphism, and K be the kernel of this homomorphism ϕ . K is free by theorem (2.4.2), since ϕ restricted to each factor is injective,

$$A_i \cap K = \{1\} (\forall i \in I).$$

Therefore G is residually solvable. □

5.9 A remark, generalized free products of nilpotent groups sometimes may fail to be residually solvable.

Gilbert Baumslag remarked in [11], by a counter example that not every subgroup of generalised free product of two finitely generated torsion-free nilpotent groups is indicable. We use this example to prove the following theorem:

Remark 5.9.1. There exist two finitely generated torsion-free non-abelian free nilpotent group A and B , such that the amalgamated product of them with abelian amalgamation,

$$G = \{A * B; C_A = C_B\},$$

is not residually solvable.

Proof. Let $A = gp(a, x)$ be a free nilpotent group of class 2, and $B = gp(b, y)$ be a free nilpotent group of class 3. $C_A = gp(a, a^x)$

and $C_B = gp(b, [b, b^y])$ are free abelian group of rank 2. If we form

$$\begin{aligned} G &= \{A * B; C_A = C_B\} \\ &= \{A * B; a = b, a^x = [b, b^y]\}, \end{aligned}$$

then

$$\begin{aligned} a &= [a, a^y]^x \\ &= [a^x, a^{y^x}], \end{aligned}$$

so a lies in every term of the derived series of G . Hence G is not residually solvable. □

5.10 Abelian amalgamating subgroup does not suffice for residual solvability. A significant example and some observations

We have shown that if the amalgamating subgroup of generalized free product of torsion-free nilpotent groups is cyclic, (theorem (5.6.1)) or central in both factors (theorem (5.2.1)) then these groups are residually solvable. One may conjecture that it would be the case when the amalgamating subgroup is abelian. The answer to this conjecture is negative by the example due to Baumslag [11] as we mentioned in the theorem (5.9.1).

Example 5.10.1. *Let $A = gp(a, x)$ be a free nilpotent group of class 2, and $B = gp(b, y)$ be a free nilpotent group of class 3. $C_A = gp(a, a^x)$ and $C_B = gp(b, [b, b^y])$ are free abelian group of*

rank 2. If we form

$$\begin{aligned} G &= \{A * B; C_A = C_B\} \\ &= \{A * B; a = b, a^x = [b, b^y]\}, \end{aligned}$$

then

$$\begin{aligned} a &= [a, a^y]^x \\ &= [a^x, a^{y^x}], \end{aligned}$$

Observation 5.10.2. $a \in G$ lies in every term of the derived series of G . Hence G is not residually solvable since

By doing some computation we obtain the following observation on this example. Some of the computations has been either checked or came out using the Magnus software [1].

Put $N = gp_G(a) \triangleleft G$, compute

$$G/N = \langle a, b, x, y; b = a = 1, a^x = [b, b^y] \rangle = \langle xN, yN \rangle .$$

C_A is not normal in A , $C_A \triangleleft A$,

C_A is not central in A , $C_A \not\leq \xi A$,

the Hirsch number of A is 3, $h(A) = 3$,

the Hirsch number of amalgamating subgroup, C_A is 2, $h(C_A) = 2$,

C_A is of infinite index of A , $[A : C_A]$ is infinite,

C_B is not normal in B , $C_B \not\triangleleft B$,

C_B is not central in B , $C_B \not\leq \xi B$,

the Hirsch number of B is 5, $h(B) = 5$,

the Hirsch number of amalgamating subgroup, C_B is 2, $h(C_B) = 2$,

the index of C_B in B is infinite, $[B : C_B]$ is infinite.

$$A = \langle a_1, a_2, a_3; [a_2, a_1]a_3^{-1}, [a_3, a_1], [a_3, a_2] \rangle$$

$$C_A = \langle a_1, a_3; [a_3, a_1] \rangle$$

$$A/C_A = gp(xC_A) = \langle a, x; a^x, a \rangle = \langle x \rangle = A/gp_A(N) = \mathbb{Z}$$

$$B = \langle b_1, b_2, b_3, b_4, b_5; [b_2, b_1]b_3^{-1},$$

$$[b_3, b_1]b_4^{-1}, [b_3, b_2]b_5^{-1}, [b_i, b_j](i = 4, 5; j = 1, 2, 3, 4) \rangle$$

$$C_B = \langle b_1, b_4; [b_1, b_4] \rangle$$

$$gp_B(C_B) = gp(b, [b, y]^{y^3}b^{y^3}, b^y, b^{y^2})$$

$$B/gp_B(C_B) = \langle b, y; b, [b, y]^{y^3}b^{y^3}, b^y, b^{y^2} \rangle = \langle y \rangle$$

$$h(gp_B(C_B)) = 4$$

$$B/gp_B(N) = \langle b, y; b, b^y, b^{y^2} \rangle = \langle y \rangle$$

$$h(gp_B(N)) = 4.$$

A way to find the Hirsch length of a nilpotent group is to find every independent basic commutators. As an example, let's calculate $h(B)$.

Chapter 6

Generalized free products of free groups

6.1 Background and a question of Peter Neumann

The object of this section is to give a very short proof of the following theorem of Ivanov and Schupp. Let H be a finitely generated subgroup of a free group F and the index $[F : H]$ infinite. Then there exists a nontrivial normal subgroup N of F such that $N \cap H = \{1\}$. As noted in above the object of this section is to give a new proof of the following theorem of S.V.Ivanov and P.E.Schupp [24].

Theorem (6.1.1) also provides a partial negative solution to the following problem of P. Neumann: *Is it possible that the free product $\{A * B; H = K\}$ with amalgamation, where A, B are free groups of finite ranks, H, K are finitely generated subgroups of A, B , respectively, is a simple group?*

Theorem 6.1.1. *Let H be a finitely generated subgroup of a free group F and the index $[F : H]$ infinite. Then there exists a non-trivial normal subgroup N of F such that $N \cap H = \{1\}$.*

6.1.1 Some preparatory lemmas

Our proof depends on two facts. The first is a theorem of M. Hall [21] which states that a finitely generated subgroup of a finitely generated free group is a free factor of a subgroup of finite index. Second, key fact, is the following simple lemma.

Lemma 6.1.2. *Let I be a normal subgroup of finite index of a group F , and let L be a normal subgroup of I . Furthermore let*

b_1, \dots, b_m be a complete set of representatives of the right coset Ig of I in F . Then

$$N = \bigcap_{i=1}^m b_i^{-1} L b_i \subseteq F.$$

Proof. Notice that conjugation induces an automorphism of I and hence $L^f \subseteq F$ for every $f \in F$. Hence each $b_i^{-1} L b_i \subseteq I$ and therefore $N \subseteq I$. In fact $N \subseteq F$. To see this, let $f \in F$. Then f can be written in the form

$$f = xb \quad (x \in I, b \in \{b_1, \dots, b_m\}).$$

Consequently

$$\begin{aligned} f^{-1} N f &= b^{-1} x^{-1} N x b \\ &= b^{-1} N b \\ &= b^{-1} \left(\bigcap_{i=1}^m b_i^{-1} L b_i \right) b \\ &= \bigcap_{i=1}^m (b_i b)^{-1} L (b_i b) \quad (1). \end{aligned}$$

Notice that if b_i' is the representative of the coset $I b_i b$ then the

map $i \rightarrow i'$ is a permutation of $\{1, \dots, m\}$. Hence we now write

$$b_i b = a_i b_i \quad (i = 1, \dots, m; a_i \in I).$$

Then it follows from (1) that

$$\begin{aligned} f^{-1} N f &= \bigcap_{i=1}^m b_i'^{-1} a_i^{-1} L a_i b_i \\ &= \bigcap_{i=1}^m b_i'^{-1} L b_i' \\ &= \bigcap_{i=1}^m b_i^{-1} L b_i \\ &= N. \end{aligned}$$

□

6.2 A new proof of a theorem of Ivanov and Schupp, a partial answer to a question of Peter Neumann

Proof. If F is not finitely generated, or H is trivial, then the theorem is trivial. So we assume that F is finitely generated and that H is not trivial. By the theorem of M.Hall cited above, there exists a subgroup $K \leq F$ of finite index $[F : K] = n$, such that K is the free product of H over a second subgroup Q :

$$K = H * Q.$$

Since K is of finite index in F , the number of conjugates of K in F is finite and their intersection I is a normal subgroup of F of finite index (see e.g. A.G. Kurosh [27] vol 1, p.) This can be proved in the same manner that we proved the lemma above. It follows directly from the proof of the subgroup theorem for free products

due to A.G.Kurosh [27] that

$$I = (I \cap H) * J.$$

Now the normal closure of J in I , $L = gp_I(J) \trianglelefteq I$. Observe that by $L \cap (I \cap H) = \{1\}$, therefore

$$L \cap H = \{1\}.$$

Now by the lemma $N = \bigcap_{j=1}^m L^{b_j} \trianglelefteq F$. Note that N is a nontrivial normal subgroup of F , since it contains the commutator subgroup of all of them. Note that N is the desired normal subgroup, since $N \cap H = \{1\}$, using the fact that $N \leq L$. \square

Corollary 6.2.1. *A finitely generated subgroup H of a free group F has infinite index if and only if there is a nontrivial normal subgroup N of F with $N \cap H = \{1\}$. [24]*

6.3 Simplicity or not?

6.3.1 A sufficient condition for generalized free products of free groups not to be simple

Corollary 6.3.1. *Suppose A, B are free groups of finite ranks and H, K are finitely generated subgroups of A, B , respectively. Then the free product $\{A * B; H = K\}$ with amalgamation is not simple provided either of indexes $[A : H], [B : K]$ is infinite. [24]*

Proof. Suppose $[A : H]$ is infinite. By theorem (6.1.1), there is a nontrivial subgroup N of A such that

$$H \cap N = \{1\}.$$

Then the quotient of $\{A * B; H = K\}$ by normal closure of N is isomorphic to

$$\{(A/N) * B; H = K\},$$

hence $\{A * B; H = K\}$ cannot be simple. □

Note that an affirmative solution to this problem of P. Neumann was announced by Burger and Mozes [14], example (6.3.4). Also note that examples of simple free products $\{A * B; H = K\}$, where A, B are free groups of infinite rank and their subgroups H, K have infinite rank, are due Ruth Camm, theorem (6.3.3), [15].

6.3.2 Amalgamated subgroup not finitely generated

Example 6.3.2. *Ruth Camm constructed an example of simple free products $G = \{A * B; H = K\}$ where A, B are free groups of finite rank and their subgroups H, K have infinite rank.*

This example can be thought of simple amalgamated products of two residually solvable groups where the amalgamation is not residually solvable. (i.e where the amalgamated subgroup fails the maximal condition)

Theorem 6.3.3. [15] *There exist continuously many non-isomorphism simple amalgamated products of two finitely generated free groups*

with not finitely generated amalgamation.

6.3.3 Amalgamated subgroup finitely generated

Note 6.3.4. *Burger and Mozes constructed an infinite family of torsion free finitely presented simple groups. For every pair (m, n) of sufficiently large even integers, they constructed a finite square complex whose universal covering is the product $T_m \times T_n$ of regular trees of respective degrees m and n and whose fundamental group $\Gamma_{m,n}$ enjoys the following properties: The group $\Gamma_{m,n}$ is simple, finitely presented and isomorphic to a free amalgamated product $\{F * F; G\}$ where F, G are finitely generated free groups, [14].*

6.4 Homomorphic mappings from doubles, significant properties of their kernels

Lemma 6.4.1. *If A is a group, C is a subgroup of A , ϕ is an isomorphic mapping of A onto a group B , and D is the amalgamated product of A and B amalgamated C with $C\phi$,*

$$D = \{A * B; C = C\phi\},$$

then there is a homomorphism from D onto one of the factors, and the kernel of this homomorphism, K , is:

$$gp(a(a\phi)^{-1} | a \in A).$$

Furthermore this map injects to each factor.

Proof. Let α be the homomorphism from A onto itself, and β be the homomorphism B onto the inverse of the isomorphic copy of A , i.e. $\beta = \phi^{-1}$. These homomorphism can be extended to a homomorphism from D onto A , by proposition (2.2.6). By the

way that α and β are defined, we realize that this homomorphism is one-to-one restricted to either A or B . \square

Lemma 6.4.2. *Let A, B, C, D, K and ϕ be as above. Then K is not central in D . In other words, $[K, D] \neq \{1\}$.*

Proof. First, note that the center of D is $\xi D = \xi A \cap \xi B \cap C$. By lemma (6.4.1), $K \cap A = \{1\}$, and hence $\xi A \cap K = \{1\}$. This implies that $[K, D] \neq \{1\}$. \square

Lemma 6.4.3. *Let A, B, C, D, K and ϕ be as above. Furthermore, let C be normal in A . Then K commutes with C , i.e. $[C, K] = 1$.*

Proof. Let $c, c' \in C$, and $a \in A$. Since $C \triangleleft A$ and $C\phi \triangleleft B$, and ϕ is an isomorphism, we can do the following computation:

$$a^{-1}ca = c' = c'\phi = (a^{-1}\phi)(c\phi)(a\phi) = (a^{-1}\phi)c(a\phi).$$

Note that $a(a\phi)^{-1} \in K$, so

$$ca(a\phi)^{-1} = a(a\phi)^{-1}c$$

$$[C, K] = \{1\}.$$

This completes the proof of lemma (6.4.3). \square

Corollary 6.4.4. *Let A, B, C, D, K and ϕ be as above. Then K is free*

Proof. Since K is one-to-one restricted to either A or B , then

$$K \cap A = \{1\} = K \cap A\phi.$$

So, by a theorem of Hanna Neumann (2.4.2), K is free. \square

6.5 Doubles of residually solvable groups

6.5.1 Meta-residual-solvability

Theorem 6.5.1. *Let A be a residually solvable group, C be a subgroup of A , and $'\bar{\cdot}'$ be an isomorphic mapping of A onto A . Then the generalized free product of A and \bar{A} amalgamated C with \bar{C} ,*

$$G = \{A * \bar{A}; C = \bar{C}\}$$

is an extension of a free group by a residually solvable group.

Proof. Let $\phi : G \rightarrow A$, then $K = \ker \phi = gp(a\bar{a}^{-1} | a \in A)$. K is free by a theorem of Hanna Neumann, (2.4.2), since

$$A \cap K = \{1\} = \bar{A} \cap \ker \phi.$$

Therefore G is an extension of a free group by a residually solvable group; that is to say G is meta-residually-solvable. \square

Corollary 6.5.2. *G is meta-residually-solvable.*

6.5.2 Effect of solubly separability on the amalgamated subgroup and residual solvability

Theorem 6.5.3. *Let $\bar{}$ be an isomorphism from a group A onto itself, C be a subgroup of A , and G be the amalgamated product of A and \bar{A} with C amalgamated:*

$$G = \{A * \bar{A}; C = \bar{C}\}.$$

If A is residually solvable then G is also residually solvable, provided that C be solubly separable in A .

Proof. Assuming C is solubly separable in A and A is residually solvable, we want to show that G is residually solvable. That is to show that for every non-trivial element $(1 \neq)d \in G$, there exists a homomorphism, ϕ , from G onto a solvable group S , $\phi : G \rightarrow S$, such that $d\phi \neq 1$. We consider two cases:

Case 1: Let $1 \neq d \in A$. There exists an epimorphism ϕ from G onto A , so that $d\phi = d$. Since A is residually solvable, there exists $\lambda \in \mathbb{N}$, such that $d \notin \delta_\lambda A$. Where $\delta_\lambda A$, is the λ -th derived group of

A. Now put $S = A/\delta_\lambda A$ a solvable group of derived length at most λ . Note that the canonical homomorphism, θ from A onto S , maps d onto a non-trivial element in S . Now consider the composition of these two epimorphism, $\theta \circ \phi$, which maps G onto S . The image of d in S is non-trivial.

$$\theta \circ \phi(d) = d\theta = d\delta_\lambda A \neq_S 1.$$

Case 2: Let $1 \neq d \notin A$ but $d \in G$. Now d can be expressed as follows:

$$d = a_1 b_1 a_2 b_2 \cdots a_n b_n \quad (a_i \in A - C \quad b_i \in \bar{A} - \bar{C}).$$

Since the equally indexed filtration $\{\delta_\lambda A\}_{\lambda \in \mathbb{N}}$, and $\{\delta_\lambda B\}_{\lambda \in \mathbb{N}}$ of A and \bar{A} are compatible, we can form G_λ :

$$G_\lambda = \{A/\delta_\lambda A * \bar{A}/\delta_\lambda \bar{A}; C\delta_\lambda A/\delta_\lambda A = \bar{C}\delta_\lambda \bar{A}/\delta_\lambda \bar{A}\}.$$

Note that G_λ is residually solvable (by mapping it to one of the factors and noting that the kernel of the map is free). Consider the canonical homomorphism θ from G onto G_λ . Since C is solvably

separable in A , i.e.

$$\bigcap_{\lambda \in \mathbb{N}} C\delta_\lambda A = C$$

i.e. $\forall a \in A$ if $a \notin C$, $\exists \lambda \in \mathbb{N}$ s.t. $a \notin \delta_\lambda A$.

$\lambda \in \mathbb{N}$ can be so chosen that

$$a_i \notin C\delta_\lambda A, \bar{a}_i \notin \bar{C}\delta_\lambda \bar{A} \text{ (for } i = 1, \dots, n \text{).}$$

Hence

$$a_1\delta_\lambda A b_1\delta_\lambda \bar{A} \cdots a_n\delta_\lambda A b_n \delta_\lambda \bar{A} \neq_{G_\lambda} 1.$$

This completes the proof of theorem (6.5.3) by using a key lemma by G. Baumslag (3.4.4). \square

6.5.3 Solvable separability is a sufficient condition for residual solvability. A negative theorem

Note that the condition of solvable separability of the amalgamated subgroup in the factors, in the case of doubles is necessary. The following negative theorem shows that the amalgamated product

of doubles is not residually solvable where the factors are residually solvable groups.

Theorem 6.5.4. *Let A be a finitely generated residually solvable group, and C be a normal subgroup of A , such that A/C be perfect. Let “ $\bar{}$ ” be an isomorphic mapping of A onto itself. Then*

$$D = \{A * \bar{A}; C = \bar{C}\}$$

*is meta-residually solvable, but **not** residually solvable.*

Proof. D is meta-residually solvable by theorem (6.5.1).

By lemma (6.4.1), there exists an epimorphism from D onto A .

Let K be the kernel of this epimorphism. Since C is normal in A , by using lemma (6.4.3)

$$[C, K] = 1. \quad (*)$$

Now we want to show that D is *not* residually solvable. We proceed by contradiction. Suppose D is *residually solvable*.

Let d be a non-trivial element in $[K, D]$. The existence of such

an element is guaranteed by lemma (6.4.2). Now assume μ be a homomorphism of D onto a solvable group S , so that $d\mu \neq 1$.

$$\mu : D \rightarrow S \text{ a solvable group, s.t. } d\mu \neq_S 1.$$

Since μ is an epimorphism, $C\mu$ is a normal subgroup of S , and by (*),

$$[C\mu, K\mu] = 1.$$

If we show that $S = C\mu$, then $[K\mu, S] = 1$. Which implies that $d\mu =_S 1$, which is a *contradiction*. We now need to show that $S = C\mu$.

$D \twoheadrightarrow S$ induces a homomorphism from D/C onto $S/C\mu$. Since A/C and \bar{A}/\bar{C} have a perfect subgroup, this induces a homomorphism from A/C to 1. So, $S/C\mu = 1$. Hence $C\mu = S$. \square

Chapter 7

Residual solvability and one-relator groups

7.1 Background and motivation

It is known that a free group is residually solvable. The question of whether a group with a single defining relation is residually solvable is coming right after this. Gilbert Baumslag showed in 1971 [5] that a group G with a single defining relation in which there are no negative exponents, is residually solvable. Of course, not every one-relator group is residually solvable. For example [5], the group

$$G = \langle a, b; a = [a, a^b] \rangle$$

is not residually solvable since $a \neq 1$ (by the Freiheitssatz (7.3.1)) and every term of the derived series of G contains a .

7.2 Baumslag-Solitar groups

$B_{n,m} = \langle a, b; b^{na} = b^m \rangle$ is residually solvable. In fact it is free-by-solvable.

$B_{1,n}$ is solvable, but not polycyclic.

7.3 When the relator is a commutator

Let G be a group that can be presented in the form

$$G_1 = \langle t, a, \dots, c; r = uw^{-1} = 1 \rangle$$

where u and w are positive words in the given generators and each generator occurs with exponent sum zero in uw^{-1} . Here a word is termed positive if only non-negative powers of t, a, \dots, c occur in v . Then G is residually solvable, by a theorem of Gilbert Baumslag

in 1981 [6]. In fact, he showed that G is free-by-cyclic. Let

$$G = \langle t, a, \dots, c; [u, w] \rangle.$$

If u and w are positive, is a special case of the introducing group G_1 , hence is free-by-cyclic and therefore residually solvable.

But if u or w are not positive in G , it is an open question, in general. G. Baumslag [6] gave examples in which G is not residually solvable. (see example (7.4.1))

We discuss the special case where u is one of the given generator of G . Note that we would like to solve the problem with Magnus Break down. In order to do this one can decompose G as follows: Let $G = \langle t, a, \dots, c; [t, w] \rangle$. Since $[t, w] = 1$, therefore $t^{-1}wt = w$. The exponent sum of t in the relator is zero. Now let $N = gp_G(a, \dots, c)$, and

$$N_0 = gp(a_{m(a)}, \dots, a_{M(a)+1}, c_{m(c)}, \dots, c_{M(c)+1}; w_0 = w_1)$$

Observe that

$$N_0 = \{A * B; H = K\}$$

where

$$A = gp(a_{m(a)+1}, \dots, a_{M(a)+1}, \dots, c_{m(c)+1}, \dots, c_{M(c)+1}, w_1)$$

$$B = gp(a_{m(a)}, \dots, a_{M(a)}, \dots, c_{m(c)}, \dots, c_{M(c)}, w_0)$$

$$H = gp(a_{m(a)+1}, \dots, a_{M(a)}, \dots, c_{m(c)+1}, \dots, c_{M(c)}, w_1)$$

$$K = gp(a_{m(a)+1}, \dots, a_{M(a)}, \dots, c_{m(c)+1}, \dots, c_{M(c)}, w_0)$$

Now the question is since we know that free groups are residually solvable, what the necessary condition on the amalgamation part is to make N_0 residually solvable.

Theorem 7.3.1. (*Freiheitssatz*) [29] *Let G be a group with a single defining relator, i.e.,*

$$G = \langle x_1, \dots, x_q; r \rangle .$$

Suppose r is cyclically reduced, i.e., the first and the last letters in r are not inverses of each other. If each of x_1, \dots, x_q actually appears in r , then any proper subset of $\{x_1, \dots, x_q\}$ freely generates a free group.

The following lemma is an immediate consequence of W. Magnus' basic breakdown of groups with a single defining relation:

Lemma 7.3.2. *Let $G = \langle b, x, \dots, c; r = 1 \rangle$ be a group with a single defining relation. Suppose that b occurs in r with exponent sum zero and that μ and ν are respectively the minimum and maximum subscripts of x occurring in r_0 . If $\mu < \nu$ and if both x_μ and x_ν occur only once in r_0 then $N = gp_G(x, \dots, c)$ is free. Moreover if G is a two-generated group with generators b and x , then N is free of rank $\nu - \mu + 1$.*

7.4 Non-positive one-relator groups, some examples and comments

Example 7.4.1. [6] $G = \langle a, b, \dots, c; [u, v] = 1 \rangle$, where

$$u = a,$$

$$v = [a, b][w, w^b]$$

$$w = [a, b]^{-1}[a, b]^a.$$

then G is not residually solvable.

It follows from Magus' solution of the word problem that $w \neq 1$.

Furthermore since $[u, v] = 1$ we find that

$$[a, b]^a [w, w^b]^a = [a, b][w, w^b]$$

Hence

$$w = [a, b]^{-1}[a, b]^a = [w, w^b]([w, w^b]^a)^{-1}.$$

So w lies in every term of the derived series of G .

Example 7.4.2. $G = \langle a, b; uw^{-1} = 1 \rangle$ where u and w are positive, and that both generators appear in uw^{-1} . If α is the exponent sum of a in uw^{-1} and if β is that of b , then G is free-by-cyclic if $\alpha > 0 > \beta$, [6].

Example 7.4.3. $G = \langle a, b; [a, [a, b]] \rangle$ is free-by-cyclic.

Proof.

$$r = [a, [a, b]] = a^{-1}[a, b]^{-1}a[a, b] = a^{-1}b^{-1}a^{-1}bab^{-1}ab$$

so $r_0 = b_1^{-1}b_2b_1^{-1}b_0$. Using the fact that $\sigma_a(r) = 0$. Observe that $\nu = 2$ and $\mu = 0$, and b_0 and b_2 occur only once in r_0 , therefore $N = gp_G(b)$ is free, by lemma (7.3.2). Since G/N is cyclic, and so solvable. Since N is free, it is residually solvable. By using lemma (7.3.2) G is residually solvable. \square

Example 7.4.4. $G = \langle a, b; [a, w] \rangle$, where $w = [a, b]^n$.

Proof. Let $N = gp_G(b)$, the normal closure of b in G . By using the

magnus break down,

$$N_0 = \langle b_0, b_1, b_2; (b_1^{-1}b_0)^n = (b_2^{-1}b_1)^n \rangle .$$

Put

$$x_0 = b_1^{-1}b_0, x_1 = b_2^{-1}b_1, y = b_1.$$

By using Tietze transformation

$$\begin{aligned} N_0 &= \langle x_0, x_1, y; x_0^n = x_1^n \rangle \\ &= \langle x_0, x_1; x_0^n = x_1^n \rangle * \langle y \rangle . \end{aligned}$$

Now let $K = \langle x_0, x_1; x_0^n = x_1^n \rangle$. K can be written as follows:

$$K = \{ \langle x_0 \rangle * \langle x_1 \rangle; \langle x_0^n \rangle = \langle x_1^n \rangle \}.$$

Observe that each factor of K is abelian, now by theorem (1.3.7)

K is residually solvable. $\langle y \rangle$ is also residually solvable since it is free. Therefore their free product of them is residually solvable, so N_0 is residually solvable.

One can easily in the similar manner prove that N_i is residually

solvable for every $i \in \mathbb{N}$. Assuming that

$$N_{i,j} = gp(N_i, N_{i+1}, \dots, N_j)$$

in the similar manner we have:

$$N_{i,j} = \langle x_i \rangle *_{\langle x_i^n \rangle = \langle x_{i+1}^n \rangle} \langle x_{i+1} \rangle * \dots *_{\langle x_{j+1}^n \rangle = \langle x_j^n \rangle} \langle x_{j+1} \rangle * \langle y \rangle .$$

By using the generalized of lemma (1.3.7). $N_{i,j}$ is residually solvable. By lemma (7.4.5), the union of them N , is residually solvable, and by using the lemma (3.3.11), therefore G is residually solvable. \square

Lemma 7.4.5. $N = \bigcup_{r < 0; s > 0} N_{r,s}$ is residually solvable.

Proof. Suppose that $\forall i, j \in \mathbb{N} (i \leq j)$, $N_{i,j}$ is residually solvable.

We want to show that N is residually solvable. Since

$$N = \bigcup_{r < 0; s > 0} N_{r,s},$$

therefore

$$\delta_i N = \delta_i \left(\bigcup_{r < 0; s > 0} N_{r,s} \right).$$

Since every element in $g \in \delta_i N$ is a finite product of commutators of elements from a (finite) subset of the $N_{r,s}$ groups. So $g \in \delta_i N_{r,s}$, for some suitably small value of $r < 0$ and large value of $s \geq 0$.

Thus

$$\delta_i N = \bigcup_{r < 0; s > 0} \delta_i N_{r,s}.$$

Now fix integers j, k . Then the infinite union above can be re-written

$$\delta_i N = \bigcup_{r < 0; s > 0} \delta_i N_{j+r, k+s}.$$

so

$$\delta_i N \cap N_{j,k} = \left(\bigcup_{r < 0; s > 0} \delta_i N_{j+r, k+s} \right) \cap N_{j,k}.$$

Distributing \cap over \cup , we get

$$\delta_i N \cap N_{j,k} = \left(\bigcup_{r < 0; s > 0} \delta_i N_{j+r, k+s} \cap N_{j,k} \right).$$

Now each term in the union can be written as

$$\delta_i N_{j+r, k+s} \cap N_{j,k} = (\delta_i N_{j, k+s} \cap N_{j,k}) \cap (\delta_i N_{j+r, k} \cap N_{j,k})$$

And because $s < 0$ and $r > 0$ ([5][pg. 175, lemma 4.3]) tells us (after conjugating all the groups in the statement of that lemma by a suitable power of d that

$$\delta_i N_{j,k+s} \cap N_{j,k} = \delta_i N_{j,k}$$

$$\delta_i N_{j+r,k} \cap N_{j,k} = \delta_i N_{j,k}.$$

So each term in the union can be re-expressed as

$$\delta_i N_{j+r,k+s} \cap N_{j,k} = \delta_i N_{j,k},$$

Note that this expression is independent of r and s ! Thus, we get

$$\delta_i N \cap N_{j,k} = \delta_i N_{j,k}.$$

([5] [pg. 175, line 16]) Now we show that $\delta_i N \cap N_{j,k} = \delta_i N_{j,k}$ implies that N is residually solvable. Let $a \in N$ be a non-trivial element.

Then since $N = \bigcup_{r < 0, s > 0} N_{r,s}$, there is an integer $j = j(a) \in \mathbb{N}$ such that $a \in N_{-j,j}$. By our hypothesis: $\forall i, j \in \mathbb{N} (i \leq j)$, $N_{i,j}$ is residually solvable, so we know that $N_{-j,j}$ is residually solvable.

By definition (3.2.2), there exists an integer $i \in \mathbb{N}$ such that $a \notin$

$\delta_i N_{-j,j}$. Since $\delta_i N \cap N_{-j,j} = \delta_i N_{-j,j}$, we see that $a \notin \delta_i N \cap N_{-j,j}$.

But $a \in N_{-j,j}$. So it must be that $a \notin \delta_i N$. Thus we have found a

normal subgroup $\delta_i N \triangleleft N$ such that $a \notin \delta_i N$ and $N/\delta_i N$ is solvable.

Thus, G satisfies definition (3.2.1), so G is residually solvable, and

this completes the proof of lemma. □

7.5 The relator is a basic commutator

Definition 7.5.1. [21]

- 1) $c_i = x_i, i = 1, \dots, r$ are the basic commutator of weight one, $w(x_i) = 1$.
- 2) Having defined the basic commutators of weight less than n , the basic commutators of weight n are $c_k = [c_i, c_j]$, where
 - (a) c_i and c_j are basic and $w(c_i) + w(c_j) = n$, and
 - (b) $c_i > c_j$, and if $c_i = [c_s, c_t]$, then $c_j \geq c_i$.
- 3) The commutators of weight n follow those of weight less than n and are ordered arbitrarily with respect to each other. basic commutators will always be numbered so that they are ordered by their subscripts.

In this paper, we study particular type of basic commutator, which are in the form of:

$$s_{k+1} = [s_k, y] \text{ where } s_1 = x$$

Theorem 7.5.1. $G = \langle x, y; r = [y, s_k] \rangle$ is free-by-cyclic.

Lemma 7.5.2. $s_k = [s_{k-1}, y] = y^{-k+2} \dots y^{k-1}$ for $k > 2$.

Proof. By induction on the weight of the commutator.

$$s_3 = x_1^{-1}x_0x_1^{-1}x_2 \quad s_3 = [s_2, y] = y^{-1}x^{-1}yxy^{-1}x^{-1}y^{-1}xy^2 [y, s_3] = (s_3^y)^{-1}s_3 = x_3^{-1}x_2x_1^{-1}x_2x_1^{-1}x_0x_1^{-1}x_2. \quad \square$$

Lemma 7.5.3. $r = [y, s_k]$ there is only one x_0 (the minimum index) and x_k (the maximum index).

Proof. By induction on the weight of the commutator, and using

$$[y, s_k] = s_k^{y^{-1}}s_k.$$

□

[8] - [9] - [?] - [36] -[19]-[16]-[34]-[20]-[17]- [21]-[28]-[37]-[2]-[30]-
[31]-[23]-[1]-[32]-[12]-[13]

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