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The pricing of interest rate dependent securities

Cakici, Nusret, Ph.D.

City University of New York, 1989

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**THE PRICING OF INTEREST RATE DEPENDENT
SECURITIES**

By

A

Nusret Cakici

A dissertation submitted to the Graduate Faculty in Business in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1989

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ABSTRACT**THE PRICING OF INTEREST RATE DEPENDENT
SECURITIES**

By

Nusret Cakici

Adviser: Professor Avner Wolf

The dissertation consists of four chapters. In the first chapter, we review the pricing of bonds and options on bonds using one factor model of the term structure of interest rates. In the second chapter, five different forms of the term structure of interest rates are used to determine bond prices, bond option prices and to evaluate optimal early exercise policies for bond options. Our numerical simulations suggest that the pricing of bonds as well as their options are quite sensitive to the specification of the behavior of interest rates. In particular, it is shown that the popular square root process for interest rates may be inappropriate for pricing bonds and options on bonds. Empirical results suggest all five interest rate specifications lead to bond price estimates which are consistently above the actual market prices. The magnitude of the overestimation increases with the bond's time to maturity. The conclusion of the second chapter is that these findings may be the result of either model misspecification or measurement errors in estimating inputs.

In the third chapter, three forms of the term structure of interest rates are utilized to estimate prices of bond futures and bond futures options. Numerical simulations suggest that bond futures prices and bond futures

option prices are quite sensitive to underlying assumption about the behavior of interest rates. We apply Black's European option pricing model and its American version (Barone-Adesi and Whaley) to value bond futures options. Several estimates of implied standard deviation are compared. Average pricing errors are compared and systematic mispricing biases are analyzed. The empirical results can be summarized as follows. First, Black's model explains actual option prices much better than stochastic interest rate models. Second, there is no difference, in terms of predictive ability, between the European and American versions of Black's model. Third, the results indicate that market prices of these options deviate significantly from their corresponding model prices and it is also shown that mispricing errors are related to time to maturity and exercise price.

In the last chapter, we test Black's European futures option model and its American version using transaction data on T-Note and T-Bond futures options. The results presented here indicates that market prices of these options deviate significantly from their corresponding model prices and it is also shown that mispricing errors are related to time to maturity and exercise price. The hedging tests provided abnormal profits only in the case of call options. Further research is required in determining the adequacy of these models.

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CONTENTS

	Page
ABSTRACT	
I INTRODUCTION AND LITERATURE REVIEW	
1. Introduction and Summary	1
2. General Valuation Model for Pricing Bonds and Options on Bonds	3
II CONTINUOUS TIME APPROACH TO PRICING BONDS AND OPTIONS ON BONDS	
1. Introduction	8
2. Implementation of Models	11
3. Statistical Estimation of stochastic interest rate processes	12
4. Estimation of λ , the market price of interest rate risk	16
5. Solution of partial differential differential equations and comparison of model values	17
6. Comparison of model values with actual market prices	20
7. Summary and Conclusions	21
III THE PRICING OF BOND FUTURES AND OPTIONS ON BOND FUTURES	
1. Introduction	33

2. Valuation of Bond Futures and Options on Bond Futures	35
3. Stochastic Interest Rate	35
4. Constant Interest Rate	40
5. European Futures Options	40
6. American Futures Options	40
7. Numerical Simulations	41
8. Empirical Tests of Pricing Models	43
9. Data	43
10. Implied Parameters	44
11. Tests of Biases in the Pricing Models	47
12. Summary and Conclusions	48
 IV EMPIRICAL TESTS OF VALUATION MODELS FOR OPTIONS ON T-NOTE AND T-BOND FUTURES	
1. Introduction	55
2. Valuation Models for Options on T-Note and T-Bond Futures ..	57
3. European Futures Options	57
4. American Futures Options	57
5. Empirical Tests of Pricing Models	58

6. Data 59

7. Implied Parameters 60

8. Tests of Biases in the Pricing Models 61

9. Market Efficiency Tests 62

10. Summary and Conclusions 63

REFERENCES 71

LIST OF TABLES

	<u>Page</u>
1. Regression Results	23
2. Regression Results	23
3. Analysis of Residuals	24
4. Analysis of Residuals	24
5. Estimated values of market price of interest rate risk	25
6. Comparison of Model values for coupon rate of 10%	26
7. Comparison of Model values for coupon rate of 6%	27
8. Comparison of Model values for coupon rate of 14%	28
9. Comparison of Model values for T-Bills	29
10. Comparison of Model values with discrete coupon payments	30
11. Comparison of optimal early exercise policies	31
12. Forecast Errors	32
13. Bond Futures Simulations	50
14. Bond Futures Options Simulations	51
15. Tests of the predictive power of different estimates of volatility .	52
16. Comparison of Market and Model Prices	53

17. Time to maturity and Exercise Price Biases in Pricing Models ..	54
18. Summary of T-Note and T-Bond Futures Options Transaction Data	65
19. Summary Statistics for Estimates of Implied Volatilities	66
20. Comparison of Market and Model Prices	67
21. Time to maturity and Exercise Price Biases in Pricing Models ..	68
22. Average Profits for Hedge Portfolios for Calls	69
23. Average Profits for Hedge Portfolios for Puts	70

Chapter I

Introduction and Literature Review

1 Introduction and Summary

The purpose of this dissertation is to link the research in two areas of finance: the term structure of interest rates, and the pricing of futures and options on interest rate instruments. The dissertation consists of four chapters. The first chapter reviews the pricing of bonds and options on bonds using one factor model of the term structure of interest rates. In the second chapter, five different forms of the term structure of interest rates are used to determine bond prices, bond option prices, and to evaluate optimal early exercise policies for bond options. Our numerical simulations suggest that the pricing of bonds as well as their options are quite sensitive to the specification of the behavior of interest rates. In particular, it is shown that the popular square root process for interest rates may be inappropriate for pricing bonds and options on bonds. Preliminary empirical results suggest that all five interest rate specifications lead to bond price estimates which are consistently above the actual prices. The magnitude of the over-

estimation increases with the bond's time to maturity. The conclusion of the second chapter is that these findings may be the result of either model misspecification or measurement errors in estimating inputs.

In the third chapter, three forms of the term structure of interest rates are utilized to estimate prices of bond futures and bond futures options. Numerical simulations suggest that bond futures prices and bond futures option prices are quite sensitive to underlying assumption about the behavior of interest rates. We apply Black's European option pricing model and its American version (Barone-Adesi and Whaley) to value bond futures options. Several estimates of implied standard deviation are compared. Average pricing errors are compared and systematic mispricing biases are analyzed. The empirical results can be summarized as follows. First, Black's model explains actual option prices much better than stochastic interest rate models. Second, there is no difference, in terms of predictive ability, between the European and American versions of Black's model. Third, the results indicate that market prices of these options deviate significantly from their corresponding model prices and it is also shown that mispricing errors are related to time to maturity and exercise price.

In the last chapter, we test Black's European futures option model and its American version using transaction data on T-Note and T-Bond futures options. The results presented here indicates that market prices of these options deviate significantly from their corresponding model prices. It is also shown that mispricing is related to time to maturity and exercise price. The hedging tests provided abnormal profits only in the case of call options. Further research is needed in determining the adequacy of these models.

2 General Valuation Model For Pricing Bonds and Options on Bonds

In this section we integrate the works of Merton [1973], Vasicek [1977], Dothan [1978], Courtadon [1982], Ball and Torous [1983], and Cox, Ingersol and Ross [1985]. Merton, and Ball and Torous derived a closed-form solution for European option prices given a specific process for the evolution of bond prices. Under the assumption that interest rates followed a particular process, Vasicek, and Dothan derived closed form formulas for pricing discount bonds. Recently, Cox, Ingersol and Ross derived a closed form formula for discount bonds and options on these bonds in a general equilibrium framework.

We shall use the following notation:

$B(t, T, r)$ =price at time t of a default-free bond paying one dollar
at time T ,

$F(t, T, r)$ =price at time t of a futures contract which will mature
at time T ,

$c(t, T, r)$ =price at time t of a European call option which will expire
at time T ,

$p(t, T, r)$ =price at time t of a European put option which will expire
at time T ,

$C(t, T, r)$ =price at time t of a American call option which will expire
at time T ,

$P(t, T, r)$ =price at time t of a American put option which will expire
at time T .

r_t =the spot interest rate from t to time $t + 1$,

In deriving a general valuation equation for bonds with continuous coupon payments, the following assumptions are made:

1. Markets are perfect and trading takes place continuously.
2. The spot rate follows a continuous Markov process. The continuity assumption implies that the spot rate process does not have jumps. A Markov process implies that the spot rate process is independent of the past. The spot rate process is completely determined by its current value. From the joint assumptions of a Markov process and continuity the spot rate process can be described by following stochastic differential equation

$$dr = f(r, t)dt + \sigma g(r, t)dz \quad (1)$$

where dz is a change in the Wiener process and dr is a change in the interest rate. The functions $f(r, t)$ and $\sigma g(r, t)$ are the instantaneous drift and standard deviation, respectively, of the process $r(t)$.

3. The price $B(t, T, r)$ of a default free coupon paying bond is determined by the spot rate process from t to maturity date, T .
4. Investors have homogenous expectations and there are no riskless arbitrage opportunities.

Under these assumptions, Vasicek shows that the basic equation for pricing bonds with continuous coupon payments, is

$$1/2\sigma^2g^2B_{rr} + (f + \sigma g\lambda)B_r - rB + B_t + i = 0 \quad (2)$$

where $\lambda(t, r)$ is the market price of interest rate risk, and is given by

$$\lambda(t, r) = \frac{\mu(t, T, r) - r}{\sigma(t, T, r)}$$

where μ is the expected rate of return on bonds. In equation (2), i is the continuous coupon rate, B_{rr} is the second derivative of B with respect to r , B_r is the first derivative of B with respect to r , and B_t is the time derivative. The bond price $B(t, T)$ is obtained by solving (2) subject to the boundary conditions

$$B(T, T, r) = 1 \text{ and}$$

$$B(t, T, \infty) = 0$$

The same assumptions and arguments lead to following differential equation for options on default free bonds.

$$1/2\sigma^2g^2W_{rr} + (f + \sigma g\lambda)W_r - rW + W_t = 0 \quad (3)$$

Since the value of European and American options depends on the value of the default-free bond, this value must be determined first in the estimation process. Also, it should be noted that, options, whether European or American, satisfy the same differential equation. Letting c be the value of an European call option, p the value of an European put option and E be the exercise price, the values of these options at maturity are:

$$c(r, T, T) = \text{Max}(B(r, t, T) - E, 0)$$

$$p(r, T, T) = \text{Max}(E - B(r, t, T), 0)$$

The value of an European call option is calculated by solving the following differential equation

$$1/2\sigma^2g^2c_{rr} + (f + \sigma g\lambda)c_r - rc + c_t = 0 \quad (4)$$

subject to

$$c(r, 0) = \max[B(r, t) - E, 0], \text{ and} \quad (5)$$

$$c(\infty, t) = 0 \quad (6)$$

The value of a European put is found by solving

$$1/2\sigma^2g^2p_{rr} + (f + \sigma g\lambda)p_r - rp + p_t = 0 \quad (7)$$

subject to

$$p(r, 0) = \max[E - B(r, T), 0], \text{ and} \quad (8)$$

$$p(\infty, t) = E \quad (9)$$

The valuation equations for American call and put options on default free bonds will be the same, except that an additional boundary condition must be satisfied to consider early exercise privileges. Letting C be the value of an American call and P be the value of an American put, the value of an American call option is found by solving the differential equation

$$1/2\sigma^2g^2C_{rr} + (f + \sigma g\lambda)C_r - rC + C_t = 0 \quad (10)$$

subject to

$$C(r, 0) = \max[B(r, t) - E, 0], \text{ and} \quad (11)$$

$$C(\infty, t) = 0 \quad (12)$$

$$C(r, t) = \max[B(r, t) - E, C(r, t)] \text{ for } t < T \quad (13)$$

The value of an American put is calculated by solving the differential equation

$$1/2\sigma^2g^2P_{rr} + (f + \sigma g\lambda)P_r - rP + P_t = 0 \quad (14)$$

subject to

$$P(r, 0) = \max[E - B(r, T), 0], \text{ and} \quad (15)$$

$$P(\infty, t) = E \quad (16)$$

$$P(r, t) = \max[E - B(r, t), P(r, t)] \quad \text{for } t < T \quad (17)$$

The finite difference methods of Brennan and Schwartz and Courtadon will be used to solve the differential equations in (2), (10) and (14). Note that the differential equations above are presented without assuming any specific model for the term structure. All one factor models of the term structure of interest rates can be obtained from the differential equations above by substituting specific forms of $f(r, t)$ and $g(r, t)$ into the differential equations.

The next chapter deals with implementation of models. In this section, it is shown that under specific assumptions, the value of a default free bond is found by solving the partial differential equation (2) subject to certain boundary conditions while the value of an option is found by solving the partial differential equation (3) subject to the appropriate boundary conditions for calls and puts.

Implementation of these valuation models requires three steps. First, the parameters of the process of the spot interest rate must be estimated. Second, the market price of risk, (λ) , must be estimated. Third, since an analytical solution does not exist, the partial differential equations must be solved with the numerical methods explained by Brennan and Schwartz and Courtadon.

Chapter II

Continuous Time Approach to Pricing Bonds and Options on Bonds

1 Introduction

The purpose of this chapter is to link the research from two areas of finance: the term structure of interest rates, and the pricing of bond options under interest rate uncertainty. The paper can be viewed as an integration of the work done by Merton [1973], Vasicek [1977], Dothan [1978], Courtaudon [1982], Ball and Torous [1983], and Cox, Ingersoll and Ross [1985]. Merton, and Ball and Torous derived a closed-form solution for European option prices given a specific process for the behavior of bond prices, assuming bond prices were given. However, the price process of bonds is not a relevant factor when pricing European options. In fact, it can be shown that Ball and Torous' closed form valuation formula is identical to that of Merton's even though they assumed a different process for bond prices. However, when valuing American options on bonds, the interest rates process becomes a very important factor.

Under the assumption that interest rates followed a particular process, Vasicek, and Dothan derived closed form formulas to price discount bonds. Recently, Cox, Ingersoll and Ross derived a closed form formula for discount bonds and options on these bonds in a general equilibrium framework.

As is clear from the aforementioned papers, the key to pricing bonds as well as options on bonds is to assume an appropriate stochastic process to describe the behavior of interest rates. Given the interest rate process, a general valuation equation can be derived and solved. In some specific cases, there are closed form formulas for bond prices as well as for European options on these bonds. However, (to the best of our knowledge) a closed form formula for pricing an American option on coupon paying bonds under uncertain interest rate has not been obtained yet. Consequently, numerical methods are required to value American options on bonds.

This paper differs from the existing literature in three respects. First, the existing literature has focused on valuation models of discount bonds and options on discount bonds. In these papers (Merton; Brennan and Schwartz; Cox, Ingersoll and Ross among others), it is usually asserted that a coupon bond is a portfolio of discount bonds. However, this approach is not free of problems and should be carefully reviewed. In practice, when an investor buys a coupon bond, he is generally expected to pay the seller not only the stated price in the market, but also any accrued interest. For example, if three months have elapsed since the last interest payment, half of the semi-annual payment is added to the purchase price to determine the total amount required. This practice implies that coupon payments are made continuously rather than at discrete times. Therefore, a coupon bond is not exactly a portfolio of discount bonds, but is a financial security which

has a fixed payoff at maturity while making continuous coupon payments throughout its life. This suggests that conclusions based on the previous theoretical and empirical research should be analyzed with caution since with continuous coupon payments, there are no closed form solutions to value bonds or options on these bonds. Second, there is, to date, little research regarding the empirical estimation of the stochastic process which governs the spot interest rate. In this paper, the interest rate process assumed by five models will be estimated using actual data (which will be described in detail later in the paper). Third, we estimated bond prices and compared them to actual bond prices of U.S. Treasury bills, bonds and notes traded from December 1985 to December 1986. Until now, none of these models have been directly tested using historical estimates of inputs. Brown and Dybvig [1986] fit the Cox, Ingersoll and Ross model to bond prices from CRSP bond files for the period of December 1952 to December 1983. They calculated implied parameters and checked whether the implied parameters were good approximations to observed values or time series estimates. They found that the model overpredicted premium issues and underpredicted discount issues suggesting a possible neglected tax effect. Our approach is not directly comparable to Brown and Dybvig's because we use historical data to estimate the parameters and test the predictive power of various models.

The rest of the chapter is organized in the following manner. Section 2 deals with the estimation of stochastic processes for interest rates and the estimation of the market price of interest rate risk. Empirical results are examined. In section 3, using estimated values of the parameters, we calculate the theoretical prices for each model and compare them to actual

prices as well as the sensitivity of each model to its parameters is analyzed. Section 3 also looks at the possible source of error that is made by treating coupon bonds as a portfolio of discount bonds rather than securities with continuous coupon payments. This has important implications for using these models in practice if one remembers that the treatment of coupon bonds as a portfolio of discount bonds reduces computer time spent to solve for the value of securities. In fact, the differential equation must be solved only once at each time period rather than once for each bond at every time period. In section 4, we analyze the results obtained by comparison of model values and market prices. Section 5 provides the summary and conclusions.

2 Implementation of models

In the previous chapter, it was shown that under specific assumptions, the value of a default free bond is found by solving the partial differential equation (2) subject to certain boundary conditions, and the value of an option is found by solving the partial differential equation (3) subject to the appropriate boundary conditions for calls and puts.

Implementation of these valuation models requires three steps. First, the parameters of the process of the spot interest rate must be estimated. Second, the market price of risk, (λ), must be estimated. Third, since an analytical solution does not exist, the partial differential equations must be solved with the numerical methods explained by Brennan and Schwartz, and Courtadon.

2.1 Statistical estimation of stochastic interest rate processes

The following five processes for spot interest rate are estimated:

$$dr = \sigma r dz \quad (1)$$

$$dr = \kappa dt + \sigma dz \quad (2)$$

$$dr = \alpha(\gamma - r)dt + \sigma dz \quad (3)$$

$$dr = \alpha(\gamma - r)dt + r\sigma dz \quad (4)$$

$$dr = \alpha(\gamma - r)dt + \sqrt{r}\sigma dz \quad (5)$$

where r is the spot interest rate, σ is the instantaneous standard deviation, κ is a constant, γ is the long run mean of the spot rate of interest, and α is the adjustment coefficient.

In estimating the parameters of the above models, differentials will be replaced by differences.¹ For convenience, we hereafter call the equations in (1) through (5) model 1, model 2, model 3, model 4 and model 5, respectively. These models were used by Dothan (equation 1), Vasicek (equation 3), Courtadon (equation 4) and Cox, Ingersol and Ross (equation 5).

Model 1 assumes that the anticipated change in the interest rate is zero and the unanticipated change is proportional to its level. In Model 2 it is assumed that the anticipated and anticipated changes in interest rates are constant. In Model 3-5 interest rate changes are assumed to be mean reverting. Mean reverting process implies that there is some long-term

¹It can be shown that a straightforward replacement of differentials with differences is not exactly right. However, we assume that the errors caused by this are negligible.

mean of interest rates to which instantaneous rates converge. The higher the value of α , the faster short term rates converge. Models 3-5 differ only in the assumptions made about unanticipated changes in the interest rates. In Model 3 it is assumed that the standard deviation of unanticipated changes does not depend on the level of interest rates. Model 4 assumes that the standard deviation of unanticipated changes are proportional to the level of interest rate. In Model 5, it is assumed that the standard deviation of unanticipated changes are proportional to the square root of the level of interest rates.

To estimate the processes of the spot interest rate, data were taken from the Citibank Data Base. The spot rate used is the effective annual yield for 3 month T-bills (FYGM3). Interest rates on Treasury Composite for 10 years and longer maturities (GL) are used as a proxy for the mean level of spot interest rates. The NYSE price index is used as a proxy for the market portfolio. The sample covers a forty year period from January 1947 through July 1987. The estimation period ends in December 1985, and the last 19 months of data are reserved for checking the predictive ability of each model. Two different forms of each model are estimated: one assumes that there is a constant term in the original process; the other assumes the absence of constant term in the original process. For empirical purposes, discrete approximations substitute the stochastic processes of the interest rates. When there is a constant term in the original equation, a discrete approximation will result in estimating the following equations:

$$(r_t - r_{t-1})/r_{t-1} = \beta_0/r_{t-1} + u_t \quad (6)$$

$$r_t - r_{t-1} = \beta_0 + u_t \quad (7)$$

$$r_t - r_{t-1} = \beta_0 + \beta_1(\gamma_{t-1} - r_{t-1}) + u_t \quad (8)$$

$$(r_t - r_{t-1})/r_{t-1} = \beta_0/r_{t-1} + \beta_1((\gamma_{t-1} - r_{t-1})/r_{t-1}) + u_t \quad (9)$$

$$(r_t - r_{t-1})/\sqrt{r_{t-1}} = \beta_0/\sqrt{r_{t-1}} + \beta_1((\gamma_{t-1} - r_{t-1})/\sqrt{r_{t-1}}) + u_t \quad (10)$$

(Equations 6-10 correspond to models 1 through 5, respectively.)

In the second case, with no constant term in the original equation, a discrete approximation will lead to the following estimation equations:

$$(r_t - r_{t-1})/r_{t-1} = \beta_0 + u_t \quad (11)$$

$$r_t - r_{t-1} = \beta_0 + u_t \quad (12)$$

$$r_t - r_{t-1} = \beta_0 + \beta_1(\gamma_{t-1} - r_{t-1}) + u_t \quad (13)$$

$$(r_t - r_{t-1})/r_{t-1} = \beta_0 + \beta_1((\gamma_{t-1} - r_{t-1})/r_{t-1}) + u_t \quad (14)$$

$$(r_t - r_{t-1})/\sqrt{r_{t-1}} = \beta_0 + \beta_1((\gamma_{t-1} - r_{t-1})/\sqrt{r_{t-1}}) + u_t \quad (15)$$

(Equations 11-15 correspond to models 1 through 5, respectively.)

Ordinary Least Squares (OLS) procedure was used to estimate the parameters. In each case, Durbin's h statistics for first order serial correlation was significant. To correct for serial correlation, the maximum likelihood (MAXL) procedure of Beach and McKinnon [3] was used. In Table 1, results for the first set of equations (those with a constant term) are reported. Table 2 shows the results for the second set of equations. Parameters were estimated for the entire sample period as well as for various subperiods.²

²Estimation is done for the following subperiods: (1) January 1947 through December 1957, (2) January 1958 through December 1968, (3) January 1969 through December 1979, (4) January 1980 through December 1985. With the exception of last period, when interest rate fluctuations were very large, results by and large were similar to those obtained for the entire sample period.

As can be seen from Tables 1 and 2, the correction for serial correlation significantly increased the explanatory power of each model. It should be noted that in terms of the adjusted R^2 and the standard errors of the regressions, Models 1 and 4 fit the data better than any other model for the entire sample as well as for the subsamples. Models 2 and 3 have the least power in explaining the interest rate changes.

From this evidence, it would be fair to conclude that the standard deviation of unanticipated changes in the spot rate of interest are proportional to the level of interest and not to the square root of this level. This implies that the previous theoretical and empirical research should be analyzed with caution if the results were based on the square root process.

The suggested stochastic processes were further analyzed by examining the residuals from the OLS and MAXL procedures. The logarithm of squared residuals from each model at time t was regressed on the logarithm of the interest rate at time $t-1$.

$$\ln \hat{u}_t^2 = \alpha_0 + \alpha_1 \ln r_{t-1} \quad (16)$$

If the process is correctly specified then the slope coefficient, α_1 should not be significantly different from 0. The estimation results are reported in table 3 and table 4. When the entire sample period is used, the null hypothesis that α_1 is equal to zero, can not be rejected for models 1 and 4. The t -values for models 1 and 4 are -1.04 and 0.34 respectively, in Table 3. From Table 4, the t -values for α_1 are 1.83 and -1.16 for models 1 and 4. α_1 for models 2, 3 and 5 is significantly different from zero. This provides further evidence that the square root process for interest rates may not be appropriate.

2.2 Estimation of λ , the market price of interest rate risk

First, for empirical purposes, it is assumed that the market price of interest rate risk is constant throughout the estimation period. However, if all investors have logarithmic utility function then λ is equal to $\rho_{rm}\sigma_m$ where ρ_{rm} is the instantaneous correlation coefficient between changes in r and the rate of return on aggregate wealth and σ_m is the standard deviation of the rate of return on aggregate wealth (Brennan and Schwartz (1979)). In this case, λ will be constant if ρ_{rm} and σ_m are constant. λ is estimated directly by estimating ρ_{rm} and σ_m . Estimated values of ρ_{rm} , σ_m and λ are reported in Table 5.

Brennan and Schwartz estimate λ by trying different values and selecting the one which minimize the root mean square of the prediction error. This value was used to test whether the model's predictions were good. By this procedure any model can make good predictions. This tells very little about the predictive ability of the model.

From Table 5, it can be seen that λ value is relatively small when compared to the Brennan and Schwartz estimate of 0.0355. When the entire sample period is considered, their value is 15 times larger than our estimate and has an opposite sign. The simulation results are discussed in the following section.

3 Solution of the partial differential equations and comparison of model values

When coupon payments are continuous, there are no known solutions to the above differential equations. Therefore, finite difference methods were applied to find the values of bonds and options on bonds. For this purpose r is replaced by $s=1/(1+r)$ in the differential equations above. Since r can range from 0 to infinity, s can vary between 0 and 1. If we let $B(r,t)=b(s,t)$ then the partial derivatives of B in terms of b is

$$B_r = -s^2 b_s \quad (17)$$

$$B_{rr} = s^4 b_{ss} + 2s^2 b_s \quad (18)$$

By substituting for the value of r and the derivatives of B , equation (2) becomes:

$$1/2\sigma^2 g^2 s^4 b_{ss} + [\sigma^2 g^2 s^2 - fs^2 - s^2 \sigma g \lambda] b_s - ((s-1)/s)b - b_t + i = 0 \quad (19)$$

Let $W(r,t) = w(s,t)$, again by substituting the partial derivatives equation (3) will become:

$$1/2\sigma^2 g^2 s^4 w_{ss} + [\sigma^2 g^2 s^2 - fs^2 - s^2 \sigma g \lambda] w_s - ((s-1)/s)w - w_t = 0 \quad (20)$$

Substituting specific forms of f and g we can solve the differential equations for the five models. The finite difference approximations of these transformed equations are obtained by calculating $b(\cdot)$ and $w(\cdot)$ at discrete intervals. The transformed equations for bond prices are:

$$1/2\sigma^2(1.0-s)^2 s^2 b_{ss} + [\sigma^2(1.0-s)^2 - \lambda\sigma(1.0-s)s] b_s - ((s-1)/s)b - b_t + i = 0 \quad (21)$$

$$1/2\sigma^2 s^4 b_{ss} + [(\sigma^2 - \lambda\sigma)s^2] b_s - ((s-1)/s)b - b_t + i = 0 \quad (22)$$

$$\begin{aligned}
1/2\sigma^2 s^4 b_{ss} + [(\sigma^2 - \gamma(\alpha - ((1.0 - s)/s)) - \lambda\sigma)s^2]b_s \\
- ((s - 1)/s)b - b_t + i = 0
\end{aligned} \tag{23}$$

$$\begin{aligned}
1/2\sigma^2 s^2(1.0 - s)^2 b_{ss} + [\sigma^2(1.0 - s)^2 - \gamma(\alpha s - (1.0 - \lambda\sigma/\gamma) \\
(1.0 - s))]b_s - ((s - 1)/s)b - b_t + i = 0
\end{aligned} \tag{24}$$

$$\begin{aligned}
1/2\sigma^2(1.0 - s)^4 b_{ss} + [\sigma^2(1.0 - s)^4/s^2 - s^2\alpha\gamma + \gamma((1.0 - s)/s)s^2 \\
- \lambda\sigma(1.0 - s)^2]b_s - ((s - 1)/s)b - b_t + i = 0
\end{aligned} \tag{25}$$

Transformed equations for options are the same except they do not have the coupon rate.

To compare model values, we solved for the value of 10 year default free bond, and the value of American call and put options with one year to expiration. We used 0.01 grid spacing in time dimension and grid spacing of 0.002 in the s direction. The results presented in Tables 6, 7 and 8 are for ten, six and fourteen percent coupon rates, respectively. Bond values as well as option values differ significantly among models. Generally, models 2 and 3 gave similar values, and models 1, 4, 5 resulted in similar values as well. In Table 9 the value of a 6 month T-Bill and options on this T-Bill with 3 months to expiration are compared. To obtain the estimates in Table 9, we used 0.005 grid spacing in time dimension and 0.002 in the s direction. T-Bill prices as well as options on it are very close for each model. We also calculated bond values in Tables 6, 7 and 8 using discrete coupon payments. These results are provided in Table 10. For this table we used the same grid spacing as before. In order to compare discrete vs. continuous coupon payments, bond values from table 10 should be compared with bond values

from Tables 6, 7 and 8 for 10, 6 and 14 percent coupon rates, respectively. The results in Table 10 suggest that unless the term structure of interest rate is flat, there is a significant difference between discrete coupon bond values and continuous coupon bond values. Table 11 shows the interest rate at which options with varying expiration dates will be exercised early. It is clear from Table 11 that the interest rate at which it is optimal to exercise the option before maturity varies significantly between the models.

The simulations can also be used to analyze the sensitivity of model values to underlying parameters. Tables 6-8 and other simulations which are not all reported here provide the following results. The bond and option prices for a given time to maturity are determined by λ , r , σ and i for Models 1 and 2. Models 3-5 include two extra parameters, an adjustment coefficient, α , and the long term mean of interest rates, γ . For each model, bond prices are an increasing function of σ , and a decreasing function of r . If λ is positive, a higher λ implies lower bond prices. For a negative λ , a greater absolute value of λ implies higher bond values. The effect of α depends on whether the term structure is decreasing or increasing. If the term structure is downward (upward) sloping a higher α implies higher (lower) bond values. Bond prices are a decreasing function of the long term mean of interest rates, γ , and increasing function of coupon rate, i .

4 Comparison of model values with actual market prices

Data on U.S. Treasury securities were taken from the CRSP bond files for the period of December 1985 through December 1986. For each month, we used data for every T-bill, note, and bond on the files with the exception of callable bonds, notes, flower bonds, and bonds with special tax provisions. Therefore, our sample includes all T-bills, non-callable notes and bonds which are fully taxable. To compare the model values with market prices, we calculated the bond values by solving the differential equations above. All of the parameter values are taken from Tables 1 and 5. At each point in time, we used the actual 10 year and longer maturity index as the proxy for the long term mean of interest rates. Market prices from CRSP bond files are calculated as

$$\text{Market price (MP)} = 1/2(\text{Bid price} + \text{Asked price}) + \text{Accrued interest}$$

In order to evaluate the predictions (PRED) of the models, we calculated the mean percent and the mean absolute percent errors, defined as

$$\text{Mean percent error (MPE)} = \frac{1}{n} \sum_{i=1}^n \frac{\text{PRED}_i - \text{MP}_i}{\text{MP}_i} * 100$$

$$\text{Mean absolute percent error (MAPE)} = \frac{1}{n} \sum_{i=1}^n \frac{|\text{PRED}_i - \text{MP}_i|}{\text{MP}_i} * 100$$

where n is the number of observations.

Table 12 provides a summary of these forecast errors.³ From Table 12 one concludes that first, the prediction of long term bond prices are sensitive to the specification of the behavior of interest rates. Second, each of the five forms of interest rate processes previously specified lead to bond prices which are consistently above the actual market prices.⁴ Third, the overprediction increases with the time to maturity.

³Monthly breakdown of forecast errors is almost identical to the results in Table 12

⁴This can be easily seen by comparing MPE and MAPE. The difference between two is almost nil.

Some Pitfalls

Can the models be rejected based on the above empirical findings? Not quite. Clearly, we made number of strong assumptions when estimating the inputs. We used the T-bill rate as a proxy for the instantenous rate. We also assumed that the NYSE index was a good proxy for the market portfolio. Last but not least, we assumed that the index of bonds with maturities longer than 10 years was a good proxy for the long term mean of short term interest rates. Obviously, this is very rough approximation. The index is a geometric average of short term rates but we are really interested in the arithmetic average of short term interest rates. This would greatly reduce bond price predictions for Models 3-5. Finally, the parameters were assumed to be constant throughout the estimation periods and the test period. Actually, they are not constant. We also assumed that our historical estimates did not contain any measurement errors.

Given all these difficulties, there remains a lot to be done empirically before these models can be rejected. The results might simply suggest that our historical estimates of the parameters involve some estimation error. Future research will help in clarifying some of the outstanding open questions not addressed in this research.

5 Summary and Conclusions

This paper uses five different forms of the term structure of interest rates to determine bond prices, bond option prices, and to evaluate the optimal early exercise policies for bond options. The results can be summarized as follows. First, the results regarding the parameters estimation suggest that

the unexpected change in the interest rate is proportional to the level of interest rate. The results also indicate that the square root process, the most widely used process does not explain actual interest rate changes. Using the estimated parameters of the term structure models, bond prices and options on these bonds are evaluated and compared. The second finding is that these numerical simulations suggest that the specification of the interest rate process is the cornerstone for the valuation of bonds and options on bonds. The values generated using estimated parameters differ greatly among the models considered in this paper. Third, it is argued that the coupon bonds cannot be treated as a portfolio of discount bonds. However, for empirical purposes it would be much more efficient if we could treat coupon bonds as a portfolio of discount bonds. We examined whether any valuation errors arose as a result of this assumption. The results show that unless the term structure is flat, there is a considerable difference between discrete versus continuous coupon payments. Fourth, the interest rate at which it is optimal to exercise the option differ significantly among the models. Fifth, the empirical results suggest that all five different forms of the interest rate processes lead to bond prices which are consistently above the actual market prices. The overprediction increases with the time to maturity. Given the difficulties associated with this type of empirical test, we conclude that the results might be due to either model misspecification or our inability to measure inputs correctly.

TABLE 1
REGRESSION RESULTS

	<i>OLS</i>				<i>MAXL</i>			
	β_0	β_1	σ	R^2	β_0	β_1	σ	R^2
Model 1	0.03596 (4.57)		0.0927	0.0313	0.04356 (3.87)		0.0873	0.1405
Model 2	0.01439 (0.61)		0.5071	0.0000	0.01422 (0.44)		0.4810	0.1003
Model 3	-0.01539 (-0.53)	0.03533 (1.75)	0.5060	0.0044	-0.11342 (-2.14)	0.15045 (3.51)	0.4727	0.1310
Model 4	-0.04327 (-1.77)	0.05430 (3.41)	0.0916	0.0529	-0.12939 (-2.96)	0.11847 (4.02)	0.0853	0.1795
Model 5	-0.01824 (0.44)	0.03235 (1.85)	0.1787	0.0091	-0.09922 (-2.26)	0.11276 (3.36)	0.1659	0.1460

Note: Figures in paranthesis are t-values.

TABLE 2
REGRESSION RESULTS

	<i>OLS</i>				<i>MAXL</i>			
	β_0	β_1	σ	R^2	β_0	β_1	σ	R^2
Model 1	0.01035 (2.37)		0.0942	0.0000	0.0103 (1.65)		0.0885	0.1169
Model 2	0.01439 (0.61)		0.5071	0.0000	0.01422 (0.44)		0.4810	0.1003
Model 3	-0.01539 (-0.53)	0.03533 (1.75)	0.5060	0.0044	-0.11342 (-2.14)	0.15045 (3.51)	0.4727	0.1310
Model 4	-0.00286 (-0.57)	0.02945 (4.91)	0.0919	0.0472	-0.00950 (-1.25)	0.04363 (4.67)	0.0862	0.1629
Model 5	-0.00417 (-0.38)	0.02904 (2.28)	0.1787	0.0089	-0.02977 (-1.67)	0.07321 (3.35)	0.1663	0.1407

Note: Figures in paranthesis are t-values.

TABLE 3

ANALYSIS OF RESIDUALS

	<i>OLS</i>			<i>MAXL</i>		
	ρ	α_0	α_1	ρ	α_0	α_1
Model 1	0.3359	-6.7306 (-30.95)	-0.1597 (-1.12)	0.0458	-6.8110 (-31.07)	-0.1505 (-1.04)
Model 2	0.3198	-7.0033 (-30.90)	1.9703 (13.20)	0.0756	-7.3493 (-31.65)	2.1374 (13.98)
Model 3	0.3345	-6.9543 (-29.68)	1.9297 (12.51)	0.0663	-6.5442 (-29.25)	1.6965 (11.51)
Model 4	0.3499	-6.9030 (-30.13)	-0.0571 (-0.38)	0.0318	-6.9486 (-30.58)	-0.0502 (0.34)
Model 5	0.3559	-6.9617 (-28.34)	0.9009 (5.58)	0.0524	-6.7768 (-30.92)	0.8314 (5.76)

Models 1-5 correspond to residuals from equations 23-27, respectively.

TABLE 4

ANALYSIS OF RESIDUALS

	<i>OLS</i>			<i>MAXL</i>		
	ρ	α_0	α_1	ρ	α_0	α_1
Model 1	0.3447	-7.1287 (-31.12)	0.0585 (0.39)	0.0475	-7.3798 (-32.33)	0.1778 (1.83)
Model 2	0.3198	-7.0033 (-30.90)	1.9703 (13.20)	0.0757	-7.3493 (-31.65)	2.1374 (13.98)
Model 3	0.3345	-6.9543 (-29.68)	1.9297 (12.51)	0.0663	-6.5442 (-29.25)	1.6965 (11.51)
Model 4	0.3432	-6.9189 (-28.26)	-0.1229 (-0.76)	0.0429	-6.8624 (-28.32)	-0.1846 (-1.16)
Model 5	0.3548	-7.0228 (-27.11)	0.9104 (5.34)	0.0566	-6.6565 (-28.79)	0.7166 (4.71)

Models 1-5 correspond to residuals from equations 28-32, respectively.

TABLE 5
ESTIMATED VALUES OF MARKET PRICE
OF INTEREST RATE RISK

Time period	ρ_{rm}	σ_m	λ
1947, 1 – 1985, 12	-0.06808	0.0341	-0.0023
1947, 1 – 1957, 12	0.06698	0.0314	0.0021
1958, 1 – 1968, 12	0.08197	0.0285	0.0023
1969, 1 – 1979, 12	-0.31509	0.0394	-0.0124
1980, 1 – 1985, 12	-0.18875	0.0377	-0.0071

TABLE 6

COMPARISON OF MODEL VALUES FOR COUPON RATE OF 10%

		Model 1	Model 2	Model 3	Model 4	Model 5
$r = 0.2500$	$B(r, 10)$	0.45991	1.28105	1.30572	0.57414	0.66119
	$C(r, 1)$	0.0	0.52749	0.47518	0.0	0.00422
	$P(r, 1)$	0.54009	0.15003	0.07872	0.42586	0.33881
$r = 0.2019$	$B(r, 10)$	0.57307	1.40349	1.39398	0.67990	0.77308
	$C(r, 1)$	0.0	0.60365	0.52860	0.0	0.01470
	$P(r, 1)$	0.42693	0.12174	0.06278	0.32010	0.22692
$r = 0.1710$	$B(r, 10)$	0.67043	1.48642	1.45205	0.76193	0.85764
	$C(r, 1)$	0.0	0.65635	0.56461	0.0	0.03075
	$P(r, 1)$	0.32957	0.10358	0.05299	0.23807	0.14747
$r = 0.1416$	$B(r, 10)$	0.78717	1.56805	1.50796	0.85196	0.94857
	$C(r, 1)$	0.0	0.70903	0.59993	0.0	0.05877
	$P(r, 1)$	0.21283	0.08629	0.04396	0.14804	0.08822
$r = 0.1211$	$B(r, 10)$	0.88618	1.62659	1.54729	0.92264	1.01878
	$C(r, 1)$	0.00038	0.74725	0.62513	0.00028	0.08930
	$P(r, 1)$	0.11382	0.07418	0.03781	0.07736	0.05664
$r = 0.1013$	$B(r, 10)$	0.99857	1.68436	1.58553	0.99779	1.09247
	$C(r, 1)$	0.01894	0.78533	0.64992	0.01152	0.13033
	$P(r, 1)$	0.01953	0.06242	0.03195	0.01295	0.03351
$r = 0.0799$	$B(r, 10)$	1.14317	1.74837	1.62725	1.08808	1.17958
	$C(r, 1)$	0.14317	0.82789	0.67728	0.08808	0.19124
	$P(r, 1)$	0.00006	0.04955	0.02568	0.00006	0.01626
$r = 0.0593$	$B(r, 10)$	1.30992	1.81128	1.66758	1.18470	1.27112
	$C(r, 1)$	0.30992	0.87010	0.70405	0.18470	0.27112
	$P(r, 1)$	0.0	0.03704	0.01971	0.0	0.00647
$r = 0.0373$	$B(r, 10)$	1.52410	1.87982	1.71077	1.29978	1.37792
	$C(r, 1)$	0.52410	0.91650	0.73308	0.29978	0.37792
	$P(r, 1)$	0.0	0.02351	0.01337	0.0	0.00160

$B(r, 10)$ is the bond value with 10 years to maturity, $C(r, 1)$ and $P(r, 1)$ are the american options with 1 year to expiration. The coupon rate and long term mean are 10 percent. σ and α are maximum likelihood estimates taken from table 1. $\lambda = -0.0023$.

TABLE 7

COMPARISON OF MODEL VALUES FOR COUPON RATE OF 6%

	Model 1	Model 2	Model 3	Model 4	Model 5
$r = 0.2500$					
$B(r, 10)$	0.31170	1.01609	1.02760	0.40727	0.48317
$C(r, 1)$	0.0	0.31109	0.24277	0.0	0.00004
$P(r, 1)$	0.68830	0.20156	0.13160	0.59273	0.51683
$r = 0.2019$					
$B(r, 10)$	0.40003	1.11551	1.09841	0.49053	0.57224
$C(r, 1)$	0.0	0.35755	0.27186	0.0	0.00027
$P(r, 1)$	0.59997	0.16414	0.10575	0.50947	0.42776
$r = 0.1710$					
$B(r, 10)$	0.47769	1.18287	1.14502	0.55577	0.64001
$C(r, 1)$	0.0	0.38975	0.29152	0.0	0.00087
$P(r, 1)$	0.52231	0.13994	0.08971	0.44423	0.35999
$r = 0.1416$					
$B(r, 10)$	0.57229	1.24919	1.18989	0.62796	0.71326
$C(r, 1)$	0.0	0.42197	0.31082	0.0	0.00259
$P(r, 1)$	0.42771	0.11676	0.07481	0.37204	0.28674
$r = 0.1211$					
$B(r, 10)$	0.65360	1.29676	1.22145	0.68503	0.77004
$C(r, 1)$	0.0	0.44537	0.32461	0.0	0.00539
$P(r, 1)$	0.34640	0.10048	0.06460	0.31497	0.22996
$r = 0.1013$					
$B(r, 10)$	0.74698	1.34371	1.25214	0.74614	0.82982
$C(r, 1)$	0.0	0.46868	0.33818	0.0	0.01070
$P(r, 1)$	0.25302	0.08461	0.05484	0.25386	0.17129
$r = 0.0799$					
$B(r, 10)$	0.86823	1.39574	1.28563	0.81996	0.90068
$C(r, 1)$	0.00001	0.49475	0.35316	0.0	0.02184
$P(r, 1)$	0.13177	0.06722	0.04435	0.18004	0.11439
$r = 0.0593$					
$B(r, 10)$	1.00854	1.44687	1.31801	0.89911	0.97536
$C(r, 1)$	0.01842	0.52061	0.36782	0.0	0.04205
$P(r, 1)$	0.01021	0.05027	0.03432	0.10144	0.06823
$r = 0.0373$					
$B(r, 10)$	1.19039	1.50259	1.35268	0.99362	1.06275
$C(r, 1)$	0.19039	0.54904	0.38371	0.00110	0.08196
$P(r, 1)$	0.0	0.03192	0.02367	0.02498	0.03038

$B(r, 10)$ is the bond value with 10 years to maturity, $C(r, 1)$ and $P(r, 1)$ are the american options with 1 year to expiration. The coupon rate and long term mean are 0.06 and 0.10, respectively. σ and α are maximum likelihood estimates taken from table 1. $\lambda = -0.0023$.

TABLE 8

COMPARISON OF MODEL VALUES FOR COUPON RATE OF 14%

	Model 1	Model 2	Model 3	Model 4	Model 5
$r = 0.2500$					
$B(r, 10)$	0.60812	1.54666	1.58403	0.74095	0.83937
$C(r, 1)$	0.0	0.75089	0.71736	0.0	0.03571
$P(r, 1)$	0.39188	0.11357	0.04841	0.25905	0.16133
$r = 0.2019$					
$B(r, 10)$	0.74607	1.69218	1.68974	0.86911	0.97418
$C(r, 1)$	0.0	0.85588	0.79355	0.00031	0.08412
$P(r, 1)$	0.25393	0.09177	0.03829	0.13089	0.07525
$r = 0.1710$					
$B(r, 10)$	0.86308	1.79071	1.75929	0.96781	1.07561
$C(r, 1)$	0.00036	0.92841	0.84482	0.01203	0.13750
$P(r, 1)$	0.13692	0.07791	0.03216	0.03219	0.04014
$r = 0.1416$					
$B(r, 10)$	1.00179	1.88769	1.82624	1.07563	1.18435
$C(r, 1)$	0.02400	1.00086	0.89503	0.07806	0.21027
$P(r, 1)$	0.02114	0.06478	0.02656	0.00043	0.01890
$r = 0.1211$					
$B(r, 10)$	1.11852	1.95723	1.87333	1.16005	1.26809
$C(r, 1)$	0.11852	1.05340	0.93085	0.16005	0.27670
$P(r, 1)$	0.00072	0.05563	0.02276	0.0	0.00987
$r = 0.1013$					
$B(r, 10)$	1.25054	2.02585	1.91911	1.24978	1.35579
$C(r, 1)$	0.25054	1.10572	0.96606	0.24978	0.35601
$P(r, 1)$	0.0	0.04676	0.01917	0.0	0.00461
$r = 0.0799$					
$B(r, 10)$	1.41942	2.10187	1.96905	1.35737	1.45921
$C(r, 1)$	0.41942	1.16420	1.00491	0.35737	0.45921
$P(r, 1)$	0.0	0.03709	0.01534	0.0	0.00165
$r = 0.0593$					
$B(r, 10)$	1.61244	2.17657	2.01732	1.47180	1.56764
$C(r, 1)$	0.61244	1.22219	1.04293	0.47180	0.56764
$P(r, 1)$	0.0	0.02771	0.01170	0.0	0.00047
$r = 0.0373$					
$B(r, 10)$	1.85863	2.25796	2.06902	1.60753	1.69384
$C(r, 1)$	0.85863	1.28595	1.08416	0.60753	0.69384
$P(r, 1)$	0.0	0.01758	0.00785	0.0	0.00007

$B(r, 10)$ is the bond value with 10 years to maturity, $C(r, 1)$ and $P(r, 1)$ are the american options with 1 year to expiration. The coupon rate and long term mean are 0.14 and 0.10 respectively. σ and α are maximum likelihood estimates taken from table 1. $\lambda = -0.0023$.

TABLE 9
COMPARISON OF MODEL VALUES FOR T-BILLS

	Model 1	Model 2	Model 3	Model 4	Model 5
$r = 0.2500$					
$B(r, 0.5)$	0.88262	0.88222	0.89282	0.88455	0.88473
$C(r, 0.25)$	0.0	0.01558	0.01727	0.0	0.00059
$P(r, 0.25)$	0.06738	0.06778	0.05718	0.06545	0.06527
$r = 0.2019$					
$B(r, 0.5)$	0.90409	0.90319	0.91248	0.90541	0.90556
$C(r, 0.25)$	0.00074	0.02050	0.02215	0.00128	0.00381
$P(r, 0.25)$	0.04591	0.04921	0.04041	0.04459	0.04444
$r = 0.1710$					
$B(r, 0.5)$	0.91816	0.91701	0.92525	0.91913	0.91924
$C(r, 0.25)$	0.00709	0.02414	0.02566	0.00781	0.00860
$P(r, 0.25)$	0.03184	0.03944	0.03185	0.03087	0.03076
$r = 0.1416$					
$B(r, 0.5)$	0.93178	0.93040	0.93746	0.93231	0.93243
$C(r, 0.25)$	0.01410	0.02793	0.02924	0.01451	0.01471
$P(r, 0.25)$	0.01822	0.03123	0.02486	0.01769	0.01757
$r = 0.1211$					
$B(r, 0.5)$	0.94133	0.93992	0.94605	0.94161	0.94173
$C(r, 0.25)$	0.01906	0.03075	0.03186	0.01927	0.01939
$P(r, 0.25)$	0.00867	0.02602	0.02052	0.00839	0.00827
$r = 0.1013$					
$B(r, 0.5)$	0.95065	0.94927	0.95441	0.95065	0.95079
$C(r, 0.25)$	0.02391	0.03361	0.03448	0.02392	0.02403
$P(r, 0.25)$	0.0	0.02133	0.01668	0.0	0.00072
$r = 0.0799$					
$B(r, 0.5)$	0.96092	0.95959	0.96354	0.96064	0.96074
$C(r, 0.25)$	0.02929	0.03685	0.03740	0.02908	0.02916
$P(r, 0.25)$	0.0	0.01654	0.01285	0.0	0.00001
$r = 0.0593$					
$B(r, 0.5)$	0.97082	0.96973	0.97240	0.97027	0.97040
$C(r, 0.25)$	0.03450	0.04009	0.04028	0.03407	0.03417
$P(r, 0.25)$	0.0	0.01214	0.00940	0.0	0.0
$r = 0.0373$					
$B(r, 0.5)$	0.98156	0.98076	0.98193	0.98068	0.98076
$C(r, 0.25)$	0.04018	0.04368	0.04341	0.03948	0.03957
$P(r, 0.25)$	0.0	0.00760	0.00591	0.0	0.0

$B(r, 0.5)$ is the bond value with 6 months to maturity, $C(r, 0.25)$ and $P(r, 0.25)$ are the american options with 3 months to expiration. Long term mean is 10 percent. σ and α are maximum likelihood estimates taken from table 1. $\lambda = -0.0023$.

TABLE 10
COMPARISON OF BOND VALUES WITH DISCRETE COUPON
PAYMENTS

		Model 1	Model 2	Model 3	Model 4	Model 5
$r = 0.2500$	$i = 0.06$	0.29858	1.01124	1.02343	0.39515	0.47199
	$i = 0.10$	0.43805	1.27309	1.29831	0.55395	0.64264
	$i = 0.14$	0.57751	1.53494	1.57319	0.71274	0.81330
$r = 0.2019$	$i = 0.06$	0.38761	1.11158	1.09491	0.47907	0.56183
	$i = 0.10$	0.55235	1.39705	1.38762	0.66083	0.75587
	$i = 0.14$	0.71709	1.68253	1.68034	0.84258	0.94990
$r = 0.1710$	$i = 0.06$	0.46593	1.17955	1.14196	0.54486	0.63021
	$i = 0.10$	0.65084	1.48100	1.44640	0.74379	0.84147
	$i = 0.14$	0.83575	1.78246	1.75084	0.94272	1.05273
$r = 0.1416$	$i = 0.06$	0.56139	1.24646	1.18723	0.61768	0.70413
	$i = 0.10$	0.76905	1.56363	1.50295	0.83495	0.93357
	$i = 0.14$	0.97671	1.88079	1.81868	1.05222	1.16301
$r = 0.1211$	$i = 0.06$	0.64348	1.29445	1.21908	0.67526	0.76143
	$i = 0.10$	0.86949	1.62287	1.54273	0.90661	1.00470
	$i = 0.14$	1.09550	1.95129	1.86638	1.13796	1.24796
$r = 0.1013$	$i = 0.06$	0.73779	1.34181	1.25003	0.73693	0.82179
	$i = 0.10$	0.98381	1.68133	1.58139	0.98300	1.07937
	$i = 0.14$	1.22983	2.02085	1.91275	1.22906	1.33696
$r = 0.0799$	$i = 0.16$	0.86035	1.39427	1.28381	0.81144	0.89335
	$i = 0.10$	1.13102	1.74609	1.62357	1.07487	1.16765
	$i = 0.14$	1.40169	2.09791	1.96333	1.33829	1.44195
$r = 0.0593$	$i = 0.06$	1.00267	1.44583	1.31645	0.89136	0.96880
	$i = 0.10$	1.30046	1.80972	1.66433	1.17295	1.26042
	$i = 0.14$	1.59824	2.17362	2.01221	1.45453	1.55205
$r = 0.0373$	$i = 0.06$	1.18704	1.50199	1.35140	0.98690	1.05713
	$i = 0.10$	1.51807	1.87904	1.70797	1.28966	1.36871
	$i = 0.14$	1.84910	2.25609	2.06454	1.59242	1.68028

For a given interest rate there are three bond values for 6, 10 and 14 percent coupon rate. All the parameters are the same as in Tables 6, 7 and 8. In order to compare discrete versus continuous coupon payments, bond values for a given interest rate should be compared with the values from tables 6, 7 and 8 for 10, 6 and 14 percent coupon rate, respectively.

TABLE 11

COMPARISON OF OPTIMAL EARLY EXERCISE POLICIES
OPTIMAL EARLY EXERCISE POLICIES FOR CALL OPTION

	Model 1	Model 2	Model 3	Model 4	Model 5
$t = 0.2$	0.0941	0.0040	0.0081	0.0941	0.0776
$t = 0.4$	0.0917	0.0	0.0020	0.0917	0.0684
$t = 0.6$	0.0917	0.0	0.0020	0.0917	0.0638
$t = 0.8$	0.0893	0.0	0.0	0.0917	0.0616
$t = 1.0$	0.0893	0.0	0.0	0.0917	0.0593

OPTIMAL EARLY EXERCISE POLICIES FOR PUT OPTION

	Model 1	Model 2	Model 3	Model 4	Model 5
$t = 0.2$	0.1086	0.5723	0.5924	0.1086	0.1628
$t = 0.4$	0.1111	0.6129	0.6287	0.1086	0.1710
$t = 0.6$	0.1111	0.6340	0.6502	0.1111	0.1765
$t = 0.8$	0.1111	0.6502	0.6611	0.1111	0.1820
$t = 1.0$	0.1136	0.6611	0.6722	0.1111	0.1848

t is the time to expiration. Optimal exercise policy gives the interest rate at which it is optimal to exercise the option before the maturity. Coupon rate is 10 percent. Optimal early exercise policies were solved simultaneously as bond and option values in Table 6 were calculated.

TABLE 12
FORECAST ERRORS

		All sample	$T < 1$	$1 < T < 5$	$5 < T < 10$	$10 < T < 20$	$20 < T < 30$
Model 1	<i>MPE</i>	4.97	0.23	2.30	8.25	20.46	28.18
	<i>MAPE</i>	4.99	0.29	2.30	8.25	20.46	28.17
Model 2	<i>MPE</i>	8.92	0.23	2.60	12.09	41.70	81.39
	<i>MAPE</i>	8.94	0.29	2.61	12.09	41.70	81.39
Model 3	<i>MPE</i>	8.77	0.23	2.59	11.96	41.03	79.14
	<i>MAPE</i>	8.79	0.29	2.60	11.96	41.03	79.14
Model 4	<i>MPE</i>	4.97	0.23	2.30	8.25	20.41	28.04
	<i>MAPE</i>	4.99	0.29	2.31	8.25	20.42	28.04
Model 5	<i>MPE</i>	5.44	0.23	2.32	8.59	22.94	35.66
	<i>MAPE</i>	5.46	0.29	2.33	8.59	22.94	35.66

Note: T is time to maturity, *MPE*=Mean percent error and *MAPE*=Mean absolute percent error.

Chapter III

The Pricing of Bond Futures and Options on Bond Futures

1 Introduction

The purpose of this chapter is two-fold. First, three different forms of the term structure of interest rates are used to determine bond futures prices and bond futures option prices through simulations. Second, empirical results on the performance of these models are provided and compared to Black's (1976) European futures option pricing formula as well as to Barone-Adesi and Whaley (1987) American model. This is the first paper that provide empirical results on the performance of stochastic interest rate models. Black[1976] was the first to provide a valuation formula for futures commodity options. Black's model has important advantages. It is simple and easy to apply. It is also independent of preferences. In Black's model interest rate is assumed to be constant. While the validity of this assumption may not be questioned in pricing stock options, currency options or index options it seems questionable in modeling bond options

and bond futures options pricing because bond price fluctuations depend only on interest rates fluctuations. It is therefore more intuitively appealing to apply models with stochastic interest rates to the pricing of bond futures options. The literature on pricing bond futures and options on bond futures with stochastic interest rates modeling is scarce. The most publicized and quoted work is that of pricing bond futures and bond futures options are those of Cox, Ingersol and Ross[1981] and Courtadon[1983].

The major reason of using models with stochastic interest rates is that they seem more realistically appealing. We know that interest rates are not constant and fluctuate over time. The disadvantage of these models is that they are much more complicated. Therefore, it is of interest to investigate these models in terms of their applicability and predictive power.

Our preliminary empirical results can be summarized as follows. First, the Black's models appears to have a better explanatory power than that of the stochastic interest rate model. This is somewhat surprising based on one's expectations of the stochastic interest rate model. This issue is addressed later in the paper. Second, there are no differences, in terms of predictive ability, between Black's European model and Barone-Adesi-Whaley's American versions of Black's model. Third, the results indicate that market prices of these options deviate significantly from their corresponding model prices and the systematic mispricing errors are related to time to maturity and exercise price.

When pricing interest rate dependent claims, the key element is to assume an appropriate stochastic process for interest rates. The attempt to price bond futures and futures options using stochastic interest rate processes almost always does not yield closed form formulas. Consequently,

numerical methods are required to value bond futures and options on bond futures.

This chapter differs from the Cox, Ingersol and Ross (CIR) and Courtadon papers in two important respects. First, both CIR and Courtadon papers assume square root process for the behavior of interest rates. We use the square root process, and two additional forms of the term structure of interest rates to examine whether assumptions about the interest rate process are significant. Second, until now, there is no empirical work regarding the ability of these models to predict bond futures prices and bond futures option prices. One of the major contribution of this paper is that we test and provide empirical results about the ability of these models to predict actual market prices.

The rest of the chapter is organized in the following manner. In section 2, we describe the model and explain its implementation. Section 3 examines the simulations. In section 4, we analyze the data and present empirical results. Section 5 provides a summary and conclusions.

2 Valuation of Bond Futures and Options on Bond Futures

2.1 Stochastic Interest Rate

The model developed here is similar to a model developed by Black, Derman and Toy [1987] for Treasury Bond options.¹ We assume that the interest

¹Black, Derman and Toy [1987] use the whole yield curve as input in their model. In our model, we use only current interest rates and underlying futures prices. Their model is

rate process is the only variable that determine bond prices, bond futures prices and bond futures option prices. Once we know today's interest rate and the future's interest rate changes, everything about bond prices, bond futures prices and bond futures option prices is known. We will assume that the interest rate r follows a diffusion process. Its dynamic is given by

$$dr = \alpha(\gamma - r)dt + \sigma g(r)dz \quad (1)$$

where dr is change in interest rate, dz is change in Wiener process, σ is the volatility parameter, γ is the long run mean of interest rates and α is the adjustment coefficient. This process is a mean reverting process where there is some long term mean rate of interest to which short term rates eventually converge. The higher the value of α is, the faster is the converging process. The terms $\alpha(\gamma - r)$ and $\sigma g(r)$ are the instantaneous drift and standard deviation, respectively, of the process r . It is assumed that g is of the form

$$g(r) = r^\beta$$

where β is equal to 0, 0.5 or 1. Unanticipated changes in the interest rate are assumed either to be constant, proportional to the square root of the level of interest rate or proportional to the level of interest rate for $\beta = 0.0$, $\beta = 0.5$ and $\beta = 1.0$, respectively. It is also assumed that all bonds are priced according to the Local Expectations Hypothesis (LEH). The LEH implies that at each instant t , the bond price $B(r, t, c)$ satisfies

$$E_t[dB] + cdt = Br(t)dt$$

different from the one tested in this paper. Therefore, the results should not be interpreted as acceptance or rejection of their model. In order to use their model we need to have whole term structure of interest rates on a daily basis. It would be interesting to test their model using the whole term structure of interest rates.

where c is the continuous coupon rate and E is the expectation operator. Given equation (1) and the assumptions above, based on CIR and Vasicek it can be shown that $B(r,t)$ satisfies equation (2).

$$1/2\sigma^2g^2B_{rr} + \alpha(\gamma - r)B_r - rB + B_t + c = 0 \quad (2)$$

In equation (2), B_{rr} is the second derivative of B with respect to r , B_r is the first derivative of B with respect to r and B_t is the time derivative. Denoting T to be the maturity date of bond, the bond price $B(t, T, r)$ is then obtained by solving (2) subject to the following two boundary conditions:

$$B(T, T, r) = 1 \quad \text{and}$$

$$B(t, T, \infty) = 0$$

In addition, we assume that Treasury Bond futures options will mature at t_1 , where the treasury bond futures contract matures at t_2 where $t_1 < t_2$. Bond futures and options on bond futures will satisfy the same partial differential equation but different boundary conditions must be employed in the solution procedure. Assuming the exact values of β , α and γ are known, we can value bonds, bond futures and bond futures options. If $F(r,t)$ represent the value of a bond futures contract, CIR show that the futures price of a Treasury bond with maturity t_2 is equal to the value of an equivalent bond which pays a continuous coupon $rF(r, t)$ and, at the maturity date of the futures contract, pays a lump sum payment equal to the treasury bond value at that time. This implies that $F(r,t)$ must satisfy the differential equation

$$1/2\sigma^2g^2F_{rr} + \alpha(\gamma - r)F_r + F_t = 0 \quad (3)$$

Futures price $F(r, t)$ is obtained by solving (3) subject to

$$F(r, t_2) = B(r, t_2)$$

If $C(r, t)$ represents the value of an American call option on the Treasury bond futures contract with time to expiration t_1 and exercise price E , then $C(r, t)$ is calculated by solving the partial differential equation in (4).

$$1/2\sigma^2g^2C_{rr} + \alpha(\gamma - r)C_r - rC + C_t = 0 \quad (4)$$

subject to

$$C(r, 0) = \max[0, F(r, t_1) - E]$$

$$C(r, t) = \max[C(r, t), F(r, t) - E] \quad \text{for } t < t_1$$

If $P(r, t)$ represents the value of an American put option on the Treasury bond futures, then $P(r, t)$ can be calculated by solving the partial differential equation in (5).

$$1/2\sigma^2g^2P_{rr} + \alpha(\gamma - r)P_r - rP + P_t = 0 \quad (5)$$

subject to

$$P(r, 0) = \max[0, E - F(r, t_1)]$$

$$P(r, t) = \max[P(r, t), E - F(r, t)] \quad \text{for } t < t_1$$

It is assumed here that there is a unique Treasury bond issue which is delivered against the expiring Treasury bond future contract. In the real world, the agent who sold the futures contract has the option of choosing the Treasury bonds for delivery with different terms of maturity and coupon rates.

The solution procedure requires to find the solution for bond prices, futures prices and bond futures option prices simultaneously. However, if the purpose is to price bond futures, one can take bond prices as given. On the other hand, when the purpose is to price bond futures options, one can

take bond futures prices as given. Therefore, any model of pricing bond futures should provide us with the correct value for bond prices. The same logic applies for bond futures options. In pricing bond futures options, our valuation models should give us the correct value for futures prices. When comparing futures (futures options) pricing models, all the models should give the same initial bond price (futures price).

Since there are not always closed form solutions to the above differential equations finite difference methods were applied to find the value of underlying securities. For the purpose of calculations r is replaced by $s = 1/(1+r)$ in the differential equations above. Since r can range from 0 to infinity, s can vary between 0 and 1. If we let $B(r, t) = b(s, t)$ then the partial derivatives of B in terms of b is

$$B_r = -s^2 b_s$$

$$B_{rr} = s^4 b_{ss} + 2s^2 b_s$$

Substituting the value of r and derivatives of B , equation (2) becomes

$$1/2\sigma^2 g^2 s^4 b_{ss} + [\sigma^2 g^2 s^2 - \gamma(1.0 - s)s]b_s - ((1.0 - s)/s)b - b_t + c = 0 \quad (6)$$

The same procedures of transformations are also applied to equations 3-5 for the solution of bond futures and bond futures options. Substituting the specific forms of $g(r) = r^\beta$ we get the differential equations to be solved.

2.2 Constant interest rate

2.2.1 European Futures Options

Black shows that under the usual perfect market assumptions and a constant interest rate, the value of a call option on a futures contract is

$$c(F, T; E) = e^{-rT} [FN(d_1) - EN(d_2)] \quad (7)$$

where $d_1 = [\ln(F/E) + 0.5\sigma^2T]/\sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$ and where $N()$ is the cumulative univariate normal distribution.

The value of a European put option on a futures contract is

$$p(F, T; E) = e^{-rT} [EN(-d_2) - FN(-d_1)] \quad (8)$$

where all notation is the same as before.

2.2.2 American Futures Options

American futures option prices are calculated using the Barone-Adesi and Whaley [1987] algorithm. The quadratic approximation method for the American call option on a futures contract as provided in Barone-Adesi and Whaley is

$$C(F, T; E) = c(F, T; E) + A_2(F/F^*)^{q_2} \quad \text{where } F < F^* \text{ and}$$

$$C(F, T; E) = F - E \quad \text{where } F \geq F^*$$

$$\text{and where } A_2 = (F^*/q_2)1 - e^{-rT}N(d_1(F^*)),$$

$$d_1(F^*) = [\ln(F^*/E) + 0.5\sigma^2T]/\sigma\sqrt{T},$$

$$q_2 = (1 + \sqrt{1 + 4k})/2, \text{ and } k = 2r/[\sigma^2(1 - e^{-rT})].$$

F^* is the critical futures price above which the American call futures option should be exercised. The critical price is determined by numerical methods by solving

$$F^* - E = c(F^*, T, E) + 1 - e^{-rT} N(d_1(F^*)) F^* / q_2$$

The quadratic approximation of the American put option on a futures contract is

$$P(F, T; E) = p(F, T; E) + A_1(F/F^{**})^{q_1} \quad \text{where } F > F^{**} \text{ and}$$

$$P(F, T; E) = E - F \quad \text{where } F \leq F^{**}$$

$$\text{and where } A_1 = -(F^{**}/q_1) [1 - e^{rT} N(-d_1(F^{**}))], \quad q_1 = (1 - \sqrt{1 + 4k})/2$$

and where all other notation is the same as for the American call option. F^{**} is the critical futures price at which the American put futures option should be exercised early and is determined numerically by solving

$$E - F^{**} = p(F^{**}, T; E) - 1 - e^{-rT} N(-d_1(F^{**})) F^{**} / q_1$$

3 Numerical Simulations

For a given coupon rate and bond prices, and assumption about the behavior of the interest rate, the prediction of futures prices and bond futures option prices are the subject of the numerical simulations in this section. Initially, we calculate the bond futures prices predicted by each model by solving equation (3). Then, for a given bond futures price, we calculate the bond futures option prices predicted by each model.

In our simulations, we start by guessing γ and use it to compute a bond price (bond futures price). We then compare this price with the given bond price (bond futures price). We keep adjusting γ until our model gives us the correct value for bonds (bond futures). Then, using this γ we calculate futures prices (bond futures option prices).

In Table 13 we have produced the bond futures prices for the following parameters:

$B=0.75, 1.0$ and 1.25

Coupon rate, $c=0.08$

Face value of bond= 1.0

$T= 16.0$

$t_1= 1.0$

Volatility parameter, $\sigma=0.10$ and 0.20

$r=0.05, 0.10$ and 0.15

$\beta=0.0, 0.5$ and 1.0

$\alpha=0.20$ and 0.40

A quick look at Table 13 (and other simulations which not all reported here) suggest the following interesting results. First, when the maturity of futures contract is short (three months or less), all three models provide almost identical results. Second, when a futures contract has more than one year to expiration, the choice of the model matters. Third, for a given bond price and interest rate, the futures prices are an increasing function of β . Forth, futures prices are increasing functions of the interest rate and the adjustment coefficient.

In Table 14, we have generated bond futures option prices for the following parameters:

$F=0.90, 1.0$ and 1.10

Coupon rate, $c=0.08$

Face value of bond= 1.0

$T=16.0$

$t_1 = 0.92$

$t_2 = 1.0$

Volatility parameters, $\sigma=0.10, 0.20$

The adjustment coefficient, $\alpha=0.10, 0.20$

$\beta=0.0, 0.5$ and 1.0

A close look at Table 14 reveals the following: bond futures option prices are very sensitive to the underlying assumptions about unanticipated changes in interest rates. For a given futures price, option prices are an increasing function of interest rate. For a given futures price and interest rate, option prices are a decreasing function of β . This result is useful because if one assumes that $\beta=1.0$ and finds that the underlying pricing model overprices the actual market prices it implies that the square root model will make even larger prediction errors.

In the next section, using data on bond futures options, we empirically test the stochastic interest rate model ($\beta = 1$) and compare the results with Black's model.

4 Empirical Tests of Pricing Models

In Table 15, we present the empirical results on the performance of the bond futures option pricing models. We follow the usual procedure. First, we describe the data and explain how the model prices are calculated. Second, we analyze the results. Third, we examine whether there exists systematic patterns in the models' prediction errors.

4.1 Data

The tests of the models for options on bond futures are based on data for bond futures options that are traded on The Chicago Board of Trade (CBOT). The data consists of all daily settlement prices from January 2,

1987 to December 31, 1987.² The original sample has 9747 calls and 8542 put option prices. However, the actual sample is smaller than this because all nominal³ prices and option prices which are less than the option's intrinsic value were eliminated. Our final sample consists of 5826 calls and 5057 puts. The interest rate is the weekly average of 3-month T-Bill rates collected from the Federal Reserve Bank publications. Theoretically, one should use the T-bill rates which correspond to the option's maturity. It is assumed that the use of weekly averages will have negligible effect on our results.

4.2 Implied Parameters

Black's European option pricing formula as well as its American version has five parameters. These are F , E , T , r and σ . The terms F , E , T and r are known or are easily observable. The only unknown parameter is σ of the underlying futures contracts. There are several ways one can estimate σ . For example, one can use historical data to calculate the standard deviation of futures price changes or one can use the implied volatilities. Implied volatilities can also be estimated in different ways. The most common approaches are:

²The problem of nonsynchronous prices with using daily settlement prices are well known. But the complexity of stochastic interest rate models does not allow us to use transaction prices. CPU time required by stochastic interest rate models is enormous. We are in the process of another research project using transaction prices to test constant interest rate models more thoroughly.

³The Chicago Board of Trade data manual refers to nominal price "a future/strike or option contract does not have a trade in it all day, will, if it has previously traded at some time since it was listed for trading, have a closing price denoted as a nominal".

1. To calculate implied volatility by using at the money option or the closest one being at the money.
2. Implied volatilities can be calculated using all the options on a given day using either equal weights or different weights.
3. Using all the option prices on a given date, and finding σ which minimizes the sum of squared residuals.

All these volatility estimates are calculated for calls and puts separately. In order to test the predictive power of these models, we also calculate historical volatilities at time t over the period of $(t-n)$ to t , where n is the number of days to the option expiration date. The number of days used for the calculation of historical volatility is equal to the number of days left to option's expiration date. Actual volatilities are calculated over the period of $(t+1)$ to $(t+n)$. To test the predictive ability of each estimate, we run the following regression

$$\sigma_{ACT} = \alpha + \beta\sigma_{EST}$$

where σ_{ACT} is the actual volatility and σ_{EST} is the estimated σ either historical or implied. Table 3 shows the results of these tests. The results indicate that, in terms of R^2 , historical volatilities outperform most implied volatility estimates. We also calculate the mean absolute percent errors and mean square errors. The results are similar to those obtained from the regression analysis.

We calculate Black's European model and Barone-Adesi and Whaley American model prices using implied standard deviations. Using all options with the same expiration date, every day we compute an at the money implied standard deviation (ISD1), an equally weighted implied standard

deviation (ISD2) and a non-linear least squares estimates of standard deviation (ISD3).

In order to generate option prices for stochastic interest rate models, there are three parameters to estimate: γ , α and σ . Since γ is determined through a process of fitting model prices to actual prices, the only remaining parameters to be estimated are α and σ . By trial and error, considerable time was spent to determine α and σ to accomplish a meaningful predictive power of the models. At the start, we assumed that the market price of interest rate risk is zero i.e. the local expectation hypothesis holds. Based on zero market price of interest rate risk, model prices were calculated and compared to actual market prices. Finally, we set $\lambda = 0.12$ for calls, $\lambda = 0.14$ for puts, $\alpha = 0.07$, and $\sigma = 0.35$ where λ , is the market price of interest rate risk (See Cox, Ingersol and Ross[1981] or Courtadon[1982]).

Table 16 presents the comparison of the model prices with market prices for the European and American futures pricing models of Black, Barone-Adesi-Whaley, and for stochastic interest rate models. European futures option pricing predictions are calculated using Black's equation. American futures options pricing predictions are calculated using Barone-Adesi and Whaley algorithms (1987).⁴

From Table 16, one can conclude that the predictive power of constant interest rate models depends on the method of estimating implied volatilities. Note that non-linear least squares estimate yields smaller pricing errors than those of the other implied standard deviations. The perfor-

⁴The model prices generated by using historical data to estimate standard deviation did not perform well in predicting actual market prices. Therefore, we report only the results for implied volatility estimates.

mance of models based on stochastic interest rate was not of a quality as we expected it to be. This might be due to restricting ourselves to one factor model of the term structure of interest rates. One could use more than one factor for modelling the behavior of interest rates.⁵ Finally, the results in Table 16 indicate that Black's European futures options pricing formula performed at least as well as the American futures option pricing model.

4.3 Tests of Biases in the Pricing Models

Previous research has shown that Black's European model as well as its American version shows systematic price biases when it is used to value foreign currency options or index options (See Shastri and Tandon [1986] and Whaley [1986] among others). Since the systematic biases imply model misspecification and/or market inefficiency, it is interesting to see whether these biases exist in the bond futures options market. Therefore, we test for the existence of time to maturity bias and moneyness bias (in, at or out of the money) for every model.

Table 17 presents the results of these tests. The hypothesis that models' mispricing is related to option maturity and the degree of moneyness is tested using the following regressions:

$$ERR = \beta_0 + \beta_1 T + \beta_2 \ln(F/E)$$

where ERR is the difference between the market price and model prediction. Table 17 shows that both time to maturity and exercise price biases

⁵See Brennan and Schwartz [5,6,7,8] for two factor models.

depend on how implied volatilities are estimated. It indicates that put options that in the money are overvalued, call options that out of money are overvalued. Time to maturity bias does not have consistent pattern among different models.

5 Summary and Conclusions

This paper uses three forms of the term structure of interest rates to determine bond futures prices and bond futures option prices. Our numerical simulations suggest that bond futures prices and bond futures option prices are quite sensitive to the underlying assumption about the behavior of interest rates. We provide empirical results on the performance of these models and compare them to Black's European futures option pricing model and Barone-Adesi and Whaley American version. Average pricing errors and systematic mispricing errors are analyzed.

Our preliminary empirical results can be summarized as follows. First, Black's model explains the actual market prices much better than the stochastic interest rate models. Second, there is no difference, in terms of predictive ability, between the European and the American version of Black's model. Third, the results indicate that market prices of these options deviate significantly from their corresponding model prices and it is also shown that systematic mispricing errors are related to time to maturity and exercise price.

The results can be interpreted as evidence for rejection of the hypothesis that the option pricing models (constant interest rate as well as stochastic interest rate models) are valid and/or that the market for bond futures

options is efficient. Practical implication of the results is that Black's European futures option pricing model and its Barone-Adesi and Whaley American version can be used to value bond futures options if the time to maturity is short and the option is at the money or close to being at the money.

TABLE 13

BOND FUTURES SIMULATIONS

 $\alpha = 0.20$ $\sigma = 0.10$ $t_1 = 1.0$ $T = 16.0$

	r	0.05	0.10	0.15
$B = 0.75$	$\beta = 0.0$	0.71123	0.74283	0.77470
	$\beta = 0.5$	0.71356	0.74603	0.77927
	$\beta = 1.0$	0.71385	0.74625	0.77977
$B = 1.0$	$\beta = 0.0$	0.96801	1.01033	1.05488
	$\beta = 0.5$	0.97332	1.01768	1.06107
	$\beta = 1.0$	0.97491	1.01706	1.06313
$B = 1.25$	$\beta = 0.0$	1.22629	1.27807	1.33298
	$\beta = 0.5$	1.23196	1.28625	1.34486
	$\beta = 1.0$	1.23310	1.28834	1.34693

 $\alpha = 0.20$ $\sigma = 0.20$ $t_1 = 1.0$ $T = 16.0$

	r	0.05	0.10	0.15
$B = 0.75$	$\beta = 0.0$	0.70915	0.73570	0.76489
	$\beta = 0.5$	0.71466	0.74590	0.77751
	$\beta = 1.0$	0.71455	0.74607	0.77936
$B = 1.0$	$\beta = 0.0$	0.96467	1.00072	1.04040
	$\beta = 0.5$	0.97362	1.01511	1.05892
	$\beta = 1.0$	0.97381	1.01691	1.06328
$B = 1.25$	$\beta = 0.0$	1.22076	1.26614	1.31388
	$\beta = 0.5$	1.23247	1.28502	1.33890
	$\beta = 1.0$	1.23255	1.28835	1.34474

 $\alpha = 0.40$ $\sigma = 0.10$ $t_1 = 1.0$ $T = 16.0$

	r	0.05	0.10	0.15
$B = 0.75$	$\beta = 0.0$	0.71483	0.74533	0.77533
	$\beta = 0.5$	0.71779	0.74655	0.77820
	$\beta = 1.0$	0.71726	0.74770	0.77742
$B = 1.0$	$\beta = 0.0$	0.97291	1.01248	1.05181
	$\beta = 0.5$	0.97605	1.01546	1.05757
	$\beta = 1.0$	0.97387	1.01596	1.05807
$B = 1.25$	$\beta = 0.0$	1.22912	1.27784	1.32884
	$\beta = 0.5$	1.23382	1.28395	1.33520
	$\beta = 1.0$	1.23460	1.28464	1.33678

TABLE 14
BOND FUTURES OPTIONS SIMULATIONS

		$\alpha = 0.20$	$\sigma = 0.10$	$t_1 = 0.92$	$t_2 = 1.0$	$T = 16.0$	$E = 1.0$
	r	0.05	0.10	0.15	0.05	0.10	0.15
$F = 0.90$	$\beta = 0.0$	0.01926	0.03930	0.04505	0.11548	0.13145	0.13574
	$\beta = 0.5$	0.0	0.00308	0.00658	0.10	0.10	0.10160
	$\beta = 1.0$	0.0	0.0	0.0	0.10	0.10	0.10
$F = 1.0$	$\beta = 0.0$	0.07439	0.08720	0.08978	0.07135	0.08324	0.08615
	$\beta = 0.5$	0.02529	0.03290	0.03973	0.02495	0.03310	0.03838
	$\beta = 1.0$	0.00714	0.01166	0.01628	0.00649	0.01085	0.01582
$F = 1.10$	$\beta = 0.0$	0.14354	0.14873	0.14835	0.04289	0.05109	0.05337
	$\beta = 0.5$	0.10198	0.10426	0.10583	0.00348	0.00765	0.01205
	$\beta = 1.0$	0.10	0.10	0.10	0.0	0.0	0.0
		$\alpha = 0.20$	$\sigma = 0.20$	$t_1 = 0.92$	$t_2 = 1.0$	$T = 16.0$	$E = 1.0$
	r	0.05	0.10	0.15	0.05	0.10	0.15
$F = 0.90$	$\beta = 0.0$	0.01087	0.04083	0.06062	0.10749	0.13177	0.14861
	$\beta = 0.5$	0.00406	0.01369	0.02262	0.10176	0.10867	0.11418
	$\beta = 1.0$	0.0	0.00021	0.00201	0.10020	0.10000	0.10090
$F = 1.0$	$\beta = 0.0$	0.07754	0.10265	0.11799	0.07345	0.09557	0.10886
	$\beta = 0.5$	0.04276	0.05561	0.06562	0.04128	0.05425	0.06349
	$\beta = 1.0$	0.01183	0.02158	0.02938	0.01260	0.02081	0.02995
$F = 1.10$	$\beta = 0.0$	0.15393	0.17240	0.18194	0.05151	0.06975	0.08035
	$\beta = 0.5$	0.11346	0.12130	0.12927	0.01609	0.02646	0.03463
	$\beta = 1.0$	0.10	0.10	0.10095	0.00014	0.00203	0.00644
		$\alpha = 0.10$	$\sigma = 0.10$	$t_1 = 0.92$	$t_2 = 1.0$	$T = 16.0$	$E = 1.0$
	r	0.05	0.10	0.15	0.05	0.10	0.15
$F = 0.90$	$\beta = 0.0$	0.03492	0.06284	0.07170	0.13021	0.15369	0.16036
	$\beta = 0.5$	0.00346	0.01149	0.01941	0.10150	0.10652	0.11181
	$\beta = 1.0$	0.0	0.0	0.00075	0.10	0.10	0.10
$F = 1.0$	$\beta = 0.0$	0.09492	0.11442	0.12000	0.09065	0.10996	0.11570
	$\beta = 0.5$	0.03686	0.04902	0.05859	0.03569	0.04850	0.05870
	$\beta = 1.0$	0.00981	0.01721	0.02525	0.00957	0.01706	0.02473
$F = 1.10$	$\beta = 0.0$	0.16458	0.17673	0.17907	0.06317	0.07785	0.08281
	$\beta = 0.5$	0.10768	0.11596	0.12158	0.01024	0.01967	0.02880
	$\beta = 1.0$	0.10	0.10	0.10079	0.0	0.0	0.00292

TABLE 15

Tests of the predictive power of different estimates of volatility

$$\sigma_{ACT} = \alpha + \beta \sigma_{EST}$$

HSD	0.2326 (23.572)	-0.2411 (-4.568)	0.0832
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CALLS

	α	β	R^2
Black - European ISD1	0.2064 (17.488)	-0.1247 (-1.494)	0.0056
Black - European ISD2	0.2229 (20.300)	-0.2342 (-3.161)	0.0394
Black - European ISD3	0.2095 (18.057)	-0.1459 (-1.796)	0.0101
Black - American ISD1	0.2066 (17.574)	-0.1263 (-1.514)	0.0059
Black - American ISD2	0.2219 (20.325)	-0.2287 (-3.090)	0.0376
Black - American ISD3	0.2197 (19.424)	-0.2201 (-2.774)	0.0297

PUTS

	α	β	R^2
Black - European ISD1	0.2171 (17.277)	-0.2021 (-2.269)	0.0186
Black - European ISD2	0.2436 (19.893)	-0.3433 (-4.549)	0.0825
Black - European ISD3	0.2226 (17.098)	-0.2348 (-2.616)	0.0260
Black - American ISD1	0.2172 (17.335)	-0.2038 (-2.289)	0.0190
Black - American ISD2	0.2461 (20.308)	-0.3613 (-4.807)	0.0917
Black - American ISD3	0.2193 (17.099)	-0.2126 (-2.394)	0.0212

σ_{ACT} is the actual volatility and σ_{EST} is the estimated volatility. HSD is the historical volatility, ISD1 is at the money σ , ISD2 is equally weighted σ and ISD3 is the least squares estimates.

Note: t values are in parenthesis.

TABLE 16
Comparison of Market and Model Prices

Model		n	Mean ERR	CALLS			
				Mean ERR > 0	Mean ERR < 0	% ERR > 0	% ERR < 0
Black – European	ISD1	5107	0.1091	0.1047	-0.1156	0.595	0.405
Black – European	ISD2	5765	0.0978	0.0984	-0.0974	0.404	0.596
Black – European	ISD3	5826	0.0853	0.0935	-0.0766	0.51	0.49
Black – American	ISD1	5107	0.1032	0.0971	-0.1124	0.598	0.402
Black – American	ISD2	5766	0.0922	0.0893	-0.0943	0.412	0.587
Black – American	ISD3	5826	0.0835	0.0903	-0.0745	0.572	0.428
Stochastic interest rate		5826	0.3377	0.3630	-0.2101	0.835	0.165
				PUTS			
Black – European	ISD1	4370	0.1578	0.1653	-0.1281	0.796	0.204
Black – European	ISD2	5004	0.1686	0.0933	-0.2355	0.47	0.53
Black – European	ISD3	5057	0.1280	0.1279	-0.1284	0.65	0.35
Black – American	ISD1	4370	0.1619	0.1683	-0.1385	0.785	0.215
Black – American	ISD2	5004	0.1712	0.0961	-0.2395	0.476	0.524
Black – American	ISD3	5057	0.1334	0.1352	-0.1299	0.663	0.337
Stochastic interest rate		5057	0.3308	0.3604	-0.2203	0.788	0.212

ERR is the difference between market price and model price. n is the number of observations, ISD1 is at the money σ , ISD2 is equally weighted σ and ISD3 is the least squares estimates. Black-European model prices calculated using Black European futures pricing formula and Black-American futures option prices are calculated using Barone-Adesi and Whaley algorithm.

TABLE 17

Time to Maturity and Exercise Price Biases in Pricing Models

$$ERR = \beta_0 + \beta_1 T + \beta_2 \ln(F/E)$$

	<i>CALLS</i>			R^2
	β_0	β_1	β_2	
Black – European ISD1	0.0758 (18.816)	-0.0953 (-7.935)	0.6381 (23.528)	0.1285
Black – European ISD2	-0.0356 (-9.921)	0.1144 (11.104)	0.3952 (15.738)	0.0506
Black – European ISD3	0.0315 (10.422)	0.0102 (1.170)	0.5319 (25.130)	0.1004
Black – American ISD1	0.0656 (16.875)	-0.0945 (-8.146)	0.4914 (18.767)	0.0931
Black – American ISD2	-0.0369 (-10.795)	0.1002 (10.226)	0.2777 (11.631)	0.0327
Black – American ISD3	0.0151 (4.823)	0.0824 (9.124)	0.4558 (20.778)	0.0719
Stochastic interest rate	0.1069 (9.970)	0.6741 (21.794)	0.9896 (13.179)	0.0857
	<i>PUTS</i>			
Black – European ISD1	-0.0116 (-2.747)	0.2377 (18.326)	1.0619 (38.412)	0.3027
Black – European ISD2	-0.1762 (-38.290)	0.0513 (3.773)	1.8713 (58.945)	0.4132
Black – European ISD3	-0.0165 (-4.729)	-0.0086 (-0.843)	1.3003 (54.074)	0.3664
Black – American ISD1	-0.0225 (-5.167)	0.2419 (18.110)	1.1943 (41.958)	0.3330
Black – American ISD2	-0.1771 (-38.280)	0.0457 (3.338)	1.9532 (61.175)	0.4308
Black – American ISD3	-0.0259 (-7.222)	0.0425 (4.005)	1.3646 (55.077)	0.3776
Stochastic interest rate	0.0964 (8.534)	0.5385 (16.156)	-0.2197 (-2.823)	0.0496

Note: t values are in parenthesis.

Chapter IV

Empirical Tests of Valuation Models for Options on T-Note and T-Bond Futures

1 Introduction

The purpose of this chapter is to test valuation models for options on T-Note and T-Bond futures. Options on interest rate futures are one of the most actively traded contracts in Chicago Board of Trade (CBOT). Options on U.S. Treasury Bond futures were first introduced in October 1982, and options on U.S. Treasury Note futures began trading in May 1985. Underlying instrument is \$100,000.00 face value CBOT U.S. Treasury Bond futures (U.S. Treasury Note futures) contract deliverable during the months of March, June, September or December. The buyer of a call option has the right, but not obligation, to assume a long T-Bond (T-Note) futures position prior to expiration at the specified exercise price. The seller of the call is obligated to assume a short T-Bond (T-Note) futures position upon exercise by the buyer. The buyer of a put option has the right but not obligation, to assume a short T-Bond (T-Note) futures position prior to

expiration at the specified exercise price. The seller of the put option is obligated to assume a long T-Bond (T-Note) futures position upon exercise by the buyer.

Black[1976] was the first to provide a valuation formula for futures options. Black's model has important advantages. It is simple and easy to apply. It is also independent of preferences. In Black's model interest rate is assumed to be constant. This might not be an important issue when dealing with stock options, currency options or index options. However, this assumption seems unacceptable for bond options and bond futures options because bond prices depend only on interest rates. Intuitively, one would expect that models with stochastic interest rates would be more appropriate for pricing bond futures options. There is scant literature and research on pricing bond futures and options on bond futures under the assumption that interest rates are stochastic. To the best of our knowledge the only work that deals with pricing bond futures or bond futures options under interest rate uncertainty are those of Cox, Ingersol, and Ross [1981] and Courtadon [1983].

The major advantage of using models with stochastic interest rates is that they are more realistic. We know that interest rates are not constant. The disadvantage of these models is that they are much more complicated and difficult to apply especially for transaction data. Therefore, we restrict our research only on constant interest rate models. Our preliminary empirical results indicates that market prices for these options deviate significantly from their corresponding model prices and it is also shown that mispricing errors are related to time to maturity and exercise price.

The rest of the chapter is organized in the following manner. In section

2, we describe the model and explain its implementation. In section 3, we analyze the data and present empirical results. Section 4 provides a summary and conclusions.

2 Valuation Models for Options on T-note and T-Bond Futures

2.1 European Futures (Forward) Options

Black shows that under the usual perfect market assumptions and a constant interest rate, the value of a call option on a futures (forward) contract is

$$c(F, T; E) = e^{-rT} [FN(d_1) - EN(d_2)] \quad (1)$$

where $d_1 = [\ln(F/E) + 0.5\sigma^2T]/\sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$ and where $N()$ is the cumulative univariate normal distribution.

The value of a European put option on a futures (forward) contract is

$$p(F, T; E) = e^{-rT} [EN(-d_2) - FN(-d_1)] \quad (2)$$

where all notation is the same as before.

2.2 American Futures Options

American futures option prices are calculated using the Barone-Adesi and Whaley [1987] algorithm. The quadratic approximation method for the American call option on a futures contract as provided in Barone-Adesi and Whaley is

$$C(F, T; E) = c(F, T; E) + A_2(F/F^*)^{q_2} \quad \text{where } F < F^* \text{ and}$$

$$C(F, T; E) = F - E \quad \text{where } F \geq F^*$$

$$\text{and where } A_2 = (F^*/q_2)1 - e^{-rT}N(d_1(F^*)),$$

$$d_1(F^*) = [\ln(F^*/E) + 0.5\sigma^2T]/\sigma\sqrt{T},$$

$$q_2 = (1 + \sqrt{1 + 4k})/2, \text{ and } k = 2r/[\sigma^2(1 - e^{-rT})].$$

F^* is the critical futures price above which the American futures option should be exercised immediately. The critical price is determined by numerical methods by solving

$$F^* - E = c(F^*, T, E) + 1 - e^{-rT}N(d_1(F^*))F^*/q_2$$

The quadratic approximation of the American put option on a futures contract is

$$P(F, T; E) = p(F, T; E) + A_1(F/F^{**})^{q_1} \quad \text{where } F > F^{**} \text{ and}$$

$$P(F, T; E) = E - F \quad \text{where } F \leq F^{**}$$

$$\text{and where } A_1 = -(F^{**}/q_1)1 - e^{rT}N(-d_1(F^{**})), \quad q_1 = (1 - \sqrt{1 + 4k})/2$$

and where all other notation is the same as for the American call option. F^{**} is the critical futures price at which the American futures option should be exercised early and is determined numerically by solving

$$E - F^{**} = p(F^{**}, T; E) - 1 - e^{-rT}N(-d_1(F^{**}))F^{**}/q_1$$

3 Empirical Tests of Pricing Models

In this section we present the empirical results on the performance of European and American futures option pricing models. We follow the usual procedure. First, we describe the data and explain how model prices are calculated. Second, we analyze the results. Third, we examine whether the market for options on bond futures is efficient.

3.1 Data

The tests of the models for options on futures are based on data for T-Note and T-Bond futures options that are traded on The Chicago Board of Trade (CBOT). The data consists of all transaction prices from January 2, 1987 to December 31, 1987.¹ These option prices are obtained from CBOT and are referred as "time and sales data". The volume of each transaction is not reported. The original sample for bonds has 595,116 bond futures prices, 138,145 call option and 138,802 put option prices. For T-Notes, there were 87,677 note futures prices, 8,958 call option and 12,575 put option prices. However, the actual sample is smaller than this because all nominal² prices and option prices which are less than the option's intrinsic value were eliminated. The futures option pricing model require simultaneous futures and option prices. To satisfy this requirement, the future price that come closest to futures option price is used. We tried to match every option price with corresponding futures price. The maximum allowed difference between option and future price is 30 seconds with 15 seconds on the average. Thus, our final sample consists of 3,643 calls and 5,106 puts for T-Notes and 62,065 calls and 60,224 puts for T-Bonds. Table 18 provides more information about the final sample. Table 18 indicates that at the money options are more active. With respect to time to maturity, the most of trading activity was in the nearest contract month. The interest rate is the daily U.S. Treasury Bill rate with one month to expiration.

¹The problem of nonsynchronous prices with using daily closing prices are well known. By using transaction data we hope to eliminate this problem.

²The Chicago Board of Trade data manual refers to nominal price "a future/strike or option contract does not have a trade in it all day, will, if it has previously traded at some time since it was listed for trading, have a closing price denoted as a nominal".

3.2 Implied Parameters

Black's European option pricing formula as well as its American version has five parameters. These are F , E , T , r and σ . F , E , T and r are known or easily observable. The only unknown parameter is σ of the underlying futures contracts. There are several ways one can estimate σ . For example, one can use historical data to calculate the standard deviation of futures price changes or one can use the implied volatilities. Implied volatilities can also be estimated in different ways. The most common approaches are:

1. One can calculate implied volatility by using at the money option or the one most close to being at the money.
2. Implied volatilities can be calculated using all the options on a given day giving them either equal weights or different weights.
3. using all option prices on a given date, and finding σ which minimizes the sum of squared residuals.

We use the third approach. All futures option prices on a given day were regressed on their corresponding model prices:

$$V_j = V_j(\sigma) + u_j \quad j = 1, \dots, N$$

where V_j is the actual market price, $V_j(\sigma)$ corresponding model price and u_j is the error term. All transaction prices for a given maturity in the final sample were used in each regression. Table 19 presents the means and standard deviations for implied standard deviations. As can be seen from this table, the means of implied volatilities for T-Bonds are higher than those for T-Notes, as expected. The standard deviations implied by call

option transaction prices are lower, on average than that implied by put option prices. Table 20 presents the comparison of model prices with market prices for Black's European and American futures option pricing models. European futures option pricing predictions are calculated using Black's equation. American futures option pricing predictions are calculated using Barone-Adesi and Whaley algorithms (see their paper, 1987). Table 20 indicates that deviation of market prices from the corresponding model prices are in general higher for the puts compared to the calls.

3.3 Tests of Biases in the Pricing Models

It is well known that Black's European model as well as its American version shows systematic mispricing when it is used to value foreign currency options or index options (See Shastri and Tandon [1986], and Whaley [1986], among others). It is interesting to check whether these biases exist in T-Note and T-Bond futures options market. Therefore, we test for the existence of time to maturity bias and for bias related to which the option is in or out of the money for every model.

Table 21 presents the results of these tests. The hypothesis that models' mispricing is related to option maturity and the degree to which the option is in the money or out of the money is tested using the following regression:

$$ERR = \beta_0 + \beta_1 T + \beta_2 \ln(F/E)$$

where ERR is the difference between the market price and model prediction. Table 21 indicates that put options in the money are overvalued, call options out of the money are overvalued. Time to maturity bias does not have consistent pattern among different models. Time to maturity and the

degree of moneyness of the option does not have much explanatory power for call options (R^2 is less than 3%). The explanatory power of time to maturity and moneyness for put options is much higher. R^2 is approximately 35% for T-Note put options, and 45% for T-Bond put options. This indicates that Black's model might be used to price call options but not put options.

3.4 Market Efficiency Tests

The existing systematic biases shown in Table 21 imply that either the futures option pricing models are misspecified or the markets for T-Note and T-Bond futures options are inefficient. One way to test this is to check whether above normal profits can be earned in those markets. The models would still be useful if they allow us to make abnormal profits. The test design, the buy and hold strategy, is the same as in Whaley [1986]. Each day implied volatility is estimated using non-linear procedure, separate for calls and puts. This estimate of volatility is used to price options on the next day. Depending on the nature of mispricing, hedge portfolio was formed at that moment. Let C_F be the partial derivative of call option with respect to futures price and P_F be the partial derivative of put option with respect to futures price. If call option is underpriced (overpriced) by the model, hedge portfolio is long (short) one futures options position and short (long) C_F futures contract. If put option is undervalued (overvalued), the hedge portfolio is long (short) one futures option position and long (short) P_F futures contract. Each hedge portfolio was held until the futures option expired or until the end of the sample whichever came first. At this point,

the hedge portfolio was liquidated and profits were calculated.³ In order to consider transaction costs, the hedge portfolios were formed only the price deviation was equal or greater than 0.05.

In Tables 22 and 23, we provide the results of these hedging tests. The results indicate that if one uses American version of Black's model, then selling overvalued calls produces abnormal profits. For puts, hedging profits, in most cases, are negative. It is not clear if these results are time specific or general nature of these markets. It is interesting to note that even though European and American models provide almost identical results, average number of futures contract positions differ greatly between them. Average number of futures contracts are the average hedge ratios for mispriced options. For example, the average hedge ratio for undervalued T-Note options is 0.39 if one were to use European model and it is 0.98 if one uses American model. This is also true for T-Bond calls, T-Note puts and T-Bond put options. The average hedge ratios are always higher for American version of models.

4 Summary and Conclusions

This paper tests Black's European futures option model and its American version using transaction data on T-Note and T-Bond futures options. The results presented here indicate that model prices for these options deviate significantly from their corresponding market prices. These deviations are much larger for put options than those for call options. The practical

³As pointed out in Whaley [1986], by holding the portfolio open until expiration, at that time actual and market prices converge to the same value (intrinsic value), there is some assurance that observed profit opportunities will be captured.

implication is that Black's model can be used to price call options on T-Note and T-Bond futures but probably not for put options on these instruments. Theoretically, this also implies that put-call parity most likely does not hold for T-Note and T-Bond futures options. From a practical point of view traders of these options should have been very careful not to take "arbitrage" trades such as conversion and reverse conversion based on the model. A possible reason for the deviation of the model price from the market price is that it is optimal to exercise the put option before the expiration date (see Merton (1973)). This may happen due to either deep discount or more likely small differences between the coupon and the short term rates. In addition the mispricing errors are related to the time to maturity and the exercise price. The implications of these findings are: (1) as the expiration date approaches the option pricing model should be used very carefully especially for very short dated options; (2) getting in and out of the money will result in larger errors while using the model to price the options. The hedging tests provided abnormal profits only in the case of call options. This result reinforces our earlier observation that the model can be used to detect mispricing for call options. It fails to detect mispricing for put options. The search for answers to this question is left for future research.

TABLE 18
SUMMARY OF T-NOTE AND T-BOND FUTURES OPTIONS
TRANSACTION DATA

T - NOTES

<i>F/X</i>	<i># of transactions</i>	<i>time to expiration</i>	<i># of transactions</i>
<i>CALL</i>	<i>PUT</i>	<i>in weeks</i>	<i>CALL</i>
<i>PUT</i>			<i>PUT</i>
$F/X < 0.90$	2	$T < 4$	912
$0.90 \leq F/X < 0.95$	123	$4 \leq T < 8$	1193
$0.95 \leq F/X < 1.00$	2411	$8 \leq T < 12$	992
$1.00 \leq F/X < 1.05$	1090	$12 \leq T < 16$	376
$1.05 \leq F/X < 1.10$	16	$16 \leq T < 20$	144
$1.10 \leq F/X$	1	$20 \leq T$	26
<i>All</i>	3643	<i>All</i>	3643

T - BONDS

$F/X < 0.90$	1427	$T < 4$	14445
$0.90 \leq F/X < 0.95$	13325	$4 \leq T < 8$	15909
$0.95 \leq F/X < 1.00$	38863	$8 \leq T < 12$	16579
$1.00 \leq F/X < 1.05$	8052	$12 \leq T < 16$	8765
$1.05 \leq F/X < 1.10$	358	$16 \leq T < 20$	4530
$1.10 \leq F/X$	40	$20 \leq T$	1837
<i>All</i>	62065	<i>All</i>	62065

TABLE 19

Summary Statistics for estimates of implied volatilities

Underlying contract	<i>CALLS</i>			
	Model	# of observations	Mean	standard deviation
T - Note	Black - European	340	0.0842	0.0294
T - Note	Black - American	340	0.0838	0.0294
T - Bond	Black - European	529	0.1284	0.0304
T - Bond	Black - American	529	0.1290	0.0336
 <i>PUTS</i>				
T - Note	Black - European	340	0.0888	0.0304
T - Note	Black - American	340	0.0883	0.0302
T - Bond	Black - European	529	0.1366	0.0311
T - Bond	Black - American	529	0.1365	0.0316

Black-European model prices calculated using Black's European futures pricing formula. Black-American futures option prices are calculated using Barone-Adesi and Whaley algorithm.

TABLE 20
Comparison of Market and Model Prices

Underlying contract	<i>CALLS</i>		# of observations	MAE	MAPE
	Model				
T - Note	Black - European		3643	0.0431	8.3146
T - Note	Black - American		3643	0.0438	8.4208
T - Bond	Black - European		62065	0.0450	8.5499
T - Bond	Black - American		62065	0.0476	10.1356
<i>PUTS</i>					
T - Note	Black - European		5106	0.0704	8.7984
T - Note	Black - American		5106	0.0720	9.1571
T - Bond	Black - European		60224	0.0887	10.3241
T - Bond	Black - American		60224	0.0921	11.2582

MAE is the mean value of the difference between market price and model price. MAPE is the mean value of the percent error. Black-European model prices calculated using Black European futures pricing formula. Black-American futures option prices are calculated using Barone-Adesi and Whaley algorithm.

TABLE 21

Time to Maturity and Exercise Price Biases in Pricing Models

$$ERR = \beta_0 + \beta_1 T + \beta_2 \ln(F/E)$$

Underlying contract	Model	CALLS			R^2
		β_0	β_1	β_2	
T - Note	Black - European	0.0048 (2.050)	0.0207 (1.424)	0.6120 (10.108)	0.0273
T - Note	Black - American	0.0033 (1.379)	0.0578 (3.882)	0.5996 (9.668)	0.0249
T - Bond	Black - European	0.0105 (20.116)	0.0064 (2.367)	0.3179 (36.424)	0.0223
T - Bond	Black - American	-0.0022 (-3.779)	0.0879 (29.737)	0.2735 (28.633)	0.0205
<i>PUTS</i>					
T - Note	Black - European	0.0108 (4.751)	-0.0535 (-4.129)	2.599 (52.838)	0.3534
T - Note	Black - American	0.0106 (4.603)	-0.0107 (-0.818)	2.6633 (53.418)	0.3593
T - Bond	Black - European	0.0027 (4.058)	-0.0924 (-25.822)	2.3370 (225.455)	0.4583
T - Bond	Black - American	-0.0054 (-7.749)	-0.0124 (-3.301)	2.3665 (217.887)	0.4463

Note: t values are in paranthesis.

TABLE 22

Average Profits from Hedge Portfolios for Calls

Underlying contract	Model	Minimum Abs.Pr.Dev.	# of observations	Average # of contracts	Average Profits
<i>Undervalued Calls</i>					
T - Note	Black - European	$ DIF \geq 0.05$	872	0.3902	0.0147
T - Note	Black - American	$ DIF \geq 0.05$	813	0.9816	-0.0024
T - Bond	Black - European	$ DIF \geq 0.05$	15318	0.3295	-0.1242
T - Bond	Black - American	$ DIF \geq 0.05$	14179	0.9806	0.1388
T - Note	Black - European	$ DIF \geq 0.10$	383	0.3879	-0.2064
T - Note	Black - American	$ DIF \geq 0.10$	363	0.9814	-0.0925
T - Bond	Black - European	$ DIF \geq 0.10$	6610	0.3419	-0.4094
T - Bond	Black - American	$ DIF \geq 0.10$	6399	0.9812	-0.0891
<i>Overvalued Calls</i>					
T - Note	Black - European	$ DIF \geq 0.05$	946	0.4432	-0.2806
T - Note	Black - American	$ DIF \geq 0.05$	1003	0.9815	-0.0664
T - Bond	Black - European	$ DIF \geq 0.05$	13883	0.3488	-0.0698
T - Bond	Black - American	$ DIF \geq 0.05$	14824	0.9790	0.4021
T - Note	Black - European	$ DIF \geq 0.10$	515	0.4240	-0.2184
T - Note	Black - American	$ DIF \geq 0.10$	541	0.9805	0.0591
T - Bond	Black - European	$ DIF \geq 0.10$	6877	0.3922	0.1046
T - Bond	Black - American	$ DIF \geq 0.10$	7399	0.9771	0.8630

TABLE 23

Average Profits from Hedge Portfolios for Puts

Underlying contract	Model	Minimum Abs.Pr.Dev.	# of observations	Average # of contracts	Average Profits
<i>Undervalued Puts</i>					
T - Note	Black - European	$ DIF \geq 0.05$	1413	-0.5829	0.1340
T - Note	Black - American	$ DIF \geq 0.05$	1332	-0.9797	0.0699
T - Bond	Black - European	$ DIF \geq 0.05$	17983	-0.5363	-0.0678
T - Bond	Black - American	$ DIF \geq 0.05$	17005	-0.9793	-0.4002
T - Note	Black - European	$ DIF \geq 0.10$	796	-0.5865	-0.1193
T - Note	Black - American	$ DIF \geq 0.10$	766	-0.9789	-0.1166
T - Bond	Black - European	$ DIF \geq 0.10$	10604	-0.5421	-0.3334
T - Bond	Black - American	$ DIF \geq 0.10$	10043	-0.9781	-0.5881
<i>Overvalued Puts</i>					
T - Note	Black - European	$ DIF \geq 0.05$	1916	-0.3904	-0.3883
T - Note	Black - American	$ DIF \geq 0.05$	1985	-0.9781	-0.4262
T - Bond	Black - European	$ DIF \geq 0.05$	22718	-0.3127	-0.0877
T - Bond	Black - American	$ DIF \geq 0.05$	23829	-0.9770	0.0126
T - Note	Black - European	$ DIF \geq 0.10$	1243	-0.3656	-0.2326
T - Note	Black - American	$ DIF \geq 0.10$	1330	-0.9771	-0.4119
T - Bond	Black - European	$ DIF \geq 0.10$	15102	-0.2926	0.0731
T - Bond	Black - American	$ DIF \geq 0.10$	16077	-0.9747	-0.0126

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