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ON CHARACTERIZING A LINE GRAPH BY THE SPECTRUM OF ITS ADJACENCY MATRIX

by

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Table of Contents

Section 1. Introduction

This thesis is concerned with the relationship between a line graph and the eigenvalues of its adjacency matrix. We begin with the necessary definitions.

A graph G with V vertices is connected if for any two vertices i_0 and i_n, there exist vertices $i_1, i_2, \ldots, i_{n-1}$ such that i_j and i_{j+1} are adjacent for $j = 0, 1, \ldots, n-1$. The valence of a vertex is the number of edges for which that vertex is an end point. A graph that has the same valence, d, at every vertex is said to be regular of valence d.

The adjacency matrix of a graph G, denoted A(G), is a square 0-1 matrix of order V whose rows and columns correspond to the vertices of G and for which $A_{ij} = 1$ if and only if i and j are adjacent. We shall consider only loopless graphs with no multiple edges, and so A(G) is a symmetric matrix with zeros on the diagonal. The eigenvalues of a graph are the eigenvalues of its adjacency matrix, and since A(G) is real and symmetric, the eigenvalues of a graph are real. Let $G(k, -2)$ be the set of all regular connected graphs that have k distinct eigenvalues and whose least eigenvalue is -2.

A graph is strongly regular if it is not only regular, but also there exist constants p_1 and p_2 such that two adjacent vertices are mutually adjacent to p_1 vertices, while two non-adjacent vertices are mutually adjacent to p_2 vertices. A graph is m by n bipartite if the vertices can be partitioned into a set of order m and a set of order n such that two vertices are not adjacent if they are in the same set. A bipartite graph is complete if, conversely, any two vertices that are not in the same set are adjacent. The complete m by n bipartite graph is denoted $K_{m,n}$. A graph with m vertices in which any pair of vertices is adjacent is called an m-clique. A vertex is isolated if its valence is zero.

An Hadamard matrix of order n is a square matrix of order n with every entry 1 or -1 such that any two different rows are orthogonal.

Given a graph G, we can form a new graph, L(G), the line graph of G, in which the vertices of L(G) correspond to the edges of G, and in which two vertices of L(G) are adjacent if and only if the corresponding edges in G have exactly one end point in common. A graph H for which there exists a graph G such that L(G) = H is called a line graph.

If we have a graph G, the addition or removal of isolated vertices will obviously leave L(G) unchanged. Thus without loss of generality we can assume that G has no isolated vertices.

A balanced incomplete block design (BIBD) consists of a set of v objects together with b subsets of these objects such that:

(1) Each subset is of order k.

(2) Each object is in r subsets.

(3) Each pair of objects appears in $\lambda > 0$ subsets together.

The subsets are customarily called blocks. A block design for which $\lambda = r$ is called trivial. A design for which $r = k$ is called symmetric and denoted SBIB; if $r \neq k$, the design is asymmetric. The graph of a BIBD is a b by v bipartite graph where the two sets of vertices correspond to the subsets and objects of the BIBD, and two vertices are adjacent if and only if they correspond to an object and a subset, and the object is contained in the subset.

$G(3, -2)$ has been completely described by Seidel [8], and in this thesis we begin the systematic study of $G(4, -2)$ and $G(5, -2)$. We show, with the help of a theorem of Hoffman and Ray-Chaudhuri [4], that if H is in $G(4, -2)$ then, with finitely many exceptions, one of the following three conditions holds:

(1) H is the line graph of a strongly regular graph.

(2) H is the line graph of a SBIB.

(3) $H = L(K_{m,n})$ with $m > n$.

The question of whether a graph that satisfies condition (2) is characterized by its eigenvalues has been settled [3]. In this thesis we study the corresponding question for graphs that satisfy condition (1) or (3). We show that, with finitely many exceptions, a graph that satisfies condition (1) is characterized by its eigenvalues unless it is a strongly regular graph of valence t^2 with $p_1 = p_2 = t$. For a graph that satisfies condition (3), we ask if it can be characterized as a regular connected graph with mn vertices which has $m+n-2$, $m-2$, $n-2$, and -2 as its distinct eigenvalues, a question first raised in a different form by J.W.Moon. We show that if H has these properties, then $H = L(K_{m,n})$ if any one of the following conditions is satisfied:

(1) H contains an m-clique.

(2) $n = 2$.

(3) $m = n + 1$.

(4) $m = 5$ and $n = 3$.

(5) $m+n > 18$, $(m,n) \neq (2t^2 + t,\ 2t^2 - t)$, $t > 1$.

On the other hand, for every $(m,n) = (2t^2 + t,\ 2t^2 - t)$, $L(K_{m,n})$ is not characterized by its distinct eigenvalues if there exists a symmetric Hadamard matrix of order $4t^2$ with constant diagonal.

Now suppose H is in G(5, -2). Then, again with only finitely many exceptions, H is the line graph of G where

(1) G is a regular graph, or

(2) G is the graph of a non-trivial asymmetric balanced incomplete block design.

Again we ask the natural question: for a given G in satisfying condition (2), is $H = L(G)$ characterized by its distinct eigenvalues among all regular connected graphs with the same number of vertices? Conditions are given under which H is characterized; in particular it is shown that the above question can be answered affirmatively if the block design is a Steiner Triple System, i.e., $k = 3$ and $\lambda = 1$.

Section 2. G(k, -2) and the Line Graph of a Balanced Incomplete Block Design.

In this section we study the relationship between BIBDs and certain graphs in $G(k,-2)$, $k = 3,4,5$. We show first that if H is in $G(4,-2)$ or $G(5,-2)$, then, with a finite number of exceptions, H is a line graph. We next show that if $H = L(G)$ is in $G(k,-2)$, then G is either regular or bipartite. We then show that if G is a bipartite graph, then

(1) L(G) is in $G(3,-2)$	if and only if	G is the graph of a BIBD which is symmetric and trivial.
(2) L(G) is in $G(4,-2)$	if and only if	G is the graph of a BIBD which is either symmetric or trivial but not both.
(3) L(G) is in $G(5,-2)$	if and only if	G is the graph of a BIBD which is neither symmetric nor trivial.

We next investigate the graphs of G which are regular. We first show the relationship between the eigenvalues of G and the eigenvalues of L(G). Finally we show that if G is regular and L(G) is in $G(4,-2)$, then G is strongly regular.

In order to show that there are only a finite number of graphs in $G(4, -2)$ and $G(5, -2)$ that are not line graphs, we first need the following theorem by Hoffman [2] : H is regular and connected if and only if there exists a polynomial $p(x)$ such that $p(A(H)) = J$ where J is the matrix with every entry 1. The degree of p is one less than the number of distinct eigenvalues, and p is called the polynomial of the graph.

A path of length n from i_0 to i_n is a sequence of vertices $i_1, i_2, \ldots, i_{n-1}$ such that i_{j+1} and i_j are adjacent for $j = 0, 1, \ldots, n-1$. It is easy to prove by induction that $A^n{}_{ij}$ is the number of paths of length n from i to j. Thus if p is of degree m, $p(A) = J$ implies that for any i and j, $A^k{}_{ij} \neq 0$ for some $k \leq m$. Thus there is a path of length less than $m+1$ from i to j for any two vertices i and j. Hoffman and Ray-Chaudhuri [5] have proved the following theorem: If H is a regular connected graph whose least eigenvalue is -2, and H is regular of valence $d > 16$, then there exists a graph G such that $L(G) = H$, or $J - A(H) = H_t$ where H_t is a square matrix of order 2t with t 2x2 matrices down the diagonal. The 2x2 matrices have every entry equal to one, and these are the only non-zero entries in H_t. Since $A(H)$ and J are commuting, symmetric matrices, they can be simultaneously diagonalized by an orthogonal matrix U. H_t is the direct sum of t copies of J where J is square and of order 2. Thus the eigenvalues of H_t are 2 and 0. Hence we have

$$\text{(2.1)} \qquad U(H_t)U^T = U(J-A(H))U^T$$
$$= UJU^T - UA(H)U^T$$
$$= J_0 - A(H)_0 \quad \text{where } J_0 = \begin{pmatrix} 2t & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & \ddots \; 0 \end{pmatrix}$$

$$\text{and } A(H)_0 = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_{2t} \end{pmatrix}$$

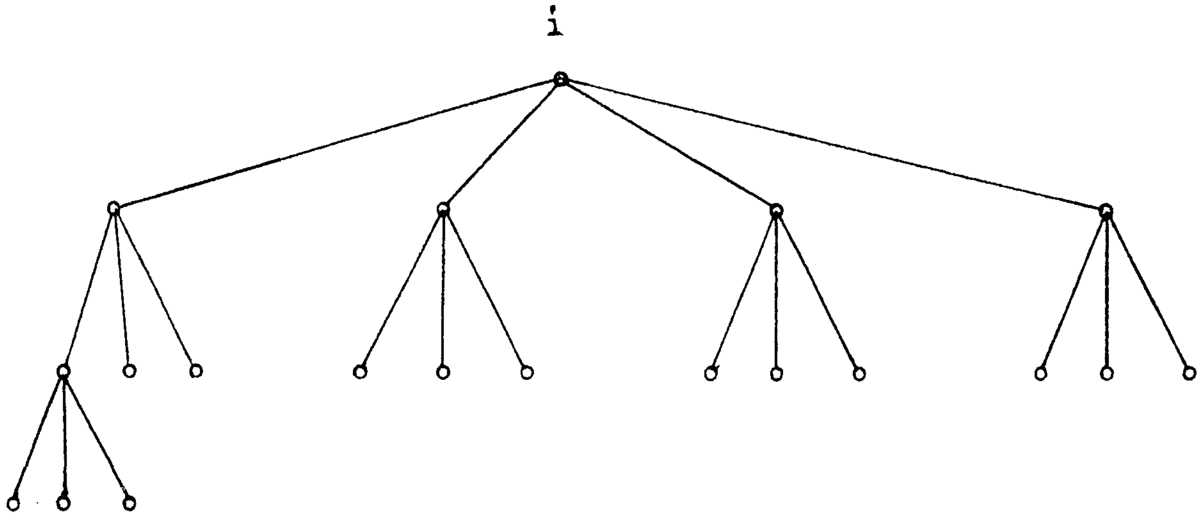
Thus $U(H_t)U^T$ is diagonalized with the eigenvalues of H_t down the diagonal. Since $u = (1, 1, \ldots, 1)$ is a common eigenvalue of J and A, and $Ju = 2tu$, and $H_t u = 2u$, $\alpha_1 = 2t-2$. For any other eigenvalue, (2.1) tells us that $2 = 0 - \alpha$, i.e., $\alpha = -2$,

or $0 = \alpha$.

Thus, since H_t has at most 3 eigenvalues, we know that if H is in $G(k,-2)$ for $k \neq 3$, and the valence of H is greater than sixteen, then H is a line graph.

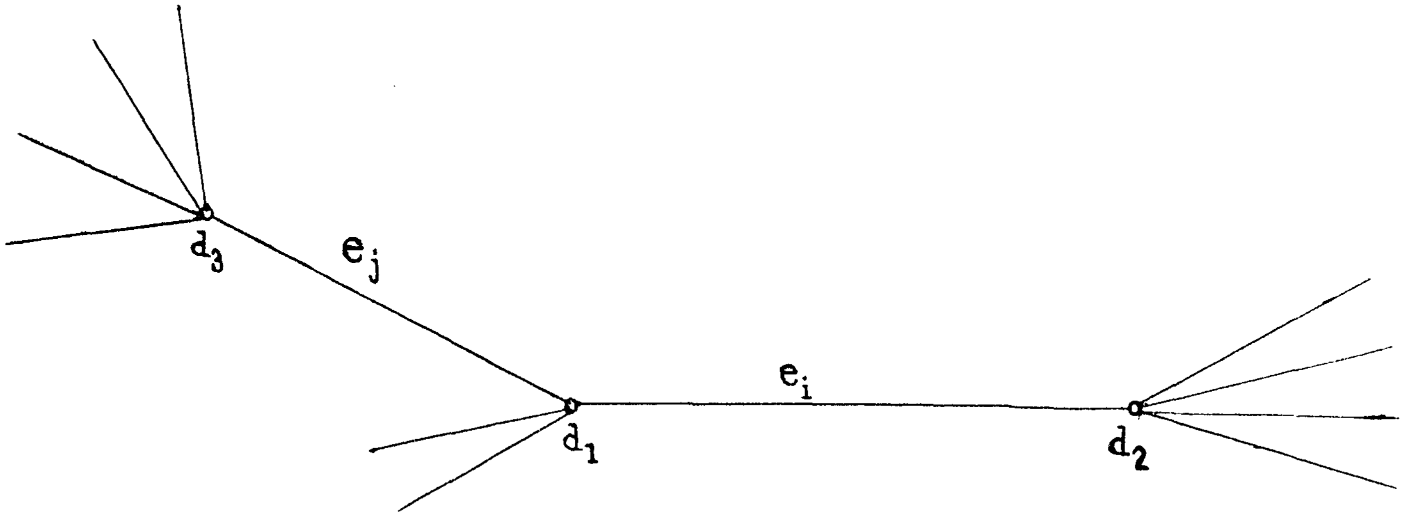
Theorem 2.2 All but a finite number of graphs in $G(4,-2)$ and $G(5,-2)$ are line graphs.

Proof: From the remarks above on the polynomial of a graph, if H is in $G(4,-2)$ or $G(5,-2)$, any two vertices in H can be joined by a path of length less than five.



If H is regular of valence d, and i is any vertex, there are d vertices which are joined to i by a path of length one, i. e., the vertices adjacent to i. Each of these vertices is adjacent to at most d-1 new vertices, and so there are at most d(d-1) vertices connected to i by a path of length two but not by a path of length one. Since each of these new vertices is adjacent to at most d-1 still uncounted vertices, there are at most $d(d-1)^2$ vertices connected to i by a path of length three but not by a path of length two or one. Thus if H is in G(4, -2), and H is regular of valence d, then H has at most $d(d-1)^2$ vertices. Similarly, if H is in G(5, -2), H has at most $d(d-1)^3$ vertices. Thus for any given d, there are at most a finite number of graphs in G(4, -2) and G(5, -2) that are regular of valence d. Thus there are only a finite number of graphs in G(4, -2) and G(5, -2) with valence less than seventeen.

Now suppose that H is in G(k, -2) and L(G) = H. What can we say about G?



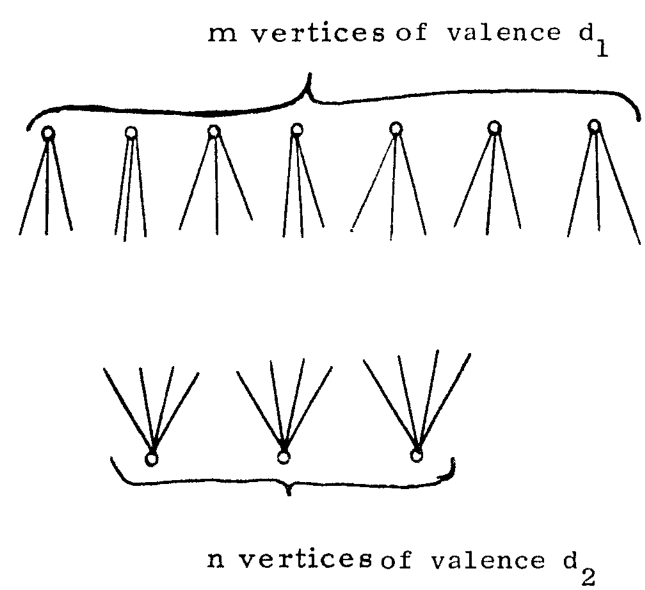
If e_i is an edge in G, and the valences of the end points of e_i are d_1 and d_2, then the valence of the vertex in H that corresponds to e_i is $d_1 + d_2 - 2$ since for each vertex adjacent to the vertex corresponding to e_i in H, there must be an edge in G that has one end point in common with e_i and there are $d_1 + d_2 - 2$ such edges. Hence H is regular of valence $d_1 + d_2 - 2$. If e_j has an end point in common with e_i, then the valences of the end points of e_j are d_1 and d_3. But then by the same reasoning as before, the valence of the vertex in H corresponding to e_j is $d_1 + d_3 - 2$. Since H is regular $d_1 + d_3 - 2 = d_1 + d_2 - 2$, or $d_3 = d_2$. Thus we see that if a vertex in G is adjacent to a vertex of valence d_1, then it is of valence d_2, and vice-versa. If G were not connected, then L(G) would not be connected and thus not in G(k, -2). Thus G is connected, and so if we label each vertex with its valence, every vertex must be labelled d_1 or d_2. There are two possible cases:

(1) $d_1 = d_2$ in which case G is regular, or

(2) $d_1 \neq d_2$ in which case we can partition the vertices into

the vertices of valence d_1 and the vertices of valence d_2. If two vertices of valence d_1 were joined by an edge, then the corresponding vertex in H would be of valence $2d_1 - 2 = d_1 + d_2 - 2$, or $d_1 = d_2$. Hence we see that by using this partition of the vertices, G is a bipartite graph.

Now let us look at case (2) a bit more closely. Without loss of generality, we can assume that $d_1 < d_2$. Let m be the number of vertices with valence d_1 and n be the number of vertices of valence d_2. Then we have the following situation:



Note that $d_1 \leq n$ and $d_2 \leq m$. Since each edge has exactly one end point in the m vertices and one end point in the n vertices, we can conclude that

$$d_1 m = |E| = d_2 n \quad \text{where E is the set of edges.}$$

Henceforth $|S|$ will be the cardinality of the set S.

The incidence matrix of a BIBD is a 0-1 $v \times b$ matrix whose rows correspond to the objects and whose columns correspond to the blocks such that the (i,j) entry is 1 if and only if the ith object is in the jth block. Note that a BIBD is trivial if and only if the incidence matrix is J.

Note that if G is the graph of a BIBD, then

$$A(G) = \left[\begin{array}{c|c} O & B \\ \hline B^T & O \end{array}\right]$$

where B is the incidence matrix of the BIBD. Note also that if the BIBD is a SBIB, then B is square. If the BIBD is trivial, then the graph of the BIBD is $K_{b,v}$.

Theorem 2.3 If G is the graph of a BIBD, and $H = L(G)$, then one of the following holds:

(1) G is the graph of a trivial and symmetric BIBD and H is in $G(3,-2)$.

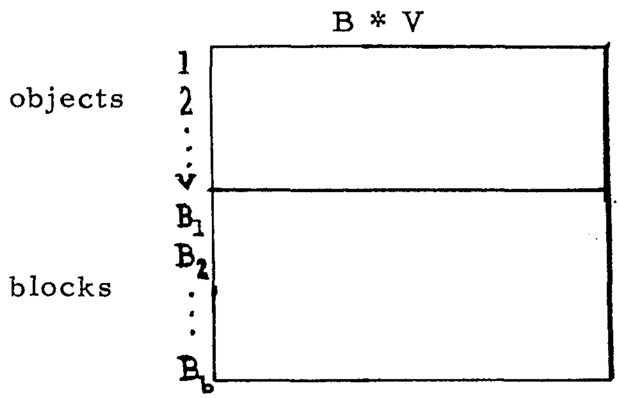
(2) G is the graph of a BIBD that is either symmetric or trivial but not both, and H is in $G(4,-2)$.

(3) G is the graph of an asymmetric and non-trivial BIBD, and H is in $G(5,-2)$.

Proof: Conclusion (1) has been shown in a somewhat different form by Shrikhande [10] and [2]. Hoffman and Ray-Chaudhuri [4] have shown that the line graph of a non-trivial SBIB is in $G(4,-2)$.

Thus we may assume that G is the graph of an asymmetric BIBD.

Define K as follows:



$B * V = \{(B, v) \mid$ B is a block; v is an object in B $\}$

$$K_{ij} = \begin{cases} 1 & \text{if the object or block in the } i^{th} \text{ row is one of the elements of the ordered pair in the } j^{th} \text{ column} \\ 0 & \text{otherwise} \end{cases}$$

Notice that $K^TK = 2I + A(H)$ while

$$KK^T = \begin{bmatrix} rI & B \\ B^T & kI \end{bmatrix}$$

where B is the incidence matrix of the BIBD and B^T is the transpose of B.

Now if $z = (x, y)$, x the first r coordinates

y the last k coordinates

$$\text{Then } KK^T(z) = \rho z \iff \begin{aligned} &rx + By = \rho x \\ \text{and} \quad &B^T x + ky = \rho y \end{aligned}$$

$$\iff \begin{aligned} &B(y) = (\rho - r)x \\ \text{and} \quad &B^T(x) = (\rho - k)y \end{aligned}$$

Case I: $x = 0$

Note that $y \neq 0$ since $z \neq 0$

$$\text{In this case } KK^T(z) = \rho z \iff \begin{aligned} &B(y) = (\rho - r)x = 0 \\ \text{and} \quad &0 = B^T(x) = (\rho - k)y \end{aligned}$$

$$\iff \begin{aligned} &B(y) = 0 \\ \text{and} \quad &\rho = k \end{aligned}$$

By Fisher's theorem $b \geq v$; if $b = v$, then $k = r$ and the design is symmetric (see Hall [1], p. 103).

Thus, since the design is asymmetric by assumption, $b > v$, i.e., B has more columns than rows. Thus there is a $y \neq 0$ such that $B(y) = 0$. Using this y, we see that k is an eigenvalue of KK^T.

Case II: $x \neq 0$

Note that if $KK^T(z) = \rho z$, then

$$BB^T(x) = B\big((\rho - k)y\big) = (\rho - k)B(y)$$
$$= (\rho - k)(\rho - r)x$$

Now let us suppose that $(\rho - k)(\rho - r)$ is an eigenvalue of BB^T, for some $\rho \neq k$, with x as an eigenvector. Then if we define $y = \dfrac{B^T(x)}{\rho - k}$, we have $B^T(x) = (\rho - k)y$ by definition, and

$$B(y) = B\left(\frac{B^T(x)}{\rho - k}\right)$$

$$= \frac{BB^T(x)}{\rho - k} = \frac{(\rho - k)(\rho - r)}{\rho - k} \quad x = (\rho - r)x$$

Putting Case I and Case II together we have the following:

$$\rho \text{ is an eigenvalue of } KK^T \Longleftrightarrow \begin{cases} \rho = k \\ \text{or } (\rho - k)(\rho - r) \text{ is an eigenvalue} \\ \text{of } BB^T \end{cases}$$

Since B is the incidence matrix of the BIBD,

$BB^T = \lambda J + (r - \lambda) I$ which has $(r - \lambda)$ and rk as its eigenvalues.

From the equation $(\rho - k)(\rho - r) = rk$, we deduce that $\rho = r + k$ and $\rho = 0$ are eigenvalues of KK^T.

Since the discriminant of $(\rho - k)(\rho - r) - (r - \lambda) = 0$ is positive, there exist ρ_1 and ρ_2, $\rho_1 \neq \rho_2$ such that

$$(\rho_i - k)(\rho_i - r) = (r - \lambda).$$

Thus KK^T has ρ_1, ρ_2, k, r + k, and 0 as eigenvalues. If $\rho = 0$ or $\rho = r + k$, then $(r - \lambda) = (\rho - r)(\rho - k) = rk$

But then BB^T has only one eigenvalue, and since BB^T is real and symmetric, it must be that $BB^T = rkI$. But then $\lambda = 0$, which is a contradiction.

Thus 0, r + k, ρ_1, ρ_2, are distinct.

Now suppose that k, 0, r + k, ρ_1, and ρ_2 are not distinct. If $\rho_1 = k$, then $(\rho_1 - r)(\rho_1 - k) = (r - \lambda) = 0$ and the BIBD is trivial. In this case $\rho_2 = r$.

Thus we see that if G is the graph of an asymmetric BIBD,

> KK^T has five distinct eigenvalues $0, k, r+k, \rho_1$, and ρ_2
>
> if the design is non-trivial, and
>
> KK^T has four distinct eigenvalues 0, k, r + k, and r
>
> if the design is trivial.

K^TK has the same non-zero eigenvalues as KK^T. In the definition of $B * V$, $|B * V| =$ kb since each of the b blocks contains k objects. We noted before that $b > v$, so the number of rows of $K = b + v < 2b \leq kb =$ the number of columns of K. Thus K has more columns than rows, and so there is a $z \neq 0$ such that $K(z) = 0$. But then $K^TK(z) = 0$, i.e. 0 is an eigenvalue of K^TK. Thus K^TK and KK^T have the same eigenvalues.

Clearly K^TK and A(H) have the same number of distinct eigenvalues. Now we can say the following:

> A(H) has five eigenvalues if the design is asymmetric and
>
> non-trivial, and
>
> A(H) has four eigenvalues otherwise.

Since $K^T K$ is positive semi-definite, 0 is the smallest eigenvalue. Thus -2 is the smallest eigenvalue of $A(H)$. This completes the proof of Theorem 2.3.

In Theorem 2.3 we saw that if G is a non-regular graph of a BIBD, then $L(G)$ is in $G(4,-2)$ or $G(5,-2)$.

We now show the converse.

<u>Theorem 2.4</u> If $H \quad G(4,-2)$ or $G(5,-2)$

$H = L(G)$

G not regular

then G is the graph of an asymmetric BIBD.

<u>Lemma 2.5</u> If A is a real symmetric matrix of order n

$Au = \lambda u$ where $u = (1 \ldots\ldots 1)$

λ a simple eigenvalue and the only other eigenvalue of A is μ

then $$A = \frac{\lambda - \mu}{n} J + \mu I$$

<u>Proof:</u> Since A is symmetric and u is an eigenvector,

$$AJ = \lambda J = JA$$

Thus there exists an orthogonal matrix U such that

$$UAU^T = A_0 = \begin{bmatrix} \lambda & & & 0 \\ & \mu & & \\ & & \mu & \\ 0 & & & \ddots \; \mu \end{bmatrix}$$

$$UJU^T = J_0 = \begin{bmatrix} n & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & \ddots \; 0 \end{bmatrix}$$

If p is any polynomial, then

$$p(A) = J \iff p(U^T A_o U) = (U^T J_o U)$$
$$\iff U^T p(A_o) U = U^T J_o U$$
$$\iff p(A_o) = J_o$$
$$\iff p(\lambda) = n$$
$$\text{and} \quad p(\mu) = 0$$

If we define $p(x) = \dfrac{n(x-\mu)}{\lambda-\mu}$

then $p(A) = J$

$$n\,\frac{(A-\mu I)}{\lambda-\mu} = J$$

$$A = \frac{\lambda-\mu}{n} J + \mu I$$

Cor. 2.6 If A is a real symmetric matrix

$Au = d_1 d_2 u$, $d_1 d_2$ a simple eigenvalue

and the only other eigenvalue is $\dfrac{d_2(n-d_1)}{n-1}$

then $A = \dfrac{d_2(d_1-1)}{n-1} J + \dfrac{d_2(n-d_1)}{n-1} I$

Now we can prove Theorem 2.4 :

Since G is not regular, it is bipartite with m vertices of valence d_1 and n vertices of valence d_2 . Let $1, 2, \ldots, m$ be the vertices of valence d_1 and $1', 2', \ldots, n'$ the vertices of valence d_2 .

$$M * N = \left\{ (i, j') \;\middle|\; i \text{ and } j' \text{ are adjacent} \right\}$$

We now define

$$\begin{array}{ccc} & & M * N \\ K & = \begin{array}{c} 1' \\ 2' \\ \vdots \\ n' \\ 1 \\ 2 \\ \vdots \\ m \end{array} & \left[\begin{array}{c} \qquad\qquad\qquad \\ \\ \\ \hline \\ \\ \\ \\ \\ \end{array}\right] \end{array}$$

as we did in Theorem 2.3.

By reasoning as we did in Theorem 2.3, we find that

$$K^T K = 2I + A(H)$$

and $KK^T = \left[\begin{array}{c|c} d_2 I & B \\ \hline B^T & d_1 I \end{array}\right]$ where $A(G) = \left[\begin{array}{c|c} O & B \\ \hline B^T & O \end{array}\right]$

Since $m > n$, i.e., B has more columns than rows, we can conclude, as in Theorem 2.3, that

ρ is an eigenvalue of $KK^T \Longleftrightarrow \rho = d_1$ or $(\rho - d_1)(\rho - d_2)$ is an eigenvalue of BB^T.

If $u = (1, \ldots, 1)$, $BB^T u = Bd_1 u = d_1 Bu = d_1 d_2 u$

and if $(\rho - d_1)(\rho - d_2) = d_1 d_2$,

then $\rho = d_1 + d_2$ and $\rho = 0$

Therefore d_1, $d_1 + d_2$, and 0 are eigenvalues of KK^T.

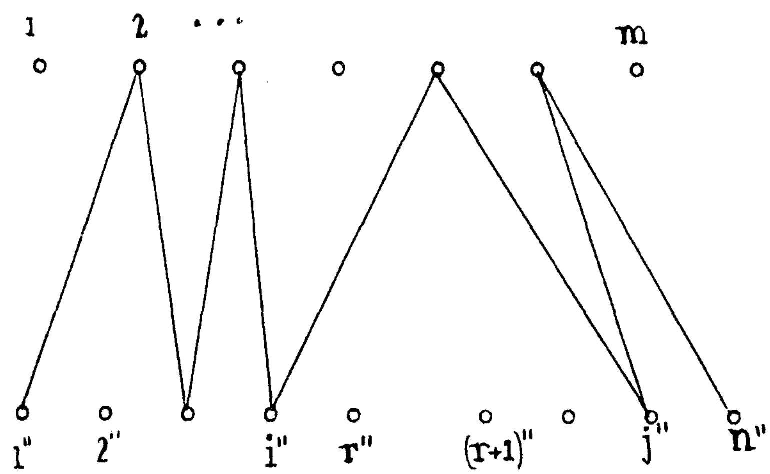
$BB^T_{ij} = 0$ implies that the inner product of the i^{th} and j^{th} row of B is zero. But then the inner product of the i^{th} and j^{th} row of A(G) is zero and, since A(G) is symmetric, $A^2_{ij} = 0$. Thus in this case there are no paths of length 2 from i to j.

Now suppose d_1d_2 is not a simple eigenvalue. Then since BB^T is symmetric, it is reducible [6, p. 122] i.e., of the form

$$\begin{array}{cc} 1'', 2'', \cdots r'' & r''+1 \cdots n'' \\ \hline \multicolumn{1}{|c|}{*} & \multicolumn{1}{c|}{0} \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{*} \\ \hline \end{array}$$

where the non-zero blocks are square.

Since G is connected, there is a path from $1''$ to n''. But this implies there is a path of length 2 from $i'' \in \{1'' \cdots r''\}$ to $j'' \in \{(r+1)'' \cdots n''\}$



But this implies that $(BB^T)_{i''j''} \neq 0$, which is a contradiction. Thus d_1d_2 is a simple eigenvalue.

We now look at two possible cases.

Case 1: BB^T has two or more distinct eigenvalues other than $d_1 d_2$.

Case 2: BB^T has one other eigenvalue, other than $d_1 d_2$.

<u>Case 1:</u> There exist μ_1 and μ_2 such that $d_1 d_2$, μ_1, and μ_2 are distinct eigenvalues of BB^T.

Suppose ρ_1 and ρ_2 are the roots of $(\rho - d_1)(\rho - d_2) - \mu_1$

$$\rho_1 = \rho_2 \text{ if and only if } (d_1 + d_2)^2 - 4(d_1 d_2 - \mu_1) = 0$$

$$(d_1 - d_2)^2 - 4\mu_1 = 0$$

$$\mu_1 = \frac{-(d_1 - d_2)^2}{4}$$

But $d_1 \neq d_2$, so that $\mu_1 < 0$. But μ_1 is an eigenvalue of a positive semi-definite matrix, which is a contradiction.

Thus, $\rho_1 \neq \rho_2$

Similarly, if ρ_3 and ρ_4 are the roots of $(\rho - d_1)(\rho - d_2) - \mu_2$

$$\rho_3 \neq \rho_4 .$$

Also $\rho_1 = \rho_3$ implies

$$\mu_1 = (\rho_1 - d_1)(\rho_1 - d_2) = (\rho_3 - d_1)(\rho_3 - d_2) = \mu_2$$

Therefore ρ_1, ρ_2, ρ_3, and ρ_4 are distinct.

Also, if $\rho_1 = d_1 + d_2$ or $\rho_1 = 0$

then $\mu_1 = (\rho_1 - d_1)(\rho_1 - d_2) = d_1 d_2$ which is a contradiction.

Thus $\rho_1, \rho_2, \rho_3, \rho_4$, $d_1 + d_2$ and 0 are distinct eigenvalues of KK^T.

Thus $\rho_1, \rho_2, \rho_3, \rho_4$, $d_1 + d_2$ are distinct eigenvalues of $K^T K$.

Since the number of rows of $K = m + n < 2m \leq d_1 m$ = the number of edges in G = the number of columns of K, there exists a $z \neq 0$ such that $K(z) = 0$. And thus $K^T K(z) = 0$ and 0 is an eigenvalue of $K^T K$. Thus $\rho_1, \rho_2, \rho_3, \rho_4, d_1 + d_2$, and 0 are six distinct eigenvalues of $K^T K$, and since $K^T K = 2I + A(L(G))$, $L(G) = H$ has at least six distinct eigenvalues. Thus case 1 is vacuous.

<u>Case 2</u>: BB^T has $d_1 d_2$ and μ as eigenvalues. In this case

$$nd_2 = \mathrm{Tr}\ BB^T = d_1 d_2 + (n-1)\mu$$

or

$$\mu = \frac{d_2 (n - d_1)}{n - 1}$$

Now we can apply Cor. 2.6 where $A = BB^T$

Thus we get $$BB^T = \frac{d_2 (d_1 - 1)}{n - 1} J + \frac{d_2 (n - d_1)}{n - 1} I$$

In addition, $JB = d_1 J$

Thus B is the incidence matrix of a BIBD with parameters

$$(v\ b\ r\ k\ \lambda) = \left(n, m, d_2, d_1, \frac{d_2 (d_1 - 1)}{n - 1} \right)$$

and G is the graph of that design.

<u>Cor. 2.7</u> If $H \in G(4, -2)$

$H = L(G)$; G not regular

then $G = K_{mn}$

By Theorem 2.4, G is the graph of a block design.

By Theorem 2.3, G is the graph of a trivial or symmetric design.

Since $k = d_1 > d_2 = r$, the design is asymmetric and thus trivial.

Since $A(G) = \left[\begin{array}{c|c} & B \\ \hline B^T & \end{array}\right]$ where $B = J$, $G = K_{m,n}$

<u>Cor. 2.8</u> If $H \in G(4, -2)$

$L(G) = H$; G not regular

Then the eigenvalues of H are
$$d_1 + d_2 - 2 = m + n - 2$$
$$d_1 - 2 = n - 2$$
$$d_2 - 2 = m - 2$$
$$\text{and } -2 = -2$$

Proof: In Theorem 2.4 we saw that

KK^T had eigenvalues $0, d_1 + d_2, d_1, \rho_1, \rho_2$

where $(\rho - d_1)(\rho - d_2) = \mu$

Since K^TK has the same eigenvalues, it must be that

$\rho_1, \rho_2, d_1 + d_2, d_1$, and 0 are not distinct.

We saw that $\rho_1 \neq \rho_2$, $\rho_1 \neq d_1 + d_2$ $\rho_1 \neq 0$

Thus $\rho_1 = d_1$ or $\rho_2 = d_1$

In either case, $\mu = 0$. Thus we can say $\rho_1 = d_1$ and
$\rho_2 = d_2$

Thus K^TK has eigenvalues $d_1 + d_2$, d_1, d_2, and 0

Since $0 = \mu = \dfrac{d_2\,(n - d_1)}{n - 1}$, $n = d_1$

and since $md_1 = |E| = nd_2$,

$m = d_2$

Cor. 2.9 If $H \in G(5, -2)$

$L(G) = H$; G not regular

G the graph of a BIBD with parameters $(v\ b\ r\ k\ \lambda)$

$$D = \left(\frac{r - k}{2}\right)^2 + r - \lambda$$

then the eigenvalues of H are $r + k - 2$, $D^{\frac{1}{2}} + \dfrac{r + k - 4}{2}$, $k - 2$, $-D^{\frac{1}{2}} + \dfrac{r + k - 4}{2}$, and -2.

Proof: $$\mu = \frac{d_2\,(n - d_1)}{n - 1}$$

$$(\rho - d_1)(\rho - d_2) = \frac{d_2\,(n - d_1)}{n - 1}$$

$$\rho^2 - (d_1 + d_2)\rho + d_1 d_2 - \frac{d_2(n - d_1)}{n - 1} = 0$$

$$\rho_1 = \frac{d_1 + d_2}{2} + 1/2\left((d_1 + d_2)^2 - 4\,d_1 d_2 + \frac{4d_2(n - d_1)}{n - 1}\right)^{\frac{1}{2}}$$

$$= \frac{d_1 + d_2}{2} + \left(\left(\frac{d_1 - d_2}{2}\right)^2 + \frac{d_2\,(n - d_1)}{n - 1}\right)^{\frac{1}{2}}$$

$$= \frac{r + k}{2} + \left(\left(\frac{r - k}{2}\right)^2 + \frac{r(v - k)}{v - 1}\right)^{\frac{1}{2}}$$

Since $r - \lambda = r - \dfrac{r(k - 1)}{v - 1} = \dfrac{r(v - k)}{v - 1}$

$$\rho_1 = \frac{r+k}{2} + D^{\frac{1}{2}}$$

$$\rho_2 = \frac{r+k}{2} - D^{\frac{1}{2}}$$

Thus the eigenvalues of $K^T K$ are $r+k$, k, $\frac{r+k}{2} + D^{\frac{1}{2}}$, $\frac{r+k}{2} - D^{\frac{1}{2}}$ and 0.

So far we have been considering the case where $L(G) = H$ and G is not regular. Now let us suppose that G is regular. We form the matrix K whose rows correspond to the vertices, and whose columns correspond to the edges. $K_{ij} = 1$ if the vertex corresponding to the i^{th} row is an end point of the edge corresponding to the j^{th} column. $K_{ij} = 0$ in all other cases. By the reasoning in Theorem 2.3 we find that

$$K^T K = 2I + A(L(G))$$

$$KK^T = dI + A(G)$$ where d is the valence of each vertex of G.

Now suppose $\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3 > -2$ are the eigenvalues of H

then $\alpha_0+2, \alpha_1+2, \alpha_2+2, \alpha_3+2$ and 0 are eigenvalues of $K^T K$

Thus $\alpha_0+2, \alpha_1+2, \alpha_2+2, \alpha_3+2$ and possibly 0 are eigenvalues of KK^T

and so $\alpha_0+2-d, \alpha_1+2-d, \alpha_2+2-d, \alpha_3+2-d$ and possibly $-d$ are eigenvalues of G

But $(-d)$ is an eigenvalue of G if and only if G is bipartite [2]

Thus the following has been proved:

Proposition 2.10

If H is regular of valence α_0

and the eigenvalues of H are α_0, α_1, α_2, α_3, and -2

L(G) = H G regular

then G is regular of valence $d = \dfrac{\alpha_0 + 2}{2}$

and the eigenvalues of G are d, $\alpha_1 - d + 2$, $\alpha_2 - d + 2$, $\alpha_3 - d + 2$.

G also has -d as an eigenvalue if and only if G is bipartite.

Proposition 2.11

If H = L(G)

G has eigenvalues d, β_1, β_2, β_3 $d > 2$

then H has eigenvalues 2d-2, $\beta_1 + d - 2$, $\beta_2 + d - 2$, $\beta_3 + d - 2$, and -2.

Proof: 2d, $\beta_1 + d$, $\beta_2 + d$, $\beta_3 + d$ are eigenvalues of KK^T

The number of columns of K = the number of edges of G $= \dfrac{Vd}{2} > V$

= the number of rows of K.

Therefore K^TK has 0 as an eigenvalue as well as

2d, $\beta_1 + d$, $\beta_2 + d$, and $\beta_3 + d$.

Theorem 2.12 If H = L(G)

H is in G(4, -2)

G regular and not bipartite

then G is strongly regular.

Proof: From Prop. 2.10, G has three distinct eigenvalues. Thus there is a polynomial p of degree 2 such that p(A) = J.

In other words

$A^2 + aA + bI = cJ$ for appropriate constants a, b, and c

Thus $p_1 = c - a$

and $p_2 = c$ and the graph is strongly regular

(vid. Introduction)

<u>Theorem 2.13</u> If $H = L(G)$

H is in $G(4, -2)$ G regular and bipartite

then G is the graph of an SBIB

Proof: Let

$$A(G) = \begin{bmatrix} O & B \\ B^T & O \end{bmatrix}$$

Since A is bipartite, if α is an eigenvalue, then $-\alpha$ is an eigenvalue. Since H has four eigenvalues, the eigenvalues of G must be of the form $\pm d$ and $\pm\alpha$. Thus the eigenvalues of BB^T are d^2 and α^2. If d^2 is not simple, then BB^T is reducible and, as in Theorem 2.4, G is not connected. Thus $BB^T\mathbf{u} = d^2\mathbf{u}$ and d^2 is a simple eigenvalue. Thus we can apply lemma 2.5 and, as in Theorem 2.4, derive that G is the graph of a BIBD with incidence matrix B. But since G is regular and bipartite, B is square and v = the number of rows = the number of columns = b, and thus $k = r$.

<u>Corollary 2.14</u> If $H = L(G)$ and H is in $G(4, -2)$

then (1) $G = K_{m,n}$ $m > n$,

or (2) G is the graph of a SBIB,

or (3) G is strongly regular.

Proof: Cor. 2.14 (1), (2), and (3) are proved respectively in Cor. 2.7, Theorem 2.13, and Theorem 2.12.

Theorem 2.15 If $H = L(G)$ is in $G(3,-2)$ and G is bipartite, then G is the graph of a trivial SBIB.

Proof: This theorem can be deduced from a proof of Shrikhande [10] but it is easier to prove it directly than it is to translate the notation. Since G is bipartite, the eigenvalues are d, $-d$, and α. But since $-\alpha$ is also an eigenvalue, it must be that $\alpha = 0$. Thus $A^2 + dA = \frac{2d^2}{V} J$. Since $A^2_{kk} = d$ and $A_{kk} = 0$, $d = \frac{2d^2}{V}$ or $V = 2d$. But the only possible bipartite graph with $V = 2d$ is $K_{d,d}$.

Theorem 2.16 If $H = L(G)$ is in $G(3,-2)$ and G is not bipartite, then G is a clique.

Proof: G has two eigenvalues. Thus G has a polynomial of degree 1, so $A + bI = cJ$. Thus if $i \neq j$, $a_{ij} = c$ for all j. Since $A_{ij} = 1$ for some (i,j), $A_{ij} = 1$ for all $i \neq j$, or A is an m-clique.

Summary

In this section the following has now been proven:

If $H = L(G)$, then

(1) H is in $G(3,-2)$ if and only if G is an m-clique $m > 4$, or $G = K_{m,m}$

(2) H is in $G(4,-2)$ if and only if G is the graph of a SBIB, or G is strongly regular but not a clique, or $G = K_{m,n}$

(3) H is in G(5, -2) if and only if G is the graph of a non-trivial

and asymmetric BIBD,

or G is a regular bipartite graph with five eigenvalues,

or G is regular, not bipartite, and has four eigenvalues.

Section 3. Some Lemmas about $L(K_{m,n})$

We have seen that if $H = L(G)$ is in $G(4, -2)$ and G is not regular, then $H = L(K_{m,n})$. Further, by Corollary 2.8, H has eigenvalues $m+n-2$, $m-2$, $n-2$, and -2. We now address ourselves to the following question: is there another graph with mn vertices in $G(4, -2)$ with the same eigenvalues as $L(K_{m,n})$? We shall now derive some lemmas useful for answering this question.

For any regular connected graph with eigenvalues $d, \beta_1, \beta_2, \ldots\ldots \beta_t$, $p(x) = |V| \prod_{k=1}^{t} \frac{x-\beta_k}{d-\beta_k}$ is the polynomial of minimum degree such that $p(A) = J$. [2] Thus if $G = L(K_{m,n})$,

$$p(x) = \frac{1}{m+n} (x-(m-2))\,(x-(n-2))\,(x+2)$$

$$= \frac{1}{m+n} (x-(m-2))\left(x^2-(n-4)x-2(n-2)\right)$$

If A is the adjacency matrix of another graph with the same eigenvalues as $L(K_{m,n})$, then $p(A) = J$ for that matrix also.

Suppose A is such a matrix. Then let us define B so that

(3.1) $$(m-n)B = A^2 - (n-4)A - 2(n-2)I - 2J.$$

Then $$\frac{1}{m+n}(A-(m-2)I)((m-n)B+2J) = J$$

$$(m-n)AB - (m-n)(m-2)B + 2AJ - 2(m-2)J = (m+n)J$$

and since $AJ = (m+n-2)J$ and $m \neq n$, we have

(3.2) $$AB - (m-2)B = J$$

Since A and J are commuting, symmetric matrices, they have a common set of eigenvectors. We can let $u_1 = (1, 1, \ldots\ldots, 1)$ and have

and have	$A(u_1) = (m+n-2)u_1$	$J(u_1) = mnu_1$
	$A(u_2) = (m-2)u_2$	$J(u_2) = 0$
	$A(u_3) = (n-2)u_3$	$J(u_3) = 0$
	$A(u_4) = (-2)u_4$	$J(u_4) = 0$

But then from (3.1),

$$(m-n)Bu_1 = A^2u_1 - (n-4)Au_1 - 2(n-2)Iu_1 - 2Ju_1$$

$$= \left((m+n-2)^2 - (n-4)(m+n-2) - 2(n-2) - 2mn\right)u_1$$

$$= m(m-n)u_1$$

Thus m is an eigenvalue of B.

Similarly, $(m-n)Bu_2 = ((m-2)^2 - (n-4)(m-2) - 2(n-2))u_2$

$$= m(m-n)u_2$$

$$(m-n)Bu_3 = ((n-2)^2 - (n-4)(n-2) - 2(n-2))u_3$$

$$= 0$$

$$(m-n)Bu_4 = (4 + (n-4)^2 - 2(n-2))u_4$$

$$= 0$$

Since the multiplicity of the eigenvalue of u_1 as an eigenvector of B is the same as the multiplicity of the eigenvalue of u_1 as an eigenvector of A, the only eigenvalues of B are m and 0. Thus B is positive semi-definite and $x^T Bx \geq 0$ for all x.

Note that $(m - n) B_{kk} = (m + n - 2) + 0 - 2(n - 2) - 2$

$= m - n$ and hence $B_{kk} = 1$ for all k.

For any $i \neq j$, let x' be the following vector:

$$x'_i = 1$$

$$x'_j = -1$$

$$x'_k = 0 \text{ for all other coordinates k.}$$

Then $0 \leq x'^T Bx' = \sum_{i,j} B_{ij} x'_i x'_j = B_{ii} + B_{jj} - B_{ij} - B_{ji}$

$= 2 - 2B_{ij}$ since B is symmetric.

Thus $B_{ij} \leq 1$

Now for any $i \neq j$, let x'' be the following vector:

$$x''_i = x''_j = 1$$

$$x''_k = 0 \text{ for all other coordinates k.}$$

Then $0 \leq x''^T Bx'' = \sum_{i,j} B_{ij} x''_i x''_j = B_{ii} + B_{jj} + B_{ij} + B_{ji}$

$= 2 + 2B_{ij}$

Thus $-1 \leq B_{ij}$

From $-1 \leq B_{ij} \leq 1$, we have, using (3.1)

$$-(m - n) \leq A^2_{ij} - (n - 4) A_{ij} - 2(n - 2) I_{ij} - 2 \leq (m - n)$$

Thus if $A_{ij} = 1$, $A^2_{ij} - (n - 4) - 2 \leq m - n$, or

<u>(3.3)</u> $$A^2_{ij} \leq m - 2.$$

Now define C such that

$$(n - m)\, C = A^2 - (m - 4)\, A - 2(m - 2)\, I - 2J.$$

By using the same u_1, u_2, u_3, and u_4 as before, we find that

$$(n - m)\, C\, u_1 = n\,(n - m)\, u_1\,,$$

$$(n - m)\, C\, u_2 = 0,$$

$$(n - m)\, C\, u_3 = n\,(n - m)\, u_3\,, \text{ and}$$

$$(n - m)\, C\, u_4 = 0$$

Thus C is also positive semi-definite.

$$(n - m)\, C_{kk} = (m + n - 2) - 2(m - 2) - 2 = (n - m)$$

Thus $C_{kk} = 1$ for all k.

If x' and x'' are as before

$$0 \leq x'^T C x' = \sum_{i,j} C_{ij}\, x'_i\, x'_j = 2 - 2C_{ij}$$

$$\text{or} \quad C_{ij} \leq 1$$

$$0 \leq x''^T Cx' = \sum_{i,j} C_{ij} x''_i\, x''_j = 2 + 2C_{ij}$$

$$\text{or} \quad -1 \leq C_{ij}$$

Thus $\quad -1 \leq C_{ij} \leq 1$

$$-1 \leq -C_{ij} \leq +1$$

$$(n - m) \leq -(n - m)\, C_{ij} \leq -(n - m)\,, \text{ or}$$

(3.4) $\quad (n - m) \leq -A^2_{ij} + (m - 4)\, A + 2\,(m - 2)\, I_{ij} + 2 \leq m - n$

If $A_{ij} = 1$

$$-A^2_{ij} + m - 4 + 2 \leq m - n$$

(3.5) $\quad n - 2 \leq A^2_{ij}$

Putting (3.3) and (3.5) together, we have

(3.6) $\quad n-2 \leq A^2_{ij} \leq m-2 \qquad \text{if } A_{ij} = 1$

Thus, if $A_{ij} = 1$, $(m-n)B_{ij} = A^2_{ij} - (n-4) - 2 \geq (n-2) - (n-4) - 2 = 0$

and so, if $A_{ij} = 1$, then $0 \leq B_{ij} \leq 1$

Since B has eigenvalues 0 and m, the minimal polynomial for B is

$$x(x-m) = x^2 - mx$$

Thus $\quad B^2 - mB = 0$

$$B^2 = mB$$

In summation, if the eigenvalues of H are the eigenvalues of $L(K_{mn})$ and B is defined as in (3.1), then we know the following:

(1) $AB - (m-2)B = J$

(2) $-1 \leq B_{ij} \leq 1 \quad$ for all i and j

(3) $B_{kk} = 1 \quad$ for all k

(4) If $A_{ij} = 1$, then $0 \leq B_{ij} \leq 1$

(5) $B^2 = mB$

(6) If $A_{ij} = 1$, then $n-2 \leq A^2_{ij} \leq m-2$

Lemma 3.7 If H is such that

(1) $|V| = mn$

(2) $d = m + n - 2$

(3) $A^2_{ij} = m-2$ or $A^2_{ij} = n-2$ for all (i,j) such that $A_{ij} = 1$

(4) $A^2_{ij} = 2 \quad$ if $A_{ij} = 0$

then there exist $n\binom{m}{2}$ pairs of adjacent vertices such that $A^2_{ij} = m-2$

and there exist $m\binom{n}{2}$ pairs of adjacent vertices such that $A^2_{ij} = n-2$

Proof: Let $(m - n)\, B = A^2 - (n - 4)\, A - 2(n - 2)\, I - 2J$

$$u = (1, \ldots\ldots, 1)$$

Since $A\, u = (m + n - 2)\; u$, by (2),

and $J\, u = mn\mathbf{u}$, by (1),

$$(m - n)\, Bu = \left((m + n - 2)^2 - (n - 4)(m + n - 2) - 2mn \right) u$$

$$= m\,(m - n)\, u$$

Thus $\quad Bu = m\, u$

Since $A^2_{kk} = m + n - 2 \quad$ and $A_{kk} = 0$,

$$B_{kk} = 1$$

If $A_{ij} = 0$ and $i \neq j$, $A^2_{ij} = 2$ implies that $B_{ij} = 0$

Because of (3), if $A_{ij} = 1$ then $B_{ij} = 0$ or $B_{ij} = 1$

Thus for any i, there exist m j's such that $B_{ij} = 1$

Hence there are $(m - 1)$ j's that are unequal to i such that $B_{ij} = 1$

Therefore there are $(m - 1)$ j's such that $A_{ij} = 1$ and $A^2_{ij} = m - 2$ while the remaining $(n - 1)$ j's such that $A_{ij} = 1$ have the property that $A^2_{ij} = n - 2$.

Summing over i, there are $mn\,(m - 1)$ ordered pairs of adjacent vertices such that $A^2_{ij} = m - 2$

and there are $mn\,(n - 1)$ ordered pairs of adjacent vertices such that

$$A^2_{ij} = n - 2.$$

Thus there are $nm \dfrac{(m - 1)}{2}$ pairs of adjacent vertices such that

$$A^2_{ij} = m - 2$$

and $mn \dfrac{(n - 1)}{2}$ pairs of adjacent vertices such that

$$A^2_{ij} = n - 2$$

<u>Lemma 3.8</u> The hypothesis of 3.7 implies that $H = L(K_{mn})$.

Proof: J.W.Moon [7] and A.J.Hoffman [3] have proved that the conclusion of 3.7 implies that $H = L(K_{mn})$.

<u>Lemma 3.9</u> A sufficient condition for H to be $L(K_{mn})$ is

<u>(3.10)</u> $A_{ij} = 0 \Longrightarrow B_{ij} \geq 0$

$$\sum_{\{j \mid A_{ij}=0\}} B_{ij} = \sum_j (J-A)_{ij} B_{ij}$$

$$= ((J-A)B)_{ii}$$

$$= (JB)_{ii} - (AB)_{ii}$$

$$= m - (J + (m-2)B)_{ii}$$

$$= m - 1 + (m-2) = 1$$

Since $B_{jj} = 1$ and $A_{jj} = 0$,

$$\sum_{\{j \mid A_{ij}=0,\, i \neq j\}} B_{ij} = 0 \qquad \text{and thus by condition (3.10)}$$

$$B_{ij} = 0 \text{ if } A_{ij} = 0, \quad i \neq j$$

Thus $0 \leq B_{ij} \leq 1$, which implies that $(B_{ij})^2 \leq B_{ij}$

with equality if $B_{ij} = 0$ or $B_{ij} = 1$

Hence $m = mB_{kk} = B^2_{kk} = \sum_j B_{kj} B_{jk} = \sum_j (B_{kj})^2 \leq \sum_j B_{kj}$

$$= (B \text{[redacted]} u)_k = m\, u_k = m$$

Thus the inequality is actually equality and B is a 0-1 matrix

such that (1) $B_{kk} = 1$

$$(2)\ B_{ij} = 0 \quad \text{if } A_{ij} = 0 \quad i \neq j$$

$$(3)\ B_{ij} = 0 \ \text{or}\ B_{ij} = 1 \quad \text{if } A_{ij} = 1$$

But from the definition of B, this means

$$(1)\ d = m + n - 2$$

$$(2)\ A^2_{ij} = 2 \quad \text{if } A_{ij} = 0 \quad i \neq j$$

$$(3)\ A^2_{ij} = m - 2 \ \text{or}\ A^2_{ij} = n - 2 \ \text{if } A_{ij} = 1$$

Since B is of order mn, we can apply (3.8) and imply that $H = L(K_{mn})$

Def. If G is a graph and W is a subset of the vertices, the subgraph generated by W has W as its vertices with two vertices adjacent if and only if they were adjacent in G.

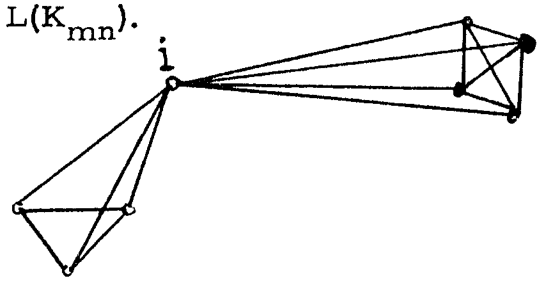
Def. The subgraph G_v is the subgraph generated by the vertices adjacent to v.

For $L(K_{mn})$, the v subgraph is the same for every vertex, i.e., a complete graph on $m - 1$ vertices and a complete graph on $n - 1$ vertices.

We now show that this property characterizes $L(K_{mn})$.

Lemma 3.11 If G is a graph such that for every v, G_v is a complete graph on $m - 1$ vertices and a complete graph on $n - 1$ vertices, then $G = L(K_{mn})$.

Proof:



m -1 clique

n -1 clique

Suppose $A_{ij} = 1$. Then there are two possibilities:

(1) j is in the $m-1$ clique, or

(2) j is in the $n-1$ clique.

In Case (1), $A^2_{ij} = m-2$ and $(m-n)B_{ij} = (m-2) - (n-4) - 2$

$$= m - n$$

In Case (2), $A^2_{ij} = n-2$ and $(m-n)\,B_{ij} = (n-2) - (n-4) - 2$

$$= 0$$

Thus we have that $B_{ij} = 0$ or 1 if $A_{ij} = 1$.

Thus $$\sum_{\{j \mid A_{ij}=1\}} (B_{ij})^2 = \sum_{\{j \mid A_{ij}=1\}} B_{ij} = \sum_j B_{ij}A_{ij}$$

$$= (AB)_{ii} = (J + (m-2)\,B\,)_{ii} = m-1$$

and $$\sum_j (B_{ij})^2 = \sum_j B_{ij}B_{ji} = B^2_{ii} = mB_{ii} = m$$

Thus
$$m = (B_{ij})^2 = \sum_{A_{ij}=1} (B_{ij})^2 + \sum_{\substack{A_{ij}=0 \\ i \neq j}} (B_{ij})^2 + B_{jj}$$

$$= m-1 + \sum_{\substack{A_{ij}=0 \\ i \neq j}} (B_{ij})^2 + 1$$

Therefore $$\sum_{\substack{A_{ij}=0 \\ i \neq j}} (B_{ij})^2 = 0$$

Therefore $A_{ij} = 0$ and $i \neq j$ implies that $B_{ij} = 0$

Thus by **Lemma 3.10**, $H = L(K_{m,n})$.

Some Impossible Subgraphs of Graphs in G(k, -2) [4]

Suppose K is a principal submatrix of A, $K(x) = -2x$

Then $A(z) = -2z$ where z is the vector which is identical with x in the coordinates for which x is defined and 0 in the other coordinates.

Proof: $$\frac{\langle Az, z\rangle}{z^T z} \geq -2$$ with equality if and only if $Az = -2z$

But $$\frac{\langle z, Az\rangle}{z^T z} = \frac{\sum_{ij} A_{ij} z_i z_j}{\sum_i z_i^2} = \frac{\sum_{ij} A_{ij} x_i x_j}{\sum_i x_i^2}$$

$$= \frac{\langle x, Kx\rangle}{\sum_i x_i^2} = \frac{-2 \,\|x\|}{\|x\|} = -2$$

Thus $Az = -2z$

Since $Au = du$, $u \cdot z = 0$ where $u = (1, \ldots\ldots, 1)$

and $\sum z_i = 0$

but then $\sum x_i = 0$

Further if j is some vertex that is not a row of K

then $(Az)_j = 0$

$$\sum A_{jk} z_k = 0$$

$$\sum A_{jk} x_k = 0$$

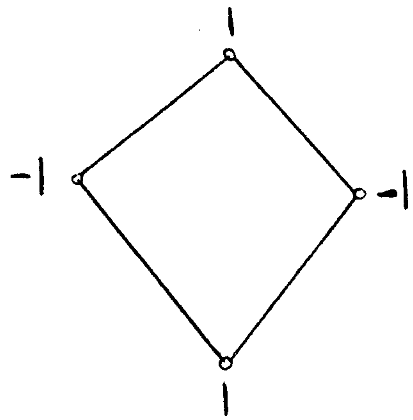
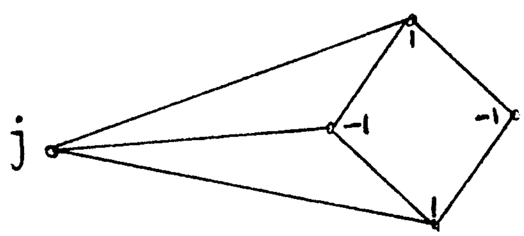


Figure 1

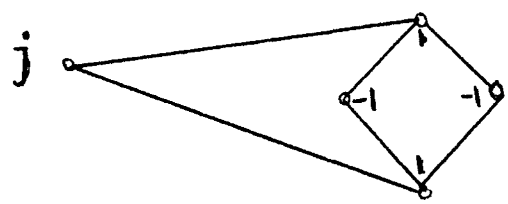
Figure 1 is a possible subgraph. The number next to a vertex i is the value of x_i, so that in this case the eigenvector z has only four non-zero coordinates.

The following are impossible subgraphs:

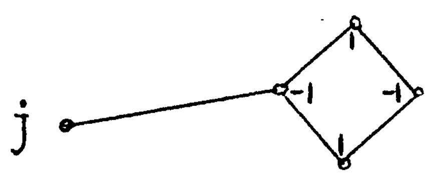
(3.12)



(3.13)



(3.14)



since in each case $\sum A_{jk} x_k \neq 0$

Notice that we are using the term subgraph here to mean the subgraph generated by some set of vertices.

Now, having proven the lemmas that are needed for answering the question raised at the beginning of this section, we proceed to the theorems.

Section 4. Some Theorems about $L(K_{m,n})$

The eigenvalues of $L(K_{m,n})$ are $m+n-2$, $m-2$, $n-2$, and -2. At the beginning of the last section, we raised the following question: is there another graph G such that $H = L(G)$, and H has the same eigenvalues as $L(K_{m,n})$ and H has mn vertices? Obviously G cannot be $K_{m'n'}$ with $(m', n') \neq (m, n)$, and Hoffman and Ray-Chaudhuri [4] have proved that if G is a SBIB, then G is characterized by its eigenvalues unless $v = 4$, $k = 3$, and $\lambda = 2$. But in this case the eigenvalues of L(G) are 6, 2, 0, and -2, and these are not the eigenvalues of $L(K_{m,n})$ for any m or n. We saw before that if $d > 16$, and H is in $G(4, -2)$, then $H = L(G)$ for some G. Thus if $m+n-2 \quad 16$, any graph with the same eigenvalues as $L(K_{m,n})$ must be the line graph of a strongly regular graph.

In this section we prove first that if any of the following conditions are satisfied, then $H = L(K_{m,n})$

(1) H contains an m-clique

(2) $n = 2$

(3) $m = n+1$

(4) $m = 5$ and $n = 3$

(5) $m+n \quad 18$, $(m,n) \neq (2t^2+t, 2t^2-t)$ $\quad t > 1$

We next show that a necessary and sufficient condition for $L(K_{m,n})$

not to be characterized by its spectrum is that there exist a symmetric Hadamard matrix with constant diagonal of order $m + n$.

Throughout this section we assume that H is a graph with eigenvalues $m + n - 2$, $m - 2$, $n - 2$, and -2.

Theorem 4.1 If $m = n + 1$

then $H = L(K_{m,n})$

Proof: Equation (3.1) now becomes

$$B = A^2 - (n-4)A - 2(n-2)I - 2J$$

Thus the entries of B are all integers , i.e., $B_{ij} = 0$ or 1 or -1

$$\sum_j B_{ij} = (Bu)_i = m$$

$$\sum_j (B_{ij})^2 = \sum B_{ij} B_{ji} = B^2_{ii} = mB_{ii} = m$$

Since $B_{ij} \leq (B_{ij})^2$ for all i, j

$$m = \sum_j B_{ij} \leq \sum_j (B_{ij})^2 = m$$

Thus

$$B_{ij} = (B_{ij})^2 \quad , \text{ i.e., } B_{ij} \text{ is } 0 \text{ or } 1$$

But then condition (3.10) is satisfied and $H = L(K_{m,n})$

Theorem 4.2 If H contains an m-clique

then $H = L(K_{m,n})$

Proof: Let $1, 2, 3, \ldots\ldots, m$ be the vertices in the m-clique.

Define $V_i = \left\{ v \text{ a vector} \;\middle|\; v \text{ adjacent to } i \quad v \notin \{1 \ldots\ldots m\} \right\}$

$i = 1 \ldots\ldots m$

$|V_i| = (m + n - 2) - (m - 1) = n - 1$

The following are true for all V_i and V_j $\quad i \neq j$

(4.2.1) $V_i \cap V_j = \emptyset$

Proof: If v is in $V_i \cap V_j$

then i and j are mutually adjacent to the m -2 other vertices in the clique as well as to v. Thus $A_{ij} = 1$ and $A^2_{ij} \geq m - 1$, contradicting (3.6).

(4.2.2) If v is in V_i $\quad$ j is in $\{1 \ldots\ldots m\}$ $\quad j \neq i$

then $A_{vj} = 0$

Proof: $v \in V_i \Rightarrow v \notin \{1 \ldots\ldots m\}$

thus $A_{vj} = 1 \Rightarrow v \in V_j$

but then $v \in V_j \cap V_i$, contradicting (4.2.1).

(4.2.3) V_i is an $(n - 1)$ clique

$x \in V_i \Rightarrow A_{xi} = 1$

Then by (3.6) $A^2_{xi} \geq n - 2$

Thus x and i are mutually adjacent to n -2 other vertices. But i is not adjacent to any of the vertices in the m-clique by (4.2.2), and the only other vertices that are adjacent to i are in V_i. There are n -2 vertices in V_i that are not x. Thus for x and i to be mutually adjacent to n -2 vertices, it must be that x is adjacent to all the other vertices of V_i. Since x was arbitrary, V_i must be complete, i. e., an n-1 clique.

Note that (4.2.3) implies that H_i is an m -1 clique and an n-1 clique for i = 1, ..., m.

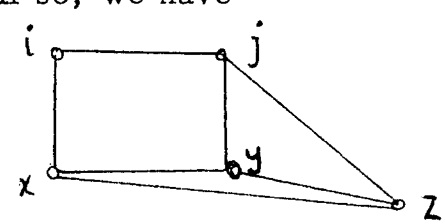
<u>(4.2.4)</u> Every vertex that is not in the m-clique is in some V_i

Proof: V_1, V_2, ..., V_m and $\{1.......m\}$ are mutually disjoint. Therefore

$$\left| V_1 \cup V_2 \cup \ldots \cup V_m \cup \{1......m\} \right| = m\,(n-1) - m = mn$$

<u>(4.2.5)</u> If x is in V_i y, z is in V_j then x is not adjacent to both y and z

Proof: If so, we have



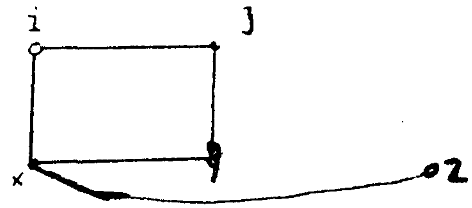
which is impossible subgraph (3.12)

Now suppose $x \notin \{1........m\}$

then x is in V_i for some i, by (4.2.4)

x is adjacent to i and the n-2 other vertices of V_i

Thus x is adjacent to (m +n-2) -(n -1) = m -1 other vertices. Let y and z be any two of these vertices. By (4.2.2), (4.2.4) and (4.2.5), y is in V_j and z is in V_k, with i, j, and k distinct. If y and z are not adjacent, we have



which is impossible subgraph (3.14)

Thus y and z must be adjacent. Since y and z were arbitrary. vertices adjacent to x and not in V_i , the m -1 vertices adjacent to x and not in V_i form an m -1 clique. None of the vertices in the m-1 clique is adjacent to a vertex in V_i by (4.2.5). Thus H_x is an n -1 clique and an m -1 clique.

Thus for any $x \notin \{1.....m\}$, H_x is an m -1 clique and an n -1 clique. As noted before, the same is true for the vertices in $\{1.......m\}$. Thus by lemma 3.11, $H = L(K_{mn})$.

Now If H is a graph, the complementary graph $\tilde{H}$ is a graph whose vertices correspond to the vertices of H, and for which two vertices are adjacent if and only if the corresponding vertices in H are not adjacent. Clearly

$$A(\tilde{H}) = J - A(H) - I$$

<u>Theorem 4.3</u> If n = 2

then $H = L(K_{m,n})$

Proof: The eigenvalues of H are m, m-2, 0 and -2.

As in the previous section, A(H) and J are commuting symmetric matrices. Thus there exist $u_1 = (1, \ldots\ldots 1)$, u_2, u_3, and u_4 such that

$$A(H)u_1 = (m+n-2)\,u_1 \qquad J(u_1) = mnu_1 = 2m\,u_1$$

$$A(H)u_2 = (m-2)\,u_2 \qquad J(u_2) = 0$$

$$A(H)u_3 = (n-2)\,u_3 \qquad J(u_3) = 0$$

$$A(H)u_4 = (-2)\,u_4 \qquad J(u_4) = 0$$

Thus
$$\begin{aligned}
A(\tilde{H})u_1 &= Ju_1 - A(H)u_1 - I(u_1) \\
&= (2m - m - 1)u_1 \\
&= (m-1)\,u_1 \\
A(\tilde{H})u_2 &= (0 - (m-2) - 1)\,u_2 \\
&= -(m-1)\,u_2 \\
A(\tilde{H})u_3 &= (-1)u_3 \\
A(\tilde{H})u_4 &= u_4
\end{aligned}$$

Thus, since the valence of H is the largest eigenvalue, we see that $d = m - 1$ and that $-d$ is an eigenvalue.

But then H is bipartite. [2]

Let S and T be the partition of the vertices. Since $|S| + |T| = 2m$, either S or T contains m vertices (in fact since H is regular, $|S| = |T| = m$).

Thus we have m vertices in $\tilde{H}$ such that no two of them are adjacent. But then the corresponding m vertices in H form an m-clique. Thus, by Theorem 4.2, $H = L(K_{m,n})$.

Now let us consider the case where $m = 5$ and $n = 3$. In this case, the eigenvalues of H are $m+n-2 = 6$, $m-2 = 3$, $n-2 = 1$, -2. As we saw at the beginning of Section 3, if

$$p(x) = \frac{1}{8}(x-3)(x-1)(x+2) = \frac{1}{8}(x^3 - 2x^2 - 5x + 6)$$

$$p(A(H)) = J$$

Thus $A^3 - 2A^2 - 5A + 6I = 8J$ where $A = A(H)$

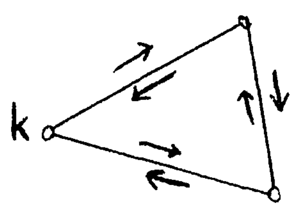
H is regular since $Au = 6u$ where $u = (1.......1)$

$$A^2_{kk} = d = 6 \quad \text{for all } k$$

Thus $A^3_{kk} - 2\cdot 6 - 5\cdot 0 + 6 = 8$ for all k

or $A^3_{kk} = 14$ for all k

Thus we see that there are 14 paths of length 3 from k to k.



For each path of length 3, k must be a vertex of a triangle. On the other hand, each triangle of which k is a vertex provides two such paths. Thus for our case, k is a vertex of 7 triangles. In other words, H_k has seven edges. The number of vertices of H_k is the valence of k which is 6. Now we wish to look at the possible H_k, and to do so we shall use the following proposition:

<u>Proposition 4.4</u>

If G is a regular graph of valence D with V vertices; the smallest eigenvalue of G is −2; k an arbitrary vertex; x, y, and z distinct vertices in G_k; $V(G_k)$ = vertices of G_k; d(x) = degree of x in G_k; d(y) and d(z) similarly defined: no two vertices of x, y, z adjacent

then $d(x) + d(y) + d(y) \geq 4D - (V + 2)$

Proof:

Define $X = \{w$ a vertex : w adjacent to x; $w \neq k$; w not adjacent to $k\}$

$Y = \{w$ a vertex : w adjacent to y; $w \neq k$; w not adjacent to $k\}$

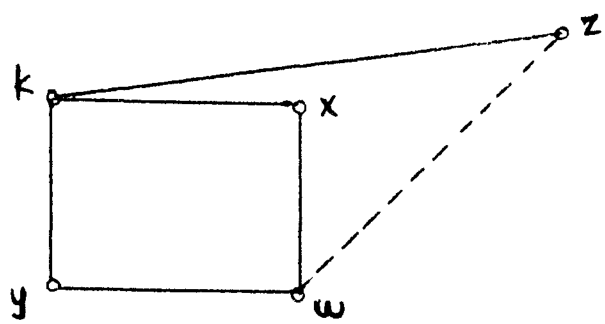
$Z = \{w$ a vertex : w adjacent to z; $w \neq k$; w not adjacent to $k\}$

Then $|X| + d(x) + 1 = D$

$|Y| + d(y) + 1 = D$

$|Z| + d(z) + 1 = D$

X, Y, and Z are pairwise disjoint. For if not, say if w is in $X \cap Y$, we have



If w and z are adjacent, we have impossible subgraph (3.13).
If w and z are not adjacent, we have impossible subgraph (3.14)

Also X, Y, and Z are disjoint from $\{k\} \cup V(G_k)$ by definition. Thus

$$|X| + |Y| + |Z| + 1 + D \leq |V|$$

$$3D - \big(d(x) + d(y) + d(z) + 3\big) + 1 + D \leq V \text{ or}$$

$$d(x) + d(y) + d(z) \geq 4D - (V + 2)$$

In our case,

$$d(x) + d(y) + d(z) \geq 4 \cdot 6 - (15 + 2) = 7$$

Now let us return to the possible G_k. As we said before, G_k has six vertices and seven edges. From (3.6) we see that if $A_{ij} = 1$,

$$1 \leq A^2_{ij} \leq 3.$$

Thus if x is a vertex in G_k, $A_{kx} = 1$ and so $1 \leq A^2_{kx} \leq 3$.

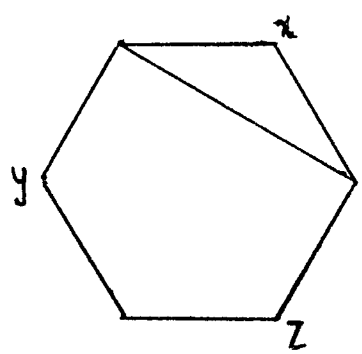
In other words,

$1 \leq d(x) \leq 3$ for all x in G_k. In particular, there are no isolated vertices in G_k. We now classify the G_k's by the largest polygon that each contains.

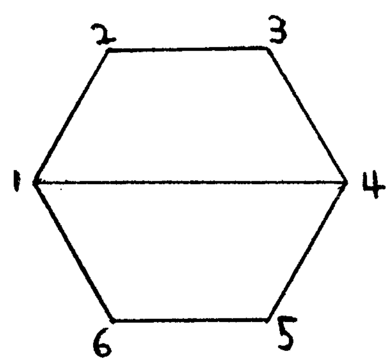
Case 1. Largest polygon a hexagon.

There are only two possibilities for the last edge:

1a



1b.



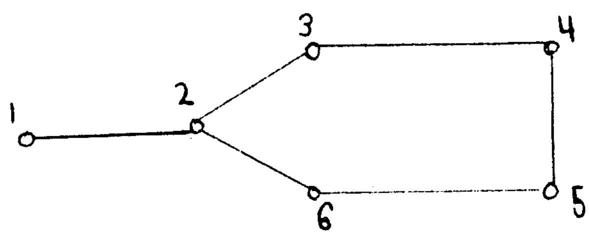
In subcase 1a, x y and z satisfy the hypothesis of Proposition 4.5 but $d(x) + d(y) + d(z) = 6$

In subcase 1b, the subgraph generated by $\{1\ 2\ 3\ 4\ 5\}$ is impossible subgraph (3.14)

Thus Case 1 is vacuous.

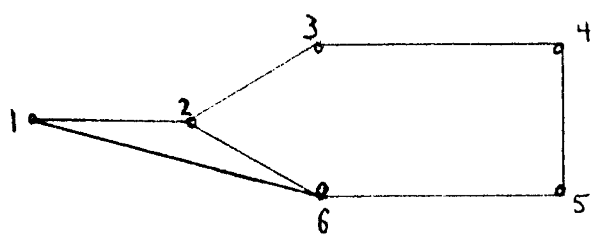
Case 2. Largest polygon a pentagon.

$d(x) \geq 1$ implies that six lines must look as follows

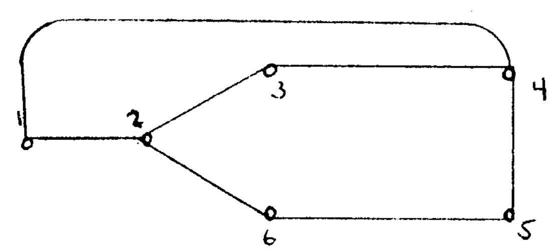


If the last line is adjacent to 1, there are two possibilities:

2a

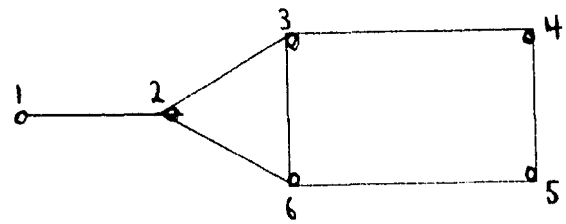


2b

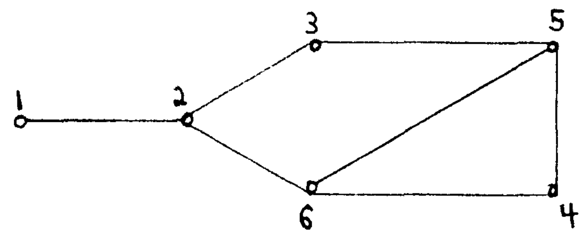


If the last line is not adjacent to 1, it is not adjacent to 2 since $d(2) = 3$. Thus the possibilities are:

2c



2d



In 2a, $\{1\ 6\ 5\ 4\ 3\ 2\}$ forms a hexagon and we are in case 1.

In 2b, $\{1\ 2\ 3\ 4\ 5\}$ generates impossible subgraph (3.14).

In 2c, if $x = 6$, $y = 4$, $z = 1$,

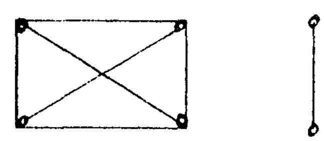
$d(x) + d(y) + d(z) = 6$, contradicting (4.4).

In 2d, $\{1\ 2\ 3\ 5\ 6\}$ generates impossible subgraph (3.14).

Thus Case 2 is vacuous.

Case 3. Largest polygon is a quadrilateral.

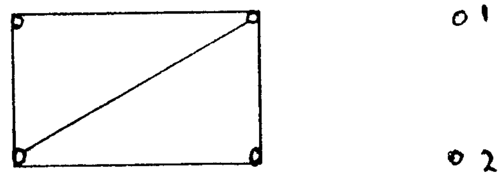
Subcase 3a. The quadrilateral has two diagonals. The last edge must join the last two vertices since no vertex is isolated.



In this case, k and the four vertices of the quadrilateral form a five-clique and by Theorem 4.2, $H \cong L(K_{3,5})$.

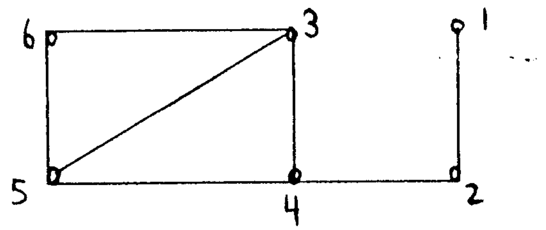
Subcase 3b. The quadrilateral has one diagonal.

(See diagram on next page)



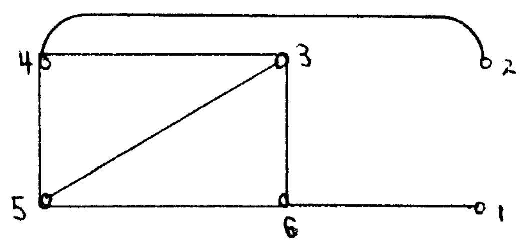
If $A_{12} = 1$ then the last edge must join an end point of the edge between 1 and 2 to the quadrilateral, and it must join it at a vertex of valence 2. Thus we get

$\underline{3b_i}$



If $A_{12} = 0$, one of the two remaining edges is adjacent to 1 and one to 2. In addition, one is adjacent to 4 and one to 6. Thus we have

$\underline{3b_{ii}}$



In $3b_i$, let $x = 4$, $y = 6$, $z = 1$. Then $d(x) + d(y) + d(z) = 6$

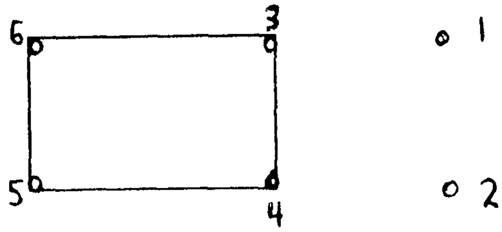
In $3b_{ii}$, let $x = 1$, $y = 2$, $z = 3$. Then $d(x) + d(y) + d(z) = 5$

In either case, this contradicts (4.4).

Thus subcase 3b is vacuous.

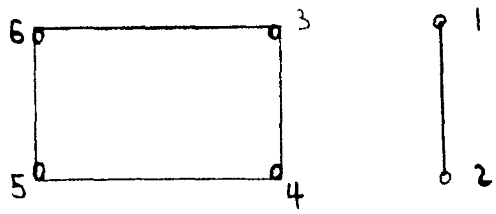
<u>Subcase 3c.</u> The quadrilateral has no diagonals.

(See diagram on next page)



If $A_{12} = 0$, then without loss of generality, we can assume $d(1) = 1$. But then $\{1\ 3\ 4\ 5\ 6\}$ generates impossible subgraph (3. 14).

Thus we can assume $A_{12} = 1$

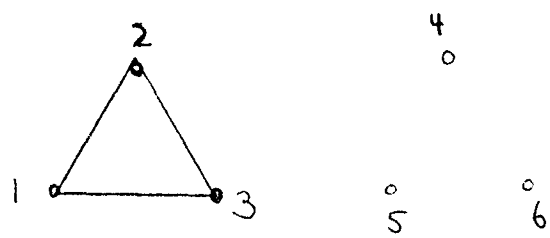


If the last two lines have 1 as an end point, either they go to adjacent vertices in which case G_k contains a pentagon, or they go to non-adjacent vertices in which case $\{1\ 3\ 4\ 5\ 6\}$ generates impossible subgraph (3. 13).

Thus one of the last two lines has 1 as an end point and one has 2 as an end point. Thus $\{1\ 3\ 4\ 5\ 6\}$ generates impossible subgraph (3. 14).

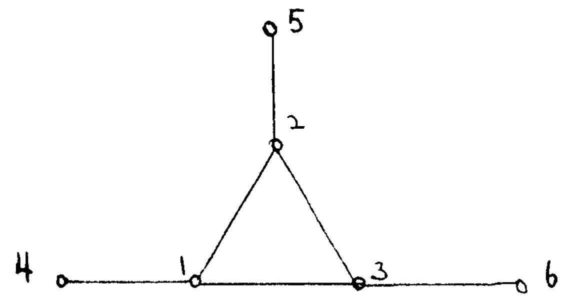
Thus subcase 3c is vacuous.

<u>Case 4</u>. Largest polygon is a triangle.



Subcase 4a. d(1) = d(2) = d(3) = 3

If the third edge adjacent to 1 and the third edge adjacent to 2 have an end point in common, then H_k contains a quadrilateral, which is case 3. Thus we must have the following:

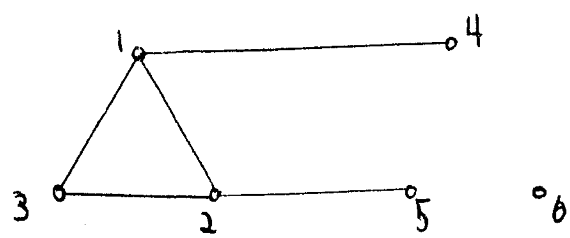


But then the last line will form a quadrilateral.

Thus subcase 4a is vacuous.

Subcase 4b. d(1) = d(2) = 3 d(3) = 2

As in subcase 4a, the third edge adjacent to 1 and the third edge adjacent to 2 can not have an end point in common. Thus we have the following:



The last two edges are in the subgraph generated by $\{4\ 5\ 6\}$

If $A_{45} = 1$, then the figure contains a pentagon.

If $A_{45} = 0$, then $A_{56} = A_{46} = 1$

and H_k contains a pentagon.

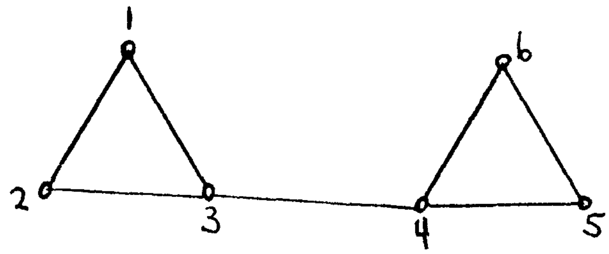
Subcase 4c. $d(1) = d(2) = d(3) = 2$

Then the subgraph generated by $\{4\ 5\ 6\}$ contains four edges, which is impossible.

Thus by symmmetry the last possible case is

Subcase 4d. $d(1) = d(2) = 2 \qquad d(3) = 3$

Then the subgraph generated by $\{4\ 5\ 6\}$ contains three edges, i.e., we have the following as the only possible H_k in case 4:



Case 5. H_k is a tree.

Then $7 = |E| = |V| - 1 = 5$

In summation, if $m = 5$, $n = 3$, and H has eigenvalues 6, 3, 1, and -2, then for any vertex v, H_v can be one of two possible graphs, i.e.,

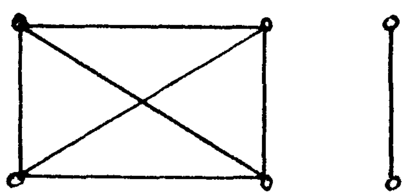


Figure 1

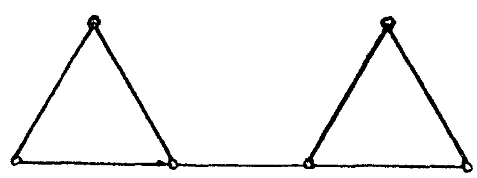
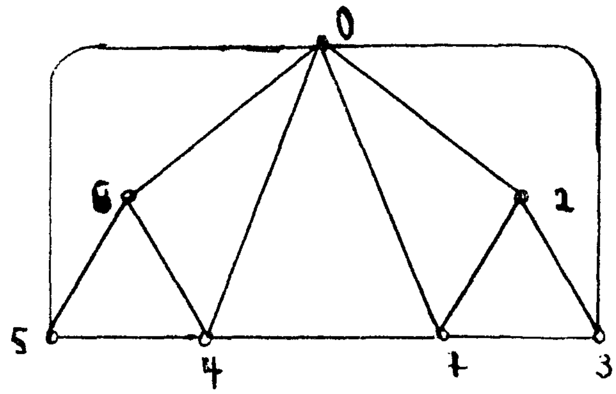


Figure 2

But if for any vertex H_v is Figure 1, then H contains an m-clique and H = $L(K_{m,n})$. Thus if there is a graph H in H (4, -2) with 15 vertices that has 6, 3, 1, and -2 as eigenvalues, then either H_v is Figure 2 for all v or H = $L(K_{m,n})$.

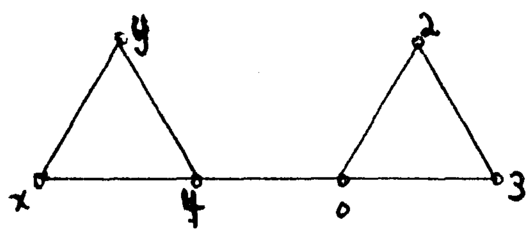
Now suppose G is such that G_v is Figure 2 for all v.

Let 0 be a vertex and 1 - 6 be the vertices adjacent to 0.



Now 4 is adjacent to two other vertices; call them 7 and 8.

G_1 looks as follows:

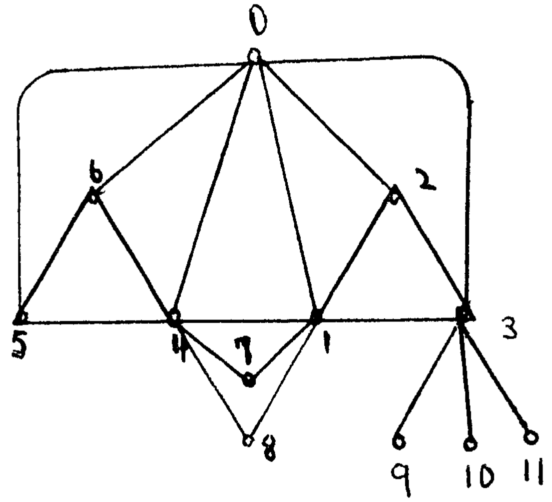


x and y are adjacent to 4 and to 1 since the above is G_1.

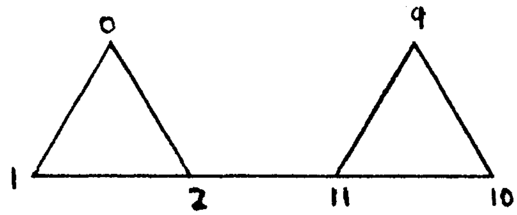
However, they are not adjacent to 0, so x and y must be 7 and 8.

Vertex 3 is adjacent to three other vertices. As can be seen in G_1, 3 is not adjacent to 7 or 8. Let 3 be adjacent to 9, 10, and 11.

Thus we have the following:

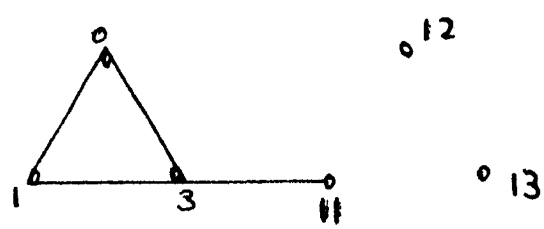


At this point G_3 is



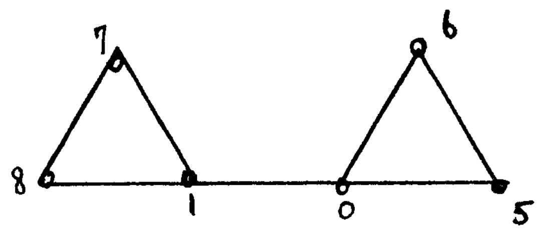
Since 0 and 1 are adjacent to six vertices already, it must be that 9, 10, and 11 form a triangle with one vertex, say 11, adjacent to 2. Two other vertices are adjacent to 2. They are not 7 or 8 because of G_1 . They are not 9 or 10 because of G_3 . Thus we can call them 12 and 13.

Now G_2 is

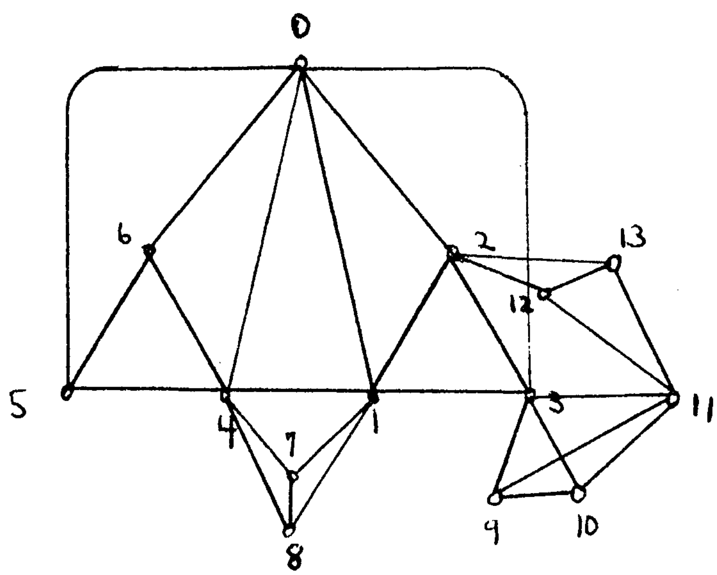


Thus 11, 12, and 13 form a triangle.

G_4 is now

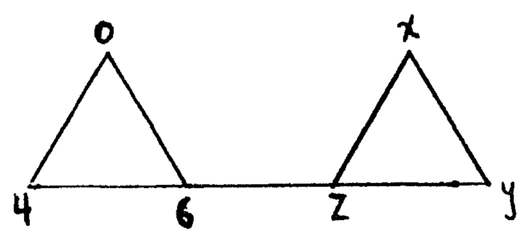


Thus we have the following graph so far:



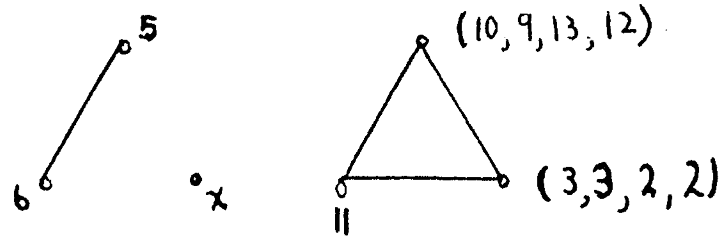
Thus 5 is adjacent to neither 7 nor 8.

Thus G_5 is of the form



since 4 and 0 are already adjacent to six vertices.

G_2 shows that 11 is adjacent to 12, 13, 3, and 2. G_3 shows that 11 is adjacent to 9 and 10 also. Thus 11 is not adjacent to 5 so that $z \neq 11$. Thus $z \in \{9, 10, 12, 13, 14\}$, the only remaining vertices. If z is respectively (9, 10, 12, 13), then G_z is



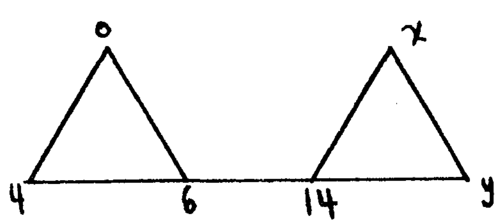
Thus x is adjacent to 5 and 6. $A^2_{56} = 3$ since 5 and 6 are mutually adjacent to z, 0, and 4, and by (3.6) $A^2_{56} \leq 3$. Thus $x = 0$ or $x = 4$.

But from G_5 we see that x and z are adjacent. From G_0 and G_4 we see that for $z = 9, 10, 12$, or 13, this is impossible.

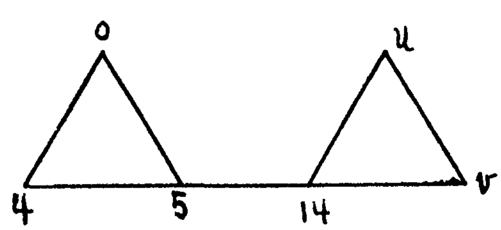
Thus $z = 14$.

Now we have the following

$\underline{G_5}$



$\underline{G_6}$

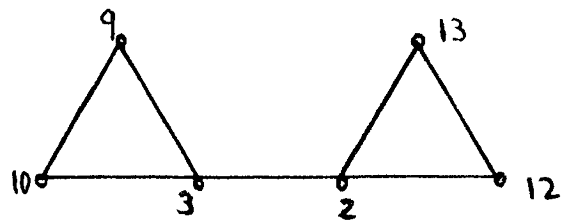


Since u and v are adjacent to 6 and not 5, and x and y are adjacent to 5 and not 6, u v x and y are distinct. So far we have 0, 1, 2, 3, 4, and 11 are adjacent to six vertices and, in particular, not adjacent to 14. In addition, G_5 and G_6 tell us that u v x and y are not adjacent to 4, so they can not be 7 or 8.

Thus by elimination,

$$\{u\ v\ x\ y\} = \{9,\ 10,\ 12,\ 13\}$$

Since G_{11} is the following



either u = 9 and v = 10

or u = 13 and v = 12

In the first case, 9 and 10 are mutually adjacent to 14, 3, 11, 6.

In the second case, 13 and 12 are mutually adjacent to 14, 2, 11, 6.

In either case we have a contradiction of (3.6).

Thus there is no G such that G_v is Figure 2 for all v,

and so G must be $L(K_{5,3})$.

Thus we can now say

<u>Theorem 4.15</u> If H has eigenvalues 6, 3, 1, and -2, and 15 vertices,

then $H = L(K_{5,3})$

As we stated in the introduction to this section, if there exists a G such that L(G) has $m+n-2$, $n-2$, $m-2$, and -2 as eigenvalues, then $G = K_{m,n}$ or G is a strongly regular graph. Now let us look at some of the properties that must be possessed by such a strongly regular G.

We can now apply Prop. 2.10 with $\alpha_0 = m+n-2$

$\alpha_1 = m-2$ and

$\alpha_2 = n-2$

Since H is regular, the largest eigenvalue of H is equal to its degree i.e., $m+n-2$. Thus $2d-2 = m+n-2$

$$\text{or} \qquad d = \frac{m+n}{2}$$

Thus G has the following eigenvalues:

$$(m+n-2) + 2 - \frac{m+n}{2} = \frac{m+n}{2}$$

$$(m-2) + 2 - \frac{m+n}{2} = \frac{m-n}{2}$$

$$\text{and} \qquad (n-2) + 2 - \frac{m+n}{2} = \frac{n-m}{2}$$

If G has $|V|$ vertices and $|E|$ edges

$$|V| \frac{m+n}{4} = V \frac{d}{2} = \text{number of vertices in } L(G)$$

$$= \text{number of vertices in } L(K_{mn})$$

$$= mn$$

Thus $|V| = \dfrac{4mn}{m+n}$

Thus $\quad p(x) = V \dfrac{\left(x - \frac{m-2}{2}\right)\left(x - \frac{n-m}{2}\right)}{\left(d - \frac{m-n}{2}\right)\left(d - \frac{n-m}{2}\right)}$

$$= \frac{4\,mn}{m+n} \; \frac{x^2 - \left(\frac{m-n}{2}\right)^2}{d^2 - \left(\frac{m-n}{2}\right)^2}$$

$$= \frac{4mn}{m+n} \left(\frac{x^2 - \left(\frac{m-n}{2}\right)^2}{mn} \right)$$

$$= \frac{4}{m+n} \quad x^2 - \left(\frac{m-n}{2}\right)^2$$

and so

$$A^2 = \frac{m+n}{4} J + \frac{(m-n)^2}{4} I$$

Thus A is the incidence (caveat: not adjacency) matrix of a SBIB

with $\lambda = \dfrac{m+n}{4}$

$$v = b = \frac{4mn}{m+n}$$

$$k = r = d = \frac{m+n}{2}$$

λ an integer implies $m+n = 4e$.

$$r = \frac{m+n}{2} = A^2_{ii} = \frac{m+n}{4} + \frac{(m-n)^4}{2}$$

Therefore $(m-n)^2 = m+n = 4e$

Therefore e is a square, say $e = t^2$

Thus $m + n = 4t^2$

$m - n = 2t$

$m = 2t^2 + t$

and $n = 2t^2 - t$

Thus we have the following:

<u>Theorem 4.6</u> If $m + n - 2 > 16$

and $(m, n) \neq (2t^2 + t, \ 2t^2 - t)$ t an integer $t > 1$

then $H = L(K_{m,n})$

<u>Theorem 4.7</u> If there exists a regular G such that L(G) has

$m + n - 2$, $m - 2$, $n - 2$, and -2 as eigenvalues

then there exists a symmetric Hadamard matrix with

constant diagonal of order $4t^2 = m + n$

A Hadamard matrix is one for which every entry is ± 1 and also $HH^T = 4eI$. It is well known that e must be an integer if the order of H is greater than 2. [1, page 205]

Proof of Theorem 4.7: We saw in Theorem 4.6 that

$$m + n = 4t^2$$

$$m - n = 2t^2$$

and $|V| = \dfrac{4mn}{m+n} = 4t^2 - 1$

Define $B = J - 2A$ Note that the entries of B are all ± 1

and let $H = \left[\begin{array}{c|c} 1 & u^T \\ \hline u & B \end{array}\right]$

Then $HH^T = \left[\begin{array}{c|c} 1+u^Tu & u^T+u^TB \\ \hline u^T+u^TB & uu^T+B^2 \end{array}\right]$

$$u^TB = u^T(J-2A) = u^TJ - 2u^TA = (4t^2-1)u^T - 2\cdot 2t^2\, u^T$$
$$= -u^T$$

$$B^2 = (J-2A)^2 = J^2 - 4JA - 4A^2 = (4t^2-1)J - 4(2t^2)J + 4(t^2J + t^2I)$$
$$= -J + 4t^2 I$$

Thus $HH^T = 4t^2 I$

H is symmetric since B is, and since $A_{kk} = 0$, $H_{kk} = 1$

<u>Theorem 4.8</u> If there exists a symmetric Hadamard matrix with constant diagonal of order $4t^2$, $t > 1$ then $L(K_{m,n})$ is not characterized by its spectrum.

Proof: Suppose H is symmetric with constant diagonal. Without loss of generality, we may assume that the diagonal entries are $+1$.

H has two eigenvalues $2e^{\frac{1}{2}}$ of multiplicity m_1 and $-2e^{\frac{1}{2}}$ of multiplicity m_2.

Then $m_1 + m_2 = 4e$

$$2e\,(m_1 - m_2) = 4e$$

$$m_1 - m_2 = 2e$$

if we let $t = \dfrac{m_1 - m_2}{2} = e$

$m_1 + m_2 \equiv 0 \bmod 2$ implies that

t is an integer and so J is of order $4t^2$

If a row or column of a Hadamard matrix is multiplied by -1, it remains a Hadamard matrix. Thus if we multiply each row by its first entry and each column by its first entry we get a normalized Hadamard matrix, i.e., the first row and column entries are all $+1$. Notice that when we normalize a symmetric Hadamard matrix with constant diagonal, we still have a Hadamard matrix that is symmetric with constant diagonal.

If we have a normalized symmetric Hadamard matrix with 1's for diagonal entries and we delete the first row and column, and then change the -1's to 1's, and the 1's to 0's, it is well known that we obtain the incidence matrix of a block design with parameters

$$v = b = 4t^2 - 1$$

$$r = k = 2t^2$$

$$\lambda = t^2 \qquad \text{[cf. 1, page 205]}$$

If we call the matrix that results from this process A, then

$$AA^T = t^2 J + t^2 I \quad \text{where } A \text{ is symmetric with 0's on the diagonal}$$

Thus $A^2 - t^2 I = t^2 J$

or if $p(x) = \frac{1}{t^2}(x^2 - t^2) = \frac{1}{t^2}(x + t)(x - t)$

$$p(A) = J$$

Since $A^2_{ii} = t^2 J_{ii} - t^2 I_{ii} = 2t^2$

the eigenvalues of A are $2t^2$ and $\pm t$

If we let G be the graph whose adjacency matrix is A,

then G is regular of valence $2t^2$

If $t > 1$ then $d = 2t^2 \geq 2$ and we can use Prop. 2.11

Thus the eigenvalues of $H = L(G)$ are:

$$2d - 2 = 4t^2 - 2$$

$$t + d - 2 = 2t^2 + t - 2$$

$$-t + d - 2 = 2t^2 - t - 2$$

and -2.

If we define $m = 2t^2 + t$ and $n = 2t^2 - t$

then the eigenvalues of H are $m + n - 2$, $m - 2$, $n - 2$, and -2.

Also, the number of vertices of H is the number of edges of G

$$= \frac{|V|\, d}{2} = \frac{(4t^2 - 1)\, 2t^2}{2} = 4t^4 - t^2 = (2t^2 - t)(2t^2 - t)$$
$$= mn$$

This complete the proof of Theorem 4.8

<u>Theorem 4.9</u> If $H = L(G)$

the spectrum of H is $m + n - 2$, $m - 2$, $n - 2$, and -2, then H is characterized by its spectrum if and only if there does not exist a Hadamard matrix of order $m + n$ which is symmetric and has constant diagonal.

We have shown that if H is a symmetric Hadamard matrix of order $4t^2$, then $L(K_{m,n})$ is not characterized by its spectrum among all line graphs in $G(4,-2)$ with mn vertices where $m = 2t^2 + t$ and $n = 2t^2 - t$. We did this by normalizing the first row and column and then using the remaining rows and columns to form an adjacency matrix. But suppose we normalized another row and column. Also, what if there is more than one Hadamard matrix of order $4t^2$? If $t = 2$, then the regular graph whose line graph has eigenvalues 14, 8, 4, and -2 must have 8, 2, and -2 as its eigenvalues by Prop. 2.10 and our previous observation that G is not bipartite. Thus G is in $G(3,-2)$, which has been completely described by Seidel [8]. He shows that the only graph in $G(3,-2)$ with 15 vertices is $T(6)$, which is the line graph of the 6-clique. Thus for $t = 2$, there is only one regular graph G with $4t^2 - 1$ vertices such that $A^2 = t^2 I + t^2 J$. This shows that there is only one Hadamard matrix of order 16 which is symmetric with constant diagonal. The case $t = 3$ yields quite a different result. Seidel and Goethals [9] have exhibited a Hadamard matrix that yields a different graph for the normalization of each of the 36 rows! The existence of at least 92 different graphs has been proven, and so there are many different Hadamard matrices that are symmetric with constant diagonal of order 36.

Section 5. Some Remarks on G(5, -2)

As we have seen in Section 2, the line graph of an asymmetric non-trivial BIBD is in G(5, -2). In this section we consider whether the line graph of a BIBD is characterized among all connected regular graphs on the same number of vertices and with the same eigenvalues. We can not hope to distinguish between non-isomorphic BIBDs with the same parameters, of course, since the eigenvalues determine only the parameters. We shall assume the graph to be a line graph (an assumption satisfied in all but a finite number of cases), and derive sufficient conditions for the graph to be characterized. In particular, it will be shown that the line graph of a Steiner Triple System satisfies these conditions, so that (with finitely many exceptions) the line graph of a Steiner Triple System is characterized by its spectrum among all regular connected graphs with the same number of vertices.

If G is the graph of a BIBD with parameters (v b r k λ), then, if we define $D = \frac{(r-k)^2}{4} + r - \lambda$ then the eigenvalues of $H = L(G)$ are $r+k-2$, $\frac{r+k}{2} + D^{\frac{1}{2}} - 2$, $k-2$, $\frac{r+k}{2} - D^{\frac{1}{2}} - 2$, and -2.

By Prop. 2.10 the eigenvalues of G are $\frac{r+k}{2}$, $D^{\frac{1}{2}}$, $\frac{k-r}{2}$, $-D^{\frac{1}{2}}$, and $-\frac{r+k}{2}$ if G is bipartite.

But if G is bipartite, and since there are an odd number of eigenvalues, $\frac{k-r}{2} = 0$, or $k = r$. But this implies that the design is symmetric, which is a contradiction. Thus if $H = L(G)$ has the same eigenvalues as the line graph of a BIBD, then G is regular and has four eigenvalues.

Note that if H is the line graph for which $r+k$ is odd, then H is characterized by its spectrum among all line graphs in $G(5, -2)$; if not, there exists a regular graph G with degree $\frac{r+k}{2}$.

If $|V|$ = the number of vertices of G

then $\frac{|V|\, d}{2}$ = the number of edges of G

= the number of vertices of $L(G)$

= the number of vertices of the line graph of the BIBD

$= r\, v$

Thus, since $d = \frac{r+k}{2}$,

$$|V| = \frac{4\, r\, v}{r+k}$$

Also, if we define $t = r - k$, t is a positive even integer since the BIBD is asymmetric.

Thus $D = \left(\frac{t}{2}\right)^2 + r - \lambda$ is an integer.

Note also that $A^2_{kk} = d = \frac{r+k}{2}$

Thus $tr\, A^2 = \frac{r+k}{2}\, |V| = \frac{r+k}{2}\, \frac{4\, r\, v}{r+k} = r\, v$

Theorem 5.1 If D is not a square

then $\frac{r+k}{r-k}$ is an integer and $\frac{t}{2}$ divides k

Proof: Let m be the multiplicity of $\frac{k-r}{2}$

Then, since D is not a square, and $tr\,A = 0$, D and -D have the same multiplicity, and

$$0 = tr\,A = \frac{r+k}{2} + m\left(\frac{k-r}{2}\right)$$

Thus
$$m = \frac{r+k}{r-k} = \frac{2k+t}{t}$$

hence t divides 2k and $\frac{t}{2}$ divides k

Theorem 5.2 Suppose D is not a square

k an odd prime

$\lambda \neq 2$

then the line graph of the BIBD with parameters (v, b, r, k, λ) is characterized by its spectrum among all line graphs in $G(5, -2)$ with rv vertices.

Proof: By Theorem 5.1

$\frac{t}{2}$ divides k

Thus $\frac{t}{2} = k$ or $\frac{t}{2} = 1$, i.e.,

$r = 3k$ or $r = k+2$

If $r = 3k$, then

$$m = \frac{r+k}{r-k} = 2$$

$$|V| = \frac{4\,r\,v}{r+k} = \frac{4 \cdot 3k\;v}{4k} = 3v$$

If D has multiplicity m'

then $2m' + m + 1 = |V|$

or $2m' = 3v - 3 = 3(v-1)$

$$D = \left(\frac{r-k}{2}\right)^2 + r - \lambda = k^2 + 3k - \lambda$$

Then we have

$$6\,k\,v = 2\,r\,v = \operatorname{tr} A^2_{(G)} > 2\,m'\,D$$

$$= 3\,(v-1)\,(k^2 + 3k - \lambda)$$

Thus $2\,v > (v-1)\left(k + 3 - \frac{\lambda}{k}\right)$

$$= 3\,(v-1) + (v-1)\left(k - \frac{\lambda}{k}\right)$$

$$\lambda < r = 3k \leq k^2 \Rightarrow k^2 - \lambda > 0$$

$$k\left(k - \frac{\lambda}{k}\right) > 0$$

$$\left(k - \frac{\lambda}{k}\right) > 0$$

Thus $2v > 3\,(v-1)$

$3 > v$, which is a contradiction.

Thus it must be that $r = k+2$.

$v\,r = b\,k$ implies that $v \equiv 0 \mod k$

Since $r \equiv 2 \mod k$, $-2 \equiv -r \equiv r\,(k-1) = \lambda\,(v-1) \equiv -\lambda \mod k$

and $\lambda \equiv 2 \mod k$

Since $0 < \lambda < r = k+2$, it must be that $\lambda = 2$

This contradicts the hypothesis and completes the proof of Theorem 5.2

Cor. 5.3 If H is the line graph of a block design

and if D is not a square

k a prime $k \geq 3$

then unless $(v, b, r, k\ \lambda) = \left(\frac{k\,(k+1)}{2}, \frac{(k+1)\,(k+2)}{2}, k+2, k, 2\right)$

H is characterized by its eigenvalues among all line graphs

in $G(5, -2)$ with $v\,r$ vertices.

Proof: By Theorem 5.2, the only possible G such that $L(G) = H$

would exist only if $\lambda = 2$ and $r = k+2$.

Proposition 5.4 If H is the line graph of a BIBD

k an odd prime

$\lambda = k-2$

then H is characterized by its spectrum among all regular

connected graphs with the same number of vertices.

Proof: $$D = \left(\frac{r-k}{2}\right)^2 + r - \lambda$$

$$= \left(\frac{r-k}{2} + 1\right)^2 + k - (\lambda + 1)$$

$$= \left(\frac{r-k}{2} + 1\right)^2 + 1$$

If D is a square, then there exist two consecutive positive

integers that are squares.

Therefore D is not a square.

Thus there is only one possible exception, as seen in Cor. 5.3

But $\lambda = 2$ implies that $k = \lambda + 2 = 4$, which is not an odd prime.

Cor. 5.5 If H is the line graph of a Steiner Triple System then H is characterized by its spectrum among all line graphs with the same number of vertices.

Proof: In this case $k = 3$ which is an odd prime and $\lambda = k - 2 = 1$

From Cor. 5,3 we see that if D is not a square and k is an odd prime, then H is characterized by its spectrum unless there exists a graph G with the following properties:

(5.6.1) G is regular with valence $k + 1$

(5.6.2) $V = k(k+2)$

(5.6.3) G has eigenvalues $k+1$, $(k+1)^{\frac{1}{2}}$, -1, and $-(k+1)^{\frac{1}{2}}$

Thus we would like to know if a graph with such propeıties exists. We can show that such graphs do not exist if k is even or $k = 1$. We can also show that there exists exactly one such graph for $k = 3$. We do not completely understand the situation for odd k, $k \geq 5$, but we shall present the proof of the uniqueness of the case $k = 3$ in such a way that the properties for general odd k are illuminated.

If $k = 1$, $|V| = 3$ and $d = 2$. Thus G is a 3-clique with eigenvalues 2 and -1, and does not possess property (5.6.3).

Now let us look for regular graphs H that have $k+1$, $\pm(k+1)^{\frac{1}{2}}$ and -1 as eigenvalues. For such a graph, it must be that

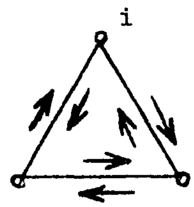
$$A^3 + A^2 - (k+1)A - (k+1)I = (k+1)J$$

Since $A^2_{ii} = d = k+1$, $A^3_{ii} = k+1$

<u>Theorem 5.7</u> If k is even

then there does not exist a graph with property (5.6.3).

Proof:



For every triangle for which i is a vertex, there are two paths of length three from i to i. Obviously every path of length three from i to i determines a triangle of which i is a vertex. Thus the number of paths from i to i is exactly twice the number of triangles of which i is a vertex. Thus A^3_{ii} is even. Since $A^3_{ii} = k+1$, property (5.6.3) implies that k is odd.

<u>Theorem 5.8</u> If a graph has property (5.6.3)

then it has no quadrilaterals.

$$A^4 + A^3 - (k+1)A^2 - (k+1)A = (k+1)JA = (k+1)^2 J$$

Therefore $A^4_{ii} = (k+1)(2k+1)$

Thus there are $(k+1)(2k+1)$ cycles of the type (i a b c i)

We shall call a cycle degenerate if the vertices in the cycle are not all distinct. Let us count the degenerate cycles of length four.

Case 1: $b = i$

There are $(k+1)^2$ such cycles since a and c may be any two vertices adjacent to i.

Case 2: $b \neq i$

then $a = c$

There are $k(k+1)$ such cycles since a may be any vertex adjacent to i, and b may be any vertex adjacent to a that is not i.

Thus the total number of degenerate four-cycles is

$$(k+1)^2 + k(k+1) = (2k+1)(k+1) = A^4_{ii}$$

Thus we see that H has no quadrilateralsand the proof of Theorem 5.8 is complete.

We can derive more properties of graphs satisfying (5.6.1) and (5.6.3). $A^5 + A^4 - (k+1)A^3 - (k+1)A^2 = (k+1)^3 J$

Therefore $A^5_{ii} = (k+1)^3 + (k+1)$

Let us count the number of degenerate five-cycles $(i\ a\ b\ c\ d\ i)$.

Case 1: $i = b$

There are $(k+1)$ choices for a and $k+1$ choices for the cycle $(i\ c\ d\ i)$. Therefore there are $(k+1)^2$ cycles of this type.

Case 2: $i = c$

There are $(k+1)^2$ cycles of this type since each one is a cycle of type 1 in the reverse direction.

<u>Case 3</u>: $i \neq a$, $i \neq b$, $i \neq c$, $i \neq d$

There are three possibilities: $a = c$, $b = d$, or $a = d$.

<u>3A</u> : $a = c$, $b \neq d$

($a \neq d$ since $a = c$ and d are adjacent)

($i\ a\ d\ i$) is a cycle; There are $(k+1)$ choices for this; b is any vertex adjacent to a but not i or d. Thus there are $(k+1)(k-1)$ of this type.

<u>3B</u> : $a = c$, $b = d$

($a \neq d$ since a and b are adjacent)

In this case the cycle is ($i\ a\ b\ a\ b\ i$). Obviously there is one of these for each three-cycle, i. e., $k+1$ in all.

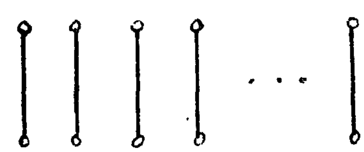
<u>3C</u> : $a \neq c$, $b = d$

($a \neq d$ since a is adjacent to b)

There are $(k+1)(k-1)$ of this type since a cycle of this type corresponds to a cycle of 3A in the reverse direction.

<u>3D</u> : $a \neq c$, $b \neq d$, i. e., $a = d$

There are $k+1$ choices for an a adjacent to i.

There are $(k-1)$ cycles $a\ b\ c$ that do not contain i. This is because H_i is 

H_i is of this form since if two edges are joined, H contains a quadrilateral; $d = k+1 =$ number of points;

$$\frac{A^3{}_{ii}}{2} = \frac{k+1}{2} = \text{number of lines}$$

in H_i. Thus the number of this type is $(k+1)(k-1)$.

Since the above cases were mutually exclusive, the number of degenerate five- cycles is

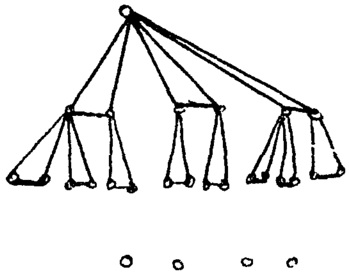
$$2(k+1)^2 - 3(k+1)(k-1) + k + 1 =$$
$$(k+1)\ (\ 2(k+1) + 3(k-1) + 1) = 5k\ (k+1)$$

Thus the number of non degenerate five-cycles from i to i is

$$\begin{aligned}(k+1)^3 + (k+1) - 5k(k+1) &= (k+1)((k+1)^2 + 1 - 5k) \\ &= (k+1)(k^2 - 3k + 2) \\ &= (k+1)(k-1)(k-2)\end{aligned}$$

Now define $L_t = \{v \text{ vertices} \mid \text{the shortest path from } v \text{ to } i \text{ of length } t.\}$

$V = L_o \cup L_1 \cup L_2 \cup L_3$ since H satisfies a polynomial of degree 3.

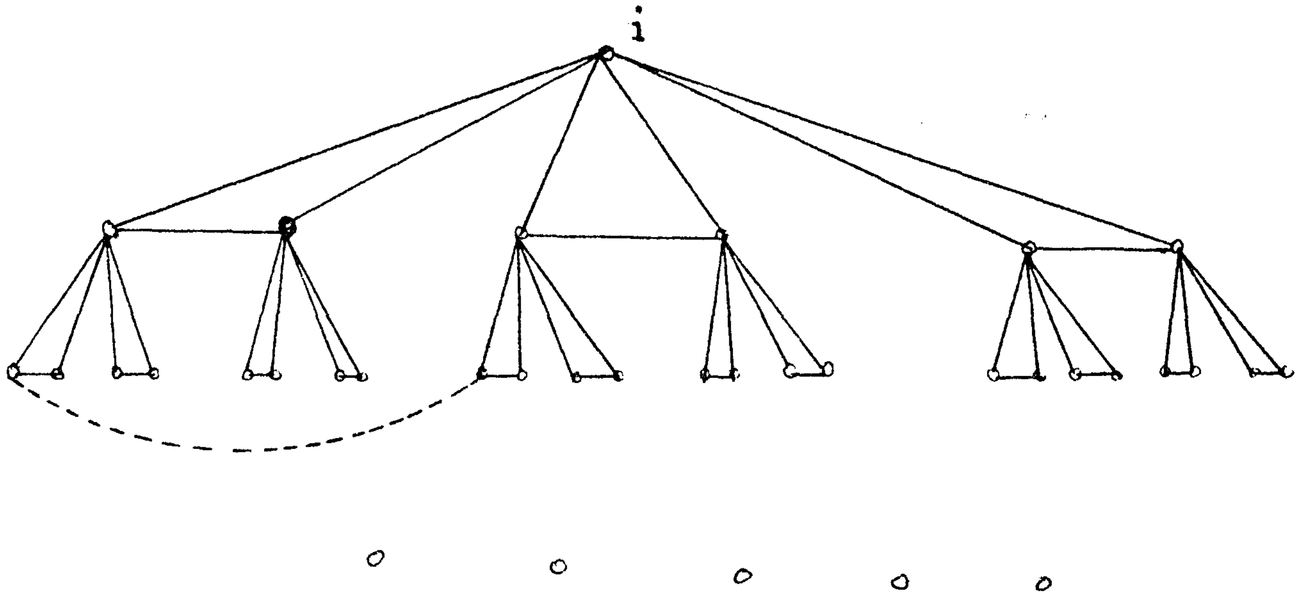


L_o 1 vertex

L_1 k +1 vertices

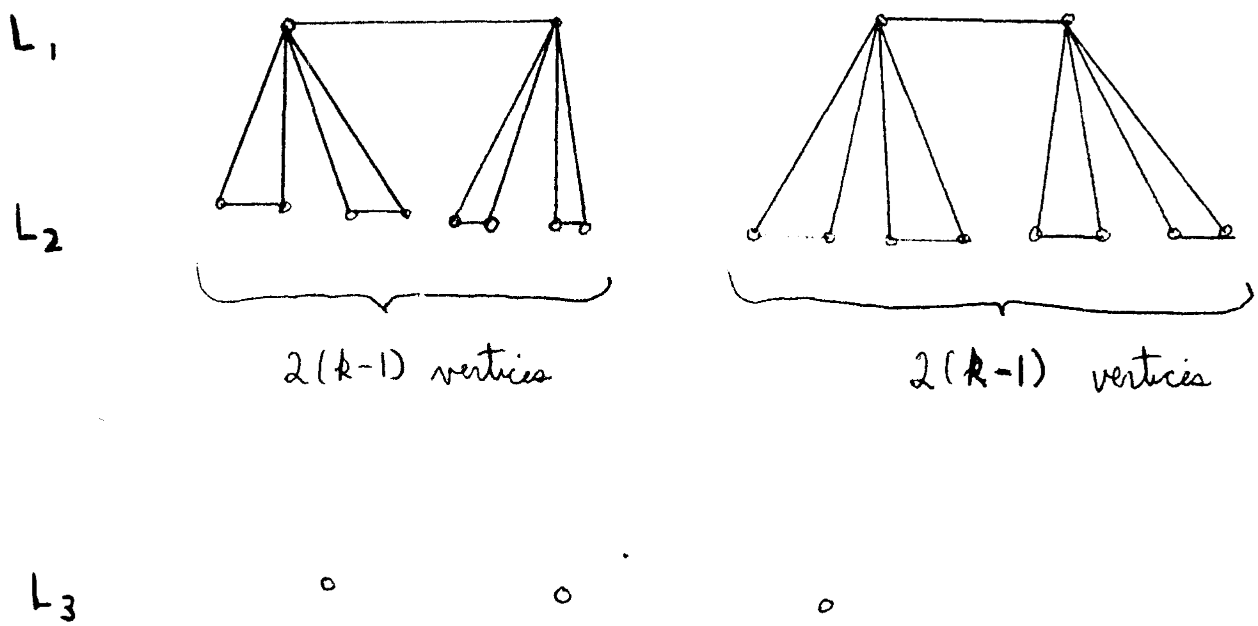
L_2 (k +1).(k -1) vertices

L_3 k -1 vertices



L_0 contains only i. In L_1 are the vertices adjacent to i, and they are adjacent in pairs, as shown, since H has no quadrilaterals. Each vertex in L_1 is adjacent to $(k-1)$ vertices in L_2 since $d = k+1$. These $(k+1)(k-1)$ vertices are distinct since H has no quadrilaterals. They are adjacent in pairs as shown for the same reason. Now suppose we add a new edge joining two vertices in L_2 in such a way that we do not form a quadrilateral. Then we form two non-degenerate five-cycles from i to i. In fact, it is clear that this is the only way such a five-cycle can be formed. Thus there are $1/2\,(k+1)(k-1)(k-2)$ such edges. Since each edge is incident to two vertices, each vertex in L_2 is, on the average, adjacent to $k-2$ such edges.

Let us look at a particular vertex v in L_2 (see diagram on next page). Now v can not be adjacent to any of the next $2k-3$ vertices to the right of v without forming a quadrilateral. Also, v can be adjacent to at most one of the next $(k-1)$ vertices for the same reason. The same is true for each successive set of $(k-1)$ vertices.



Thus, on the average, each edge is adjacent to one vertex in L_3, i.e., there are $(k+1)(k-1)$ edges from a vertex in L_2 to a vertex in L_3. But there are a total of

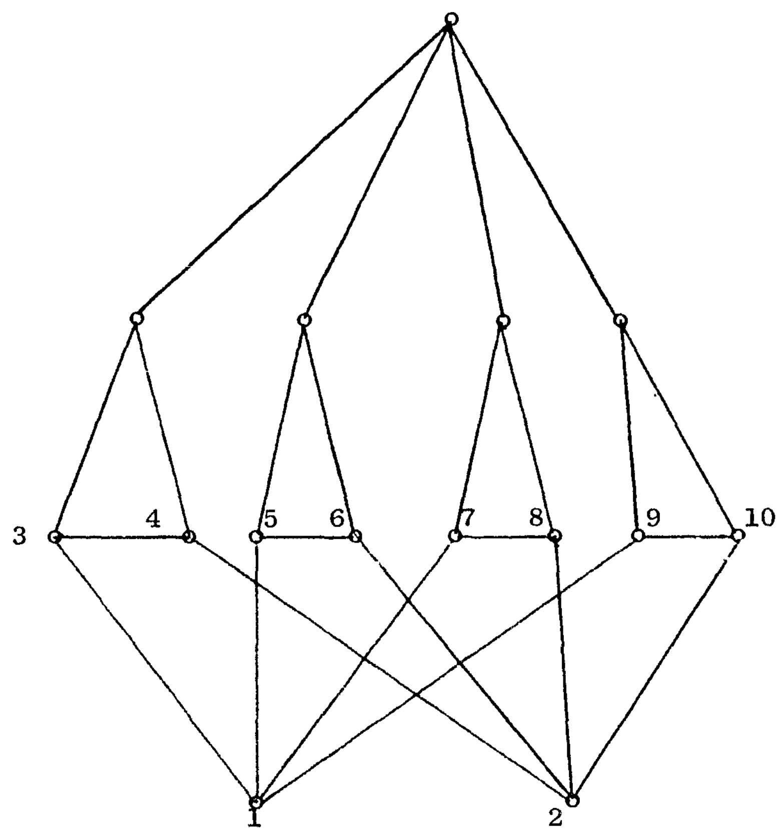
$$k(k+2) - (k+1)(k-1) - (k+1) - 1 = k-1 \text{ vertices in } L_3 .$$

Thus, since the valence of H is $k+1$, every edge incident to a vertex in L_3 is incident to a vertex in L_2, i.e. no two vertices in L_3 are adjacent.

<u>Proposition 5.9</u> There is at most one G that satisfies (5.6.1) and (5.6.3) if $k = 3$

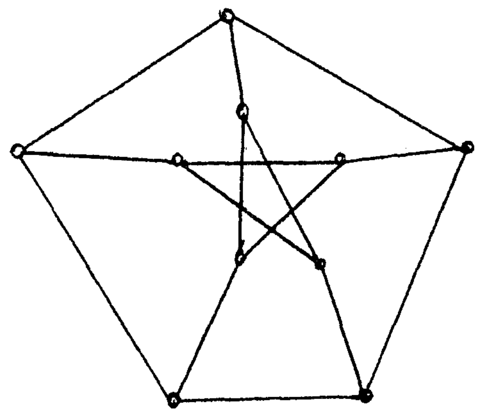
In this case $d = k+1 = 4$

Thus we have the following

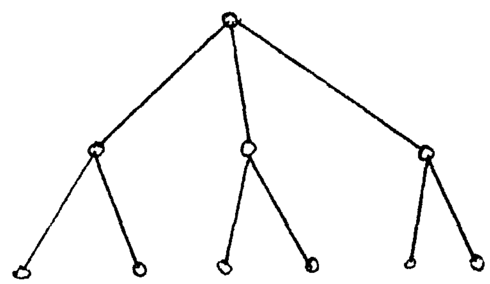


Vertex 1 is adjacent to four vertices in L_3. Without loss of generality we can say that they are 3, 5, 7, and 9. If 1 and 2 are mutually adjacent t a vertex, say 3, then 1 and 2 are adjacent because of H_3. Thus 2 is adjacent to 4, 6, 8, and 10. Vertex 3 is adjacent to another vertex in L_2; because of G_1 it must be 5, 7, or 9. It can not be 5 since that would form a quadrilateral. Vertex 7 and 9 are symmetric, so say that 3 is adjacent to 7. Then, because of G_1, 5 and 9 are adjacent. Now 4 is adjacent to 6, 8, or 10; if it were 6 or 8, a quadrilateral would be formed. Thus 4 is adjacent to 10 and 6 to 8 also. Now every vertex is of valence 4, so we have all of our edges. Thus this graph is the only one with $k+1$, $(k+1)^{\frac{1}{2}}$, $-(k+1)^{\frac{1}{2}}$, and -1 as eigenvalues and 15 vertices.

The figure below is the Peterson graph:



Notice that there are no quadrilaterals or triangles. Thus if we take any vertex i, it is adjacent to three vertices which are mutually non-adjacent since there are no triangles. Thus each of these vertices are adjacent to two new vertices and these six new vertices are distinct since there are no quadrilaterals.



We now have ten vertices, the total number. It is now clear that if $A_{ij} = 1$, then $A^2_{ij} = 0$, and if $A_{ij} = 0$, then $A^2_{ij} = 1$.

Thus if $p(x) = x^2 + x - 2 = (x+2)(x-1)$,

then $p(A) = J$. Thus the eigenvalues of the Peterson graph are $d = 3, 1$, and -2. Then, by Theorem 2.10, the line graph of the Peterson graph has eigenvalues 4, 2, -1, and -2.

Thus, if $k = 3$, the eigenvalues are $k+1$, $\pm (k+1)^{\frac{1}{2}}$, and -1.

Theorem 5.10 If $k = 3$ there is a unique graph with eigenvalues $k+1$, $\pm(k+1)^{\frac{1}{2}}$, and -1.

Cor. 5.11 A line graph of a BIBD with parameters $(v\ b\ r\ k\ \lambda) = (6, 10, 5, 3, 2)$ is not characterized by its eigenvalues among all graphs in $G(5, -2)$ with 30 vertices.

Proof: If P is the Peterson Graph

$L(L(P))$ has 30 vertices and eigenvalues 6, 4, 1, 0 and -2

by Prop. 2.10

But in this case $D^{\frac{1}{2}} = 2$ and $r + k - 2 = 6$

$$\frac{r+k}{2} + D^{\frac{1}{2}} - 2 = 4$$

$$k - 2 = 1$$

$$\frac{r+k}{2} - D^{\frac{1}{2}} - 2 = 0$$

and -2.

Summary

Thus we have shown that the line graphs of Steiner Triple Systems are characterized by their eigenvalues among all regular connected graphs with $r\,v$ vertices. In addition, we have given an example showing that not all asymmetric non-trivial BIBDs are so characterized; and we have indicated where additional such examples may be found.

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Autobiographical Statement

Michael Doob was born in New York City on November 14, 1942. He received his early education in the public school system of the District of Columbia and in 1960 graduated from Woodrow Wilson High School in Washington D.C. In 1964 he received a Bachelor of Arts degree from Columbia University, New York City, after which he attended the City College of New York. He received his Master of Science degree from City College and in 1966 was awarded a graduate fellowship at the City University of New York. In 1967 he married the former Judith Sherman.