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The Existence of σ-finite Invariant Measures for Certain Markov Processes and Related Results

by

Arnold H. Fischthal

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ABSTRACT

The Existence of σ-finite Invariant Measures for Certain Markov Processes and Related Results

by

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Adviser: Professor Richard E. Isaac

Several theorems to show the existence of a σ-finite measure for a general Markov process with stationary transition probability are exhibited. A general hypothesis that is assumed is there exists a set F with $m(F) > 0$ such that $P(X_n \in F \text{ infinitely often} | X_0 = x) = 1$ for a.e. $x \in F$ and that for each x there exists an i such that $P^i(x,F) > 0$.

In addition to the above, the first theorem uses the condition that for this set F

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{X_k \in A}}{\sum_{k=0}^{n} 1_{X_k \in F}} \quad \text{exists a.e. } \forall A \subset F .$$

A similar condition to the one above except using transition probability functions instead of indicator functions is used in the second result.

Another existence theorem assumes there exists a δ, $0 < \delta < 1$, for this set F such that for all $A \subset F$ with $m(A) > \delta$ we have

$$\bar{m}\left(\omega\colon \liminf_{n} \frac{\sum_{k=0}^{n} 1_{X_k \in A}(\omega)}{\sum_{k=0}^{n} 1_{X_k \in F}(\omega)} > 0, X_0 \in F\right) > 0$$

where $\bar{m}$ is the measure induced on sequence space by the initial measure m on state space. The final existence theorem uses a condition similar to the one above except transition probability functions are used instead of indicator functions.

Also shown is the following result on the finiteness of the invariant measure, which involves the expected first recurrence time to a given set:

$$E_{\pi}(T_1 | X_0 \in \Lambda) = \infty \Leftrightarrow \pi(X) = \infty .$$

A theorem on the uniqueness of a σ-finite invariant measure as well as results on the mixing properties of a set in an infinite space are also exhibited.

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§S (Summary):

This paper deals with the existence of a σ-finite invariant measure for a general, discrete time M.P. Although we are looking for σ-finite measures, some criteria will be given to determine when such a measure will in fact be finite. Several conditions will be discussed that insure the existence of the desired measure.

Much work has been done to show the existence of finite invariant measures. A broad summary of results in this area can be found in a paper by Ito [15]. One classic and well-known result on infinite measures is from a paper by Harris [10] where the existence of a σ-finite invariant measure is guaranteed by the imposition of his condition that the initial measure m satisfies $m(A) > 0 \Rightarrow P(X_n \in A$ infinitely often $| X_0 = x) = 1$ for all $x \in X$. We believe the material to be presented here is more general than work already done on the existence of an infinite invariant measure. Also to be discussed is the uniqueness of such a measure as well as results on mixing properties of translates of sets.

We begin in Section 0 by giving the necessary definitions. In Chapter 1 a σ-finite invariant measure is shown to exist if the following condition is satisfied:

$$\exists \text{ a set } F \in \Sigma,\ m(F) > 0,\ \text{s.t. } \forall A \subset F,\ A \in \Sigma$$

$$\text{we have } \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{X_k \in A}(\omega)}{\sum_{k=0}^{n} 1_{X_k \in F}(\omega)} \text{ exists a.e.}(\bar{m}) .$$

The technique used here is found in [10] and is related to a procedure

used by Halmos [8] which involves proving a finite invariant measure exists on a "sub-process" and then extending this measure to a σ-finite invariant measure on the original process.

A similar assumption is made in Chapter 2; however, there we will be dealing with transition probabilities rather than indicator functions. The condition to be imposed there is

$$\exists \text{ a set } F \in \Sigma,\ m(F) > 0, \text{ and } \exists \text{ an } x \in X$$

$$\text{s.t. } \forall A \subset F,\ A \in \Sigma \text{ the following is true}$$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} P^{k}(x,A)}{\sum_{k=0}^{n} P^{k}(x,F)} \text{ exists.}$$

The method used in Chapter 2 is different than in Chapter 1. Here we begin by defining a finitely additive set function on a subset of Σ which will be extended to a σ-finite invariant measure on all of Σ .

In Chapter 3 we exhibit two additional conditions that insure the existence of a σ-finite invariant measure. The first is

$$\exists \text{ a set } F \in \Sigma \text{ and a } \delta,\ 0 < \delta < 1, \text{ s.t.}$$

$$\forall A \in \Sigma,\ A \subset F \text{ with } m(A) > \delta \text{ we have}$$

$$\bar{m}\left(\omega : \underline{\lim}_{n} \frac{\sum_{k=0}^{n} 1_{X_k \in A}(\omega)}{\sum_{k=0}^{n} 1_{X_k \in F}(\omega)} > 0,\ X_0 \in F\right) > 0 .$$

To use this condition it will be shown, as in Chapter 1, that a finite invariant measure exists on the F-process (this will be done by an entirely different procedure than in Chapter 1) and then, again, this measure will be extended to all of X. The second condition is

$$\exists \text{ a set } F,\ m(F) > 0, \text{ and a } \delta,\ 0 < \delta < 1, \text{ s.t.}$$

$$\forall A \in \Sigma, \text{ with } A \subseteq F,\ m(A) > \delta \text{ we have}$$

$$m\left(x : \frac{\lim}{n} \frac{\sum_{k=0}^{n} P^k(x,A)}{\sum_{k=0}^{n} P^k(x,F)} > 0,\ x \in F\right) > 0 .$$

The proof utilizing this condition follows almost precisely the technique of Chapter 2 and is outlined in Chapter 3 with the necessary modifications. Also in Chapter 3 we show how these four conditions are related.

In a paper by Feldman [5], some results similar to the main results of the first two chapters are shown. However, his technique is different and relies heavily on the initial measure. Also, since we are essentially free of the condition of conservativity which Feldman imposes, our technique yields a slightly different result. The results of Chapter 3 cannot be gotten by his technique.

Chapter 4 deals with a theorem determining when an invariant measure will be finite or infinite by using the relationship of finiteness with the expected first recurrence time to a fixed set.

Chapter 5 exhibits a result on the uniqueness of an invariant measure and Chapter 6 proves some interesting mixing results of translates of a set.

§0. Preliminaries

We begin by defining what is meant by a discrete time Markov Process (M.P.). Let X be an arbitrary set called the state space of the process, and let Σ be a separable (i.e. countably generated) σ-algebra of subsets of X with $X \in \Sigma$. The elements of Σ will sometimes be referred to as "events". Let m be a non-negative, countably-additive, real-valued set function defined on Σ; m is called a measure. Let $P_n(x,A)$, $n=0,1,2,3,\ldots$, be a sequence of functions satisfying the following for each n

(0.1) $P_n(x,\cdot)$ is a probability measure (i.e. a measure of total mass 1) for each $x \in X$.

(0.2) $P_n(\cdot,A)$ is a Σ-measurable real-valued function for each $A \in \Sigma$.

The functions P_n are called the transition probability functions (t.p.f.). The quadruple $(X,\Sigma,m,\{P_n\})$ completely defines a discrete time M.P.

Another way to characterize a M.P. is by random variables. A random variable defined on X is simply a real-valued, Σ-measurable function. A sequence of random variables $X_0, X_1, X_2, \ldots$ defined on (X,Σ,m) is called a discrete time M.P. (The subscript "n" in P_n or X_n will sometimes be referred to as "time n" or "the n^{th} step") if they satisfy

$$(0.3) \qquad P(X_n \in A \mid X_{n-1} = x_{n-1}, x_{n-2} = x_{n-2}, \ldots, X_0 = x_0) = P(X_n \in A \mid X_{n-1} = x_{n-1})$$
$$[\text{a.e.}(\tilde{m}) .$$

The abbreviation a.e.(m) (read: almost everywhere with respect to the measure m) means that, e.g., the equation in (0.3) is true for all x_{n-1} except possibly those in a set of m-measure equal to zero. The notation of conditional probability in (0.3) stands for a function measurable w.r.t. the σ-field generated by the random variables $X_{n-1}, X_{n-2}, \ldots, X_0$ s.t.

(0.4)
$$\bar{m}(X_n \in A, X_{n-1} \in A_{n-1}, X_{n-2} \in A_{n-2}, \ldots, X_0 \in A_0) = \int_{\{X_{n-1} \in A_{n-1}, \ldots, X_0 \in A_0\}} P(X_n \in A \mid X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) d\bar{m}$$

where $\bar{m}$ is the measure on product space to be introduced shortly.

Essentially the characteristic of a M.P. described by (0.3) is that if we are after the probability of an event in the present (time n+1) and the complete history of the process is known (times n,n-1,...,0) then all that is actually needed to calculate the required probability is the most recent past (time n).

If there is a single t.p.f. $P(x,A)$ that satisfies

(0.5) $$P(x,A) = P_n(x,A) \quad \text{a.e.}(m) \ \forall n$$

then $P(x,A)$ is called a stationary t.p.f. It is this type of M.P. which will be dealt with in this paper. $P(x,A)$ gives the probability of moving from the state x into the set A in one step. ($P_n(x,A)$ is the probability of moving from state x into the set A in one step but occurring at the n^{th} step).

All of the t.p.f. defined so far have been one-step functions. The probability of moving from the state x into the set A in n steps is given by

(0.6) $$P^n(x,A) = P(X_{n+k} \in A \mid X_k = x) .$$

If $P(x,A)$ is stationary then so are the $P^n(x,A)$. One may arrive at the $P^n(x,A)$ recursively by the formula

(0.7) $$P^n(x,A) = \int_X P^{n-1}(y,A) P(x,dy) .$$

For completeness $P^0(x,A)$ is defined to be $1_A(x)$, the indicator function, which is defined as 1 if $x \in A$ and 0 if $x \notin A$ for $x \in X$ and $A \in \Sigma$.

Quite often in this paper we will be dealing with the sequence space of a process, which is the M.P. $(\bar{X}, \Sigma_\infty, \bar{m}, \bar{P})$ derived from (X, Σ, m, P) by looking at the state space $\bar{X} = X \times X \times X \times X \times X \times \ldots$ with the induced σ-algebra and measure. In many instances the discussion of the process on state space and the process on sequence space can and will be used interchangeably.

The measure m defines the weight of each event in Σ at the start of the process (i.e. at time zero). After one step the masses of the events will change according to the formula

$$(0.8) \qquad m_1(A) = \int_X P(x,A)m(dx) \ .$$

Similarly after the second step the new weights are

$$(0.9) \qquad m_2(A) = \int_X P(x,A)m_1(dx) \ ,$$

and so on. The measures m_i, $i=0,1,2,\ldots,(m_0 = m)$ are also the coordinate measures gotten from the projections of $\bar{m}$ on the coordinate spaces, e.g., $\bar{m}(A X X X X X \ldots) = m_0(A)$ and $\bar{m}(X X X X A X X X \ldots) = m_2(A)$. The interesting question to be asked is under what conditions does there exist an initial measure π that satisfies

$$(0.10) \qquad \pi(A) = \int_X P(x,A)\pi(dx) \quad \forall A \in \Sigma$$

i.e., the distribution of the weights remains unchanged through time. Any measure satisfying (0.10) is called an invariant measure so that we have $\pi(A) = \pi_1(A) = \pi_2(A) = \ldots \ \forall A$. A measure is called σ-finite if there exists a countable collection $\{A_i\} \subset \Sigma$ s.t. $\cup A_i = X$ and each A_i has finite mass.

On sequence space $\bar{X}$, a transformation θ can be defined, called the shift transformation, as follows: for $\omega = (x_0, x_1, x_2, \ldots) \in \bar{X}$ let $\theta(\omega) = (x_1, x_2, x_3, \ldots)$. θ also defines an operator on functions of $\bar{X}$ by $\theta(f(\omega)) \equiv f(\theta(\omega))$. A measure $\bar{m}$ on sequence space is called invariant if $\pi(\theta^{-1}A) = \pi(A)$, $\forall A \in \Sigma_\infty$.

Any non-invariant initial measure to be used in this paper will be assumed to be equal to 1, since any non-zero, σ-finite measure can be replaced by an equivalent measure of unit mass. Hence, the measure induced on sequence space by a unit measure will also have mass one. Without undue loss of generality, we will assume that X is the set of real numbers and the random variables will be the coordinate representation of the process, i.e., $X_i(\omega) = x_i$.

Note that the two characterizations of Markov processes are related in the following way: $P_n(x,A)$ of the first characterization equals $P(X_n \in A | X_{n-1} = x)$ of the second.

General probabilistic results and techniques used in this paper can be found in [1],[6], and [16].

CHAPTER 1: Indicator Limit Theorem

The main result of this chapter is

Theorem 1.1: Assume $\exists$ a set $F \in \Sigma$, $m(F) > 0$, satisfying the following three conditions:

$$(1.2) \qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{X_k \in A}(\omega)}{\sum_{k=0}^{n} 1_{X_k \in F}(\omega)} \quad \text{exists} \quad \text{a.e.}(\bar{m}) \quad \forall A \subset F,\ A \in \Sigma$$

$$(1.3) \qquad P(X_n \in F \text{ infinitely often} \mid X_0 = x) = 1 \quad \text{for} \quad \text{a.e. } x \in F$$

$$(1.4) \qquad \text{for each } x \in X \ \exists \text{ an } i \text{ s.t. } P^i(x,F) > 0 .$$

Then $\exists$ a σ-finite invariant measure for the M.P. (X,Σ,m,P).

The basic idea of the proof of Theorem 1.1 will be to show there exists a finite invariant measure for the "process on F" (to be defined shortly) and then to extend this measure to a σ-finite invariant measure on all of X, which will be invariant for the original process. In a paper by Harris [10], he shows that a finite invariant measure exists on the F-process under his condition C: $m(E) > 0$ implies $P(X_n \in E$ infinitely often $| X_0 = x) = 1$ $\forall x \in X$. He then shows how to extend this measure to all of X. The assumptions made in Theorem 1.1 will be shown to be sufficient even though weaker than Harris'. In some instances our procedures will be similar to those of Harris.

We now define what is meant by the process on F. Condition (1.3) says that for almost all $\omega = (x_0, x_1, x_2, \ldots)$ in sequence space, with $x_0 \in F$, we have infinitely many of the x_i in F. Therefore we have the following well-defined random variables $Y_0, Y_1, Y_2, \ldots$ on these ω defined as follows:

$Y_0(\omega) = x_0$

$Y_1(\omega) = x_{i_1}$ where i_1 is the first subscript greater than zero s.t. $x_{i_1} \in F$

$Y_2(\omega) = x_{i_2}$ where i_2 is the first subscript greater than i_1 s.t. $x_{i_2} \in F$

$\vdots$

so that $Y_i(\omega) \in F \ \forall \ i$. The $\{Y_i\}$ restricted to F is easily shown to be a M.P. (the F-process). The state space is F. Its σ-algebra Σ_F is the sub-σ-algebra of Σ consisting of all subsets of F that are in Σ. The initial measure is $m_F(A) = m(A)/m(F)$ for all $A \in \Sigma_F$, and a transition probability function which can be derived from $P(x,A)$ by the formula

$$(1.5) \quad P_F(x,A) = P(x,A) + \int_{F^c} P(x,dy)P(y,A) + \int_{F^c}\int_{F^c} P(x,dy)P(y,dz)P(z,A) + \ldots$$

where F^c denotes the compliment of the set F. $P_F(x,A)$ satisfies the requirements (0.1) and (0.2) in the definition of a t.p.f.

We now begin the process of defining a measure on the F-process. Almost all $\omega \in \bar{X}$ with $x_0 \in F$ yield a point $\omega_F = (x_0, x_{i_1}, x_{i_2} \ldots) \in \bar{F}$, the sequence space of the F-process. Let $\sum_{k=0}^{n} 1_{X_k \in F}(\omega) = j_n(\omega)$ so that for $A \subset F$ we can write

$$(1.6) \quad \sum_{k=0}^{n} 1_{X_k \in A}(\omega) = \sum_{k=0}^{j_n(\omega)} 1_{Y_k \in A}(\omega) .$$

This is true since $Y_k \in A$ if and only if some corresponding $X_j \in A$.

Using (1.6) we have

$$(1.7)\qquad \frac{\sum_{k=0}^{n} 1_{X_k\in A}(\omega)}{\sum_{k=0}^{n} 1_{X_k\in F}(\omega)} = \frac{\sum_{k=0}^{j_n(\omega)} 1_{Y_k\in A}(\omega)}{j_n(\omega)} .$$

Condition (1.3) implies $j_n(\omega) \to \infty$ for a.e. ω, with $x_0 \in F$, and it goes to infinity through the natural numbers. Hence $j_n(\omega)$ can be replaced by r so the right side of (1.7) becomes

$$(1.8)\qquad \frac{\sum_{k=0}^{r} 1_{Y_k\in A}(\omega_F)}{r} .$$

Notice that the ω can now be replaced by ω_F. Condition (1.2) says that the left side of (1.7) converges a.e. $(\bar{m})$ hence necessarily does the right side, which implies (1.8) converges a.e. $(\bar{m}_F)$ i.e.

$$(1.9)\qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{j_n(\omega)} 1_{Y_k\in A}(\omega)}{j_n(\omega)} = \lim_{r\to\infty} \frac{\sum_{k=0}^{r} 1_{Y_k\in A}(\omega_F)}{r} .$$

For $A \subset F$ let

$$(1.10)\qquad f_A(\omega_F) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{Y_k\in A}(\omega_F)}{n+1} .$$

By (1.9) f_A is defined for a.e. ω_F thus allowing the following integration

$$(1.11)\qquad \int_{\bar{F}} f_A \, d\bar{m}_F = \int_{\bar{F}} \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{Y_k\in A}}{n+1} \, d\bar{m}_F .$$

Since the ratio on the right side of (1.11) is bounded by the integrable function 1 (recall that $\bar{m}_F$ is a finite measure) the bounded convergence theorem can be applied to (1.11) yielding

$$(1.12)\qquad \int_{\bar{F}} f_A \, d\bar{m}_F = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \int_{\bar{F}} 1_{Y_k \in A} \, d\bar{m}_F}{n+1} .$$

But $1_{Y_k \in A}$ is one precisely at those points where $Y_k \in A$; hence (1.12) can be written as

$$(1.13)\qquad \int_{\bar{F}} f_A \, d\bar{m}_F = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \bar{m}_F(Y_k \in A)}{n+1} .$$

Definition 1.14: Let the measure μ be defined as follows

$$\mu(A) = \int_{\bar{F}} f_A(\omega_F)\bar{m}_F(d\omega_F)$$

which is well-defined for all $A \in \Sigma_F$.

Each finite ratio on the right in (1.13) i.e.

$$(1.15)\qquad \frac{\sum_{k=0}^{n} \bar{m}_F(Y_k \in A)}{n+1}$$

is a probability measure on Σ_F , hence μ is also a finite measure on Σ_F. (That the setwise limit of a sequence of finite measures is a finite measure has been shown, see e.g. [3]). Since (1.15) equals one, for all n , when $A = F$, we see that μ is non-trivial.

The next step is to show that μ is an invariant measure w.r.t. the F-process, i.e. to show

$$(1.16)\qquad \int_F P_F(y,A)\mu(dy) = \mu(A) \quad \forall A \in \Sigma_F .$$

To accomplish this we first prove two lesser results. Let

$\mu_n(A) = \frac{1}{n+1}\left[\bar{m}_F(Y_0 \in A) + \bar{m}_F(Y_1 \in A) + \dots + \bar{m}_F(Y_n \in A)\right]$ so that

$$\mu_n(dy) = \frac{1}{n+1}\left[\bar{m}_F(dy_0) + \bar{m}_F(dy_1) + \dots + \bar{m}_F(dy_n)\right] .$$

Lemma 1.17: For all $A \in \Sigma_F$ we have

$$\lim_{n\to\infty} \int_F P_F(y,A)\mu_n(dy) = \mu(A) .$$

Proof: The measures $\bar{m}_F(dy_i)$ are the coordinate measures in sequence space gotten by restricting $\bar{m}_F$ to each coordinate random variable. Using the definition of $\mu_n(dy)$ above we write

$$\int_F P_F(y,A)\mu_n(dy) = \frac{1}{n+1}\left[\int_F P_F(y,A)\bar{m}_F(dy_0) + \dots + \int_F P_F(y,A)\bar{m}_F(dy_n)\right]$$

$$= \frac{1}{n+1}\left[\bar{m}_F(Y_1 \in A) + \bar{m}_F(Y_2 \in A) + \dots + \bar{m}_F(Y_{n+1} \in A)\right]$$

$$= \mu_n(A) - \frac{\bar{m}_F(Y_0 \in A)}{n+1} + \frac{\bar{m}_F(Y_{n+1} \in A)}{n+1} .$$

Now taking the limit as n → ∞ yields the desired result. Q.E.D.

The function $P_F(y,A)$, for fixed A , can be approximated by simple functions $s_i(y)$ s.t. $0 \le s_1 \le s_2 \le \dots \le P_F(y,A)$ and $\lim_{i\to\infty} s_i(y) = P_F(y,A)$ for each y .

Lemma 1.18: Let $\{s_i(y)\}$ be the simple functions just described, for some fixed A . Then

$$(1.19) \qquad \lim_{n\to\infty} \int_F s_i(y)\mu_n(dy) = \int_F s_i(y)\mu(dy) .$$

Proof: Writing s_i as $\sum_{j=0}^{i} c_{ij} 1_{A_{ij}}$ and integrating yields

$$\int_F s_i d\mu_n = \sum_{j=0}^{i} c_{ij}\mu_n(A_{ij}) .$$

Taking the limit as $n \to \infty$ yields (1.19). Q.E.D.

The next two more important lemmas will be sufficient to prove our theorem.

Lemma 1.20: Under the same hypotheses as Theorem 1.1, a finite invariant measure exists for the process on F .

Proof: Since $P_F(y,A)$ is a bounded function (it is less than or equal to one) the s_i can be chosen uniformly convergent (see e.g. [17]) i.e. $\forall\ \epsilon > 0\ \exists$ an N s.t. $\forall\ i,k > N$ and all y we have $|s_i(y) - s_k(y)| < \epsilon$. Therefore choose any $\epsilon > 0$ and let N be as above. Let $k \geq i > N$; since we are specifying $k \geq i$ and the s_i are non-decreasing functions, absolute values will not be needed in the following. For fixed j we have

$$0 \leq \int_F (s_k - s_i)d\mu_j \leq \int_F \epsilon d\mu_j \qquad \text{or}$$

$$(1.21) \qquad 0 \leq \int_F s_k d\mu_j - \int_F s_i d\mu_j \leq \epsilon\mu_j(F) = \epsilon$$

recalling that μ_j is a probability measure on F for each j . First taking the limit in (1.21) as $k \to \infty$ and using the monotone convergence theorem yields

$$(1.22) \qquad 0 \leq \int_F P_F(y,A)\mu_j(dy) - \int_F s_i\, d\mu_j \leq \epsilon$$

and now taking the limit in (1.22) as $j \to \infty$ and using Lemmas 1.17 and 1.18 gives

$$(1.23) \qquad 0 \le \mu(A) - \int_F s_i \, d\mu \le \epsilon .$$

Finally, taking the limit as $i \to \infty$ and again using the monotone convergence theorem, results in

$$(1.24) \qquad 0 \le \mu(A) - \int_F P_F(y,A)\mu(dy) \le \epsilon .$$

But (1.24) is true for all $\epsilon > 0$, hence

$$\int_F P_F(y,A)\mu(dy) = \mu(A) \quad \forall\, A \in \Sigma_F . \qquad \text{Q.E.D.}$$

<u>Lemma 1.25</u> (Harris): Under the hypotheses of Theorem 1.1, since a finite invariant measure exists for the process on F, a σ-finite invariant measure, π, exists for the original process.

Although the assumptions made in Theorem 1.1 are much weaker than the assumption made by Harris (mentioned at the beginning of this section), conditions (1.3) and (1.4) are sufficient to completely carry through the proof of Lemma 1.25 given by Harris [10; Lemma 1] without any changes. He used his condition for two purposes in this Lemma, (1) to insure the existence of the F-process but this is accomplished by our condition (1.3), and (2) to show that the space could be broken up into a countable number of sets of finite π measure in the following way:

$$(1.26) \qquad S_{ij} = \left\{x \in F^c : P^i(x,F) > 1/j\right\}$$

and this is accomplished by condition (1.4).

The measure π, whose existence is assured by Lemma 1.25, can be defined as follows: the definition of $P_F(y,A)$ for $A \in \Sigma_F$ (see 1.5) shows that the function P_F can be extended to all of Σ (however it would not be a t.p.f. in this instance) and then let

$$\pi(A) = \int_F P_F(y,A)\mu(dy) \quad \forall A \in \Sigma . \tag{1.27}$$

The definition (1.27) shows that $\pi(F) = 1$ and that $\pi(S_{ij}) \leq 1$ for all i,j .

CHAPTER 2: Transition Probability Limit Theorem

In this section we again consider a condition under which it will be shown that a σ-finite invariant measure exists for a general Markov process (X,Σ,m,P) ; The condition here will be similar to (1.2) of Chapter 1, but instead of indicator functions, we will be using the transition probability function.

Condition 2.1: There exists a set $F \in \Sigma$, and an $x_0 \in X$ such that for all $A \in \Sigma$, $A \subset F$, the following limit exists:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} P^k(x_0,A)}{\sum_{k=0}^{n} P^k(x_0,F)} .$$

Recall that $P^k(x,A)$ is the k^{th} step transition probability function, i.e. $P(X_{n+k} \in A \mid X_n = x)$, the probability of arriving in A after k steps given that we start at the point x . Also, since we are always assuming the function $P(x,A)$ is stationary (see Section 0), this k^{th} step function is independent of n .

The first result to be established is a general one. By a corollary of the Hahn-Banach Theorem, there exists a generalized limit (which we will call a Banach Limit (B.L.)) defined on all bounded sequences of real numbers $a_1, a_2, a_3, \ldots$ which will be denoted by $\operatorname{Lim}_n a_n$ [see e.g. 4 , p.73]. Some properties of Lim that will be needed in this and the following chapters are

B.L. 1: linearity: $\operatorname{Lim}_n (\alpha a_n + \beta b_n) = \alpha \operatorname{Lim}_n a_n + \beta \operatorname{Lim}_n b_n$

B.L. 2: positivity: $a_i \geq 0 \ \forall_i \Longrightarrow \operatorname{Lim}_n a_n \geq 0$

B.L. 3: $\liminf a_n \le \underset{n}{\mathrm{Lim}}\, a_n \le \limsup a_n$

B.L. 4: if $\lim_{n\to\infty} a_n$ exists then $\lim_{n\to\infty} a_n = \underset{n}{\mathrm{Lim}}\, a_n$.

Let $\mu_1, \mu_2, \mu_3, \ldots$ be a sequence of finite measures on (X, Σ) such that $\limsup \mu_n(X) < \infty$. This immediately implies $\limsup \mu_n(A) < \infty$, $\forall A \in \Sigma$ and hence $\mu_1(A), \mu_2(A), \mu_3(A), \ldots$ is a bounded sequence for each A . We will assume that a B.L. has been defined on the set of all bounded sequences of real numbers (there are, in general, more than one possible B.L.).

Definition 2.2.: Let $\mu(A) = \underset{n}{\mathrm{Lim}}\, \mu_n(A)$.

If $f(x)$ is a bounded, measurable function, then the sequence

$$\int_X f(x)\mu_1(dx),\ \int_X f(x)\mu_2(dx),\ \ldots \tag{2.3}$$

is a bounded sequence of real numbers so that the B.L. mentioned above can be taken of all such sequences (2.3) i.e., we can consider $\underset{n}{\mathrm{Lim}} \int_X f(x)\mu_n(dx)$. Note that saying that $f \in L_1(\mu_n)$ for all n would not suffice since that would not imply (2.3) is bounded.

Theorem 2.4: Let $\mu_1, \mu_2, \mu_3, \ldots$ be a sequence of finite measures on the measure space (X, Σ) such that $\limsup \mu_n(X) < \infty$. Let $f(x)$ be a bounded, measurable, real-valued function on X . Then

$$\underset{n}{\mathrm{Lim}} \int_X f(x)\mu_n(dx) = \int_X f(x)\mu(dx) .$$

(Note that by B.L. 1, μ is only a finitely additive set function. However, integration with respect to it can still be defined [see e.g., [4], pp. 95-119].)

Proof: The desired result will be arrived at in several steps.

(A) First the theorem will be shown to hold for the case of an indicator function i.e., let $f(x) = 1_A(x)$ for some set $A \in \Sigma$. We have

$$\int_X f d\mu_n = \int_X 1_A d\mu_n = \int_A d\mu_n = \mu_n(A) .$$

Taking the B.L. yields

$$\operatorname*{Lim}_n \int_X f d\mu_n = \operatorname{Lim} \mu_n(A) = \mu(A) ,$$

but $\mu(A)$ is just $\int_X 1_A d\mu$ hence the theorem is valid for indicator functions.

(B) Now let $f(x)$ be a simple function i.e. a linear sum of a finite number of indicator functions. We have $f(x) = \sum_{i=1}^{m} a_i 1_{A_i}(x)$ with $A_i \cap A_j = \emptyset$ for $i \neq j$. Integrating as in part (A) leads to

$$\int_X f d\mu_n = \int_X \sum_{i=1}^{m} a_i 1_{A_i} d\mu_n$$

$$= \sum_{i=1}^{m} a_i \int_X 1_{A_i} d\mu_n$$

$$= \sum_{i=1}^{m} a_i \mu_n(A_i)$$

and as before, taking the B.L. (and using the finite addivity of Lim) allows us to write

$$\operatorname*{Lim}_n \int_X f d\mu_n = \operatorname*{Lim}_n \sum_{i=1}^{m} a_i \mu_n(A_i)$$

$$= \sum_{i=1}^{m} a_i \mu(A_i)$$

$$= \int_X f \, d\mu \, .$$

Hence the theorem holds for simple functions.

(C) We can and will assume, without loss of generality, that $f(x)$ is non-negative. Also, since f is a bounded function, it can be approximated uniformly by an increasing sequence of simple functions $0 \leq s_1 \leq s_2 \leq s_3 \leq \ldots \leq f$ such that $\lim_{i\to\infty} s_i(x) = f(x)$ for each x.

As was done in Chapter 1, we have that for any $\epsilon > 0$ there exists a positive integer N such that for all $i \geq j > N$ we have $0 \leq s_i(x) - s_j(x) \leq \epsilon$ for all x. Note again that absolute values are not needed since we specify $i \geq j$. We can now write

$$0 \leq \int_X s_i \, d\mu_k - \int_X s_j \, d\mu_k \leq \epsilon\mu_k(x) \, .$$

Taking the limit here as $i \to \infty$ and applying the monotone convergence theorem yields

$$0 \leq \int_X f \, d\mu_k - \int_X s_j d\mu_k \leq \epsilon \, \mu_k(X) \, . \tag{2.5}$$

In (2.5) we take $\operatorname*{Lim}_k$ to get

$$0 \leq \operatorname*{Lim}_k \int_X f \, d\mu_k - \operatorname*{Lim}_k \int_X s_j \, d\mu_k \leq \epsilon \, \mu(X) \, , \tag{2.6}$$

and using part (B) of this theorem on (2.6) results in

$$0 \leq \operatorname*{Lim}_k \int_X f \, d\mu_k - \int_X s_j \, d\mu \leq \epsilon \, \mu(X) \, . \tag{2.7}$$

We cannot take the limit in (2.7) as $j \to \infty$ and apply the monotone convergence since μ is not countably additive. However, this difficulty is easily bypassed.

Using the fact that the s_i are uniformly convergent to f we can write

$$(2.8) \qquad 0 \le \int_X f\, d\mu - \int_X s_j\, d\mu \le \epsilon\, \mu(X) .$$

Taking the limit as $j \to \infty$ in (2.8) yields

$$(2.9) \qquad 0 \le \int_X f\, d\mu - \lim_{j\to\infty} \int_X s_j\, d\mu \le \epsilon\, \mu(X)$$

recalling that $s_j \uparrow$ so this limit exists. Since $\mu(X) < \infty$ (by B.L. 3, we have $\mu(X) = \operatorname{Lim}_n \mu_n(X) \le \limsup \mu_n(X) < \infty$) and (2.9) is true for all ϵ we have

$$\int_X f\, d\mu = \lim_{j\to\infty} \int_X s_j d\mu .$$

Now continuing with the proof by taking the limit as $j \to \infty$ in (2.7) we can use this result to get

$$(2.10) \qquad 0 \le \operatorname{Lim}_k \int_X f\, d\mu_k - \int_X f\, d\mu \le \epsilon\, \mu(X) .$$

Since (2.10) is true for all $\epsilon > 0$ $(\mu(X) < \infty)$ we have

$$\operatorname{Lim}_k \int_X f\, d\mu_k = \int_X f\, d\mu . \qquad \text{Q.E.D.}$$

Note that this theorem is also true with the region of integration being any set A as long as $\limsup \mu_n(A) < \infty$. The main result of this chapter is

<u>Theorem 2.11</u>: Let (X, Σ, m, P) be a M.P. Assume there exists a set $F \in \Sigma$, $m(F) > 0$, and an $x_0 \in X$ such that

$$(2.12) \qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{n} P^k(x_0, A)}{\sum_{k=0}^{n} P^k(x_0, F)}$$

exists for all $A \subset F$, $A \in \Sigma$. Also let this set F satisfy $\sum_{k=0}^{\infty} P^k(x,F) = \infty$ for all $x \in X$. Then there exists a σ-finite measure π satisfying $\int_X P(x,A)\pi(dx) = \pi(A)$ for all $A \in \Sigma$.

The idea of the proof of Theorem 2.11 will be to define a content, μ, on a subset of Σ and then to build from μ an invariant measure on all of Σ. Let $\mu_n(x_0,A)$ be the ratio in (2.12). Let $\bar{\Sigma}$ be the subset of Σ containing all sets $A \subset X$ such that $\limsup \mu_n(x_0,A) < \infty$; note that by hypothesis this implies all $A \in \Sigma$ with $A \subset F$ are in $\bar{\Sigma}$. For $A \in \bar{\Sigma}$, therefore, we have that the sequence $\mu_1(x_0,A), \mu_2(x_0,A), \ldots$ is bounded and hence assuming a B.L. has been defined on the set of all bounded sequences, we have

Definition 2.13: Let $\mu(x_0,A) = \underset{n}{\mathrm{Lim}}\, \mu_n(x_0,A)$ for $A \in \bar{\Sigma}$.

Since, by Theorem 2.11, we are only considering this one value x_0, we will write $\mu(x_0,A)$ and $\mu_n(x_0,A)$ simply as $\mu(A)$ and $\mu_n(A)$ respectively. μ is only defined for sets in $\bar{\Sigma}$ and its extension to the invariant measure π will be accomplished in two steps. First we extend μ to all of Σ by

Definition 2.14:

$$\text{Let } \mu^*(A) \begin{cases} = \mu(A) & \text{for } A \in \bar{\Sigma} \\ = \infty & \text{for } A \notin \bar{\Sigma} \end{cases}$$

However μ^* as defined need not be a measure. We now show

Lemma 2.15: μ^* is a finitely additive set function on Σ.

Proof: First it will be shown that $\bar{\Sigma}$ is closed under finite unions; for $A,B \in \bar{\Sigma}$ it must be true (by definition) that $\limsup \mu_n(A) < \infty$ and $\limsup \mu_n(B) < \infty$, so certainly it is true that $\limsup \mu_n(A \cup B) < \infty$ hence $A \cup B \in \bar{\Sigma}$.

To prove the lemma, let $A,B \in \bar{\Sigma}$ with $A \cap B = \emptyset$. Therefore,

$\mu^*(A) = \mu(A)$ and $\mu^*(B) = \mu(B)$ and since μ is finitely additive (B.L. 1), we have $\mu^*(A \cup B) = \mu(A \cup B) = \mu(A) + \mu(B) = \mu^*(A) + \mu^*(B)$.

If either A or B (or both) is not in $\bar{\Sigma}$, say $A \notin \bar{\Sigma}$, then by definition of $\bar{\Sigma}$, $\limsup \mu_n(A) = \infty$; hence $\limsup \mu_n(A \cup B) = \infty$ so $A \cup B \notin \bar{\Sigma}$ thus $\mu^*(A \cup B) = \infty = \mu^*(A) + \mu^*(B)$. Q.E.D.

Now we move from a finitely additive function to one that is countably additive (i.e. a measure).

Definition 2.16: Let $\pi(A)$ be the infimum over the set $\left\{ \sum_{i=1}^{\infty} \mu^*(A_i) : \bigcup_{i=1}^{\infty} A_i \supset A \right\}$ over all such countable unions containing A.

It has been shown by Isaac [12] that π as defined here is indeed a countably additive measure on Σ. We now prove a result concerning μ which will be needed later to show π is a sub-invariant measure.

Theorem 2.17: Assume the conditions of Theorem 2.11 are in effect. Let $A \in \bar{\Sigma}$, then

$$\lim_n \int_X P^i(x,A)\mu_n(dx) = \mu(A), \quad i = 1,2,3,\ldots .$$

Proof: The proof for the case $i = 1$ will be given, the technique for the other values of i will be similar. By the definition of $\mu_n(A)$ we have

$$\mu_n(dx) = \frac{P^0(x_0,dx) + P^1(x_0,dx) + \ldots + P^n(x_0,dx)}{\sum_{k=0}^{n} P^k(x_0,F)} .$$

We perform the following integration (noting that $P^0(x_0,dx)$ is simply $1_{x_0}(dx)$)

$$\int_X P(x,A)\mu_n(dx) = \frac{\sum_{k=0}^{n} \int_X P(x,A)P^k(x_0,dx)}{\sum_{k=0}^{n} P^k(x_0,F)}$$

$$= \frac{\sum_{k=1}^{n+1} P^k(x_0,A)}{\sum_{k=0}^{n} P^k(x_0,F)}$$

$$(2.18) \qquad = \mu_n(A) + \frac{P^{n+1}(x_0,A) - P^0(x_0,A)}{\sum_{k=0}^{n} P^k(x_0,F)} .$$

Taking the B.L. of both sides of this last result yields

$$\operatorname*{Lim}_n \int_X P(x,A)\mu_n(dx) = \operatorname*{Lim}_n \mu_n(A) = \mu(A) .$$

To arrive at this we have used B.L. 1, and the fact that $\sum_{k=0}^{\infty} P^k(x,F) = \infty$ for all x (x_0 in particular here) forces the fraction in (2.18) to go to 0 . Q.E.D.

Condition 2.1 has not been used yet. Certainly if π is the trivial measure 0 it will be invariant (and very uninteresting). Condition 2.1 will show that this is not the case.

Theorem 2.19: π is non-trivial.

Proof: It will be shown that $\pi(F) = 1$. Condition 2.1 implies μ is countably additive on F . To see this note that for $A \subset F$, $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ (by B.L. 4) i.e., μ on F equals the usual limit of a sequence of countably additive finite measures $\left(\mu_n\Big|_F\right)$ and the Vitali-Hahn-Saks Theorem [see, e.g. [7] ; p. 32] then shows that μ is a measure (on F) and hence is countably additive (on F) . (In our specific situation it is not difficult to show μ is countably additive by direct and simpler methods than used in the proof of the VHS Theorem.)

Let $\{A_i\}$ be any countable covering of F ; let $B_i = A_i \cap F$ so that $\bigcup_{i=1}^{\infty} B_i = F$; let $C_1 = B_1$, $C_2 = B_2 - B_1$, $C_3 = B_3 - \bigcup_{i=1}^{2} B_i$, and in general, $C_n = B_n - \bigcup_{i=1}^{n-1} B_i$ so that the C_i are disjoint and still satisfy $\bigcup_{i=1}^{\infty} C_i = F$. Recall that all measurable subsets of F are in $\bar{\Sigma}$ so that $\mu^*(C_i) = \mu(C_i)$ by definition of μ^* . Since μ^* and μ agree on all measurable subsets of F , μ^* is also countably additive on F . Using this countable additivity we have $\sum_{i=1}^{\infty} \mu^*(C_i) = \mu^*(F) = \mu(F) = 1$ ($\mu(F) = 1$ by (2.12) and 2.13). Also $\mu^*(A_i) \geq \mu^*(C_i)$ since $A_i \supset C_i$ by construction and in general A_i need not be in $\bar{\Sigma}$. Therefore, $\sum_{i=1}^{\infty} \mu^*(A_i) \geq \sum_{i=1}^{\infty} \mu^*(C_i) = 1$.

By the preceeding construction it is possible to obtain such a sequence $\{C_i\}$ for each such countable covering $\{A_i\}$ of F , hence for each such covering we have shown $\sum_{i=1}^{\infty} \mu^*(A_i) \geq 1$. So necessarily, by the definition of π being the infimum of these summations over all of these coverings, we have $\pi(F) \geq 1$. In fact, there exists a cover such that $\sum_{i=1}^{\infty} \mu^*(A_i) = 1$, namely F itself. Q.E.D.

The following combinatoric lemma will be needed for the next theorem.

Lemma 2.20: Let a_{k_n} be a countable collection of non-negative numbers. Then for any Banach limit, Lim, we have $\operatorname{Lim}_{n} \sum_{k=1}^{\infty} a_{k_n} \geq \lim_{j \to \infty} \operatorname{Lim}_{n} \sum_{k=1}^{j} a_{k_n}$.

Proof: We have, for fixed n , $\sum_{k=1}^{\infty} a_{k_n} - \sum_{k=1}^{j} a_{k_n} \geq 0 \ \forall k,j$. Using B.L. 2 and then B.L. 1 yields

$$(2.21) \qquad \operatorname{Lim}_{n} \sum_{k=1}^{\infty} a_{k_n} - \operatorname{Lim}_{n} \sum_{k=1}^{j} a_{k_n} \geq 0 .$$

Taking the limit as $j \to \infty$ and using the fact that (2.21) is true for all j gives

$$\operatorname*{Lim}_n \sum_{k=1}^{\infty} a_{k_n} - \lim_{j\to\infty} \operatorname*{Lim}_n \sum_{k=1}^{j} a_{k_n} \geq 0$$

which is the desired result. Q.E.D.

Theorem 2.22: $\int_X P^i(x,A)\pi(dx) \leq \pi(A)$, $A \in \Sigma$, $\forall\ i \geq 0$.

(Note that this is equivalent to saying that π is a subinvariant measure.)

Proof: First we assume $A \in \overline{\Sigma}$. Let $B^i = \{x\colon P^i(x,A) > 0\}$ and let $B_k^i = \{x\colon \frac{1}{k} \geq P^i(x,A) > \frac{1}{k+1}\}$ for $k = 1,2,3,\ldots$; note that $\bigcup_{k=1}^{\infty} B_k^i = B^i$ and $B_k^i \cap B_j^i = \emptyset$ for $k \neq j$. We now show that $B_k^i \in \overline{\Sigma}$ for every i,k . We have

$$\begin{aligned}\frac{1}{k+1}\mu_n\left(B_k^i\right) &= \int_{B_k^i} \frac{1}{k+1}\mu_n(dx)\\ &\leq \int_{B_k^i} P^i(x,A)\mu_n(dx)\\ &\leq \int_X P^i(x,A)\mu_n(dx)\ .\end{aligned}$$

Since we have assumed $A \in \overline{\Sigma}$, the B.L. of this last integral exists (by Theorem 2.17) and equals $\mu(A) < \infty$. Therefore $\limsup_n \mu_n\left(B_k^i\right) < \infty$ implying $B_k^i \in \overline{\Sigma}$ $\forall\ i,k$. Hence, $\mu^*\left(B_k^i\right) = \mu\left(B_k^i\right) < \infty$.

Using the countable addivity of μ_n we write

$$\int_X P^i(x,A)\mu_n(dx) = \int_{B^i} P^i(x,A)\mu_n(dx)$$

$$= \sum_{k=1}^{\infty} \int_{B_k^i} P^i(x,A)\mu_n(dx) \ ,$$

so
$$\operatorname*{Lim}_n \int_X P^i(x,A)\mu_n(dx) = \operatorname*{Lim}_n \sum_{k=1}^{\infty} \int_{B_k^i} P^i(x,A)\mu_n(dx)$$

$$\text{(2.23)} \qquad \geq \lim_{j\to\infty} \operatorname*{Lim}_n \sum_{k=1}^{j} \int_{B_k^i} P^i(x,A)\mu_n(dx) \ .$$

To get (2.23) we have used Lemma 2.20 with a_{k_n} equal to $\int_{B_k^i} P^i(x,A)\mu_n(dx)$. Continuing from (2.23) we use the finite addivity of Lim to get (2.24)

$$\text{(2.24)} \qquad \geq \lim_{j\to\infty} \sum_{k=1}^{j} \operatorname*{Lim}_n \int_{B_k^i} P^i(x,A)\mu_n(dx)$$

$$\text{(2.25)} \qquad = \lim_{j\to\infty} \sum_{k=1}^{j} \int_{B_k^i} P^i(x,A)\mu(dx)$$

and (2.25) comes from Theorem 2.4 (see the note at the end of the proof of 2.4). Since $\mu^* = \mu$ on $\overline{\Sigma}$ we have (2.26)

$$\text{(2.26)} \qquad = \lim_{j\to\infty} \sum_{k=1}^{j} \int_{B_k^i} P^i(x,A)\mu^*(dx)$$

$$\text{(2.27)} \qquad \geq \lim_{j\to\infty} \sum_{k=1}^{j} \int_{B_k^i} P^i(x,A)\pi(dx)$$

with (2.27) true since $\pi \leq \mu^*$ by definition of π . Continuing from (2.27) using the disjointness of the B_k^i (for different k) we have

$$\text{(2.28)} \qquad = \lim_{j\to\infty} \int_{\bigcup_{k=1}^{j} B_k^i} P^i(x,A)\pi(dx)$$

$$= \int_{B^i} P^i(x,A)\pi(dx) \tag{2.29}$$

and (2.29) comes from the countable addivity of π. So we have thusfar shown

$$\operatorname*{Lim}_n \int_X P^i(x,A)\mu_n(dx) \geq \int_{B^i} P^i(x,A)\pi(dx) \tag{2.30}$$

$$= \int_X P^i(x,A)\pi(dx) .$$

The left side of (2.30) is $\mu(A)$ by Theorem 2.17 and since $A \in \overline{\Sigma}$ we have $\mu^*(A) = \mu(A)$, so for any $A \in \overline{\Sigma}$, we have

$$\mu^*(A) \geq \int_X P^i(x,A)\pi(dx) . \tag{2.31}$$

However, if $A \notin \overline{\Sigma}$ then by definition we have $\mu^*(A) = \infty$ so (2.31) is certainly true for these sets also, hence (2.31) is true for all $A \in \Sigma$. But $\int_X P^i(x,A)\pi(dx)$ is a measure on Σ, and it has been shown [12; p. 988] that the measure π as defined is the largest measure less than or equal to μ^*, therefore it must be true that

$$\pi(A) \geq \int_X P^i(x,A)\pi(dx) .$$ Q.E.D.

Theorem 2.32: π is a σ-finite measure.

Proof: Let $W_{ij} = \left\{x: \frac{1}{j} \geq P^i(x,F) > \frac{1}{j+1}\right\}$ $i = 0,1,2,3,\ldots$, $j = 1,2,3,\ldots$.

By the hypothesis $\sum_{k=0}^{\infty} P^k(x,F) = \infty$ for all $x \in X$ we have $\bigcup_{i,j} W_{ij} = X$.

Therefore it will suffice to show $\pi(W_{ij}) < \infty$ for all i,j. We have

$$\int_X P^i(x,F)\pi(dx) \geq \int_{W_{ij}} P^i(x,F)\pi(dx)$$

$$\geq \int_{W_{ij}} \frac{1}{j+1} \pi(dx)$$

$$= \frac{1}{j+1} \pi(W_{ij})$$

and using Theorem 2.22 allows us to write

$$\pi(F) \geq \frac{1}{j+1} \pi(W_{ij})$$

but $\pi(F) = 1$ hence $\pi(W_{ij}) < \infty \ \forall \ i,j$. Q.E.D.

Note that the hypothesis $\sum_{k=1}^{\infty} P^k(x,F) = \infty$ for all x is stronger than really necessary here. Although this condition will be needed in the next theorem, all that was actually required for Theorem 2.32 was the existence, for each x , of an i s.t. $P^i(x,F) > 0$. This condition could not be weakened to hold only for almost all x since there would be no assurance that $\pi\{x: P^i(x,F) = 0 \ \forall \ i\} < \infty$, which would be necessary for σ-finiteness.

Now the final lemma whose conclusion is the desired result.

Lemma 2.33: π is an invariant measure, i.e.,

$$\int_X P(x,A)\pi(dx) = \pi(A) \qquad \forall \ A \in \Sigma .$$

Proof: Let

$$\pi P(A) = \int_X P(x,A)\pi(dx)$$

so that

$$\pi P(dy) = \int_X P(x,dy)\pi(dx) .$$

Theorem 2.22 says $\pi(A) \geq \pi P(A)$. To show $\pi(A) = \pi P(A)$ for all A we use a proof by contradiction and assume there exists a set $A \in \Sigma$ s.t. $\pi(A) > \pi P(A)$. We now show that it suffices to assume $\pi(A) < \infty$. By σ-finiteness of π , there exists a countable collection

$\{A_i\}$ disjoint such that $\cup A_i = A$ and $\pi(A_i) < \infty$. One of these A_i must satisfy $\pi(A_i) > \pi P(A_i)$, for if not we would have $\pi(A_i) = \pi P(A_i)$ for each i. Then, since $\lim_{n\to\infty} \pi\left(\bigcup_{i=1}^{n} A_i\right) = \pi(A)$ and $\lim_{n\to\infty} \pi P\left(\bigcup_{i=1}^{n} A_i\right) = \pi P(A)$, $\pi(A) = \infty$ would imply $\pi P(A) = \infty$, but A was chosen s.t. $\pi(A) > \pi P(A)$. Hence, even if $\pi(A) = \infty$ an A_i could be found, $\pi(A_i) < \infty$, that would still satisfy $\pi(A_i) > \pi P(A_i)$.

Let $\{B_i\}$ be a countable disjoint decomposition of X into sets of finite measure for both π and πP, such that $B_1 = A$ of the last paragraph. Then for fixed N

$$
\begin{aligned}
1 = \pi(F) &\geq \pi P(F) \\
&\geq \int_X P(x,F)\pi(dx) - \int_X P^N(x,F)\pi P(dx) \\
&= \int_X \sum_{k=1}^{N} P^k(x,F)\pi(dx) - \int_X \sum_{k=1}^{N} P^k(x,F)\pi P(dx)
\end{aligned}
$$

$$
(2.34) \qquad = \sum_{i=1}^{\infty} \int_{B_i} \sum_{k=1}^{N} P^k(x,F)\pi(dx) - \sum_{i=1}^{\infty} \int_{B_i} \sum_{k=1}^{N} P^k(x,F)\pi P(dx)
$$

$$
(2.35) \qquad = \sum_{i=1}^{\infty} \int_{B_i} \sum_{k=1}^{N} P^k(x,F)(\pi - \pi P)(dx) .
$$

Note that (2.35) is well-defined since π and πP are finite on each B_i so $(\pi - \pi P)(dx)$ makes sense. Also note that each of the two infinite sums in (2.34) being finite in value implies that the entire collection of elements in (2.34) is absolutely convergent. Hence the rearrangement of terms yielding (2.35) gives the same sum. Since $\pi \geq \pi P$, each element in (2.35) is non-negative so (2.35) is

$$(2.36) \qquad \geq \int_{B_1} \sum_{k=1}^{N} P^k(x,F)(\pi - \pi P)(dx) \ .$$

But by hypothesis $\sum_{k=1}^{N} P^k(x,F) \to \infty$ as $N \to \infty$ $\forall\, x$ and B_1 was chosen s.t. $(\pi - \pi P)(B_1) > 0$ so (2.36) necessarily goes to infinity. However, this is a contradiction since (2.36) is bounded above by 1 .

Hence only equality can hold in $\pi(A) \geq \pi P(A)$ giving the desired result. Q.E.D.

Notes: (1) In Chapter 1 the plan of attack was to define the F-process and then show a finite invariant measure existed for this process. In this chapter we could still have defined the process on F , but because the denominator in condition 2.1 was the sum of transition probability functions instead of indicator functions (as in Chapter 1), conversion of 2.1 to a condition on the F-process was not as simple as before.

(2) In 2.1 it was assumed that the limit existed for some fixed value x_0 . If, however, 2.1 was true for more than one value of x then the definition of μ in 2.13 could depend on the x chosen when taking the B.L. Fixing a different value of x might result in a different invariant measure. The uniqueness of an invariant measure will be discussed in a later chapter.

(3) The measure m in Chapter 1 was used in a very weak sense but in Chapter 2 it was not used at all.

CHAPTER 3: Liminf Theorems

In this chapter we will exhibit two conditions similar to (1.2) and (2.1) for which we will again show a σ-finite invariant measure necessarily exists for the M.P. (X,Σ,m,P). We begin with

Condition 3.1: There exists a set $F \in \Sigma$, $m(F) > 0$, and a δ, $0 < \delta < 1$, such that for all $A \subset F$, $A \in \Sigma$ with $m(A) > \delta$ we have

$$\bar{m}\left(\omega: \liminf_{n} \frac{\sum_{k=0}^{n} 1_{X_k \in A}(\omega)}{\sum_{k=0}^{n} 1_{X_k \in F}(\omega)} > 0,\ X_0 \in F\right) > 0 .$$

(Recall that a bar over the measure indicates that we are refering to sequence space.)

To show that Condition 3.1 implies the existence of the desired measure we will again prove that a finite invariant measure exists for the process $\{Y_i\}$ on F. To do this we will assume

$$P(X_n \in F \text{ infinitely often} \mid X_0 = x) = 1 \quad \text{a.e.} \quad x \in F$$

to make sure that we can indeed talk about the F-process.

Before proceeding to the proof of our statement, we would like to remark on the similarity of Condition 3.1 with Condition 3.2 below, found in a paper by Ito [15; Condition III]:

Condition 3.2: $m(A) > 0$ implies

$$\liminf_{n} \frac{\sum_{k=0}^{n} m_F(Y_k \in A)}{n+1} > 0 .$$

We make note of the similarity in the following way; if we were to state a condition analogous to 3.1 but only involving the F-process, it would be

Condition 3.3: There exists a δ, $0 < \delta < 1$, such that for all $A \in \Sigma_F$ with $m_F(A) > \delta$ we have

$$\bar{m}_F(\omega_F: \liminf_{n} \frac{\sum_{k=0}^{n} 1_{Y_k \in A}(\omega_F)}{n+1} > 0) > 0 .$$

(Recall that Σ_F and m_F are the σ-algebra and measure respectively of the F-process introduced in Chapter 1.)

The denominator in 3.1 has become $n+1$ in 3.3 because in the F-process $\sum_{k=0}^{n} 1_{Y_k \in F}(\omega_F)$ would just be counting the total number of steps made. Using 3.3 and Fatou's Lemma we can write

$$0 < \int_{\bar{F}} \liminf_{n} \left(\frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1}\right) d\bar{m}_F$$

$$\leq \liminf_{n} \int_{\bar{F}} \frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1}\, d\bar{m}_F$$

$$= \liminf_{n} \left(\frac{\sum_{k=0}^{n} \int_{\bar{F}} 1_{Y_k \in A}\, d\bar{m}_F}{n+1}\right)$$

$$(3.4) \qquad = \liminf_{n} \left(\frac{\sum_{k=0}^{n} \bar{m}_F(Y_k \in A)}{n+1}\right) .$$

Therefore we have, by (3.4), that $m_F(A) > \delta$ implies

$$\liminf_{n} \left(\frac{\sum_{k=0}^{n} m_F(Y_k \in A)}{n+1}\right) > 0 .$$

Hence we can say that Condition 3.3 (and 3.1) is more general than Ito's Condition 3.2 in two respects. First, and most obvious, is that his condition must be satisfied for all sets of positive measure whereas our requirement is that the condition be valid only for all sufficiently large sets, i.e. $m(A) > \delta > 0$. Second, we have that 3.3 is true for the sum of indicator functions rather than the more specialized m_F function.

Although Ito stops when he produces an invariant measure that is necessarily finite, and equivalent to the initial measure m, we do not require such a strong result. It will suffice here to show the existence of some finite invariant measure for the F-process which will eventually yield a σ-finite invariant measure for the original process.

We now state the first main theorem, the proof of which will comprise the first half of this chapter.

Theorem 3.5: Let Condition 3.1 hold. Let

$$P(X_n \in F \text{ infinitely often} \mid X_0 = x) = 1 \quad \text{for a.e. } x \in F\ ;$$

assume for each x there exists an i s.t. $P^i(x,F) > 0$. Then there exists a finite invariant measure for the F-process.

The proof will be given in several parts:

I: A finite content μ will be defined on (F,Σ_F).

II: We show this content is invariant.

III: An invariant measure will be extracted from this content.

I: For $A \subset F$, let

$$M_n(A) = \int_{\bar{F}} \frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1}\, d\bar{m}_F = \frac{\sum_{k=0}^{n} \bar{m}_F(Y_k \in A)}{n+1}\ .$$

Without loss of generality we assume $m(F) = 1$, so $M_n(A) \leq 1$ for each n. Therefore we can assume that a Banach limit has been defined on sequences of the form $M_1(A), M_2(A), M_3(A), \ldots$ for all $A \in \Sigma_F$. (Note that our assumption that $m(F) = 1$ does not lead to any difficulties; i.e., if $m(F) < \infty$ the measure m can be adjusted to give

$m(F) = 1$ (in fact m_F is such a measure). If, however, $m(F) = \infty$, then m can be replaced by an equivalent (even finite) measure m' such that $m'(F) = 1$. Since the initial measure plays a somewhat minor role, an equivalent measure replacing the initial measure would not introduce any complications.)

Definition 3.6: Let $\mu(A) = \operatorname{Lim}_n M_n(A)$, $A \in \Sigma_F$.

We see immediately that μ is non-trivial since $\lim_{n\to\infty} M_n(F) = 1$ making $\mu(F) = 1$ by B.L. 4. By B.L. 1 we see that μ is finitely additive on Σ_F, therefore by definition it is a finite content.

II: The technique that shows μ is invariant is almost identical to the one used in Chapter 1 to show there that the measure μ was invariant. The only difference for the present situation is that in several instances the B.L. would be taken instead of the regular limit, but this will not affect the results obtained. For completeness, we state the required results here without proof.

Lemma 3.7: (See Lemma 1.17).

$$\operatorname{Lim}_n \int_F P_F(y,A)M_n(dy) = \mu(A) .$$

Lemma 3.8: (See Lemma 1.18).

$$\operatorname{Lim}_n \int_F s_i(y)M_n(dy) = \int_F s_i(y)\mu(dy) .$$

The proofs of these two Lemmas carry over completely since B.L. is finitely additive.

Lemma 3.9: (See Lemma 1.20). Under the same hypotheses as Theorem 3.5, μ, as defined in 3.6, is an invariant content.

Here too the proof carries over completely except going from (1.22) to (1.23) where a B.L. is now required.

III: We now proceed to show the existence of the invariant measure. To do this we will need several lemmas. The first two are due to Yosida and Hewitt [18].

Definition 3.10: A content μ is called purely finitely additive (p.f.a.) if the relation $0 \le \alpha \le \mu$ for a measure (i.e. countably additive) α implies that $\alpha \equiv 0$. Here the inequality is the standard lattice ordering of set functions.

Lemma 3.11: If μ is a finite content and is p.f.a., and if m is a finite measure with μ and m defined on a σ-algebra Σ, then for all $\epsilon > 0$ there exists an $S \in \Sigma$ such that $\mu(S^c) = 0$ and $m(S) < \epsilon$.

Lemma 3.12: If μ is a finite content on a σ-algebra Σ, then there exists a unique decomposition $\mu = \mu_c + u_f$ where μ_c is a measure and μ_f is a p.f.a. content.

Note that the measure μ_c of 3.12 is maximal in the sense of the following lemma.

Lemma 3.13: If α is any measure such that $\mu \ge \alpha$, then $\mu_c \ge \alpha$.

Proof: Since $\mu \ge \alpha$ we can write $\mu = \alpha + \mu_\alpha$. If μ_α is a p.f.a. content, then by the uniqueness of the decomposition of Lemma 3.12 we must have $\mu_c = \alpha$.

If, however, μ_α is not p.f.a., then by Lemma 3.11 it can be decomposed, to wit, $\mu_\alpha = \mu_{\alpha c} + \mu_{\alpha f}$ (where $\mu_{\alpha c}$ is the measure and $\mu_{\alpha f}$ is p.f.a.). We are now able to write $\mu = \alpha + \mu_{\alpha c} + \mu_{\alpha f}$, but $\alpha + \mu_{\alpha c}$ is a measure and again by uniqueness $\mu_f = \mu_{\alpha f}$; or, more significantly, it must be true that $\mu_c = \alpha + \mu_{\alpha c}$ yielding $\mu_c \ge \alpha$. Q.E.D.

A fact that will be needed later is $\bar{m}(\omega: Y_i \in A) = \bar{m}_F(\omega_F: Y_i \in A)$ for $A \subset F$. Recall that the variables $\{Y_i\}$ were originally defined on the space X so both sets make sense. These sets originate from different spaces and although it is perhaps obvious that the equality should hold, a proof will be given for completeness.

Lemma 3.14: For $A \subset F$, $\bar{m}(\omega\colon Y_i \in A) = \bar{m}_F(\omega_F\colon Y_i \in A)$.

Proof: The method will be illustrated for the case when $i = 1$ (for $i > 1$ the technique is similar). The set $\{\omega\colon Y_1 \in A\}$ can be broken up into the following disjoint pieces

$$A_1 = \{\omega\colon x_0 \in F,\ x_1 \in A\}$$

$$A_2 = \{\omega\colon x_0 \in F,\ x_1 \in F^c,\ x_2 \in A\}$$

$$A_3 = \{\omega\colon x_0 \in F,\ x_1 \in F^c,\ x_2 \in F^c,\ x_3 \in A\}$$

$$\vdots$$

by looking at only those possibilities where the first entry into F (after x_0) occurs in A . Note that the sets A_i are in Σ . We have

$$\begin{aligned}
\bar{m}(Y_1 \in A) &= \bar{m}(A_1) + \bar{m}(A_2) + \dots \\
&= \int_F P(x,A)m(dx) + \int_{F^c}\int_F P(x,dy)P(y,A)m(dx) + \dots \\
&= \int_F \Big[P(x,A) + \int_{F^c} P(x,dy)P(y,A) + \\
&\qquad \int_{F^c}\int_{F^c} P(x,dy)P(y,dz)P(z,A) + \dots\Big]m(dx)
\end{aligned}$$

$$= \int_F P_F(x,A)m(dx) \tag{3.15}$$

$$= \int_F P_F(x,A)m_F(dx) = \bar{m}_F(Y_1 \in A) . \tag{3.16}$$

We used the definition of $P_F(x,A)$ to get (3.15) and the facts that $m(F) = 1$ and $m_F(A) = m(A)/m(F)$ to get (3.16). Q.E.D.

Lemma 3.17: $m_F(A) > \delta$ implies $\mu(A) > 0$, for the δ in Condition 3.1.

Proof: Note that $m_F(A) > \delta$ implies $m(A) > \delta$ and Condition 3.1 allows us to say that, for such a set A,

$$\bar{m}\left(\liminf_n \frac{\sum_{k=0}^{n} 1_{X_k \in A}}{\sum_{k=0}^{n} 1_{X_k \in F}} > 0 \,,\ X_0 \in F \right) > 0 \,.$$

Letting $\tilde{F} = FXXXXX\ldots$ we can write

$$(3.18) \qquad 0 < \int_{\tilde{F}} \liminf_n \frac{\sum_{k=0}^{n} 1_{X_k \in A}}{\sum_{k=0}^{n} 1_{X_k \in F}} \, d\bar{m}$$

$$(3.19) \qquad = \int_{\tilde{F}} \liminf_n \frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1} \, d\bar{m}$$

$$(3.20) \qquad \leq \liminf_n \int_{\tilde{F}} \frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1} \, d\bar{m}$$

$$= \liminf_n \left(\frac{\sum_{k=0}^{n} \int_{\tilde{F}} 1_{Y_k \in A} \, d\bar{m}}{n+1} \right)$$

$$= \liminf_n \left(\frac{\sum_{k=0}^{n} \bar{m}(Y_k \in A)}{n+1} \right)$$

$$(3.21) \qquad = \liminf_n \left(\frac{\sum_{k=0}^{n} \bar{m}_F(Y_k \in A)}{n+1} \right)$$

(3.22) $\leq \mu(A)$.

The reasoning for these steps are as follows: The sequence of ratios of which we are taking the liminf in (3.18) might contain duplicate adjacent members for a fixed ω (this would occur when $X_n \notin F$ making $1_{X_k \in A}$ and $1_{X_k \in F}$ zero simultaneously). These duplications are eliminated when we move to the sequence of ratios in (3.19), hence the value of the liminf is unchanged. Fatou's lemma gives (3.20); (3.21) comes from using Lemma 3.14; step (3.22) from the definition of μ as a Banach limit and using B.L. 3. Q.E.D.

Lemma 3.23: μ is not p.f.a.

Proof: For all $A \in \Sigma_F$ we have $m_F(A) + m_F(A^c) = 1$. If for the set A^c it is true that $m_F(A^c) > \delta$, then $m_F(A) + \delta < 1$ or equivalently $m_F(A) < 1-\delta$. Since $m_F(A^c) > \delta$, applying the result of Lemma 3.17 to A^c yields $\mu(A^c) > 0$, but if μ is p.f.a. , this contradicts the conclusion of Lemma 3.11.

More precisely, we have shown there exists a positive ϵ (namely $1-\delta$) such that for all sets A with $m_F(A) < \epsilon$ we have $\mu(A^c) > 0$. Hence μ is not p.f.a. Q.E.D.

Finally, the result we want:

Lemma 3.24: There exists a finite invariant measure for the process on F under the conditions in Theorem 3.5.

Proof: Applying Lemma 3.12 to μ , we write $\mu = \mu_c + \mu_f$. We will show that μ_c is the required measure, i.e., it will be shown that

$$\int_F P_F(x,A)\mu_c(dx) = \mu_c(A)$$

for all $A \in \Sigma_F$. Note that Lemma 3.23 implies μ_c is non-trivial. We have

$$(3.25)\qquad \int_F P_F(x,A)\mu_c(dx) \le \int_F P_F(x,A)\mu(dx)$$

$$(3.26)\qquad = \mu(A)\ .$$

The left side of (3.25) is a measure; (3.25) is true since $\mu_c \le \mu$, (3.26) is true since μ is an invariant content (Lemma 3.9). However, (3.25) says the measure $\int_F P_F(x,A)\mu_c(dx)$ is less than or equal to μ , but by the maximality of the component μ_c established in Lemma 3.13, it must be true that

$$(3.27)\qquad \int_F P_F(x,A)\mu_c(dx) \le \mu_c(A)\ .$$

Assume there exists a set, A , yielding strict inequality in (3.27), i.e.,

$$(3.28)\qquad \int_F P_F(x,A)\mu_c(dx) < \mu_c(A)\ .$$

But it must also be true that

$$(3.29)\qquad \int_F P_F(x,A^c)\mu_c(dx) \le \mu_c(A^c)\ .$$

Adding lines (3.28) and (3.29) gives

$$\int_F P_F(x,F)\mu_c(dx) < \mu_c(F) \qquad \text{or}$$

$$\mu_c(F) < \mu_c(F)\ .$$

A contradiction results, therefore we must have equality in (3.27) for all $A \in \Sigma_F$. Q.E.D.

Now applying Lemma 1.16 results in a σ-finite invariant measure for the original process.

The second half of this chapter involves the same type of change in hypothesis as was made when going from Chapter 1 to Chapter 2. That is, we go from a condition on indicator functions to a similar condition on transition probability functions.

Condition 3.30: There exists a set F, $m(F) > 0$, and a δ, $0 < \delta < 1$, such that for all $A \subset F$, $A \in \Sigma$ with $m(A) > \delta$, we have

$$m\left(x: \liminf_n \frac{\sum_{k=0}^{n} P^k(x,A)}{\sum_{k=0}^{n} P^k(x,F)} > 0 \ , \ x \in F\right) > 0 \ .$$

Again we are weakening a condition of Ito [15 ; condition VIII] with the same results as mentioned previously.

As we found in Chapter 2, the proof using transition probability functions is not as easy. In the first part of this chapter, as in Chapter 1, we were able to show a finite invariant measure for the F-process existed, and then this measure could be extended to the required measure on all of X. Fortunately, however, much of the technique to show there exists a σ-finite invariant measure from Condition 3.30 will come directly from Chapter 2. The proof for this situation will be outlined, putting in the necessary changes. First, we state the main result to be established.

Theorem 3.31: Let Condition 3.30 be in effect. Let $\sum_{k=0}^{\infty} P^k(x,F) = \infty$ for all $x \in X$. Then there exists a σ-finite invariant measure for the Markov process.

As before, we begin by defining a content μ. Let

$$M_n(A) = \int_F \frac{\sum_{k=0}^{n} P^k(x,A)}{\sum_{k=0}^{n} P^k(x,F)} \, m(dx)$$

and let $\bar{\Sigma} = \{A \in \Sigma: \limsup M_n(A) < \infty\}$. We can now talk about a Banach limit of sequences of the form $M_1(A), M_2(A), \ldots$ for $A \in \bar{\Sigma}$. Although the definition of $\bar{\Sigma}$ is slightly different here than in Chapter 2, this difference will not interfere with the mechanics of the proofs.

Definition 3.32: Let $\mu(A) = \operatorname*{Lim}_{n} M_n(A)$.

Following the reasoning from Chapter 2:

Theorem 3.33: Let $A \in \bar{\Sigma}$, $\sum_{k=0}^{\infty} P^k(x,F) = \infty$ for all $x \in X$, then

$$\operatorname*{Lim}_{n} \int_X P(x,A) M_n(dx) = \mu(A) .$$

Proof: Same as Theorem 2.17 except with these slight alterations due to the different definition of μ .

$$M_n(dy) = \int_F \frac{\sum_{k=0}^{n} P^k(x,dy)}{\sum_{k=0}^{n} P^k(x,F)} m(dx)$$

so

$$\int_X P(y,A) M_n(dy) = \int_X P(y,A) \left[\int_F \frac{\sum_{k=0}^{n} P^k(x,dy)}{\sum_{k=0}^{n} P^k(x,F)} m(dx) \right]$$

which becomes, after several simplifications

$$(3.34) \qquad = \int_F \frac{\sum_{k=0}^{n} P^k(x,A)}{\sum_{k=0}^{n} P^k(x,F)} m(dx) + \int_F \frac{P^{n+1}(x,A) - P^0(x,A)}{\sum_{k=0}^{n} P^k(x,F)} m(dx)$$

hence taking the Banach limit of both sides in (3.34) yields the desired result. Q.E.D.

As before we define μ^* on all of Σ as follows:

Definition 3.35: $$\mu^* = \begin{cases} \mu(A) & \text{for } A \in \overline{\Sigma} \\ \infty & \text{for } A \notin \overline{\Sigma} \end{cases}$$

Lemma 3.36: μ^* is a finitely additive set function.

Proof: The proof of Lemma 2.15 carries over completely.

Definition 3.37: Let $\pi(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu^*(A_i) : \bigcup_{i=1}^{\infty} A_i \supset A \right\}$ over all such countable unions containing A .

The next lemma, to prove π is non-trivial, is somewhat different than in Chapter 2 since there we needed the fact that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} P^k(x,A)}{\sum_{k=0}^{n} P^k(x,F)}$$

existed for all $A \subset F$.

Lemma 3.39: π is non-trivial.

Proof: We first show π is not purely finitely additive on F . Let A be any set in Σ_F such that $m(A) > \delta$. Then by 3.30 we have

$$0 \leq \int_F \liminf_n \frac{\sum_{k=0}^{n} P^k(x,A)}{\sum_{k=0}^{n} P^k(x,F)} \, m(dx)$$

$$\leq \liminf_n \int_F \frac{\sum_{k=0}^{n} P^k(x,A)}{\sum_{k=0}^{n} P^k(x,F)} \, m(dx)$$

$$\leq \operatorname{Lim}_n M_n(A) = \mu(A) .$$

This last step by B.L. 3, hence $m(A) > \delta \Rightarrow \mu(A) > 0$.

By the same reasoning as in Lemma 3.23, μ cannot be purely finitely additive.

Using Lemma 3.12, μ can be written as $\mu_c + \mu_f$. It must be true that $\mu_c(F) > 0$ since otherwise $\mu \equiv \mu_f$ i.e. μ would be p.f.a. on F. Hence $\mu^*(F) = \mu(F) \geq \mu_c(F) > 0$. Therefore, for any countable covering $\{C_i\}$ of F, by the finite addivity of μ^*, we have

$$\mu^*(C_i) \geq \mu^*(C_i \cap F) \geq \mu(C_i \cap F) \geq \mu_c(C_i \cap F) . \tag{3.40}$$

Hence $\sum_{i=1}^{\infty} \mu^*(C_i) \geq \sum_{i=1}^{\infty} \mu_c(C_i \cap F) = \mu_c(F) > 0$, so by the definition of π as the inf over all such sums we have $\pi(F) > 0$. Q.E.D.

Note that by using the p.f.a. property of μ this proof was much simpler than the corresponding proof of Theorem 2.19.

Lemma 3.41: $\int_X P^i(x,A)\,\pi(dx) \leq \pi(A) \qquad \forall i$.

Proof: Same as Theorem 2.22 with μ_n replaced by M_n.

Lemma 3.42: π is a σ-finite measure.

Proof: Proof of Theorem 2.32 carries over completely.

Lemma 3.43: π is an invariant measure.

Proof: Same as for Theorem 2.33.

Thus we have proven Theorem 3.31.

We have shown several conditions implying the existence of a σ-finite invariant measure. Before ending the discussion on them, an interesting question arises. Does the similarity of (1.2) with 3.1, and 2.1 with 3.30 indicate they are somehow related? As they are now stated, (1.2) and 3.1 are in a sense independent of each other.

In the general Markov process, if one of these conditions is true, the other need not be. For example, for (1.2) to hold, all that is needed is the existence of the limit, which might be zero on a set B (in Σ_F) with $m(B) = 1$. This would imply there does not exist a δ satisfying 3.1. On the other hand, if 3.1 holds, then certainly the limit need not exist anywhere, so (1.2) would not have to be true.

However, under the following condition we will show that (1.2) does indeed imply 3.1.

Definition 3.44: A Markov process is said to be non-singular with respect to the measure m if $m(A) = 0$ implies $P(x,A) = 0$ a.e. (m).

Theorem 3.45: Let the Markov process (X, Σ, m, P) be non-singular and satisfy hypotheses (1.2), (1.3), and (1.4) of Theorem 1.1. Then there exists an F and a δ satisfying Condition 3.1.

Proof: The first step will be to show there exists a δ such that $m(A) > \delta \Rightarrow \mu(A) > 0$. Recall that the measure, μ, we are using here comes from Chapter 1 since we are assuming the hypotheses of Theorem 1.1 are in effect.

For purposes of a proof by contradiction, assume that for each δ, $0 < \delta < 1$, there exists a set $A_\delta \subset F$ such that $m(A_\delta) > \delta$ with $\mu(A_\delta) = 0$ (we are still assuming $m(F) = 1$). Therefore, there exists a sequence $\{\delta_i\}$ such that $\delta_i \uparrow 1$ with $\mu(A_{\delta_i}) = 0$. Since $m\left(\bigcup_{i=1}^{n} A_{\delta_i}\right) = \delta_n$, we have $m\left(\bigcup_{i=1}^{\infty} A_{\delta_i}\right) = 1$. By the non-singularity of $P(x,A)$, we then have $P_F^k\left(x, \bigcup_{i=1}^{\infty} A_{\delta_i}\right) = 1$ for a.e. $(m_F)x \in F$, for all k.

(Note that $P(x,A) = 0$ a.e.(m) $\Rightarrow P_F(x,A) = 0$ a.e. m_F by the definition of $P_F(x,A)$; see (1.5).) Therefore,

$$\bar{m}_F\left(Y_k \in \bigcup_{i=1}^{\infty} A_{\delta_i}\right) = \int_F P_F^k\left(x, \bigcup_{i=1}^{\infty} A_{\delta_i}\right) m_F(dx)$$

$$= \int_F 1 \; m_F(dx)$$

$$= m_F(F) = 1 \quad \text{for all} \quad k \; .$$

Using (1.13) and Definition 1.14, we have $\mu\left(\bigcup_{i=1}^{\infty} A_{\delta_i}\right) = 1$; but μ is countably additive on F so $\mu\left(\bigcup_{i=1}^{\infty} A_{\delta_i}\right) = \sum_{i=1}^{\infty} \mu(A_{\delta_i}) = 0$ since each $\mu(A_{\delta_i}) = 0$ by assumption. This is a contradiction. Hence we have shown there must exist a δ such that $m(A) > \delta \Rightarrow \mu(A) > 0$, and in fact one of the δ_i must work. Recall that $\bar{F}$ is the sequence space of the F-process, and $\tilde{\bar{F}} = FXXXXX\ldots$.

We now complete the proof. We have a δ such that

$$m(A) > \delta \Rightarrow \mu(A) > 0$$

$$\Rightarrow \int_{\bar{F}} \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1} \, d\bar{m}_F > 0 \tag{3.46}$$

$$\Rightarrow \lim_{n\to\infty} \int_{\bar{F}} \frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1} \, d\bar{m}_F > 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \bar{m}_F(Y_k \in A)}{n+1} > 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \bar{m}(Y_k \in A)}{n+1} > 0 \tag{3.47}$$

$$\Rightarrow \lim_{n\to\infty} \int_{\tilde{F}} \frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1}\, d\bar{m} > 0$$

$$\Rightarrow \int_{\tilde{F}} \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{Y_k \in A}}{n+1}\, d\bar{m} > 0$$

$$\Rightarrow \int_{\tilde{F}} \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{X_k \in A}}{\sum_{k=0}^{n} 1_{X_k \in F}}\, d\bar{m} > 0 \tag{3.48}$$

$$\Rightarrow \bar{m}\left(\omega\colon \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{X_k \in A}}{\sum_{k=0}^{n} 1_{X_k \in F}} > 0,\ X_0 \in F\right) > 0$$

$$\Rightarrow \bar{m}\left(\omega\colon \liminf_{n} \frac{\sum_{k=0}^{n} 1_{X_k \in A}}{\sum_{k=0}^{n} 1_{X_k \in F}} > 0,\ X_0 \in F\right) > 0\ .$$

We get (3.46) by using the definition of μ (see 1.14), (3.47) comes from Lemma 3.14, and (3.48) by using the fact that the subscript of Y_k actually counts the number of entries into F and following the reasoning in the proof of Lemma 3.17 the limit is unchanged. Q.E.D.

Notes: As with (1.2) and 3.1, it can be shown there is the same general independence of Conditions 2.1 and 3.30. However, even with the imposition of non-singularity we cannot get an implication here. The main obstacle is that the limit in 2.1 need only be defined for a single value of x .

It can be shown, as was done in Theorem 3.45, that $\exists$ a δ such that $m(A) > \delta \Rightarrow \mu(A) > 0$, however the integration in 3.45 may not be possible since liminf of the ratio in 2.1 might be zero for all other values of x .

CHAPTER 4: Finiteness

In this chapter we continue our discussion of invariant measures by establishing some criteria for deciding if the measure is finite or infinite.

A Markov process will be said to satisfy Condition A if, for each $B \in \Sigma$, $P(X_n \in B \text{ infinitely often} | X_0 = x) = 1$ for a.e. $x \in B$. A Markov process is said to be ergodic if $\pi(\theta^{-1}A \Delta A) = 0 \Rightarrow \pi(A) = 0$ or $\pi(A^c) = 0$. In this chapter we assume the Markov process is ergodic and satisfies Condition A, and π to be the measure on sequence space.

For the invariant measure π, le A be a set such that $0 < \pi(A) < \infty$. For ω in sequence space define

$$T_0(\omega) = \text{first index } i \text{ such that } X_i(\omega) \in A$$

$$T_1(\omega) = \text{second index } i \text{ such that } X_i(\omega) \in A$$

$$T_2(\omega) = \text{third index } i \text{ such that } X_i(\omega) \in A$$

$$\vdots$$

The quantities $T_i - T_{i-1}$, for $i = 1,2,3,\ldots$ represent the number of steps (possibly different for each ω) from the first step out of A (which was entered for the <u>first</u> time at the T_0 step) until reentering A (T_1 step). Condition A and ergodicity assures us that the $T_i - T_{i-1}$ are well defined for a.e. ω.

<u>Definition 4.1</u>: The expected time of first recurrence to the set A is defined as

$$E_\pi\left[T_1(\omega) - T_0(\omega)\right] = \int_{X} (T_1 - T_0)\,d\pi .$$

For our purposes the criteria to be shown will work under the restriction

$T_0 \equiv 0$, i.e., it will be assumed $X_0 \in A$. Under this restriction the expected time of first recurrence will be written as

$$E_\pi(T_1 - T_0) = E_\pi(T_1 \mid X_0 \in A) = E_{\pi_A}(T_1) \quad ,$$

where π_A represents the normalized measure induced by A on π as follows:

$$\pi_A(B) = \pi(B, X_0 \in A)/\pi(A) \ .$$

Note that $\pi_A(X) = 1$. The main result of this chapter is

Theorem 4.2: Let π be an invariant measure. Let the Markov process satisfy Condition A and be ergodic. Let A be a set in Σ such that $0 < \pi(A) < \infty$. Then $E_{\pi_A}(T_1) = \infty \Leftrightarrow \pi(X) = \infty$.

Proof: Part I. It can be shown [1 ; 6.38] that if $\pi(X) < \infty$ (without loss of generality we will assume in this case that $\pi(X) = 1$) we have $E_{\pi_A}(T_1) = 1/\pi(A)$. Hence this gives one-half of the desired result, i.e., $\pi(X) < \infty \Rightarrow E_{\pi_A}(T_1) < \infty$ or equivalently $E_{\pi_A}(T_1) = \infty \Rightarrow \pi(X) = \infty$.

Part II: We now assume $\pi(X) = \infty$. To prove the second half of this theorem we first note that in that same result [1 ; 6.38], it was shown that the variables $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are stationary under the measure π_A . Stationarity allows us to apply the ergodic theorem to

$$\frac{\sum_{i=1}^{n} (T_i - T_{i-1})}{n} \tag{4.3}$$

so that the limit as $n \to \infty$ of (4.3) is $E_{\pi_A}(T_1)$ a.e. (π_A) . This fact will be needed later.

Let $\theta^i 1_A$ be $1_A(\theta^i \omega)$. It will be shown that the ergodic theorem can be applied to the ratio

$$(4.4) \qquad \frac{\sum_{i=T_0}^{T_n} \theta^i 1_A}{T_n - T_0 + 1}$$

even if we do not keep the restriction $T_0 \equiv 0$. First, for fixed ω, we let $T_0(\omega) = m$ and (4.4) becomes

$$(4.5) \qquad \frac{\sum_{i=m}^{T_n} \theta^i 1_A}{T_n - m + 1} .$$

For i such that $0 \leq i < m$ we know $\theta^i 1_A = 0$ (since T_0 is the <u>first</u> index i with $\theta^i(\omega) \in A$) so (4.5) is

$$(4.6) \qquad \frac{\sum_{i=0}^{T_n} \theta^i 1_A}{T_n - m + 1} .$$

Since for a.e. ω, T_n is an increasing sequence of integers, we are able to say that for $n = 0,1,2,\ldots$ the ratio (4.6) (and hence (4.4)) is a subsequence of

$$(4.7) \qquad \frac{\sum_{i=0}^{k} \theta^i 1_A}{k - m + 1}$$

for $k = 1,2,3,\ldots$ (note that $k \geq m$). Since m is fixed and finite for a.e. ω, the ergodic theorem can be applied directly to (4.7) [see e.g. 9 ; p. 18] and this ratio must converge to zero a.e. (π) (and hence a.e. (π_A)). This is true since the limit must be an integrable constant and the only such constant on an infinite measure space is zero.

We now approach the ratio in (4.4) another way by inverting it to give

$$(4.8) \qquad \frac{T_n - T_0 + 1}{\sum_{i=T_0}^{T_n} \theta^i 1_A} .$$

We manipulate (4.8) as follows: we know that $\theta^i 1_A = 1$ only when $X_i(\omega) \in A$ and this occurs for the first time when $i = T_0$, the second time when $i = T_1$, etc. Therefore the statement " $\theta^i 1_A = 1$ " will be true precisely $n+1$ times as i ranges from T_0 to T_n, so (4.8) becomes

$$(4.9) \qquad \frac{T_n - T_0 + 1}{n+1} .$$

Writing $T_n - T_0$ as the telescoping sum

$$T_n - T_{n-1} + T_{n-1} - T_{n-2} + \dots + T_2 - T_1 + T_1 - T_0$$

or $\sum_{k=1}^{n} (T_k - T_{k-1})$, changes (4.9) into

$$(4.10) \qquad \frac{\sum_{k=1}^{n} (T_k - T_{k-1}) + 1}{n+1} .$$

Since $\pi(A) < \infty$ we have $\pi_A(X) = 1$. Therefore if $E_{\pi_A}(T_1) < \infty$, then, as was shown at the beginning of Part II, (4.10) would converge to $E_{\pi_A}(T_1)$ a.e. (π_A). But (4.7) was shown to converge to zero a.e. (π_A). Since (4.7) and (4.10) are essentially inverses of each other, this is a contradiction. Hence $\pi(X) = \infty \Rightarrow E_{\pi_A}(T_1) = \infty$. Q.E.D.

Notes: The results of Theorem 4.2 contain a well-known result by Kac [cf. 7, page 12] as well as the discrete case for infinite measures [cf. 6]. However we believe the result for the infinite case for general state spaces is new.

CHAPTER 5: Uniqueness

The main result of this chapter will be a theorem on the uniqueness of an invariant measure.

Definition 5.1: A set $A \in \Sigma$ will be called closed if $P(x,A) = 1$, $\forall\, x \in A$, i.e., an element in A must stay in A.

Definition 5.2: A Markov process will be called indecomposable, if Σ does not contain two disjoint non-empty closed sets.

The theorem to be proven is

Theorem 5.3: Let π_1 and π_2 be two non-trivial, invariant, σ-finite measures, satisfying Condition A [see Chapter 4], on the same indecomposable Markov process. Then there exists a constant c s.t. $\pi_1 = c\pi_2$.

The proof will be accomplished via a sequence of lemmas.

Lemma 5.4: If π is a non-trivial, σ-finite, invariant measure and if $A \in \Sigma$ is such that $\pi(A) = 0$ then there exists a non-empty closed set C contained in A^c such that $\pi(C^c) = 0$.

Proof: Let

$$\begin{aligned} A_0 &= A \\ A_1 &= \{x: P(x,A_0) > 0\} \\ A_2 &= \{x: P(x,A_1) > 0\} \\ &\ \vdots \end{aligned}$$

By the invariance of π we have

$$(5.5)\qquad 0 = \pi(A) = \int_X P(x,A)\pi(dx) = \int_{A_1} P(x,A)\pi(dx)\ .$$

But A_1 consists of these x's with $P(x,A) > 0$ so the only way to get the integral on the right in (5.5) equal to zero is if $\pi(A_1) = 0$. Replacing A in (5.5) with A_1, and A_1 with A_2 we see that by the same reasoning $\pi(A_2) = 0$, and so on. Let $A^* = \bigcup_{i=1}^{\infty} A_i$ so that $\pi(A^*) = 0$.

If $x \in A^{*^c}$ then by the definition of the A_i we have $P(x,A_i) = 0$ for all i , hence $P(x,A^*) = 0$ or equivalently $P(x,A^{*^c}) = 1$. By Definition 5.1 A^{*^c} is closed. Since $\pi(A^*) = 0$, $\pi(A^{*^c}) > 0$ hence A^{*^c} is not empty. So $C = A^{*^c}$ is the desired set. Q.E.D.

Lemma 5.6: Let π be an invariant measure on an indecomposable process. If C is a non-empty closed set then $\pi(C) > 0$.

Proof: If $\pi(C) = 0$ then by Lemma 5.4 there exists a closed set $D \subset C^c$ with D non-empty. But C and D being two closed, non-empty, disjoint sets contradicts indecomposability, hence $\pi(C) > 0$. Q.E.D.

Lemma 5.7: Let π_1 and π_2 be two non-trivial invariant measures on an indecomposable process. Then there do not exist two disjoint sets A_1 and A_2 such that $A_1 \cup A_2 = X$ and $\pi_1(A_2) = 0$ and $\pi_2(A_1) = 0$, i.e. the supports of π_1 and π_2 cannot be disjoint.

Proof: Assume the contrary, i.e., assume there exists A_1, A_2 such that $A_1 \cup A_2 = X$, $A_1 \cap A_2 = \emptyset$ and $\pi_1(A_2) = 0$, $\pi_2(A_1) = 0$. By Lemma 5.4 we have that $\pi_1(A_2) = 0$ implies there exists a non-empty closed set $C_1 \subset A_2^c$, and also $\pi_2(A_1) = 0$ implies there exists a non-empty closed set $C_2 \subset A_1^c$. But C_1 and C_2 are disjoint, contradicting indecomposability. Q.E.D.

Lemma 5.8: Let π_1 and π_2 be two non-trivial invariant measures, satisfying Condition A, on an indecomposable process. Then $\pi_1(A) = 0$ implies $\pi_2(A) = 0$.

Proof: Let A^* be as in the proof of Lemma 5.4. A^{*^c} is closed and non-empty so by Lemma 5.6 we have $\pi_2(A^{*^c}) > 0$. We note that the set $\left(A^{*^c}\right)^{*^c}$ is closed and disjoint from A^{*^c} hence by indecomposability

$\left(A^{*^c}\right)^{*^c}$ must be empty. Therefore for each x there must exist an n such that $P^n(x, A^{*^c}) > 0$. If $\pi_2(A) > 0$ there exists a set A' in A with $\pi_2(A') > 0$ such that for some N and $\epsilon > 0$ we have $P^N(x, A^{*^c}) > \epsilon$ for all $x \in A'$.

Condition A implies that $P(X_n \in A^*$ for some $n > N | X_0 = x) = 1$ for a.e. $x \in A^*$ so we have

$$P(X_n \in A^* \text{ for some } n > N | X_0 = x)$$

$$= \int_X P(X_n \in A^* \text{ for some } n > N | X_N = y) P^N(x, dy)$$

$$= \int_{A^*} P(X_n \in A^* \text{ for some } n > N | X_N = y) P^N(x, dy) \tag{5.9}$$

$$+ \int_{A^{*^c}} P(X_n \in A^* \text{ for some } n > N | X_N = y) P^N(x, dy) .$$

Since A^{*^c} is closed, we have $P(X_n \in A^*$ for some $n > N | X_N = y) = 0$ for $y \in A^{*^c}$ so the right integral above is zero allowing us to write (5.9) as

$$\leq \int_{A^*} P^N(x, dy) + 0$$

$$= P^N(x, A^*)$$

$$\leq 1 - \epsilon$$

for $x \in A'$. But by Condition A this cannot be true except on a set of measure zero, contradicting the assumption $\pi_2(A) > 0$. Q.E.D.

Lemma 5.10: Under the hypotheses of Lemma 5.8 the measures π_1 and π_2 are equivalent.

Proof: Follows immediately from Lemma 5.8.

The final result will be proven by showing a relationship between an indecomposable process on state space and an ergodic process on sequence space. Once ergodicity is shown, then the following theorem [from 13; Theorem 1] proves the desired result. As before, $\bar{\pi}_1$ and $\bar{\pi}_2$ are the measures induced on sequence space by π_1 and π_2 respectively.

Recall the following facts: a measure is ergodic if for any invariant set A we have $\pi(A) = 0$ or $\pi(A^c) = 0$; a set $A \in \Sigma_\infty$ is invariant if $\pi(\theta^{-1}A \Delta A) = 0$; a set $B \in \Sigma$ is stochastically closed if $P(x,B) = 1$ for $a.e.(\pi) x \in B$; and a measure α is absolutely continuous with respect to the measure β (written $\beta \gg \alpha$) if $\beta(E) = 0 \Rightarrow \alpha(E) = 0$.

Theorem 5.11: If $\bar{\pi}_1 \gg \bar{\pi}_2$ and $\bar{\pi}_1$ is ergodic, then $\bar{\pi}_1 = c\bar{\pi}_2$ for some constant c .

To show that the hypotheses of Theorem 5.3 are sufficient to imply the ergodicity of $\bar{\pi}_1$ we will need the following result [14; Theorem 1].

Theorem 5.12: If the Markov process satisfies Condition A , then there is a 1-1 correspondence λ (up to equivalence) between the class of stochastically closed sets $\{V\}$ in state space and invariant sets $\{V_\alpha\}$ in sequence space; and if $V_\alpha = \lambda(V)$, then $\bar{\pi}_1(V_\alpha) = \pi_1(V)$.

Theorem 5.13: If π_1 satisfies the hypotheses of Theorem 5.3, then $\bar{\pi}_1$ is ergodic.

Proof: Let V_1 be any invariant set in Σ_∞ . By the proof of Theorem 5.12 [14; Theorem 1], we see that $V_1 = \{\omega: x_0 \in V\}$ for some stochastically closed set V . Also, V_1 invariant implies V_1^c is invariant hence $V_1^c = \{\omega: x_0 \in U\}$ for some stochastically closed set U . The sets V and U must be disjoint otherwise V_1 and V_1^c would not be disjoint.

Let $U_0 = U$

$U_1 = \{x \in U_0 : P(x, U_0) = 1\}$

$U_2 = \{x \in U_1 : P(x, U_1) = 1\}$

$\vdots$

letting $U' = \bigcap_{i=0}^{\infty} U_i$ we claim that U' is closed (under Definition 5.1) since $x \in U' \Rightarrow P(x, U_i) = 1$ for all i, which implies $P(x, \bigcap_{i=0}^{\infty} U_i) = 1$ and hence $P(x, U') = 1$ for all $x \in U'$.

Note that since U is stochastically closed we have $\pi_1(U) = \pi_1(U')$. By a similar construction using V, we get V' closed such that $\pi_1(V) = \pi_1(V')$. But $V' \cap U' = \emptyset$ hence by indecomposability, one of V' or U' must be empty, say it's V'. Then $\pi_1(V) = 0$, hence, $\bar{\pi}_1(V_1) = 0$ by the final statement in Theorem 5.12. Q.E.D.

Now applying Theorem 5.11 proves Theorem 5.3 since the measures in state space and sequence space are essentially in 1-1 correspondence.

Two examples will show that without either Condition A or indecomposability, there need not exist a unique measure. The first example will satisfy Condition A only.

Let $X = \{a, b\}$, $P(a, \{a\}) = 1$, $P(b, \{b\}) = 1$. Here there exists three distinct non-equivalent invariant measures:

$\pi_1(a) = 1 \qquad \pi_1(b) = 0$,

$\pi_2(a) = 0 \qquad \pi_2(b) = 1$, and

$\pi_3(a) = \frac{1}{2} \qquad \pi_3(b) = \frac{1}{2}$.

Note that there are two disjoint closed sets, namely $\{a\}$ and $\{b\}$.

The next example is a bit more complicated. Here we only have indecomposability. Let X = integers, $P(a,\{a+1\}) = \frac{2}{3}$, $P(a,\{a-1\}) = \frac{1}{3}$. The two non-equivalent measures

$$\pi_1(a) = 1 \quad \forall\ a \quad \text{and}$$

$$\pi_2(a) = 2^a$$

are both invariant: for π_1 we have

$$(5.14) \qquad \pi_1(a) = P(a-1, \{a\})\pi_1(a-1) + P(a+1,\{a\})\pi_1(a+1)$$

$$= \tfrac{2}{3}(1) + \tfrac{1}{3}(1) = 1$$

and for π_2 we have (using 5.14)

$$\pi_2(a) = \tfrac{2}{3}\left(2^{a-1}\right) + \tfrac{1}{3}\left(2^{a+1}\right)$$

$$= 2^{a-1}\left(\tfrac{2}{3} + \tfrac{1}{3}\cdot 4\right)$$

$$= 2^{a-1}(2) = 2^a .$$

However it might not be so obvious that this example does not satisfy Condition A.

We first note that the Borel-Cantelli Lemma gives the following implication: $P(X_n \in A \text{ infinitely often} | X_0 = x) = 1 \Rightarrow \sum_{k=1}^{\infty} P^k(x,A) = \infty$. For our situation this translates into $\sum_{k=1}^{\infty} P^k(a,\{a\}) = \infty$ must be true, if Condition A is true. We now show, however, that $\sum_{k=1}^{\infty} P^k(a,\{a\}) < \infty$ for all a. We choose for example $a = 0$ (other values of a produce similar results).

$$P^0(0,\{0\}) = 1$$

$$P(0,\{0\}) = 0$$

$$P^2(0,\{0\}) = 4/9$$

$$P^3(0, \{0\}) = 0$$

$$P^4(0, \{0\}) = 6(2/9)^2$$

$$\vdots$$

$$P^{2n}(0, \{0\}) = C(2n, n)(\tfrac{2}{3})^n(\tfrac{1}{3})^n$$

and a little algebra will show that term by term the non-zero elements in the series $\sum_{k=0}^{\infty} P^k(0, \{0\})$ are less than or equal to the terms in $1 + 4(2/9) + 16(2/9)^2 + 64\,(2/9)^3 + \ldots$; but this series can easily be shown to converge (it is a geometric series with $r = 8/9$) . Hence, $\sum_{k=0}^{\infty} P^k(0, \{0\}) < \infty$ implying Condition A is not satisfied.

CHAPTER 6: Mixing

As before, π represents an invariant measure on state space, $\bar{\pi}$ the induced invariant measure on sequence space, and θ represents the shift transformation on sequence space. Note that $\bar{\pi}$ invariant means $\bar{\pi}(A) = \bar{\pi}(\theta^{-1}A)$ for all $A \in \Sigma_\infty$.

The stationary process $X_0, X_1, X_2, \ldots$ can be embedded in a bilateral stationary process $\ldots, X'_{-1}, X'_0, X'_1, \ldots$ such that $X_0, X_1, X_2, \ldots$ and $X'_0, X'_1, X'_2, \ldots$ have the same distribution. So without loss of generality we can talk about a bilateral stationary process [see 1 ; proposition 6.5]. Here θ will be assumed to be invertible (i.e. θ maps $\bar{X}$ one-to-one onto $\bar{X}$) and non-singular (i.e. $\pi(A) = 0 \Rightarrow \pi(\theta A) = \pi(\theta^{-1}A) = 0$) .

For $\bar{\pi}(\bar{X}) = 1$ and $\bar{\pi}$ invariant and ergodic, it has been shown [see, e.g., 7 ; page 61] that for θ defined on bilateral space and any sets A, C we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \bar{\pi}(\theta^{-k}A \cap C) = \bar{\pi}(A)\bar{\pi}(C)$$

hence it follows easily that for $\bar{\pi}(B) > 0$ and $\bar{\pi}(C) > 0$ we have

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \bar{\pi}(\theta^{-k}A \cap C)}{\sum_{k=0}^{n} \bar{\pi}(\theta^{-k}B \cap C)} = \frac{\bar{\pi}(A)}{\bar{\pi}(B)} . \tag{6.1}$$

The question to be investigated here is what can be said in the case of $\bar{\pi}(\bar{X}) = \infty$. In general, (6.1) may not hold if $\bar{\pi}(\bar{X}) = \infty$. The first part of the answer to our question will show how strong a result similar to (6.1) can be proven without adding any further conditions. We first state a result conjectured by Hopf [11] and proven by Chacon and Ornstein [see e.g. 2].

Lemma 6.2: Let $\bar{\pi}(A) < \infty$, $0 < \bar{\pi}(B) < \infty$ with the measure $\bar{\pi}$ invariant and ergodic. Let the process satisfy Condition A. Then

$$(6.3) \qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{\theta^{-k}A}(\omega)}{\sum_{k=0}^{n} 1_{\theta^{-k}B}(\omega)} = \frac{\bar{\pi}(A)}{\bar{\pi}(B)} \quad \text{a.e. } (\bar{\pi}) .$$

This will be used to prove

Theorem 6.4: Let $A, B, \bar{\pi}$ be as in Lemma 6.2. Let N be any positive integer. Then there exists a set C , with $\bar{\pi}(C) > N$, such that

$$(6.5) \qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \bar{\pi}(\theta^{k}A \cap C)}{\sum_{k=0}^{n} \bar{\pi}(\theta^{k}B \cap C)} = \frac{\bar{\pi}(A)}{\bar{\pi}(B)} .$$

Proof: By Lusin's Theorem [see16; p. 139 #6], since we have convergence a.e. in (6.3) and $\bar{\pi}$ is σ-finite, there exists a partition $\{A_i\}$ of X-M (where M is the set of divergence of (6.3)) such that (6.3) converges uniformly on every element A_i . Therefore, given $\epsilon > 0$, for each A_i there exists an $n_i(\epsilon)$ such that

$$(6.6) \qquad \left| \frac{\sum_{k=0}^{n} 1_{\theta^{-k}A}}{\sum_{k=0}^{n} 1_{\theta^{-k}B}} - \frac{\bar{\pi}(A)}{\bar{\pi}(B)} \right| \le \epsilon$$

for all $\omega \in A_i$ as long as $n > n_i$. We can write (6.6) as

$$(6.7) \qquad -\epsilon \sum_{k=0}^{n} 1_{\theta^{-k}B} + \frac{\bar{\pi}(A)}{\bar{\pi}(B)} \sum_{k=0}^{n} 1_{\theta^{-k}B} \le \sum_{k=0}^{n} 1_{\theta^{-k}A} \le \epsilon \sum_{k=0}^{n} 1_{\theta^{-k}B} + \frac{\bar{\pi}(A)}{\bar{\pi}(B)} \sum_{k=0}^{n} 1_{\theta^{-k}B} .$$

Since $\bar{\pi}\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty$, we can find (many) j such that $\bar{\pi}\left(\bigcup_{i=1}^{j} A_i\right) > N$ of the theorem. We first examine the right inequality in (6.7)

$$(6.8) \qquad \sum_{k=0}^{n} 1_{\theta^{-k}A} \le \epsilon \sum_{k=0}^{n} 1_{\theta^{-k}B} + \frac{\bar{\pi}(A)}{\bar{\pi}(B)} \sum_{k=0}^{n} 1_{\theta^{-k}B} .$$

We note that (6.8) is true for all $\omega \in \bigcup_{i=1}^{j} A_i$ if $n > n' = \max(n_1, n_2, \ldots, n_j)$.

Letting $\bigcup_{i=1}^{j} A_i = C$ we multiply (6.8) by 1_C yielding

$$(6.9) \qquad 1_C \sum_{k=0}^{n} 1_{\theta^{-k}A} \le \epsilon \, 1_C \sum_{k=0}^{n} 1_{\theta^{-k}B} + \frac{\bar{\pi}(A)}{\bar{\pi}(B)} 1_C \sum_{k=0}^{n} 1_{\theta^{-k}B} .$$

This is true for <u>all</u> ω (since $1_C = 0$ for $\omega \notin C$), $n > n'$, hence integrating (6.9) over X gives

$$\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}A) \le \epsilon \sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}B) + \frac{\bar{\pi}(A)}{\bar{\pi}(B)} \sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}B)$$

or equivalently

$$(6.10) \qquad \frac{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}A)}{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}B)} \le \epsilon + \frac{\bar{\pi}(A)}{\bar{\pi}(B)} .$$

Now keeping C fixed (and equal to $\bigcup_{i=1}^{j} A_i$), if we were to take a smaller ϵ , forcing n' higher, (6.10) would still be true as long as $n > n'$ i.e. for big enough n . Taking limsup of (6.10) yields

$$\limsup_{n} \frac{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}A)}{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}B)} \le \epsilon + \frac{\bar{\pi}(A)}{\bar{\pi}(B)}$$

which is true for every ϵ hence allowing us to write

$$(6.11)\qquad \limsup_{n} \frac{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}A)}{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}B)} \le \frac{\bar{\pi}(A)}{\bar{\pi}(B)} .$$

Now, proceeding along the same lines with the left inequality in (6.7) and taking liminf yields

$$(6.12)\qquad \liminf_{n} \frac{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}A)}{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}B)} \ge \frac{\bar{\pi}(A)}{\bar{\pi}(B)}$$

but (6.11) and (6.12) imply

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}A)}{\sum_{k=0}^{n} \bar{\pi}(C \cap \theta^{-k}B)} = \frac{\bar{\pi}(A)}{\bar{\pi}(B)} . \qquad \text{Q.E.D.}$$

Theorem 6.4 assures us of the existence of as large a finite set C as we want such that (6.1) holds. However, if the set C is already fixed, we have the following theorem.

Theorem 6.13: Let A, B, $\bar{\pi}$ be as in Lemma 6.2. Let C be any set, $\bar{\pi}(C) > 0$, and N any positive integer. Then there exists a set D, $D \subset C$, such that $\bar{\pi}(D) > N$ if $\bar{\pi}(C) = \infty$ or $\pi(C-D) < 1/N$ if $\bar{\pi}(C) < \infty$ and

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \bar{\pi}(\theta^{-k}A \cap D)}{\sum_{k=0}^{n} \bar{\pi}(\theta^{-k}B \cap D)} = \frac{\bar{\pi}(A)}{\bar{\pi}(B)} .$$

Proof: The result follows by applying the technique of the proof of Theorem 6.4 as follows: partition $X-M$ as before into $\{A_i\}$, let

$B_i = A_i \cap C$; if $\bar{\pi}(C) = \infty$, then $\bar{\pi}\left(\bigcup_{i=0}^{\infty} B_i\right) = \infty$ hence just replace A_i with B_i in the proof of 6.4. If $\bar{\pi}(C) < \infty$, then, since $\bigcup_{i=0}^{\infty} B_i = C$ (except possibly for a set of measure zero) we have $\bar{\pi}\left(\bigcup_{i=0}^{n} B_i\right) \uparrow \bar{\pi}(C)$. Therefore, replacing A_i with B_i in such a way to have $\bar{\pi}\left(C - \bigcup_{i=0}^{n} B_i\right) < 1/N$ will suffice. **Q.E.D.**

Theorem 6.13 says that if the set C is already fixed, we can find a 'large' subset of C satisfying (6.1). However, to get the stronger result of having (6.1) be true for more general A,B,C we impose a further condition. Let

$$\mu_n(A) = \frac{\sum_{k=0}^{n} \bar{\pi}(\theta^{-k}A \cap C)}{\sum_{k=0}^{n} \bar{\pi}(\theta^{-k}B \cap C)} .$$

<u>Condition 6.14:</u> For sets B,C such that $\bar{\pi}(B) > 0$, $\bar{\pi}(C) > 0$ and $\sum_{k=0}^{\infty} \bar{\pi}(\theta^{-k}B \cap C) = \infty$ and for all sequences of sets $A_1, A_2, A_3, \ldots$ such that $A_i \downarrow \emptyset$ we have $\limsup_i(\limsup_n \mu_n(A_i)) = 0$.

The first result to be proven is

<u>Theorem 6.15:</u> For all A, B, C, such that $0 < \bar{\pi}(B) < \infty$, $0 < \bar{\pi}(C)$, $A \subset B$ assume Condition 6.14 is true, and $\bar{\pi}$ is invariant and ergodic. Then

$$\lim_{n\to\infty} \mu_n(A) = \frac{\bar{\pi}(A)}{\bar{\pi}(B)} . \tag{6.16}$$

<u>Proof:</u> The proof will be accomplished in several steps.

I. An invariant, ergodic measure $\bar{\mu}$ will be produced.

II. $\bar{\mu}$ will be shown to be countably additive on B and hence non-trivial.

III. It will be shown that $\bar{\mu}(A) = \dfrac{\bar{\pi}(A)}{\bar{\pi}(B)}$.

IV. (6.16) will be shown to be true.

I. Let n_i be any increasing sequence of positive integers. Let $\bar{\Sigma} = \{A \in \Sigma\colon \limsup_{n_i} \mu_{n_i}(A) < \infty\}$. Define a Banach limit, Lim (which might depend on the sequence n_i), on the sequences $\mu_{n_1}(A), \mu_{n_2}(A), \ldots$ for $A \in \bar{\Sigma}$. As was done in Chapter 2, $\mu(A) = \mathrm{Lim}\, \mu_{n_i}(A)$ can be extended to a σ-finite measure $\bar{\mu}$ on all of Σ . Recall this was done as follows: first let

$$\mu^*(A) = \begin{cases} \mu(A) & \text{for } A \in \bar{\Sigma} \\ \infty & \text{for } A \notin \bar{\Sigma} \end{cases}$$

and then let

$$\bar{\mu}(A) = \inf\left\{ \sum_{i=1}^{\infty} \mu^*(A_i)\colon \bigcup_{i=1}^{\infty} A_i \supset A \right\}$$

over all such countable unions.

To show $\bar{\mu}$ is an invariant measure, we first show μ is an invariant content on $\bar{\Sigma}$. For $A \in \bar{\Sigma}$ we have

$$\mu(\theta^{-1}A) = \mathrm{Lim}_{n_i} \frac{\sum_{k=0}^{n_i} \bar{\pi}\,[\theta^{-k}(\theta^{-1}A) \cap C]}{\sum_{k=0}^{n_i} \bar{\pi}\,[\theta^{-k}B \cap C]}$$

$$= \mathrm{Lim}_{n_i} \left[\frac{\sum_{k=0}^{n_i} \bar{\pi}\,(\theta^{-k}A \cap C)}{\sum_{k=0}^{n_i} \bar{\pi}\,(\theta^{-k}B \cap C)} - \frac{\bar{\pi}\,(A \cap C)}{\sum_{k=0}^{n_i} \bar{\pi}(\theta^{-k}B \cap C)} + \frac{\bar{\pi}(\theta^{-n_i-1}A \cap C)}{\sum_{k=0}^{n_i} \bar{\pi}\,(\theta^{-k}B \cap C)} \right]$$

$$= \mu(A) - 0 + 0 \,. \tag{6.17}$$

The finite additivity of Lim was used to get (6.17). The zeroes in (6.17) come from the fact the denominators go to infinity but the numerators are bounded. Note that this also shows that for $A \in \bar{\Sigma}$ we have $\theta^{-k}A \in \bar{\Sigma}$ for each k . So μ is invariant.

By the definition of μ^* , $\mu^*(\theta^{-1}A) = \mu(\theta^{-1}A) = \mu(A) = \mu^*(A)$ for $A \in \bar{\Sigma}$. If $A \notin \bar{\Sigma}$, then $\theta^{-1}A \notin \bar{\Sigma}$; therefore, $\mu^*(A) = \infty = \mu^*(\theta^{-1}A)$. Hence, μ^* is invariant. Finally, by the definition of $\bar{\mu}$:

$$\bar{\mu}(\theta^{-1}A) = \inf\left\{ \sum_{i=1}^{\infty} \mu^*(A_i) : \bigcup_{i=1}^{\infty} A_i \supset \theta^{-1}A \right\}$$

$$= \inf\left\{ \sum_{i=1}^{\infty} \mu^*(\theta A_i) : \bigcup_{i=1}^{\infty} A_i \supset \theta^{-1}A \right\} \tag{6.18}$$

$$= \inf\left\{ \sum_{i=1}^{\infty} \mu^*(\theta A_i) : \bigcup_{i=1}^{\infty} \theta A_i \supset A \right\} \tag{6.19}$$

$$= \inf\left\{ \sum_{i=1}^{\infty} \mu^*(B_i) : \bigcup_{i=1}^{\infty} B_i \supset A \right\} \tag{6.20}$$

$$= \bar{\mu}(A) \,.$$

The invariance of μ^* is used to get (6.18). In (6.19) we use the invertibility of θ , i.e., $\left(\bigcup_{i=1}^{\infty} A_i \supset \theta^{-1}A \right) \Leftrightarrow \left(\theta\left(\bigcup_{i=1}^{\infty} A_i \right) \supset A \right)$. Also, by invertibility, any set B_i can be represented in the form θA_i which yields (6.20). So $\bar{\mu}$ is invariant.

To show that $\bar{\mu}$ is ergodic, it need only be noted that for an invariant set A , it must be true that $\bar{\pi}(A) = 0$ or $\bar{\pi}(A^c) = 0$, since $\bar{\pi}$ is ergodic. By invariance of $\bar{\pi}$, $\bar{\pi}(A) = 0 \Rightarrow \bar{\pi}(\theta^{-k}A) = 0$ for all k . Hence, $\bar{\pi}(\theta^{-k}A \cap C) = 0$ for all k . Therefore, $A \in \bar{\Sigma}$ since $\sup \mu_{n_i}(A) = 0$, so that $\mu(A) = \mu^*(A) = \bar{\mu}(A) = 0$ (or $\bar{\mu}(A^c) = 0$).

II. We will use Condition 6.14 to show that μ (and hence μ^* and $\bar{\mu}$) is countably additive on B (for fixed B). Let $\{A_i\}$ be any countable collection of disjoint subsets of B . Note that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mu(B) = 1$. Let $B_i = \bigcup_{j=i}^{\infty} A_j$ so $B_i \downarrow \emptyset$ and by 6.14 $\limsup_i(\limsup_n \mu_n(B_i)) = 0$. By definition, $\mu(B_i) = \operatorname{Lim}_{n_j} \mu_{n_j}(B_i)$ and by the properties of Lim we have $\operatorname{Lim}_{n_j} \mu_{n_j}(B_i) \leq \limsup_{n_j} \mu_{n_j}(B_i)$.

Taking limsup over i of this inequality yields

$$(6.21) \qquad \limsup_i \mu(B_i) \leq \limsup_i\left(\limsup_{n_j} \mu_{n_j}(B_i)\right) .$$

The right side of (6.21) is zero by Condition 6.14 (note that the limsup of a subsequence is less than or equal to the limsup of the entire sequence), hence, $\limsup_i \mu(B_i) = 0$. But $\mu(B_i) \geq 0$ for all i (by B.L.2), so $\lim_{i\to\infty} \mu(B_i) = 0$. Therefore

$$0 = \lim_{i\to\infty} \mu\left(\bigcup_{j=i}^{\infty} A_j\right)$$

$$= \lim_{i\to\infty} \mu\left(\bigcup_{j=1}^{\infty} A_j - \bigcup_{j=1}^{i-1} A_j\right)$$

$$(6.22) \qquad = \mu\left(\bigcup_{j=1}^{\infty} A_j\right) - \lim_{i\to\infty} \sum_{j=1}^{i-1} \mu(A_j)$$

$$(6.23) \qquad = \mu\left(\bigcup_{j=1}^{\infty} A_j\right) - \sum_{j=1}^{\infty} \mu(A_j) .$$

We get (6.22) by the finite additivity of μ (B.L.1) and (6.23) is the desired result of countable additivity. This also shows that $\bar{\mu}(A) = \mu(A)$ for all $A \subset B$ so that $\bar{\mu}$ is non-trivial since $\mu(B) = 1$ (see Lemma 2.19 for the proof of $\mu(B) = 1$).

III. There are now two invariant ergodic measures, $\bar{\pi}$ and $\bar{\mu}$.

By applying Lemma 6.2 to both measures we have

$$\lim_{n\to\infty} \frac{\sum\limits_{k=0}^{n} 1_{\theta^{-k}A}(\omega)}{\sum\limits_{k=0}^{n} 1_{\theta^{-k}B}(\omega)} = \frac{\bar{\pi}(A)}{\bar{\pi}(B)} \quad \text{a.e. } (\bar{\pi}) \tag{6.24A}$$

$$= \frac{\bar{\mu}(A)}{\bar{\mu}(B)} \quad \text{a.e. } (\bar{\mu}) . \tag{6.24B}$$

By the definition of $\bar{\mu}$ we have $\bar{\pi} \gg \bar{\mu}$ hence if $\{\omega\colon (6.24A) \text{ converges}\} \cap \{\omega\colon (6.24B) \text{ converges}\} = \emptyset$ then $\bar{\pi}\{\omega\colon (6.24B) \text{ converges}\} = 0$ since the empty intersection says $\{\omega\colon (6.24B) \text{ converges}\} \subset \{\omega\colon (6.24A) \text{ diverges}\}$. The absolute continuity of $\bar{\mu}$ with respect to $\bar{\pi}$ then implies $\bar{\mu}\{\omega\colon (6.24B) \text{ converges}\} = 0$ which contradicts (6.24B) (recall that $\bar{\mu}(B) = 1$). Hence, (6.24A) and (6.24B) must have a non-empty intersection implying $\bar{\mu}(A) = \dfrac{\bar{\pi}(A)}{\bar{\pi}(B)}$.

IV. Finally (6.16) will be shown to be true. For a fixed set $A, A \subset B$, let n_j be any sequence as in Part I such that $\lim_{n_j\to\infty} \mu_{n_j}(A)$ exists. There must be at least one such sequence since $\mu_n(A) \leq 1$ for all n . However, by Parts I-III we must have

$$r = \liminf \mu_{n_j}(A) \leq \operatorname{Lim} \mu_{n_j}(A) \leq \limsup \mu_{n_j}(A) = r$$

so $\mu(A) = r$. But $\mu(A) = \bar{\mu}(A)$ for $A \subset B$ so $\bar{\mu}(A) = r = \dfrac{\bar{\pi}(A)}{\bar{\pi}(B)}$.

However, this last result must be true for all suitable sequences n_j hence all limits of $\mu_{n_j}(A)$ must be identical, namely $\dfrac{\bar{\pi}(A)}{\bar{\pi}(B)}$. Since $\{\mu_n(A)\}$ is bounded and has only this one cluster point, the limit of $\mu_n(A)$ must exist and equal $\dfrac{\bar{\pi}(A)}{\bar{\pi}(B)}$. Q.E.D.

So far (6.16) has been proven only for sets A in B . To show there are many more sets satisfying (6.16) we prove

Theorem 6.25: Let A be any set in Σ that can be broken up into a finite number of pieces $A_1, A_2, \ldots, A_n$ such that for each i there exists a k such that $\theta^{-k}A_i \subset B$. Then (6.16) holds for this set A . Assume the same hypotheses as Theorem 6.15 (except A need not be in B).

Proof: First μ , as defined in the last theorem, is countably additive on each A_i . Note that the hypothesis of the theorem implies that $A \in \bar{\Sigma}$ since each $A_i \in \bar{\Sigma}$. Let $\{B_j^i\}$ be any disjoint collection of subsets of A_i . $\theta^{-k}A_i \subset B$ implies $\theta^{-k}B_j^i \subset B$ for each j . We have

$$(6.26) \qquad \mu\left(\bigcup_{j=1}^{\infty} B_j^i\right) = \mu\left(\theta^{-k}\left(\bigcup_{j=1}^{\infty} B_j^i\right)\right) \qquad \text{for each } k$$

$$= \mu\left(\bigcup_{j=1}^{\infty} \theta^{-k}B_j^i\right)$$

$$(6.27) \qquad = \sum_{j=1}^{\infty} \mu\left(\theta^{-k}B_j^i\right)$$

$$(6.28) \qquad = \sum_{j=1}^{\infty} \mu\left(B_j^i\right) .$$

(6.26) is gotten using the invariance of μ on $\bar{\Sigma}$, (6.27) by the countable addivity of μ on B , and (6.28) by the invariance of μ on $\bar{\Sigma}$. This is the countable addivity of μ on A_i .

To show the countable addivity of μ on A , let $\{D_i\}$ be any disjoint collection of subsets of A . Let $D_i^j = D_i \cap A_j$ so that

$$\mu\left(\bigcup_{i=1}^{\infty} D_i\right) = \mu\left(\bigcup_{i=1}^{\infty}\left(\bigcup_{j=1}^{n} D_i^j\right)\right)$$

$$= \mu\left(\bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{\infty} D_i^j\right)\right)$$

$$(6.29) \qquad = \sum_{j=1}^{n} \mu\left(\bigcup_{i=1}^{\infty} D_i^j \right)$$

$$(6.30) \qquad = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \mu\left(D_i^j\right)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{n} \mu\left(D_i^j\right)$$

$$(6.31) \qquad = \sum_{i=1}^{\infty} \mu(D_i) \; .$$

We get (6.29) by the finite addivity of μ on $\overline{\Sigma}$, (6.30) by the countable additivity of μ on A_j , and (6.31) by the finite addivity of μ on $\overline{\Sigma}$. This then is the countable addivity of μ on A .

As was mentioned at the end of Part II of Theorem 6.15, the technique in Lemma 2.19 to prove $\overline{\mu}$ non-trivial, also proves $\mu(A) = \overline{\mu}(A)$ for any set A on which μ is countably additive. Hence by the remarks in Part IV of 6.15, since $A \in \overline{\Sigma}$, we have $\sup \mu_n(A) < \infty$ and as was shown there, (6.16) is implied by this fact. Q.E.D.

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