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by

Ennio Gozzi

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FUNCTIONAL INTEGRAL APPROACH TO PARISI-WU STOCHASTIC QUANTIZATION AND RELATED PROBLEMS.

by

Ennio Gozzi

ADVISOR: Prof.B.SAKITA

ABSTRACT

A path-integral formulation is given of the Parisi-Wu method of stochastic quantization. The connection between the Langevin and Fokker-Planck equations is rederived in functional form bringing to light new aspects of these equations. The relation between this functional method and the recent approach by De-Alfaro-Fubini-Furlan is clarified. In particular we present the non-local transformation that connects the two formulations. In this dissertation we also investigate the hidden supersymmetry recently discovered by Parisi and Sourlas in Langevin processes. The Kelvin relations, for diffusion processes out of equilibrium, are shown to be a necessary and sufficient condition to have supersymmetry.This proves that this symmetry is a manifestation, at the path-integral level, of the

Onsager principle of microreversibility present in the stochastic process. We analyze, in particular, the interesting interplay between forward and backward Fokker-Planck dynamics and, as an application, we rederive the fluctuation-dissipation theorem as a Ward-identity of this supersymmetry.

Hunc igitur terrorem animi tenebrasque necessest
non radii solis neque lucida tela dici discuntiant,
sed naturae species ratioque.

Lucretius

(De Rerum Naturae II 58-61)

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HIDDEN SUPERSYMMETRY IN STOCHASTIC PROCESSES

INTRODUCTION AND SUMMARY.

In the last ten years we have witnessed a great advancement in our understanding of quantum field theory. For the first time physicists have tried to go beyond perturbation theory to which they were accustomed from quantum electrodynamics.

New approximation techniques have been developed like WKB method, large-N expansion, strong coupling expansion, lattice and many more. These new methods have revealed that field theories possess a richer structure than what one suspects in ordinary perturbation theory. Still, even with all these new methods, we are far from having an exact quantitative understanding of the physics involved in many processes. It is for this reason that physicists continue to struggle to develop new and more powerful tools to treat field theories. It was within this effort that, three years ago, G.Parisi and Y.S.Wu[1] proposed a new and interesting method for the quantization of physical systems. Their idea was to introduce a fifth "time" and postulate a stochastic Langevin dynamics for the system. These authors showed that, at least at the perturbative level, the usual quantum theory is reproduced in the equilibrium limit ($\tau \to \infty$) of that dynamics.

The first advantage of this method is the possibility to quantize gauge theories without fixing the gauge. We know that the procedure of gauge fixing and insertion of Faddeev-Popov ghosts is necessary in usual perturbation theory, but it breaks down in a non-perturbative regime where the phenomenon of the Gribov ambiguities[18] appears. The hope of Parisi-Wu was to give a new perturbative procedure free of the Gribov problem. Much work has already been done in this direction to settle this point[17].

Another recent application of the Parisi-Wu method is the use of the Langevin equation for the numerical simulation of lattice field theory models [20]. The computer simulation of physical systems is usually done using the traditional Montecarlo methods, but Parisi pointed out that the Langevin approach has some potential advantages, like the control over convergence and the possibility of arranging for cancellation of statistical fluctuations. The recent computer calculations of hadronic masses[20] in QCD have been achieved mainly thanks to these properties of the Langevin simulation.

The third advantage of the stochastic quantization, and we hope not the last, has been a better and deeper understanding[7] of the so called quenching procedure for the large N expansion of gauge theory models.

In view of all these applications, and hoping for more to come, we have reformulated the Parisi-Wu method in a path integral form[13,2,3,5] and we present it in section I of this dissertation. The hope is to be able to use all the techniques developed for functional integration in recent years[10], and bring out the rich content, still undiscovered, in this new approach of Parisi-Wu. We also hope to be able to put this new technique on the same firm ground as the two previous methods of quantization: canonical and path integral one.

One of the first thing we have done, using path-integral in stochastic processes, is a rederivation of the Fokker-Planck equation from the Langevin one. The relation between these two equations was well-known[9] in the literature but never before derived in path integral form. Our derivation helps in clarifying many aspects (stationarity, boundary conditions etc) that were somehow hidden in the Langevin approach (I-A).

Around the same time of the Parisi-Wu proposal, other authors (De-Alfaro,Fubini,Furlan)[12] also advanced the idea of introducing a fifth "time" τ and a functional formalism in this extra time. The connection between this method and the one of Parisi-Wu was not clear at that time. It has been fully clarified in some work we have done[3] and which is the subject of paragraph I-B. The relation between these two approaches also helps in understanding, at least at a formal level, the mechanism of gauge fixing in stochastic quantization (I-C).

The second section of this dissertation is devoted to the study of another phenomenon recently found in Langevin processes. Parisi and Sourlas [13] discovered in 1982 that the generating functional associated with Langevin equations, manifests an hidden supersymmetry. This interesting phenomenon has attracted a lot of interest especially among the experts in supersymmetry[13,15,16,23]. Their hope was to be able to convert any supersymmetric generating functional for non-trivial models (Wess-Zumino, SuperYang-Mills,etc) in a stochastic equation or a system of them. Succeeding in doing this would have meant having at their disposal all the tools that mathematicians have developed, during the last 60 years, for stochastic equations. This high hope has been frustrated lately and not much progress has been made in this problem.

Our interest in this topic lies in a different direction. Somehow our approach is the opposite of the current one: we want to understand which feature of the stochastic process is responsible for this hidden supersymmetry. We have found the answer analyzing the multicomponent Langevin equation.

$$\dot{\varphi}_i = -F_i + \eta_i$$

These equations are the ones that describe transport phenomena of not one only physical quantity φ , but of many of them at a time, like charge density, heat density, matter density and so on. It is well known, in fact, that even under one single drift force, like electric field for example, we have diffusion of not only charge, but also of heat, matter and so on.

Building the generating functional associated with these equations[5] (see II-B), we found that it manifests a supersymmetric invariance only if the following relations are satisfied by the drift forces.

$$\frac{\partial F_i}{\partial \varphi_j} = \frac{\partial F_j}{\partial \varphi_i}$$

These relations are known in the literature[24] on non-equilibrium thermodynamics as Kelvin relations. They reveal a sort of cross-phenomenon typical of these multidiffusion processes: an example is the Peltier-Seebeck effect.

The Kelvin relations, in our scheme, become sufficient and necessary conditions to have supersymmetry in the Langevin process. The Kelvin relations have been tested experimentally long ago[24],and long ago Onsager[14] tried to understand theoretically their origin. He found the answer in the so called "microreversibility principle" nowadays known as Onsager principle He proved[14], in two remarkable papers, that

assuming time reversibility for the microscopic equations of motion, the Kelvin relations could directly be obtained. (see II-C). This means that this microreversibility gives signals of itself even at the level of the macroscopic-irreversible processes and the signals are the Kelvin relations.
From our analysis it is clear that the Kelvin relations are a common feature of supersymmetry and of microreversibility thus suggesting that our supersymmetry might be a manifestation on the macroscopic variables φ_i of the Onsager principle. This idea is further confirmed in (II-D) where the Ward identities, stemming from this symmetry, are derived. They can be read, once the Fermionic variables have been integrated away, as symmetry relations between forward and backward Fokker-Planck dynamics, thus confirming the nature of this supersymmetry as a time symmetry. This answers our original question of which feature of the stochastic process is responsible for the supersymmetry.

As an application we derive (II-D) the fluctuation-dissipation theorem as a Ward identity in the equilibrium limit. This derivation throws new light (II-F) on this important theorem. We think that the connection we have made between supersymmetry and Onsager principle might help in further clarifying many still obscure problems in non-equilibrium thermodynamics[24]. We feel that the work presented in this second section of the dissertation is another interesting example of the cross-fertilization between field theory and statistical mechanics.

In the last two paragraphs of this thesis (II-E,F) we show that, even in any quantum mechanical model and not only in Langevin processes, we have a hidden supersymmetry. Some speculations on the meaning of this phenomenon and its relation to the Nicolai mapping are presented in the last paragraph (II-F).

-SECTION-I-

FUNCTIONAL METHODS FOR STOCHASTIC PROCESSES

I-A):Path-integral approach to Parisi-Wu stochastic quantization

The quantum correlation functions for a physical system, described by an Euclidean action $S[\varphi]$, are given by

$$\langle 0|T\varphi(x_1)\cdots\varphi(x_\ell)|0\rangle = \frac{\int \mathcal{D}\varphi\,[\varphi(x_1)\cdots\varphi(x_\ell)]\,e^{-S[\varphi]}}{\int \mathcal{D}\varphi\, e^{-S[\varphi]}}. \qquad \text{(I-1)}$$

Parisi and Wu [1] proposed the following alternative method to obtain the quantum averages:

i) Introduce a fifth "time" τ, in addition to the usual four space-time x^μ, and postulate the following Langevin equation for the dynamics of the field φ in this extra time

$$\frac{\partial\varphi(x^\mu,\tau)}{\partial\tau} = -\frac{\partial S[\varphi]}{\partial\varphi} + \eta(x^\mu,\tau) \qquad \text{(I-2)}$$

with η a Gaussian noise

$$\langle \eta(x,\tau)\rangle_\eta = 0$$

$$\langle \eta(x,\tau)\,\eta(x',\tau')\rangle_\eta = 2\,\delta(x-x')\,\delta(\tau-\tau')$$

$$\langle \eta(x_1\tau_1)\cdots\,\eta(x_{2n+1},\tau_{2n+1})\rangle_\eta = 0$$

$$\langle \eta(x_1\tau_1)\cdots\,\eta(x_{2n},\tau_{2n})\rangle_\eta = \sum_{\substack{\text{Possible}\\ \text{Pair}\\ \text{comb.}}}\ \prod_{\text{Pair}} \langle \eta(x_i\tau_i)\,\eta(x_j\tau_j)\rangle_\eta . \qquad \text{(I-3)}$$

The angular brackets $\langle\ \ \rangle_\eta$ denote Gaussian averages with respect to the random variable η .

ii) Evaluate the stochastic averages of fields φ_η satisfying eq. (I-2) , that are

$$\langle \varphi_\eta(x_1\tau_1)\,\varphi_\eta(x_2\tau_2)\cdots\cdots\,\varphi_\eta(x_\ell\tau_\ell)\rangle_\eta . \qquad \text{(I-4)}$$

iii) Put $\tau_1=\tau_2=\cdots=\tau_\ell$ in (I-4) and take the limit $\tau_1\to\infty$.

It is then possible to prove, at least perturbatively, that

$$\lim_{\tau_1\to\infty}\langle \varphi_\eta(x_1\tau_1)\cdots\cdots\,\varphi_\eta(x_\ell\tau_1)\rangle_\eta = \frac{\int \mathcal{D}\varphi\,[\varphi(x_1)\cdots\varphi(x_\ell)]\,e^{-S[\varphi]}}{\int \mathcal{D}\varphi\ \ e^{-S[\varphi]}} \qquad \text{(I-5)}$$

We will try first to understand this equation perturbatively and later on through probabibilistic arguments. For simplicity let us suppose $S[\varphi]$ is a free scalar theory. The Langevin eq. is then

$$\frac{\partial\varphi}{\partial\tau} = \partial^2\varphi - m^2\varphi + \eta . \qquad \text{(I-6)}$$

Its solution can be written as

$$\varphi(x,\tau) = \int_{0^-}^{\tau} d\tau' \int dy\, G(x-y\,,\tau-\tau')\,\eta(y,\tau') \tag{I-7}$$

where G is the retarted Green function which satisfies

$$\frac{\partial G(x,\tau)}{\partial \tau} = (\partial^2 - m^2)\, G(x,\tau) + \delta(x)\,\delta(\tau).$$

$$G(x,\tau) = 0 \qquad \tau < 0 \tag{I-8}$$

In (I-7) we have chosen $\varphi(x,0)=0$.This choice does not affect the final result thanks to the Markoff character of the stochastic process (I-6)[9] (see pag.11).

The solution to (I-8) is (x 1-dim.for simplicity)

$$G(x,\tau) = \int \frac{dk}{2\pi}\, \exp\{-\tau(k^2+m^2) + ikx\}\, \theta(\tau) \tag{I-9}$$

The next step is to use (I-7) and (I-9) to calculate the correlation functions

$$\langle \varphi_\eta(x,\tau)\, \varphi_\eta(x',\tau') \rangle_\eta \equiv D(x-x'\,;\tau,\tau') \tag{I-10}$$

We find

$$D(x-x'\,;\tau,\tau') = 2\int_0^\infty d\tau'' \int dy\, G(x-y\,,\tau-\tau'')\, G(x'-y\,,\tau'-\tau'')$$

that, in momentum space, is

$$D(k\,;\tau,\tau') = \frac{\exp[-(k^2+m^2)(\tau-\tau')]}{k^2+m^2} \left\{ 1 - \exp[-2(k^2+m^2)\tau'] \right\} \tag{I-11}$$

Following the steps ii) and iii)(pag.12) of the Parisi-Wu prescription, we see that

$$\lim_{\tau\to\infty} D(k;\tau,\tau) = \frac{1}{k^2+m^2} \qquad \text{(I-12)}$$

and this confirms the relation (I-5), at least for the free case. It is of course possible to treat also the interacting case, developing the perturbation theory from eq.(I-2), and order by order checking that (I-5) holds.

As the φ_η , solutions of (I-2), are stochastic variables another alternative description of their evolution is through the concept of probability. We can define all a set of "Conditional probability functions "

$$P_1(\varphi,\tau)$$
$$P_2(\varphi^0 0 \mid \varphi^1 \tau_1)$$
$$P_3(\varphi^0 0, \varphi^1 \tau^1 \mid \varphi^2 \tau^2)$$

where $P_1(\varphi,\tau)$ is the probability to be in configuration φ at time τ ; $P_2(\varphi^0 0 \mid \varphi^1 \tau^1)$ is the probability to be in configuration φ^1 at time τ_1 , if the system started at 0 in configuration φ_0. In the same way $P_3(\varphi^0 0, \varphi^1 \tau^1 \mid \varphi^2 \tau^2)$. is the probability to be in φ^2 at time τ^2 if it was in φ^0 at $\tau=0$ and in φ^1, τ^1. The connection between the φ_η that appears in (I-4), and the probability P_2.

is through the relation:

$$\int F[\varphi_\eta(\tau)]\, e^{-\int_0^\tau \eta^2/4\, d\tau'}\, \tilde{\mathcal{D}}\eta = \int F[\varphi(\tau)]\, P_2(\varphi_0 0|\varphi\tau)\, \mathcal{D}\varphi(\tau)$$

$$\tilde{\mathcal{D}}\eta = \lim_{N\to\infty} \prod \mathcal{D}\eta(\tau_i)$$

where the φ_η is solution of the Langevin eq. with initial configuration φ_0.

The process described by the Langevin equation is of Markoff type[9], that means

$$\frac{P_q(\varphi^0 0 \cdots \varphi_{q-2}\tau_{q-2}\,|\,\varphi_{q-1}\tau_{q-1})}{P_{q-1}(\varphi^0 0 \cdots \varphi_{q-3}\tau_{q-3}\,|\,\varphi_{q-2}\tau_{q-2})} = P_2(\varphi_{q-2}\tau_{q-2}\,|\,\varphi_{q-1}\tau_{q-1}) \qquad \text{(I-13)}$$

The essence of this relation is that the system forgets its initial conditions , and all the conditional probabilities can be derived from P_2 . From now on we will call P_2 , for simplicity, P . For this function P an equation exists that describes its evolution in time τ. It is called the Fokker-Planck (F.P.) equation and it has been derived many times in the literature[9].

$$\frac{\partial P}{\partial \tau} = \frac{\partial^2 P}{\partial \varphi^2} + \frac{\partial}{\partial \varphi}\left(P \frac{\partial S}{\partial \varphi}\right). \qquad \text{(I-14)}$$

This equation is nothing else than a differential version of the well-known Smoluchowski relation

$$\int \mathcal{D}\varphi\, P(\varphi_0 0|\varphi\tau)\, P(\varphi\tau|\varphi_1\tau_1) = P(\varphi_0 0|\varphi_1\tau_1) \qquad \text{(I-15)}$$

which, on the other side, is nothing else than an expression of the Markoff character (I-13).

The F.P.equation (I-14) can be recast in a Schrodinger type form

$$\frac{\partial \psi}{\partial \tau} = -2\hat{H}^{F.P}\psi . \qquad \text{(I-16)}$$

where $\psi \equiv P e^{S/2}$.

and $\hat{H}^{F.P} = -\frac{1}{2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{8}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \frac{1}{4}\frac{\partial^2 S}{\partial \varphi^2}$.

Because of the form of eq.(I-16), we will call $\hat{H}^{F.P}$ the Fokker-Planck "Hamiltonian". It is a positive semi-definite operator

$$\hat{H}^{F.P} = \frac{1}{2} Q Q^{+} \quad , \quad Q = \left(\frac{\partial}{\partial \varphi} + \frac{1}{2}\frac{\partial S}{\partial \varphi}\right).$$

$$\hat{H}^{F.P}\psi_n = E_n \psi_n \quad ; \quad E_n \geq 0 \qquad \text{(I-17)}$$

whose ground-state $E_0 = 0$ is $\psi_0 = e^{-S/2}$. The general solution of eq. (I-16) is

$$\psi[\varphi,\tau] = \sum_n c_n \psi_n e^{-2E_n\tau} \qquad \text{(I-18)}$$

c_n normalizing constants. The probability, from eq.(I-18), is

$$P[\varphi,\tau] = e^{-S/2} \sum_n c_n \psi_n e^{-2E_n\tau}.$$

The only term that does not go to zero, in the limit $\tau \to \infty$, is ψ_0 and so we have

$$\lim_{\tau\to\infty} P[\varphi,\tau] = c_0\, e^{-S/2}\, e^{-S/2} = e_0\, e^{-S[\varphi]}.$$

This"formally"proves (I-5).

As we said in the introduction, our goal in this section is to build a generating functional (that we are going to call $Z^{F.P.}$ for Fokker-Planck generating functional) from which the correlation functions (I-4) can be derived

$$\langle \varphi_\eta(x_1\tau_1) \cdots \varphi_\eta(x_\ell\tau_\ell)\rangle_\eta \equiv \left.\frac{\delta^\ell Z^{F.P.}[J]}{\delta J(x_1\tau_1)\cdots\delta J(x_\ell\tau_\ell)}\right|_{J=0} \qquad \text{(I-19)}$$

We feel that this reformulation of the stochastic quantization may offer new insights and, in particular, may help in bringing to light new features, still undiscovered, of the Parisi-Wu method.

The $Z^{F.P.}$, in (I-19) ,can be built if we retrace the steps i),ii) iii)(pag.8) of the Parisi-Wu prescription:

$$Z^{F.P.}[J] = N\int \tilde{\mathcal{D}}\varphi\,\tilde{\mathcal{D}}\eta\, P(\varphi(0))\,\delta(\varphi-\varphi_\eta)\exp\left[-\int_0^\tau J\varphi\, d\tau'\right]\exp\left[-\int_0^\tau \eta^2/4\, d\tau'\right].$$

$$\tilde{\mathcal{D}}\varphi \equiv \lim_{N\to\infty}\prod_{i=0}^{N}\mathcal{D}\varphi(\tau_i). \qquad \text{(I-20)}$$

φ_η that appears in (I-20) is the solution of the Langevin eq.(I-2) solved with initial probability $P(\varphi(0))$; N is a normalizing constant and $\tilde{\mathcal{D}}\varphi = \lim_{N\to\infty}\prod_{i=0}^{N}\mathcal{D}\varphi(\tau_i)$ where $\varphi(\tau_i)$are the field configuration at the time τ_i , having sliced the interval $0-\tau$ in N infinitesimal parts ε. This measure is a product of the usual four-dimensional path integral measures .

The $\delta(\varphi-\varphi_\eta)$ in (I-20) is a "formal" Dirac $\delta(\ )$ that can be written as

$$\delta(\varphi-\varphi_\eta) = \delta(\dot{\varphi} + \frac{\partial S}{\partial\varphi} - \eta) \| \frac{\delta\eta}{\delta\varphi} \| \qquad \text{(I-21)}$$

where $\|\frac{\delta\eta}{\delta\varphi}\|$ is the Jacobian of the transformation $\eta \to \varphi$ in (I-2)

$$\| \frac{\delta\eta}{\delta\varphi} \| = \det \left[\left(\partial_\tau + \frac{\partial^2 S}{\partial\varphi(\tau)\partial\varphi(\tau')} \right) \delta(\tau-\tau') \right]$$

With formal manipulations, we can write this as

$$\| \frac{\delta\eta}{\delta\varphi} \| = \exp \left[\mathrm{tr} \ln \left(\partial_\tau + \frac{\partial^2 S}{\partial\varphi(\tau)\partial\varphi(\tau')} \right) \delta(\tau-\tau') \right] =$$

$$= \exp \left[\mathrm{tr} \ln \partial_\tau \left(\delta(\tau-\tau') + \partial_{\tau'}^{-1} \frac{\partial^2 S}{\partial\varphi(\tau)\partial\varphi(\tau')} \right) \right]. \qquad \text{(I-22)}$$

where $\partial_{\tau'}^{-1}$ is just to indicate the Green function $G(\tau-\tau')$ solution of

$$\partial_\tau G(\tau-\tau') = \delta(\tau-\tau') \qquad \text{(I-23)}$$

The solutions are $G(\tau-\tau') = \theta(\tau-\tau')$ if we choose propagation forward in time, or $G(\tau-\tau') = -\theta(\tau'-\tau)$ for propagation backward in time. It is also possible to choose $G(\tau-\tau') = \frac{\theta(\tau-\tau') - \theta(\tau'-\tau)}{2}$ but we will concentrate on the first two. In the first case (propagation forward in time, that is the one chosen by Parisi-Wu) we get

$$\| \frac{\partial\eta}{\delta\varphi} \| = \exp(\mathrm{tr} \ln \partial_\tau) \exp \left[\mathrm{tr} \ln \left(\delta(\tau-\tau') + \theta(\tau-\tau') \frac{\partial^2 S}{\partial\varphi(\tau)\delta\varphi(\tau')} \right) \right].$$

the term $\mathrm{tr}\,\ln\partial_\varphi$ can be dropped, as it cancels with the same term in the denominator of (I-20) once we normalize $\hat{Z}^{F.P} = \dfrac{Z^{F.P}[J]}{Z^{F.P}[0]}$, so we are left with

$$\left\|\frac{\delta\eta}{\delta\varphi}\right\| = \exp\left[\mathrm{tr}\,\ln\left(\delta(\tau-\tau')+\theta(\tau-\tau')\frac{\delta^2 S}{\delta\varphi(\tau)\,\partial\varphi(\tau')}\right)\right].$$

With the usual "formal" expansion for the ln, we obtain

$$\left\|\frac{\delta\eta}{\delta\varphi}\right\| = \exp\left[\mathrm{tr}\left\{\theta(t-t')\frac{\partial^2 S}{\partial\varphi(\tau)\partial\varphi(\tau')} + \theta(\tau-\tau')\,\theta(\tau'-\tau)\frac{\partial^2 S}{\partial\varphi(\tau)\partial\varphi(\tau')}\,\frac{\partial^2 S}{\partial\varphi(\tau')\partial\varphi(\tau)} + \cdots\right\}\right]$$

$$= \exp\left[\int d\tau\,\theta(0)\frac{\partial^2 S}{\partial\varphi^2(\tau)} + \int d\tau'\,d\tau\,\theta(\tau-\tau')\,\theta(\tau'-\tau)\frac{\partial^2 S}{\partial\varphi(\tau)\partial\varphi(\tau')}\cdot\frac{\partial^2 S}{\partial\varphi(\tau')\partial\varphi(\tau)} + \cdots\right]$$

The second term in this expression is zero, because $\int\theta(t-t')\,\theta(t'-t)\,dt' = 0$ and the same for all subsequent terms. The only one left is the first term and regularizing $\theta(0)=\frac{1}{2}$ * we get

$$\left\|\frac{\delta\eta}{\delta\varphi}\right\| = \exp\left[\frac{1}{2}\int_0^T d\tau'\,\frac{\partial^2 S}{\partial\varphi^2}\right]. \tag{I-24}$$

Inserting (I-24) and (I-21) back into (I-20) and performing the integration over η , we have

$$Z^{F.P}[J] = \int \mathcal{D}\varphi\, P(\varphi(0))\exp\left\{-\int_0^T\left[\frac{1}{4}\left(\dot\varphi+\frac{\partial S}{\partial\varphi}\right)^2-\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\right]d\tau' + J(\tau_1)\,\varphi(\tau_1)\right\} \tag{I-25a}$$

The regularization $\theta(0)=1/2$ corresponds to a mid-point prescription for the path-integral (I-20) (see ref.10 capt.5)

In (I-25a) we have restricted $J(\tau)$ to $J(\tau_1)$ just because we are interested, in (I-5), only in correlations at the same time τ_1. In (I-25a) we have neglected the normalizing constant $\mathcal{N}$ and all the usual four-space integration .It is easy to reinstate them when necessary.

The Lagrangian in (I-25a), which we call the F.P. lagrangian, does not seem to have any relation to the hamiltonian in (I-16). It is not difficult to see the connection :Let us first in (I-25a) perform the integration of the term $\int_0^\tau \frac{1}{2}\dot{\varphi}\frac{\partial S}{\partial \varphi} d\tau' = \frac{1}{2}[S(\varphi(\tau)) - S(\varphi(0))]$ and second let us rescale the time $\tau' \rightarrow \tau'/2$, so that we get

$$Z^{F.P}[J] = \int \mathcal{D}\varphi(0)\, P(\varphi(0))\, e^{+S[\varphi(0)]/2}\, \mathcal{D}\varphi(2\tau)\, e^{-S[\varphi(2\tau)]/2}\, \tilde{\mathcal{D}}''\varphi\, e^{-\int_0^{2\tau}\mathcal{L}^{F.P} d\tau'} \tag{I-25b}$$

where $$\tilde{\mathcal{D}}''\varphi = \lim_{N\to\infty} \prod_{i=1}^{N-1} \mathcal{D}\varphi(\tau_i).$$

and $$\int_0^{2\tau}\mathcal{L}^{F.P} d\tau' = \int_0^{2\tau}\left[\dot{\varphi}^2/2 + \frac{1}{8}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \frac{1}{4}\frac{\partial^2 S}{\partial \varphi^2}\right] d\tau'$$

Now it is clear why we called $\mathcal{L}^{F.P.}$ Fokker-Planck Lagrangian: It is the "Euclidean" Lagrangian for the Hamiltonian $\hat{H}^{F.P.}$ in (I-16) and, for this same reason, we also call $Z^{F.P}$ the generating functional of the F.P.dynamics.

Going back to the derivation of (I-25a) we want to stress what has been done: we have integrated away the η and replaced the role it plays in the Langevin eq. with a sort of "effective" action $\mathcal{L}^{F.P}$ .This Lagrangian might look very complicated but it contains only the field φ as a dynamical variable (Fokker-Planck dynamics) and the noise has disappeared .

With the generating functional (I-25a) we can develop perturbation theory[8] using the Feynman rules dictated by $\mathcal{L}^{F.P.}$.
The number of graphs is very large, due to the extra vertices contained in $\frac{1}{8}\left(\frac{\partial S}{\partial\varphi}\right)^2-\frac{1}{4}\frac{\partial^2 S}{\partial\varphi^2}$
The careful reader might ask what happens if in (I-24) we had chosen $G(\tau-\tau')=-\theta(\tau'-\tau)$ .Then the action in (I-25) would have been

$$\int_0^{2\tau}\mathcal{L}^{F.P}d\tau'=\int_0^{2\tau}\left[\frac{1}{2}\dot{\varphi}^2+\frac{1}{8}\left(\frac{\partial S}{\partial\varphi}\right)^2+\frac{1}{4}\frac{\partial^2 S}{\partial\varphi^2}\right]d\tau$$

The only difference from (I-25) is in the sign of the third term.The corresponding Hamiltonian is known in the literature[9] as the Kolmogoroff-Fokker-Planck backward Hamiltonian :

$$\hat{H}^{F.P}_{\text{back}}=-\frac{1}{2}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{8}\left(\frac{\partial S}{\partial\varphi}\right)^2+\frac{1}{4}\frac{\partial^2 S}{\partial\varphi^2}=\frac{1}{2}Q^+Q. \qquad \text{(I-26)}$$

Before concluding, we want to make a remark concerning the Jacobian (I-21). All the steps from (I-21) to (I-25), that we have done to derive the generating functional $Z^{F.P.}$, are possible only if the Jacobian is not identically zero. If this happens the $Z^{F.P}$ itself is zero. The same Langevin equation starting point of the stochastic quantization, loses all its meaning. Infact $\left\| \frac{\delta \eta}{\delta \varphi} \right\| = 0$ means that there is no field associated through (I-2) to a particular η . In technical language this can be expressed by saying that the winding number of the transformation $\eta \rightarrow \varphi$ is zero. This number Δ has been studied in great detail in ref. 13 and 15 and it is known as the Witten Index.

Up to know, of the random process (I-2) we have only used the property that it is Gaussian. Besides this property, there is another very interesting one: the stochastic process (I-3) is stationary.

By stationary[9] we mean a process whose "momenta" $C(\tau'_1, \tau'_2 \cdots \tau'_\ell)$.

$$C(\tau'_1, \tau'_2 \cdots \tau'_\ell) = \langle \eta(\tau'_1) \ldots \eta(\tau'_\ell) \rangle_\eta ,$$

are functions only of the differences $(\tau'_i - \tau'_j)$. The process (I-2), (I-3) has exactly this feature. A question that raises naturally is if also the correlation functions

$$\langle \varphi_\eta(x_1 \tau_1) \cdots \varphi_\eta(x_\ell \tau_\ell) \rangle_\eta . \qquad \text{(I-27)}$$

manifest this property. The answer is generally no. Infact the averages that we perform in (I-20) are not only in η, but also on the initial configuration $\varphi(0)$ for which we give the $P(\varphi(0))$. It is the form of this $P(\varphi(0))$ that determines if (I-27) is a function only of the differences $\tau_i-\tau_j$. The choice of Parisi-Wu[1] was $P(\varphi(0))=\delta(\varphi(0)-\varphi_1)$ (with φ_1 a definite configuration, even zero) and the perturbative calculation done by them (see I-11) showed that their choice of $P(\varphi(0))$ does not make (I-27) stationary. To find out which is the proper one, let us start supposing that (I-27) is stationary

$$\langle \varphi_\eta(x_1\tau_1)\cdots\varphi_\eta(x_q\tau_q)\rangle_{\eta,P(\varphi(0))} = f(\tau_1-\tau_2,\tau_2-\tau_3-\cdots) \tag{I-28}$$

where we use the notation $\langle\quad\rangle_{\eta,P}$, to remind us of the average over η and $P(\varphi(0))$. From (I-28) we see that, if we rescale all the τ_i of an amount T, nothing changes on the R.H.S., so

$$\langle \varphi_\eta(x_1\tau_1)\cdots\varphi_\eta(x_q\tau_q)\rangle_{\eta,P} = \langle \varphi_\eta(x_1,\tau_1+T)\cdots\varphi_\eta(x_q,\tau_q+T)\rangle_{\eta,P}$$

Let us first put on both sides $\tau_1=\tau_2=\cdots=\tau_q$.

$$\langle \varphi_\eta(x_1\tau_1)\cdots\varphi(x_q,\tau_1)\rangle_{\eta,P} = \langle \varphi_\eta(x_1,\tau_1+T)\cdots\varphi_\eta(x_q,\tau_1+T)\rangle_{\eta}.$$

and second let us take the limit of $T\to\infty$

$$\lim_{T\to\infty}\langle \varphi_\eta(x_1\tau_1)\cdots\varphi_\eta(x_q\tau_1)\rangle_{\eta,P} = \lim_{T\to\infty}\langle \varphi_\eta(x_1,\tau_1+T)\cdots\varphi_\eta(x_q,\tau_1+T)\rangle_{\eta,P}$$

The L.H.S. does not depend on T, while the R.H.S. (see I-5) is the "quantum" correlation function, so we have

$$\langle \varphi_\eta(x_1,\tau_1) \cdots\cdots \varphi_\eta(x_q,\tau_1)\rangle_{\eta,P} = \frac{\int \mathcal{D}\varphi[\varphi\cdots\varphi]\, e^{-S[\varphi]}}{\int \mathcal{D}\varphi\, e^{-S[\varphi]}} \qquad \text{(I-29)}$$

As the L.H.S is stationary, we can rescale all the fields backward of τ_1, that means

$$\langle \varphi_\eta(x_1,0) \cdots\cdots \varphi_\eta(x_q,0)\rangle_{\eta,P} = \langle \varphi_\eta(x_1,\tau_1) \cdots\cdots \varphi_\eta(x_q,\tau_1)\rangle_{\eta,P}$$

and using (I-29) we conclude that

$$\langle \varphi_\eta(x_1,0) \cdots\cdots \varphi_\eta(x_q,0)\rangle_{\eta,P} = \frac{\int \mathcal{D}\varphi[\varphi\cdots\varphi]\, e^{-S[\varphi]}}{\int \mathcal{D}\varphi\, e^{-S[\varphi]}} \qquad \text{(I-30)}$$

From this expression we can explicitly derive the form of $P(\varphi(0))$: the L.H.S of (I-30) is at $\tau=0$, so we do not have any random effect caused by η (η has not been switched on yet). The only average is with respect to $P(\varphi(0))$, that means the L.H.S of (I-30) is

$$\int \mathcal{D}\varphi(0)\, P(\varphi(0))\, [\varphi(x_1,0) \cdots\cdots \varphi(x_q,0)]$$

Comparing with its R.H.S, we get

$$P(\varphi(0)) = \frac{e^{-S[\varphi(0)]}}{\int \mathcal{D}\varphi(0)\, e^{-S[\varphi(0)]}} \qquad \text{(I-31)}$$

From (I-29) we can derive a second conclusion: With that particular form of $P(\varphi(0))$, for which the correlation functions are stationary, we do not need to take the limit $\tau_1 \to \infty$. At every time τ_1 , we have that the stochastic average is already the quantum one.

The physical meaning of this is very clear: from the beginning we put the system in the equilibrium distribution $P(\varphi) = e^{-S[\varphi]}$ and the presence of the Langevin dynamics $\dot{\varphi} = -\frac{\partial S}{\partial \varphi} + \eta$ does not modify this. On the contrary, in the case of the choice $P(\varphi(0)) = \delta(\varphi(0) - \varphi_1)$ we started with every field in configuration φ^1. and then the Langevin dynamics was able to spread them to the equilibrium form as the time $\tau \to \infty$. This gives us an understanding of how the Fokker-Planck dynamics works. This was first clarified by Onsager-Machlup and the $\mathcal{L}^{F.P}$ is known, in thermodynamics out-of-the-equilibrium , as Onsager-Machlup function[8]:it is a thermodynamics potential related to the rate of entropy production.To understand this let us recall the definition of entropy S

$\Delta S = -\ln \Delta P$.   (this is in infinitesimal form)

From (I-25a) we have $\Delta P = e^{-\mathcal{L}^{F.P}\Delta\tau}$ and $-\ln \Delta P = \mathcal{L}^{F.P}\Delta\tau$

so

$$\frac{\Delta S}{\Delta \tau} = \mathcal{L}^{F.P}.$$

That explains why $\mathcal{L}^{F.P}$ is related to the rate of entropy production.

Going now back to (I-31) and inserting this in (I-25a), we obtain

$$Z^{F.P}_{\substack{\mathrm{vacu}\\ \mathrm{vacu}}} = \int \mathcal{D}\varphi(0)\, e^{-S(\varphi(0))}\, {}'\tilde{\mathcal{D}}\varphi\, \exp\left[-\int_0^{\tau}\left[\frac{1}{4}\left(\dot{\varphi}+\frac{\partial S}{\partial\varphi}\right)^2-\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\right]d\tau'\right] \qquad \text{(I-32a)}$$

$${}'\tilde{\mathcal{D}}\varphi = \lim_{N\to\infty} \prod_{i=1}^{N} \mathcal{D}\varphi(\tau_i)$$

or inserting it in (I-25b) (with the time rescaled) we get

$$Z^{F.P}_{\substack{\mathrm{vacu}\\ \mathrm{vacu}}}[J] = \int \mathcal{D}\varphi(0)\, e^{-S[\varphi(0)]/2}\, \mathcal{D}\varphi(2\tau)\, e^{-S[\varphi(2\tau)]/2}\, \tilde{\mathcal{D}}''\varphi\, e^{-\int_0^{2\tau}\mathcal{L}^{F.P}d\tau} \qquad \text{(I-32b)}$$

We like to call this the vacuum-vacuum generating functional of the Fokker-Planck dynamics. The reason is clear if we remember that the ground-state (vacuum) of $\hat{H}^{F.P}$ is $\psi_0 = e^{-S/2}$. Another way (but less transparent) to get stationary correlation functions is to start from (I-25a) and take the limit of integration from $\int_0^{\tau}$ to $\int_{-\infty}^{\tau}$. What happens is that the Fokker-Planck dynamics builds up a probability, between $-\infty$ and 0 equal to e^{-S}, that means equal to the one we inserted at $\tau=0$ by (I-31).

Before concluding a word of caution is needed: stochastic quantization does not compel us to choose (I-31) . Any normalizable form of $P(\varphi(0))$ is acceptable: the result, the limit of $\tau_1 \to \infty$, is independent of $\varphi(0)$. The particular choice (1-31) has the advantage that it avoids step iii)(pag. 8) of the Parisi-Wu prescription. Somehow this $Z^{F.P}_{\substack{\mathrm{vacu}\\ \mathrm{vacu}}}$ is another manner to represent the traditional quantum generating functional. Its advantages will be described in the next paragraph.

I-B): The De-Alfaro-Fubini-Furlan functional approach and its connection to Parisi-Wu method.

Around the same time of the Parisi-Wu proposal, De-Alfaro-Fubini Furlan[12] advanced also the idea of introducing an extra time τ. and a corresponding generating functional. The connection with the work of Parisi-Wu was not clear at that time. This paragraph is aimed at clarifying this[3].

The method of quantization proposed by De-Alfaro et al.[12] can be summarized as follows:

i) Introduce a fifth time τ and consider the fields as functions of the four Euclidean variables and of τ.

ii) The conventional Euclidean Lagrangian $\mathcal{L}$ is interpreted as a "potential" energy term for a new "Lagrangian" L.

$$L = \frac{1}{2}\left(\frac{\partial \varphi}{\partial \tau}\right)^2 - \mathcal{L}. \tag{I-33}$$

The "Hamiltonian" associated to this L is

$$H = \frac{1}{2}\pi^2 + \mathcal{L}.$$

where

$$\pi = \frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial \varphi}{\partial \tau}.$$

and the associated "canonical" structure is

$$\{\varphi(x,\tau), \pi(y,\tau)\}_{P.B} = \delta(x-y)$$

If we take $\hbar = 1$ the dimensions of τ and π are respectively $[\tau] = \ell$; $[\pi] = \ell^{-2}$, ℓ is for length.

Using this H , the generating functional can now be written as the well-known Gibbs-average for equilibrium statistical mechanics

$$Z[J] = \int \mathcal{D}\pi \mathcal{D}\varphi \exp\left[-\int d^4x \, (H + J\varphi)\right] \tag{I-34}$$

π and φ in the measure $\mathcal{D}\pi\mathcal{D}\varphi$ are configurations at an

sure $\mathcal{D}\pi\mathcal{D}\varphi$ry time τ_1 . Because of the Liouville theorem, the measure $\mathcal{D}\pi\mathcal{D}\varphi$ and Z itself are completely independent of τ_1 . This form of the generating functional is not only formally very elegant because of its unifying character with statistical mechanics, but has also some technical advantages over the usual formulation.The measure of integration $\mathcal{D}\pi\mathcal{D}\varphi$ is, infact, canonical invariant and this allows us to perform canonical transformation in Z without ever having to introduce a Jacobian. One might still object that the addition of the momenta to the formalism is just a useless complication, because they can always be integrated away as they enter only as Gaussian variables . This objection is really at the root of the problem and will help in clarifying the issue:let us infact perform a transformation of variables

$$\varphi = F[\Phi].$$

$$\frac{\partial \varphi}{\partial \tau} = \frac{\partial F}{\partial \Phi} \frac{\partial \Phi}{\partial \tau}. \tag{I-35}$$

The Lagrangian $\mathcal{L}$ becomes

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial F}{\partial \Phi}\right)^2\left(\frac{\partial \Phi}{\partial \tau}\right)^2 - \mathcal{R}.$$

The new momenta π_Φ is

$$\pi_\Phi = \left(\frac{\partial F}{\partial \Phi}\right)^2 \frac{\partial \Phi}{\partial \tau}.$$

and the new "Hamiltonian"

$$H = \left(\frac{\partial F}{\partial \Phi}\right)^{-2} \pi_\Phi^2 + \mathcal{R}.$$

As the measure of integration does not change in (I-34) under (I-35), we can write Z as

$$Z = \int \mathcal{D}\pi_\Phi \, \mathcal{D}\Phi \; e^{-\int\left[\left(\frac{\partial F}{\partial \Phi}\right)^{-2} \pi_\Phi^2 + \mathcal{R}\right] d^4x}$$

If we now integrate over π_Φ, we get

$$Z = \int \mathcal{D}\Phi \left[\det \frac{\delta F}{\delta \Phi}\right] \exp\left\{-\int d^4x \, \mathcal{R}\right\}.$$

This means that the definition (I-34) of Z has automatically produced the functional determinant of the transformation from φ to Φ. In doing this transformation we should, of course, take care of evaluating the Jacobian in the discretized form of the path integral, and also remember that we are using (see pag.16) a mid-point prescription. All this will produce, for non linear F, the well-known[11] extra potential ΔV in H first derived by J.L.Gervais and A.Jevicki.

The features, that we have just described for the generating functional (I-34), will help us in understanding why for this Z we do not need the introduction of the Faddev-Popov determinant[18] to quantize gauge theories. In general the form

(I-34) is a very powerful tool to get the correct Feynman rules for systems with constrains like gauge theories.

One might ask, at this point, why the extra Gaussian variables that enter in (I-34) are momenta. In the brief exposition done so far, one understands that these Gaussian variables are momenta because they are "derived" from a Lagrangian L in the usual fashion. This Lagrangian, on the other end, has been postulated "ad hoc"[12] so to get that particular form of Z . We would like, on the contrary, to obtain the generating functional (I-34) and understand the Gaussian variables π as momenta not by postulating L , but deriving all the formalism from more basic principles. We found[3] the way to do this in the stochastic quantization approach by Parisi-Wu.

In the previous paragraph we have derived a generating functional for the Parisi-Wu stochastic method and it would be interesting to compare this $Z^{F.P}$ with the Z in (I-34). The Lagrangian that appears in (I-25a) gives momenta that are

$$\tilde{\pi} = \frac{1}{2}\left(\dot{\varphi} + \frac{\partial S}{\partial \varphi}\right) \cdot \qquad \text{(I-37)}$$

but the dimensions of $\tilde{\pi}$ and τ are different from the ones of De-Alfaro et al.[12] and the reader might wonder how our generating functional $Z^{F.P.}$ can be related to (I-34). It is true that in both of them there is a fifth time τ , but there are also major differences:

1) The path integration in $Z^{F.P}$ is a product of four-dimensional path integrations, each at a particular fifth time slice τ_i, while the path integration in (I-34) is just one at an arbitrary fifth time τ_1.
2) In $Z^{F.P}$ (I-25a) the Lagrangian is integrated in $\int d\tau$ and $\int d^4x$ while in (I-34) the exponent $\int H d^4x$ is only integrated in $\int d^4x$.
3) Using $Z^{F.P}$ we have to send $\tau_1 \to \infty$ to recover the quantum averages, while using (I-34) we directly get them without taking any limit.

It is clear that we have to specialize $Z^{F.P}$ so that its τ_1 and τ dependence disappears. We proved in the previous paragraph (I-A) that the trick to achieve this is to choose $P(\varphi(0)) = e^{-S[\varphi(0)]}$ so that $Z^{F.P}$ becomes stationary or what we called the vacuum-vacuum generating functional of the Fokker-Planck dynamics (I-32a)

$$Z^{F.P}_{\substack{vacuum\\vacuum}} = \int \mathcal{D}\varphi(0)\, \exp[-S(\varphi(0))] \int \tilde{\mathcal{D}}\varphi\, \exp\left[-\int_0^{\tau}\left[\frac{1}{4}\left(\dot{\varphi}+\frac{\partial S}{\partial \varphi}\right)^2 - \frac{1}{2}\frac{\partial^2 S}{\partial \varphi^2}\right] d\tau' + J(\tau_1)\varphi(\tau_1)\right)$$

$$\tilde{\mathcal{D}}\varphi = \lim_{N\to\infty} \prod_{i=1}^{N} \mathcal{D}\varphi(\tau_i) \qquad \text{(I-38)}$$

With this choice, we have seen previously that:

i) The correlation-functions $\langle \varphi_{n} \cdots \varphi_{n} \rangle_{n}$ derived from (I-38) are stationary, that means they depend only on the differences $\tau_i - \tau_j$. As we are interested only in correlation functions at the same time τ_1, the stationarity implies that they are independent of τ_1. If so, we do not need to take the limit $\tau_1 \to \infty$ to recover the quantum averages. This means that, at any intermediate time τ_1 , the correlation functions derived from (I-38) are already the quantum ones, as in the Z of De-Alfaro-Fubini-Furlan.

ii) Because of this property, the expression (I-38) not only does not depend on τ_1, but also does not depend on τ at least for theories without internal symmetries.

These two properties i) and ii) are crucial for the the proof that (I-38) is equivalent to (I-34) but still (I-38) looks much different than (I-34).

The first step to do in (I-38), as it is independent of , is to choose $\tau_1 = 0$, so we have :

$$Z^{F.P}_{\substack{vacu\\vacu}} = \int \mathcal{D}\varphi(0) e^{-S[\varphi]} \int \tilde{\mathcal{D}}\varphi \exp\left\{-\int_0^\tau \left[\frac{1}{4}\left(\dot{\varphi} + \frac{\partial S}{\partial \varphi}\right)^2 - \frac{1}{2}\frac{\partial^2 S}{\partial \varphi^2}\right] d\tau' + J(0)\varphi(0)\right\}$$

$$\tilde{\mathcal{D}}\varphi = \lim_{N \to \infty} \prod_{i=1}^{N} \mathcal{D}\varphi(\tau_i) \qquad \text{(I-39)}$$

The second step is the following change of variables

$$\dot{X} = \dot{\varphi} + \frac{\partial S}{\partial \varphi}. \qquad \text{(I-40)}$$

Before proceeding, we should observe from (I-40) and the definition of momentum (I-37) that $\tilde{\pi} = \dot{X}/2$.

The transformation (I-40) is known in supersymmetry as the Nicolai transformation[16] and in this form it has been written for the first time by Von Holten [16]. This transformation is non local: infact if we integrate it, we get

$$\dot{X}(\tau) = \varphi(\tau) + \int_0^\tau \frac{\partial S}{\partial \varphi}[\varphi(\tau')] d\tau'$$

$$\text{(I-41)}$$

In any case, the important point is that $X(0)=\varphi(0)$ so that in (I-39) the current is still coupled to $\varphi(0)$ and not to some complicated non-local object. This is what we wanted to achieve when we moved $\tau_1 \to 0$. In (I-39) we can limit ourselves to analyze only the part

$$\int {}'\tilde{\mathcal{D}}\varphi \, \exp\left[-\int_0^{\tau}\left[\frac{1}{4}\left(\dot{\varphi}+\frac{\partial S}{\partial \varphi}\right)^2-\frac{1}{2}\frac{\partial^2 S}{\partial \varphi^2}\right]d\tau'\right] \tag{I-42}$$

To implement the change of variables (I-41), we need to calculate the Jacobian

$$J = \left\| \frac{\delta X(\tau)}{\delta \varphi(\tau')} \right\| .$$

This can be easily done, as in the previous paragraph, and the result is

$$J = \exp \int_0^{\tau} \frac{1}{2}\,\frac{\partial^2 S}{\partial \varphi^2}\, d\tau' \tag{I-43}$$

Inserting (I-43) in (I-42) we get

$$\begin{aligned}\int {}'\tilde{\mathcal{D}}\varphi \, \|J\| \exp\left[-\int_0^{\tau} \dot{X}^2/4 \, d\tau'\right] &= \int {}'\tilde{\mathcal{D}}\varphi \left\| \frac{\delta X}{\delta \varphi} \right\| \exp -\int_0^{\tau} \dot{X}^2/4 \, d\tau' = \\ &= \int {}'\tilde{\mathcal{D}} X \exp\left[-\int_0^{\tau} \dot{X}^2/4 \, d\tau'\right].\end{aligned} \tag{I-44}$$

In the new variables X we have a free dynamics. We should not be misled by this : as it is pointed out in ref.16, X is an extended object in terms of φ , so the free dynamics of is a free dynamics of extended objects.

The expression (I-44) can now be used in (I-34) and we obtain

$$Z^{F.P}_{\substack{\text{voem}\\ \text{voem}}} = \int \mathcal{D}\varphi(0) \exp[-S(\varphi(0))] \mathcal{D}X_N \, \tilde{\mathcal{D}}'' X \exp\left\{\left[-\int_0^{\tau} \dot{X}^2/4\right] + J(0)\varphi(0)\right\}.$$

$$X_N \equiv X(\tau_N)$$

The path integration $"\tilde{\mathcal{D}} X$ can be done, as it is a free one, and we get

$$Z^{F.P}_{\substack{\text{voem}\\ \text{voem}}} = \mathcal{N} \int \mathcal{D}\varphi(0) \exp[-S[\varphi(0)]] \mathcal{D}X_N \exp\left[-(X_N - X_0)^2/4\tau\right] \frac{1}{\sqrt{\tau}} \qquad \text{(I-45)}$$

(I-45) starts looking more like the generating functional by De-Alfaro-Fubini-Furlan (I-34). First of all we have in (I-45) as in (I-34) just path integration over two variables $\varphi(0)$ and X_N. and not, as before, all a chain of path integration; second, the exponent in (I-45) is just $S[\varphi]$ plus a Gaussian term in X_N.

From (I-45) we can also observe that $Z^{F.P}_{\substack{\text{voem}\\ \text{voem}}}$ is independent of τ, and because of this we can choose τ to be infinitesimal Δ. Remembering (I-39)

$$\tilde{\pi} = \lim_{\Delta \to 0} \frac{X(i+1) - X(i)}{\Delta} \, \frac{1}{2}. \qquad \text{(I-46)}$$

and reinserting $\int d^4x$, we can write (I-45) as

$$Z^{F.P}_{\substack{\text{voem}\\ \text{voem}}} = \mathcal{N} \int \mathcal{D}\varphi(0) \exp[-S(\varphi(0)) + J(0)\varphi(0)] \mathcal{D}X_N \, e^{-\int \frac{(X_N - X_0)^2}{4\Delta} d^4x} \frac{1}{\sqrt{\Delta}} = \qquad \text{(I-47)}$$

$$= \mathcal{N} \int \mathcal{D}\varphi(0) \exp[-S[\varphi(0)] + J(0)\varphi(0)] \mathcal{D}\tilde{\pi}(0) \exp\left[-\int \tilde{\pi}^2 \Delta \, d^4x\right] \Delta^{1/2}$$

Since this expression is independent of Δ , we can rescale $\tilde{\pi}$, $\tilde{\pi}\sqrt{\frac{\Delta}{2}} \doteq \hat{\pi}$ so that (I-47) becomes

$$Z^{FP}_{\substack{\text{vacuu}\\ \text{vacuu}}} = \int \mathcal{D}\varphi(0)\, \mathcal{D}\hat{\pi}(0)\, \exp - \int [\tfrac{1}{2}\hat{\pi}^2 + \mathcal{L} + J(0)\,\varphi(0)]\, d^4x$$

This is exactly the form of the generating functional (I-34) of De-Alfaro-Fubini-Furlan and we got it directly from stochastic quantization, without having to postulate any new lagrangian or other similar things, moreover it is now clear why the π are momenta and which is their exact derivation.

A word of explanation is needed for the rescaling $\tilde{\pi}\sqrt{\frac{\Delta}{2}} = \hat{\pi}$ that we have done. First of all $\tilde{\pi}$ has not the same dimension as the momenta that appear in the De-alfaro-Fubini-Furlan expression, while $\hat{\pi}$ has the same dimension; second, from (I-46) we can see that stochastically $\tilde{\pi}$ is not a well-behaved object. We know, infact, that $\langle \dot{X} \rangle \sim \frac{1}{\sqrt{\Delta}}$ and so $\langle \tilde{\pi} \rangle \sim \frac{1}{\sqrt{\Delta}}$ which means $\tilde{\pi}$ is not defined in the limit $\Delta \to 0$, while $\hat{\pi} \sim O(1)$ is a well behaved object.

I-C): Gauge fixing in stochastic quantization.

Having now established the equivalence of these two formulations we will use it to further clarify other aspects of the Parisi-Wu approach. According to these authors one of the advantages of the stochastic quantization is the possibility to quantize gauge theories without fixing the gauge. A lot of work has already been done in this direction [7,17] but we intend here to further clarify it using the functional methods just developed.

Let us deal, for simplicity, with abelian gauge theory

$$S[A] = \frac{1}{2}\int A_\mu(x)\left(-\Box\,\delta_{\mu\nu} + \partial_\mu\partial_\nu\right) A_\nu(x)\, d^4x.$$

It is well-known that the projector operator $\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu$ is singular so that we cannot explicitly invert it and obtain the propagator, unless we fix the gauge in $S[A]$.

The stochastic quantization[1] offers an alternative way to obtain the propagator. Repeating the same steps as in (I-2), we have the Langevin eq. for abelian gauge theory, that is

$$\frac{\partial A_\mu}{\partial \tau} = \left(\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu\right) A_\nu + \eta_\mu$$

$$\langle \eta_\mu \rangle_\eta = 0$$

$$\langle \eta_\mu(x,\tau)\,\eta_\nu(y,\tau')\rangle_\eta = 2\,\delta_{\mu\nu}\,\delta(\tau-\tau')\,\delta^4(x-x').$$

or in Fourier space

$$\dot{A}_\mu(k,\tau) = -k^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) A_\nu(k,\tau) + \eta_\mu(k,\tau)$$

$$\langle \eta_\mu(k,\tau)\rangle_\eta = 0 \qquad \text{(I-48)}$$

$$\langle \eta_\mu(k,\tau)\,\eta_\nu(k',\tau')\rangle_\eta = 2\,\delta_{\mu\nu}\,\delta^4(k+k')\,\delta(\tau-\tau').$$

Solving (I-48) with the initial conditions $A_\mu(k,0)=0$ (we follow Namiki et al.ref.17) we have

$$A_\mu(k,\tau)=\int_0^\infty G_{\mu\nu}(k;\tau-\tau')\,\eta_\nu(k,\tau')\,d\tau'$$

where $G_{\mu\nu}$ is the Green function solution of eq.(I-48)

$$\left[\delta_{\mu\rho}\frac{\partial}{\partial\tau}+(k^2\delta_{\mu\rho}-k_\mu k_\rho)\right]G_{\rho\nu}(k;\tau-\tau')=\delta_{\mu\nu}\delta(\tau-\tau'). \tag{I-49}$$

Choosing the initial conditions $G_{\mu\nu}(k+0)=\delta_{\mu\nu},\ G(-0)=0$ we get that

$$G_{\mu\nu}(k;\tau-\tau')=\left\{\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)e^{-k^2|\tau-\tau'|}-\frac{k_\mu k_\nu}{k^2}\right\}\theta(\tau-\tau')$$

The propagator in stochastic quantization (we call it from now on "stochastic propagator") will be

$$\langle A_\mu(k,\tau)A_\nu(k',\tau')\rangle_\eta=\int_0^\infty d\tau''\int_0^\infty d\tau'''\,G_{\mu\rho}(k,\tau-\tau'')\,G_{\nu\lambda}(k',\tau'-\tau''')\langle\eta_\rho(k,\tau'')\eta_\lambda(k,\tau''')\rangle_\eta$$

$$\equiv\delta^4(k+k')\,D_{\mu\nu}(k,\tau-\tau'). \tag{I-50}$$

with $D_{\mu\nu}(k;\tau-\tau')=\Delta^T_{\mu\nu}(k)\left[e^{-k^2|\tau-\tau'|}-e^{-k^2(\tau+\tau')}\right]+2\tau_<\frac{k_\mu k_\nu}{k^2}$

$\tau_<$ is the smaller between τ and τ'

and $\Delta^T_{\mu\nu}=\frac{1}{k^2}(\delta_{\mu\nu}-k_\mu k_\nu/k^2)$.

$\Delta^T_{\mu\nu}$ is nothing else than the ordinary propagator in the Landau gauge. For $\tau=\tau'\Rightarrow\infty$ the $D_{\mu\nu}$ becomes

$$D_{\mu\nu}(k,\tau,\tau)\longrightarrow\left[\Delta^T_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}2\tau\right]. \tag{I-51}$$

so we see that it has a transverse part $\Delta^T_{\mu\nu}$ and a longitudinal one $\frac{k_\mu k_\nu}{k^2}$ that apparently diverges with τ. According to Parisi-Wu this divergence should cancel in gauge invariant quantities.

The above approach seems really to permit the quantization of gauge theory without fixing the gauge. The reason for this can be understood from the character of the operator in (I-49). Contrary to the gauge theory operator $\left(\delta_{\mu\nu} - \frac{K_\mu K_\nu}{K^2}\right)$, this new operator is not singular and hence its inverse $G_{\mu\nu}(K,\tau-\tau')$ can be explicitly built. We can see this if we also Fourier-transform τ in $G_{\mu\nu}(K;\tau-\tau')$.

$$\tilde{G}_{\mu\nu}(K^\mu,\omega) = \int G_{\mu\nu}(K,\tau)\, e^{-i\omega\tau}\, d\tau.$$

we have

$$\left[(-i\omega+K^2)\delta_{\mu\rho} - K_\mu K_\rho\right]\tilde{G}_{\rho\nu}(K,\omega) = \delta_{\mu\nu}.$$

The presence of $i\omega$ in the matrix $\left[(-i\omega+K^2)\delta_{\mu\rho} - K_\mu K_\rho\right]$ acts as a mass term for the gauge field and this is what allows us to get a propagator.

At first sight everything seems quite clear, however we have to pay particular attention to the special choice that we have made of the initial conditions in (I-48) $A_\mu(K,0)=0$. It was this choice that gave origin to the Landau propagator for the transverse part of $D_{\mu\nu}$. It is infact easy to prove that different initial conditions produce, in the limit $\tau\to\infty$, different transverse propagator .[7,17] The careful reader might wonder how an initial condition can give a signal of itself in the propagator. After all we get the propagator sending $\tau\to\infty$ and, if the process is Markoffian, it should not bear any memory of the initial conditions. That is true for most of the variables entering the

the action $S[A]$,but not for the longitudinal components A_μ^L.
The Langevin eq. in momentum space is

$$\dot{A}_\mu(k,\tau) = -k^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)A_\nu(k,\tau) + \eta_\mu(k,\tau). \qquad \text{(I-52)}$$

Introducing the transverse and longitudinal part for A_μ.

$$A_\mu^T(k,\tau) = \left(\delta_{\mu\nu} - k_\mu k_\nu/k^2\right)A_\nu(k,\tau)$$

$$A_\mu^L(k,\tau) = \frac{k_\mu k_\nu}{k^2}A_\nu(k,\tau)$$

and the same for $\eta_\mu(k,\tau)$. ,we get that (I-52) becomes

$$\dot{A}_\mu^T = -k^2 A_\mu^T + \eta_\mu^T \qquad \text{(I-53a)}$$

$$\dot{A}_\mu^L = \eta_\mu^L. \qquad \text{(I-53b)}$$

The solutions of eq.(I-53a) ,with an arbitrary initial condition $A_\mu^T(k,0)$,is

$$A_\mu^T(k,\tau) = A_\mu^{T(0)}(k,\tau) + A_\mu^T(k,0)\,e^{-k^2\tau}.$$

where $A_\mu^{T(0)}(k,\tau)$ is the solution with initial condition $A_\mu^T(k,0)=0$ .We see that for $\tau \to \infty$, A_μ^T tends to $A_\mu^{T(0)}(k,\tau)$ irrespective of the of $A_\mu^T(k,0)$ value. This means that the transverse part of the gauge field does not bear any memory of the initial condition,the reason being that in (I-53a) there is a damping force $-k^2A_\mu^T$ .For A_μ^L. ,on the contrary,there is no damping force,so if we start with an initial configuration $A_\mu^L(k,0)=k_\mu\Phi$,we have

$$A_\mu^L(k,\tau) = k_\mu\Phi(k,0) + \int_0^\tau d\tau''\,\theta(\tau-\tau'')\,\eta_\mu^L(k,\tau'') \qquad \text{(I-54)}$$

and for $\tau \to \infty$ there is no way that, on the R.H.S. of (I-54), might disappear. This phenomena in stochastic quantization always happens (See J.Alfaro and B.Sakita ref.7) when the action has a symmetry and, as consequence of that, there are ignorable variables Ω associated with the generators of the symmetry. These variables, even if necessary to describe the dynamics of the system, never enter the action and so in the Langevin eq. the drift force $\frac{\partial S}{\partial \Omega}$. for them is zero.

For $k=0$ even the force on the transverse variables (I-53a) is zero :this is related to the infrared problem and we will not treat it here.

Going back to gauge theory, if we had calculated the $\langle A_\mu A_\nu \rangle_\eta$. using (I-54), we would have obtained

$$\langle A_\mu(k,\tau) A_\nu(k',\tau') \rangle_\eta = \frac{k_\mu k'_\nu}{k^2 k'^2} \Phi(k) \Phi(k') + D_{\mu\nu}(k,\tau,\tau').$$

The piece $\frac{k_\mu k'_\nu}{k^2 k'^2} \Phi(k) \Phi(k')$, that is independent of τ', modifies the transverse part $\Delta^T_{\mu\nu}$ of the "stochastic propagator".

$$\Delta^T_{\mu\nu} = \delta(k+k') \frac{1}{k^2} \left\{ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right\} + \frac{k_\mu k'_\nu}{k^2 k'^2} \Phi(k) \Phi(k').$$

We see that this is not any more in the Landau form. We can conclude that the gauge fixing really enters through the initial conditions of the longitudinal part of the Langevin eq. A second problem that is still open is the role of the divergent piece $2\tau \frac{k_\mu k'_\nu}{k^2 k'^2}$ in (I-51). Parisi-Wu [1] claimed that this divergence disappears when we calculate gauge invariant quantity. A detailed calculation done by Namiki et al.[17] showed that this is true at least to second order in perturbation theory. Moreover they proved that, using the "stochastic propagator", also a contribution identical to the one of the Faddeev-Popov ghosts arises. This was something crucial to prove. Infact, as the gauge fixing somehow enters through the initial conditions, then also the Faddeev-Popov contribution has to appear in a way or another to have unitarity.

The proof of Namiki et al.[17] was a perturbative one, we will present here a "formal" proof based on the functional methods developed in the previous paragraphs (I-A,B).

Starting from the Langevin eq. (I-53a,b) in position space

$$\dot{A}_\mu^T(x,\tau) = -\Box A_\mu^T + \eta_\mu^T(x,\tau) \qquad \text{(I-55)}$$

$$\dot{A}_\mu^L = \eta_\mu^L$$

with

$$A_\mu^T(x,\tau) = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)A_\nu(x,\tau)$$

$$A_\mu^L(x,\tau) = \frac{\partial_\mu\partial_\nu}{\Box}A_\nu(x,\tau)$$

we can write, as in paragraph (I-A) the corresponding $Z^{F.P.}$

$$Z^{F.P} = \int \tilde{\mathcal{D}}A_\mu^T\, \tilde{\mathcal{D}}A_\mu^L\, \tilde{\mathcal{D}}\eta_\mu^T\, \tilde{\mathcal{D}}\eta_\mu^L\, P(A_\mu^T(0))\, P(A_\mu^L(0))\, \delta(A_\mu^T - A^T_{\mu,\eta}) \cdot \delta(A_\mu^L - A^L_{\mu,\eta})\, e^{-\int \eta_\mu^{T\,2}/4}\, e^{-\int \eta_\mu^{L\,2}/4}$$

The Jacobian of the transformation $\eta_\mu^T \to A_\mu^T$ is the usual one as before

$$J = \left\| \frac{\delta \eta_\mu^T}{\delta A_\mu^T} \right\| = e^{-\int_0^\tau \frac{1}{2} \frac{\partial^2 S}{\partial \varphi^2} d\tau'}$$

this is due to the fact that $S[\,]$ is a function only of A_μ^T.

$$S = -\int A_\mu^T \Box A_\mu^T\, d^4x$$

On the other side, the Jacobian $\eta_\mu^L \to A_\mu^L$ is just $\det(\partial_t \delta(t-t'))$ and can be dropped in $\mathcal{Z}^{F.P}$, so the total result for $\mathcal{Z}^{F.P.}$ is

$$\mathcal{Z}^{F.P} = \int \tilde{\mathcal{D}} A_\mu^T \, P(A_\mu^T(0)) \exp\left\{-\int_0^\tau \left[\dot{A}_\mu^{T\,2}/2 + \frac{1}{8}\left(\frac{\partial S}{\partial A_\mu^T}\right)^2 - \frac{1}{4}\frac{\partial^2 S}{\partial A_\mu^{T\,2}}\right] d\tau'\right\} \cdot \int \tilde{\mathcal{D}} A_\mu^L \, P(A_\mu^L(0)) \, e^{-\int_0^\tau (\dot{A}_\mu^L)^2/2 \, d\tau'} \tag{I-56}$$

We could, of course, develop perturbation theory from this generating functional, and order by order check that it contains both the gauge fixing and the Faddeev-Popov ghosts, but we will limit ourselves to a more "formal"proof.

Let us first of all remember that the system does not remember the initial condition on A_μ^T, so in the first term in (I-56) we can choose $P(A_\mu^T(0)) = e^{-S}$. This means, as we explained before, that we make the generating functional stationary. Having done that we can perform the transformation (I-41) which brings it to the De-Alfaro-Fubini-furlan form.

The final result will be

$$\mathcal{Z}^{F.P} = \left[\int \mathcal{D} A_\mu^T(0) \mathcal{D} \hat{\pi}_\mu^T \, e^{-\int \hat{\pi}_\mu^2 d^4x} e^{-S[A(0)]}\right]\left[\int \tilde{\mathcal{D}} A_\mu^L \, P(A_\mu^L(0)) \, e^{-\int_0^\tau (\dot{A}_\mu^L)^2/2 \, d\tau' d^4x}\right] \tag{I-57}$$

The second term in (I-57) is apparently more difficult to handle. We showed before that there is no choice of initial conditions $P(A_\mu^L(0))$, that the system can forget, so there is no choice of P that can make the $\mathcal{Z}^{F.P}$ stationary. Nevertheless the dynamics of A_μ^L is a free one, so the the path integral can be done :

$$\int \mathcal{D} A_\mu^L \, P(A_\mu^L(0)) \, e^{-\int_0^\tau (\dot{A}_\mu^L)^2/2 \, d\tau'} = \int \mathcal{D}A_\mu^L(0) \mathcal{D}A_\mu^L(N) P(A_\mu^L(0)) \, e^{-\int \frac{[A_\mu^L(N) - A_\mu^L(0)]^2}{\tau} d^4x} \frac{1}{\sqrt{2\pi\tau}}$$

$$A_\mu^L(N) = A_\mu^L(\tau). \qquad \text{(I-58)}$$

As we said before the dynamics in A_μ^L is a free one, that means $\pi_\mu^L = \frac{A_\mu^L(N) - A_\mu^L(0)}{\tau}$. This allows us to substitute in (I-57,58) $A_\mu^L(N)$ with $\pi_\mu^L(0)$, so that we get

$$\int \mathcal{D}A_\mu^L(0) \mathcal{D}A_\mu^L(N) \, P(A_\mu^L(0)) \, e^{-\int \frac{[A_\mu^L(N) - A_\mu^L(0)]^2}{2} d^4x} = \int \mathcal{D}A_\mu^L(0) \, P(A_\mu^L(0)) \mathcal{D}\pi_\mu^L(0) \, e^{-\int \frac{\pi_\mu^{L2}\tau}{2} d^4x} \sqrt{\tau/2\pi}. \qquad \text{(I-59)}$$

Here we have not integrated away the τ , because this part of $Z^{F.P}$ is not stationary and so a dependence on τ. appears for particular correlation-functions.

Inserting (I-59) back into (I-56) and integrating the $\widehat{\pi}^T$ away, we obtain

$$Z^{F.P} = \int \mathcal{D}A_\mu^T(0) \, e^{-S[A(0)]} \mathcal{D}A_\mu^L(0) \, P(A_\mu^L(0)) \mathcal{D}\pi_\mu^L(0) \, e^{-\int \frac{\tau \pi_\mu^{L2}(0)}{2} d^4x} \sqrt{\tau/2\pi}. \qquad \text{(I-60)}$$

From this expression it is clear why $P(A_\mu^L(0))$ acts as a gauge fixing. We can, infact, define

$$F(A_\mu^L(0)) = -\sqrt{\ln P(A_\mu^L(0))}. \qquad \text{(I-61)}$$

and rewrite (I-60) as

$$Z^{F.P} = \frac{1}{\sqrt{2\pi}} \int \mathcal{D}A_\mu^T(0) \, e^{-S[A(0)] - F^2(A_\mu^L(0))} \mathcal{D}A_\mu^L(0) \mathcal{D}\pi_\mu^L(0) \, e^{-\int \frac{\pi_\mu^{L2}\tau}{2} d^4x} \sqrt{\tau}. \qquad \text{(I-62)}$$

We see that $F^2(A_\mu^L(0))$, with its dependence on $A_\mu^L(0)$, breaks the gauge invariance of $S[A]$, and acts as the usual gauge-fixing term that we are accustomed to [18] (see B.W.Lee Les Houches '75 lecture notes).

Before proceeding we should remember that $F(A_\mu^L(0))$ does not have to be just dependent on $A_\mu^L(0)$, it can contain also a dependence on $A_\mu^T(0)$. The next step is to bring to light the Faddev-Popov ghosts. Let us first of all recall that, being A_μ^L longitudinal, we can write it as

$$A_\mu^L \equiv \partial_\mu A^L.$$

A^L is nothing else than the group-parameter, on our case the phase for $U(1)$. Second, let us make the following transformation from A^L to a new $\hat{A}^L$ defined through the relation

$$A^L = F[(\partial_\mu \hat{A}^L)].$$

the corresponding momenta $\Pi_\mu^L, \hat{\Pi}_\mu^L$. change as

$$\hat{\Pi}_\mu^L = \frac{\delta(\partial_\nu F)}{\delta(\partial_\nu \hat{A}_L)} \Pi_\mu^L = \frac{\delta F}{\delta \hat{A}_L} \Pi_\mu^L. \tag{I-63}$$

Note that $\frac{\delta F}{\delta \hat{A}_L}$. is exactly the Faddeev-Popov determinant for abelian theory with gauge fixing F^2. (B.W.Lee ref.18).

Inserting now (I-63) in (I-62) we get

$$Z^{F.P} = N \int \mathcal{D}A_\mu^T(0)\, e^{-S[A(0)] - F^2}\, \mathcal{D}\hat{A}_L(0)\, \mathcal{D}\hat{\Pi}_\mu^L(0)\, e^{-\int \tau \frac{\hat{\Pi}_\mu^{L\,2}}{2} \left(\frac{\partial F}{\partial \hat{A}_L}\right)^{-2}} \sqrt{\tau}. \tag{I-64}$$

Integrating away the   we obtain

$$Z^{F.P} = N\int \mathcal{D}A_\mu^T(0)\, e^{-S(A(0))-F^2}\, \mathcal{D}\hat{A}_L(0)\det\left(\frac{\delta F}{\delta \hat{A}^L}\right). \tag{I-65}$$

that is the usual generating functional we are accustomed with gauge fixing and Faddeev-Popov determinant.

In (I-64) the $\hat{\pi}_\mu^L$ played the role of the ghosts. We want to stress that the $\left(\frac{\delta F}{\delta \hat{A}_L}\right)$ has come out naturally just because we had the $\hat{\pi}_\mu^L$, and these $\hat{\pi}_\mu^L$ were there because of the Fokker-Planck dynamics. We feel that this proof clarifies once and for all the mechanism that is responsible for the generation of gauge-fixing in stochastic quantization.

It is now easy to understand why gauge invariant quantity do not have divergences with τ : if we had to calculate, for example, $\langle G(A_\mu^T)\rangle$

$$\langle G(A_\mu^T)\rangle = \int G(A_\mu^T)\,\mathcal{D}A_\mu^T(0)\,\mathcal{D}\hat{A}^L(0)\,\mathcal{D}\hat{\pi}_\mu^L(0)\, \underset{\sqrt{\tau}}{e^{-S(A)-F^2-\int \hat{\pi}_\mu^L\left(\frac{\partial F}{\partial \hat{A}_L}\right)^{-2}\tau}}$$

we can integrate away the "ghosts" and obtain

$$\langle G(A_\mu^T)\rangle = \int G(A_\mu^T)\,\mathcal{D}A_\mu^T(0)\,\mathcal{D}\hat{A}^L(0)\, e^{-S(A)-F^2} \det\left(\frac{\delta F}{\delta \hat{A}^L}\right).$$

We see that the τ dependence has disappeared and this very much depends on the way τ enter in $Z^{F.P.}$ (and this is dictated by the Fokker-Planck dynamics).Let us now see what happens if we had to calculate gauge-non-invariant quantity like $\langle G(A_\mu^T, A_\mu^L)\rangle$ In this case, the presence of A_μ^L in G brings down in (I-57) a dependence on γ, so

$$\langle G\,(A_\mu^T, A_\mu^L)\rangle \sim g(\gamma)$$

The steps that we have done from (I-52) to (I-64) rely very much on the possibilities of defining a transverse A_μ^T and longitudinal A_μ^L part for A_μ (I-52). We know that this is not so easy for non-abelian theory where the Gauss-Law is non linear. We think, anyhow, that it should still be possible to repeat the same steps as here, defining not transverse variables but gauge invariant ones, and group variables Ω in place of the longitudinal ones.

SECTION II

HIDDEN SUPERSYMMETRY IN STOCHASTIC PROCESSES

II-A):Supersymmetric Fokker-Planck dynamics and dimensional reduction in parabolic stochastic equations.

In this section we want to study a different form for $Z^{F.P}$. The expression for the Jacobian that we have derived in (I-22) is not the only manner to write it. Another way is by using anti-commuting variables: $\psi, \bar\psi$

$$\left\|\frac{\delta\eta}{\delta\varphi}\right\| = \int \tilde{\mathcal{D}}\psi\, \tilde{\mathcal{D}}\bar\psi \exp\left\{+\int_0^{\tau} \bar\psi \left(\frac{\partial}{\partial\tau'} + \frac{\partial^2 S}{\partial\varphi^2}\right)\psi\, d\tau'\right\}. \tag{II-1}$$

With this form for the Jacobian, and rescaling the time $\tau \to \tau/2$, the $Z^{F.P}$ becomes

$$Z^{F.P}_{S.S} = \int \tilde{\mathcal{D}}\varphi\, \tilde{\mathcal{D}}\psi\, \tilde{\mathcal{D}}\bar\psi\, P(\varphi(0)) \exp\left\{-\int_0^{2\tau}\left[\frac{1}{2}\left(\dot\varphi + \frac{1}{2}\frac{\partial S}{\partial\varphi}\right)^2 - \bar\psi\left(\partial_\tau + \frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\right)\psi\right] d\tau' + J\varphi\right\}$$

The reason for the index S.S. is that, with a proper choice of b.c for φ and $\psi, \bar\psi$, this $Z^{F.P}$ reveals a hidden supersymmetry recently discovered by Parisi and Sourlas[13].

Let us take $P(\varphi(0)) = \delta(\varphi(0) - \varphi(2\tau))$ and periodic boundary conditions for $\psi, \bar\psi$, then $Z^{F.P}$ can be written as

$$Z^{F.P}_{SS} = \int {}'\tilde{\mathcal{D}}\varphi\, {}'\tilde{\mathcal{D}}\psi\, {}'\tilde{\mathcal{D}}\bar\psi \exp\left\{-\int_0^{2\tau}\left[\frac{1}{2}\dot\varphi^2 + \frac{1}{8}\left(\frac{\partial S}{\partial\varphi}\right)^2 - \bar\psi\left(\partial_{\tau'} + \frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\right)\psi\right] d\tau'\right.$$

$${}'\tilde{\mathcal{D}}\varphi = \lim_{N\to\infty} \prod_{i=1}^{N} \mathcal{D}\varphi(\tau_i). \tag{II-2}$$

The invariance of $Z^{F.P}_{S.S}$ is (in infinitesimal form)

$$\delta\varphi = \varepsilon\psi + \bar{\varepsilon}\bar{\psi}$$
$$\delta\psi = \bar{\varepsilon}\left(\dot{\varphi} + \frac{1}{2}\frac{\partial S}{\partial\varphi}\right)$$
$$\delta\bar{\psi} = \varepsilon\left(\dot{\varphi} - \frac{1}{2}\frac{\partial S}{\partial\varphi}\right)$$

$\varepsilon, \bar{\varepsilon}$ infinitesimal (II-3)
anticommmuting param.

This symmetry is a "sort" of supersymmetry in the sense that it mixes commuting with anticommuting variables $\varphi \rightarrow \psi, \bar{\psi}$ and for not breaking the symmetry we have to introduce the same b.c. for φ and $\psi, \bar{\psi}$.

The reader might be puzzled by our choice of b.c.. The one on the bosonic field φ , $\delta\left(\varphi(0) - \varphi(2T)\right)$ seems to constrain the field in $T=0$ to be equal to $\varphi(2T)$, an apparent contradiction for stochastic processes, but it is not so . We are, infact, only interested in the limit of $T \to \infty$ and we know that the system will not remember the initial probability, so it does not matter the form of $P(\varphi(0))$, it is only important that $\int P(\varphi(0))\, d\varphi(0) = 1$.

The Lagrangian that appears in (II-2) was first introduced by E.Witten[21] as an example of non-relativistic supersymmetry. In particular he studied the corresponding Hamiltonian

$$H^{F.P}_{S.S} = -\frac{1}{2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{8}\left(\frac{\partial S}{\partial\varphi}\right)^2 - \frac{1}{4}\left[\bar{\psi},\psi\right]\frac{\partial^2 S}{\partial\varphi^2} \tag{II-4}$$

The generators of the symmetry (II-3) are

$$Q_\psi = \left[-\frac{\partial}{\partial\varphi} + \frac{1}{2}\frac{\partial S}{\partial\varphi}\right]\psi \quad ; \quad Q_{\bar{\psi}} = \bar{\psi}\left[\frac{\partial}{\partial\varphi} + \frac{1}{2}\frac{\partial S}{\partial\varphi}\right]$$

and the $H^{F.P}_{S.S}$ can be written as

$$H^{F.P}_{S.S} = \frac{1}{2}\left\{Q_\psi, Q_{\bar{\psi}}\right\}.$$

moreover, as Q_ψ and $Q_{\bar{\psi}}$ are generators of a symmetry, they commute with $H^{F.P}_{S.S}$; $\left[Q_\psi, H^{F.P}_{S.S}\right] = 0 = \left[Q_{\bar{\psi}}, H^{F.P}_{S.S}\right]$

When we quantize the system associated with $H^{F.P}_{S.S}$, we have to require that

$$\{\bar\Psi, \psi\} = 1 \qquad \psi^2 = \bar\Psi^2 = 0$$

A representation of $\psi, \bar\Psi$ that satisfies this algebra is

$$\bar\Psi = \sigma^+ \qquad \psi = \sigma^-$$

with $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

In this representation $H^{F.P}_{S.S.}$ is

$$H^{F.P}_{S.S} = -\frac{1}{2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{8}\left(\frac{\partial S}{\partial\varphi}\right)^2 - \frac{\sigma^3}{4}\frac{\partial^2 S}{\partial\varphi^2} = \tag{II-5}$$
$$= \begin{pmatrix} H^{F.P}_{Forw}, & 0 \\ 0, & H^{F.P}_{Back} \end{pmatrix}$$

We see that this $H^{F.P}_{SS}$ contains the two Hamiltonian (I-17) and (I-26) for the forward and backward Fokker-Planck dynamics, and this is the reason why we called this paragraph "supersymmetric Fokker Planck dynamics".

Both $H^{F.P}_{Forw}$ and $H^{F.P}_{Back}$ are positive semi-definite operators, infact $H^{F.P}_{Forw} = \frac{1}{2} Q Q^+$ (see I-17) with $Q = \left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\frac{\partial S}{\partial\varphi}\right)$ and $H^{F.P}_{Back} = \frac{1}{2} Q^+ Q$

As we already explained before, the ground state of $H^{F.P}_{Forw}$ is at $E = 0$ and it has the form $\Psi_0^{Forw} = e^{-S/2}$. Also $H^{F.P}_{Back}$ has a state at $E_0 = 0$ and it is $\Psi_0^{Back} = e^{S/2}$

If we assume that $\Psi_0^{Forw} = e^{-S/2}$ is normalizable (and we assumed that to garantee that the system had a sensible equilibrium distribiution, as discussed in I-5) then $\Psi_0^{Back} = e^{S/2}$ is not normalizable and it cannot be accepted as part of the spectrum of $H^{F.P}_{Back}$. This means that the spectrum of $H^{F.P}_{Back}$ is positive - definite. These conditions can also be expressed by using the language of supersymmetry[21]: there is only one ground state for $H^{F.P}_{SS}$ that is

$$\psi_0^{SS} = \begin{pmatrix} e^{-S/2} \\ 0 \end{pmatrix}.$$

The other one $\begin{pmatrix} 0 \\ e^{+S/2} \end{pmatrix}$ cannot be accepted. The $\psi_0^{S.S}$ is an invariant state under (II-3). That means that supersymmetry is unbroken[21].

Let us now go back to $Z^{F.P}_{S.S}$ (II-2).

$$Z^{F.P}_{S.S} = \int \tilde{\mathcal{D}}\varphi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; e^{-\int_0^{2T}\left[\dot{\varphi}^2/2 + \frac{1}{8}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \bar{\psi}\left(\partial_t + \frac{1}{2}\frac{\partial^2 S}{\partial \varphi}\right)\psi\right] d\tau'} \tag{II-6}$$

We will drop the ' on $\tilde{\mathcal{D}}\varphi$ for convenience of notation from now on. We can in this expression formally perform the integration over the $\psi, \bar{\psi}$ and what we will get is just $\det\left|\left(\partial_t + \frac{1}{2}\frac{\partial^2 S}{\partial \varphi^2}\right)\delta(\tau-\tau')\right|$ This determinant can be evaluated in a different way than the one used in SECT.I. In particular it can be evaluated as a regularized product of its eigenvalues $\alpha^{(n)}$.

$$\det\left|\left(\partial_\tau + \frac{1}{2}\frac{\partial^2 S}{\partial \varphi^2}\right)\delta(\tau-\tau')\right| = \prod_{n=-\infty}^{+\infty} \alpha^{(n)}.$$

where

$$\left(\partial_t + \frac{1}{2}\frac{\partial^2 S}{\partial \varphi^2}\right)\psi_n = \alpha^{(n)}\psi_n. \tag{II-7}$$

the solution to this eq.is

$$\psi_n(\tau) = \left[\exp \int_0^{\tau} d\tau'\left(\alpha^{(n)} - \frac{1}{2}\frac{\partial^2 S}{\partial \varphi^2}\right)\right]\psi(0) \tag{II-8}$$

As we have imposed periodic b.c. on $\psi(t)=\psi(0)$, we get for $\alpha^{(n)}$.

$$\alpha^{(n)}= i\,\frac{2\pi n}{\tau}+\frac{1}{\tau}\int_0^\tau d\tau'\,\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}.$$

$$\prod_{n=-\infty}^{+\infty}\alpha^{(n)}=\prod_{n=-\infty}^{+\infty}\left[i\,\frac{2n\pi}{\tau}+\frac{1}{\tau}\int_0^\tau\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\,d\tau'\right]=$$

$$=\left[\prod_{n=-\infty}^{+\infty}{}'\left[i\,\frac{2n\pi}{\tau}\right]\right]\left[\frac{1}{\tau}\int_0^\tau\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\right]\prod_{n=-\infty}^{+\infty}{}'\left[1+\frac{\int_0^\tau\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\,d\tau'}{2\pi n i}\right]=$$

$$=\mathcal{C}'\left[\frac{1}{\tau}\int_0^\tau\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\,d\tau'\right]\prod_{n=-\infty}^{+\infty}\left[1+\frac{\int_0^\tau\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\,d\tau'}{2n\pi i}\right].$$

($\mathcal{C}'$ is to indicate that we exclude the term $n=0$ from the product)

$$\prod_{n=-\infty}^{+\infty}\alpha^{(n)}=\mathcal{C}'\sinh\frac{1}{4}\int_0^\tau d\tau'\,\frac{\partial^2 S}{\partial\varphi^2} \tag{II-9}$$

If we had used antiperiodic b.c. for $\psi(t)$,(but we would break the supersymmetry in this way) we would have got

$$\prod_{n=-\infty}^{+\infty}\alpha^{(n)}=\mathcal{C}''\cosh\frac{1}{4}\int_0^\tau d\tau'\,\frac{\partial^2 S}{\partial\varphi^2} \tag{II-10}$$

Inserting (II-9) in $Z^{F.P}_{S.S}$, we get

$$Z^{F.P}_{S.S}=\mathcal{C}/2\left\{\left[\int\tilde{\mathcal{D}}\varphi\,\exp\left[-\int_0^\tau\left[\dot\varphi^2/2+\frac{1}{8}\left(\frac{\partial S}{\partial\varphi}\right)^2-\frac{1}{4}\frac{\partial^2 S}{\partial\varphi^2}\right]d\tau'\right]\right]-\right.$$
$$\left.-\left[\int\tilde{\mathcal{D}}\varphi\,\exp\left[-\int_0^\tau\dot\varphi^2/2+\frac{1}{8}\left(\frac{\partial S}{\partial\varphi}\right)^2+\frac{1}{4}\frac{\partial^2 S}{\partial\varphi^2}\right]\right]\right\}. \tag{II-11}$$

Remembering the form of $\mathcal{L}^{F.P}_{Forw}$ and $\mathcal{L}^{F.P}_{Back}$. from SECT.I (I-25), (I-26a) ,we can write (II-11) as

$$Z^{F.P}_{SS}=Z^{F.P}_{Forw}-Z^{F.P}_{Back}. \tag{II-12}$$

If we had chosen antiperiodic b.c., we would have got

$$Z^{F.P}_{SS}=Z^{F.P}_{Forw}+Z^{F.P}_{Back}. \tag{II-13}$$

We see that also at the path integral level, it appears the nice interplay of forward and backward dynamics already present at the operatorial level in $H^{F.P}_{S.S.}$. The presence of both dynamics is very amusing, but it might bother the careful reader who knows that the Parisi-Wu[1] prescription was to choose the forward one. It should be remembered, anyhow, that in the stochastic quantization we have to take the limit of integration $\tau\to\infty$ in (II-11) to infinity. In this limit only the $Z^{F.P}_{FORW}$ is left in (II-12), infact

$$\lim_{\tau\to\infty} Z^{F.P}_{S.S} = \lim_{\tau\to\infty} \left[Z^{F.P}_{FORW} - Z^{F.P}_{BACK} \right] =$$

$$= \lim_{\tau\to\infty} \left[\mathrm{tr}\, e^{-\hat{H}^{F.P}_{FORW}\tau} - \mathrm{tr}\, e^{-\hat{H}^{F.P}_{BACK}\tau} \right] =$$

$$= \lim_{\tau\to\infty} \left(\sum_n e^{-E_n\tau} - \sum_m e^{-\tilde{E}_m\tau} \right) = \lim_{\tau\to\infty} Z^{F.P}_{FORW}.$$

In the last step we have dropped the $e^{-\tilde{E}_m\tau}$ because all the $\tilde{E}_m$ are positive, as we proved before. It is also clear from (II-13) that this limit is insensible to the b.c., infact also starting from (II-13), we would have got $Z^{F.P}_{FORW}$ in the limit .

In paragraph II-D we will further analyze the significance of the presence of both dynamics in $Z^{F.P}_{S.S}$: the interested reader is referred to that paragraph.

Having established this basic notation, we can bring the formalism a step further with the use of superfields. Let us first rewrite (II-6) with an auxiliary field ω (that in statistical mechanics is known as response field)[29]

$$Z^{F.P}_{S.S}=\int\tilde{\mathcal{D}}\varphi\,\tilde{\mathcal{D}}\omega\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\exp\left\{\int_0^{\mathcal{T}}\left[\omega^2/2-\dot{\varphi}^2/2+\frac{1}{2}\omega\frac{\partial S}{\partial\varphi}+\bar{\psi}\left(\frac{\partial}{\partial\tau'}+\frac{1}{2}\frac{\partial^2 S}{\partial\varphi^2}\right)d\tau'\right]\right\}$$

(II-14

Second, let us introduce the scalar supersfield[22]

$$\Phi=\varphi+\bar{\alpha}\psi+\alpha\bar{\psi}+\alpha\bar{\alpha}\omega\cdot$$

$\alpha,\bar{\alpha}$ are elements of a Grassman algebra

$$\int d\alpha=0=\int d\bar{\alpha}\qquad\int d\alpha\,\alpha=1=\int d\bar{\alpha}\,\bar{\alpha}$$

Using Φ the action in (II-14) can be written as

$$\int d\tau\,\mathcal{L}^{F.P}_{S.S}=\int d\tau\,d\alpha\,d\bar{\alpha}\left[K[\Phi]-S[\Phi]\right]$$

(II-15)

where $$K[\Phi]=D_\alpha\Phi\,D_{\bar{\alpha}}\Phi\qquad:\ D_\alpha=\partial_\alpha-\bar{\alpha}\partial_t\cdot$$

and $S[\Phi]\cdot$ is the usual Euclidean action of our system ,but where the place of the scalar field φ is taken by the superfield Φ In (II-15) the space of integration (apart the usual d^4x) has been enlarged to $\tau,\alpha,\bar{\alpha}$: this new space is called, in supersymmetric jargon, superspace and the transformation (II-3) can be seen as a transformation on the "component" fields $\omega,\varphi,\psi,\bar{\psi}$ induced by the following "supertranslation"[22] in the superspace

$$\delta\alpha=\varepsilon$$
$$\delta\bar{\alpha}=\bar{\varepsilon}$$
$$\delta\tau=-(\bar{\alpha}\varepsilon-\bar{\varepsilon}\alpha)\qquad\text{(II-16)}$$

Using (II-15) we can formally write $Z^{F.P}_{S.S}$ as

$$Z^{F.P}_{s.s} = \lim_{\tau \to \infty} \int \tilde{\mathcal{D}}\Phi \; e^{-\int_{-\tau}^{\tau} d\tau \int d\alpha \, d\bar{\alpha} \left\{ \left[K[\Phi] - S[\Phi] \right] + \tilde{J}\Phi \right\}} \tag{II-17}$$

where $\tilde{J}(x,\alpha,\bar{\alpha},\tau) = \tilde{J}(x)\,\delta(\tau)\delta(\alpha)\delta(\bar{\alpha})$.

In (II-17) we have moved the origin of integration in $\int d\tau$ from 0 to $-\tau$, so that when we send $\tau \to \infty$, the origin $-\tau$ goes to $-\infty$. As a consequence we can presume that at $\tau = 0$ the system is already relaxed to the equilibrium. It is for this reason that we have restricted the current in (II-17) to be at $\tau = 0$. We have also introduced in $\tilde{J}$, $\delta(\alpha)$ and $\delta(\bar{\alpha})$ because we are interested only in correlations of φ fields.

Using only the supersymmetric invariance (II-3), we are now going to prove that the $Z^{F.P}_{s.s}$ in (II-17) is the same as the usual quantum generating functional.

Let us follow Cardy [19] and introduce an interpolating generating functional

$$Z^{F.P}_{s.s_\lambda} = \int \tilde{\mathcal{D}}\Phi \; \exp - \int d\tau \int d\alpha \int d\bar{\alpha} \left[\left[\lambda + (1-\lambda)\delta(\tau)\delta(\bar{\alpha})\delta(\alpha) \right] S[\Phi] - K[\Phi] \right] \tag{II-18}$$

It is immediate to see that for $\lambda = 1$ this is the supersymmetric generating functional (II-17), while for $\lambda = 0$ it is the "quantum" one $Z = \int \mathcal{D}\varphi \, e^{-S(\varphi)}$ infact

$$Z^{F.P}_{s.s_{\lambda=0}} = \int \tilde{\mathcal{D}}\Phi \; e^{-S[\varphi(0)] + \int d\tau \int d\alpha \, d\bar{\alpha} \, K[\Phi]} = \tag{II-19}$$

$$= N \int \mathcal{D}\varphi(0) \, e^{-S[\varphi(0)]}$$

In the last step of (II-19) we have just performed (remembering the form of the path integral measure) N-1 path integration for the φ field (avoiding the one in $\tau=0$) and all the N path integrations for the ψ, ψ, ω.

$$\prod_{i=1}^{N} \mathcal{D}\psi(\tau_i) \quad ; \quad \prod_{i=1}^{N} \mathcal{D}\bar{\psi}(\tau_i) \quad ; \quad \prod_{i=1}^{N} \mathcal{D}\omega(\tau_i) \qquad \begin{matrix} \tau_0 = -\tau \\ \tau_N = +\tau . \end{matrix}$$

These last integrations can be easily done as $\psi, \bar{\psi}, \omega$ enter only in $K[\Phi]$ and this is just the free Lagrangian

$$\int d\tau \int d\alpha \int d\bar{\alpha}\, K[\Phi] \propto \int [\omega^2/2 + \omega\dot{\varphi} - \bar{\psi}\dot{\psi}]\, d\tau$$

We should also notice that $Z^{F.P}_{S.S_\lambda}$ is still invariant under the supersymmetry transformation (II-3), because $S[\Phi]$ and $K[\Phi]$ are separately invariant, and moreover $\int d\tau\, d\alpha\, d\bar{\alpha}\, \delta(\bar{\alpha})\delta(\alpha)\delta(\tau)$ (in (II-18)) can be written as $\int d\tau\, d\alpha\, d\bar{\alpha}\, \theta(\tau+\bar{\alpha}\alpha)$ that is also invariant.

The next step is to show that the correlation functions derived from $Z^{F.P}_{S.S_\lambda}$ are independent of λ .Let us discuss for simplicity the 2-point function

$$C^{\lambda}(x,\tau=0,\alpha=0=\bar{\alpha}) = \langle \Phi(x,\tau=0=\alpha=\bar{\alpha})\, \Phi(0) \rangle^{\lambda}$$

Its derivative with respect to λ is

$$\frac{\partial C^{\lambda}}{\partial \lambda} = -\int dx' \int d\tau' \int d\alpha' \int d\bar{\alpha}' \left[1-\delta(\tau')\delta(\bar{\alpha}')\delta(\alpha')\right] \langle \Phi(x,0)\, \Phi(0)\, S[x';\tau',\alpha',\bar{\alpha}'] \rangle^{\lambda} \tag{II-20a}$$

Because of the invariance (II-3) $\langle \Phi(x,0)\Phi(0) S[x';\tau',\alpha',\bar{\alpha}'] \rangle^{\lambda}$ can only have the form[19] $f(x,x';\tau'+\bar{\alpha}'\alpha')$ infact the only combination invariant under (II-16) is $(\tau+\bar{\alpha}\alpha)$.(also $\bar{\varepsilon}\alpha+\varepsilon\bar{\alpha}$ is invariant but it is zero once integrated over $\int d\alpha\, d\bar{\alpha}$)

Making now use of the well-known property[19]

$$\int dx \int_a^b d\tau \int d\alpha\, d\bar{\alpha}\, g(x,\tau+\bar{\alpha}\alpha) = \int dx \int_a^b d\tau\, g'(x,\tau) =$$

$$= \int dx \left[g(x,b) - g(x,a) \right].$$

we obtain for (II-20a)*

$$\frac{\partial e^{\lambda}}{\partial \lambda} = -\int dx' \int_{-\infty}^{+\infty} d\tau'\, d\alpha'\, d\bar{\alpha}' \left[1 - \delta(\tau')\delta(\bar{\alpha}')\delta(\alpha')\right] \theta(-\tau)\, g(x,x',\tau'+\bar{\alpha}'\alpha') =$$

$$= -\int dx' \int_{-\infty}^{+\infty} d\tau'\, \theta(-\tau')\, g'(x,x',\tau') + \int dx'\, g(x,x',0) =$$

$$= -\int dx' \int_{-\infty}^{0} d\tau'\, g'(x,x',\tau') + \int dx'\, g(x,x',0) = \int dx'\, g(x,x',-\infty)$$

The fields φ at $\tau=-\infty$ can always be chosen to be of any value (because of the Markoffian character of the Langevin process) even zero, so that $g(x,x',-\infty)=0$ and thus

$$\frac{\partial e^{\lambda}}{\partial \lambda} = 0 \qquad \text{(II-20b)}$$

This means that the correlation functions are independent of λ , so the ones derived from $Z^{F.P}_{S.S\,\lambda=1}$ (that is II-17) are the same as the ones derived from $Z^{F.P}_{S.S\,\lambda=0}$ (that is the quantum one).

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*The $\theta(-\tau)$ in the first step appears because we choose, as in the Parisi-Wu prescription, the forward green function

We would like to point out that we have derived this result only using arguments based on supersymmetry and never made use of the usual Fokker-Planck arguments (like in section I). This proof might prove useful when no information is available on the spectrum of the Fokker-Planck Hamiltonian, or when it is too hard too deal with the operator formalism.

We like to call the phenomenon we have just described a "dimensional reduction",because we have gone from a generating functional $Z^{F.P}_{S.S}$ in 4+2 dim.(4 x^{μ} and 2 for τ) to one Z in 4 dim. A similar phenomena of dimensional reduction also appears for elliptic stochastic equations (that are equation of the form $\frac{\partial S}{\partial \varphi}=\eta$)[19]. Parisi and Sourlas were the first to point it out. In their case the "dimensional reduction" was due to an invariance in superspace that had the character of a "superrotation" leaving invariant a quadratic form $x^2+\bar{\alpha}\alpha$. The invariance we have here for Langevin eq.(parabolic stochastic eq.) is a "supertranslation" in superspace (II-16),but that leaves invariant a similar "quadratic form"$(\tau+\bar{\alpha}\alpha)$.

II-B):Multicomponent Langevin equation and Kelvin relations.

After the discovery of this hidden supersymmetry by Parisi and Sourlas, a lot of work has been done by different groups (see for example ref.13,15,23). The main line of research has centered on the attempt to associate a stochastic process to any supersymmetric generating functional (like Wess-Zumino model or supersymmetric Yang-Mills theories,etc).
This association should in principle be possible thanks to the Nicolai theorem [23] (for details see paragraph II-F). The success of this line of research has been ,anyhow,very limited.

Our interest, on the contrary, lies in a different direction: we would like to find out which symmetry, or general feature, of the stochastic process is responsible for the hidden supersymmetry of the generating functional $Z^{F.P}_{s.s.}$. We feel that this supersymmetry is not just a spurious artifact originating from formal manipulations, but instead it is a signal of something more fundamental intrinsic of the stochastic process. We have found the answer to this analyzing the multicomponent generalization of the Langevin equation.
Many processes[24] in nature are described ,at least phenomenologically,by multicomponent Langevin equations:

$$\frac{\partial \varphi_i}{\partial \tau} = -F_i(\varphi) + \eta_i$$

(II-21

φ_i is to indicate the physical <u>macroscopic</u> quantity of which we study the diffusion. The index (i) is for the different variables that φ may represent: heat density, charge density, matter density, etc. F_i is the drift force responsible for the diffusion: temperature gradient, electric field, gravitational field, concentration gradient, etc.

η_i is a Gaussian stochastic noise

$$\langle \eta_i(t)\, \eta_j(t') \rangle = 2\,\delta_{ij}\,\delta(t-t') \tag{II-22}$$

We will restrict ourselves to study quantities φ_i even under time reversal

$$t \to -t$$
$$\varphi_i \to \tilde{\varphi}_i = \varphi_i \tag{II-23a}$$

and drift forces F_i invariant under $t \to -t$

$$F_i \to \tilde{F}_i = F_i \tag{II-23b}$$

These kinds of forces are called, in the literature[24], irreversible forces and the even variables φ_i are known as " " type variables.

As for as the single component case we can derive the evolution eq. for the transition probability $P_2(\{\varphi_i^0\}0 / \{\varphi_i^1\}t_1)$

$$\frac{\partial P_2}{\partial t} = -\sum_i \frac{\partial}{\partial \varphi_i} F_i P_2 + \sum \frac{\partial^2}{\partial \varphi_i^2} P_2 \equiv \hat{L}\, P_2 \tag{II-24}$$

and it is, as before, called Fokker-Planck eq. Under time-reversal, we can define a corresponding probability $\tilde{P}_2$

$$P_2(\{\varphi_0\}0|\{\varphi_i^1\}t_1) \Rightarrow P_2(\{\tilde{\varphi}_i^0\}0|\{\varphi_i^1\}-t_1) \equiv \tilde{P}_2(\{\tilde{\varphi}_i^0\}0|\{\tilde{\varphi}_i^1\}t_1)$$

and remembering (II-23), we have

$$\tilde{P}_2(\{\varphi_i^0\}0|\{\tilde{\varphi}_i^1\}t_1) = \tilde{P}_2(\{\varphi_i^0\}0|\{\varphi_i^1\}t_1). \tag{II-25}$$

the evolution eq. for $\tilde{P}_2$ can be easily derived from (II-24)

$$\frac{\partial \tilde{P}_2}{\partial t} = \sum_i F_i \frac{\partial \tilde{P}_2}{\partial \varphi_i} + \sum_i \frac{\partial^2 \tilde{P}_2}{\partial \varphi_i^2} \equiv \hat{L}^+ \tilde{P}_2 \tag{II-26}$$

It is called, as before, backward Fokker-Planck eq.

Parallel to all this we can, of course, derive the corresponding functional formalism as we have done before.

The form of $Z_{s.s}^{F.P}$ is

$$Z_{s.s}^{F.P}[J] = N \int \prod_i \tilde{\mathcal{D}}\varphi_i \tilde{\mathcal{D}}\eta_i P(\{\varphi_i(0)\}) \delta(\varphi_i(t) - \varphi_i(t)_\eta) e^{-\int_0^t [\sum \eta_i^2/4 + J_i \varphi_i] d\tau'} \tag{II-27}$$

and repeating the usual steps, we get

$$Z_{ss}^{F.P}[J] = \int \prod_i \tilde{\mathcal{D}}\varphi_i \mathcal{D}\psi_i \mathcal{D}\bar{\psi}_i P(\{\varphi_i^0\}) e^{-\int_0^{2t} \mathcal{L}_{s.s}^{F.P} - \int_0^{2t} \sum F_i \dot{\varphi}_i - \int_0^{2t} \frac{1}{2} J_i \varphi_i dt'} \tag{II-28}$$

where

$$\int \mathcal{L}_{s.s}^{F.P} dt = \int \sum_{i,j} \left[\frac{1}{2} \dot{\varphi}_i^2 + \frac{1}{8} F_i^2 - \bar{\psi}_i (\partial_t \delta_{ij} + \frac{1}{2} \frac{\partial F_i}{\partial \varphi_j}) \psi_j\right] \tag{II-29}$$

In the single component case this Lagrangian reduces to

$$\mathcal{L}_{s.s}^{F.P} = \frac{1}{2} \hat{\varphi}^2 + \frac{1}{8} F^2 - \bar{\psi} (\partial_t + \frac{1}{2} \frac{\partial F}{\partial \varphi}) \psi$$

and, as we know, it is invariant under the transformation

$$\delta\varphi = \varepsilon\psi + \bar{\varepsilon}\bar{\psi}$$
$$\delta\bar{\psi} = \varepsilon(\dot{\varphi} + \tfrac{1}{2}F)$$
$$\delta\psi = \bar{\varepsilon}(\dot{\varphi} - \tfrac{1}{2}F).$$

Naively we would say that, if this is so for the single component case, then the multicomponent should be invariant under

$$\delta\varphi_i = \varepsilon\psi_i + \bar{\varepsilon}\bar{\psi}_i$$
$$\delta\bar{\psi}_i = \varepsilon(\dot{\varphi}_i + \tfrac{1}{2}F_i)$$
$$\delta\psi_i = (\dot{\varphi}_i - \tfrac{1}{2}F_i)\bar{\varepsilon} \qquad \text{(II-30)}$$

Unfortunately this is not the case unless

$$\frac{\partial F_i}{\partial \varphi_J} = \frac{\partial F_J}{\partial \varphi_i} \qquad \text{(II-31)}$$

From the structure of the variation $\delta\mathcal{L}^{F.P}_{S.S}$, it is easy to see that the (II-31) are not only sufficient conditions to have supersymmetry,but also necessary ones.

These relations, new for us,are well-known in the literature on stochastic processes [25] and are called "potential conditions". They garantee, infact, the existence of a potential function V from which F_i can be derived.

$$F_i = \frac{\partial V}{\partial \varphi_i} \qquad \text{(II-32)}$$

This potential V is what is called, in supersymmetric jargon, superpotential for the Lagrangian (II-29).

The reader may argue, at this point, that we still have to take care of the extra spurious term $-\int_0^{2t}\sum F_i\dot{\varphi}_i\,dt'$ in (II-28). In the single component case, this term was just a surface term and a proper choice of boundary conditions would eliminate it. In the multicomponent case $-\int_0^{2t} \sum F_i \dot{\varphi}_i \, dt'$ is not a surface term, but thanks to (II-32) it becomes one. So (II-32) does not only garantee the supersymmetry of (II-28), but it is also necessary to make the spurious term a surface one. To definitively eliminate this term, we need periodic b.c. on the path integral (II-28) as before.

In most of the phenomenological cases[24], the drift forces in (II-21) are linear in the variables φ_i.

$$F_i = \sum_j M_{ij} \varphi_j . \qquad M_{ij} \text{ constant matrix}$$

In this case the conditions (II-31) become

$$M_{ij} = M_{ji} \tag{II-33}$$

The (II-33) have a well-known physical meaning[26]:they are the celebrated "Kelvin reciprocity relations".
As we said at the beginning,the indices i,j stay for the different macroscopic physical quantities (heat, charge, matter densities, etc) of which we study the diffusion. Indicating for example 1 for the heat density , 2 for electric charge density, the relation (II-33)

$$M_{12} = M_{21} .$$

means that the coefficient M_{12}, that gives the response of the charge to a temperature gradient, is the same as M_{21}. the coefficient for the heat response to an electric field (Peltier and Seebck effects). These identities have been verified experimentally[24] long ago.

Onsager in 1931[14],and before him Bohr[26]asked himself which was the theoretical origin of the Kelvin relations (II-33) or in general of the potential conditions (II-31).

He found [14] the answer in the so-called "Principle of microreversibility". This principle,as we will see in the next paragraph,is the sole and unique hypothesis we need to derive the Kelvin relations ,and in our picture,it becomes the hypothesis we need to have the supersymmetric invariance (II-30) for the $Z^{F.P}_{S.S}$ in (II-28).

It is for this reason that we can say that the "Principle of microreversibility" is really at the origin of the supersymmetry of $Z^{F.P}_{S.S}$.

II-C):Onsager principle of microscopic reversibility and supersymmetry.

The variables φ_i that we have used up to now are usually referred as macroscopic variables. These are gross variables sufficient to describe the thermodynamics of the system and they are a small finite number. They can, of course,only give an average or gross behaviour of our system, it is for this that their equation is a stochastic one .

In principle, to have a better description of the system, we should not use the macroscopic variables but instead the microscopic ones $\{m_i\}$ of each individual particle, and from their law of motion we should be able to rederive all the features of the system as a whole. This task is, of course, beyond man and computer capacity but it is the basis of statistical mechanics both for equilibrium and non-equilibrium phenomena.

One of the properties of microscopic dynamics, that might have a reflection also at the macroscopic level, is the fact that the fundamental equations governing the motion of the individual particles are symmetric with respect to past and future:this is the Onsager principle of microreversibility.

This microreversibility will give signals of itself even in macroscopic irreversible processes:here is the power of this principle. How it manifests itself at the macroscopic level is through some symmetry between phenomenological coefficients like the "Kelvin relations.

Following ref.27, let us introduce a complete set $\{m_i\}$ of microscopic variables describing the whole system. As there are so many of these variables we need to introduce also here the concept of probability $\mathcal{P}_2(\{m^0\}0|\{m^1\}t_1)$. This is the conditional probability that $\{m^1\}$ is realized if $\{m^0\}$ was realized t_1. instants earlier. Due to the reversibility of the microscopic equations of motion, this probability must be invariant under time reversal

$$\mathcal{P}_2(\{m^0\}0|\{m\}t) = \mathcal{P}_2(\{\tilde{m}_0\}0|\{\tilde{m}\}-t). \qquad \text{(II-34)}$$

$\{\tilde{m}\}$ are the transformed variables under $t \to -t$. Having $\mathcal{P}_2$. ,we can, of course, derive an expression for the macroscopic joint probability $W(\{\varphi_i^0\}0|\{\varphi_i\}t)$. Let us, first of all, recall the definition of joint probability

$$W(\{\varphi_i^0\}0|\{\varphi_i\}t) = P_1(\{\varphi_i^0\}0)P_2(\{\varphi_i^0\}0|\{\varphi_i\}t).$$

It is easy to understand that

$$W(\{\varphi_i^0\}0|\{\varphi_i\}t)\,d\{\varphi_i^0\}\,d\{\varphi\} = \int_{\{\varphi\}}^{\{\varphi+d\varphi\}} d\{m\}\int_{\{\varphi_0\}}^{\{\varphi_0+d\varphi_0\}} d\{m_0\}\,\mathcal{P}_2(\{m_0\}0|\{m\}t)\,\mathcal{P}_1(\{m_0\}) \qquad \text{(II-35)}$$

In (II-35) $\mathcal{P}_2(\{m_0\})$ is the initial distribution for the variables The integration in (II-35) cover all values of the microscopic variables $\{m\}$ which fulfill the conditions

$$\{\varphi_i\} \le \{\varphi_i(\{m\})\} \le \{\varphi_i + d\varphi_i\}.$$

$$\{\varphi_i^0\} \le \{\varphi_i^0(\{m_0\})\} \le \{\varphi_i^0 + d\varphi_i^0\}.$$

Changing in (II-15) the integration variables $m \to \tilde{m}$, $m_0 \to \tilde{m}_0$ and using the symmetry property (II-34), we can easily obtain

$$W(\{\varphi_i^0\}0|\{\varphi_i\}t) = W(\{\tilde{\varphi}_i^0\}0|\{\tilde{\varphi}_i\}-t). \tag{II-36}$$

If we remember the definitionof joint probability, we can rewrite (II-36) as

$$P_1(\{\varphi_i^0\})\, P_2(\{\varphi_i^0\}0|\{\varphi_i\}t) = P_1(\{\varphi_i\})\, P_2(\{\varphi_i^0\}0|\{\varphi_i\}-t)$$

$$P_1(\{\varphi_i^0\})\, P_2(\{\varphi_i^0\}0|\{\varphi_i\}t) = P_1(\{\varphi_i\})\, \tilde{P}_2(\{\varphi_i^0\}0|\{\varphi_i\}t). \tag{II-37}$$

These relations are the cornerstone for the proof of the potential conditions (II-31). From (II-37) we have for

$$P_2(\{\varphi_i^0\}0|\{\varphi_i\}t) = \frac{P_1(\{\varphi_i\})\, \tilde{P}_2(\{\varphi_i^0\}0|\{\varphi_i\}t).}{P_1(\{\varphi_i^0\})} \tag{II-38}$$

Inserting the R.H.S. of (II-38) in the forward Fokker-Planck eq. (II-24), we get

$$\frac{\partial P_1}{\partial t}\tilde{P}_2 + P_1\frac{\partial \tilde{P}_2}{\partial t} = -\sum_i \frac{\partial}{\partial \varphi_i}(F_i P_1)\tilde{P}_2 - \sum_i F_i P_1 \frac{\partial \tilde{P}_2}{\partial \varphi_i} + \sum_i \frac{\partial^2 P_1}{\partial \varphi_i^2}\tilde{P}_2 + P_1\sum_i \frac{\partial^2 P}{\partial \varphi_i^2} + 2\sum_i \frac{\partial P_1}{\partial \varphi_i}\frac{\partial \tilde{P}_2}{\partial \varphi_i}.$$

where P_1 stays for $P_1(\{\varphi_i^0\})$ and $\tilde{P}_2$ for $\tilde{P}_2(\{\varphi_0\}0|\{\varphi_i\}t)$.

Using now the fact that P_1 itself satisfies the forward F.P. eq., we obtain

$$P_1 \frac{\partial \tilde{P}_2}{\partial t} = -\sum_i (F_i P_1)\frac{\partial \tilde{P}_2}{\partial \varphi_i} + P_1 \sum_i \frac{\partial^2 \tilde{P}_3}{\partial \varphi_i^2} + \sum_i 2\left(\frac{\partial P_1}{\partial \varphi_i}\right)\frac{\partial \tilde{P}_2}{\partial \varphi_i}$$

$\tilde{P}_2$ obeys the backward F.P.eq., so we can further simplify this expression and reduce it to

$$2\, P_1 \left(\sum_i F_i \frac{\partial}{\partial \varphi_i} \tilde{P}_2\right) + 2 \sum_i \left(\frac{\partial P_1}{\partial \varphi_i}\right)\left(\frac{\partial \tilde{P}_2}{\partial \varphi_i}\right) = 0$$

Let us now multiply this on the left for an arbitrary function $g(\{\varphi\})$ and integrate over $\{\varphi\}$

$$\int g(\{\varphi\}) \left[P_1 \sum_i F_i \frac{\partial}{\partial \varphi_i} + \frac{\partial P_1}{\partial \varphi_i} \frac{\partial}{\partial \varphi_i}\right] \tilde{P}_2 \qquad \text{(II-39)}$$

This eq.holds for any time τ even for $\tau = 0$. P_2 as $\tilde{P}_2$ at $\tau = 0$ are equal to $\delta(\{\varphi_i\} - \{\varphi_i^0\})$ for the definition itself of conditional probability. Inserting this in (II-39) and integrating by parts, we obtain an equation involving $g(\{\varphi\})$ and $g'(\{\varphi\})$. To satisfy this equation, as $g(\{\varphi\})$ and $\{\varphi\}$ are arbitrary functions, we have to have the coefficients of $g(\{\varphi\})$ and of its first and second derivative separately equal to zero.

The coefficients of $\frac{\partial g}{\partial \varphi}$ is

$$P_1 F_i + \frac{\partial P_1}{\partial \varphi_i} = 0$$

From this relation it is clear, as $F_i = -\frac{1}{P_1}\frac{\partial P_1}{\partial \varphi_i}$, that

$$\frac{\partial F_i}{\partial \varphi_J} = \frac{\partial F_J}{\partial \varphi_i} \qquad \text{(II-40)}$$

and these are the Kelvin or potential conditions.

To summarize this long proof,we want to remind the reader that we started from (II-34) that is the Onsager principle .

Of course this is not a proof that the Kelvin relations can only be proved assuming (II-34), but no other proof has appeared in the last 50 years, that does not make use of the microreversibility principle.

In the introduction to this section we stated our goal of discovering which symmetry of the stochastic process is behind the supersymmetry of $Z^{F.P}_{s.s}$. After the analysis of the previous pages we can answer that the symmetry is the microreversibility. It is true, of course, that to generate the typical supersymmetric form (II-28) for the action, we needed other ingredients such as:

1) to start from an eq. first order in ∂_t.(Langevin eq.)

2) with a Gaussian noise η_i.

These are two ingredients that the Nicolai theorem[23] seems to indicate as necessary feature of any supersymmetric action. They are anyhow structural properties and not symmetry properties of the stochastic process. The analysis we have done revealed that a third ingredient was needed, that are the potential conditions, and these conditions bring in the signal of a symmetry. A pure stochastic Gaussian process without the (II-40) would not have generated the supersymmetry of $Z^{F.P}_{s.s}$.

We can, at this point, state with confidence that the supersymmetry is a manifestation, at the level of macroscopic variables, of the time reversibility present at the level of the microscopic variables.

We can now even substitute the "Onsager principle" with the "Supersymmetry principle". The Onsager principle was, after all, introduced just to explain the Kelvin relations(II-40), but the same conditions we get if we impose the invariance (II-30) on the generating functional of the Fokker-Planck dynamics. This latter request is a symmetry request on macroscopic variables and not a symmetry request on "hidden" microscopic variables as the Onsager principle is. To further tie up the microreversibility with supersymmetry, we are going to analyze ,in the next paragraph, the Ward identities that stem from the invariance (II-30). These identities, once the Fermionic degree of freedom have been integrated away, reveal themselves as symmetry relations between the forward and the backward Fokker-Planck dynamics thus unveiling the real character of this symmetry as a time-symmetry.

II-D): Ward identities and fluctuation-dissipation theorem.

As we did in (II-A) for the single component case, we can develop the operator formalism for the supersymmetric Fokker-Planck dynamics associated with $Z^{F.P}_{S.S}$ (II-28).
The $H^{F.P}_{S.S}$ for $\mathcal{L}^{F.P}_{S.S.}$ (II-29) is

$$H^{F.P}_{S.S} = \sum_{i,j} \left\{ -\frac{1}{2}\frac{\partial^2}{\partial \varphi_i^2} + \frac{1}{8}F_i - \frac{[\bar{\Psi}_i, \Psi_j]}{4}\frac{\partial F_i}{\partial \varphi_j} \right\}. \tag{II-41}$$

Quantizing the system we have to require that

$$\{\bar{\Psi}_i, \Psi_j\} = \delta_{ij} \qquad \Psi_i^2 = \bar{\Psi}_i^2 = 0$$

A representation for Ψ_i and $\bar{\Psi}_i$ that satisfies this algebra is

$$\bar{\Psi}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sigma_+ \\ 0 \\ 0 \end{pmatrix} \qquad \Psi_j = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma_- \\ 0 \\ 0 \end{pmatrix}. \tag{II-42}$$

where the entries of these column "vector" are $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
The product of two "vectors" is done multiplying each element as matrix. It is easy to check that

$$[\bar{\Psi}_j, \Psi_i] = \delta_{ij}\,\sigma_3 \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and so in this representation $H^{F.P}_{S.S.}$ can be written as

$$H^{F.P}_{S.S} = \begin{pmatrix} H^{F.P}_{\text{Forw}} & \\ & H^{F.P}_{\text{Back}} \end{pmatrix} \tag{II-43}$$

Parallel to what we have done for the single component case we can also represent $Z^{F.P}_{S.S.}$ (II-28) as in (II-12)

$$Z^{F.P}_{s.s} = Z^{F.P}_{Ferm} - Z^{F.P}_{Bose} \qquad \text{(II-44)}$$

where $Z^{F.P}_{Ferm} = \int \prod_i \tilde{\mathcal{D}}\varphi_i \, e^{-\int_0^{2t} \sum_i [\dot{\varphi}_i^2/2 + F_i^2/8 - (\frac{\partial F_i}{\partial \varphi_i})^2 \frac{1}{4}] dt}$

and

$$Z^{F.P}_{Bose} = \int \prod_i \tilde{\mathcal{D}}\varphi_i \, e^{-\int_0^{2t} \sum_i [\dot{\varphi}_i^2/2 + F_i^2/8 + (\frac{\partial F_i}{\partial \varphi_i})^2 \frac{1}{4}] dt} \qquad \text{(II-45)}$$

Up to now we have only rewritten the $Z^{F.P}_{s.s}$ in different forms, but not jet made full use of the symmetry that it has.

We know that any time we have a symmetry we can derive what are called the Ward-identities. To derive these relations from we have introduce Fermionic currents $J_\psi, J_{\bar{\psi}}$ in $Z^{F.P}_{s.s}$ besides J_φ. and also ,in a perfect supersymmetric fashion, an auxiliary field ω and its current. The form of $Z^{F.P}_{s.s.}$ will be

$$Z^{F.P}_{s.s} = \int \prod_i \tilde{\mathcal{D}}\varphi_i \tilde{\mathcal{D}}\omega_i \prod \tilde{\mathcal{D}}\psi_i \prod \mathcal{D}\bar{\psi}_i \, e^{-\int_0^{2t} [\mathcal{L}^{F.P}_{s.s} + \varphi \cdot J_\varphi + \omega \cdot J_\omega + J_\psi \cdot \psi + \bar{\psi} \cdot J_{\bar{\psi}}] dt'}$$

(II-46)

where $J \cdot \varphi \equiv \sum J_i \varphi_i$ and $\mathcal{L}^{F.P}_{s.s} = [\sum_{i,j} \dot{\varphi}_i^2/2 - \frac{\omega_i F_i}{2} - \frac{\omega_i^2}{2} - \bar{\psi}_i(\partial_t \delta_{ij} + \frac{1}{2}\frac{\partial F_i}{\partial \varphi_j})\psi_j]$

The set of currents $J_\varphi, J_\omega, J_\psi, J_{\bar{\psi}}$ must have very particular transformation laws under (II-30), so that the full action in (II-46) and $Z^{F.P}_{s.s.}$ is invariant. It is easy to find that, corresponding to (II-30), the transformations of the currents must be

$$\begin{aligned} \delta J_{\varphi_i} &= \varepsilon \dot{J}_{\bar{\psi}_i} + \dot{J}_{\psi_i} \bar{\varepsilon} \\ \delta J_{\omega_i} &= \varepsilon J_{\bar{\psi}_i} - J_{\psi_i} \bar{\varepsilon} \\ \delta J_{\bar{\psi}_i} &= \bar{\varepsilon} J_\varphi + \bar{\varepsilon} \dot{J}_{\omega_i} \\ \delta J_{\psi_i} &= -\varepsilon J_{\varphi_i} + \varepsilon \dot{J}_{\omega_i} \end{aligned} \qquad \text{(II-47)}$$

Under this transformation

$$\delta \cdot Z^{F.P}_{S.S}\left[J_\varphi, J_\omega, J_\psi, J_{\bar\psi}\right] = 0 \qquad \text{(II-48)}$$

In the expression (II-46) we can, as we did before, integrate away the Fermionic variables and we will be left with

$$Z^{F.P}_{S.S} = \left\{ Z^{F.P}_{Forw}\left[J_\varphi, J_\omega\right] F\left[J_\psi, J_{\bar\psi}\right] - Z^{F.P}_{Boek}\left[J_\varphi, J_\omega\right] F\left[J_\psi, J_{\bar\psi}\right] \right\} \qquad \text{(II-49)}$$

where

$$F\left[J_\psi, J_{\bar\psi}\right] = e^{-\int \sum_{i,j} J_{\psi_i}(t)\,\Delta^{-1}(t-t')\,J_{\bar\psi_j}(t')\,dt\,dt'}$$

$$\Delta_{ij} = \left(\partial_t \delta_{ij} + \frac{1}{2}\frac{\partial F_i}{\partial \varphi_j}\right)\delta(t-t')$$

The invariance (II-48) can now be read as

$$\delta Z^{F.P}_{S.S} = 0 = \delta\left(Z^{F.P}_{Forw} F\right) - \delta\left(Z^{F.P}_{Forw} F\right)$$

or

$$\delta\left(Z^{F.P}_{Forw} F\right) = \delta\left(Z^{F.P}_{Forw} F\right). \qquad \text{(II-50)}$$

We can, at this point, apply any operator $\hat{O}\left(\frac{\delta}{\delta J_\varphi}, \frac{\delta}{\delta J_{\omega_i}}, \frac{\delta}{\delta J_{\psi_i}}, \frac{\delta}{\delta J_{\bar\psi_i}}\right)$ on the R.H.S. of (II-48) and we get

$$\hat{O}\,\delta Z^{F.P}_{S.S} = 0 \qquad \text{(II-51)}$$

Making use of (II-47) and fixing the currents to zero in (II-51), it is easy to see that (II-51) becomes

$$\int G\left[\varphi(t_1)\,\omega_j(t_2)\,\psi_h(t_3)\,\bar\psi_l(t_4)\right] e^{-\int \mathcal{L}^{F.P}_{S.S}} = 0 \qquad \text{(II-52)}$$

where $G[\;]$ are combinations of Green functions.

In field theory language (II-52) are nothing else than the Ward identities associated with the supersymmetry (II-30).

Let us now repeat the same steps we have done to derive (II-56), starting this time from (II-50). Applying the operator $\hat{O}$ to both sides of (II-50) we have

$$\hat{O}\left(\delta \mathcal{L}^{F\cdot P}_{\text{Forw}} F\right) = \hat{O}\left(\delta \mathcal{L}^{F\cdot P}_{\text{Back}} F\right)$$

and using, once again, (II-47) we get

$$\int G[\ ]\, e^{-\int \mathcal{L}^{F\cdot P}_{\text{Forw}} d\tau'} = \int G[\ ]\, e^{-\int \mathcal{L}^{F\cdot P}_{\text{Back}} d\tau'} \tag{II-53}$$

Due to the structure of (II-49), single Fermion fields never enter in G . In their place we have Δ (entering for $\psi, \bar{\psi}$). So we can formally write (II-53) as

$$\int G[\varphi, \omega, \Delta^{-1}]\, e^{-\int \mathcal{L}^{F\cdot P}_{\text{Forw}}} = \int G[\varphi, \omega, \Delta^{-1}]\, e^{-\int \mathcal{L}^{F\cdot P}_{\text{Back}}} \tag{II-54}$$

These relations are the most general and complete identities we can derive when we fully exploit the supersymmetric invariance of .

We see that they express nothing else than a time reversal invariance ,thus confirming that this symmetry is a manifestation of the Onsager principle of microreversibility.

Before concluding we want to analyze the Ward-identities in the equilibrium limit $\tau \to \infty$ of the Fokker-Planck dynamics because, after all, that is the physical regime in which we are interested.

Borrowing the formalism (II-43) we can write (II-54) as

$$\mathrm{tr}\left[G(\,)\,e^{-2\hat{H}^{F.P}_{Forw}t}\right] = \mathrm{tr}\left[G(\,)\,e^{-2\hat{H}^{F.P}_{Boeh}t}\right] \tag{II-55}$$

In the limit $t\to\infty$ we have

$$\lim_{t\to\infty}\sum_{n}\langle n|G(\,)|n\rangle e^{-E_n t} = \lim_{t\to\infty}\sum_{\tilde{n}}\langle\tilde{n}|G(\,)|\tilde{n}\rangle e^{-\tilde{E}_n t}. \tag{II-56}$$

where $|n\rangle, |\tilde{n}\rangle$ and $E_n, \tilde{E}_n$ are eigenstates and eigenvalues of, respectively, $\hat{H}^{F.P}_{Forw}$ and $H^{F.P}_{Boeh}$. We have proved before that

$$E_n \geq 0 \qquad \tilde{E}_n > 0$$

so we get from (II-56)

$$\langle 0|G(\,)|0\rangle = 0 \tag{II-57}$$

where $|0\rangle$ is the ground state of $H^{F.P}_{Forw}$ and in the $|\varphi\rangle$ representation it is $\langle\varphi|0\rangle = e^{-V(\varphi)/2}$.

In path integral formalism (II-57) can be read as

$$\int G(\,)\,e^{-S\,\ell^{F.P}_{Forw}} = 0 \tag{II-58}$$

The relations (II-57,58) are extremely important:they indicate that, as a consequence of supersymmetry, there are a set of operators whose vacuum expectation value (vacuum of the Fokker-Planck dynamics or equilibrium average)is zero. These relations could not have been derived if we had just analyzed the forward dynamics.Of course the functions $G(\,)$ are not any Green functions, but very particular ones: they are the only combinations, which enter in the Ward-identities (II-52).

Let us now explicitely derive one of this relation and analyze its meaning. Starting from (II-50) and (II-47) we have

$$\delta \mathcal{Z}^{F.P.}_{s.s} = 0 = \sum_i \Big[\frac{\partial \mathcal{Z}^{F.P}_{s.s}}{\partial J_{\varphi_i}} [\varepsilon \dot{J}_{\bar\psi_i} + \dot{J}_{\psi_i} \bar\varepsilon] + \frac{\partial \mathcal{Z}^{F.P}_{s.s}}{\partial J_{\omega_i}} [\varepsilon J_{\bar\psi_i} - J_{\psi_i} \bar\varepsilon] - \frac{\partial \mathcal{Z}^{F.P}_{s.s}}{\partial J_{\bar\psi_i}} [\bar\varepsilon J_{\varphi_i} + \bar\varepsilon \dot{J}_{\omega_i}] + [-\varepsilon J_{\varphi_i} + \varepsilon \dot{J}_{\omega_i}] \frac{\partial \mathcal{Z}^{F.P}}{\partial J_{\psi_i}} \Big] \tag{II-59}$$

Let us choose the operator $\hat{O}$ as $\frac{\delta}{\delta J_{\psi_i}(t_1)} \frac{\delta}{\delta J_{\varphi_k}(t_2)}$ and apply it on the R.H.S. of (II-59). Equating to zero the coefficients of ε and $\bar\varepsilon$ we obtain (after putting $J=0$)

$$\begin{cases} -\frac{\partial}{\partial t_1} \left(\frac{\partial^2 \mathcal{Z}^{F.P}_{s.s}}{\partial J_{\varphi_k}(t_2) \partial J_{\varphi_J}(t_1)} \right)\Big|_{J=0} - \frac{\delta^2 \mathcal{Z}^{F.P}_{s.s}}{\delta J_{\varphi_k}(t_2) \delta J_{\omega_J}(t_1)}\Big|_{J=0} - \frac{\delta^2 \mathcal{Z}^{F.P}_{s.s.}}{\delta J_{\psi_J}(t_1) \delta J_{\bar\psi_k}(t_2)}\Big|_{J=0} = 0 \\ \frac{\delta^2 \mathcal{Z}^{F.P}_{s.s}}{\delta J_{\psi_J}(t_1) \delta J_{\psi_k}(t_2)}\Big|_{J} = 0 \end{cases}$$

This can be formally written as

$$-\frac{\partial}{\partial t_1} \langle \varphi_k(t_2) \varphi_J(t_1) \rangle - \langle \varphi_k(t_2) \omega_J(t_1) \rangle = \langle \psi_J(t_1) \bar\psi_k(t_2) \rangle$$

$$\langle \psi_J(t_1) \psi_k(t_2) \rangle = 0 \tag{II-60}$$

As we said before, we are only intersted in the equilibrium limit $t \to \infty$ of (II-60) that is

$$-\frac{\partial}{\partial t_1} \langle 0 | \varphi_k(t_2) \varphi_J(t_1) | 0 \rangle - \langle 0 | \varphi_k(t_2) \omega_J(t_1) | 0 \rangle = \langle 0 | \psi_J(t_1) \bar\psi_k(t_2) | 0 \rangle \tag{II-61a}$$

It is possible to show that

$$<0|\psi_K(t_2)\,\bar{\psi}_J(t_1)|0> = 0 \qquad \text{(II-61b)}$$

Infact $<0|\psi_K(t_2)\,\bar{\psi}_J(t_1)|0> = <0|\psi_K(0)\,U_{KJ}(t_2-t_1)\,\bar{\psi}_J(0)|0>$ where the evolution operator $U_{KJ}(t-t')$ can be represented as an NxN diagonal matrix (remember the form of $H^{F.P}_{S.S}$ (II-42)) whose elements are the 2x2 matrices

$$U(t_2-t_1) = e^{-H^{F.P}_{S.S}(t_2-t_1)} \begin{pmatrix} I & & \\ & I & \\ & & \ddots \; I \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Going back to (II-61b), it is immediate to see why that expression is zero when $J \neq K$ .In the case $J = K$, on the other end, we have

$$<0|\psi_K(t_1)\,\bar{\psi}_K(t_2)|0> = \overbrace{e^{-V/2}\;0}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-H^{F.P}_{Forw}(t_2-t_1)}, & 0 \\ 0, & e^{-H^{F.P}_{Back}(t_2-t_1)} \end{pmatrix}\begin{pmatrix} 0 & \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e^{-V/2} \\ 0 \end{pmatrix}.$$

that is

$$<0|\psi_K(t_1)\,\bar{\psi}_K(t_2)|0> = \overbrace{e^{-V/2}\;0}\begin{pmatrix} 0 & 0 \\ 0 & e^{-H^{F.P}_{Forw}(t_2-t_1)} \end{pmatrix}\begin{pmatrix} e^{-V/2} \\ 0 \end{pmatrix} = 0$$

So in (II-61a) we obtain

$$-\frac{\partial}{\partial t_1}<0|\varphi_K(t_2)\,\varphi_J(t_1)|0> = <0|\varphi_K(t_2)\,\omega_J(t_1)|0> \qquad \text{(II-62)}$$

This relation is well known to people working in thermodynamics out-of-the-equilibrium , where it is called the <u>fluctuation-dissipation theorem</u> [28]. It relates, infact, the fluctuation $<\varphi\varphi>$ of a system with another function $<\varphi\,\omega>$ whose meaning we are going to explain.

Let us suppose that we perturb our original Langevin system (II-21) with an external force δF_i ,switched on at the instant t_0. The system will react to this, and all the "averages" will be changed

$$\delta\langle\varphi_i(t)\rangle = \langle\varphi_i(t)\rangle^{pert} - \langle\varphi_i(t)\rangle^{unpert.}$$

We can, if the response of the system is linear, write the following relation between $\delta\langle\varphi\rangle$ and δF

$$\delta\langle\varphi_i(t)\rangle = \sum_j \int_{t_0}^{t} K_{ij}(t_0-t')\,\delta F_j(t')\,dt' \qquad \text{(II-63)}$$

$K_{ij}(t-t')$ is what is called the <u>response function</u>[29] of the system. K_{ij} or better its Fourier transform is related to the absorption spectrum[24] of our system: it tells us, infact, how the system reacts to an external pertubation, and so it tells us how it absorbs or diffuses energy scattered on it. It is, of course, also related to the rate of entropy production (dissipation[24]) that accours while we drive the system out of its equilibrium with the perturbation δF^i .

Introducing the concept of probability, the L.H.S of (II-63) can be written as

$$\delta\langle\varphi_i(t)\rangle = \int\varphi_i\, P^{pert}(\varphi)\,d\varphi - \int\varphi_i\, P^{eq}\,d\varphi_i \qquad \text{(II-64)}$$

where P^{pert} is the probability after the system has been perturbed by δF_i. It is possible to have a formal expression for $P^{pert.}$ if we go back to the original Fokker-Planck eq.It is easy to see,from (II-24), that

$$\delta\hat{L} = \sum_i \cdot \delta F_i \frac{\delta}{\delta\varphi_i} \tag{II-65}$$

and so

$$\begin{aligned} P^{\text{pert}} &= \left(\exp\int_{t_0}^{t} dt' \left[\hat{L} + \delta\hat{L}\right] P^{eq}\right) = \\ &= \left[e^{-\hat{L}(t-t_0)} + \int_{t_0}^{t} dt'\, e^{\hat{L}(t-t')} \hat{\delta L}\, e^{L(t-t_0)} + \dots\right] P_{eq} \\ &= P_{eq} + \int_{t_0}^{t} dt'\, e^{\hat{L}(t-t')}\, \delta\hat{L}(t')\, P_{eq}. \end{aligned} \tag{II-66}$$

Inserting this in (II-64) and using the definition (II-63) of $K_{ij}(t-t')$ we obtain

$$K_{ij} = \Theta(t-t') \int \mathcal{D}\varphi \left[\varphi_i\, e^{\hat{L}(t-t')} \left(-\frac{\delta}{\delta\varphi_j}\right) P^{eq}\right. \tag{II-67}$$

Reminding the reader that $P^{eq} = |\psi_0|^2$ (ψ_0 ground state of $H^{F.P}_{Forw}$) we can rewrite (II-67) as

$$K_{ij}(t-t') = \Theta(t-t') \langle 0 | \hat{\varphi}_i(t) \frac{\partial V}{\partial\varphi_j}(t') | 0 \rangle \tag{II-68}$$

From the form of the Lagrangian (II-16), we can rewrite this relation as

$$K_{ij}(t-t') = -2\,\Theta(t-t') \langle 0 | \hat{\varphi}_i(t)\, \hat{\omega}_j(t') | 0 \rangle. \tag{II-69}$$

From this equation it is now clear the physical meaning of the correlation $\langle 0|\varphi_i \cdot \omega_j|0\rangle$: it is the response function, and the fluctuation-dissipation theorem (II-62) can be read (once the time has been rescaled back) as

$$K_{ij}(t_2-t_1) = \Theta(t_2-t_1) \frac{\partial}{\partial t_1} \langle 0 | \varphi_i(t_2)\, \varphi_j(t_1). \tag{II-70}$$

The importance of this theorem, for the system for which it is valid, lies in the fact that it relates an experimentally measurable quantity as K_{iJ} (through the absorption spectrum of the system) to the fluctuation $\langle \varphi\varphi \rangle$. Theoretically this theorem is a powerful tool to get information on the correlation functions of a systems, when we have no knowledge of its stochastic or quantum dynamics, but only a knowledge of how it responds to an external perturbation. In this respect, it has been widely used recently in lattice field theory. There are,of course,other manners[24,28] to demonstrate this theorem. We feel that ours is, anyhow, very amusing as it was derived from a symmetry principle and this might throw a new light on it .(see II-F).

II-E): "Ground-state-wave-function representation". Hidden supersymmetry in quantum mechanics.

In this paragraph we want to show that the hidden supersymmetry, that we have studied for stochastic processes, is actually a more general phenomenon[4].

Let us start, for simplicity, with a one-dimensional quantum mechanical system

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + U(x) \cdot$$

and let us suppose that it has a discrete non-degenerate ground state ψ_0

$$\hat{H}\psi_0 = E_0\psi_0 \cdot \tag{II-71}$$

One of the main properties of the ground-state-wave-function is its absence of nodes: that means $\psi_0 \cdot$ has always the same sign (for simplicity we take $\psi_0 \cdot$ real and positive, if it is negative we can rotate it by $e^{i\pi}$). Being positive-definite we can always define the $\ln \psi_0 \cdot$

$$V \equiv -\ln \psi_0 \tag{II-72}$$

that means

$$\psi_0 = e^{-V} \tag{II-73}$$

Inserting (II-72) in (II-71) we have that

$$-\frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}+\mathcal{U}(x)=E_0.$$

so $\mathcal{U}(x)$ can be expressed as

$$\mathcal{U}(x)=E_0+\frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2-\frac{1}{2}\frac{\partial^2 V}{\partial x^2}. \qquad \text{(II-74)}$$

and the whole Hamiltonian can be put in the form

$$\hat{H}=-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2-\frac{1}{2}\frac{\partial^2 V}{\partial x^2}+E_0. \qquad \text{(II-75)}$$

This is basically another way of writing $\hat{H}$ once Ψ_0 is known. We like to call it "<u>Ground-state-wave-function-representation</u>"4. Ideas somehow similar to these have been expressed long time ago by Coester and Haag [31] and from this reference we derived the title of this paragraph.

Subtracting E_0 from $\hat{H}$, we can define a new Hamiltonian

$$\tilde{H}_{-V}=\hat{H}-E_0.$$

One realizes the similarity of $\tilde{H}_{-V}$ with the Fokker-Planck Hamiltonian (I-16). This new $\tilde{H}_{-V}$ is a positive semi-definite operator as it can be written (like the F.P.case) in the form $\tilde{H}_{-V}=\frac{1}{2}QQ^{+}$ with $Q=\left(\frac{\partial}{\partial x}-\frac{\partial V}{\partial x}\right)$, and its ground-state is $\tilde{\Psi}_{0,-V}=e^{-V}$ at $E_0=0$.

As $\tilde{H}_{-V}$ differs from $\hat{H}$ only for the subtraction of a constant E_0, it is clear that the vacuum-vacuum expectation value of operators $\hat{O}$ is the same for both $\tilde{H}_{-V}$ and $\hat{H}$. Infact, going to Euclidean time, we get

$$\langle \tilde{\Psi}_{0-V} | \hat{O} | \tilde{\Psi}_{0-V} \rangle = \lim_{\tau \to \infty} \mathrm{tr}\, \exp(-\tilde{H}_{-V}\tau)\, \hat{O} \Big/ \mathrm{tr}\, \exp(-\tilde{H}_{-V}\tau) =$$

$$= \lim_{\tau \to \infty} \mathrm{tr}\, \exp(-\hat{H}\tau + E_0\tau)\, \hat{O} \Big/ \mathrm{tr}\, \exp(-\hat{H}\tau + E_0\tau)$$

We can cancell $\exp E_0\tau$ between numerator and denominator as it is a constant and we have

$$\langle \tilde{\Psi}_{0-V} | \hat{O} | \tilde{\Psi}_{0-V} \rangle = \lim_{\tau \to \infty} \frac{\mathrm{tr}\, \exp(-\hat{H}\tau)\, \hat{O}}{\mathrm{tr}\, \exp(-\hat{H}\tau)} = \langle \tilde{\Psi}_0 | \hat{O} | \tilde{\Psi}_0 \rangle$$

This identity can also be expressed by saying that the vacuum-vacuum (ground state-ground state) generating functional associated with $\tilde{H}_{V}$.

$$\tilde{Z}_{-V} = \int \mathcal{D}x \, \exp\left(-\int_{-\infty}^{+\infty} \left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 - \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\right] d\tau'\right)$$

is the same as the one associated with $\hat{H}$.

$$Z = \int \mathcal{D}x \, \exp\left(-\int_{-\infty}^{+\infty} \left[\frac{1}{2}\dot{x}^2 + U(x)\right] d\tau'\right).$$

that means

$$\tilde{Z}_{-V}[J] = Z[J]. \tag{II-76}$$

As in the Fokker-Planck case, $\tilde{H}_{-V}$ can be generalized to a more complex one proposed by Witten[21]

$$H^{SS} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\partial V}{\partial x} - [\bar{\psi}, \psi]\, \frac{\partial^2 V}{\partial x^2}. \tag{II-77}$$

where ψ and $\bar{\psi}$ are anticommuting variables

$$\bar{\psi}^2 = \psi^2 = 0 \qquad \{\bar{\psi}, \psi\} = 1.$$

and they, as usual, can be expressed as

$$\bar{\psi} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Using this form for $\psi, \bar{\psi}$, $H^{S.S.}$ can be written as a 2x2 matrix of operators

$$H^{S.S} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 - \frac{1}{2}\sigma^3 \frac{\partial^2 V}{\partial x^2} = \begin{pmatrix} \tilde{H}_{-V}, & 0 \\ 0, & \tilde{H}_{+V} \end{pmatrix}.$$

with

$$\hat{H}_{+V} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}. \tag{II-78}$$

Note the simililarity of this $\hat{\tilde{H}}_{+V}$ with the backward Fokker-Planck Hamiltonian (I-26).

As we know the index S.S. is due to the fact that $H^{S.S.}$ manifests a non-relativistic supersymmetry whose conserved charges are

$$Q_\psi = \left(-\frac{\partial}{\partial x} + \frac{\partial V}{\partial x}\right)\psi \quad ; \quad Q_{\bar{\psi}} = \bar{\psi}\left(\frac{\partial}{\partial x} + \frac{\partial V}{\partial x}\right).$$

In the Hamiltonian $H^{S.S}$ the function V , related through (II-72) to the ground-state-wave-function of $\hat{H}$,plays the role of the superpotential.The ground state (at $E_0=0$) of this $H^{S.S}$ is

$$\Psi_0^{S.S} = \begin{pmatrix} e^{-V} \\ 0 \end{pmatrix}. \tag{II-79}$$

It is a normalizable state because $\Psi_0 = e^{-V}$ was normalizable being the ground-state of $\hat{H}$. Ψ_0^{SS} is annihilated by Q_ψ ; $Q_{\bar{\psi}}$ and so the supersymmetry of this $H^{S.S.}$ is unbroken.

All this is by now familiar to the reader and we will not spend much time on it.

Once again the quantities we are interested in calculating for $H^{s.s}$ are the vacuum-vacuum expectation values of operators. Before doing that we have to adapt the operators $\hat{O}$ to the supersymmetric structure of the system, in the sense that every operator has to be transformed in a 2x2 matrix of operators as the $\hat{H}$ itself was in (II-77). The consistent way to do this has been explained in great detail in ref.33: for bosonic operators it consists simply in the replacement

$$\hat{O} \longrightarrow \begin{pmatrix} \hat{O} & \\ & \hat{O} \end{pmatrix} = \sigma$$

Using now the ground state (II-79), we have that

$$\langle \Psi_0^{s.s} | \sigma | \Psi_0^{s.s} \rangle = \langle \Psi_{0-V} | \hat{O} | \Psi_{0-V} \rangle \tag{II-80}$$

This equality also holds for correlation functions and not only for expectation values of operators at the same time. That means that (II-80) can be expressed by saying that the generating functional associated with $H^{s.s}$.

$$Z^{s.s}[J] = \int \mathcal{D}x\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\; \exp\left(-\int_{-\infty}^{+\infty}\left[\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 - \bar{\psi}\left(\frac{\partial}{\partial \tau'} + \frac{\partial^2 V}{\partial x^2}\right)\psi\right]d\tau'\right)$$

is the same as $\tilde{Z}_{-V}[J]$.

$$Z^{s.s}[J] = \tilde{Z}_{-V}[J]$$

On the other side we know from (II-76) that $Z[J] = \tilde{Z}_{-V}[J]$, so we can conclude that

$$Z^{s.s}[J] = Z[J] \tag{II-81}$$

We should remember that we started from a quantum mechanical model without any symmetry, and we have discovered that, at the level of vacuum-vacuum correlations, it is the same as a supersymmetric model whose superpotential is given by the ln of the graund-state of H' . We found this very amusing !. Of course only the vacuum-vacuum correlations are the same in the two models but we know that in field theory this is all we actually need to build the dynamics of the system.

We have pointed out before the similarity of $H-V'$ with the Fokker-Planck Hamiltonian (I-16). The equality (II-76) in particular means that we can reduce any quantum mechanical generating functional to the F.P.form (I-25). This means, on the other side, that it must be possible to associate a stochastic process to any H . It is easy to see which is the stochastic process that we have to associate to H if $\tilde{Z}_{-V'}$ is the corresponding F.P.generating functional. Repeating backwards the steps (I-25) to (I-2), we get that the process is

$$\dot{x} = -\frac{\partial V}{\partial x} + \eta .$$

that is

$$\dot{x} = \frac{\partial \ln \psi_0}{\partial x} + \eta . \qquad \text{(II-82)}$$

This is a well-known stochastic process: It is the one[32] E.Nelson associated to any quantum mechanical model in his form of stochastic quantization.

Nelson formulation[32] is, of course, much more detailed.In particular he does not restrict himself to just the ground state or to real ψ_0. as we have done here. But, with these restrictions in mind, we can say that (II-82) is the Nelson stochastic process for the vacuum. This process is actually the only one that has been studied during the years, and it has been used widely to prove many important theorems in constructive field theory[32]. The $\tilde{Z}_{-V}$, that we have got, can now be interpreted as the functional integral form of Nelson method of quantization. F.Guerra said in ref.33 that"a path integral form for the Nelson stochastic quantization was still missing, after so many years from its formulation". Now we have it through $\tilde{Z}_{-V}$. We think this might help, in the future, to get better insights on Nelson procedure.

II-F): Open problems.

In this last paragraph we would like to briefly mention some problems that are still open and could be attacked with the methods developed in this dissertation.

The first one is based on the idea of implementing the Nicolai mapping[23] in the "Ground-state-wave-function representation" just described. The Nicolai mapping is based on a theorem proved by H.Nicolai in 1980,and it can be summarized in this way: Given a supersymmetric generating functional

$$\mathcal{Z}^{S.S} = \int \mathcal{D}\varphi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; e^{-\int \mathcal{L}^{B} + \mathcal{L}^{F.}} \tag{II-82}$$

$\mathcal{L}^{B}$ Bosonic part of the Lagrangian

$\mathcal{L}^{F.}$ Fermionic part of the Lagrangian

with unbroken supersymmetry, a mapping exists (Nicolai mapping) for the Bosonic variables $\varphi \rightarrow X$ such that its Jacobian $\left\| \frac{\delta\varphi}{\delta X} \right\|$ is the inverse of the Matthew-Salam determinant (this is the determinant we get integrating out the Fermions in $\mathcal{Z}^{S.S.}$) this means

$$\mathcal{Z}^{S.S} = \int \mathcal{D}X \left\| \frac{\delta\varphi}{\delta X} \right\| \Delta_{MS}\; e^{-\int \mathcal{L}^{B}[\varphi[X]]} = \int \mathcal{D}X\; e^{-\int \mathcal{L}^{B}[\varphi[X]]} \tag{II-83}$$

Moreover Nicolai proved that, for that particular mapping, $\mathcal{L}^{B}[\varphi[X]] \sim K \dot{X}^{2}$:in the new variables X the dynamics is a free one!. We should not be misled by this: the Nicolai transformation is a non-local transformation, so $X(t)$ are extended objects in term of $\varphi(t)$. For example the Nicolai transformation for the $Z^{F.P}_{S.S.}$ (II-2) is

$$\dot{X} = \dot{\varphi} + \frac{\partial S}{\partial \varphi}$$

$$X(t) = \varphi(t) + \int_0^t \frac{\partial S}{\partial \varphi}[\varphi(t')]\, dt' \qquad \text{(II-84)}$$

We have explained in II-E that any quantum mechanical model can be reduced to a supersymmetric one through the "Ground state wave function representation"., as a consequence we can apply now the Nicolai mapping to any quantum mechanical generating functional. Infact from (II-81) we have

$$Z[J] = \int \mathcal{D}\varphi\, e^{-\int[\dot{\varphi}^2/2 + U(\varphi)]\,dt} = Z^{S.S.}$$

Applying the Nicolai mapping to $Z^{S.S.}$ we get

$$Z^{S.S} = \int \mathcal{D}\varphi\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\, e^{-\int[\dot{\varphi}^2/2 + \frac{1}{2}(\frac{\partial V}{\partial \varphi})^2 + \bar{\psi}(\partial_t + \frac{\partial^2 V}{\partial \varphi^2})\psi]\,dt'} = \int \mathcal{D}x\, e^{-\int k \dot{x}^2/2} \quad \text{where} \quad \dot{X} = \dot{\varphi} + \frac{\partial V}{\partial \varphi}$$

and so

$$Z[J] = \int \mathcal{D}x\, e^{-\int k \dot{x}^2/2}.$$

If this is true it means that ,instead of studying the usual quantum mechanics in term of the local interacting φ ,we can study it as a gas of free extended objects X . It is not clear to us, at the moment, if this has any profound meaning.We feel that we need a better understanding of the nature of the extended objects represented by X

The second open problem that we would like to mention is related to the fluctuation-dissipation theorem presented in II-D. We wonder what happens if the hidden supersymmetry, from which it was derived, is spontaneously broken. We know[21] that this is the case, for example, for quadratic drift forces $F=K\varphi_i^2$ in (II-21). If the symmetry is spontaneously broken the Ward identities and the detailed balance (II-36) and the Kelvin relations (II-40) will still be valid but not the fluctuation dissipation theorem. Infact a crucial ingredient in the proof of this theorem (II-62) was that in (II-61) $\langle 0|\psi_i \cdot \bar\psi_J|0\rangle$ was equal to zero. This result was dependent on the fact that the ground state had the form $\begin{pmatrix} e^{-V/2} \\ 0 \end{pmatrix}$. When the supersymmetry is spontaneously broken, the ground state has not this form anymore[21]. In general both components are different from zero, so its form is $\begin{pmatrix} \psi_0^{Ferm} \\ \psi_0^{Boch} \end{pmatrix}$ where ψ_0^{Ferm} is the new ground state of $H^{F.P}_{Ferm}$, while ψ_0^{Boch} is the one of $H^{F.P}_{Boch}$. In this case $\langle 0|\psi_i \cdot \bar\psi_J|0\rangle$ would be different from zero, and this implies that we have a violation to the fluctuation-dissipation of the form

$$K_{iJ} = -\frac{\partial}{\partial t_1}\langle 0|\varphi_i(t_2)\varphi_J(t_1)|0\rangle + [\langle 0|\psi_i \cdot \bar\psi_J|0\rangle].$$

The phenomena of breaking of the fluctuation-dissipation is not new. It appeared before in spin-glasses in connetion with the breaking of a "formal" symmetry known as replica symmetry[30]. In our case it might have a more profound character as the symmetry involved is not a formal one but microreversibility.

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