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The variety of representations of certain classes of groups

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THE VARIETY OF REPRESENTATIONS OF CERTAIN CLASSES OF GROUPS

by

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Abstract

THE VARIETY OF REPRESENTATIONS OF CERTAIN CLASSES OF ONE RELATOR GROUPS

by

Sal Liriano

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Let G be a finitely generated group. The set of homomorphisms from G into SL(2,$\mathbb{C}$) inherits the structure of a complex affine variety denoted by R(G). R(G) is an invariant of the group G. In the sixties G. Baumslag created a class of groups he termed Parafree. A group G is Parafree of rank k if 1): G is residually nilpotent (the lower central series intersects in the Identity). 2): There exists a free group F of rank k with $\frac{G}{\gamma_n G} \cong \frac{F}{\gamma_n F}$ for all n ≥1. Where $\gamma_n G$ denotes the n^{th} element of the lower central series. Notice that free groups are Parafree. The following groups

$$G_{pqt} = < x_1,x_2,x_3 ; x_1^p x_2^q = x_3^t >$$

are also Parafree of rank 2, for p, q , t >1 and with no common devisors other than one. More generally, the groups

$$Gp_1\ldots p_n = <a_1\ldots a_n ; a_1^{p_1} a_2^{p_2} \ldots a_{n-1}^{p_{n-1}} = a_n^{p_n}>$$

are Parafree of rank n-1, when $p_1, p_2, \ldots, p_{n-1}, p_n$ have no common devisors except one. In my thesis the next two theorems are proven.

<u>THEOREM 1</u>: Dim $R(G_{pqt})=6$, and $R(G_{pqt})$ is a reducible variety.

<u>THEOREM 2</u>: For $n \geq 3$, $\mathrm{Dim}(G_{p_1 \ldots p_n})=3(n-1)$.

Theorems 1 and 2 are especially striking when one learns that $R(G)$, for G free of rank n is an irreducible affine variety of dimension 3n.

Wilhelm Magnus, first in a paper titled "The Uses of Two by Two Matrices in Combinatorial Group Theory", and then the joint book with the mathematics historian Bruce Chandler "The History of Combinatorial Group Theory" (1982) asks whether any of the Parafree groups

$$S^*_{i,j}=\langle a,b,c;\ a=[a,c^i][b,c^j]\rangle$$

are ever different for different integer parameters $i,j > 1$. In the thesis the following is shown:

<u>THEOREM 3</u>: The groups $S^*_{1,1}$ and $S^*_{30,30}$ are not isomorphic.

Other results appear in the thesis; we mention a few:

1) A proof that free groups are Hopfian using naive algebraic Geometry.
2) A proof that if G is a group having a presentation with n generators and m relations, and also a presentation with n-m generators, that then G is free.

The proof of 2 was effected using naive arguments from algebraic geometry. In the book "The History of Combinatorial Group Theory" we are told that Magnus proved this theorem using lower central series arguments in 1939, and that In 1967 U. Stammbach gave a proof involving homological methods.

This work is dedicated to:

Μαριος Σεργιου, 𓇋𓈖𓊪𓅱𓃣, The 9th Century Indian Philosopher Śankaŕa, The String Quartets of Bela Bartok.

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TABLE OF CONTENTS

NOTATION

$Hom(G,\Gamma)$	Set of homomorphisms from G to the group Γ
$R(G)$	Representation variety of G in some predetermined algebraic group.
K	An algebraically closed field.
SL_2C	Group consisting of two by two matrices over the complex numbers having determinant equal to one.
A^n	Set of all n-tuples of elements of the field k.
A	The algebra $K[x_1,\ldots,x_n]$.
$Z(f)$	Set of zeros of some polynomial f of A in A^n.
$Z(T)$	Set of zeros of some set T of polynomials of A.
$I(Y)$	Vanishing ideal of some subset Y of A^n.
$R(\alpha)$	Radical ideal of some ideal α.
$h(p)$	Height of some prime ideal p.
f^*	Map induced on coordinate ring by some regular map f.
$A(V)$	Coordinate ring of the affine set V.
$K(V)$	Quotient field associated to the coordinate ring A(V).
$Dim(V)$	Dimension of the closed algebraic set V.
$Dim(R)$	Krull dimension of the ring R.
F_n	The free group of rank n usually with free basis $\{x_1,\ldots,x_n\}$.
$Cl(E)$	The closure of the algebraic set E.
$V \times W$	The product of the algebraic closed set V with the algebraic closed set W.

$G=<x_1,x_2,\ldots,x_n;r_1,\ldots>$

The presentation of some group G on generators $x_1,x_2,\ldots,x_n$ subject to the relations $r_1,r_2,\ldots$.

Orb $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ The $SL_2\mathbb{C}$ or orbit of some given matrix $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$.

$\Omega(p,M)$ Set given by $\{A \mid A \in SL_2\mathbb{C}$ and $A^p=M\}$, M is a matrix in $SL_2\mathbb{C}$.

$\Omega(p,\pm\rho(W))$ Set given by $\{A \mid A \in SL_2\mathbb{C}$ and $A^p=\pm\rho(W)\}$, W is a word in some specified free group F_n and $\rho \in R(F_n)$.

Cent(M) The $SL_2\mathbb{C}$ centralizer of the matrix M.

***

INTRODUCTION

1.1

In 1983 Culler and Shalen [CS] reintroduced to the study of finitely generated groups algebraic geometric methods whose origins go back to the work of Poincare. More precisely Culler and Shalen took the set of all representations of some compact 3 manifold group G into $SL_2\mathbb{C}$. This set can be endowed with the structure of a complex affine algebraic variety whose coordinate functions are the matrix entries of generators of G. They then showed that the space of characters of representations of G in $SL_2\mathbb{C}$ has the structure of an algebraic variety K(G), which they christened "the character variety"; they then obtained a theorem guaranteeing that the group G would split as an HNN extension or a free product with amalgamation based on properties of the "character variety". More precisely they proved a variant of the following theorem:

<u>THEOREM</u> <u>A</u>:

Let G be a finitely presented group. Suppose that the character variety of representations of G into $SL_2\mathbb{C}$ is positive dimensional. Then G has a nontrivial decomposition as a free product with amalgamation, or G has an HNN decomposition [CS].

In the same paper they prove the following useful result making possible the application of the above theorem once the dimension of a component of R(G) is known (see section 1.2 for the definition of R(G)).

THEOREM B:

Let T be an irreducible component in the representation variety of G in $SL_2\mathbb{C}$ and suppose that T contains an irreducible representation. Then the dimension of the character variety of representations of G in $SL_2\mathbb{C}$ is of dimension larger of equal to (Dim (T)-3).

In fact knowing that the dimension of the character variety is positive offers a very direct way of showing that a group is infinite. This is precisely the method employed by G. Baumslag and Shalen to show that certain groups are infinite [BS].

1.2

We recall that if G is a finitely generated group generated by X = $\{x_1, x_2, \ldots, x_n\}$, then the set Hom(G, $SL_2\mathbb{C}$) of all homomorphisms of G into the group $SL_2\mathbb{C}$ of all 2 by 2 matrices of determinant one over the field $\mathbb{C}$ of complex numbers can be endowed with the structure of an affine algebraic set (see body of thesis for details). For ease of notation we will denote this set by R(G). It is important to notice that R(G) does not depend on the choice of the finite set X of generators for G. In other words, if we choose a second finite set of generators X' for G then the resulting affine algebraic set of representations is isomorphic to the initial one (see body of thesis for details). It follows that R(G) is an invariant of the group G, i.e., isomorphic groups have isomorphic affine sets of representations.

1.3

In the early sixties G. Baumslag [B2] introduced a class of groups he termed parafree. We recall that a group G is termed parafree if

1) G is residually nilpotent (the lower central series intersects in the Identity).

2) There exists a free group F with the property that for all $n \geq 1$

$$\frac{G}{\gamma_n G} \cong \frac{F}{\gamma_n F} \tag{1.3.1}$$

where $\gamma_n G$ denotes the n^{th} element of the lower central series of G.

Denote by $\mu(G)$ the minimal number of generators of a group G. Now define the rank, $\rho(G)$, of G to be $\rho(G) = \mu(G/\gamma_2 G)$, and the deviation, $\delta(G)$, of the group G to be $\delta(G) = \mu(G) - \rho(G)$. A parafree group G is of rank r if the free group in (1.3.1) is also of rank r. Notice that a free group is parafree of deviation zero and conversely a parafree group of deviation zero is free by a theorem of Wilhelm Magnus which states that if a group has the same lower central factors as a free group and is generated by the same finite number of generators as the free group then it is free [WM2].

In the same paper G. Baumslag proved the following result which becomes quite useful in manufacturing parafree groups [B2].

<u>PROPOSITION C</u>:

Let H be parafree of rank r, let (x) be the infinite cyclic group on x, let $h \in H$ and let l be a positive integer which is not divisible by the prime n. If h generates its own centralizer in H and if $h = h_1^l$ mod

$\gamma_2 H$, where h_1 is not a proper power mod $\gamma_2 H$, then the generalized free product

$$G = \{ H_*(x) \; ; \; h = x^n \}$$

is parafree of rank r.

It should be mentioned that a free group of rank n is parafree of the same rank. Using the above result it is possible to show that the following two types of groups are parafree.

A)

$$G_{pqt} = < x_1, x_2, x_3 \; ; \; x_1^p \, x_2^q = x_3^t > \tag{1.3.2}$$

for p,q and t both larger than one and having no common divisors other than one. These groups are parafree of rank 2 and deviation one.

B)

$$Gp_1 \ldots p_n = < a_1 \ldots a_n \, ; a_1^{p_1} \, a_2^{p_2} \ldots a_{n-1}^{p_{n-1}} = a_n^{p_n} >$$

when $p_1, p_2, \ldots, p_n$ have no common divisor except. These groups are parafree of rank (n-1) and deviation one.Notice that the class A is contained in B.

Observe that in the above classes A and B we can assume, without loss of generality, that all the exponents in the relations are positive except for those in the right of the relations in A and B. These two exponents we can assume to be negative. This assumption poses no problems since the resulting groups are isomorphic to the ones where the opposite assumption is made. In 1975 Meskin [M] on work based in part on work of Rosenberger [R] proved the following theorem:

THEOREM(Meskin, 1975) :

Let $G_{p_1 \dots p_n} = \langle a_1 \dots a_n ; a_1^{p_1} a_2^{p_2} \dots a_{n-1}^{p_{n-1}} a_n^{p_n} = 1 \rangle$ where all the exponents are all greater than one, then any one-relator group with a relator of the same form is isomorphic to G iff the sequence of its exponents is a permutation of $P_1, P_2, \dots, P_n$.

It turns out that non-free parafree groups have many properties in common with free groups [B3]. An intriguing question is the relation between the varieties of representation of a non free parafree group of a given rank, say n, and the representation variety of the free group of the same rank n.

1.4

As part of a thesis problem the computation of the dimension of the affine algebraic set $R(G_{223})$ was undertaken. From statements made above it should be clear that this group is a parafree group of rank two. The goal was to compare the dimension of $R(G_{223})$ with that of $R(F_2)$. In fact it was hoped that the dimension would differ. First an attempt was made to determine the dimension using ideal theory, but it was abandoned. Then a computer with a Grobner basis implementation was employed, but the polynomials involved were too big. Finally via matrix methods, and completely by hand the following result emerged.

THEOREM 1:

Dim $(R(G_{223})) = 6$, and $R(G_{223})$ is a reducible variety.

The result above is more striking when one learns that if F_n is the free group of rank n then $R(F_n)$ is irreducible and Dim $(R(F_n))=3n$. The proof of Theorem 1 involved a method that could be generalized. Its generalization eventually gave rise to the following, seemingly rather technical, result:

THEOREM 4.1:

Let $G = \langle x_1,\ldots,x_n,y \;; W(x_1,\ldots,x_n) = y^k \rangle$, where $k \geq 3$ and W is a freely reduced word, not the identity, in $x_1,\ldots,x_n$. Let

$$\varphi(S_+) = \left\{\rho \mid \rho \in R(F_n), \text{ and } \rho(W(x_1,\ldots,x_n)) = I\right\}$$

$$\varphi(S_-) = \left\{\rho \mid \rho \in R(F_n), \text{ and } \rho(W(x_1,\ldots,x_n)) = -I\right\}$$

then

$$\mathrm{Dim}(R(G)) = \mathrm{Max}\left\{\mathrm{Max}\{\mathrm{Dim}(\varphi(S_+)),\ \mathrm{Dim}(\varphi(S_-))\}+2,\ 3n\right\} \leq 3n+1$$

COROLLARY 4.20:

Let $G = \langle x_1,\ldots,x_n,y \;; W(x_1,\ldots,x_n) = y^k \rangle$, where $k \geq 3$ and W is a freely reduced word, not the identity, in $x_1,\ldots,x_n$. Let

$$\mathrm{Max}\{\mathrm{Dim}(\varphi(S_+)),\ \mathrm{Dim}(\varphi(S_-))\}+2 = 3n$$

then R(G) is reducible and G is not free.

COMMENT:

The Main theorem is true even when k=2, but it has a more difficult formulation (see main body of thesis). Theorem 4.20 together with its corollary despite the arresting appearance have a number of concrete consequences.

THEOREM 4.21:

Let G have presentation $G = \langle a,b; a^p = b^t \rangle$ where $p,t \geq 2$, then Dim R(G) =4, and R(G) is reducible.

LEMMA 6.1:

Let $G_{pqt} = \langle x_1, x_2, x_3 \ ;\ x_1^p\, x_2^q = x_3^t \rangle$ where p, q, are negative integers and t is a positive integer equal to one or zero, and not all 2, then Dim $R(G_{pqt})$=6, and $R(G_{pqt})$ is a reducible variety.

COMMENT 1:

If all the integers are two we can only say that the dimension of the algebraic variety is 6.

COMMENT 2:

The group obtained when p, q are taken to be positive and t negative is isomorphic to the the one where the opposite assumption is made.

We return next to the groups

$$G_{p_1 \ldots p_n} \text{ with } n \geq 3.$$

We are not able to say as much for this class as for the other classes we investigated. As we have already noticed this class contains the previous class of groups. The question of the reducibility status of this class in general is as of now still open. All we can prove is

THEOREM 6.51:

For $n \geq 3$, $\mathrm{Dim}(G_{p_1 \ldots p_n}) = 3(n-1)$.

In his paper 1967 G. Baumslag proves the following results [B2].

RHEOREM (G. Baumslag, 1967):

For every pair of integers r and d satisfying $r \geq 2$ and $d \geq 1$ there exist infinitely many nonisomorphic parafree groups of rank r and having deviation d.

The next theorem is a direct result of Lemma 6.1

THEOREM 7.3:

There exist infinitely many non isomorphic parafree groups of rank two and deviation one with R(G) reducible, and with Dim(R(G)=6.

THEOREM 7.4:

For any integer $r \geq 2$ there exist an infinite number of parafree groups of rank r and deficiency one having representation variety of dimension 3r.

One might be tempted to conjecture that all parafree groups of rank n have representation variety of dimension 3n. As the next theorem will show this is not the case.

THEOREM 7.41:

Given any integer $k > 1$ there exists a non-free parafree group of rank 2 having representation variety of dimension larger or equal to 4+2k.

Thus regarding to the question relating the rank of parafree groups with the dimension of the representation variety of their corresponding free group little more can be said. Yet it is true that the above groups are not of deviation one. A more sensible question is whether all non-free parafree group of rank n and deficiency one have representation varieties of dimension 3n. We shall consider this question at another time.

1.5

An intriguing question in the theory of parafree groups is the isomorphism problem. Is there an algorithm which decides whether or not a pair of parafree groups in some recursive class of finitely presented parafree groups are isomorphic?

Whilhem Magnus first in a paper titled "The Uses of Two by Two Matrices in Combinatorial Group Theory" [WM], and then in his joint book with the mathematics historian Bruce Chandler "The History of Combinatorial Group Theory" [CM], to illustrate the difficulty of the isomorphism problem in group theory makes mention of a class of groups constructed by G. Baumslag in 1969. We quote from the book:

"We shall denote these groups by $S^*_{i,j}$, where i,j are nonzero integers. They are a special class of groups called parafree, and they are defined as having three generators a,b,c and a single defining relation $a=[a,c^i][b,c^j]$. The groups $S^*_{i,j}$ have the following properties with the free group F of rank 2.

1) $S^*_{i,j}$ has a normal subgroup with an infinite cyclic quotient group.

2) Every two generator subgroup of $S^*_{i,j}$ is free.

3) Let $\gamma_n G$ denote the n^{th} group of the lower central series of the group G. Then the intersection of all the $\gamma_n S^*_{i,j}$ is the unit element, and for all n

$$\frac{S^*_{i,j}}{\gamma_n S^*_{i,j}} \cong \frac{F}{\gamma_n F}.$$

4)

$$\frac{S^*_{i,j}}{\gamma_n S^*_{i,j}}'' \cong \frac{F}{\gamma_n F}''$$

where G'' means the second derived group of G. The great difficulties of the isomorphism problem for torsion free one relator groups are illus- trated by the fact that until now (1980) there exists no proof showing that any two of the groups $S^*_{i,j}$ are nonisomorphic."

In 1990 we became interested in the question of differentiating two of these groups, so we undertook the task of counting the number of points in the representation varieties over some finite linear group for two of these parafree groups.

<u>THEOREM</u> <u>7.5</u>:

The groups, $S^*_{1,1}$ and $S^*_{30,30}$, are not isomorphic.

<u>COMMENT</u>:

The demonstration was first effected by counting the number of representations of the two groups in $SL_2(Z_5)$ via a program in Basic using a 20 Mhz computer purchased for the purpose. The calculation ran for two days. Later a simple program in the group theory language Cayley was written. Via Cayley it was possible to count the number of representation of $S^*_{1,1}$ in the affine group PSL(2,5) over the Galois field of 5 elements. This set naturally inherits the structure of an affine variety having the property that each representation is given by a point in R(G). G. Baumslag, B. Johnson, and H. Short in a computational group theory seminar the following summer wrote a C program capable of differentiating many of these groups.

1.6

In the early thirties H. Hopf proved by topological methods that free groups of finite rank were not isomorphic to any of their proper quotients. He was unaware that in a paper of ten years earlier by Nielsen a combinatorial proof of this result lay buried (see "The History of Combinatorial Group Theory" [CM]). In this thesis a proof is given of the fact that free groups of finite rank are never isomorphic to any of their proper quotients using quite elementary algebro geometric arguments. Using similar arguments it is possible to prove a result of W. Magnus which he obtained by an argument involving the lower central series (see "The History of Combinatorial Group Theory",[CM]):

THEOREM (Magnus, 1939):

Suppose that a group G is given by a finite presentation

$$G = \langle x_1,\ldots,x_n;\ r_1,\ldots,r_m \rangle$$

If G can be generated by n-m elements then G is free.

Proof:

If G can be generated by n-m elements, then Dim (R(G)) < 3(n-m), or G is free of rank n-m. Now the fact that G has presentation as above implies that dim R(G)≥ 3(n-m), since G has presentation of deficiency n-m. We know however that a free group of rank n-m has dimension 3(n-m). We conclude then that G is free of rank (n-m).

In the book "The History of Combinatorial Group Theory" [CM] we are told that In 1967 U. Stammbach gave a homological proof of the above theorem [US].

***

AFFINE ALGEBRAIC VARIETIES

SECTION ONE

In this section we will give a general introduction to the basic theory of affine varieties. This material is being include here for completeness; emphasis will be placed on ideas relevant to the core of this work. As a result of the standard nature of this material proofs will not be given; instead, the reader is referred to the standard books on the subject [HR], [SI].

Let k be an algebraically closed field. We define affine n-space over k, denoted A^n, to be the set of all n-tuples of elements of k. An element $P=(n_1,\ldots,n_n) \in A^n$ will be called a point. Let $A=K[x_1,\ldots,x_n]$ be the polynomial ring in n variables. The elements of A can be interpreted as functions from A^n to k in the obvious way. Given a polynomial $f \in A$, we can talk about the zeros of f, namely $Z(f)=\{ P \in A^n | \ f(P) = 0\}$. More generally, if T is any subset of A, we can define the zero set of T to be the common zeros of all elements of T, namely

$$Z(T)= \{P \in A^n | \ f(P) = 0 \text{ for all } f \in T\}.$$

If α is the ideal of A generated by T, then $Z(T) = Z(\alpha)$. In addition, since A is a noetherian ring, any ideal α has a finite set of generators $f_1,\ldots,f_m$. Thus Z(T) can be expressed as the common zeros of a finite set of polynomials $f_1,\ldots,f_m$.

<u>DEFINITION</u>:

A subset $V \subset A^n$ is called an affine algebraic variety, or a closed algebraic set, or an affine variety if there are polynomials $f_1,\dots,f_m$ $\in A = K[x_1,\dots,x_n]$ such that V is the solution set of the system of equations $f_1=0,\dots,f_m = 0$. The following two propositions are useful in dealing with varieties.

PROPOSITION 1.1:

Let $V \subset A^n$, $W \subset A^m$ be two affine varieties. Then $VxW \subset A^{n+m}$ is an affine variety.

PROPOSITION 1.2:

The union of two algebraic varieties is an algebraic variety. The intersection of any countable family of algebraic varieties is also an algebraic variety. The empty set and the whole space are algebraic varieties.

The above proposition permits one to introduce a topology on A^n called the Zariski topology. This topology is generally not Housedorf.

THE ZARISKI TOPOLOGY:

We define the Zariski Topology on A^n by taking the open subsets to be complements of algebraic sets. This is a topology since the intersection of two open sets is open and the union of any family of open sets is open.

DEFINITION:

A non-empty subset Y of a topological space X is irreducible if it can not be expressed as the union $Y= Y_1 \cup Y_2$ of two proper closed subsets of Y.

It should be added that the irreducibility of an algebraic variety is a computable invariant of the variety [BH]. A free group, as will be seen, has an irreducible variety of representations in $SL_2\mathbb{C}$.

PROPOSITION 1.3:

Let V and W be irreducible affine varieties; then VxW is also an irreducible affine variety.

The following proposition underlines the special nature of irreducible affine varieties.

PROPOSITION 1.4:

let Y be an irreducible subset of X and suppose that $Y \subseteq C_1 \cup C_2 \cup \ldots \cup C_m$, where the C_j 's are closed; then $Y \subseteq C_i$ for some i.

DEFINITION:

We say that a proper subset S of an irreducible affine variety V is dense in V if its Zariski closure $\bar{S}$ is the whole variety V.

The following proposition, the result of the fact that open sets are dense in the Zariski topology, will become useful in a crucial lemma.

PROPOSITION 1.5:

Let X be irreducible and let O_i, (i=1,2), be non-empty open sets; then $O_1 \cap O_2$ is a non-empty open set.

Given some arbitrary subset of affine space it is reasonable to ask for all the polynomials that vanish on the set. Clearly the vanishing set of all those polynomials will contain the initial set and it will be a closed algebraic set.

DEFINITION:

Let $Y \subseteq A^n$; then $I(Y) = \{f \in A \mid f(P)=0 \text{ for all } P \in Y\}$.

A strong connection exists between the vanishing ideal, I(Y), of a set Y, and the zero set, Z(T), of a set of polynomials, T, as is made apparent by the next proposition.

PROPOSITION 1.6:

a) If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.

b) If $Y_1 \subseteq Y_2$ are subsets of A^n, then $I(Y_1) \supseteq I(Y_2)$.

c) For any two subsets $Y_1 \cup Y_2$ of A^n we have

$$I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2).$$

d) For any subset $Y \subseteq A^n$, $Z(I(Y)) = \bar{Y}$, the closure of Y.

DEFINITION:

Let α be some ideal; then $\mathrm{Rad}(\alpha) = \{ f \mid \exists\, n \in Z_+ \text{ with } f^n \in \alpha \}$.

THEOREM 1.7 (Hilbert's Nullstellensatz):

For any ideal $\alpha \subseteq A$, $I(Z(\alpha)) = \mathrm{Rad}(\alpha)$.

COROLLARY 1.8:

Z and I set up a bijective correspondence between the set of closed algebraic subsets of A^n and the set of ideals $\alpha \subset A$ having the property that $\alpha = \mathrm{Rad}(\alpha)$.

In other words, given a closed algebraic set V of A^n we obtain an ideal $\alpha \in A$, by putting $\alpha = I(V)$, with the property that $\alpha = \mathrm{Rad}(\alpha)$. On the other hand given an ideal α of A with the property that $\alpha = \mathrm{Rad}(\alpha)$ we obtain a closed algebraic set V on putting $V = Z(\alpha)$.

DEFINITION:

If $Y \subseteq A^n$ is an affine algebraic set we define the affine coordinate ring $A(Y)$ of Y to be $A/I(Y)$.

If $Y \subseteq A^n$ is an irreducible affine variety, then $A(Y)$ is an integral domain. Furthermore, $A(Y)$ is a finitely generated k-algebra. Conversely, any finitely generated k-algebra B which is an integral domain is the affine coordinate ring of some irreducible affine variety.

DEFINITION:

One says that a topological space V is noetherian if for every descending chain $C_1 \supseteq C_2 \supseteq \ldots$ of closed subsets, there is an integer p such that $C_p = C_{p+1} = \ldots$

A^n is a noetherian topological space. If $C_1 \supseteq C_2 \supseteq \ldots$ is a descending chain of closed subsets, then $I(C_1) \subseteq I(C_2) \subseteq \ldots$ is an ascending chain of ideals in $A = K[x_1,\ldots,x_n]$. Since A is a noetherian ring, this chain of ideals is eventually stationary. But for each i, $C_i = Z(I(C_i))$, so the chain C_i is eventually stationary.

<u>PROPOSITION</u> <u>1.9</u>:

In a noetherian topological space X any non-empty closed subset Y can be expressed as a finite union $Y = V_1 \cup V_2 \cup \ldots \cup V_m$ of irreducible closed subsets V_i with the property that for different i, j, V_i is not contained in V_j. This decomposition can be achieved uniquely. The $V_{i's}$ are called the irreducible components of X.

<u>COROLLARY</u> <u>1.10</u>:

Every close algebraic set in A^n can be expressed uniquely as a union of irreducible components.

<u>LEMMA</u> <u>1.11</u>:

Let $V_1,V_2,\ldots,V_r$ be the irreducible components of an affine algebraic set V, and $W_1,\ldots,W_s$ be the irreducible components of an affine algebraic set W, then the sets $V_i \times W_j$ where $1 \leq i \leq r$, $1 \leq j \leq s$ are the irreducible components of V x W.

<u>DEFINITION</u>:

Let V be an irreducible algebraic variety. A quasi-affine variety is a zariski open set of V.

DEFINITION:

If X is a noetherian topological space we define the dimension of X to be the supremum of all integers n such that there exists a chain

$$C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n$$

of distinct irreducible closed subsets of X. One defines the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

THEOREM 1.13:

Ley Y and X be algebraic varieties. If $Y \subset X$, then Dim $Y \leq$ Dim X. If X is irreducible, and Y is closed in X, and Dim Y = Dim X, then X = Y.

DEFINITION:

The Krull dimension of an affine algebra $B \neq 0$ is the supremum of the lengths n of all prime ideal chains

$$p_0 \subseteq p_1 \subseteq \ldots \subseteq p_n \qquad (p_{i+1} \neq p_i)$$

DEFINITION:

The height, h(p), of an ideal p is the supremum of the lengths of all chains as above with $p = p_n$.

DEFINITION:

Let B be a ring and q an arbitrary ideal of B we call any prime ideal p of B containing q a prime divisor of q. A prime divisor p is called a minimal prime divisor of q if p'= p for any prime divisor p' of q with $p' \subset p$.

PROPOSITION 1.14:

If Y is an affine algebraic set, then the dimension of Y is equal to the Krull dimension of its affine coordinate ring A(Y).

THEOREM 1.15:

Let K be a field and let B be an integral domain which is a finitely generated k-algebra. Then

a) The dimension of B is is equal to the transcendence degree of the quotient field K(B) of B over K.
b) For any prime ideal p in B, we have height(p) + Dim B/p = Dim B.

Proof: See [MH].■

THEOREM 1.16 (Krull's Principal Ideal Theorem):

Let B be a noetherian ring and (a) a principal ideal of B. Then $h(p) \leq 1$ for any minimal prime divisor p of (a), and h(p)=1 if a is not a zero divisor of B.

Proof: See [KE].■

DEFINITION:

A hyppersurface in A^n is a subvariety defined by a single polynomial equation $f(x_1,\ldots,x_n) = 0$.

COROLLARY 1.17:

In A^n let V be an irreducible algebraic variety of dimension d, and let H be a hyppersurface. If $V \cap H \neq \emptyset$ and V is not contained entirely in H, then the irreducible components of $V \cap H$ have dimension d-1.

Proof: See [KE].■

DEFINITION:

Let V be some affine variety in A^n. A function f given on V and taking values in K is called regular if there exists a polynomial F in A with coefficients in K such that f(x) = F(x) for all points $x \in X$.

DEFINITION:

Let $V \subset A^n$ and $W \subset A^m$ be closed algebraic sets. A mapping $f: V \longrightarrow W$ is called regular if there exist m regular functions $f_1, \ldots, f_m$ on V such that $f(x) = (f_1(x), \ldots, f_m(x))$ for all x in V.

DEFINITION:

Let $V \subset A^n$ and $W \subset A^m$ be closed algebraic sets. A regular mapping $f: V \longrightarrow W$ is called an isomorphism if it has an inverse, in other words, if there exists a regular mapping g from W to V such that fg=1 and gf=1. Under these circumstances the closed sets V and W are also called isomorphic.

Given a regular mapping $f: V \longrightarrow W$, where V and W are regular we can associate a mapping $f^*: A(W) \longrightarrow A(V)$. Thus every regular mapping between closed algebraic sets leads to a morphism f^* on the respective algebra of regular functions. The reverse situation is also true: every morphism between A(W) and A(V) leads to a regular mapping between W and V. In the event that V and W are isomorphic this morphism of algebras is an isomorphism (see [HR]).

THEOREM 1.18:

Let V and W be two closed algebraic sets, then V and W are isomorphic iff A(V) and A(W) are isomorphic.

THEOREM 1.19 (chevalley):

If f: V $\longrightarrow$ W is a regular map of closed algebraic sets, with V irreducible, and f(V) dense in W (hence W irreducible), then f(V) contains a non-empty open set Z of W.

Proof: See [AR].■

DEFINITION:

Let V and W be closed algebraic sets. We say that a map ϕ:V$\longrightarrow$ W is almost surjective if the closure of ϕ(V) is dense in W.

THEOREM 1.20:

Suppose that ϕ: V$\longrightarrow$ W is an almost surjective regular map and that V is irreducible, and that K is an algebraically closed field. Then there exists a non-empty open subset T of W, with the following properties: for any closed irreducible subset Y which meets T, ϕ^{-1}(Y) has a component whose dimension is equal to

$$\mathrm{Dim}\,(Y) + \mathrm{Dim}(V) - \mathrm{Dim}\,(W).$$

Proof: See [DGN].■

COROLLARY 1.21:

Suppose that y is a point of T, where T is as in Theorem 1.20. Then $\phi^{-1}(\{y\})$ has a component of dimension Dim(V) - Dim (W).

Next we will state some conditions which will be used in Theorem 1.22. This theorem will play an important role in the proof of our main results (Theorem 4.1 and Theorem 5.1).

CONDITIONS:

Let $\phi : V \longrightarrow W$ be a regular map of two closed algebraic sets, where W is irreducible. Also assume that Dim(W) = n. Let V_1 and W_1 be two proper closed subvarieties of V and W, respectively, such that the restricted map $\phi : V^{\circ} \longrightarrow W^{\circ}$, where $V^{\circ} = V - V_1$ and $W^{\circ} = W - W_1$, is such that :

1) $\phi : V^{\circ} \longrightarrow W^{\circ}$ is onto.

2) ϕ has zero dimensional fiber above each point of W°.

3) $\phi^{-1}(W^{\circ}) = V^{\circ}$.

THEOREM 1.22:

Under the above conditions $\mathrm{Dim}(\mathrm{Cl}(W^{\circ})) = \mathrm{Dim}(\mathrm{Cl}\,(V^{\circ}))$

Proof:

Let $V = \mathcal{V}_1 \cup \mathcal{V}_2 \cup ... \cup \mathcal{V}_m$ be the unique expression of V as a union of maximal irreducible algebraic sets no one of which is contained in the other (see proposition 1.9).

$$\text{Let } V_o^1 = \mathcal{V}_1 - (V_1 \cap \mathcal{V}_1)$$
$$V_o^2 = \mathcal{V}_2 - (V_1 \cap \mathcal{V}_2)$$
$$\vdots$$
$$V_o^m = \mathcal{V}_m - (V_1 \cap \mathcal{V}_m)$$

Notice $\phi^{-1}(W^\circ) = \{V_o^J \cup ... \cup V_o^k\}$. Thus there exists at least one V_o^l of dimension larger or equal to n since V° maps onto W°. Let V_o^l in fact be an element of $\{V_o^J \cup ... \cup V_o^k\}$ of maximal dimension. Now restrict ϕ to V_o^l. This map is finite to one in the image of its restriction. Also $Cl(V_o^l) = \mathcal{V}_l$ (here by "Cl" is meant the closure) is a closed irreducible subvariety of V having dimension larger or equal to n.

Notice $Cl(\phi(Cl(V_o^l)) = W_o^l \subset W$. Clearly W_o^l is an irreducible closed subvariety of W. Let $OW_o^l = \{W_o^l - (W_o^l \cap W_1)\} \subseteq W_o^l$. Clearly, OW_o^l is a possibly open subset of W_o^l with the property that given $w \in OW_o^l$, then $Dim(\phi^{-1}(w))=0$. Let $E = \{W_o^l \cap \phi(Cl(V_o^l))\}$; clearly $E \subseteq W_o^l$ and $Cl(E) = W_o^l$.

By Chevalley's Theorem there exists a Zariski open set $O \subseteq E$ so that O is non-empty. Let $O'=\{O \cap (\{W_o^l - W_1)\}$; Clearly $O' \subseteq O$, and $\phi^{-1}(O') \subseteq \phi^{-1}(O)$. These two open sets, $\phi^{-1}(O')$, and $\phi^{-1}(O)$ are open in $Cl(V_o^l)$. It is also true that given $u \in O'$; $Dim(\phi^{-1}(u) = 0$. Now $\phi^{-1}(O') \cap V_o^l$

$\subseteq V_o^1 \subseteq Cl(V_o^1)$. $\phi^{-1}(O') \cap V_o^1$ is open in the irreducible variety $Cl(V_o^1)$. thus $Cl(\{\phi^{-1}(O') \cap V_o^1\}) = Cl(V_o^1)$. Let T be the open subset of W_o^1 alluded by Theorem 1.20. Notice that $O'' =\{T \cap O'\}$ is open in W_o^1.

Now, $\Delta = \{\phi^{-1}(O'') \cap \phi^{-1}(O') \cap V_o^1\}$ is a non-empty and open in $Cl(V_o^1)$. Clearly $\Delta \subseteq \phi^{-1}(O'')$ and $\Delta \subseteq V_o^1$. In particular it follows that since Δ is dense in $Cl(V_o^1)$, $Dim(\Delta) = Dim(Cl(V_o^1))$. For any point p in Δ we have that $Dim(\phi^{-1}(\phi(p))) = 0$, since $\phi^{-1}(\phi(p))$ consists of a finite number of points. Let $\phi(p) = q$. Now q is a single point in O'' and by Theorem 1.20 we get that

$$Dim(\phi^{-1}(q)) = Dim(q) + Dim(Cl(V_o^1) - Dim(W_o^1).$$

Since $\phi^{-1}(q)$ is a finite number of points, and q is a single point we get

$$0 = 0 + Dim(Cl(V_o^1) - Dim(W_o^1).$$

This in turn implies that

$$Dim(Cl(V_o^1) = Dim(W_o^1).$$

Thus $Dim(W_o^1) \geq n$ .But W_o^1 is a closed sub-variety of W which happens to be an irreducible variety of dimension n. It follows thus that W_o^1 = W, and consequently that $Dim(Cl(V_o^1)=n$.■

AFFINE ALGEBRAIC GROUPS:

An affine algebraic group, or an affine group, is an algebraic variety Γ which at the same time is a group for which the following conditions hold: the mappings

$$\varphi : \Gamma \longrightarrow \Gamma,\ \varphi(g) = g^{-1},\ \text{and}\ \psi : \Gamma x \Gamma \longrightarrow \Gamma,\ \psi(g_1, g_2) = g_1 g_2$$

are regular.

Affine algebraic groups are always isomorphic to some subgroup of a general linear group; consequently, they are all linear. In this work the main affine algebraic group we will consider is the algebraic group obtained from the set of 2x2 matrices with entries from the complex numbers, $\mathbb{C}$, and having determinant one. This group is commonly denoted by $SL_2\mathbb{C}$.

PRESENTATION OF GROUPS:

Given a group G, a presentation of G is an epimorphism π from a free group F to G. For briefness assume that F is freely generated by the set X. Thus if $R = \ker(\pi)$, $F/R \cong G$. Let S be any subset of R such that R is generated by the conjugates of the elements of S. If a presentation π exists such that F can be chosen finitely generated, then we say that G is finitely generated. If a presentation π exists such that the set S can be chosen finite, then we say that G is finitely related. The presentation π, together with the choice of X and S, determines a set of generators and defining relators for G. In symbols

$$G = < X \; ; \; S >.$$

DEFINITION:

Let G be a finitely generated group. A representation of G is a homomorphism of G into some affine algebraic group Γ.

In other words, a representation ρ of G is nothing but an element of $HOM(G,\Gamma)$. Throughout this work we will be concerned with representations of G in the algebraic group $SL_2\mathbb{C}$.

DEFINITION:

We say that two representations ρ and ρ' are equivalent if $\rho = J\rho'$ where J is an inner automorphism of $SL_2\mathbb{C}$.

In terms of a set of generators $\{x_1,x_2,\ldots,x_n\}$ for G we define the set $R(G) \subset (SL_2\mathbb{C})^n \subset \mathbb{C}^{4n}$ to be the set of all n tuples of 2x2 matrices $(\rho(x_1),\rho(x_2),\ldots,\rho(x_n))$ where ρ is a representation of G.

THEOREM 1.23:

R(G) is a closed algebraic set in $\mathbb{C}^{4n}$.

Proof:

The defining polynomial of R(G) arise from any set of defining relations of any presentation of G. R(G) is nothing but the vanishing set of these polynomial equations in the affine space $\mathbb{C}^{4n}$, where n is the number of generators of the presentation of G.■

THEOREM 1.24 :

Let G be a group with presentation <X ; S> where the number of generators in X is finite, but the number of relations in S is infinite. Then only a finite number of relations of S are needed to obtain R(G).

Proof:

This is a direct result of the Hilbert Basis Theorem.■

COROLLARY 1.24:

There exist non-isomorphic groups G and G' with R(G) and R(G') isomorphic algebraic varieties.

Proof:

Let G be a finitely generated infinitely presented group and G' be the group having presentation consisting of the generators of G and those relations that suffice to define R(G). Then $R(G) \cong R(G')$, yet $G \ncong G'$.■

THEOREM 1.25 :

There is a one to one correspondence between points of R(G) and the representations of G in $SL_2\mathbb{C}$.

Proof:

Let ρ be a representation of G in $SL_2\mathbb{C}$ then $(\rho(x_1),\ldots,\rho(x_n))$ is a point associated with ρ. Conversely, given a point $(m_1,\ldots,m_n)$ of R(G) let τ be the representation of G given by $\tau : x_i \longrightarrow m_i$. Because the matrices $m_1,\ldots,m_n$ are $SL_2\mathbb{C}$ matrices which satisfy the relations of G,

τ is a representation of G in $SL_2\mathbb{C}$.■

The above theorem suggests that we can identify points with their corresponding representations and refer to R(G) as the space of representations of G in $SL_2\mathbb{C}$. Because a group G may have several presentations, the question arises as to whether R(G) depends on the presentation chosen. The next theorem and its corollary will address this question.

<u>THEOREM</u> <u>1.26</u>:

Given two finite sets of generators for G, then the corresponding affine algebraic sets of representations are isomorphic.

Proof:

Let that G= $gp(x_1,\dots,x_n)$ and that G = $gp(y_1,\dots,y_m)$, where m, n < ∞. There exist words w_i, v_j such that

$$x_i = w_i(y_1,\dots,y_m) \qquad (i=1,\dots,n)$$
$$y_j = v_j(x_1,\dots,x_n) \qquad (j=i,\dots,m).$$

Let R and R' denote the space of representations of G in $SL_2\mathbb{C}$ corresponding to the two different sets $\{x_1,\dots,x_n\}$, $\{y_1,\dots,y_m\}$ of generators respectively. Thus

$$R = \{(\rho(x_1),\dots,\rho(x_n) \mid \rho \text{ is a representation of G in } SL_2\mathbb{C} \},$$
$$R' = \{(\rho(y_1),\dots,\rho(y_m) \mid \rho \text{ is a representation of G in } SL_2\mathbb{C} \}.$$

The words $w_1,\dots,w_n$ define a morphism $\varphi : R' \longrightarrow R$ given as follows

$(\rho(y_1),\dots,\rho(y_m)) \longrightarrow (w_1(\rho(y_1),\dots,\rho(y_m)),\dots,w_n(\rho(y_1),\dots,\rho(y_m)))$.

A similar remark holds for $v_1,\dots,v_m$. It is not hard to see that the maps involved are polynomial maps and that they are inverses. Hence $R(G)' \cong R(G)$ as desired.■

CORROLARY 1.27:

Given a finitely generated group G, R(G) does not depend on the choice of a finite set of generators for G.

For any algebraic variety V, its dimension and reducibility status are computable invariants which can be calculated via algorithms involving a Grobner basis calculation for the vanishing ideal of V [BTH]. It follows that the dimension and the reducibility of the affine algebraic set R(G) is an invariant of a finitely generated group G. An objective in this thesis is to study these invariants for certain classes of finitely presented groups. At this point we consider the case of an infinite cyclic group.

EXAMPLE 1.27:

Let G=<x>. Then any $m \in SL_2\mathbb{C}$ gives rise to a representation of G in $SL_2\mathbb{C}$. Consequently, $R(G)=SL_2\mathbb{C}$. Because $SL_2\mathbb{C}$ is the vanishing set in $\mathbb{C}^4$ given by a single polynomial over $\mathbb{C}[x_1,\dots,x_4]$. It follows that $\mathrm{Dim}(SL_2\mathbb{C})=3$. The single polynomial defining $SL_2\mathbb{C}$ is irreducible, $SL_2\mathbb{C}$ is irreducible. Because $SL_2\mathbb{C}$ is an algebraic group $SL_2\mathbb{C}$ is nonsingular.

The following lemma will prove quite useful in the thesis.

LEMMA 1.28:

Let G and G' be two finitely generated groups and G*G' their free product. Suppose G and G' have representation varieties R(G) and R(G') respectively. Then R(G*G')=R(G)xR(G').

Proof:

Given two finitely generated presentations of G and G', respectively, the relations as well as the generators of G*G' relative to the respective presentations are obtained from those of G together with those of G'. This implies then that if ρ is a representation of G and ρ' is a representation of G' then $\rho x \rho'$ is a representation of G*G'. In other words, $\rho x \rho'$ is a point in R(G*G'). Conversely, for any point of R(G*G') we can find a representation of G*G'.■

COROLLARY 1.29:

Let F_n be a free group of rank n. Then $R(F_n) = SL_2\mathbb{C}$ x ...x $SL_2\mathbb{C}$ where the product of $SL_2\mathbb{C}$ is taken n times.

COROLLARY 1.30:

The representation variety of a free group of finite rank is irreducible.

Proof:

This follows directly from Example 1.1 and the fact that the product of two varieties have irreducible components consisting of the product of the irreducible components of the factors (Proposition 1.9).■

COROLLARY 1.40:

Let F_n be a free group of finite rank n. Then, Dim($R(F_n)$)=3n.

Proof:

This follows from the fact that the dimension of a product of varieties is the sum of the dimensions of the factors.■

<u>DEFINITION</u> (Markov, 1950):

Let P be a property of finitely presented groups which is preserved under isomorphisms. The property P is said to be a 'Markov Property' if:

1) There exists a finitely presented group G_1 with P.

2) There exist a finitely presented group G_2 which can not be embedded in any finitely presented group which has P.

It can be easily seen that freeness is a Markov Property since a non-free group can not be embedded into a free group. Recall that by a theorem of O. Schreier subgroups of free groups are free.

<u>THEOREM</u> (Adyan, Rabin, 1958):

Let P be a Markov property of finitely presented groups. Then there is no algorithm which decides whether or not finitely presented groups have the property P.

Proof: [RMO].■

<u>COMMENT</u>:

Thus given a finitely presented group it is undecidable whether the group is free or not. Consequently, the fact that a finitely generated free group has an irreducible representation variety in $SL_2\mathbb{C}$ is a positive result. We will see later that a class of groups failing to be distinguished by a infinite number of invariants from free groups, can be readily distinguished by the reducibility of their representation variety.

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MATRICES

SECTION TWO

The following section of our introduction is intended to reacquaint the reader with some basic matrix theoretic facts which will be used in the course of this work. Proofs will not be presented, instead the reader is encouraged to consult the appropriate sections of the work of Gantmacher [G].

In this section the letters A, and B shall stand for nxn matrices, and the letter I shall stand for the nxn identity matrix. Which n will be made clear in the context. We shall also adopt the following notation: given $A \in SL_2\mathbb{C}$, Orb(A) shall denote the set $\{BAB^{-1} \mid B \in SL_2\mathbb{C}\}$. In other words, Orb(A) is orbit of A under $SL_2\mathbb{C}$ conjugation.

<u>DEFINITION:</u>

The characteristic matrix of an nxn matrix A is the matrix $(\lambda I - A)$, where λ is an indeterminate. This matrix is commonly denoted as $A(\lambda)$.

<u>DEFINITION:</u>

The polynomial obtained by taking the determinant of $(\lambda I - A)$ is known as the characteristic polynomial of A. This polynomial is denoted as $\Delta(A)(\lambda)$.

<u>PROPOSITION</u> <u>2.1</u> (Cayley-Hamilton):

$\Delta(A)(\lambda)=0$.

Proof: See [G].■

In other words, evaluating A in its characteristic polynomial gives the zero matrix.

DEFINITION:

The spectrum of an nxn matrix A is the set of distinct roots of the characteristic equation. We refer to these roots as the characteristic values of A.

THEOREM 2.2:

a)

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the characteristic values of a matrix A, and f(x) a complex valued polynomial function. Then $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$ are the characteristic values of f(A), with repetition depending on multiplicity.

b)

If the derivative of f(x) is nonzero on the spectrum of of A, then in going from A to f(A) the elementary divisors do not split up. That is, if A has elementary divisors $(\lambda - \lambda_1)^{p_1}, \ldots, (\lambda - \lambda_u)^{p_u}$, then the matrix f(A) has elementary divisors $(\lambda - f(\lambda_1))^{p_1}, \ldots, (\lambda - f(\lambda_u))^{p_u}$.

Proof: See [G].■

COMMENT:

What the above theorem states is that the canonical form of the matrix f(A) is determined by the image of the spectrum under f(x) and the canonical form of the matrix A.

Two by two matrices of determinant one over the complex number field form a group under matrix multiplication. This group is known as SL_2C. The condition that the determinant be one for any matrix in SL_2C gives rise to a polynomial relation defining each element of the group. Thus $SL_2\mathbb{C}$ becomes endowed with the structure of an affine variety. $SL_2\mathbb{C}$ is consequently an affine algebraic group.

THEOREM 2.3:

$\mathrm{Dim}(SL_2\mathbb{C})=3$.

Proof:

$SL_2\mathbb{C}$ is the vanishing set in $\mathbb{C}x\mathbb{C}x\mathbb{C}x\mathbb{C}$ of a single polynomial sitting in $\mathbb{C}[x,y,z,w]$ obtained via the determinant condition.■

THEOREM 2.4:

$SL_2\mathbb{C}$ is an irreducible variety.

Proof:

The polynomial whose vanishing set gives rise to $SL_2\mathbb{C}$ is an irreducible polynomial, and consequently generates a prime ideal in the polynomial algebra $\mathbb{C}[x,y,z,w]$.■

DEFINITION:

We define the trace, tr(A), of any nxn matrix A to be the sum of the entries in the main diagonal.

LEMMA 2.5:

Let A and B be two nxn matrices. Then tr(AB)=tr(BA).

COROLLARY 2.6:

The trace of an nxn matrix is invariant under $GL_n(C)$ conjugation.

DEFINITION:

We say that nxn matrix A is diagonalizable if its conjugate to an nxn matrix all whose nonzero entries appear along the main diagonal.

LEMMA 2.7:

Given A in $SL_2\mathbb{C}$, its characteristic polynomial, $\Delta(A)(\lambda)$, is the polynomial $\lambda^2 - tr(A)\lambda + 1$.

LEMMA 2.8:

Every matrix A in $SL_2\mathbb{C}$ is similar to a matrix of one of the following form.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

where $a \neq \pm 1$.

Proof:

Let $A=\begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Since $A \in SL_2\mathbb{C}$, $\det\begin{pmatrix} x-\lambda & y \\ z & w-\lambda \end{pmatrix} = \lambda^2-(x+w)\lambda+1$ gives $\Delta(A)(\lambda)$, by Lemma 2.7. We desire to find the eigenvalues of A. Consequently we find the roots of the polynomial equation $\lambda^2-(x+w)\lambda+1=0$. They are easily found to be

$$\lambda = \frac{(x+w)\pm[(x+w)^2-4]^{1/2}}{2}.$$

Observe that $[(x+w)^2-4]=0$ only for $x+w=2$, or $x+w=-2$. If $x+w=2$, then $\lambda=1$ is an eigenvalue of multiplicity two and A is similar to either

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

depending on its minimal polynomial. If $x+w=-2$, then A is similar to either

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

depending on its minimal polynomial. Now suppose that $[(x+w)^2-4]\neq 0$. Then A has two distinct eigenvalues α and β and is thus diagonalizable. That is, there exists a non-singular matrix B such that

$$A = B\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}B^{-1},$$

where $\alpha\beta=1$ (since $A \in SL_2\mathbb{C}$). This implies that $\beta=1/\alpha$ and that A is similar to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

This completes the proof.■

<u>COROLLARY</u> <u>2.9</u>:

If $tr(A)\neq \pm 2$, then A is diagonalizable.

THEOREM 2.10:

Let B be a matrix of trace b, where $b \neq \pm 2$. A matrix $A \in SL_2\mathbb{C}$ lies in Orb(B) iff it lies in the $SL_2\mathbb{C}$ vanishing set of x_1+x_4-b.

Proof:

Let A be in Van $(x_1+ x_4-b)$. Then by Corollary 2.9 the matrix A is diagonalizable since its trace is b. So $A = T\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}T^{-1}$ where $a+a^{-1}=$ b. This implies that $a^2-ba+1 = 0$, and by application of the quadratic formula we get that $a = \frac{b\pm(b^2-4)^{1/2}}{2}$. Clearly $b^2 \neq 4$, since $b \neq \pm 2$. So A is similar to one of the following matrices:

$$\begin{pmatrix} \frac{b+(b^2-4)^{1/2}}{2} & 0 \\ 0 & \left(\frac{b+(b^2-4)^{1/2}}{2}\right)^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{b-(b^2-4)^{1/2}}{2} & 0 \\ 0 & \left(\frac{b-(b^2-4)^{1/2}}{2}\right)^{-1} \end{pmatrix}.$$

Notice however that the matrix on the right is similar to that on the left. So clearly it lies in Orb(B), since there are only two matrices having trace b and they are similar. Conversely, suppose $A \in$ Orb(B). Then it lies in the $SL_2\mathbb{C}$ vanishing set of the polynomial (x_1+x_4-b). This completes the proof.■

COROLLARY 2.11:

Let $A \in SL_2\mathbb{C}$ be any matrix of a given trace $b \neq \pm 2$. Then any matrix B in $SL_2\mathbb{C}$ having trace b is similar to A.

COROLLARY 2.12:

Let $A \in SL_2\mathbb{C}$ be of trace $b \neq \pm 2$. Then the orbit of A under $SL_2\mathbb{C}$ conjugation is an affine algebraic set of dimension 2.

Proof:

It suffices to say that it is the vanishing set of $(x_1 + x_4 - b)$ in $SL_2\mathbb{C}$, which we know is an irreducible affine variety of dimension 3. Clearly this polynomial does not vanish on all $SL_2\mathbb{C}$. Thus by Corollary 1.17 the dimension of the vanishing set is 2.■

COMMENT:

$SL_2\mathbb{C}$ is an affine variety sitting in $\mathbb{C}^4$. The polynomial $(x_1 + x_4 - b)$ lives in the algebra of regular functions (the coordinate ring) for the affine variety $SL_2\mathbb{C}$.

MATRICES, THE SPECIAL LINEAR GROUP AND BASIC ALGEBRAIC GEOMETRY

SECTION THREE

In the next section several result pertaining to dimensional theoretic aspects of solutions to matrix equations over $SL_2\mathbb{C}$ will be presented. These results will be used to prove Theorems 4.1, and Theorem 5.1. Most of the following results, though not new with high likelihood, were created by the author as the need arose.

Given a matrix $M \in SL_2\mathbb{C}$ we will denote by Cent(M) the $SL_2\mathbb{C}$ centralizer of M. Given a matrix $M \in SL_2\mathbb{C}$, and p some positive integer, the notation $\Omega(p,M)$ will be adopted to designate the set $\{A \mid A \in SL_2\mathbb{C}$ and $A^p = M\}$.

It is easy to verify that for any choice of M, $\Omega(n,M)$ is an algebraic variety sitting in four dimensional affine complex space. The variety $\Omega(n,M)$ is the solution to the matrix equation $X^p = M$ over the affine algebraic group $SL_2\mathbb{C}$. When M is I, the 2x2 identity matrix, we have the following lemma.

<u>LEMMA</u> <u>3.1</u>:

$\Omega(p,I) = R(Z_p)$ where Z_p is the finite cyclic group having n elements.

Proof: Left to the reader.■

<u>THEOREM</u> <u>3.2</u>:

i) Dim $\Omega(p,I)=0$ if $p=2$.

ii) Dim $\Omega(p,I)=2$ if $p>2$.

Proof:

(i):

Suppose p=2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^2 = I$ implies $A = A^{-1}$. So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Clearly this implies that a=d, b=-b, c=-c, d=a. Consequently, b=-b implies b=0, and c=-c implies c=0. The fact that A is in $SL_2\mathbb{C}$ with the condition that a=d implies that a=1 or a=-1. Putting all this together completes the proof of (i). In fact, because the set consists of two points we get that for p=2, $\Omega(p,I)$ is a reducible affine variety.

(ii):

Assume that p >2 is given. Then any matrix A of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ such that $a \neq \pm 1$ is a p^{th} root of unity is a solution to the equation $X^P = I$. Observe that If p > 2, then there exit p^{th}roots of unity a with the property that $a \neq \pm 1$. Since $a \neq \pm 1$ we have $tr(A) \neq \pm 2$ so we can apply Corollary 2.12 to obtain that Orb(A) is a closed algebraic set of dimension two. Dim $\Omega(p,I)$ is precisely two since $SL_2\mathbb{C}$ is an irreducible variety, and by Theorem 1.13 no variety inside $SL_2\mathbb{C}$ and different from $SL_2\mathbb{C}$ can have dimension three.■

<u>COROLLARY</u> <u>3.3</u>:

The close algebraic set $\Omega(p,I)$ is reducible.

Proof:

If p=2, then $\Omega(p,I)$ consists of two points and is thus reducible.

Assume that $p \geq 3$. Then by Theorem 2.10 for any p^{th} root of unity $a \neq \pm 1$, Orb(A), where $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, is a closed subvariety of $SL_2\mathbb{C}$ having dimension 2 and consisting entirely of all the matrices similar to A (See Thrm 2.10, Cor 2.11, Cor 2.12). ■

LEMMA 3.4:

Let G be the group given by the presentation:

$$G = \langle x_1, x_2, \ldots x_n;\ x_1^{p_1}=1,\ x_2^{p_2}=1, \ldots, x_n^{p_n}=1\rangle.$$

Then Dim $(R(G))$=Dim $\Omega(p_1,I)$ + Dim $\Omega(p_2,I)$ + ...+ Dim $\Omega(p_n,I)$.

Proof:

By Lemma 3.1: $\Omega(p,I) = R(Z_p)$. By Lemma 1.28 the free product of two groups has representation variety the product of their corresponding representation varieties. The dimension of the product of two varieties is the sum of the dimensions of the factors. This completes the proof. ■

COROLLARY 3.5:

Let G be as in Lemma 3.4. Then R(G) is reducible.

Proof:

By Lemma 1.11 the irreducible components of the product of two algebraic varieties is obtained by taking the product of the irreducible components of the factors. By Lemma 3.3 each of the $\Omega(p,I)$ is reducible. It follows that their product is reducible.■

THEOREM 3.6:

Dim $\Omega(p,-I)$=2, for $p\geq 2$.

Proof:

For p=2 the matrix $A=\begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}$ is a solution to the equation $X^p= -I$; so are all matrices in Orb(A). However, by Corollary 2.12 the dimension of Orb(A) is two. Thus Dim $\Omega(p,-I) \geq 2$. That it is strictly smaller than three follows from the fact that $\Omega(p,-I) \neq SL_2\mathbb{C}$ and an application of Theorem 1.13.■

<u>COROLLARY</u> <u>3.7</u>:

$\Omega(p,-I)$ is reducible for $p > 3$.

Proof:

This follows by the same methods involved in the proof of Corollary 3.3.■

The next theorem we shall prove is one that will help in determining the dimension of the varieties $\Omega(p,M)$, for M a matrix in $SL_2\mathbb{C}$ other than $\pm I$. We recall that $\Omega(p,M)$ is the set of solutions in $SL_2\mathbb{C}$ to the equation $X^p=M$.

<u>THEOREM</u> <u>3.8</u>:

Let $A \in \mathbf{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \neq \pm 1 \right\}$.

Then $\Omega(p,M)$, where $M=TAT^{-1}$ is given by the set

$$S_c = \left\{ TC_A A_r C_A^{-1} T^{-1} \mid C_A \in \text{Cent}(A),\ A_r \in \Omega(p,A) \right\}.$$

Proof:

Let $A_r \in \Omega(p,A)$ be given. Then $(A_r)^p = A$, and given C_A in Cent(A) we have that $(TC_AA_rC_A^{-1}T^{-1})^p = TAT^{-1}$. Thus for any A_r we get possibly many solutions of the equation $X^p = TAT^{-1}$. On the other hand, let S be a solution of $X^p = TAT^{-1}$, we shall show that S is an element of S_c. Clearly, $S^p = TAT^{-1}$ implies there exists a W in $SL_2\mathbb{C}$ such that $(W^{-1}S\ W)^p = A$. This implies that $(W^{-1}S\ W)$ lies in the set $\Omega(p,A)$. Thus there exists A_r in $\Omega(p,A)$ such that $A_r = W^{-1}SW$. So $S=W\ A_rW^{-1}$; consequently, $(W\ A_rW^{-1})^p = TAT^{-1}$, and thus $WAW^{-1}= TAT^{-1}$ implies $T^{-1}WAW^{-1}T = A$. Thus $T^{-1}W = C_A$, where C_A is in Cent(A). Consequently, $W = TC_A$. But $S = WA_rW^{-1}$ implies $S =TC_AA_rC_A^{-1}T^{-1}$. So S is an element of S_c. This is the result we wanted.■

<u>THEOREM</u> <u>3.9</u>:

Let p be any integer ≥ 2, and $M \in \mathrm{Orb}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, where $a \neq \pm 1$. Then $\mathrm{Dim}\ \Omega(p,M)=0$.

Proof:

In order to apply Theorem 3.8, we begin by finding the solutions to the equation $X^p=\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. The elementary divisors of $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ are $(\lambda - a)$, $(\lambda - a^{-1})$. Clearly then, by Theorem 2.2 , the elementary divisors of X have to be of the form $(\lambda - \varepsilon_1)$, $(\lambda - \varepsilon_2)$ where $\varepsilon_1^p = a$ and $\varepsilon_2^p = a^{-1}$. Now $\varepsilon_2^p = a^{-1}$ implies $a =(\frac{1}{\varepsilon_2})^p$; the condition that X be in $SL_2\mathbb{C}$ forces $\varepsilon_2 = \frac{1}{\varepsilon_1} = \varepsilon_1^{-1}$. Thus $(\lambda - \varepsilon_1)$, $(\lambda - \varepsilon_1^{-1})$ are the elementary divisors of X. A consequence of this is that there are only a finite number of conjugacy classes of matrices X such that $X^p=M$. By

Theorem 3.8 we have that the solutions the equation $X^p=M$, where $M=TAT^{-1}$, and $A=\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, is given by the set:

$$S_c = \left\{ TC_A A_r C_A^{-1} T^{-1} \mid C_A \in \mathrm{Cent}(A), A_r \in \Omega(p,A) \right\}. \blacksquare$$

At this point the we shall need the following two lemmas.

<u>LEMMA</u> <u>3.10</u>:

Let $a \neq \pm 1$ be given. Then

$$\mathrm{Cent}\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = \left\{ \begin{pmatrix} c_1 & 0 \\ 0 & c_4 \end{pmatrix} \middle| \; c_1,\ c_4 \in \mathbb{C},\ c_1 c_4 \neq 0 \right\}.$$

Proof: Left to the reader.■

<u>LEMMA</u> <u>3.11</u>:

Let $a \neq \pm 1$ be given. Then $\mathrm{Cent}\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)$ is abelian.

Proof:

Obvious since diagonal matrices commute.■

By applying the two previous lemmas we complete the proof of Theorem 3.9 since there are only finitely many diagonal matrices $A_r \in \Omega(p,A)$. Consequently $\mathrm{Dim}(\Omega(p,M)) = 0$.■

Next we investigate the algebraic variety $\Omega(p,M)$, where M is in $\mathrm{Orb}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.

THEOREM 3.12:

Let $B=\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, and Suppose $M \in Orb(B)$.

i) Then $\Omega(p,M) = \phi$, for p any even integer $p \geq 2$.

ii) Dim $\Omega(p,M)=0$ for p any odd integer ≥ 3.

Proof:

i)

The elementary divisor of M is the polynomial $(\lambda + 1)^2=(\lambda - (-1))^2$. By Theorem 2.2 the elementary divisors of a solution X to the equation $X^p = M$ are of the form $(\lambda - (r_1))^2$, where $r_1^p=-1$. That is r_1 is a p^{th} root of -1. The condition that X be in $SL_2\mathbb{C}$ brings to be further restrictions on X, namely that $r_1^2 = 1$. In summary, we require that $r_1^p=-1$ and that $r_1^2 = 1$. The only solutions to this simultaneous system of equations is $r_1=-1$ and this happens only when p is an odd integer.∎

ii)

In the course of proving the rest of Theorem 3.12 we shall find necessary the following proposition.

PROPOSITION 3.13:

Let $B=\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, and p odd larger than 2. Then $\Omega(p,B)$ has finite order.

Proof:

Let R be in $\Omega(p,B)$. Then R^p has elementary divisor $(\lambda -(- 1))^2$. This implies by Theorem 2.2 that the elementary divisors of R have to be of the form $(\lambda -(r_1))^2$, where $r_1^p =-1$.

Thus the matrix R is similar to the matrix B. So $R=T\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}T^{-1}$. Now For p odd $B^P=\begin{pmatrix}-1 & p\\ 0 & -1\end{pmatrix}$, and consequently $R^P=T\begin{pmatrix}-1 & p\\ 0 & -1\end{pmatrix}T^{-1}=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$.

We need to prove that T is upper triangular. To this end we shall let T be the matrix of indeterminates $\begin{pmatrix}x_1 & x_3\\ x_2 & x_4\end{pmatrix}$ and then solve the system of linear equations that arise from the matrix equation

$$\begin{pmatrix}x_1 & x_3\\ x_2 & x_4\end{pmatrix}\begin{pmatrix}-1 & p\\ 0 & -1\end{pmatrix}=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1 & x_3\\ x_2 & x_4\end{pmatrix}.$$

The solutions to this equation are $T = \begin{pmatrix}r & s\\ 0 & pr\end{pmatrix}$ where s and r are two arbitrary numbers and r is non-zero. Thus T is upper-triangular and consequently R is also upper-triangular.

Using the fact that R is upper triangular together with the fact that R must have -1 along its main diagonal, we obtain that $R =\begin{pmatrix}-1 & y\\ 0 & -1\end{pmatrix}$ where y is unknown. We desire to find the value of y. It is easy to show that for p odd $R^P=\begin{pmatrix}-1 & y\\ 0 & -1\end{pmatrix}^P= \begin{pmatrix}-1 & py\\ 0 & -1\end{pmatrix}$. Since R is in $\Omega(p,B)$ we obtain the relation $py=1$ which implies that $y = 1/p$. Thus

$$R=\begin{pmatrix}-1 & 1/p\\ 0 & -1\end{pmatrix}.$$

is the single element of $\Omega(p,B)$ and the proof of Proposition 3.13 is complete.■

In the course of completing the proof of part (ii) we will find the need for two additional lemmas.

LEMMA 3.14:

Let B be as above. Then $\mathrm{Cent}(B)=\left\{ \begin{pmatrix} c_4 & c_2 \\ 0 & c_4 \end{pmatrix} \;\middle|\; c_2, c_4 \in \mathbb{C},\ c_4 \neq 0 \right\}$.

proof: Left to the reader.■

LEMMA 3.15:

Cent(B) is abelian.

Proof:

The matrices in Cent(B) commute with themselves.■

We now continue with the proof of part (ii) of Theorem 3.12. By Theorem 3.8 all the solutions to the equation $X^p=M$, where $M = TBT^{-1}$ are given by the following set

$$S_c = \left\{ TC_A A_r C_A^{-1} T^{-1} \,\middle|\, C_A \in \mathrm{Cent}(B), A_r \in \{\Omega(p,B)\} \right\},$$

where B is the matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. We however determined that $\Omega(p,B)$ consists of a single matrix R, which by Lemma 3.14 lies in the centralizer of A. This renders the set S_c finite, and thus of dimension zero. In fact, we have also shown that it is an irreducible algebraic variety this completes the proof of Theorem 3.12.■

The next objective will be to investigate the variety $\Omega(p,M)$, where M lies in $\mathrm{Orb}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

THEOREM 3.16:

Let $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $M \in \mathrm{Orb}(B)$. Then Dim $\Omega(p,M)=0$, p any integer ≥ 2.

Proof:

In the proof of Theorem 3.16 we will need the following proposition which will be proven first.

PROPOSITION 3.17:

Let $B=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

i) Dim $\Omega(p,B)=0$, for even $p \in Z+$.

ii) Dim $\Omega(p,B)=0$, for odd $p \in Z+$.

Proof (i):

Suppose p is even and let $R \in \Omega(p,B)=0$. Then R^p has elementary divisor $(\lambda - 1)^2$. This implies by Theorem 2.2 that the elementary divisors of R have to be of the form $(\lambda - r_1)^2$, where $r_1^p =1$ and $r_1^2 =1$. These two equations have as solution the values $r_1=\pm 1$. The condition on the elementary divisors of R imposed by Theorem 2.2, as well as the conditions imposed by the two equations $r_1^p =1$ and $r_1^2 =1$ on the characteristic values of R imply that:

a): $R = T\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}T^{-1}$, or b): $R = T\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}T^{-1}$.

In Both cases T is assumed to be an $SL_2\mathbb{C}$ matrix. We shall deal with each of these cases separately.

Case a):

If $R=T\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}T^{-1}$, then $R^p=T\begin{pmatrix}1 & p\\ 0 & 1\end{pmatrix}T^{-1}=B$. By similar arguments to those used in the proof of Proposition 3.13, T is an upper triangular matrix, and consequently so is R. So $R=\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}$, where y is unknown. Our goal is to determine y. Since $R^p=B$, this implies that $y=1/p$ and Case a leads to the single solution $R=\begin{pmatrix}1 & 1/p\\ 0 & 1\end{pmatrix}$. Thus Dim $\Omega(p,B)=0$.

Case b):

If $R=T\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}T^{-1}$, then for even integers p we have that $R^p = T\begin{pmatrix}1 & -p\\ 0 & 1\end{pmatrix}T^{-1}=B$. By similar arguments to those used in the proof of Proposition 3.13 we can show that T is upper triangular. Thus R is also upper triangular and consequently $R=\begin{pmatrix}-1 & y\\ 0 & -1\end{pmatrix}$ for some unknown y. We desire to know the value of y. For p any given even integer $R^p=\begin{pmatrix}1 & -py\\ 0 & 1\end{pmatrix}=B$, consequently $-py=1$. Thus $y = -1/p$. So $R=\begin{pmatrix}1 & -1/p\\ 0 & 1\end{pmatrix}$. Consequently Dim $\Omega(p,B)=0$.

Thus we have established part (i) of Proposition 3.17. Notice that for p even, $\Omega(p,B)$ contains only two elements and these two elements lie in Cent(B). This will be quite useful.

Proof (ii):

We now consider the case where p is an odd number. Let R $\in \Omega(p,B)$. The conditions imposed by Theorem 2.2 on the elementary divisors of R together with the two conditions imposed by the equations $r_1^p = 1$, and $r_1^2 = 1$, imply that $R=T\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}T^{-1}$. Now $R^p=T\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}T^{-1}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By similar arguments to those used in the proof of Proposition 3.13 we can determine that T is upper triangular and consequently so is R. So R $=\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$. Now, it is easy to check that $R^p= \begin{pmatrix} 1 & py \\ 0 & 1 \end{pmatrix}$. Since R $\in \Omega(p,B)$ this implies that py=1. Thus y=1/p. So $R=\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}$. Thus Dim $\Omega(p,B)=0$. This completes the proof of (ii).

Notice again that for p odd, $\Omega(p,B)$ contains only one element and that this element lies in Cent(B). This completes the proof of Proposition 3.17.

For reasons already apparent the next lemma will be also necessary for the proof of Theorem 3.16.

<u>LEMMA 3.18</u>:

$$\text{Cent}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} c_4 & c_2 \\ 0 & c_4 \end{pmatrix} \,\middle|\, c_2,\ c_4 \in \mathbb{C},\ c_4 \neq 0 \right\}.$$

Proof: Left to the reader.■

<u>COMMENT</u>:

Observe that the matrices $\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1/p \\ 0 & 1 \end{pmatrix}$ obtained in Prop 3.17

live in Cent(B).

Proof of Theorem 3.16:

By Theorem 3.8 all the elements of the set $\Omega(p,M)$ are given by the following set.

$$S_c = \left\{ TC_A A_r C_A^{-1} T^{-1} \mid C_A \in \mathrm{Cent}(B),\ A_r \in \Omega(p,B) \right\},$$

where $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We however determined that $\Omega(p,B)$ consists of a finite set of matrices which by Lemma 3.18 lie in Cent(B). This renders the set S_c finite . Thus $\mathrm{Dim}(\Omega(p,M))=0.$■

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MAIN SECTION

SECTION FOUR

We introduce here our main result, Theorem 4.1. Although its formalization is rather technical, it turns out to be particularly useful in the computation of dimensions of affine algebraic sets of representations of various one relator groups.

We denote the free group on n free generators $x_1,\ldots,x_n$ by F_n. In other words $F_n = \langle x_1,\ldots,x_n\rangle$. As usual, $R(F_n)$ shall denote the representation variety of this free group in $SL_2\mathbb{C}$. Notice that

$$R(F_n)=\left\{(m_1,\ldots,m_n)\,|\,m_i\in SL_2\mathbb{C}\right\}.$$

In other words each ρ in $R(F_n)$ is identified with an n-tuple of 2x2 matrices from the algebraic group SL_2C .

In order to formulate Theorem 4.1 in a more general form we will need to introduce two integer valued functions $f(x)$ and $q(x)$ defined as follows:

$$f(x) = \begin{cases} 1 \text{ if } x \neq 2 \\ 0 \text{ if } x = 2 \end{cases} \qquad q(x) = \begin{cases} 1 \text{ if } x \neq -1 \\ 0 \text{ if } x = -1 \end{cases}$$

We can now state Theorem 4.1.

<u>THEOREM</u> <u>4.1</u>:

Let $G = \langle x_1,\ldots,x_n,y \;; W(x_1,\ldots,x_n) = y^k\rangle$, where $k \geq 2$ and $W\neq 1$ is a freely reduced word in $x_1,\ldots,x_n$. Let

$$P_+=\left\{\rho \,\middle|\, \rho \in R(F_n), \text{ and } \rho(W(x_1,\ldots,x_n))= I\right\}$$

$$P_-=\left\{\rho \,\middle|\, \rho \in R(F_n), \text{ and } \rho(W(x_1,\ldots,x_n))=-I\right\}$$

then

$$\mathrm{Dim}(R(G)) = \mathrm{Max}\left\{\mathrm{Max}\left\{\mathrm{Dim}(P_+) + 2\, f(k),\; 2\, q(\mathrm{Dim}(P_-)) + \mathrm{Dim}(P_-)\right\},\; 3n\right\} \le 3n+1$$

As already noticed, the formulation of Theorem 4.1 is rather technical. In the event that $k \geq 3$ the conclusion can be simplified a little. Indeed in this case we have that

$$\mathrm{Dim}(R(G)) = \mathrm{Max}\left\{\mathrm{Max}(\mathrm{Dim}(P_+),\; \mathrm{Dim}(P_-))+2,\; 3n\right\} \le 3n+1.$$

In the face of it this theorem is not particularly illuminating; however with its help calculations which were originally quite difficult can be achieved in acts of great swiftness. We list a few of its applications. The proofs these results will be given later.

<u>THEOREM</u>:

Let G have presentation $G = \langle a,b; a^p = b^t \rangle$ where $p,t \geq 2$, then $\mathrm{Dim}\ R(G)=4$, and $R(G)$ is reducible.

<u>THEOREM</u>:

Let $G_{p_1\ldots p_n} = \langle a_1 \ldots a_n ; a_1^{p_1} a_2^{p_2} \ldots a_{n-1}^{p_{n-1}} a_n^{p_n} = 1 \rangle$, where the exponents are integers all greater than one. Then $n \geq 3$, $\mathrm{Dim}\ R(G_{p_1\ldots p_n})=3(n-1)$.

We shall refer to the word $W(x_1,\dots,x_n)$ in the statement of Theorem 4.1 simply as W. Notice that

(2)

$$R(G)=\left\{(\rho(x_1),\dots,\rho(x_n),\sigma) \mid \rho \in R(F_n),\sigma \in \Omega(k,\ \rho(W)\neq\phi\right\}$$

We shall now define a projection φ of R(G) into $R(F_n)$ given as follows

(2.1)

$$\varphi(m_1,\dots,\ m_{n+1})=(m_1,\dots,m_n).$$

In fact when k is an odd number,for any representation ρ of $R(F_n)$ a family of representations of R(G) is obtained. This family is the fibre above ρ of the projection map φ. We recall that an element of R(G) consists of an n+1 tuple of two by two matrices satisfying the relations of G.

Not surprisingly, when k is even we will be faced with special difficulty (see Theorem 4.12), but finally similar results will be obtained.

Let $S \subset R(G)$ be defined as follows:

(3)

$$S = \left\{(\rho(x_1),\dots,\rho(x_n),\sigma) \mid \rho \in R(F_n),\sigma \in \Omega(k,\rho(W),\ \rho(W)= \pm I\right\}.$$

Let S_+, S_- be defined as follows:

$$S_+ = \left\{(\rho(x_1),\ldots,\rho(x_n),\sigma) \mid \rho \in R(F_n),\ \sigma \in \Omega(k,\rho(W)),\ \rho(W)= +I\right\}, \tag{4}$$

$$S_- = \left\{(\rho(x_1),\ldots,\rho(x_n),\sigma) \mid \rho \in R(F_n),\ \sigma \in \Omega(k,\rho(W)),\ \rho(W)= -I\right\}. \tag{5}$$

Notice that

$$S = S_+ \cup S_- . \tag{6}$$

Also notice that

$$P_+ = \varphi(S_+), \quad P_- = \varphi(S_-). \tag{6.1}$$

Thus:

$$\varphi(S) = P_+ \cup P_- . \tag{6.2}$$

COMMENT:

S is a closed subvariety of R(G) and so are S_+ and S_-. Notice that S is not empty since the trivial representation lies in S_+. It is also true that $\varphi(S)$ is a non-empty subvariety of $R(F_n)$.

The next lemma will play an important role in the proof of Theorem 4.1.

LEMMA 4.2:

Let V be some algebraic variety and S a subvariety. Then Dim(V)= Max{Dim (V-S), Dim(S)}.

Proof:

If Dim(S)= Dim(V); then one has nothing to prove. Assume that Dim(S) < Dim (V); Then irreducible components of S are contained in the irreducible components of V (see Proposition 1.4). Suppose that S' is an irreducible component of S having maximal dimension. then $S' \subset V'$, where V' is an irreducible component of V. Now if Dim $(V') \neq$ Dim (S'), Dim(V'-S')= Dim (V') (see Theorem 1.13). Thus the dimension of V is not affected. Suppose that S contains a component of maximal dimension in V, then Dim(V)=Dim(S), and the theorem is trivially true.∎

Let F_n be the free group of rank n freely generated by $\{x_1,\ldots,x_n\}$. suppose that W is a non trivial word freely reduced in F_n; Let N(W) denote the normal subgroup in F_n generated by the word W.

<u>THEOREM</u> <u>4.3</u>:

$\mathrm{Dim}(R(F_n/N(W)) < 3(n)$.

Proof:

A well known fact is that the free group of rank n has many faithful representations in $SL_2\mathbb{C}$ (see [SN], [MN]). Let $(m_1,\ldots,m_n)$ be the n-tuple of matrices obtained from a faithful representation ρ of F_n in $SL_2\mathbb{C}$. Then, since W is a freely reduced word in F_n, $\rho(W) \neq 1$. Consequently $R(F_n/N(W)) \neq R(F_n)$. However $R(F_n/N(W)) \subseteq R(F_n)$. It follows then by Theorem 1.13 that $\mathrm{Dim}.(R(F_n/N(W))) < 3(n)$.∎

The implications of the above theorem are rather profound. To illustrate this we present the following theorem which is a direct corollary of it. The reader is encouraged to consult [CM] to learn the beautiful history of this next theorem, or corollary if you will.

THEOREM (Nielson 1921, Hopf 1931):

Finitely generated free groups are Hopfian.

Proof:

This is a direct result of Theorem 4.3 and the fact that if N is a normal subgroup of G, then $\mathrm{Dim}(R(G)) \geq \mathrm{Dim}\ \mathrm{Dim}(R(G/N))$. The very minor details are left to the reader.∎

COROLLARY 4.4:

Let G be as in the statement of Theorem 4.1; then $\mathrm{Dim}(R(G))<3(n+1)$.

Proof:

In Theorem 4.1 W is clearly freely reduced. Thus so is $W^{-1}y^k$. Theorem 4.3 applies here, and consequently $\mathrm{Dim}(R(G))<3(n+1)$.∎

LEMMA 4.5:

Let G,k, and W be as in the statement of Theorem 4.1, and let k be odd; then $\mathrm{Dim}(R(G) \geq 3n$.

Proof:

Let G, k, and W be as in the statement of Theorem 4.1 . Then for any arbitrarily chosen ρ in $R(F_n)$ it is true that $\Omega(k,\rho(W)\neq\phi$. Thus the natural projection defined in (2.1) is a regular map onto $R(F_n)$. Now $R(F_n)$ is an irreducible variety of dimension 3n by Corollary 1.30, and

Corollary 1.40 respectively. It follows that the induced map φ^* between the coordinate rings $A(R(F_n))$ and $A(R(G))$ is an imbedding. Thus we have that the Krull Dimension of $A(R(G))$ is larger or equal to the Krull dimension of $A(R(F_n))$. Now by Proposition 1.14 the Krull dimension of the coordinate ring $A(R(F_n))$ is equal to $Dim(R(F_n)$. Thus $Dim\ (R(G)) \geq 3n$.■

Let S be as in (6), and φ the projection map from $R(G)$ to $R(F_n)$ introduced in (2.1).

<u>LEMMA</u> <u>4.6</u>:

$Dim(\varphi\ (S)) \leq 3n-1$.

Proof:

Recall that $S = S_+ \cup S_-$, where S_+ and S_- are the close algebraic sets in (4) and (5), respectively. Obviously, since the map φ is a map regular in the whole of $R(G)$, it can be successfully restricted to any subvariety of $R(G)$, in particular to S. Notice

$$P_+ = \varphi(S_+) = \left\{\rho \mid \in R(F_n),\ \rho(W) = I\right\} \tag{7}$$

$$P_- = \varphi(S_-) = \left\{\rho \mid \in R(F_n),\ \rho(W) = -I\right\}. \tag{8}$$

Now because F_n is the free group of rank n freely generated by the generators $\{x_1,\ldots,x_n\}$, and because there are faithful representations of F_n in $SL_2\mathbb{C}$, it is true that there exists a representation ρ' in $R(F_n)$ such that $\rho'(W) \neq \pm I$. In other words, the polynomials given rise by the equations $W=\pm I$ are not all in $I((SL_2\mathbb{C})^n)$. Recall that $\pm I$ is in the center of $SL_2\mathbb{C}$ and that free groups have trivial centers. Since $R(F_n)$ is irreducible it follows by Theorem 1.13 that any proper subvariety of it is of dimension smaller than it. Thus $\mathrm{Dim}(\varphi(S)) < \mathrm{Dim}\, R(F_n)$. Consequently,

$$\mathrm{Dim}(\varphi(S)) \leq 3n-1. \blacksquare \tag{9}$$

COMMENT:

Please notice that Lemma 4.6 makes no assumptions on the parity of k.

COROLLARY 4.7:

$\mathrm{Dim}(P_+) \leq 3n-1$, and that $\mathrm{Dim}(P_-) \leq 3n-1$.

COROLLARY 4.8:

$\mathrm{Dim}(S) \leq 3n+1$.

Proof:

Via the interpretation made possible by (4) and (5) it becomes apparent that:

$$S_+ = P_+ \times \Omega(k,I), \tag{9.1}$$

and if $\varphi(S_-)$ is non-empty one gets that:

$$S_- = P_- \times \Omega(k,-I). \tag{9.2}$$

Now, in Theorems 3.2, and Theorem 3.6 we established the following:

a) Dim $\Omega(k, I) = 2$, for $k > 2$.

b) Dim $\Omega(k, I) = 0$, If $k = 2$.

c) Dim $\Omega(k,-I) = 2$, if $k \geq 2$.

Recall that the dimension of a product of two closed algebraic sets is the sum of the dimensions of the factors, and that the dimension of a finite union of algebraic varieties has dimension that of the variety of largest dimension in the union. Thus in case where $k > 2$ in the statement of Theorem 4.1 the following lies established:

$$\mathrm{Dim}(S) = \mathrm{Max}\left\{\mathrm{Dim}(P_+),\ \mathrm{Dim}(P_-)\right\} +2. \tag{10}$$

If on the other hand $k \geq 2$, the expression in (10) takes the form:

$$\mathrm{Dim}(S)=\mathrm{Max}\left\{\mathrm{Dim}(P_+) + 2\ f(k),\ 2\ q(\mathrm{Dim}(P_-)) + \mathrm{Dim}(P_-)\right\}. \tag{10.1}$$

In light of Lemma 4.6, this directly implies the important inequality:

$$\mathrm{Dim}(S) \leq 3n+1. \blacksquare \tag{11}$$

Let φ be the projection map introduced in 2.1 and S the algebraic variety introduced in (6).

<u>LEMMA</u> <u>4.9</u>:

$(R(F_n) - \varphi(S))$ is a Zariski open subset of $R(F_n)$.

Proof:

Obviously $\varphi(S)$ contains the trivial representation of F_n. Thus $\varphi(S)$ is non empty. Since by Lemma 4.6, $Dim(\varphi(S))\leq 3n-1$, it follows that $\varphi(S)$ is not all of the irreducible affine set $R(F_n)$. Thus $(R(F_n)-\varphi(S))$ is a Zariski open subset of $R(F_n)$.■

<u>COROLLARY</u> <u>4.10</u>:

$Dim((R(F_n)-\varphi(S)))=3n$

Proof:

The dimension of Zariski open set is the dimension of its closure. Observe that $Cl((R(F_n)-\varphi(S)))=R(F_n)$. Thus, $Dim(R(F_n)-\varphi(S))=Dim(R(F_n))=3n$.■

<u>LEMMA</u> <u>4.10</u>:

Let k in Theorem 4.1 be odd, φ the projection map in (2.1), and S as in (6). Then

i) $\varphi((R(G)-S))=(R(F_n)-\varphi(S))$.

ii) for any point $m \in (R(F_n)-\varphi(S))$, $Dim(\varphi^{-1}(m))= 0$.

Proof (i):

This is obviously the case since the map φ is onto when k is odd, and R(G) in such a case is nothing but the fibre of φ over $R(F_n)$.■

Proof (ii):

In Theorems 3.2, and Theorem 3.6 it was established that $Dim\ \Omega(k,\pm I)= 2$. For $k > 2$. In Section 3 we learnt that when M is I or -I is the only time $Dim\ \Omega(k,M)>1$. For any other matrix $M \in SL_2\mathbb{C}$, $Dim\ \Omega(k,M)\leq 0$.

Notice however that in (2), $R(G)=\left\{(\rho(x_1),\ldots,\rho(x_n),\ \sigma) \mid \rho \in R(F_n),\ \sigma \in \Omega(k,\rho(W))\right\}$. Thus, by extracting from $R(F_n)$ those representations ρ with the property that $\rho(W))= \pm I$ we would leave behind a subset where the projection map φ in (2.1) has finite dimensional fibre above each point. In fact, the set left behind is precisely $(R(F_n) - \varphi(S))$.■

LEMMA 4.12:

Let k in Theorem 4.1 be odd, and φ the projection map in (2.1), and S as in (6). Then $Dim(R(G)-(S))=Dim(R(F_n)-\varphi(S))$.

Proof:

By Lemma 4.11 φ maps $(R(G)-S)$ onto $(R(F_n)-\varphi(S))$, and has finite fibre above each point $m \in (R(F_n) - \varphi(S))$. Notice that $\varphi^{-1}((R(F_n)-\varphi(S))) = (R(G)-(S))$. Thus Theorem 1.22 goes into effect since its conditions are met; we obtain that $Dim(R(G)-S)=Dim(R(F_n)-\varphi(S))$.■

THEOREM 4.13:

Let k in the statement of Theorem 4.1 be odd, φ the projection map in (2.1), and S as in (6). Then Theorem 4.1 is true.

Proof:

Notice that $R(G)=(R(G)-S) \cup S$. Consequently we can apply Lemma 4.2 to obtain that $Dim(R(G))=Max\{Dim(R(G)-S),Dim(S)\}$. We established in Lemma 4.12 that $Dim(R(F_n)-\varphi(S))=Dim(R(G)-S))$. In Corollary 4.10 it was established that $Dim(R(F_n)-\varphi(S))=3n$. Thus $Dim(R(G)-S)=3n$. In Corollary 4.8 it was established that $Dim(S)\leq 3n+1$. Thus it is true that

$$\mathrm{Dim}(R(G))=\mathrm{Max}\{\mathrm{Max}\{\mathrm{Dim}\ (S)+2,\ 3n\ \}\le 3n+1. \tag{12}$$

Since k>2 using (10) this can expressed as

$$\mathrm{Dim}(R(G))=\mathrm{Max}\{\mathrm{Max}\{\mathrm{Dim}(P_+),\ \mathrm{Dim}(P_-)\}+2,\ 3n\}\le 3n+1. \tag{12.1}$$

This completes the proof of Theorem 4.13.■

The Next objective is to lay down the necessary work to prove Theorem 4.1 when k is even.

<u>OBSERVATION</u> <u>4.14</u>:

Suppose that k is even. Again, the projection map φ in (2.1) fails to be onto $R(F_n)$; for consider ρ in $R(F_n)$ such that $\rho(W) \in \mathrm{Orb}(B)$, where B is the matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. By Theorem 3.12 we have that $\Omega(k,\rho(W)) =\phi$. Thus $\varphi(R(G))\cap(\rho(x_1),\ldots,\rho(x_n))=\phi$.

<u>LEMMA</u> <u>4.15</u>:

Let k in the statement of Theorem 4.1 be even, and B the matrix in Observation 4.14. Then

$$R(G)=\left\{(\rho(x_1),\ldots,\rho(x_n),\sigma)\,|\,\rho \in R(F_n),\ \sigma \in \Omega(k,\rho(W)),\ \rho(W) \notin \ \mathrm{Orb}(B)\right\}. \tag{13}$$

Proof:

This a direct result of the fact that representations of the n+1 generated group G are obtained by sending the first n generators $x_1,\dots,x_n$ of G to an n-tuple of $SL_2\mathbb{C}$ matrices $(m_1,\dots,m_n)$ having the property that $W(m_1,\dots,m_n)$ has a k^{th} root in $SL_2\mathbb{C}$. By Observation 4.14 this can never happens when $W(m_1,\dots,m_n)$=M, and M is in Orb(B). On the other hand, by the work done in Section 3, we know that if M is not in Orb(B) we can find a k^{th} root of M which lies in $SL_2\mathbb{C}$.■

Let

$$-2T = \{ \rho \mid \rho \in R(F_n) \text{ and } Tr(\rho(W)=-2\}. \tag{14}$$

Notice that -2T is an affine variety containing $\varphi(S_-)$ as well as the set $T=\{\rho \mid \rho\in R(F_n), \text{ and } \rho(W) \in Orb(B)\}$. (15)

LEMMA 4.16:

The set $\{R(F_n)- \{\varphi(S) \cup -2T\}\}$ is a non-empty quasi-affine variety of $R(F_n)$.

Proof:

The equation Tr(W)=-2 in (14) gives rise to a polynomial function g in the algebra of regular functions $A(R(F_n))$. Notice that g does not vanish entirely on $R(F_n)$ since the trivial representation ρ lies in $R(F_n)$, and clearly $Tr(\rho(W))\neq-2$. Recall that $R(F_n)$ is an irreducible variety sitting inside $\mathbb{C}^{4n}$. Clearly Z(g), the vanishing set of g in $\mathbb{C}^{4n}$, is a hyppersurface, H, in $\mathbb{C}^{4n}$. We have here the necessary conditions of Corollary 1.7. Consequently $R(F_n)\cap H$ has dimension 3n-1. By Lemma 4.6 we know that $Dim(\varphi\ (S))\leq$ 3n-1. Notice that $\varphi(S)$ is non-empty. This completes the proof.■

COROLLARY 4.17:

$\text{Dim } \{R(F_n)- \{\varphi(S) \cup -2T\}\} = 3n.$

Proof:

The dimension of a quasi-affine variety is the dimension of its closure.■

COROLLARY 4.18:

$\text{Dim}\{-2T\}=3n-1.$

LEMMA 4.18:

Let k in Theorem 4.1 be even, φ the projection map in (2.1). Then

i) $\varphi(R(G)-S)=\{R(F_n)-\{\varphi(S) \cup -2T\}\}.$

ii) For any point $m \in \{R(F_n)-\{\varphi(S) \cup -2T\}\}$, $\text{Dim}(\varphi^{-1}(m)) = 0.$

Proof (i):

This is a direct result of Lemma 4.15.■

Proof (ii):

Notice that all the representations ρ having infinite fibre lie in the affine algebraic set $\varphi(S)$ which is one of the sets being extracted from $R(F_n)$.Thus any point $m \in \{R(F_n)-\varphi(S)\}$ has the property that $\text{Dim}(\varphi^{-1}(m)) \leq 0$. Observe however $\{R(F_n)-\{\varphi(S) \cup -2T\}\} \subseteq \{R(F_n)-\varphi(S)\}$. Now for $m \in \{R(F_n)-\{\varphi(S) \cup -2T\}\}$, $\{\varphi^{-1}(m)\} \neq \phi$. Thus $\text{Dim}(\varphi^{-1}(m))=0$.■

LEMMA 4.19:

Let k in Theorem 4.1 be even, and φ the projection map in (2.1). Then $Dim(R(G)-(S))=Dim(\{R(F_n)-\{\varphi(S) \cup -2T\}\})$.

Proof:

By Lemma 4.18 φ maps (R(G)-S) onto $\{R(F_n)-\{\varphi(S) \cup -2T\}\}$, and has finite fibre above each point $m \in \{R(F_n)-\{\varphi(S) \cup -2T\}\}$. Notice that $\varphi^{-1}(\{R(F_n)-\{\varphi(S) \cup -2T\}\})=(R(G)-(S))$. Since the conditions of Theorem 1.22 are met, we obtain that $Dim(R(G)-S)=Dim(\{R(F_n)-\{\varphi(S) \cup -2T\}\})$.■

THEOREM 4.19:

Let k in the statement of Theorem 4.1 be even, φ the projection map in (2.1), and S as in (6). Then Theorem 4.1 is true.

Proof:

Notice that $R(G)=(R(G)-S) \cup S$. Consequently we can apply Lemma 4.2 to obtain that $Dim(R(G))=Max\{Dim(R(G)-S),Dim(S)\}$. We established in Lemma 4.12 that $Dim(\{R(F_n)-\{\varphi(S) \cup -2T\}\})=Dim(R(G)-S))$. In Corollary 4.17 it was established that $Dim(\{R(F_n)-\{\varphi(S) \cup -2T\}\})=3n$. Thus $Dim(R(G)-S)=3n$. In Corollary 4.8 it was established that $Dim(S)\le 3n+1$. Thus it is true that $Dim(R(G))=Max\{Dim(R(G)-S), Dim(S)\} \le 3n+1$. Recall that $Dim(S)=Max\left\{Dim(P_+) + 2\ f(k),\ 2\ q(Dim(P_-)) + Dim(P_-)\right\}$. In other words for the case $k \ge 2$ and even we have proven that:

(16)

$$\mathrm{Dim}(R(G))=\mathrm{Max}\left\{\mathrm{Max}\left\{\mathrm{Dim}(P_+) + 2\,f(k),\ 2\,q(\mathrm{Dim}(P_-)) + \mathrm{Dim}(P_-)\right\},\ 3n\right\} \le 3n+1. \blacksquare$$

<u>OBSERVATION</u>:

In the case that k=2 and (P_-) is empty we have proven that Dim(R(G)) =3n.

<u>COROLLARY 4.20</u>:

Let $G = \langle x_1,\ldots,x_n,y\ ;W(x_1,\ldots,x_n) = y^k \rangle$, where $k \ge 2$ and W is a freely reduced word in $\{x_1,\ldots,x_n\}$. Suppose

(20)

$$\mathrm{Max}\left\{\mathrm{Dim}(P_+) + 2\,f(k),\ 2q(\mathrm{Dim}(P_-)) + \mathrm{Dim}(P_-)\right\} = 3n$$

then R(G) is reducible and G is not free.

Proof:

If this is the the case then there exists a proper closed subvariety of R(G) having the same dimension as R(G). Consequently by Theorem 1.13 R(G) is reducible. Thus G is not free since the variety of represent-ations of a free groups is irreducible.

The next objective in this section will be to compute the dimension and freeness status of the groups having presentations:

(21)

$$G = \langle a,b;a^p = b^t \rangle$$

THEOREM 4.21:

Let G have presentation as in (21). where $p,t \geq 2$, then Dim R(G)=4 and R(G) is reducible.

Proof:

let (22)

$$\varphi(S_+)=\{\rho \mid \rho\in R(F_1) \text{ and } \rho(a^P)= I\},$$

(23)

$$\varphi(S_-)=\{\rho \mid \rho\in R(F_1) \text{ and } \rho(a^P)=-I\}.$$

By Theorem 3.2 , Dim $(\varphi(S_+))=2$ if $p > 2$, else if $p=2$, Dim $(\varphi(S_+))=0$. By Theorem 3.6, $\mathrm{Dim}(\varphi(S_-))=2$. Using the notation $P_+=\varphi(S_+))$, $P_-=\varphi(S_-)$ via the main theorem we obtain that

(24)

$$\mathrm{Max}\{\mathrm{Max}\{\ \mathrm{Dim}(P_+) + 2\ f(t),\ 2\ q(\mathrm{Dim}(P_-)) + \mathrm{Dim}(P_-)\},\ 3\}$$

ie

$$\mathrm{Dim}(R(G)) = \mathrm{Max}\{4,\ 3\}.$$

Clearly, R(G) is reducible since R(G) =$\{$ Cl(R(G)-S) $\cup$ S $\}$, where by Cl(R(G)-S) is meant the closure of (R(G)-S). Clearly, Dim(Cl(R(G)-S))= 3, by Theorem 4.1. The closed set S is of dimension 4, by Theorem 4.1. We are done.■

COROLLARY 4.22:

The groups as in (21) are not free.

Proof:

The variety of representations is reducible.■

This concludes the main section.

***

SECTION FIVE

It will become apparent that the Theorem 4.1 in Section Four is not entirely capable of handling our needs. Thus the need arises for the main theorem of this section, Theorem 5.1. First however we will need some additional clarifications.

We denote the freely generated free group on $\{x_1,\ldots,x_n\}$ by F_n. In other words $F_n = \langle x_1,\ldots,x_n\rangle$. As usual, $R(F_n)$ shall denote the representation variety of F_n in $SL_2\mathbb{C}$. Notice that

$$R(F_n)=\left\{(m_1,\ldots,m_n)\,|\,m_i \in SL_2\mathbb{C}\right\}. \tag{1}$$

In other words each ρ in $R(F_n)$ is identified with an n-tuple of 2x2 matrices from the group SL_2C .

Let $W(x_1,\ldots,x_n)\neq 1$ be a freely reduced word in the free generators $\{x_1,\ldots,x_n\}$ of F_n. Given an integer $k \geq 2$, and indeterminate y consider the equation in n+1 indeterminates obtained from

$$W(x_1,\ldots,x_n)=-y^k \tag{1.1}$$

in the group $SL_2\mathbb{C}$. The set of all solutions to this equation is an algebraic variety sitting in $\mathbb{C}^{4(n+1)}$. We shall denote this set of solutions as $\Sigma_{n+1}(n,k)$; for the sake of brevity many times we will refer to the word $W(x_1,\ldots,x_n)$ in (1.1) simply as W, and to $\Sigma_{n+1}(n,k)$ simply as Σ_{n+1}.

DEFINITION:

We say that a freely reduced word W in the free group F_n has property ξ if there exits a representation $\rho \in R(F_n)$ with the property that $Tr(\rho(w)) \neq 2$.

In order to formulate Theorem 5.1 in a more general form we will need to introduce the integer valued function $f(x)$ defined as follows:

$$f(x) = \begin{cases} 1 \text{ if } x \neq 2 \\ 0 \text{ if } x = 2 \end{cases} \tag{2}$$

<u>THEOREM</u> <u>5.1</u>:

Let $\Sigma_{n+1}(n,k)$ denote the set of solutions in $SL_2\mathbb{C}$ to the equation $W(x_1,\dots,x_n)=-y^k$, where $k \geq 2$, and $W\neq 1$ is a freely reduced word in the free generators $\{x_1,\dots,x_n\}$ such that if k is even W has property ξ. Let

$$P_+=\left\{\rho \middle| \rho \in R(F_n), \text{ and } -\rho(W)= I\right\} \tag{3}$$

$$P_-=\left\{\rho \middle| \rho \in R(F_n), \text{ and } -\rho(W)=-I\right\}. \tag{4}$$

Then

$$Dim(\Sigma_{n+1}(n,k)) = Max\left\{Max\left\{Dim(P_+) + 2\ f(k),\ Dim(P_-)+2\right\},\ 3n\right\} \leq 3n+1.$$

The statement of Theorem 5.1 seems rather technical; in the event that $k\geq 3$ the conclusion can be simplified a little. Indeed in this case it can be expressed as

$$\mathrm{Dim}(\Sigma_{n+1}(n,k)) = \mathrm{Max}\left\{\mathrm{Max}(\mathrm{Dim}(P_+),\ \mathrm{Dim}(P_-))+2,\ 3n\right\} \le 3n+1.$$

The reader may feel that this theorem is not particularly illuminating; however, with its help results that were initially very difficult to obtain using Theorem 4.1 alone were achieved with great ease. I list one of its applications whose proof will be given later.

THEOREM:

Let $G_{p_1 \ldots p_n} = \langle a_1 \ldots a_n; a_1^{p_1}\ a_2^{p_2} \ldots a_{n-1}^{p_{n-1}} a_n^{p_n} = 1\rangle$ where all the exponents are all greater than one. If $n \ge 3$, $\mathrm{Dim}\ R(G_{p_1 \ldots p_n}) = 3(n-1)$.

We now proceed to lay down the necessary work for a proof of Theorem 5.1. Notice that

$$\Sigma_{n+1}(n,k) = \left\{(\rho(x_1),\ldots,\rho(x_n),\ \sigma) \mid\ \rho \in R(F_n),\ \sigma \in \Omega(k,-\rho(W)) \neq \phi\right\}. \tag{5}$$

Clearly, a point in $\Sigma_{n+1}(n,k)$ looks like an $(n+1)$-tuple of 2x2 matrices, $(m_1,\ldots,\ m_{n+1})$, which satisfy the equation $W=-y^k$ in (1.1). We shall now define a projection φ of $R(G)$ into $R(F_n)$ given as follows

$$\varphi(m_1,\ldots,\ m_{n+1})=(m_1,\ldots,m_n). \tag{6}$$

In fact, when k in Theorem 5.1 is an odd number, for any representation ρ of $R(F_n)$ a family of representations of $R(G)$ is obtained. This family is the fibre above ρ of the projection map φ.

Not surprisingly, when k is even we will be faced with special difficulties (see Theorem 3.12). finally, similar results will be obtained.

Let $S \subset \Sigma_{n+1}(n,k)$ be defined as follows

$$S = \left\{ (\rho(x_1),\ldots,\rho(x_n),\sigma) \,\middle|\, \rho \in R(F_n),\ -\rho(W)= \pm I, \sigma \in \Omega(k,-\rho(W)) \right\}, \tag{7}$$

and let S_+, S_- be defined as follows:

$$S_+ = \left\{ (\rho(x_1),\ldots,\rho(x_n),\sigma) \,\middle|\, \rho \in R(F_n),\ -\rho(W)= I,\ \sigma \in \Omega(k,-\rho(W)) \right\}, \tag{8}$$

$$S_- = \left\{ (\rho(x_1),\ldots,\rho(x_n),\sigma) \,\middle|\, \rho \in R(F_n),\ -\rho(W)=-I,\ \sigma \in \Omega(k,-\rho(W)) \right\}. \tag{9}$$

Notice that

$$S = S_+ \cup S_-, \tag{10}$$

and that

$$P_+ = \varphi(S_+), \quad P_- = \varphi(S_-). \tag{11}$$

Thus:

$$\varphi(S) = P_+ \cup P_-. \tag{12}$$

COMMENT 12.1:

S is a closed subvariety of Σ_{n+1} and so are S_+ and S_-. Notice that S is not empty since the point (I, ...I, -I) lies in S if k is odd, and (I,..,I,R) where $R \in \Omega(k,-I)$ lies in S if K is even. It is also quite easy to check that $\varphi(S)$ is a non-empty subvariety of $R(F_n)$.

LEMMA 5.2:

Let $\Sigma_{n+1}(n,k)$ be as in the statement of Theorem 5.1; then

$\mathrm{Dim}(\Sigma_{n+1}(n,k))<3(n+1)$.

Proof:

Notice that $\mathrm{Dim}(\Sigma_{n+1})\leq 3(n+1)$, since it lies in $(SL_2\mathbb{C})^{n+1}$, which by Corollary 1.40 has dimension 3(n+1). In fact, since I^{n+1} is not in Σ_{n+1}, but is in $(SL_2\mathbb{C})^{n+1}$, it is true that $\mathrm{Dim}(\Sigma_{n+1})<3(n+1)$, by Theorem 1.13.■

LEMMA 5.3:

Let k be odd. Then $\mathrm{Dim}(\Sigma_{n+1}(n,k)) \geq 3n$.

Proof: The proof of this lemma is the same as the proof of Lemma 4.5 in the previous section. The only difference is that in the proof of Lemma 4.5, R(G) should be replaced with Σ_{n+1}, and $\Omega(k,\rho(W))$ should be replaced with $\Omega(k,-\rho(W))$.■

LEMMA 5.4:

$\mathrm{Dim}(\varphi\ (S)) \leq 3n-1$.

Proof:

Recall that $S = S_+ \cup S_-$. Where S_+ and S_- are the close algebraic sets in (8) and (9), respectively. Obviously, since the map φ is a map regular in the whole of Σ_{n+1} it can be successfully restricted to any subvariety of R(G), in particular to S. Recall that

$$P_+ = \varphi(S_+) = \left\{\rho \mid\ \in R(F_n),\ \rho(W)=-I\right\}, \tag{13}$$

$$P_- = \varphi(S_-) = \left\{\rho \mid\ \in R(F_n),\ \rho(W)=\ I\right\}. \tag{14}$$

Since F_n is the free group of rank n freely generated by the generators $\{x_1,\ldots,x_n\}$, and since there are faithful representations of F_n in $SL_2\mathbb{C}$, it is true that there exists a representation ρ' in $R(F_n)$ such that $\rho'(W)\neq\pm I$. In other words, the polynomials given rise by the equations $W=\pm I$ are not all in $I((SL_2\mathbb{C})^n)$. Recall that $\pm I$ is in the center of $SL_2\mathbb{C}$ and that free groups have trivial centers. Since $R(F_n)$ is irreducible, it follows by Theorem 1.13 that any proper subvariety of it is of dimension smaller than it. Thus $\mathrm{Dim}(\varphi(S)) < \mathrm{Dim}\ R(F_n)$. Consequently,

$$\mathrm{Dim}(\varphi(S)) \leq 3n-1. \blacksquare \tag{15}$$

COMMENT:

Please notice that Lemma 5.4 makes no assumptions on the parity of k.

COROLLARY 5.5:

$\mathrm{Dim}(P_+) \leq 3n-1$, and that $\mathrm{Dim}(P_-) \leq 3n-1$.

COROLLARY 5.6:

$\mathrm{Dim}(S) \leq 3n+1$.

Proof:

Via the very definition of S_+ and S_- (see (8) and (9)) we have

$$S_- = P_- \times \Omega(k, I), \tag{16}$$

and if $\varphi(S_+)$ is non-empty one gets that

$$S_+ = P_+ \times \Omega(k,-I). \tag{17}$$

In Theorems 3.2, and Theorem 3.6 we established the following:

a) $\mathrm{Dim}\ \Omega(k, I) = 2$, for $k > 2$,

b) $\mathrm{Dim}\ \Omega(k, I) = 0$, If $k = 2$,

c) $\mathrm{Dim}\ \Omega(k,-I) = 2$, if $k \geq 2$.

Recall that the dimension of a product of two algebraic varieties is the sum of the dimensions of the factors, and that the dimension of a finite union of algebraic varieties has dimension that of the algebraic variety of largest dimension in the union. Thus when $k > 2$ in Theorem 5.1 the following is true:

$$\mathrm{Dim}(S) = \mathrm{Max}\left\{\mathrm{Dim}(P_+),\ \mathrm{Dim}(P_-)\right\} + 2. \tag{18}$$

If on the other hand $k \geq 2$, the expression in (18) takes the form:

$$\mathrm{Dim}(S)=\mathrm{Max}\left\{\mathrm{Dim}(P_+) + 2\,f(k),\ \mathrm{Dim}(P_-) + 2\right\}. \tag{19}$$

In light of Lemma 5.4, this directly implies the important inequality:

$$\mathrm{Dim}(S) \leq 3n+1. \blacksquare \tag{20}$$

<u>COROLLARY</u> <u>5.7</u>:

$\mathrm{Dim}((R(F_n)-\varphi(S)))=3n$

Proof:

The dimension of Zariski open set is the dimension of its closure. Observe that $\mathrm{Cl}((R(F_n)-\varphi(S)))=R(F_n)$. Thus, $\mathrm{Dim}(R(F_n)-\varphi(S))=\mathrm{Dim}(R(F_n))$ $=3n.\blacksquare$

<u>LEMMA</u> <u>5.8</u>:

Let k in Theorem 5.1 be odd. Then

i) $\varphi(((\Sigma_{n+1}(n,k))-S))=(R(F_n)-\varphi(S))$.

ii) for any point $m \in ((\Sigma_{n+1}(n,k)-\varphi(S))$, $\mathrm{Dim}(\varphi^{-1}(m))= 0$.

Proof i):

This is obviously the case since the map φ is onto when k is odd, and

$R(G)$ in such a case is nothing but the fibre of φ over $R(F_n)$. ■

Proof ii):

In Theorems 3.2, and Theorem 3.6 it was established that Dim $\Omega(k,\pm I)=2$. For $k > 2$. In Section 3 we learnt that when $M \in \{I, -I\}$ is the only time Dim $\Omega(k,M)>1$. For any other matrix $M \in SL_2\mathbb{C}$, Dim $\Omega(k,M)\leq 0$. Notice however that in (5), $\Sigma_{n+1}=\left\{(\rho(x_1),\ldots,\rho(x_n),\sigma)\,|\,\rho\in R(F_n),\ \sigma\in \Omega(k,-\rho(W))\neq\phi\right\}$. Thus, by extracting from $R(F_n)$ those representations ρ with the property that $-\rho(W))=\pm I$ we would leave behind a subset where the projection map φ has finite dimensional fibre above each point. In fact, the set left behind is precisely $(R(F_n) - \varphi(S))$.■

LEMMA 5.9:

Let k in Theorem 5.1 be odd. Then $\mathrm{Dim}((\Sigma_{n+1})-S)=\mathrm{Dim}(R(F_n)-\varphi(S))$.

Proof:

By Lemma 5.8 φ maps $((\Sigma_{n+1})-S)$ onto $(R(F_n)-\varphi(S))$, and has finite fibre above each point $m \in (R(F_n)-\varphi(S))$. Notice that $\varphi^{-1}((R(F_n)-\varphi(S))=((\Sigma_{n+1})-S)$. Thus by Theorem 1.22 we get that $\mathrm{Dim}(R(G)-S)=\mathrm{Dim}(R(F_n)-\varphi(S))$.■

THEOREM 5.10:

Let k in the statement of Theorem 5.1 be odd. Then Theorem 5.1 is true.

Proof:

Notice that $\Sigma_{n+1}=((\Sigma_{n+1})-S)\cup S$. Consequently we can apply Lemma 4.2 to obtain that $Dim(\Sigma_{n+1})=Max\{Dim((\Sigma_{n+1})-S),\ Dim(S)\}$. We established in Lemma 5.9 that $Dim(R(F_n)-\varphi(S))=Dim((\Sigma_{n+1})-S))$. In Corollary 5.7 it was established that $Dim(R(F_n)-\varphi(S))=3n$. Thus $Dim((\Sigma_{n+1})-S)=3n$. In Corollary 5.6 it was established that $Dim(S)\leq 3n+1$. Thus it follows that

$$Dim(\Sigma_{n+1})=Max\{Dim(S),\ 3n\}\leq 3n+1. \tag{21}$$

Since k>2 using (10) this can expressed as

$$Dim(\Sigma_{n+1})=Max\{Max\{Dim(P_+),\ Dim(P_-)\}+2,\ 3n\}\leq 3n+1. \tag{21.1}$$

This completes the proof of Theorem 5.10. ∎

Our next goal is to lay down the necessary work to prove Theorem 5.1 when k is even. Recall that when k is even we are assuming that the work W has property ξ.

<u>OBSERVATION</u> <u>5.11</u>:

If k is even, the projection map φ in (6) fails to be onto $R(F_n)$; for consider ρ in $R(F_n)$ such that $-\rho(W)\in Orb(B)$, where $B=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$. By Theorem 3.12 we have $\Omega(k,-\rho(W))=\phi$. So $\varphi(\Sigma_{n+1})\cap(\rho(x_1),\ldots,\rho(x_n))=\phi$.

LEMMA 5.12:

Let k in the statement of Theorem 5.1 be even, and B the matrix in Observation 5.11. Then

$$\Sigma_{n+1}=\left\{(\rho(x_1),\ldots,\rho(x_n),\sigma)\,|\,\rho\in R(F_n),\text{ and }-\rho(W)\notin \mathrm{Orb}(B),\ \sigma\in\Omega(k,-\rho(W)\right\}. \tag{22}$$

Proof:

This a direct result of the fact that a point m of $\Sigma_{n+1}(n,k)$ is obtained by taking any n-tuple of $SL_2\mathbb{C}$ matrices $(m_1,\ldots,m_n)$ having the property that $-W(m_1,\ldots,m_n)$ has a k^{th} root in $SL_2\mathbb{C}$. By Observation 5.12, this can never happens when $-W(m_1,\ldots,m_n)=M$, and M is in Orb(B). On the other hand, by the work done in Section 3 we know that if M is not in Orb(B) we can find a k^{th} root of M which lies in $SL_2\mathbb{C}$ ∎

Let

$$-2T = \{\ \rho|\ \ \rho\in R(F_n) \text{ and } Tr(-\rho(W))=-2\}. \tag{23}$$

Notice that -2T is an affine variety containing $\varphi(S_+)$ as well as the set $T=\{\rho\,|\,\rho\in R(F_n),\text{ and }-\rho(W)\in \mathrm{Orb}(B)\}$. (24)

LEMMA 5.13:

Let W have property ξ. Then the set $\{R(F_n)-\{\varphi(S)\cup -2T\}\}$ is a non-empty quasi-affine variety of $R(F_n)$.

Proof:

The equation $Tr(-\rho(W))=-2$ in (23) gives rise to a polynomial function g in the algebra of regular functions $A(R(F_n))$. Notice that g does not vanish entirely on $R(F_n)$ since W has property ξ and thus there exists some representation $\rho \in R(F_n)$ such that $Tr(-\rho(W))\neq-2$. Recall that $R(F_n)$ is an irreducible variety sitting inside $\mathbb{C}^{4n}$. Clearly $Z(g)$, the vanishing set of g in $\mathbb{C}^{4n}$, is a hyppersurface, H, in $\mathbb{C}^{4n}$. We have here the necessary conditions of Corollary 1.7. Consequently $R(F_n)\cap H$ has dimension 3n-1. By Lemma 5.4 we know that $Dim(\varphi\ (S))\leq 3n-1$. Notice that $\varphi(S)$ is non-empty. This completes the proof.■

<u>COROLLARY</u> <u>5.14</u>:

Let W have property ξ. Then $Dim\ \{R(F_n)-\ \{\varphi(S) \cup -2T\}\} = 3n$.

Proof:

The dimension of a quasi-affine variety is the dimension of its closure ■

<u>COROLLARY</u> <u>5.15</u>:

Let W have property ξ. Then $Dim\{-2T\}=3n-1$.

<u>LEMMA</u> <u>5.16</u>:

Let k in Theorem 5.1 be even, and W have property ξ. Then

i) $\varphi(\Sigma_{n+1}-S)=\{R(F_n)-\{\varphi(S) \cup -2T\}\}$,

ii) For any point $m \in \{R(F_n)-\{\varphi(S) \cup -2T\}\}$, $Dim(\varphi^{-1}(m)) = 0$.

Proof i):

This is a direct result of Lemma 5.12 and the definition of the set S.■

Proof ii):

Notice that all the representations ρ having infinite fibre lie in the affine algebraic set $\varphi(S)$ which is one of the sets being extracted from $R(F_n)$. Thus any point $m \in \{R(F_n)-\varphi(S)\}$ has the property that $\mathrm{Dim}(\varphi^{-1}(m)) \leq 0$. Observe however $\{R(F_n)-\{\varphi(S) \cup -2T\}\} \subseteq \{R(F_n)-\varphi(S)\}$. Now for $m \in \{R(F_n)-\{\varphi(S) \cup -2T\}\}$, $\{\varphi^{-1}(m)\} \neq \phi$. Thus $\mathrm{Dim}(\varphi^{-1}(m))=0$.■

LEMMA 5.17:

Let k in Theorem 5.1 be even, and W have property ξ.

$\mathrm{Dim}(\Sigma_{n+1} - S)=\mathrm{Dim}(\{R(F_n)-\{\varphi(S) \cup -2T\}\})$.

Proof:

By Lemma 5.16 φ maps $(R(G)-S)$ onto $\{R(F_n)-\{\varphi(S)\cup-2T\}\}$, and has finite fibre above each point $m \in \{R(F_n)-\{\varphi(S)\cup-2T\}\}$. Notice that $\varphi^{-1}(\{R(F_n)-\{\varphi(S)\cup-2T\}\}) = (R(G)-(S))$. Since the conditions of Theorem 1.22 are met, we obtain that $\mathrm{Dim}(R(G)-S)= \mathrm{Dim}(\{R(F_n)-\{\varphi(S) \cup -2T\}\})$.■

THEOREM 5.18:

Let k in the statement of Theorem 5.1 be even, and W let have property ξ. Then Theorem 5.1 is true.

Proof:

Notice that $\Sigma_{n+1}=(\Sigma_{n+1}-S)\cup S$. Consequently we can apply Lemma 4.2 to obtain that $\mathrm{Dim}(\Sigma_{n+1})=\mathrm{Max}\{\mathrm{Dim}(\Sigma_{n+1}-S),\mathrm{Dim}(S)\}$. We established in Lemma 5.17 that $\mathrm{Dim}(\{R(F_n)-\{\varphi(S) \cup -2T\}\})=\mathrm{Dim}(\Sigma_{n+1}-S)$. In Corollary 5.14 it was established that $\mathrm{Dim}(\{R(F_n)-\{\varphi(S) \cup -2T\}\})=3n$. Thus $\mathrm{Dim}(\Sigma_{n+1}-S)=3n$. In Corollary 5.6 it was established that $\mathrm{Dim}(S)\le 3n+1$. Thus it is true that $\mathrm{Dim}(\Sigma_{n+1})= \mathrm{Max}\{\mathrm{Dim}(\Sigma_{n+1}-S), \mathrm{Dim}(S)\} \le 3n+1$. Recall that $\mathrm{Dim}(S) =\mathrm{Max}\left\{\mathrm{Dim}(P_+) + 2 f(k), \mathrm{Dim}(P_-)+ 2\right\}$. In other words for the case $k \ge 2$ and even we have proven that:

$$\mathrm{Dim}(\Sigma_{n+1})=\mathrm{Max}\left\{\mathrm{Max}\left\{ \mathrm{Dim}(P_+) + 2 f(k), \mathrm{Dim}(P_-) +2 \right\}, 3n\right\} \le 3n+1. \tag{25}$$

This completes the proof of Theorem 4.19.∎

<u>COROLLARY 5.19</u>:

Let $k \ge 3$ be even and let W have property ξ. Then Theorem 5.19 has the simpler expression:

$$\mathrm{Dim}(\Sigma_{n+1}) = \mathrm{Max}\left\{\mathrm{Max}\{\mathrm{Dim}(P_+), \mathrm{Dim}(P_-)\}+2, 3n\right\} \le 3n+1.$$

<u>COROLLARY 5.20</u>:

If in Theorem 5.1, $k = 2$, P_+ is empty, and W has property ξ, then $\mathrm{Dim}(\Sigma_{n+1})=3n$.

COROLLARY 5.21:

If W in Theorem 5.1 does not have property ξ and k is even, then

$$\mathrm{Dim}(\Sigma_{n+1}) = \mathrm{Max}\left\{\mathrm{Dim}(P_{+})+2\ f(k),\ \mathrm{Dim}(P_{-})+2\ \right\}.$$

Proof:

This is a direct result of the fact that under these conditions $\{R(F_{n})-\{\varphi(S) \cup -2T\}\}$ is empty and clearly S is not (see Comment 12.1).■

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SECTION SIX

In this section we will apply Theorems 4.1 and Theorem 5.2 in order to arrive at results that will be used in Section Seven. Our goal in this is to study the class of groups:

$$G_{p_1 \ldots p_n} = \langle a_1 \ldots a_n ; a_1^{p_1} \ a_2^{p_2} \ldots a_n^{p_n} = 1 \rangle, \tag{1}$$

where all the p_i are positive integers strictly greater than one.

We will introduce three lemmas whose proof will be given at the end of this section.

<u>LEMMA</u> <u>6.1</u>:

Let

$$G_{pqt} = \langle \, x_1, x_2, x_3 \; ; \; x_1^p \, x_2^q = x_3^t \, \rangle \tag{2}$$

where p,q≤2 are integers, and t≥2 is an integer. Then

i) Dim $(R(G_{pqt})) = 6$,

ii) $R(G_{pqt})$ is a reducible algebraic variety when at least one of the values of p, q, t is strictly larger than two.

LEMMA 6.2:

Denote by $\Sigma_2(1,-q)$ the solutions over $SL_2\mathbb{C}$ to the matrix equation $x_1^p x_2^q = -I$ where p and q carry the same signs and assumptions as in Lemma 6.1. Then $\Sigma_2(1,-q)$ is an affine variety and

i) $\mathrm{Dim}(\Sigma_2(1,-q))=4$, if $|p|>2$ or $|q|>2$,

ii) $\mathrm{Dim}(\Sigma_2(1,-q))=3$, if $|p|=2$ and $|q|=2$.

LEMMA 6.3:

Denote by $\Sigma_3(2,t)$ the solutions over $SL_2\mathbb{C}$ to the matrix equation $x_1^p x_2^q = -x_3^t$ where p, q and t carry the same signs and assumptions as in Lemma 6.1. Then $\Sigma_3(2, t)$ is an affine variety, and $\mathrm{Dim}(\Sigma_3(2, t))=6$.

OBSERVATION 6.4:

Clearly $G_{p_1 \ldots p_n}$ in (1) is isomorphic with the group $G_{-p_1 \ldots -p_n}$ via the isomorphism $\iota(a) \longrightarrow a^{-1}$. Thus we can assume that the p_i's are negative in (1). We thus obtain the groups:

$$G_{-p_1 \ldots -p_n} = \langle a_1 \ldots a_n ; a_1^{-p_1} a_2^{-p_2} \ldots a_n^{-p_n} = 1 \rangle. \tag{3}$$

Since the representation variety of a group is an invariant of the the presentation of the group chosen, (1) and (3) give rise to the same varieties of representations. Clearly the relation.

$$a_1^{-p_1} a_2^{-p_2} \ldots a_n^{-p_n} = I \tag{4}$$

gives rise to an algebraic variety in $\mathbb{C}^{4n}$ consisting of the solutions over $SL_2\mathbb{C}$ to the matrix equation in n indeterminates obtained from (4). In fact, the solutions to the equation (4) are contained in the solutions over $SL_2\mathbb{C}$ to the equation in n variables obtained from

$$a_1^{-p_1} a_2^{-p_2} \ldots a_{n-1}^{-p_{n-1}} = \pm a_n^{p_n}. \tag{5}$$

NOTATION:

Denote by Λ_n the solutions over $SL_2\mathbb{C}$ to the equation in (5). Clearly Λ_n is an algebraic variety. We denote by F_{n-1} the group freely generated by $\{a_1,a_2,\ldots,a_{n-1}\}$. $R(F_{n-1})$ will denote the representation variety of F_{n-1} in SL_2C. Recall that $R(F_{n-1})$ can be thought of as consisting of (n-1)-tuples of 2x2 matrices from SL_2C. In other words each representation ρ in $R(F_{n-1})$ can be thought of as a point $(m_1,m_2,\ldots,m_{n-1})$. The same holds true for Λ_n; each solution to the matrix equation in (5) is an n-tuple, $(m_1,m_2,\ldots,m_n)$ of 2x2 matrices from SL_2C. The letter W will stand for the word $a_1^{-p_1}a_2^{-p_2}\ldots a_{n-1}^{-p_{n-1}}$ in (5). Again, we denote by φ the projection map $\varphi : \Lambda_n \longrightarrow R(F_{n-1})$ given by

$$\varphi(m_1,\ m_2\ ,\ldots,m_n) = (m_1,\ m_2,\ldots,m_{n-1}). \tag{5.1}$$

OBSERVATION 6.41:

Λ_n is the union of two algebraic varieties, namely the algebraic variety of SL_2C solutions the equation in n variables obtained from $a_1^{-p_1}\ a_2^{-p_2}\ \ldots a_{n-1}^{-p_{n-1}} = -a_n^{p_n}$, and the the representation variety of the group $G_{-p_1 \ldots -p_n}$ in (3). Notice that

$$\Lambda_n = \{(\rho(a_1),\rho(a_2),\ldots,\rho(a_{n-1}),\sigma) \mid \rho \in R(F_{n-1}), \sigma \in \Omega(p_n, \pm\rho(W)) \neq \phi\}. \tag{5.2}$$

Λ_n Is the union of the two algebraic varieties:

$$\{(\rho(a_1),\rho(a_2),\ldots,\rho(a_{n-1}),\sigma) \mid \rho \in R(F_{n-1}), \sigma \in \Omega(p_n, \rho(W)) \neq \phi\} \tag{5.3}$$

$$\{(\rho(a_1),\rho(a_2),\ldots,\rho(a_{n-1}),\sigma) \mid \rho \in R(F_{n-1}), \sigma \in \Omega(p_n, -\rho(W)) \neq \phi\}. \tag{5.4}$$

<u>LEMMA</u> <u>6.42</u>:

The map φ in 5.1 maps Λ_n onto $R(F_{n-1})$.

Proof:

We only need to consider the case when p_n is even, since otherwise the map is onto. Suppose p_n is even; Then if $\rho(W) \in Orb(B)$, where B is the matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ we obtain that $\Omega(p_n, \rho(W)) = \phi$ in (5.3). However, for this very same choice of ρ, $-\rho(W)$ does not lie in Orb(B). and consequently $\Omega(p_n, -\rho(W)) \neq \phi$ in (5.4). Thus the map φ is always onto since Λ_n is the union of the algebraic varieties given in (5.3) and (5.4).■

<u>THEOREM</u> <u>6.5</u>:

For $n \geq 3$, $Dim(\Lambda_n) = 3(n-1)$.

Proof:

It will proceed by induction on $n \geq 3$. This has already been established in Lemma 6.3 and Lemma 6.1 for n = 3. Assume this theorem true for n-1, where n >3. We shall then show it true for n. Notice that

$$\Lambda_n=\{(\rho(a_1),\rho(a_2),\ldots,\rho(a_{n-1}),\sigma)\,|\,\rho\in R(F_{n-1})\},\sigma \in \Omega(p_n,\pm\rho(W))\neq\phi\} \tag{6}$$

Given any representation $\rho= (m_1,\ldots,m_{n-1})$ in $R(F_{n-1})$ with the property that

$$\pm m_1^{-p_1}\, m_2^{-p_2}\ldots m_{n-1}^{-p_{n-1}} =\pm I \tag{7}$$

then it is true that Dim $\varphi^{-1}(\rho) = 2$, by by Theorem 3.2 or Theorem 3.6, depending on whether +I or -I is chosen. Thus the map φ fails in general to have finite fibre. Let

$$U = \{(m_1,m_2,\ldots,m_n)\,|\,(m_1,m_2,\ldots,m_n) \in \Lambda_n \text{ and } \pm I m_1^{p_1}\ldots\, m_{n-1}^{p_{n-1}} = \pm I\}. \tag{8}$$

U is a proper subvariety of Λ_n. In fact U can be expressed as follows

$$U = (\Lambda_{n-1}) \times \Omega(p_n,\pm I\ ). \tag{9}$$

Where Λ_{n-1} is the algebraic variety consisting of solutions over SL_2C to the matrix on (n-1) variables given by

(10)

$$a_1^{-p_1} a_2^{-p_2} \dots a_{n-2}^{-p_{n-2}} = \pm a_{n-1}^{p_{n-1}}.$$

Notice that by the induction hypothesis

(11)

$$\mathrm{Dim}\ (\Lambda_{n-1}) = 3(n-2).$$

Consequently, by Theorem 3.2 and Theorem 3.6, and the fact that the dimension of a product of algebraic varieties has dimension the sum of the dimension of each individual factor we get that

(12)

$$\mathrm{Dim}\ (U) \le 3(n-2)+2.$$

Notice that,

(13)

$$3(n-2)+2 < 3(n-1).$$

Thus U is a proper subvariety of Λ_n since $\mathrm{Dim}(\Lambda_n) \ge 3(n-1)$, given that Λ_n contains the representation variety of a one relator group (3), which presentation of deficiency (n-1). Notice that $\{\Lambda_n - U\}$ is mapped by φ onto $\{(R(F_{n-1})) - \varphi(U)\}$, and that $\varphi(U)$ is a subvariety of $R(F_{n-1})$. In fact $\varphi(U) = (\Lambda_{n-1})$. Consequently by (11) we get that

(14)

$$\mathrm{Dim}(\varphi(U)) = 3(n-2).$$

Thus $\{R(F_{n-1}) - \varphi(U)\}$ is a quasi-affine variety, and by Theorem 1.13

$$\mathrm{Dim}(\{R(F_{n-1})- \varphi(U)\})=3(n-1). \tag{15}$$

For every point $m \in \{R(F_{n-1})- \varphi(U)\}$, the fibre $\varphi(m)^{-1}$ has zero dimension, and is thus finite. We thus have the conditions necessary for Theorem 1.22. Consequently,

$$\mathrm{Dim}(\{R(F_{n-1})- \varphi(U)\}) = \mathrm{Dim}(\{\Lambda_n - U\})=3(n-1). \tag{16}$$

Notice that

$$\Lambda_n=\{\{\Lambda_n - U\} \cup U \}. \tag{17}$$

Thus by Theorem 4.2

$$\mathrm{Dim}(\Lambda_n)=\mathrm{Max}\{\mathrm{Dim}\ (\{\Lambda_n - U\}),\ \mathrm{Dim}\ (U)\} \tag{18}$$

$$=\mathrm{Max}\{\ 3(n-1),\ 3n-4\ \}=\ 3(n-1).$$

This completes the proof of the theorem when p_n is odd ≥ 3.■

<u>COROLLARY</u> <u>6.51</u>:

$\mathrm{Dim}(R(Gp_1 \ldots p_n))=3(n-1)$.

Proof:

$R(Gp_1 \ldots p_n) \subset (\Lambda_n)$. Thus $\mathrm{Dim}\ (R(Gp_1 \ldots p_n)) \leq 3(n-1)$ by Theorem 6.5. Notice however that $Gp_1 \ldots p_n$ has a presentation of deficiency n-1 and consequently $\mathrm{Dim}(R(Gp_1 \ldots p_n)) \geq 3(n-1)$. Thus $\mathrm{Dim}(R(Gp_1 \ldots p_n)) = 3(n-1)$.■

LEMMA 6.1:

Let

$$G_{pqt} = \langle x_1, x_2, x_3 \ ; \ x_1^p \ x_2^q = x_3^t \rangle \tag{19}$$

where p, q≤2 are integers, and t≥2 is an integer. Then

i) $\mathrm{Dim}\ (R(G_{pqt})) = 6$,

ii) $R(G_{pqt})$ is a reducible algebraic variety when at least one of the absolute values of p, q, t is strictly larger than two.

Proof (i):

Let $\varphi(S_+)$ and $\varphi(S_-)$ be the subvarieties of $R(G_{pqt})$ given by the following

$$P_+ = \varphi(S_+) = \left\{ \rho \,\middle|\, \rho \in R(F_2), \text{ and } \rho(x_1^p x_2^q) = I \right\} \tag{20}$$

$$P_- = \varphi(S_-) = \left\{ \rho \,\middle|\, \rho \in R(F_2), \text{ and } \rho(x_1^p x_2^q) = -I \right\} . \tag{21}$$

Now by the Theorem 4.1

$$\mathrm{Dim}\ (R(G_{pqt})) = \mathrm{Max}\left\{ \mathrm{Max}\left\{ \mathrm{Dim}(P_+) + 2\ f(t),\ \mathrm{Dim}(P_-) + 2 \right\},\ 3n \right\} . \tag{22}$$

Where f is the integer valued functions introduced in Section 5.

Using Theorem 4.21 we get that $\mathrm{Dim}(\varphi(S_+)=4$. We need now to compute $\mathrm{Dim}(\varphi(S_-))$. We use Lemma 6.2 to conclude that $\mathrm{Dim}(\varphi(S_-))=3$ or 4 depending on the absolute values of p and q. Thus, by Theorem 4.1 we get

$$\mathrm{Dim}\ (R(G_{pqt}))=\mathrm{Max}\left\{\mathrm{Dim}(\varphi(S_+)\ +2\ f(t),\ \mathrm{Dim}(\varphi(S_-)\ +2,\ 6\right\} \leq 7 \tag{23}$$

$$=\mathrm{Max}\{4+2f(t),3+2 \text{ or } 4+2,\ 6\}=6.$$

Proof ii):

Without loss of generality we can assume that t is the integer whose absolute value is greater than 2. Then in (23) we get that since t is not two, $\mathrm{Dim}(\varphi(S_+)\ +2\ f(t)=6$. By Corollary 4.20 it follows that $R(G_{pqt})$ is reducible.■

<u>LEMMA</u> <u>6.2</u>:

Denote by $\Sigma_2(1,-q)$ the solutions over $SL_2\mathbb{C}$ to the matrix equation $x_1^p x_2^q$ $=-I$ where p and q carry the same signs and assumptions as in Lemma 6.1. Then $\Sigma(1,-q)$ is an affine variety and

i) $\mathrm{Dim}(\Sigma_2(1,-q))=4$ if $|p|>2$ or $|q|>2$.

ii) $\mathrm{Dim}(\Sigma_2(1,-q))=3$, if $|p|=2$, and $|q|=2$.

Proof i):

We have assumed that p and q in $x_1^p x_2^q = -I$ are negative. Thus the resulting equation $x_1^p = -x_2^{-q}$ has the property that -q is positive. Let

$$P_+ = \left\{\rho \,\middle|\, \rho \in R(F_1), \text{ and } -\rho(x_1^p) = I\right\},$$

$$P_- = \left\{\rho \,\middle|\, \rho \in R(F_1), \text{ and } -\rho(x_1^p) = -I\right\}.$$

Notice that if $|p|>2$; by Theorem 3.6 we get that Dim $(P_+)=2$, and by Theorem 3.2 we get that Dim $(P_-)=2$. By Theorem 5.1 we get

$$\mathrm{Dim}(\Sigma_2(1,-q)) = \mathrm{Max}\left\{\mathrm{Max}\left\{\mathrm{Dim}(P_+)+2f(-q),\ \mathrm{Dim}(P_-)+2\right\},\ 3\right\} \le 3+1.$$

Thus $\mathrm{Dim}(\Sigma_2(1,-q))=4$.

If on the other hand $|p|=2$ we get by Theorem 3.2 and Theorem 3.6 that Dim $(P_-)=0$, and Dim $(P_+)=2$. By theorem 5.1 we get

$$\mathrm{Dim}(\Sigma_2(1,-q)) = \mathrm{Max}\left\{\mathrm{Max}\left\{\mathrm{Dim}(P_+)+2f(-q),\ \mathrm{Dim}(P_-)+2\right\},\ 3\right\} \le 3+1.$$

Thus $\mathrm{Dim}(\Sigma_2(1,-q))=4$, since $-q>2$. This completes the proof of (i).■

Proof ii):

Suppose p has absolute value equal to two. Notice that

$$P_+=\left\{\rho \mid \rho \in R(F_1), \text{ and } -\rho(x_1^p)= I\right\},\ P_-=\left\{\rho \mid \rho \in R(F_1), \text{ and } -\rho(x_1^p)=-I\right\}.$$

By Theorem 3.6 $\mathrm{Dim}(P_+)=2$, and by Theorem 3.2 $\mathrm{Dim}(P_-)=0$. By Theorem 5.1 we get

$$\mathrm{Dim}(\Sigma_2(1,-q)) = \mathrm{Max}\left\{\mathrm{Max}\left\{\mathrm{Dim}(P_+)+2f(-q),\ \mathrm{Dim}(P_-)+2\right\},\ 3\right\} \leq 3+1.$$

Thus $\mathrm{Dim}(\Sigma_2(1,-q))=3$, since $-q=2$. This completes the proof of (ii). ∎

<u>LEMMA</u> <u>6.3</u>:

Denote by $\Sigma_3(2,t)$ the solutions over $SL_2\mathbb{C}$ to the matrix equation $x_1^p x_2^q =-x_3^t$, where p, q and t carry the same signs and assumptions as in Lemma 6.1. Then $\Sigma_3(2,t)$ is an affine variety, and $\mathrm{Dim}(\Sigma_3(2,t))=6$.

Proof:

We will employ Theorem 5.1. Notice that

$$P_+=\left\{\rho \mid \rho \in R(F_1), \text{ and } -\rho(x_1^p x_2^q)= I\right\},\ P_-=\left\{\rho \mid \rho \in R(F_1), \text{ and } -\rho(x_1^p x_2^q)=-I\right\}.$$

By Lemma 6.2 , $\mathrm{Dim}(P_+)= 4$ or 3 depending on $|p|$ and $|q|$. We look now to P_-. By Theorem 4.21 $\mathrm{Dim}(P_-)= 4$. Now by Theorem 5.1

$$\mathrm{Dim}(\Sigma_3(2,t)) = \mathrm{Max}\left\{\mathrm{Max}\left\{\mathrm{Dim}(P_+)+2f(t),\ \mathrm{Dim}(P_-)+2\right\},\ 6\right\} \leq 6+1$$

Thus $\mathrm{Dim}(\Sigma_3(2,t))=6$. This completes the proof of Lemma 6.3.∎

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SECTION SEVEN

As stated earlier, the initial goal of my investigation of representation varieties for groups was the comparison between the representation variety of a free group of a given rank and the representation variety of a parafree group of that same rank. The discovery parafree groups was made by Gilbert Baumslag in the 1960's (see [CM]).

<u>DEFINITION</u>:

Let x,y be elements of some group G. We define the commutator of x and y, denoted by (x,y), to be $(x,y)= x^{-1}y^{-1}xy$.

Given a group G we can consider the subgroup generated by the set of all its commutators and denote this subgroup by [G,G]. We can then look at the subgroup generated by commutators of commutators...so on. Thus inductively we arrive at the following notion.

<u>DEFINITION</u>:

The lower central series of a group G is defined inductively to be

$G_1= G,\ G_2= [G,G],\ G_3= [G_2,G_1],\ G_4= [G_3,G_1],\ldots$

Clearly, $G_i \triangleright G_{i+1}$ for all i. Here $\triangleright$ denotes a normal subgroup.

<u>DEFINITION</u>:

The lower central sequence of a group G is given by

$G/G_2,\ G/G_3\ ,\ G/G_4,\ \ldots.$

DEFINITION:

One says that two groups G and G' have isomorphic lower central sequences, if there is an isomorphism Ψ_k: $G/G_k \longrightarrow G'/G'_k$ such that Ψ_k induces Ψ_{k-1} on $G/G_{k-1} \longrightarrow G'/G'_{k-1}$ for k=2,3,4,...

Let G be isomorphic to G', then G an G' have isomorphic lower central sequences. The converse of this is not true as attested by the existence of non free parafree groups. However it remains quite surprising that two non-isomorphic groups as in the case of an absolutely free group and a non-free parafree group, can agree on an infinite number of invariants. A historical comment is in order here. It was W. Magnus who introduced and popularized this method of discriminating groups.

DEFINITION:

We say that a group G is parafree if:

1) G is residually nilpotent (The lower central series intersects in the Identity).
2) G has the same lower central sequence as that of a free group.
   The rank of the particular free group is defined to be the rank of the para-free group G.

We denote by $\mu(G)$ the minimal number of generators of a group G. We shall now define the rank of a group G.

DEFINITION:

We define the rank, $\rho(G)$, of G to be $\rho(G)= \mu(G/G_2)$.

DEFINITION:

We define the deviation, $\delta(G)$, of a group G to be $\delta(G) = \mu(G)-\rho(G)$.

Having reached this point, we can state some results of Baumslag in the theory of parafree groups. The proof of these results will be omitted. Instead the reader is encouraged to see [B2].

THEOREM 7.1 (G. Baumslag 1967):

For every pair of integers r and d satisfying $r\geq 2$ and $d\geq 1$ there exist infinitely many parafree groups of rank r and deviation d.

Proof: See [B2]. ■

PROPOSITION 7.2 (G. Baumslag 1967):

Let H be parafree of rank r, let (x) be the infinite cyclic group on x, let $h \in H$ and let t be a positive integer which is not divisible by the prime n. If h generates its own centralizer in H and if $h=h_1^t$ Mod H_2, where h_1 is not a proper power mod H_2, then

$$G = \{ H*(x); h = x^n \}$$

is parafree of rank r.

Proof: See [B2]. ■

Using Proposition 7.2 it is clear that the groups

(1)

$$G_{pqt} = < x_1, x_2, x_3 \; ; \; x_1^p \, x_2^q = x_3^t >$$

are parafree since for p,q an t both larger than one and having no common divisors other than one. $(x_1^p \, x_2^q)$ generates its own centralizer in the free group $F = < x_1, x_2 >$. The other conditions of proposition 3 are met and thus we have that the groups G_{pqt} are parafree of rank two; clearly these groups are of deviation one. Infinitely many of these groups are non-isomorphic (see [M]).

By Lemma 6.1 it is known that the groups G_{pqt} have reducible representation varieties and that the dimension of their representation variety is six, in contrast to the representation variety of a free group of rank two with which they share the lower central sequence; recall that that the representation variety of a free group of rank two is irreducible and of dimension 6. Thus we have established the following result:

THEOREM 7.3:

There exist infinitely many non-isomorphic parafree groups of rank two and deviation one having a reducible representation variety of dimension 6.

Next We look at the class of groups:

(2)

$$G_{p_1 \ldots p_n} = < a_1 \ldots a_n ; a_1^{p_1} \, a_2^{p_2} \ldots a_n^{p_n} = 1 >$$

when $p_1, p_2, \ldots, p_n$ are positive and have no common divisor except one. These groups are parafree of rank (n-1) by Proposition 7.2, and their representation variety in $SL_2\mathbb{C}$ by corollary 6.51 has dimension 3(n-1). It is important to note that by a Theorem of S. Meskin $G_{p_1 \ldots p_n}$ is isomorphic to $G_{k_1 \ldots k_n}$ iff $\tau(p_1, \ldots, p_n) = (k_1, \ldots, k_n)$, where τ is an element of the symmetric group S_n (see [M]). The next theorem follows immediately.

<u>THEOREM</u> <u>7.4</u>:

Given an integer $n \geq 2$ there exist an infinite number of non-isomorphic non-free parafree groups of rank n and deviation one having representation variety of dimension 3n.

The question remains whether for any $n \geq 4$ the groups in (2) have reducible representation varieties. This is left for future work. One might be tempted to conjecture that all parafree groups of rank n have representation varieties of dimension 3n, where n is the rank of the absolutely free group with which they share the lower central sequence. In fact, it was once believed that this would be true. The next theorem will give a negative answer to this.

<u>THEOREM</u> <u>7.4.1</u>

There exist a non-free parafree group of rank 2 having representation variety of dimension larger or equal to 4+2k, where k is an arbitrary positive integer larger than one.

Proof:

Choose an arbitrary number $k \geq 1$ and choose distinct odd primes $r_1, r_2, \ldots, r_k$. Clearly, the following subset of $(SL_2\mathbb{C})^{k+2}$ is an algebraic variety which by Theorems 3.2 and Theorem 3.6 is of dimension 4+2k:

$$\{\Omega(2,-I) \times \Omega(2,-I) \times \Omega(r_1,+I) \times \Omega(r_2,+I) \times \ldots \times \Omega(r_k,+I)\}. \tag{2.1}$$

The algebraic variety in (2.1) lies inside the representation variety of the group

$$G = \langle a,b,x_1,x_2, \ldots, x_k;\ a^2b^2 = x_1^{r_1},\ a^{2^2}b^{2^2} = x_2^{r_2} \ \ldots\ a^{2^k}b^{2^k} = x_k^{r_k} \rangle. \tag{3}$$

By results of G. Baumslag [B2] these groups are parafree of rank two whenever $r_1, r_2, \ldots, r_k$ are distinct odd primes. This completes the proof of the theorem. ∎

It should be pointed out that the groups in (3) are not of deviation one. A more appropriate question is whether all non-free parafree group of rank n and deviation one have representation varieties of dimension 3n.

An intriguing question in the theory of parafree groups is the isomorphism problem. Is there an algorithm which decides whether or not a pair of parafree groups in some recursive class of finitely presented parafree groups are isomorphic?

W. Magnus, first in a paper titled "The Uses of Two by Two Matrices in Combinatorial Group Theory" [WM], and then in his joint book with the mathematics historian Bruce Chandler "The History of Combinatorial Group Theory" [CM], to illustrate the difficulty of the isomorphism problem in group theory makes mention of a class of groups constructed by G. Baumslag in 1969:

"We shall denote these groups by $S^{*}_{i,j}$, where i,j are nonzero integers. They are a special class of groups called parafree, and they are defined as having three generators a,b,c and a single defining relation $a=(a,c^{i})(b,c^{j})$. The groups $S^{*}_{i,j}$ have the following properties with the free group F of rank 2.

1) $S^{*}_{i,j}$ has a normal subgroup with an infinite cyclic quotient group.
2) Every two generator subgroup of $S^{*}_{i,j}$ is free.
3) Let $\gamma_n G$ denote the n^{th} group of the lower central series of the group G. Then the intersection of all the $\gamma_n S^{*}_{i,j}$ is the unit element, and for all n

$$\frac{S^{*}_{i,j}}{\gamma_n S^{*}_{i,j}} \cong \frac{F}{\gamma_n F}$$

4)

$$\frac{S^{*}_{i,j}}{\gamma_n S^{*\prime\prime}_{i,j}} \cong \frac{F}{\gamma_n F^{\prime\prime}}$$

where G'' means the second derived group of G. The great difficulties of the isomorphism problem for torsion free one relator groups are illustrated by the fact that until now (1980) there exists no proof showing that any two of the groups $S^*_{i,j}$ are nonisomorphic."

In 1990 We became interested in the question of differentiating two of these groups, so the task of counting the number of points in the representation varieties over some finite linear group for two of these was undertaken. Eventually the next result was obtained.

<u>THEOREM 7.5</u>:

The groups, $S^*_{1,1}$ and $S^*_{30,30}$, are not isomorphic.

Proof:

Via Cayley it is possible to count the number of representation of $S^*_{1,1}$ in the affine group PSL(2,5). The total number of representations is 2760. The group PSL(2,5) is the simple group of order 60; This group is of exponent 30. This fact can be established very readily by a rather short series of commands in Cayley. This is essentially the proof for we can now conclude that $S^*_{1,1}$ is not isomorphic to $S_{30,30}$ since the number of representations of $S_{30,30}$ in PSL(2,5) is 60^2. This follows from the fact that the relation $(a,c^{30})(b,c^{30})=a$ of $S^*_{30,30}$ together with the fact that every element of PSL(2,5) is of exponent 30 allows us to assign b and c to arbitrary matrices m, n in PSL(2,5), and forces us to select identity matrix I as the only choice for assigning a. ∎

This actually shows more. It shows that the set of all representations of the group $S^{*}_{30,30}$ is isomorphic to the affine group PSL(2,5)xPSL(2,5), and that as a result of having a group structure it is a smooth affine variety sitting in affine space over the galois field with five elements.

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