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VELOCITY SPACE DIFFUSION AND LANDAU DAMPING OF PLASMAS WITH COLLISIONS IN QUASI-LINEAR REGIME

by

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ABSTRACT

The statistical behavior of a fully ionized plasma is investigated by means of a kinetic equation, including collision terms, of the form of the Krook and the Fokker-Planck models. These equations, coupled with the Poisson equation, are solved by the method of characteristics using the quasi-linear approximation technique. Dispersion relations for longitudinal oscillations of the plasma are obtained; Landau damping coefficients of the waves as well as diffusion coefficients in velocity space are explicitly evaluated. These coefficients reveal effects due to the inclusion of both non-linear and collisional terms in the kinetic equations. Temporal development of electrostatic energy fluctuations and that of the particle distribution are then examined. Application is made to a plasma with a background distribution possessing a bump. Wave growth is shown to increase wave diffusion, and consequently, the bump disappears gradually; as a result, the electrostatic energy decreases to zero, so that the distribution tends to an equilibrium limit, as described by the original kinetic equation. This result is not predicted by the quasi-linear theory of a Vlasov plasma in which a physically unrealistic plateau is formed.

VITA

Mr. Kwok-Leung Li was born in Hong Kong, where he grew up and received his formal education.

He was granted the degree of Bachelor of Science in 1958 and the Diploma of Education in 1959 by the University of Hong Kong. Subsequently, he taught and worked as the head of the Department of Science in a high school.

In 1964 he came to North America as a student, where he received the degree of Master of Education in Theory of Education at the University of Toronto, and of Master of Arts in Physics at The City College of the City University of New York. He then joined the graduate faculty of the City University of New York and completed his dissertation for the degree of Doctor of Philosophy in 1970.

I. INTRODUCTION

The study of collective longitudinal oscillations in a plasma is of great importance and interest for many physical and technical problems. In the theoretical study of a plasma, the general approach is based on the use of kinetic equations describing the evolution of the distribution functions of the plasma particles, and Maxwell's equations defining the fields caused by external sources as well as the particles of the plasma itself. A solution to the coupled equation was first given by Landau [1] and subsequently extended by van Kampen [2] and Case [3]. In recent years, the solution has been further extended in two significant directions: The consideration of effects due to collisions, and the investigation of effects due to non-linear terms.

In the linearized regime, Lenard and Bernstein [4] widened the theory by including small-angle collision terms into the kinetic equations and solved them by a generalization of the classical Fourier transform method of Landau. The effect of collisions was represented by a diffusion in velocity space. In the same regime, Karpman [5] was able to show that even very slight collisions brought the distribution to Maxwellian through this velocity space diffusion.

By including certain selected non-linear terms in the kinetic equations, Drummond and Pines [6], Vendenov, Velikhov and Sagdeev [7] were able to follow the instabilities predicted by the linearized theory into a non-linear regime for collisionless plasmas. These calculations indicated that non-linear effects tend to modify the spatially homogeneous part of the distribution in the resonance region in which a bump appears. Consequently, if the bump is small enough, a plateau is formed in the vicinity of the phase velocity, resulting in a state of steady oscillations with zero Landau damping. Recent numerical investigations of non-linear effects by Gary [8] and Armstrong [9] predict the same results.

Combined non-linear and collisional effects have been studied numerically, in the one dimensional case, by Rand [10]. Similar plateau was found to be formed when the damping rate of the waves is negligible compared with the relaxation rate of the distribution in the resonance region.

As each of the above results has its own contribution towards better understanding of the behavior of plasma oscillations, it is interesting to find an analytical solution including both effects due to collisions and those due to inclusion of non-linear terms in the kinetic equations. In this connection, further extension of the existing quasi-linear theory of plasma oscillations to the case where collision effects are not neglected is attempted in the present dissertation.

For illustrating the effects of collisional process, complicated collision operator of the Boltzmann type have been avoided; instead, the Krook model and the Fokker-Planck model for collisions are chosen here. These models represent certain essential features of collisions in plasmas, and hence provide the possibilities of treating non-linear initial-value problems of plasmas with collision effects included.

In Section II, the fundamental kinetic equations with collision operator included are investigated. A brief review of the basic assumptions of the quasi-linear theory is given and the kinetic equations rewritten in the mathematically manageable forms.

In Section III, the Krook model is discussed. Adopting this model, the kinetic equation coupled to the Poisson equation is solved by the conventional Fourier analysis method. The equation showing diffusion of the plasma in velocity space is obtained. This equation contains the collision frequency and the Landau damping coefficient as parameters which are also calculated explicitly.

In Section IV, a brief account of the Fokker-Planck model is given. The collision terms suggested by this model are then inserted into the coupled kinetic equations and the latter solved by the method of characteristics in the Fourier transformed configurations and velocity spaces. Transformed back into the original spaces, the Landau damping coefficient and the diffusion coefficients are evaluated by an energy decay method. The equation representing the growth of electrical energy and that showing the dissipation of the plasma in velocity space are then obtained.

In Section V, the detailed development of the energy fluctuations and the diffusion of the trapped particles are investigated to illustrate the quasi-linear as well as the collisional effects on velocity space diffusion

Finally, in Section VI, the main results of the present dissertation are summarized and compared with those obtained by other authors.

II. FUNDAMENTAL EQUATIONS

Consider a plasma consisting of an infinitely extended, fully ionized gas, in which the ions are uniformly distributed to form a background of positive charges. Such a system can be described by a one-particle distribution function $f(\vec{x}, \vec{v}, t)$ of the electrons, satisfying the kinetic equation:

$$\frac{\partial f}{\partial t} + (\vec{v}\cdot\nabla)f - \frac{e}{m}\vec{E}\cdot\nabla_v f = \left(\frac{\partial f}{\partial t}\right)_c \tag{2.1}$$

Here, $\vec{E}(\vec{x}, t)$ is the self-consistent field produced by the charged particles and is determined by the Poisson equation:

$$\nabla\cdot\vec{E} = 4\pi ne\left[1 - \int d^3v\, f\right] \tag{2.2}$$

where n is the average electron or ion density. The limits of the integration are understood to be from $-\infty$ to $+\infty$.

In Eq. (2.1), the term $e\vec{E}$ designates the force on a given electron arising from all other particles at distances greater than the Debye length λ_D, while the term $\left(\frac{\partial f}{\partial t}\right)_c$ known as the collision integral, represents the instantaneous change of the distribution function arising from closer collisions. Physically this collision term comes from the multiple small-angle Coulomb collisions of the plasma particles with all others within λ_D.

Of particular interest in the collective behavior of a plasma system is the fact that it possesses a peculiar dielectric property which permits the propagation of longitudinal electrostatic waves. These propagating plasma oscillations bear a certain similarity to acoustic waves in ordinary gases, except that the former is transmitted via the collective space-charge field instead of via individual particle oscillations. An individual particle moving in the plasma ensemble sets up an electrical perturbation and therefore excites some of these longitudinal oscillations. In other words, the particle emits waves. Conversely, the particles also, in their random motion, continuously absorb energy from the wave field and hence dampen the waves. The net result of this particle-wave interaction is a diffusion of the distribution function in velocity space. Since an individual particle interacts with other particles within as well as beyond the Debye length, one expects the plasma diffusion to exist even in a collisionless plasma.

Closely related to this phenomenon of velocity space diffusion is the problem of Landau damping. Particles moving parallel to a longitudinal wave with velocities close to the phase velocity of the wave couple with it and produce a resonant transfer of energy, the slower particles picking up energy from the wave while the faster ones deliver energy to it. If the particles which travel slightly slower outnumber those which go a little faster, one has Landau damping. On the other hand, where the reverse is true, trapping instabilities occur.

The magnetic field has been left out in our problem since we are interested only in the longitudinal disturbances of the plasma system

To obtain an oscillatory solution for Eqs. (2.1) and (2.2), the distribution function f is split up into two parts:

$$f(\vec{x},\vec{v},t) = f_0(\vec{v},t) + F(\vec{x},\vec{v},t)$$

Here f_0 is a slow-varying, space independent background distribution, while F represents a rapidly fluctuated, small perturbation quantity describing deviations from the smooth background. We assume that the fluctuation of F is such that its space average vanishes; that is

$$\langle F \rangle = 0$$

As a consequence of this assumption, we have

$$\langle f \rangle = 0$$

On making Fourier series transform on Eq. (2.1), one obtains

$$\frac{\partial f_k}{\partial t} + i\,\vec{k}\cdot\vec{v}\,f_k - \frac{e}{m}\sum_j \left(\vec{E}_{k-j} \cdot \nabla f_j \right) = \mathcal{L}(f_k) \qquad (2.3)$$

where f_k, $\vec{E}_k$, and $\mathcal{L}(f_k)$ are given by

$$f(\vec{x},\vec{v},t) = \sum_k f_k(\vec{v},t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\vec{E}(\vec{x},t) = \sum_k \vec{E}_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\left(\frac{\partial f}{\partial t}\right)_c = \sum_k \mathcal{L}(f_k)\, e^{i\vec{k}\cdot\vec{x}}$$

and $\mathcal{L}(f_k)$ represents the collision term, which will be taken in a linear form (Krook and Fokker-Planck models).

Now, Eq. (2.3) is valid for all values of k and hence can be split into two parts, one for $k = 0$ and the other for $k \neq 0$.

For $k = 0$, we have

$$\frac{\partial f_0}{\partial t} - \frac{e}{m}\sum_j \left(\vec{E}_{-j} \cdot \nabla_v f_j \right) = \mathcal{L}(f_0) \tag{2.4}$$

For $k \neq 0$,

$$\frac{\partial F_k}{\partial t} + i\vec{k}\cdot\vec{v}\,F_k - \frac{e}{m}\sum_j \left(\vec{E}_{k-j} \cdot \nabla_v f_j \right) = \mathcal{L}(F_k) \tag{2.5}$$

where F_k is given by

$$F(\vec{x},\vec{v},t) = \sum_k F_k(\vec{v},t)\, e^{i\vec{k}\cdot\vec{x}}$$

If the $j = 0$ term is singled out from Eq. (2.5), it becomes

$$\frac{\partial F_k}{\partial t} + i\vec{k}\cdot\vec{v}\,F_k - \frac{e}{m}\vec{E}_k\cdot\nabla_v f_0 - \frac{e}{m}\sum_{j\neq 0}\left(\vec{E}_{k-j}\cdot\nabla_v F_j\right)$$
$$= \mathcal{L}(F_k) \qquad , \quad k \neq 0$$

The terms $\sum_{j\neq 0} \vec{E}_{k-j} \cdot \nabla_v F_j$ are referred to by Drummond and Pines [6] as mode-coupling terms and are considered to be small. They are thus neglected in the quasi-linear theory. Consequently, this equation reduces to

$$\frac{\partial F_k}{\partial t} + i\vec{k}\cdot\vec{v}\,F_k - \frac{e}{m}\vec{E}_k\cdot\nabla_v f_0 = \mathcal{L}(f_k) \quad , k \neq 0 \tag{2.6}$$

We shall further put

$$\vec{E}_0 = \vec{E}_k \Big|_{k=0} = 0$$

since there is no external electric field. As a result, Eq. (2.4) reads

$$\frac{\partial f_0}{\partial t} - \frac{e}{m}\sum_{j\neq 0}\left(\vec{E}_j \cdot \nabla_v F_j\right) = \mathcal{L}(f_0) \qquad (2.7)$$

On making Fourier series transform on Eq. (2.2), we obtain

$$i\,\vec{k}\cdot\vec{E}_k = -4\pi n e\int d^3v\; F_k$$

As the Fourier component $\vec{E}_k$ is obviously parallel to the wave vector $\vec{k}$, this equation can be rewritten as

$$\vec{E}_k = \frac{4\pi n e}{k^2}\, i\,\vec{k}\int d^3v\; F_k$$

Equations (2.6), (2.7), and (2.8) together form the basic equations describing the collective longitudinal oscillation behavior of a plasma in the quasi-linear regime. The question arises as to the explicit mathematical form to take for the collision integral, $(\frac{\partial f}{\partial t})_c$, which is physically meaningful. At collision distances less than the interparticle distance, binary collision terms of the Boltzmann type represent a good approximation. However, the mathematics of it is too complicated to handle in practice because of the intractable nature of the Boltzmann integral. We shall, instead, assume that the particles obey a model kinetic equation which permits simpler mathematical treatment. In the two sections that follow, the Krook and the Fokker-Planck models are chosen for this purpose, where the solutions to the above mentioned basic equations are investigated.

III. THE KROOK MODEL

In this section, we shall investigate the solution to the fundamental equations of our plasma system, adopting the Krook model [11] for collisions.

In this model, the collision integral is replaced by a term of the form

$$\left(\frac{\partial f}{\partial t}\right)_c = -\nu\left[f - f_{eq}\right] \tag{3.1}$$

where $f_{eq}(\vec{v})$ is the equilibrium value of the distribution function and ν the constant collision frequency.

This model has the defect that particles are not conserved. But, since it expresses that collisions tend to relax the distribution function to an equilibrium value f_{eq}, it serves to describe the effect of collisions on Landau damping in a simplified fashion.

Using this kinetic model, the fundamental equations for the plasma system reduce to the following set of three equations

$$\vec{E}_k = \frac{4\pi n e}{k^2}\, i\vec{k}\int d^3v\; F_k \tag{3.2}$$

$$\frac{\partial F_k}{\partial t} + i\vec{k}\cdot\vec{v}F_k - \frac{e}{m}\vec{E}_k\cdot\nabla f_0 = -\nu F_k \tag{3.3}$$

and

$$\frac{\partial f_0}{\partial t} - \frac{e}{m}\sum_j\left(\vec{E}_{-j}\cdot\nabla F_j\right) = -\nu\left(f_0 - f_{eq}\right) \tag{3.4}$$

Equation (3.3) can be integrated readily with respect to the time variable to give

$$F_k = \frac{e}{m} \int_{-\infty}^{t} d\tau \, e^{-(i\vec{k}\cdot\vec{v} + \omega)(t-\tau)} \vec{E}_k(\tau) \cdot \nabla_v f_0(\vec{v}, \tau) \qquad (3.5)$$

As a result of the interaction between longitudinal waves and particles in the plasma, a gradual decrease of wave amplitudes takes place, known as Landau damping, as well as a change in the particle distribution function, known as velocity-space diffusion. These occur even in the absence of collisions. Collisions, however, do contribute to the mechanism for dissipation of electrostatic field energy which further damps the waves and consequently affects the particle distribution. We are interested in the combined effects and shall, in subsequent paragraphs, (a) and (b), calculate the generalized Landau damping coefficient as well as the diffusion coefficient in the quasi-linear regime.

(a) Damping Coefficient

The Landau damping can be studied by means of two methods: The usual method is to introduce a Laplace or Fourier transform of the system of Eqs. (3.2) to (3.4), giving a dispersion relation, with the imaginary frequency representing the growth rate. This is the method used by Landau [1]. This method is not practical for nonlinear problems. Therefore, in the present work we shall use the method of energy decay, introduced by Tchen [12]. This method consists of calculating the time rate of change of the electrostatic energy as proportional to the energy itself, with the proportionality coefficient equal to the combined Landau damping and collisional damping.

For this purpose, we shall now evaluate the damping coefficient by an energy decay method.

Substituting Eq. (3.5) into Eq. (3.2),

$$\vec{E}_k(t) = \frac{\omega_p^2}{k^2}\, i \int d^3v \int_{-\infty}^{t} d\tau\, e^{-(i\vec{k}\cdot\vec{v}+\nu)(t-\tau)}\, \vec{E}_k(\tau)\,\vec{k}\cdot\nabla_v f_0(\vec{v},\tau) \tag{3.6}$$

where ω_p is the plasma frequency given by

$$\omega_p^2 = \left(\frac{4\pi n e^2}{m}\right)^{\frac{1}{2}}$$

Multiplying Eq. (3.6) by $\vec{E}_{-k}(t)$ and taking ensemble average,

$$\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t)\rangle = \frac{\omega_p^2}{k^2}\, i \int d^3v \int_{-\infty}^{t} d\tau\, e^{-(i\vec{k}\cdot\vec{v}+\nu)(t-\tau)} \langle \vec{E}_{-k}(t)\cdot\vec{E}_k(\tau)\rangle\,\vec{k}\cdot\nabla_v f_0(\vec{v},\tau) \tag{3.7}$$

On differentiating Eq. (3.7) with respect to time, it can be shown [Appendix A] that the only term that survives is

$$\frac{\partial}{\partial t}\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t)\rangle = -i\,\frac{\omega_p^2}{k^2}\int d^3v\,(i\vec{k}\cdot\vec{v}+\nu) \int_{-\infty}^{t} d\tau\, e^{-(i\vec{k}\cdot\vec{v}+\nu)(t-\tau)} \langle \vec{E}_{-k}(t)\cdot\vec{E}_k(\tau)\rangle\,\vec{k}\cdot\nabla_v f_0(\vec{v},\tau) \tag{3.8}$$

We assume that f_0 is so slowly varying in time that it can be put in front of the time integral, then

$$\frac{\partial}{\partial t}\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t)\rangle = -i\,\frac{\omega_p^2}{k^2}\int d^3v\,(i\vec{k}\cdot\vec{v}+\omega)\,\vec{k}\cdot\nabla f_0(\vec{v},t)\int_0^\infty ds\, e^{-(i\vec{k}\cdot\vec{v}+\omega)s}\,\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t-s)\rangle$$

where

$$s = t - \tau$$

With the further assumption that the correlation for a monochromatic wave be

$$\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t-s)\rangle = \langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t)\rangle\, e^{i\omega s}$$

we obtain

$$\frac{\partial}{\partial t}\mathcal{E}_k(t) = -i\,\frac{\omega_p^2}{k^2}\int d^3v\,(i\vec{k}\cdot\vec{v}+\omega)\,\vec{k}\cdot\nabla f_0(\vec{v},t)\int_0^\infty ds\, e^{i(\omega-\vec{k}\cdot\vec{v})s-\omega s}\,\mathcal{E}_k(t)$$

where $\mathcal{E}_k(t)$ is the electrostatic energy defined by

$$\mathcal{E}_k(t) = \frac{1}{8\pi}\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t)\rangle$$

But, it can be shown [Appendix B] that the damping of electrostatic energy occurs at twice the rate of the damping of momentum γ_k, hence the growth rate of the wave is given by

$$\gamma_k = -\frac{i}{2}\frac{\omega_p^2}{k^2}\int d^3v\,(i\vec{k}\cdot\vec{v}+\nu)\,\vec{k}\cdot\nabla_v f_0(\vec{v},t)\int_0^\infty ds\, e^{i(\omega-\vec{k}\cdot\vec{v})s-\nu s} \tag{3.9}$$

Equation (3.9) is an explcit expression for the total damping coefficient and represents the growth rate of longitudinal plasma oscillations for the Krook Model. The expression includes the classical Landau damping and the effect of collisions. On assuming that

$$\gamma_k \ll \omega \quad , \quad \nu \ll \omega$$

and the mean square velocity

$$\overline{V^2} \ll (\omega/k)^2$$

series expansion is permitted, and we find [Appendix C]

$$\gamma_k = \gamma_{kL} + \gamma_{kc} \tag{3.10}$$

where γ_{kL} is the classical Landau damping coefficient given by

$$\gamma_{kL} = \frac{\pi}{2}\frac{\omega_p^2}{k^2}\int d^3v\,(\vec{k}\cdot\nabla_v f_0)\,\vec{k}\cdot\vec{v}\;\delta(\vec{k}\cdot\vec{v}-\omega) \tag{3.11}$$

and γ_{kc} represents the collisional damping given by

$$\gamma_{kc} = -\nu\left[1 + 3\frac{k^2\overline{V^2}}{\omega_p^2}\right] \tag{3.12}$$

(b) Diffusion Coefficient

On multiplying Eq. (3.5) by $\vec{E}_{-k}(t)$ and taking the ensemble average, we obtain

$$\langle \vec{E}_{-k}(t) F_k(\vec{v},t) \rangle = \frac{e\vec{k}}{mk^2} \int_{-\infty}^{t} d\tau \, e^{-(i\vec{k}\cdot\vec{v}+\omega)(t-\tau)} \vec{k}\cdot\nabla f_0(\vec{v},\tau) \langle \vec{E}_{-k}(t)\cdot\vec{E}_k(\tau) \rangle$$

Changing the variable from τ to $s = t-\tau$, and assuming f_0 to be slowly varying in time so that it may be taken in front of the integration, we have

$$\langle \vec{E}_{-k}(t) F_k(\vec{v},t) \rangle = \frac{e\vec{k}}{mk^2} \vec{k}\cdot\nabla f_0(\vec{v},t) \int_0^{\infty} ds \, e^{-(i\vec{k}\cdot\vec{v}+\omega)s} \langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t-s) \rangle$$

We may further take

$$\vec{E}_k(t-s) = \vec{E}_k(t) \, e^{i\Omega s}$$

where Ω is the complex frequency of the wave given by

$$\Omega(k) = \omega(k) + i\gamma_k$$

Consequently,

$$\langle \vec{E}_{-k}(t) F_k(\vec{v},t) \rangle = \frac{8\pi e\vec{k}}{mk^2} \mathcal{E}_k(t) \vec{k}\cdot\nabla f_0 \int_0^{\infty} ds \, e^{i(\Omega-\vec{k}\cdot\vec{v})s-\omega s}$$

When this expression is substituted into the equation of evolution of the background distribution function, namely, Eq. (3.4), we find

$$\frac{\partial f_0}{\partial t} = \nabla_v \cdot \left(\overleftrightarrow{D} \cdot \nabla_v f_0 \right) - \nu \left(f_0 - f_{eq} \right) \tag{3.13}$$

where $\overleftrightarrow{D}$ is the diffusion coefficient given by

$$\overleftrightarrow{D} = \frac{8\pi e^2}{m^2} \sum_k \frac{\vec{k}\vec{k}}{k^2} \mathcal{E}_k \int_0^\infty ds \, e^{i(\Omega - \vec{k}\cdot\vec{v})s - \nu s} \tag{3.14}$$

With the assumption of small collision frequency ν, and small mean square velocity $\overline{v^2} \ll (\omega/k)^2$. Eq. (3.14) can be integrated explicitly [Appendix D] to give

$$\overleftrightarrow{D} = \frac{8\pi^2 e^2}{m^2} \sum_{k \approx \omega/k} \frac{\vec{k}\vec{k}}{k^2} \mathcal{E}_k \, \delta(\vec{k}\cdot\vec{v} - \omega) + \frac{8\pi e^2}{m^2} \sum_{k \neq \omega/k} \frac{\vec{k}\vec{k}}{k^2} \mathcal{E}_k \frac{\gamma_k + \nu}{(\omega - \vec{k}\cdot\vec{v})^2} \tag{3.15}$$

It is to be noted that the diffusion coefficient is made up of the sum of two terms representing its values in two different regime of particle velocities, depending on whether the particles are in resonance ($\omega \approx \vec{k} \cdot \vec{v}$) with the waves or not ($\omega \neq \vec{k} \cdot \vec{v}$).

IV. THE FOKKER-PLANCK MODEL

This section is devoted to the study of the statistical behavior of the longitudinal oscillations of a plasma described by the Fokker-Planck model in which the collision integral is represented by

$$\left(\frac{\partial f}{\partial t}\right)_c = \beta \nabla_v \cdot \left[RT \nabla_v f + \vec{v} f \right] \qquad (4.1)$$

where β is a frequency representative of friction. R is the ratio of Boltzmann's constant to the mass of an electron and T the absolute temperature. This model was originally proposed by Zakharov and Karpmann [13] and is similar to that used in the theory of Brownian motion [14].

The collision terms in the model consist of two parts with distinct physical significance: the part $\beta \nabla_v \cdot (RT \nabla_v f)$ represents the spreading out of phase points in velocity space, the remaining part $\beta \nabla_v \cdot (\vec{v} f)$ signifies the drag of the particles along the direction of motion.

Applying this model, the fundamental equations (2.6), (2.7), and (2.8) of the plasma in the quasi-linear regime, when the coordinate space is Fourier-transformed, become

$$\vec{E}_k = \frac{4\pi n e}{k^2} i \vec{k} \int d^3v \; F_k \qquad (4.2)$$

$$\frac{\partial F_k}{\partial t} + i \vec{k} \cdot \vec{v} F_k - \frac{e}{m} \vec{E}_k \cdot \nabla_v f_0 = \beta \nabla_v \cdot \left[RT \nabla_v F_k + \vec{v} F_k \right] \qquad (4.3)$$

and

$$\frac{\partial f_0}{\partial t} - \frac{e}{m} \sum_j \left(\vec{E}_{-j} \cdot \nabla_v F_j \right) = \beta \nabla_v \cdot \left[RT \nabla_v f_0 + \vec{v} f_0 \right] \qquad (4.4)$$

To solve Eq. (4.3), we Fourier-transform the equation in velocity space so that it becomes

$$\frac{\partial \hat{F}_k}{\partial t} - (\vec{k} - \beta\vec{p}) \cdot \frac{\partial \hat{F}_k}{\partial \vec{p}} + \beta RT p^2 \hat{F}_k = \frac{ie}{m} \frac{\vec{p}\cdot\vec{k}}{k^2} \vec{k}\cdot\vec{E}_k \hat{f}_0 \qquad (4.5)$$

where

$$\hat{F}_k(\vec{p}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3v \; e^{-i\vec{p}\cdot\vec{v}} F_k(\vec{v}, t)$$

and

$$\hat{f}_0(\vec{p}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3v \; e^{-i\vec{p}\cdot\vec{v}} f_0(\vec{v}, t)$$

To solve Eq. (4.5), it is convenient to set

$$\hat{F}_k(\vec{p}, t) = \hat{G}(\vec{k}, \vec{p}, t)\, e^{-\frac{1}{2}RTp^2 - \frac{RT}{\beta}\vec{p}\cdot\vec{k}}$$

so that the term p^2 is eliminated from the coefficient of $\hat{F}_k$.

With this expression substituted into Eq. (4.5), the latter reads

$$\frac{\partial \hat{G}}{\partial t} + (\beta\vec{p} - \vec{k}) \cdot \frac{\partial \hat{G}}{\partial \vec{p}} + \frac{RT}{\beta} k^2 \hat{G} = Q \qquad (4.6)$$

where

$$Q(\vec{k}, \vec{p}, t) = \frac{ie}{mk} E_k(t) \hat{f}_0(\vec{p}, t)\, \vec{p}\cdot\vec{k}\; e^{\frac{1}{2}RTp^2 + \frac{RT}{\beta}\vec{p}\cdot\vec{k}}$$

and

$$E_k(t) = \frac{\vec{k}\cdot\vec{E}_k(t)}{k}$$

Equation (4.6) can be solved by the method of characteristics [Appendix E], yielding the result

$$\hat{G}(\vec{k},\vec{p},t) = e^{-\frac{RT}{\beta}k^2 t}\int_{-\infty}^{t} d\tau\; Q(\vec{k},\vec{p}_\tau,\tau)\, e^{\frac{RT}{\beta}k^2\tau} \tag{4.7}$$

where

$$\vec{p}_\tau = \vec{p}\, e^{-\beta(t-\tau)} + \frac{\vec{k}}{\beta}\left[1 - e^{-\beta(t-\tau)}\right]$$

Consequently

$$\hat{F}_k(\vec{p},t) = e^{-\frac{1}{2}RTp^2 - \frac{RT}{\beta}\vec{p}\cdot\vec{k}}\int_{-\infty}^{t} d\tau\; Q(\vec{k},\vec{k}_\tau,\tau)\, e^{-\frac{RT}{\beta}k^2(t-\tau)} \tag{4.8}$$

It is to be remarked that Eq. (4.8) can be rewritten in a dimensionless form, taking the following reference or unit quantities: the thermal velocity v_T, the Debye length λ_D, and the plasma frequency ω_p. Then β will appear in the form $\beta\omega_p^{-1}$ which will be assumed in the present treatment to be a small dimensionless quantity. Hence we may expand Eq. (4.8) into series and linearize with respect to β, and without going into the details, we obtain

$$\hat{F}_k(\vec{p},t) = \frac{ei}{mk^2}\int_0^\infty ds\ \vec{k}\cdot\vec{E}_k(t-s)\,\hat{f}_0(\vec{p}',t-s)\,\vec{p}'\cdot\vec{k}\left\{1-\beta RT\left[p^2 s+\vec{p}\cdot\vec{k}s^2+\frac{1}{3}k^2s^3\right]\right\}$$

where

$$s = t-\tau$$

and

$$\vec{p}' = \vec{p}\,(1-\beta s) + \vec{k}\,(1-\frac{\beta s}{2})s$$

By taking Fourier inversion of $\hat{F}_k$ with respect to velocity space, we obtain

$$F_k(\vec{v},t) = \frac{e}{mk^2}\,\frac{i}{(2\pi)^{3/2}}\int_0^\infty ds\ \vec{k}\cdot\vec{E}_k(t-s)\int_{-\infty}^{\infty} d^3p\ e^{i\vec{p}\cdot\vec{v}}\,\hat{f}_0(\vec{p}',t-s)\,\vec{p}'\cdot\vec{k}\left[1-\beta RT p^2 s-\beta RT\vec{p}\cdot\vec{k}s^2-\frac{1}{3}\beta RT k^2s^3\right] \quad (4.9)$$

The integration with respect to p can be performed explicitly [Appendix F], yielding

$$F_k(\vec{v},t) = -\frac{ie}{mk^2}\int_0^\infty ds\ e^{-i\vec{k}\cdot\vec{v}s}\,\vec{k}\cdot\vec{E}_k(t-s)\,\Phi(s) \quad (4.10)$$

where

$$\Phi(s) = i\,(\vec{k}\cdot\nabla_v f_0)\,(1 + 3\beta s - \tfrac{1}{3}\beta RT k^2 s^3)$$
$$+(\vec{k}\cdot\nabla_v f_0)(\tfrac{1}{2}\beta\,\vec{k}\cdot\vec{v}\,s^2) + \vec{k}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_0)(\beta RT s^2)$$
$$+ i\,\vec{v}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_0)(\beta s) + i\,\nabla_v^2(\vec{k}\cdot\nabla_v f_0)(\beta RT s)$$

The expression (4.10) gives the fluctuation F_k in a plasma with a distribution f_0. It consists of a collisionless and a collisional part proportional to β, and enables us to calculate the generalized Landau damping coefficient as well as the diffusion coefficient, paralleling the approach of Section III in connection with the Krook Model.

(a) Damping Coefficient

We shall now evaluate the damping coefficient for small values of k by the method of energy decay [12]. This method, valid for small k, is equivalent to the classical Fourier transform method used by Landau [1].

Substituting Eq. (4.10) into Eq. (4.2), we obtain

$$\vec{E}_k(t) = \frac{\omega_p^2}{k^2}\int d^3v \int_0^{\infty} ds\; e^{-i\vec{k}\cdot\vec{v}s}\;\vec{E}_k(t-s)\;\Phi(s) \tag{4.11}$$

Multiplying Eq. (4.11) by $\vec{E}_{-k}(t)$ and taking ensemble average,

$$\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t)\rangle = \frac{\omega_p^2}{k^2}\int d^3v \int_0^{\infty} ds\; e^{-i\vec{k}\cdot\vec{v}s}\,\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t-s)\rangle\,\Phi(s) \tag{4.12}$$

On differentiating Eq. (4.12) with respect to time t, it can be shown [Appendix A] that the equation becomes

$$\frac{\partial}{\partial t}\left\langle \vec{E}_{-k}(t)\cdot\vec{E}_{k}(t)\right\rangle = \frac{\omega_p^2}{k^2}\int d^3v \int_0^{\infty} ds\; e^{-i\vec{k}\cdot\vec{v}s}\left\langle \vec{E}_{-k}(t)\cdot\vec{E}_{k}(t-s)\right\rangle \Phi_1(s) \qquad (4.13)$$

where

$$\begin{aligned}\Phi_1(s) &= -i\,\vec{k}\cdot\vec{v}\;\Phi(s) + \frac{\partial}{\partial s}\Phi(s)\\ &= (\vec{k}\cdot\nabla_v f_0)\left[(\vec{k}\cdot\vec{v}) + 4\beta(\vec{k}\cdot\vec{v})s - \tfrac{1}{3}\beta\kappa T(\vec{k}\cdot\vec{v})k^2s^3\right]\\ &\quad + (\vec{k}\cdot\nabla f_0)\left[i3\beta - i\beta\kappa T k^2s^2 - i\tfrac{1}{2}\beta(\vec{k}\cdot\vec{v})^2s^2\right]\\ &\quad + \vec{k}\cdot\nabla(\vec{k}\cdot\nabla f_0)\left[2\beta\kappa Ts - i\beta\kappa T(\vec{k}\cdot\vec{v})s^2\right]\\ &\quad + \vec{v}\cdot\nabla(\vec{k}\cdot\nabla f_0)\left[\beta(\vec{k}\cdot\vec{v})s + i\beta\right]\\ &\quad + \nabla_v^2(\vec{k}\cdot\nabla f_0)\left[\beta\kappa T(\vec{k}\cdot\vec{v})s + i\beta\kappa T\right]\end{aligned}$$

We shall again assume as before that the correlation $<\vec{E}_{-k}(t)\cdot\vec{E}_k(t-s)>$ takes the form

$$\left\langle \vec{E}_{k}(t)\cdot\vec{E}_{k}(t-s)\right\rangle = \left\langle \vec{E}_{-k}(t)\cdot\vec{E}_{k}(t)\right\rangle e^{i\omega s}$$

Consequently Eq. (4.13) simplifies to

$$\frac{\partial \mathcal{E}_k(t)}{\partial t} = \frac{\omega_p^2}{k^2} \int d^3v \int_0^{\infty} ds\, e^{i(\omega - \vec{k}\cdot\vec{v})s}\, \Phi_k(s)\, \mathcal{E}_k(t)$$

where $\mathcal{E}_k(t)$ is the electrostatic energy of the k^{th} mode.

The damping rate of the wave, being one half that of the decay rate of energy, is thus given by

$$\gamma_k = \frac{1}{2} \frac{\omega_p^2}{k^2} \int d^3v \int_0^{\infty} ds\, e^{i(\omega - \vec{k}\cdot\vec{v})s}\, \Phi_k(s) \tag{4.14}$$

With the assumption that f_0 is so slowly varying in time that it can be taken out in front of the integral with respect to s, the integration can be performed explicitly [Appendix G] to yield the real solution

$$\gamma_k = \gamma_{kL} + \gamma_{kc} \tag{4.15}$$

$$\gamma_{kL} = \frac{\pi}{2} \frac{\omega_p^2}{k^2} \int d^3v\, (\vec{k}\cdot \nabla f_0)\, \vec{k}\cdot\vec{v}\, \delta(\vec{k}\cdot\vec{v} - \omega) \tag{4.16}$$

$$
\begin{aligned}
\gamma_{kc} = \frac{\beta}{2}\,\frac{\omega_p^2}{k^2}\,P\int d^3v\,\Bigg\{ & (\vec{k}\cdot\nabla f_0)\left[-\frac{4(\vec{k}\cdot\vec{v})}{(\vec{k}\cdot\vec{v}-\omega)^2} - \frac{2RTk^2(\vec{k}\cdot\vec{v})}{(\vec{k}\cdot\vec{v}-\omega)^4}\right] \\
& + (\vec{k}\cdot\nabla f_0)\left[\frac{3}{(\vec{k}\cdot\vec{v}-\omega)} + \frac{2RTk^2}{(\vec{k}\cdot\vec{v}-\omega)^3} + \frac{(\vec{k}\cdot\vec{v})^2}{(\vec{k}\cdot\vec{v}-\omega)^3}\right] \\
& + \vec{k}\cdot\nabla(\vec{k}\cdot\nabla f_0)\left[-\frac{2RT}{(\vec{k}\cdot\vec{v}-\omega)^2} + \frac{2RT(\vec{k}\cdot\vec{v})}{(\vec{k}\cdot\vec{v}-\omega)^3}\right] \\
& + \vec{v}\cdot\nabla(\vec{k}\cdot\nabla f_0)\left[-\frac{(\vec{k}\cdot\vec{v})}{(\vec{k}\cdot\vec{v}-\omega)^2} + \frac{1}{(\vec{k}\cdot\vec{v}-\omega)}\right] \\
& + \nabla_v^2(\vec{k}\cdot\nabla f_0)\left[-\frac{RT(\vec{k}\cdot\vec{v})}{(\vec{k}\cdot\vec{v}-\omega)^2} + \frac{RT}{(\vec{k}\cdot\vec{v}-\omega)}\right]\Bigg\}
\end{aligned}
\tag{4.17}
$$

where P indicates the principal values of the integrals.

Equation (4.15) is an explicit form of the generalized Landau damping coefficient since the integration over v can be performed for a given background distribution function $f_0(\vec{v}, t)$

It is noticed that γ_k consists of the sum of two parts, γ_{kL} and γ_{kc}. The former part is identical to the classical coefficient of Landau damping and the latter part, being proportional to β, represents the collisional damping from the Fokker-Planck equation in the quasi-linear regime. Terms involving third order derivatives are kept in the expression as they are in general not negligible as compared with those involving lower order derivatives.

In the special case where f_0 is Maxwellian, the expression (4.17) for γ_{kc} can be considerably simplified [Appendix G], yielding, in one dimensional case, the result

$$\gamma_{kc} = -\frac{\beta}{2}\left(\frac{\omega_p}{\omega}\right)^2\left[1 + 16\,\frac{k^2 v_T^2}{\omega^2}\right] \tag{4.18}$$

where v_T is the thermal velocity given by

$$v_T^2 = \frac{\kappa T}{m}$$

Now, the dispersion relation of the plasma under consideration can be readily obtained by putting

$$\vec{E}_k(t) = \vec{E}_k(0)\, e^{-i\Omega t} = \vec{E}_k(0)\, e^{-i\omega t + \gamma t}$$

in Eq. (4.1[illegible]) and solving for ω as a function of real k, yielding [Appendix H] the approximate result

$$\omega^2 = \omega_p^2 + 3k^2 v_T^2 \tag{4.19}$$

Using this relation, the expression (4.18) for γ_{kc} further reduces to

$$\gamma_{kc} = -\frac{\beta}{2}\left(1 + 13\,\frac{k^2 v_T^2}{\omega_p^2}\right) \tag{4.20}$$

(b) <u>Diffusion Coefficient</u>

When Eq. (4.10) is multiplied by $\vec{E}_{-k}(t)$ and the ensemble average taken, we have

$$\left\langle \vec{E}_{-k}(t)\, F_k(\vec{v},t)\right\rangle = -\frac{ie\vec{k}}{mk^2}\int_0^{\infty} ds\; e^{-i\vec{k}\cdot\vec{v}s}\left\langle \vec{E}_{-k}(t)\cdot\vec{E}_k(t-s)\right\rangle \bar{\Phi}(s)$$

With the assumption

$$\vec{E}_k(t-s) = \vec{E}_k(t)\, e^{i\Omega s}$$
$$= \vec{E}_k(t)\, e^{i\omega s - \gamma_k s} \qquad , \; \Omega = \omega + i\gamma_k$$

this equation becomes

$$\langle \vec{E}_{-k}(t) f_k(\vec{v},t)\rangle = \frac{8\pi e\vec{k}}{m k^2}\, \mathcal{E}_k(t) \int_0^{\infty} ds\; e^{i(\Omega - \vec{k}\cdot\vec{v})s} \left[-i\,\bar{\Phi}(s)\right] \tag{4.21}$$

where $\mathcal{E}_k(t)$ is the electrostatic energy of the k^{th} mode.

For slowly varying f_o with respect to time, only the first term in its Taylor expansion is kept, so that it may be taken out in front of the integral. The integration can then be performed explicitly [Appendix J] to yield the real solution

$$\langle \vec{E}_{-k}(t)\, f_k(\vec{v},t)\rangle = \frac{8\pi e\vec{k}}{m k^2}\, \Theta\, \mathcal{E}_k(t) \tag{4.22}$$

where

$$\Theta = (\vec{k}\cdot\nabla_v f_o)\left[\pi\delta(\vec{k}\cdot\vec{v}-\omega) + \frac{\gamma_k}{(\omega-\vec{k}\cdot\vec{v})^2} - \frac{3\beta}{(\omega-\vec{k}\cdot\vec{v})^2} - \frac{\beta\,\vec{k}\cdot\vec{v}}{(\omega-\vec{k}\cdot\vec{v})^3}\right.$$
$$\left. - \frac{2\beta\kappa T k^2}{(\omega-\vec{k}\cdot\vec{v})^4}\right] + \vec{k}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_o)\left[-\frac{2\beta\kappa T}{(\omega-\vec{k}\cdot\vec{v})^3}\right]$$
$$+ \vec{v}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_o)\left[-\frac{\beta}{(\omega-\vec{k}\cdot\vec{v})^2}\right] + \nabla_v^2(\vec{k}\cdot\nabla_v f_o)\left[-\frac{\beta\kappa T}{(\omega-\vec{k}\cdot\vec{v})^3}\right]$$

We have assumed, in the derivation of this expression, that γ_k is small compared with ω_p.

On substituting this into Eq. (4.4) for the evolution of background distribution function, the latter becomes

$$\frac{\partial f_0}{\partial t} = \frac{8\pi e^2}{m^2} \nabla_v \cdot \sum_k \frac{\vec{k}}{k^2} \mathcal{E}_k(t) \, \Theta + \beta \nabla_v \cdot \left[kT \nabla_v f_0 + \vec{v} f_0 \right] \tag{4.23}$$

This equation may be rewritten in the form

$$\frac{\partial f_0}{\partial t} = \nabla_v \cdot \left(\overleftrightarrow{D}_0 \cdot \nabla_v f_0 \right) + \nabla_v \cdot \left(\overleftrightarrow{D}_1 \cdot \nabla_v f_0 \right) + \nabla_v \cdot \vec{D}_2 + \nabla_v \cdot \vec{D}_3 \tag{4.24}$$

where

$$\overleftrightarrow{D}_0 = \frac{8\pi^2 e^2}{m^2} \sum_{k \approx \omega/v} \frac{\vec{k}\vec{k}}{k^2} \mathcal{E}_k(t) \, \delta(\vec{k}\cdot\vec{v} - \omega)$$

and

$$\overleftrightarrow{D}_1 = \frac{8\pi e^2}{m^2} \sum_{k \neq \omega/v} \frac{\vec{k}\vec{k}}{k^2} \mathcal{E}_k(t) \frac{\gamma_k}{(\omega - \vec{k}\cdot\vec{v})^2}$$

are the quasi-linear contribution of the diffusion coefficient in the resonance and non-resonance region respectively.

$$\vec{D}_2 = \frac{8\pi e^2}{m^2}\,\beta \sum_{k \neq \omega/v} \mathcal{E}_k \frac{\vec{k}}{k^2} \left\{ \left(\left[-\frac{3}{(\omega-\vec{k}\cdot\vec{v})^2} - \frac{\vec{k}\cdot\vec{v}}{(\omega-\vec{k}\cdot\vec{v})^3} \right.\right.\right.$$

$$\left. - \frac{2RTk^2}{(\omega-\vec{k}\cdot\vec{v})^4}\right](\vec{k}\cdot\nabla_v f_0) + \left[-\frac{2RT}{(\omega-\vec{k}\cdot\vec{v})^3}\right]\vec{k}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_0)$$

$$\left. + \left[-\frac{1}{(\omega-\vec{k}\cdot\vec{v})^2}\right]\vec{v}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_0) + \left[-\frac{RT}{(\omega-\vec{k}\cdot\vec{v})^2}\right]\nabla_v^2(\vec{k}\cdot\nabla_v f_0)\right\}$$

represents the contribution to the diffusion coefficient from both the quasi-linear and the collisional effect in the non-resonance region. And finally

$$\vec{D}_3 = \beta\left[RT\,\nabla_v f_0 + \vec{v}\, f_0\right]$$

is the original linearized Fokker-Planck collision terms representing a diffusion and a dynamical friction.

V. DEVELOPMENT OF PARTICLE DISTRIBUTION AS A RESULT OF QUASI-LINEAR AND COLLISIONAL EFFECTS

In this section, we investigate the detailed development of the energy fluctuations and the diffusion of the trapped particles in the one-dimensional case of the Fokker-Planck Model. Assume that, at the initial instant, the background distribution has a small trapping interval $(V_0 - u, V_0)$ of positive slope in the velocity space, otherwise the slopes are negative. In this interval, the particles are said to be in resonance with the waves and interact with the latter by surrendering their kinetic energy. Oscillations begin to build up and the spectral energy density of these oscillations increases in accordance with the law

$$\frac{\partial \mathcal{E}_k}{\partial t} = 2 \gamma_k \mathcal{E}_k$$

The growth of waves or oscillations increases the diffusion of the trapped particles according to Eq. (4.24).

Outside the trapped interval where the first derivatives of the distribution function f_0 with respect to velocity v are negative at all points, there is no growth of longitudinal oscillations; and we shall assume the energy fluctuations or noise there to be negligibly small.

Consequently, we obtain the following system of equations governing the evolution of energy fluctuations and of the diffusion of the distribution function of the plasma particles

$$\frac{\partial \mathcal{E}}{\partial t} = \alpha v \mathcal{E} \frac{\partial f_0}{\partial v} - \gamma_0 \mathcal{E} \tag{5.1}$$

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left[\frac{\lambda \mathcal{E}}{v} \frac{\partial f_0}{\partial v} + \beta R T \frac{\partial f_0}{\partial v} \right] + \frac{\partial}{\partial v}\left[\beta v f_0 \right] \tag{5.2}$$

where

$$\alpha = \frac{4\pi^2 n e^2}{m k}$$

$$\lambda = \frac{4\pi e^2}{m^2}$$

$$\gamma_0 = \beta \left(1 + 13 \frac{k^2 \overline{v^2}}{\omega_p^2} \right)$$

We shall now represent the distribution function in the trapping interval $(V_0 - u, V_0)$ approximately by the equation

$$f_0(\vec{v}, t) = A(t) \sin\left[\frac{\pi}{2}\left(\frac{v - V_0}{u}\right)\right] + g_0 \tag{5.3}$$

with $A(0) > 0$ and $g_0 > 0$.

On substituting Eq. (5.3) into Eq. (5.2) and integrating the latter within the trapping interval $(V_0 - u, V_0)$, we obtain

$$\frac{dA}{dt} = -\frac{\pi^2}{4u^2}\left[\lambda \frac{\mathcal{E}(V_0, t)}{V_0} + \beta R T \right] A$$

$$- \beta \frac{\pi}{2}\left(\frac{V_0}{u} - 1 \right) A - \beta \frac{\pi}{2} g_0 \tag{5.4}$$

Similarly, substitution of Eq. (5.3) into Eq. (5.1) yields

$$\frac{\partial \mathcal{E}}{\partial t} = \alpha V \mathcal{E} \left[\frac{\pi}{2u} \cos \frac{\pi}{2}\left(\frac{V-V_0}{u}\right) \right] A - \gamma_0 \mathcal{E} \tag{5.5}$$

By virtue of the smallness of β, Eqs. (5.3) through (5.5) can be solved simultaneously [Appendix K] to give

$$A(t) = (p-q)\left[\frac{A_0-q}{A_0-p}\, e^{-\theta_1(p-q)t} - 1 \right]^{-1} + p \tag{5.6}$$

$$\mathcal{E}(v,t) = \frac{\mathcal{E}_0(v)\left[\frac{A_0-q}{A_0-p} - 1 \right]^{\frac{V}{V_0}\mu} e^{\frac{\pi\alpha V}{2}\mu q t - \gamma_0 t}}{\left[\frac{A_0-q}{A_0-p}\, e^{-\theta_1(p-q)t} - 1 \right]^{\frac{V}{V_0}\mu}} \tag{5.7}$$

$$\frac{\partial f_0}{\partial v} = \left\{ (p-q)\left[\frac{A_0-q}{A_0-p}\, e^{-\theta_1(p-q)t} - 1 \right]^{-1} + p \right\} \frac{\pi}{2u}\mu \tag{5.8}$$

where

$$A_0 = A(0)$$

$$\mathcal{E}_0(v) = \mathcal{E}(v,0)$$

$$\mu = \cos\left[\frac{\pi}{2}\left(\frac{V-V_0}{u}\right) \right]$$

$$p = \left[-\theta_2 + \sqrt{\theta_2^2 - 4\theta_1\theta_3} \right] / 2\theta_1$$

$$q = \left[-\theta_2 - \sqrt{\theta_2^2 - 4\theta_1\theta_3} \right] / 2\theta_1$$

$$\theta_1 = \frac{\pi}{2u} \alpha V_0$$

$$\theta_2 = -\frac{\pi}{2u} \alpha V_0 A_0 - \frac{\pi^2}{4u^2} \frac{\lambda}{V_0} \varepsilon(V_0, 0) - \frac{\pi^2}{4u^2} \beta R T - \frac{\pi}{2u} \beta V_0$$

$$\theta_3 = -\frac{\pi}{2} \beta g_0$$

With a little algebraic manipulation, it can be shown that, at t = 0,

$$\varepsilon(v,t)\Big|_{t=0} = \varepsilon_0(v)$$

$$\frac{\partial f_0}{\partial v}\Big|_{t=0} = A_0 \frac{\pi}{2u} \cos\left[\frac{\pi}{2}\left(\frac{V-V_0}{u}\right)\right]$$

and at $t \to \infty$,

$$\varepsilon(v,t)\Big|_{t\to\infty} = 0$$

$$\frac{\partial f_0}{\partial v}\Big|_{t\to\infty} = g \frac{\pi}{2u} \cos\left[\frac{\pi}{2}\left(\frac{V-V_0}{u}\right)\right] < 0$$

for the interval $(V_0 - u < V < V_0)$.

Furthermore, it can be readily demonstrated that $\varepsilon(v, t)$ and $\frac{\partial f}{\partial v}$ are bounded for $(V_0 - u < V < V_0)$ and $(0 \le t \le \infty)$, the upper limits of which being given respectively by

$$\left(\frac{\partial f_0}{\partial v}\right)_{max} = A_0$$

$$\mathcal{E}_{max} > \mathcal{E}_0(v)$$

From these results, we conclude that, within the trapping interval, the positive slopes $(\frac{\partial f}{\partial v})$ gradually decreases from the positive quantity $A_0 \frac{\pi}{2u} \cos\left(\frac{V-V_0}{u}\right)\frac{\pi}{2}$ to a transient plateau, and thereafter to a negative value $q \frac{\pi}{2u} \cos\left(\frac{V-V_0}{u}\right)\frac{\pi}{2}$ in the asymptotic limit, as shown in Fig. 1 and Fig. 2 on pages 34 and 35.

From Eq. (5.8) it is readily estimated that the time for the positive slope to develop into the transient plateau is

$$t_p = \frac{1}{\theta_1(p-q)} \ln \left| \frac{p}{q} \frac{A_0 - q}{A_0 - p} \right|$$

We shall take the characteristic time t_n for development into the final equilibrium state as the time when the slope is within one percent of the final value.

During this course of time, the energy spectrum ε increases from a small initial value $\varepsilon(v, 0)$ to a maximum at the instant when the plateau is formed in the distribution function, and then decreases gradually to zero, as shown in Fig. 2 and Fig. 3 on page 35, when the distribution becomes stabilized. At the edges and outside the trapping interval, $\mathcal{E}$ maintains its zero value since it is not affected by instabilities.

It is noticed that in the special case where collisions are negligible, or where $\beta = 0$, all expressions in this appendix still hold true except that q is zero instead of a negative quantity. Consequently, the positive slopes in the distribution function develop only into a plateau and do not continue evolving beyond it into negative slopes as in the collisional case. At the same time, in the absence of collisions, the energy spectrum will reach an unstable maximum instead of converging to zero in the asymptotic limit. A comparison of various characteristics times and limits of convergence is shown for a typical plasma situation in Table 1 on page 36.

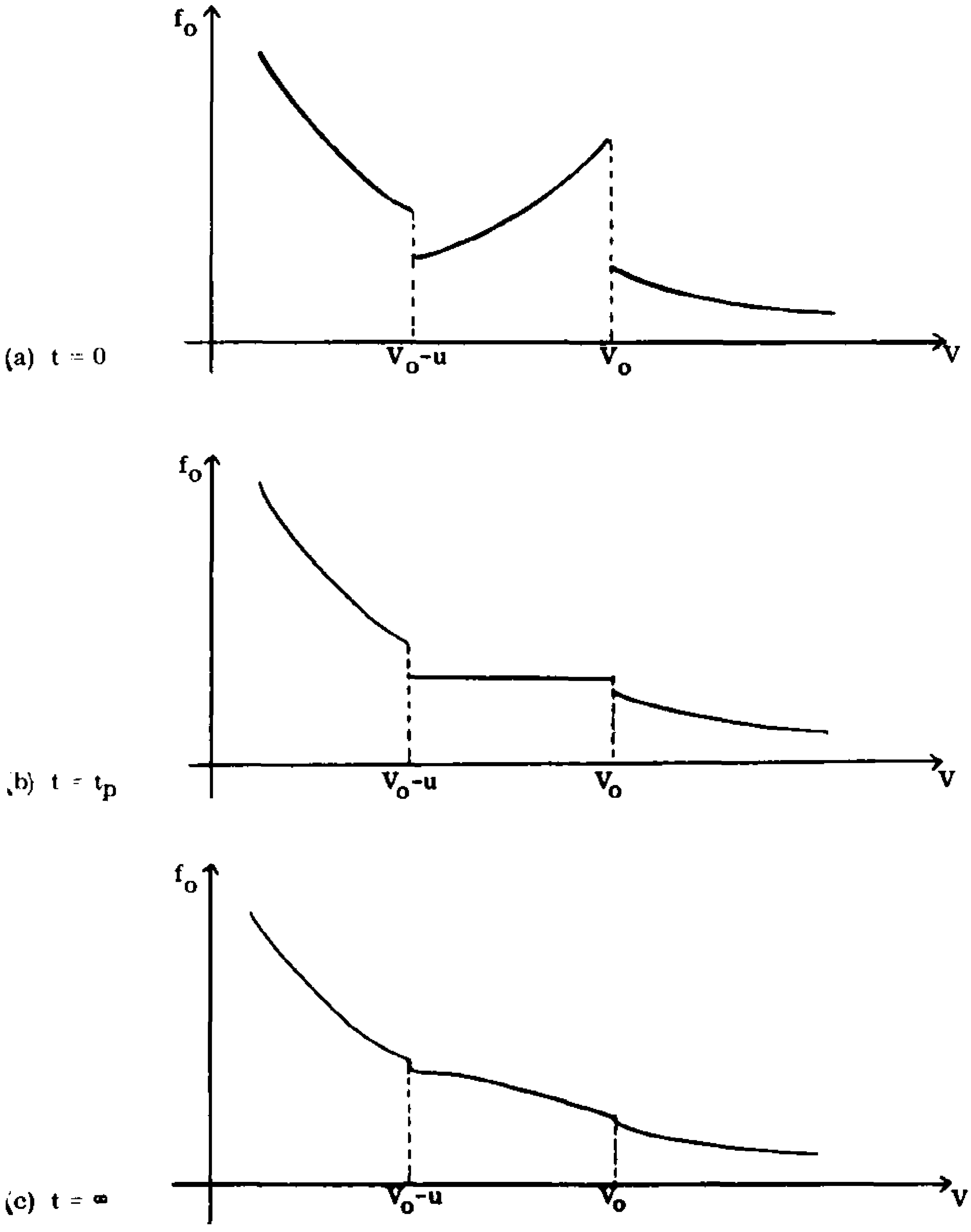

FIG. 1 DEVELOPMENT OF NEGATIVE SLOPES IN PARTICLE DISTRIBUTION FUNCTION IN TRAPPING INTERVAL

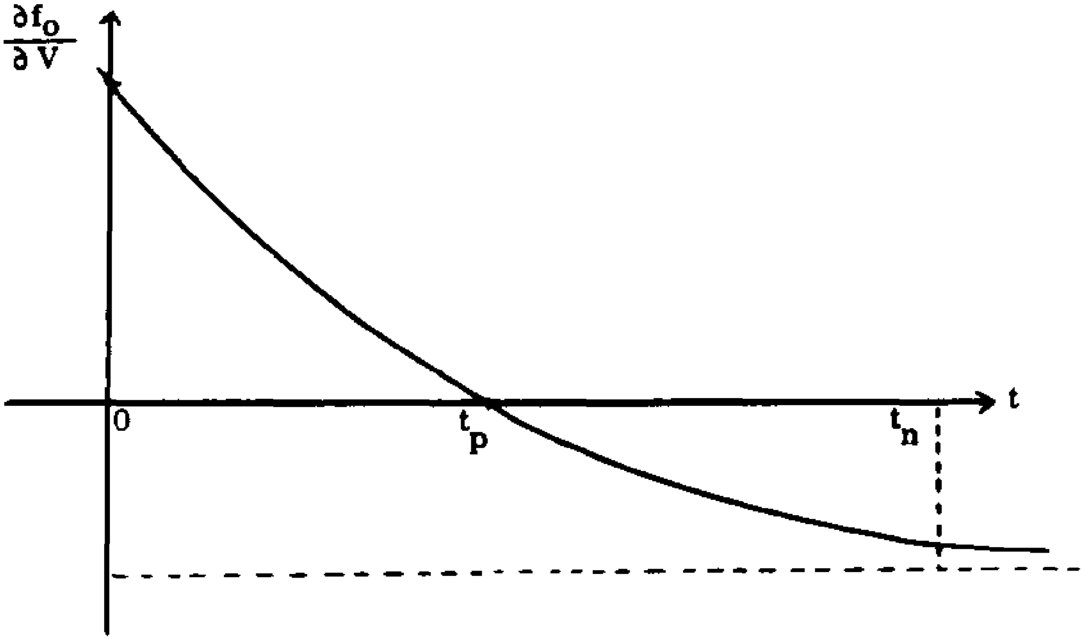

FIG. 2 TIME EVOLUTION OF SLOPES OF DISTRIBUTION FUNCTION IN TRAPPING INTERVAL

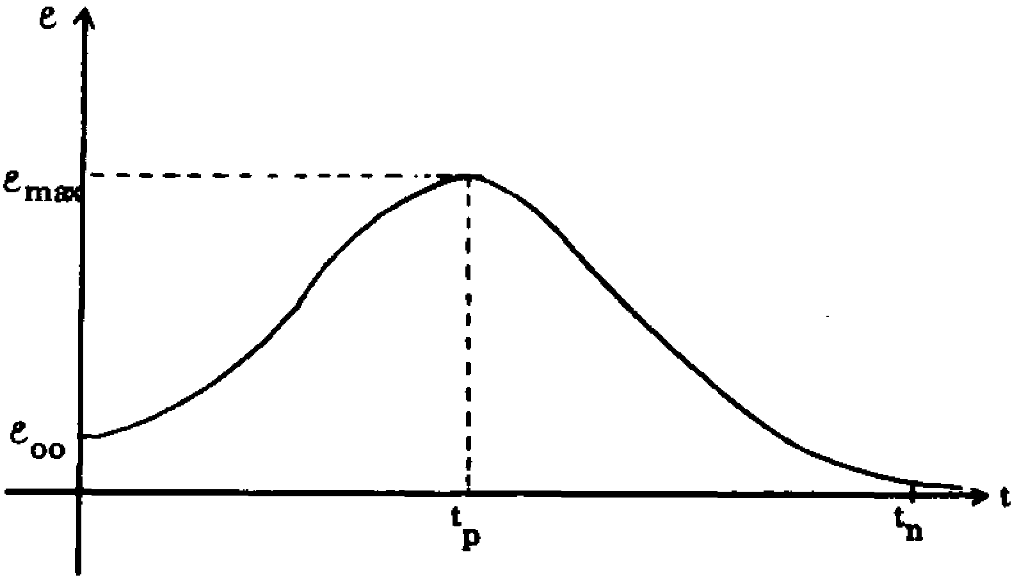

FIG. 3 TIME EVOLUTION OF ENERGY DENSITY OF OSCILLATIONS IN TRAPPING INTERVAL

Cases of collisions **characteristic times and slopes**	$\beta = 0$	$\beta = 0.005\omega_p$	$\beta = 0.02\omega_p$
t_p	$\sim 1.6 \times 10^3 \omega_p^{-1}$	$\sim 9.3 \times 10^0 \omega_p^{-1}$	$\sim 2.0 \times 10^{-1} \omega_p^{-1}$
t_n	$\sim 1.6 \times 10^3 \omega_p^{-1}$	$\sim 1.5 \times 10^1 \omega_p^{-1}$	$\sim 3.1 \times 10^0 \omega_p^{-1}$
AVERAGE ASYMPTOTIC SLOPE	0	$\sim 1.0 \times 10^7 n/V_T^2$	$\sim 1.0 \times 10^7 n/V_T^2$

TABLE 1 - SOME TYPICAL VALUES OF CHARACTERISTIC TIMES AND SLOPES

VI. CONCLUSION AND DISCUSSIONS

The basic results for the combined collective and collisional behavior of longitudinal oscillations of a plasma will be summarized in this section. All equations are written in one dimension to facilitate discussions.

In a collisional plasma, quantities like damping coefficient and diffusion coefficient have both contributions from the collective as well as the collisional effects. It is therefore of great physical interest to study the combined and coupled effects by means of a kinetic equation of the Krook and the Fokker-Planck models.

The Krook model is basically a relaxation model which expresses the hypothesis that collisions tend to relax the distribution function to an equilibrium value. The Fokker-Planck model, however, represents collisions by a diffusion term and a frictional term.

In Section III and Section IV of this dissertation, we have attempted to solve the Krook and the Fokker Planck equations in the quasi-linear regime. The basic results can be summarized as follows:

The equation governing the temporal behavior of the electrostatic energy fluctuations is given by

$$\frac{\partial \mathcal{E}_k}{\partial t} = 2 \gamma_k \mathcal{E}_k \tag{6.1}$$

where it is found that the damping coefficient γ_k will combine the collective and collisional effects in a complicate form. As an approximation sum of two parts, it can be written as

$$\gamma_k = \gamma_{kL} + \gamma_{kc}$$

Here, γ_{kL} is the classical Landau damping coefficient without collision given by

$$\gamma_{kL} = \frac{2\pi^2 n e^2}{m k^2} \omega \left. \frac{\partial f_0}{\partial v} \right|_{v = \omega/k} \tag{6.2}$$

This coefficient represents the growth rate or the damping rate due to the collective interaction of waves and particles. It is noted that the sign of γ_{kL} depends only on that of

$$\left. \frac{\partial f_0}{\partial v} \right|_{v = \omega/k}$$

since all other factors are positive. The remaining part γ_{kc} is given by

$$\gamma_{kc} = - \nu \left[1 + 3 \frac{k^2 \overline{v^2}}{\omega_p^2} \right] \tag{6.3}$$

for the Krook model, and

$$\gamma_{kc} = - \frac{\beta}{2} \left[1 + 13 \frac{k^2 \overline{v^2}}{\omega_p^2} \right] \tag{6.4}$$

for the Fokker-Planck model.

Here

$$\overline{v^2} = \int_{-\infty}^{\infty} dv \; v^2 f_0(v,t)$$

, is the mean square velocity and f_0 is the background distribution, which may depend on the wave energy in the present quasi-linear approximation.

There have been a few attempts to calculate the joint effect of collisions and of nonlinearity on Landau damping [15 - 22]. It is found by almost all these authors that, in the long wave length limit, the collisional damping has a k^2 dependence, although they all obtain considerably different numerical results.

Wu and Klevans [15] started with the first two equations in the B.B.G.K.Y. hierarchy, linearized them and substituted an approximate solution into the integral terms. They obtained solution of the form

$$\gamma_{kc} = - \frac{1}{15\pi^{3/2}} \frac{\omega_p^4}{n V_T} \left(\frac{k^2 V_T^2}{\omega_p^2}\right) \ln\left(0.707 \frac{m V_T^2}{e^2 \omega_p}\right)$$

which agree with our results in the k^2 dependence.

Matsuda [16] employed the Rostoker test particle method to obtain solution for the same problem, yielding the result

$$\gamma_{kc} = - \frac{1}{6\pi^{3/2}} \frac{\omega_p^3}{n V_T^3} \left(\frac{k^2 V_T^2}{\omega_p^2}\right) \ln\left(\frac{m V_T^2}{e^2 \omega_p}\right)$$

in which the k^2 dependence is also seen.

The logarithmetic factor is the result of an arbitrary cut-off for the integral to prevent a close collision divergence in the integration. It should be remarked that β in the Fokker-Planck model can be written in the form [10]

$$\beta = 12\pi \frac{n e^4}{m^2 V_T^3} \left(\frac{\omega_p}{\omega}\right)^3 \ln\left(\frac{3\sqrt{2}}{2} \frac{m V_T^2}{e^2 \omega_p}\right)$$

which contains the same logarithmetic factor.

Comisar [17] used a modified Fokker-Planck type term for the collision integral and predicted theoretically that the collision damping is given by

$$\gamma_{kc} = -\frac{\beta}{3\sqrt{\pi}}\left[1 + 3.85 \frac{k^2 V_T^2}{\omega_p^2} \right]$$

which can be compared with our results shown in Eqs. (6.3) and (6.4). It is seen that they agree in algebraic form but different only in the numerical coefficients which is to be expected since different models have been used. The leading terms, which are independent of k , are essentially the electron self-collision frequency due exclusively to the dynamical friction mechanism.

Shanny, Dawson and Greene [18] investigated the collisional damping of longitudinal electron oscillation on a sheet model for a Lorentz plasma. The results of their computer experiment show the k^2 dependence in the above mentioned form to be correct.

The time evaluation of the particle distribution are found to be governed by the following equations:

a) for the Krook model,

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(D_0 \frac{\partial f_0}{\partial v}\right) + \frac{\partial}{\partial v}\left(D_1 \frac{\partial f_0}{\partial v}\right) - \nu \left(f_0 - f_{eq} \right) \qquad (6.5)$$

where

$$D_0 = \frac{8\pi^2 e^2}{m^2} \sum_{k \approx \omega/v} \mathcal{E}_k \, \delta(kv - \omega) \qquad (6.6)$$

$$D_1 = \frac{8\pi e^2}{m^2} \sum_{k \neq \omega/v} \mathcal{E}_k \frac{\gamma_k}{(\omega - kv)^2} \qquad (6.7)$$

b) for the Fokker-Planck model,

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(D_0 \frac{\partial f_0}{\partial v}\right) + \frac{\partial}{\partial v}\left(D_1 \frac{\partial f_0}{\partial v}\right) + \frac{\partial}{\partial v} D_2 + \frac{\partial}{\partial v} S$$

where

$$D_2 = \frac{8\pi e^2}{m^2} \beta \sum_{k \neq \omega/v} \mathcal{E}_k \left\{ \left[\frac{-1}{(\omega-kv)^2} - \frac{kv}{(\omega-kv)^3} - \frac{2RTk^2}{(\omega-kv)^4} \right] \frac{\partial f_0}{\partial v} + \left[\frac{-v}{(\omega-kv)^3} - \frac{2RTk}{(\omega-kv)^3} \right] \frac{\partial^2 f_0}{\partial v^2} + \left[\frac{-RT}{(\omega-kv)^2} \right] \frac{\partial^3 f_0}{\partial v^3} \right\} \tag{6.9}$$

$$S = \beta \left[RT \frac{\partial f_0}{\partial v} + v f_0 \right] \tag{6.10}$$

Here $\partial S/\partial v$ is the original Fokker-Planck collision term. Further D_0 represents the quasi-linear contribution to the diffusion coefficients in the resonance region, while D_1 and D_2 represent the contributions in the non-resonance region. When the collisional coefficient β vanishes, D_2 becomes zero, but D_1 reduces to a finite value for the diffusion in the non-resonant form.

We have investigated in Section V by means of a model, the detailed development of the particle distribution function f_0 and that of the energy fluctuations $\mathcal{E}_k$. It is shown that for any gentle bump with a positive slope on the distribution function, the combined diffusion and damping effects are to wipe it out gradually until it reduces to a negative slope beyond the plateau formation. This is in contradiction with the results [6,7] obtained for a collisionless plasma where a plateau is formed as an asymptotic limit. During the course of time, the energy fluctuation is found to reach zero when the final equilibrium in the particle distribution is reached. This is again more realistic than the simpler treatment [6,7] of collisionless plasmas where the final energy is a large positive constant.

APPENDIX A

In this appendix, we shall prove Eq. (3.8) for the Krook model and Eq. (4.13) for the Fokker-Planck model.

For the Krook model, we differentiate Eq. (3.7) with respect to t, which appears in both the integrand and in the upper limit, and obtain

$$\frac{\partial}{\partial t}\left\langle \vec{E}_{-k}(t)\cdot\vec{E}_{k}(t)\right\rangle = I_A + I_B + I_C$$

where

$$I_A = i\,\frac{\omega_p^2}{k^2}\int d^3v\,\left\langle \vec{E}_{-k}(t)\cdot\vec{E}_{k}(t)\right\rangle \vec{k}\cdot\nabla_v f_0(\vec{v},t)$$

$$I_B = i\,\frac{\omega_p^2}{k^2}\int d^3v\int_{-\infty}^{t} d\tau\, e^{-(i\vec{k}\cdot\vec{v}+\nu)(t-\tau)}\,\frac{\partial}{\partial t}\left\langle \vec{E}_{k}(t)\cdot\vec{E}_{k}(\tau)\right\rangle \vec{k}\cdot\nabla_v f_0(\vec{v},\tau)$$

$$I_C = i\,\frac{\omega_p^2}{k^2}\int d^3v\int_{-\infty}^{t} d\tau\, e^{-(i\vec{k}\cdot\vec{v}+\nu)(t-\tau)}\left\langle \vec{E}_{k}(t)\cdot\vec{E}_{k}(\tau)\right\rangle \left(i\vec{k}\cdot\vec{v}+\nu\right)\vec{k}\cdot\nabla_v f_0(\vec{v},\tau)$$

Now since

$$\int_{-\infty}^{\infty} d^3v\;\vec{k}\cdot\nabla_v f_0(\vec{v},t) = 0$$

we have

$$I_A = 0$$

We now assume that $<\vec{E}_{-k}(t)\cdot\vec{E}_k(\tau)>$ is so slowly varying in time that

$$\left|\frac{1}{\langle\vec{E}_{-k}(t)\cdot\vec{E}_k(t)\rangle}\frac{\partial}{\partial t}\langle\vec{E}_{-k}(t)\cdot\vec{E}_k(\tau)\rangle\right| \ll \left|\frac{1}{e^{-(i\vec{k}\cdot\vec{v}+\omega)(t-\tau)}}\frac{\partial}{\partial t}e^{-(i\vec{k}\cdot\vec{v}+\omega)(t-\tau)}\right|$$

Consequently,

$$I_s \ll I_c$$

Thus we are left with

$$\frac{\partial}{\partial t}\langle\vec{E}_k(t)\cdot\vec{E}_k(t)\rangle = I_c$$

which is equivalent to Eq. (3.8).

For the Fokker-Planck model, we first change the independent variable from s to $t = s+\tau$ in Eq. (4.12) and then differentiate this equation with respect to t. With similar arguments with which we proved Eq. (3.8), we will readily arrive at Eq. (4.13).

APPENDIX B

We shall prove that electrostatic energy of wave decays at twice the rate of damping of momentum.

We assume solution of the form

$$F_k(\vec{v},t) = F_k(\vec{v},0)\, e^{-i\Omega t}$$

for the fluctuation of the distribution function. Then, Poisson's equation can be written as

$$E_k(t) = \frac{4\pi m e}{k}\, i\, e^{-i\Omega t} \int_{-\infty}^{\infty} d^3v \;\; F_k(\vec{v},0)$$

On taking absolute value and squaring each side of this equation, with Ω substituted by its real and imaginary parts given by $\Omega = \omega + i\gamma_k$, we have

$$\left| E_k \right|^2 = \frac{16\pi^2 m^2 e^2}{k^2}\, e^{2\gamma_k t} \left| \int_{-\infty}^{\infty} d^3v \;\; F_k(\vec{v},0) \right|^2$$

Differentiating this equation with respect to t, we obtain

$$\frac{\partial}{\partial t} \left| E_k \right|^2 = 2\gamma_k \left| E_k \right|^2$$

Now, the electrostatic energy in the kth mode is given by

$$\mathcal{E}_k(t) = \frac{\left| E_k(t) \right|^2}{8\pi}$$

Hence

$$\frac{\partial \mathcal{E}_k}{\partial t} = 2\gamma_k\, \mathcal{E}_k$$

APPENDIX C

We shall evaluate the integral given by Eq. (3.9).

Employing the standard integral

$$\int_0^{\infty} dt\; e^{-\alpha t - i\lambda t} = \frac{1}{\alpha + i\lambda}$$

the integration of s in Eq. (3.9) can be performed readily, yielding

$$\gamma_k = \frac{1}{2}\frac{\omega_p^2}{k^2}\int d^3v\,(i\vec{k}\cdot\vec{v} + \nu)(\vec{k}\cdot\nabla_v f_0)\,\frac{1}{(\omega - \vec{k}\cdot\vec{v}) + i\nu}$$

As ν is assumed to be small, we may apply the Dirac Formula

$$\lim_{\zeta \to 0}\int_{-\infty}^{\infty}\frac{F(\theta)\,d\theta}{\theta + i\zeta} = P\int_{-\infty}^{\infty}\frac{F(\theta)\,d\theta}{\theta + i\zeta} - \pi i\int_{-\infty}^{\infty}F(\theta)\,\delta(\theta)\,d\theta$$

Consequently

$$\gamma_k = \frac{1}{2}\frac{\omega_p^2}{k^2}\,P\int_{-\infty}^{\infty} d^3v\,\frac{(i\vec{k}\cdot\vec{v} + \nu)\,\vec{k}\cdot\nabla_v f_0}{(\omega - \vec{k}\cdot\vec{v}) + i\nu} + \frac{\pi}{2}\frac{\omega_p^2}{k^2}\int d^3v\,(\vec{k}\cdot\nabla_v f)\,\vec{k}\cdot\vec{v}\,\delta(\vec{k}\cdot\vec{v} - \omega)$$

Taking the real part of this expression

$$\gamma_k = \gamma_{kL} + \gamma_{kC}$$

$$\gamma_{kL} = \frac{\pi}{2}\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} d^3v\,(\vec{k}\cdot\nabla_v f_0)\,\vec{k}\cdot\vec{v}\;\delta(\vec{k}\cdot\vec{v} - \omega)$$

$$\gamma_{kC} = \frac{\nu}{2}\frac{\omega_p^2}{k^2}\,P\int_{-\infty}^{\infty} d^3v\;\frac{\omega(\vec{k}\cdot\nabla_v f_0)}{(\omega - \vec{k}\cdot\vec{v})^2}$$

The last expression can be integrated by parts to yield

$$\gamma_{kc} = -\nu \omega_p^2 P \int_{-\infty}^{\infty} d^3v \frac{\omega f_0}{(\omega - \vec{k}\cdot\vec{v})^3}$$

With the assumption that the average value $\overline{V^2}$ of the square of the particle velocity is much smaller than the ratio ω^2/k^2, this expression can be expanded to give

$$\gamma_{kc} = -\nu \left(\frac{\omega_p}{\omega}\right)^2 \int_{-\infty}^{\infty} d^3v \, f_0(\vec{v}) \left[1 + 3\left(\frac{\vec{k}\cdot\vec{v}}{\omega}\right) + 6\left(\frac{\vec{k}\cdot\vec{v}}{\omega}\right)^2 + \cdots\right]$$

For isotropic distribution f_o, this reduces to

$$\gamma_{kc} = -\nu \left(\frac{\omega_p}{\omega}\right)^2 \left[1 + 6\frac{k^2\overline{v^2}}{\omega^2}\right]$$

Now, it can be readily shown that the dispersion relation for a Krook plasma is approximately given by

$$\omega^2 = \omega_p^2 + 3k^2\overline{v^2}$$

Hence, the expression for γ_{kc} can be rewritten as

$$\gamma_{kc} = -\nu \left[1 + 3\frac{k^2\overline{v^2}}{\omega_p^2}\right]$$

APPENDIX D

We shall evaluate the integral given by Eq. (3.14), namely

$$I_D = \int_0^\infty ds\, e^{i(\Omega - \vec{k}\cdot\vec{v})s - \nu s}$$

ν can be absorbed in Ω so that the integral reads

$$I_D = \int_0^\infty ds\, e^{i(\Omega' - \vec{k}\cdot\vec{v})s}$$

where

$$\Omega' = \omega + i\gamma_k + i\nu$$

Now, the integral I_D can be integrated readily for stable plasmas as

$$I_D = -\frac{1}{i(\Omega' - \vec{k}\cdot\vec{v})}$$

On applying Dirac Formula to this expression, we have, formally

$$I_D = \frac{-1}{i(\Omega' - \vec{k}\cdot\vec{v})} = P\,\frac{-1}{i(\Omega' - \vec{k}\cdot\vec{v})} - \pi\,\delta(\vec{k}\cdot\vec{v} - \omega)$$

Consequently, the real value of $\overleftrightarrow{D}$ in Eq. (3.14) is given by

$$\overleftrightarrow{D} = \frac{8\pi^2 e^2}{m^2}\sum_{k=\omega/v}\frac{\vec{k}\vec{k}}{k^2}\,\mathcal{E}_k\,\delta(\vec{k}\cdot\vec{v} - \omega)$$

$$+ \frac{8\pi e^2}{m^2}\sum_{k\neq\omega/v}\frac{\vec{k}\vec{k}}{k^2}\,\mathcal{E}_k\,\frac{\gamma_k + \nu}{(\omega - \vec{k}\cdot\vec{v})^2}$$

APPENDIX E

We shall solve, by the method of characteristics, the equation

$$\frac{\partial \hat{G}}{\partial t} + (\beta \vec{p} - \vec{k}) \cdot \frac{\partial \hat{G}}{\partial \vec{p}} + P\hat{G} = Q \tag{E1}$$

where

$$P = \frac{RT}{\beta} k^2$$

$$Q(\vec{k}, \vec{p}, t) = \frac{ie}{m} \frac{\vec{p} \cdot \vec{k}}{k} E_k \hat{f}_0 \, e^{\frac{1}{2}RTp^2 + \frac{RT}{\beta} \vec{p} \cdot \vec{k}}$$

By the method of characteristics, a solution to Eq. (E1) is equivalent to one to the set

$$\frac{dt}{1} = \frac{dp_x}{\beta p_x - k_x} = \frac{dp_y}{\beta p_y - k_y} = \frac{dp_z}{\beta p_z - k_z}$$

$$= \frac{d\hat{G}}{Q - P\hat{G}} \tag{E2}$$

where p_x, p_y, p_z, are the coordinate components of $\vec{p}$, and k_x, k_y, k_z, those for $\vec{k}$.

From Eq. (E2), we may extract a set of four equations:

$$dt = \frac{dp_x}{\beta p_x - k_x} \tag{E3}$$

$$dt = \frac{dp_y}{\beta p_y - k_y} \tag{E4}$$

$$dt = \frac{dp_z}{\beta p_z - k_z} \tag{E5}$$

$$dt = \frac{d\hat{G}}{Q - P\hat{G}} \tag{E6}$$

The solution of Eq. (E3) may be obtained by integrating over appropriate limits.

If, at $t = \tau$, $\vec{p}$ attains the value $\vec{p}_\tau$, then, from Eq. (E3), we have

$$\int_{\tau}^{t} dt = \int_{p_{x\tau}}^{p_x} \frac{dp_x}{\beta p_x - k_x}$$

Performing the integration explicitly,

$$(t - \tau) = \frac{1}{\beta} \ln (\beta p_x - k_x) \Big|_{p_x = p_{x\tau}}^{p_x = p_x}$$

Consequently,

$$p_{x\tau} = (p_x - \frac{k_x}{\beta}) e^{-\beta(t-\tau)} + \frac{k_x}{\beta} \qquad \text{(E7)}$$

Similarly, if Eqs. (E4) and (E 5) are integrated, we shall obtain, respectively,

$$p_{y\tau} = (p_y - \frac{k_y}{\beta}) e^{-\beta(t-\tau)} + \frac{k_y}{\beta} \qquad \text{(E8)}$$

$$p_{z\tau} = (p_z - \frac{k_z}{\beta}) e^{-\beta(t-\tau)} + \frac{k_z}{\beta} \qquad \text{(E9)}$$

Combining Eqs. (E7), (E8) and (E9) gives

$$\vec{p}_\tau = (\vec{p} - \frac{\vec{k}}{\beta}) e^{-\beta(t-\tau)} + \frac{\vec{k}}{\beta} \qquad \text{(E10)}$$

We shall next solve Eq. (E6). It is obvious that this is a linear equation having the solution

$$\hat{G}(\vec{k},\vec{p},t) = e^{-I}\left[\int^{\tau=t} d\tau \; Q(\vec{k},\vec{p}_\tau,\tau)\, e^{+I} + C\right] \tag{E11}$$

where I is the integration factor given by

$$I = \int^{t} P \, dt \tag{E12}$$

and C is the constant of integration.

It is noted that $\vec{p}$ and t in the argument of Q have been replaced by $\vec{p}_\tau$ and τ respectively since we are integrating along the characteristic curve.

Integrating Eq. (E12) explicitly, we have

$$I = \frac{RT}{\beta} k^2 t \tag{E13}$$

Substituting Eq. (E13) into Eq. (E11), we find

$$\hat{G}(\vec{k},\vec{p},t)\, e^{\frac{RT}{\beta}k^2 t} = \int^{\tau=t} d\tau \;\; Q(\vec{k},\vec{p}_\tau,\tau)\, e^{\frac{RT}{\beta}k^2 \tau} + C$$

Now, we have, at the point t_0 on the characteristic curve,

$$C = \hat{G}(\vec{k},\vec{p}_{t_0},t_0)\, e^{\frac{RT}{\beta}k^2 t_0} - \int^{\tau=t_0} d\tau \; Q(\vec{k},\vec{p}_\tau,\tau)\, e^{\frac{RT}{\beta}k^2 t}$$

Therefore

$$\hat{G}(\vec{k},\vec{p},t) = e^{-\frac{kT}{\zeta}k^2 t}\int_{t_0}^{t} d\tau \; Q(\vec{k},\vec{p}_\tau,\tau)\, e^{\frac{kT}{\zeta}k^2\tau}$$

$$+ \; \hat{G}(\vec{k},\vec{p}_0,t_0)\, e^{-\frac{kT}{\zeta}k^2(t-t_0)} \qquad \text{(E14)}$$

If $t_0 \rightarrow \infty$, the last term of Eq. (E14) vanishes, and the equation becomes

$$\hat{G}(\vec{k},\vec{p},t) = e^{-\frac{kT}{\zeta}k^2 t}\int_{-\infty}^{t} d\tau \; Q(\vec{k},\vec{p}_\tau,\tau)\, e^{\frac{kT}{\zeta}k^2\tau} \qquad \text{(E15)}$$

APPENDIX F

We perform the integration over p given by Eq. (4.9), namely

$$I = \frac{i}{(2\pi)^3}\int_{-\infty}^{\infty} d^3p\, e^{i\vec{p}\cdot\vec{v}}\, \hat{f}_0(\vec{p})\, \vec{p}\cdot\vec{k}\left[1-\beta RT\left(p^2 s+\vec{p}\cdot\vec{k}s^2+\tfrac{1}{3}k^2s^3\right)\right] \tag{F1}$$

where

$$\vec{p}\,' = \vec{p}\,(1-\beta s) + \vec{k}\,\left(1-\frac{\beta s}{2}\right)s$$

On changing the independent variable from p to p' and linearizing with respect to β, we obtain, after a little algebraic manipulation,

$$I = e^{-i\vec{k}\cdot\vec{v}s}\int_{-\infty}^{\infty} d^3p\, e^{i\vec{v}\cdot\vec{p}\,'}\, f_0(\vec{p}\,')\, \vec{k}\cdot\vec{p}\,' \left\{\left[1+3\beta s-\tfrac{1}{3}\beta RTk^2s^3 - i\tfrac{1}{2}\beta\vec{k}\cdot\vec{v}s^2\right] + \left[\beta RTs^2\vec{k}\cdot\vec{p}\,' + i\beta s\vec{v}\cdot\vec{p}\,'\right] - \beta RTs\,p'^2\right\} \tag{F2}$$

Now, in order to perform this integration, we shall use the Fourier transform formula

$$\phi(\vec{v}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3p\, e^{i\vec{v}\cdot\vec{p}}\, \hat{\phi}(\vec{p})$$

from which the following formulas can be readily obtained by differentiations

$$-i(\vec{k}\cdot\nabla_v\phi) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3p\, e^{i\vec{v}\cdot\vec{p}}\,\vec{k}\cdot\vec{p}\;\hat{\phi}(\vec{p})$$

$$-\vec{v}\cdot\nabla_v(\vec{k}\cdot\nabla_v\phi) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3p\, e^{i\vec{v}\cdot\vec{p}}\,(\vec{k}\cdot\vec{v})(\vec{k}\cdot\vec{p})\,\hat{\phi}(\vec{p})$$

$$-\vec{k}\cdot\nabla_v(\vec{k}\cdot\nabla_v\phi) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3p\, e^{i\vec{v}\cdot\vec{p}}\,(\vec{k}\cdot\vec{p})^2\,\hat{\phi}(\vec{p})$$

$$i\,\nabla_v^2(\vec{k}\cdot\nabla_v\phi) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d^3p\, e^{i\vec{v}\cdot\vec{p}}\,p^2(\vec{k}\cdot\vec{p})\,\hat{\phi}(\vec{p})$$

By applying these results to Eq. (F 4), we obtain immediately

$$I = -\,e^{-i\vec{k}\cdot\vec{v}s}\;\Phi(s)$$

where

$$\begin{aligned}\Phi(s) = {} & i(\vec{k}\cdot\nabla_v f_0)\left(1 + 3\beta s - \tfrac{1}{3}\beta RT k^2 s^3\right) \\ & + (\vec{k}\cdot\nabla_v f_0)\left(\tfrac{1}{2}\beta\,\vec{k}\cdot\vec{v}s\right) + \vec{k}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_0)(\beta RT s^2) \\ & + i\,\vec{v}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_0)(\beta s) + i\,\nabla_v^2(\vec{k}\cdot\nabla_v f_0)(\beta RT s)\end{aligned}$$

APPENDIX G

We shall evaluate the integral given by Eq. (4.14), namely

$$\gamma_k = \frac{1}{2}\frac{\omega_k^2}{k^2}\int d^3v \int_0^\infty ds\, e^{i(\omega - \vec{k}\cdot\vec{v})s}\, \Phi_1(s) \tag{G1}$$

We assume that f_0 is so slowly varying in time that it can be taken out in front of the s integral.

In order to perform this integration, we resort to the Dirac formula

$$\lim_{\zeta\to 0}\int_{-\infty}^{\infty}\frac{F(\theta)d\theta}{\theta + i\zeta} = \lim_{\zeta\to 0} P\int_{-\infty}^{\infty}\frac{F(\theta)d\theta}{\theta + i\zeta} - \pi i\int_{-\infty}^{\infty}F(\theta)\delta(\theta)d\theta \tag{G2}$$

the standard integral

$$\int_0^\infty ds\, e^{i(\omega - \vec{k}\cdot\vec{v})s} = \frac{1}{i(\vec{k}\cdot\vec{v} - \omega)} \tag{G3}$$

and the identity

$$\int_0^\infty ds\, s^n e^{-i\lambda s} = (i)^n\frac{\partial^n}{\partial\lambda^n}\int_0^\infty ds\, e^{-i\lambda s} \tag{G4}$$

Combining Eqs. (G2), (G3) and (G4), we have

$$\int_{-\infty} d^3v\, F(v)\int_0^\infty ds\, s^n e^{i(\omega - \vec{k}\cdot\vec{v})s}$$

$$= (-i)^{n+1} P\int_{-\infty}^{\infty} d^3v\,\frac{n!\,F(\vec{v})}{(\vec{k}\cdot\vec{v} - \omega)^{n+1}} + (i)^n\pi\int_{-\infty}^{\infty} dv\,F(\vec{v})\,\delta^n(\vec{k}\cdot\vec{v} - \omega) \tag{G5}$$

This formula can be applied directly to Eq. (G1). Since γ_k is real, we keep only the real values. It is found that the only real part containing δ-function is

$$\gamma_{kL} = \frac{\pi}{2}\frac{\omega_p^2}{k^2}\int d^3v\,(\vec{k}\cdot\nabla_v f_0)\,\vec{k}\cdot\vec{v}\;\delta(\vec{k}\cdot\vec{v}-\omega) \tag{G6}$$

while the real part containing the principal values are

$$\begin{aligned}\gamma_{kc} = \frac{\beta}{2}\frac{\omega_p^2}{k^2} P\int d^3v \Bigg\{ &(\vec{k}\cdot\nabla_v f_0)\left[\frac{-4\,\vec{k}\cdot\vec{v}}{(\vec{k}\cdot\vec{v}-\omega)^2} - \frac{2RT\,k^2(\vec{k}\cdot\vec{v})}{(\vec{k}\cdot\vec{v}-\omega)^4}\right] \\ &+(\vec{k}\cdot\nabla_v f_0)\left[\frac{3}{(\vec{k}\cdot\vec{v}-\omega)} + \frac{2RTk^2}{(\vec{k}\cdot\vec{v}-\omega)^3} + \frac{(\vec{k}\cdot\vec{v})^2}{(\vec{k}\cdot\vec{v}-\omega)^3}\right] \\ &+\vec{k}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_0)\left[\frac{-2RT}{(\vec{k}\cdot\vec{v}-\omega)^2} + \frac{2RT(\vec{k}\cdot\vec{v})}{(\vec{k}\cdot\vec{v}-\omega)^3}\right] \\ &+\vec{v}\cdot\nabla_v(\vec{k}\cdot\nabla_v f_0)\left[\frac{-\vec{k}\cdot\vec{v}}{(\vec{k}\cdot\vec{v}-\omega)^2} + \frac{1}{(\vec{k}\cdot\vec{v}-\omega)}\right] \\ &+\nabla_v^2(\vec{k}\cdot\nabla_v f_0)\left[\frac{-RT(\vec{k}\cdot\vec{v})}{(\vec{k}\cdot\vec{v}-\omega)^2} + \frac{RT}{(\vec{k}\cdot\vec{v}-\omega)}\right]\Bigg\}\end{aligned} \tag{G7}$$

Hence

$$\gamma_k = \gamma_{kL} + \gamma_{kc} \tag{G8}$$

The expression for γ_{kc} is considerably simplified if f_0 is given Maxwellian in one dimension

$$f_0 = f_m = \frac{1}{\sqrt{2\pi}\, V_T}\, e^{-V^2/2V_T^2}$$

where V_T is the thermal velocity given by

$$V_T^2 = \frac{kT}{m}$$

All derivatives of f_0 can now be expressed in terms of f_m, and for small k, rational functions can be expanded in series. Further applying the formulas

$$\int_{-\infty}^{\infty} dv\, f_m = 1$$

$$\int_{-\infty}^{\infty} dv\, v^m f_m = (2m-1)!!\, V_T^{2m}$$

We finally simplify Eq. (G7) to

$$\gamma_{kc} = -\frac{\beta}{2}\left(\frac{\omega_e}{\omega}\right)^2 \left[1 + 16\, \frac{k^2 V_T^2}{\omega^2} \right] \tag{G9}$$

where only terms up to the second order of k have been kept.

APPENDIX H

In this section, we shall derive the dispersion relation of the plasma using the Fokker-Planck model.

We put

$$\vec{E}_k(t) = \vec{E}^\circ_k \, e^{-i\Omega t} - \vec{E}^\circ_k \, e^{-i\omega t + \gamma_k t}$$

in Eq. (4.11), obtaining

$$1 = \frac{\omega_p^2}{k^2} \int d^3v \int_0^\infty ds \; e^{i(\Omega - \vec{k}\cdot\vec{v})s} \; \Phi(s) \tag{H1}$$

where Φ (s) is given by Eq. (4.10).

Taking k to be real, Eq. (H1) gives a relation between ω and k:

$$\begin{aligned} 1 = \frac{\omega_p^2}{k^2} \int d^3v \Big\{ & (\vec{k}\cdot\nabla f_0) \Big[\frac{1}{(\vec{k}\cdot\vec{v}-\omega)} - 3\beta \mathbb{T} \delta'(\vec{k}\cdot\vec{v}-\omega) \\ & - \frac{1}{3}\beta R T k^2 \mathbb{T} \delta'''(\vec{k}\cdot\vec{v}-\omega) - \frac{1}{2}\beta \vec{k}\cdot\vec{v} \, \mathbb{T} \delta''(\vec{k}\cdot\vec{v}-\omega) \Big] \\ & + \vec{k}\cdot\nabla(\vec{k}\cdot\nabla f_0) \Big[-\beta R T \mathbb{T} \delta''(\vec{k}\cdot\vec{v}-\omega) \Big] \\ & + \vec{v}\cdot\nabla(\vec{k}\cdot\nabla f_0) \Big[-\beta \mathbb{T} \delta'(\vec{k}\cdot\vec{v}-\omega) \Big] \\ & + \nabla_v^2(\vec{k}\cdot\nabla_v f_0) \Big[-\beta R T \delta'(\vec{k}\cdot\vec{v}-\omega) \Big] \Big\} \end{aligned}$$

Part of this integration can be performed explicitly since the Dirac delta-function appears in some of the terms. We shall perform the integration and write the result in one dimension for simplicity

$$1 = \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} dv \, \frac{\left(\frac{\partial f_0}{\partial v}\right)}{kv-\omega}$$

$$+ \beta \pi \frac{\omega_p^2}{k^3} \frac{\partial^2 f_0}{\partial v^2} + \beta \frac{\pi}{2} \frac{\omega_p^2 v}{k^3} \frac{\partial^3 f_0}{\partial v^3} + \beta \frac{\pi}{3} \frac{\omega_p^2 kT}{k^3} \frac{\partial^4 f_0}{\partial v^4}$$

We take f_0 to be Maxwellian

$$f_0 = f_m = \frac{1}{\sqrt{2\pi}\, V_T} e^{-v^2/2V_T^2}$$

and express all derivatives of f_0 in terms of f_m.

Consequently, up to second order of k, we have

$$1 = \frac{\omega_p^2}{\omega^2}\left[1 + \frac{3k^2 V_T^2}{\omega^2}\right] + \beta\sqrt{\frac{\pi}{2}}\left(\frac{\omega_p^2}{k^5 V_T}\right)\left[\frac{1}{2}\frac{\omega^2}{k^2 V_T^2} - \frac{1}{6}\frac{\omega^4}{k^4 V_T^4}\right] e^{-\omega^2/2k^2V_T^2}$$

Now, it can be shown readily that the second term on the right hand side is much smaller than the first for small k, hence the dispersion relation is given approximately by

$$1 = \frac{\omega_p^2}{\omega^2}\left[1 + \frac{3k^2 V_T^2}{\omega^2}\right]$$

Solving this relation for ω, we finally have

$$\omega^2 = \omega_p^2\left[1 + \frac{3k^2 V_T^2}{\omega_p^2}\right]$$

APPENDIX J

We shall evaluate the integral given by Eq. (4.21), namely

$$\left\langle \vec{E}_k(t) F_k(\vec{v},t) \right\rangle = \frac{8\pi e \vec{k}}{m k^2} \mathcal{E}_k(t) \int_0^\infty ds \; e^{i(\omega - \vec{k}\cdot\vec{v})s - \gamma_k s} \left[- i \Phi(s) \right] \tag{J1}$$

Again, we assume that f_0 is so slowly varying in time that it can be taken out in front of the s integral.

To perform the integration, we shall make use of the following formulas

$$\lim_{\zeta \to 0} \int_{-\infty}^{\infty} \frac{F(\theta)d\theta}{\theta - i\zeta} = \lim_{\zeta \to 0} P \int_{-\infty}^{\infty} \frac{F(\theta)d\theta}{\theta - i\zeta} + \pi i \int_{-\infty}^{\infty} F(\theta)\, \delta(\theta)\, d\theta \tag{J2}$$

$$\int_0^\infty ds \; e^{-i\lambda s - \gamma s} = - \frac{i}{\lambda - i\gamma} \tag{J3}$$

$$\int_0^\infty ds \; s^n \, e^{-i\lambda s - \gamma s} = (i)^n \frac{\partial^n}{\partial \lambda^n} \int_0^\infty ds \; e^{-i\lambda s - \gamma s} \tag{J4}$$

Combination of Eqs. (J2), (J3), and (J4) yields

$$\int_0^\infty ds \, s^n \, e^{-i\lambda s - \gamma s} = P \frac{(-i)^{n+1} \, n!}{(\lambda - i\gamma)^{n+1}} + \pi (i)^n \, \delta^n(\lambda) \tag{J5}$$

This expression can be applied directly to Eq. (J1), yielding the real result

$$\left\langle \vec{E}_{-k}(t)\, F_k(\vec{v},t) \right\rangle = \frac{8\pi e \vec{k}}{m k^2}\, \mathcal{E}_k(t) \Bigg\{ (\vec{k}\cdot\nabla_v f_0)\, \pi\, \delta(\vec{k}\cdot\vec{v}-\omega)$$

$$+ (\vec{k}\cdot\nabla_v f_0)\, R_e \left[\frac{-i}{[(\vec{k}\cdot\vec{v}-\omega)-i\gamma_k]} - \frac{3\beta}{[(\vec{k}\cdot\vec{v}-\omega)-i\gamma_k]^2} \right.$$

$$- \frac{2\beta R T k^2}{[(\vec{k}\cdot\vec{v}-\omega)-i\gamma_k]^4} + \frac{\beta\, \vec{k}\cdot\vec{v}}{[(\vec{k}\cdot\vec{v}-\omega)-i\gamma_k]^2}$$

$$+ \vec{k}\cdot\nabla_v (\vec{k}\cdot\nabla_v f_0)\, R_e \left[\frac{2\beta R T}{[(\vec{k}\cdot\vec{v}-\omega)-i\gamma_k]^3} \right]$$

$$+ \vec{v}\cdot\nabla_v (\vec{k}\cdot\nabla_v f_0)\, R_e \left[\frac{-\beta}{[(\vec{k}\cdot\vec{v}-\omega)-i\gamma_k]^2} \right]$$

$$+ \nabla_v^2 (\vec{k}\cdot\nabla_v f_0)\, R_e \left[\frac{-\beta R T}{[(\vec{k}\cdot\vec{v}-\omega)-i\gamma_k]^2} \right]$$

For small γ_k compared with ω, this expression readily reduces to

$$\left\langle \vec{E}_k(t)\, F(\vec{v},t) \right\rangle = \frac{8\pi e \vec{k}}{m k^2}\, \Theta\, \mathcal{E}_k(t)$$

where

$$\Theta = (\vec{k}\cdot\nabla_v f_0) \left[\pi\, \delta(\vec{k}\cdot\vec{v}-\omega) + \frac{\gamma_k}{(\omega-\vec{k}\cdot\vec{v})^2} - \frac{3\beta}{(\omega-\vec{k}\cdot\vec{v})^2} - \frac{\beta\, \vec{k}\cdot\vec{v}}{(\omega-\vec{k}\cdot\vec{v})^3} \right.$$

$$\left. - \frac{2\beta R T k^2}{(\omega-\vec{k}\cdot\vec{v})^4} \right] + \vec{k}\cdot\nabla_v (\vec{k}\cdot\nabla_v f_0) \left[\frac{-2\beta R T}{(\omega-\vec{k}\cdot\vec{v})^3} \right]$$

$$+ \vec{v}\cdot\nabla_v (\vec{k}\cdot\nabla_v f_0) \left[\frac{-\beta}{(\omega-\vec{k}\cdot\vec{v})^2} \right] + \nabla_v^2 (\vec{k}\cdot\nabla_v f_0) \left[\frac{-\beta R T}{(\omega-\vec{k}\cdot\vec{v})^2} \right]$$

APPENDIX K

In this appendix, we attempt to solve the following equations

$$f_0(v, t) = A(t)\sin\left[\frac{\pi}{2}\left(\frac{v-v_0}{u}\right)\right] + g_0 \tag{K1}$$

$$\begin{aligned}\frac{dA}{dt} &= -\frac{\pi^2}{4u^2}\left[\lambda\frac{\mathcal{E}(v_0,t)}{v_0} + \beta RT\right]A \\ &\quad - \beta\frac{\pi}{2}\left[\frac{v_0}{u}-1\right]A - \beta\frac{\pi}{2}g_0\end{aligned} \tag{K2}$$

$$\frac{\partial\mathcal{E}}{\partial t} = \alpha v\,\mathcal{E}\,\frac{\pi}{2u}\cos\left[\frac{\pi}{2}\left(\frac{v-v_0}{u}\right)\right]A - \gamma_0\mathcal{E} \tag{K3}$$

In virtue of the smallness of β and u, the function $\mathcal{E}(V_0, t)$ appearing in Eq. (K2) can be solved by combining Eqs. (K2) and (K3) to give

$$\mathcal{E}(v_0, t) = -\frac{\alpha}{\lambda}\frac{\pi}{2}v_0^2 u\,(A-A_0) + \mathcal{E}(v_0, 0) \tag{K4}$$

When this is substituted back into Eq. (K2) we have

$$\begin{aligned}\frac{dA}{dt} &= \frac{\pi}{2u}\alpha v_0 A^2 + \left\{-\frac{\pi}{2u}\alpha v_0 A_0 - \frac{\pi^2}{4u^2}\frac{\lambda}{v_0}\mathcal{E}_{00}\right. \\ &\quad \left. - \frac{\pi^2}{4u^2}\beta RT - \frac{\pi}{2u}\beta v_0\right\}A - \frac{\pi}{2}\beta g_0\end{aligned} \tag{K5}$$

where

$$A_0 = A(0)$$

and

$$\mathcal{E}_{00} = \mathcal{E}(v_0, 0)$$

To evaluate A(t) in Eq. (K5), it is necessary to know the roots to the auxiliary algebraic equation

$$\theta_1 A^2 + \theta_2 A + \theta_3 = 0 \tag{K6}$$

with

$$\theta_1 = \frac{I}{2\mu} \alpha V_0$$

$$\theta_2 = -\frac{I}{2\mu} \alpha V_0 A_0 - \frac{I^2}{4\mu^2} \frac{\lambda}{V_0} \varepsilon_\infty - \frac{I^2}{4\mu^2} \rho RT - \frac{I}{2\mu} \rho V_0$$

$$\theta_3 = -\frac{I}{2} \rho g_0$$

The two roots of Eq. (K6) are given by

$$p = \frac{-\theta_2 + \sqrt{\theta_2^2 - 4\theta_1\theta_3}}{2\theta_1} \tag{K7}$$

$$q = \frac{-\theta_2 - \sqrt{\theta_2^2 - 4\theta_1\theta_3}}{2\theta_1} \tag{K8}$$

Knowing the values of p and q, Eq. (K5) can be solved by quadrature to give

$$\frac{1}{\theta_1(p-q)} \ln \left| \frac{A-p}{A-q} \right| + \theta_0 = t$$

where θ_0 is the constant of integration.

By inserting the initial conditions this equation can be rewritten in the form

$$A(t) = (p-q)\left[\frac{A_0-q}{A_0-p}\, e^{-\theta_1(p-q)t} - 1\right]^{-1} + p \tag{K9}$$

To obtain an expression for $\mathcal{E}$ (v, t), we may substitute Eq. (K9) into Eq. (K3) to obtain

$$\frac{\partial \mathcal{E}}{\partial t} = \alpha v \frac{\pi}{2u} \cos\left[\frac{\pi}{2}\left(\frac{v-V_0}{u}\right)\right] \mathcal{E} \left\{ (p-q)\left[\frac{A_0-q}{A_0-p}\, e^{-\theta_1(p-q)t} - 1\right]^{-1} + p \right\} - \gamma_0 \mathcal{E}$$

Again, this equation can be integrated by quadrature to yield

$$\mathcal{E}(v,t) = \frac{C(v)\, e^{\alpha v\left[\frac{\pi}{2u}\cos\left(\frac{\pi}{2}\frac{v-V_0}{u}\right)\right] q t - \gamma_0 t}}{\left| \frac{A_0-q}{A_0-p}\, e^{-\theta_1(p-q)t} - 1 \right|^{\frac{V}{V_0}\cos\left(\frac{\pi}{2}\frac{V-V_0}{u}\right)}}$$

where C(V) is an arbitrary function of V resulting from integration.

If the initial values of $\mathcal{E}$ (v, t) are given for all V as $\mathcal{E}$ (v, o), the latter can be inserted as initial conditions so that the equation can be rewritten as

$$\mathcal{E}(v,t) = \frac{\mathcal{E}(v,0)\left[\frac{A_0-q}{A_0-p} - 1\right]^{\frac{V}{V_0}\cos\left(\frac{\pi}{2}\frac{V-V_0}{u}\right)} e^{\alpha v \frac{\pi}{2}\cos\left(\frac{\pi}{2}\frac{V-V_0}{u}\right) q t - \gamma_0 t}}{\left[\frac{A_0-q}{A_0-p}\, e^{-\theta_1(p-q)t} - 1\right]^{\frac{V}{V_0}\cos\left(\frac{\pi}{2}\frac{V-V_0}{u}\right)}} \tag{K10}$$

Equation (K10), together with Eq. (K9) form the basic set of expressions describing the evolution in time of the distribution function in the trapping interval $(V_0 - u,\ V_0)$ and the corresponding energy fluctuation in the same region.

The slope of the distribution function is given by

$$\frac{\partial f_0}{\partial V} = A(t)\,\frac{\pi}{2u}\cos\left[\frac{\pi}{2}\left(\frac{V-V_0}{u}\right)\right] \tag{K11}$$

$$= \left\{(p-q)\left[\frac{A_0-q}{A_0-p}\,e^{-\theta_1(p-q)} - 1\right]^{-1} + p\right\}\frac{\pi}{2u}\cos\left[\frac{\pi}{2u}\left(\frac{V-V_0}{u}\right)\right]$$

At t = 0, Eq. (K11) reduces to

$$\left.\frac{\partial f_0}{\partial V}\right|_{t=0} = A_0\cos\left[\frac{\pi}{2}\left(\frac{V-V_0}{u}\right)\right]\frac{\pi}{2u}$$

which is the slope in the trapping interval at the initial time. This is positive since $\cos\left(\frac{V-V_0}{u}\right)\frac{\pi}{2}$ is bounded between 0 and 1 in this interval, and A_0 is given positive.

Let us now investigate the time–asymptotic behavior of $\partial f/\partial V$ and $\mathcal{E}$.

From Eqs. (K7) and (K8), it is readily seen that q is the smaller of the two roots. Hence (p−q) is positive. Now, θ_3/θ_1, which is the product of the two roots of Eq. (K6), is negative; thus, we have one positive and one negative root. q, being the smaller of the two roots, must therefore be negative. This renders the arguments of all the exponential functions in Eqs. (K10) and (K11) negative, and hence the exponential functions approach zero as t approaches infinity. Consequently, $\partial f/\partial V$ and $\mathcal{E}$ approach finite limits irrespective of the values of V which varies only from V_0-u to V_0, as given by

$$\operatorname*{Lim}_{t \to \infty} \left(\frac{\partial f_0}{\partial V} \right) = q \frac{\pi}{2u} \cos\left[\frac{\pi}{2} \left(\frac{V - V_0}{u} \right) \right] < 0$$

and

$$\operatorname*{Lim}_{t \to \infty} \varepsilon = 0$$

It is left to be shown that both $\partial f_0 / \partial V$ and ε remain bounded as t varies from 0 to ∞.

With a little algebraic manipulation, it can be shown that

$$A_0 < p$$

Hence

$$\frac{A_0 - q}{A_0 - p} < 1$$

since q is negative.

Consequently the factor $\left[\frac{A_0 - q}{A_0 - p} e^{-\theta (p-q)t} - 1 \right]$ in Eqs. (K10) and (K11) is never zero. Thus, no terms in the two equations blow up. $\partial f_0 / \partial V$ and ε are therefore bounded for $0 \le t < \infty$, the upper limit of which being given respectively by

$$\left(\frac{\partial f_0}{\partial V} \right)_{max} = A_0$$

$$\varepsilon_{max} > \varepsilon(V, 0)$$

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