Dynamical Shafarevich results for rational maps.

by

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Abstract

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Given a number field K and a finite set S of places of K, this dissertation studies rational maps with prescribed good reduction at every place $v \notin S$. The first result shows that the set of all quadratic rational maps with the standard notion of good reduction outside S is Zariski dense in the moduli space $\mathcal{M}_2$. The second result shows that if the notion of good reduction is strengthened by requiring a double unramified fixed point structure or an unramified two cycle, then one obtains a non-Zariksi density statement. The next result proves the existence of global minimal models of endomorphisms on $\mathbb{P}^n$ defined over the fractional field of principal ideal domain. This result is used to prove the last main theorem- the finiteness of twists of a rational maps on $\mathbb{P}^n$ over K with good reduction outside S.

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Contents

Introduction

Let K be a number field, and let S be a finite set of places of K which includes all of the Archimedean places. In 1963, Shafarevich proved that, up to K-isomorphism, there exist only finitely many elliptic curves over K having good reduction at all places $v \notin S$ (see [18] § IX.6). He conjectured a generalization of the result to abelian varieties, and this was proved in 1983 by Faltings [8] as part of his proof of Mordell's conjecture.

Motivated by an analogy between elliptic curves and dynamical systems on the projective line, Szpiro and Tucker [21] have asked whether there is a similar finiteness result for the set $\mathrm{Rat}_d(K)$ of rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of a given degree d defined over K. As they point out, however, if one uses the standard notions of isomorphism and good reduction for rational maps on $\mathbb{P}^1$, then simple counterexamples preclude a finiteness result of this type. For example, a rational map defined by a monic integral polynomial has everywhere good reduction, and for each fixed degree $d \geq 2$ one can easily find infinite families of such maps which are non-isomorphic.

By using a weaker notion of isomorphism defined by separate precomposition and postcomposition actions of PGL_2 on the set of rational maps of degree d, and by suitably altering the notion of good reduction, Szpiro-Tucker obtained a finiteness result of Shafarevich type for a certain class of rational maps. More recently, Petsche [13] has proved a different Shafarevich-type finiteness theorem along certain families of critically separable rational maps, using a notion of isomorphism defined via PGL_2-conjugation.

The coarse moduli space of elliptic curves is the affine j-line, where points of the affine line correspond to isomorphism classes with given j-invariant. One could restate Shafarevich's theorem in terms of Zariski nondensity in the moduli space for elliptic curves. Motivated by this equivalent statement of Shafarevich's original theorem for elliptic curves, the first chapter of this dissertation studies the distribution of quadratic rational maps with good reduction outside S within the moduli space $\mathcal{M}_2$.

The first result of this chapter is that for a fixed set of places S, the set of points $\langle \phi \rangle \in \mathcal{M}_2(K)$ which have good reduction outside S is Zariski dense. We give two proofs- one which works whenever the group of S-units is infinite, the other which holds over $\mathbb{Z}$. The two proofs are sufficiently different to warrant the inclusion of both. Whether this theorem holds in higher degrees is unknown, although we conjecture it to be true, and either of the two proofs may shed light on a successful generalization.

The second main theorem of Chapter 2 investigates a rigidification of this problem. Shafarevich's theorem is false if one considers curves of genus one (without marked rational point) rather than elliptic curves (see Mazur's discussion in [11]). We therefore require that the quadratic rational map under consideration have two unramified fixed points and strengthen the notion of K-isomorphism and good reduction to respect this structure. We then prove that the set of quadratic rational maps with two unramified fixed points which have good reduction at all places not in S is Zariski nondense in $\mathcal{M}_2$. The proof follows almost *mutatis mutandis* for quadratic rational maps with unramified 2-cycle.

The next chapter is a technical result intended to be used as a tool in Chapter 3, although we believe the result to useful in a broad context. Given a rational morphism of projective space which has good reduction outside S, then for each place $v \notin S$ one may have to conjugate to a different rational map in the conjugacy class, by definition, in order to realize good reduction at each v. The same is true for elliptic curves, but in this case there exists a *global minimal model* whose discriminant realizes the minimal valuation at every place. This gives one Weiertrass model which realizes the good reduction for every place at which the elliptic curve has good reduction. In the case of rational maps, the existence of a global affine model for the morphism which realizes the minimal valuation of the resultant at all places simultaneously was a question originally posed by Silverman (see [17]). The case of rational maps on $\mathbb{P}^1$ was recently proved by Bruin and Molnar (see [3]). Here we extend that result to rational morphisms on $\mathbb{P}^n$ for a rational map defined over the fraction field of a principal ideal domain. In particular, this implies the existence of a global S-minimal models for rational maps when $\mathcal{O}_S$ is a PID. Our method of proof is different than Bruin and Molnar, depending on the factorization of the adelic linear group $\mathrm{GL}_n(\mathbb{A}_R)$.

The last chapter of this dissertation returns to the subject of rational maps which have prescribed good reduction. The first chapter investigated the distribution of points in $\mathcal{M}_2$ which have good reduction outside S; this chapter investigates the number of rational maps with good reduction outside S within a single K-conjugacy class. The principal theorem of this chapter states that there are only finitely many K-isomorphism classes of twists of rational morphisms on $\mathbb{P}^n$ which have good reduction outside S. The context of this question is essentially orthogonal to the question of Chapter 1. In Chapter 3 we ask if a Shafarevich result holds within a fiber of the quotient map $\mathrm{Hom}_d^n \to \mathcal{M}_d^n$. The proof of this theorem relies on a result of Everste and Győry on finiteness of decomposable homogeneous forms in n variables with a given discriminant ideal and an investigation of the pre-periodic points of the rational map with size of forward orbit bounded by M.

We will fix the notation and definitions regarding the dynamics of rational morphisms on projective space; for more details see [17].

Fix coordinates $(X_0 : \ldots : X_n)$ of $\mathbb{P}^n(\bar{K})$. An arbitrary rational morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ defined over $\bar{K}$ is given by an $n+1$ tuple

$$\phi(X_0 : \ldots, X_n) = (F_0(X_0, \ldots, X_n), \ldots, F_n(X_0, \ldots, X_n))$$

where $F_i(X_0, \ldots, X_n)$ is a homogeneous polynomial of degree d for $i = 0, \ldots n$ and $F_0, \ldots, F_n$ have no nontrivial common solutions.

Using a multi-index $j = (j_0, \ldots, j_n)$ where each $0 \leq j_k \leq d$ and $j_0 + \cdots + j_n = d$ we may write

$$F_i = \sum_j a_{ij} X^j$$

with coefficients $a_{ij} \in \bar{K}$. There are $\binom{n+d}{d}$ monomials of degree d in $n+1$ variables and so ϕ can be identified with a point $(a_{0j} : \ldots : a_{nj}) \in \mathbb{P}^N$ where $N = N(n,d) = (n+1)\binom{n+d}{d} - 1$. Conversely, any point of $\mathbb{P}^N$ determines a rational map $\phi : \mathbb{P}^n \to \mathbb{P}^n$, although this map may not be a morphism. The requirement that ϕ be a morphism is equivalent to the nonvanishing of the resultant polynomial, i.e. that $\mathrm{Res}(\phi) \neq 0$, where the resultant polynomial is a multihomogeneous polynomial over $\mathbb{Z}$ in the coefficients a_{ij}.

There is a natural $\mathrm{PGL}_{n+1}(\bar{K})$ action on rational morphisms which sends $f \in \mathrm{PGL}_{n+1}(\bar{K})$ and ϕ to $\phi^f = f \circ \phi \circ f^{-1}$. When ϕ is a rational morphism it may be iterated and we write

$$\phi^n = \phi \circ \phi \circ \cdots \circ \phi$$

to denote the n^{th} iterate of ϕ. The action of conjugation is compatible with iteration in the sense that

$$(\phi^f)^n = (\phi^n)^f$$

There are natural sets of points in $\mathbb{P}^n$ which can be associated to a rational morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$. A point $P \in \mathbb{P}^n(\bar{K})$ is *periodic* if $\phi^m(P) = P$ for some positive integer $m \geq 1$ and *pre-periodic* if some iterate $\phi^m(P)$ is periodic. Equivalently, P is pre-periodic if its forward orbit is finite.

We use the notations

$$\mathrm{Per}(\phi), \mathrm{PrePer}(\phi)$$

to denote the set of all periodic points or pre-periodic points, respectively, for a fixed rational map ϕ. We also use the notation

$$\mathrm{PrePer}(\phi, M)$$

to denote the set of pre-periodic points with forward orbit of length at most M. If $M = 1$ the points are called *fixed* and we denote $\mathrm{PrePer}(\phi, 1)$ as $\mathrm{Fix}(\phi)$.

Suppose ϕ, ψ are rational maps of degree d defined over K on $\mathbb{P}^n$. We say that ϕ and ψ are $\bar{K}$-*isomorphic* if $\psi = \phi^f$ for some $f \in \mathrm{PGL}_{n+1}(\bar{K})$ and that they are K-*isomorphic* if $\psi = \phi^f$ for some $f \in \mathrm{PGL}_{n+1}(K)$. We denote the sets of $\bar{K}$-isomorphic and K-isomorphic rational maps by

$$= \{\phi^f | f \in \mathrm{PGL}_2(\bar{K})\}$$

$$[\phi]_K = \{\phi^f | f \in \mathrm{PGL}_2(K)\}$$

Definition. Let ϕ and ψ be two rational maps of degree d over K. We say that ψ is a K-*twist* of ϕ if ϕ and ψ are $\bar{K}$-isomorphic when, indeed, this is an abuse of notation. We define a *twist* of ϕ to be a K-equivalence class of rational maps $[\phi]_K$ of rational maps defined over K such that over $\bar{K}$ $[\phi] = [\psi]$. The set of twists of ϕ is denoted by $\mathrm{Twist}(\phi/K)$.

Two rational maps ϕ, ψ defined over K which are twists have identical geometric properties as maps on $\mathbb{P}^n(\bar{K})$ but may have significantly different arithmetic properties as maps on $\mathbb{P}^n(K)$. Let Hom_d^n denote the parameter space of rational morphisms of degree d on $\mathbb{P}^n$. Then if ϕ, ψ are twists they descend to the same point in the moduli space $\mathcal{M}_d^n$ under the quotient map

$$\mathrm{Hom}_d^n \to \mathrm{Hom}_d^n/\mathrm{PGL}_{n+1} = \mathcal{M}_d^n.$$

In [17] Silverman has given the following cohomological description of the set of twists

$$\mathrm{Twist}(\phi/K) = \left\{\xi \in H^1(G_K, \mathrm{Aut}(\phi)) | \xi = 1 \text{ in } H^1(G_K, \mathrm{PGL}_{n+1}(\bar{K}))\right\}$$

where G_K denotes the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$ and $\mathrm{Aut}(\phi)$ the stabilizer subgroup for the PGL_{n+1} action on rational morphisms.

Let M_K denote the places of the number field K. For any place $v \in M_K$ let $|\cdot|_v$ denote any absolute value on K associated to v. If v is non-Archimedean, let K_v denote the completion of K with respect to v and

$$\mathcal{O}_v = \{x \in K_v | |x|_v \leq 1\}$$

$$\mathcal{O}_v^\times = \{x \in K_v | |x|_v = 1\}$$

denote the subring of v-integral elements and the group of v-adic units in $\mathcal{O}_v$, respectively. $\mathcal{O}_v$ is a discrete local ring with maximal ideal $\mathfrak{m}_v = \{x \in K | |x|_v < 1\}$. Let $\mathcal{O}_v \to k_v = \mathcal{O}_v/\mathfrak{m}_v$ be the reduction map on to the residue field k_v. For $x \in \mathcal{O}_v$ we denote the image of this map by $\tilde{x}_v$ or just $\tilde{x}$ if v is understood.

For a rational morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ defined over K and $v \in M_K$ we can defined the *reduction* of ϕ at the place v in the following manner. Following the natural embedding $K \to K_v$ one can consider ϕ to be a rational map over K_v. As $K_v = \mathrm{Frac}(\mathcal{O}_v)$ and $\mathcal{O}_v$ is a PID, one can choose homogeneous coefficients for $\phi = (a_{0j} : \ldots : a_{nj})$ as a point in $\mathbb{P}^N$ such that $|a_{ij}|_v \leq 1$ and $\max_{i,j} |a_{ij}|_v = 1$.

Definition. The *reduction of ϕ at v* is the rational map

$$\tilde{\phi}_v = (\tilde{a_{0j}} : \ldots : \tilde{a_{nj}}) \in \mathbb{P}^N(k_v)$$

This reduction is independent of the choice of homogeneous coordinates. The reduction of a morphism may or may not be a morphism of the same degree over the residue field.

Definition. For a rational map $\phi : \mathbb{P}^n \to \mathbb{P}^n$ of degree d we say that ϕ has *good reduction* at v if $\tilde{\psi}_v$ has degree d for $\psi = \phi^f$ for some $f \in \mathrm{PGL}_N(K)$ and *bad reduction* at v otherwise.

This definition implies that the notion of good reduction is well defined for the K-isomorphism class $[\phi]_K$.

The reduction of a point $P \in \mathbb{P}^n(K)$ at a place v is defined in the analogous way. Reduction of a point behaves well with reduction of a map. Specifically, for any place v

$$\widetilde{\phi(P)} = \tilde{\phi}(\tilde{P}),$$

and this implies that if $P \in \mathrm{Fix}(\phi)$, then $\tilde{P} \in \mathrm{Fix}(\tilde{\phi})$. A similar statement holds for periodic and pre-periodic points of ϕ, although the length of the forward orbit may go down. It follows that reduction defines a map of sets

$$\mathrm{PrePer}(\phi, M) \to \mathrm{PrePer}(\tilde{\phi}, M)$$

For the rest of the document S denotes a finite subset of M_K which includes all of the archimedean places, $\mathcal{O}_S$ the S-integers of K, $\mathcal{O}_S^{\times}$ the S-unit group of K. More specifically,

$$\mathcal{O}_S = \{x \in K | |x|_v \leq 1 \text{ for all } v \notin S\}$$

$$\mathcal{O}_S^{\times} = \{x \in K | |x|_v = 1 \text{ for all } v \notin S\}$$

Definition. The *absolute S-integers of $\bar{K}$* will consist of all elements of $\bar{K}$ which are w-integral for every place w of $\bar{K}$ whose restriction to K is not in S. The absolute S-integers of $\bar{K}$ are denoted by $\overline{\mathcal{O}}_S$.

For a point $P = (p_0 : \ldots : p_n) \in \mathbb{P}^n(K)$ and a place v we say that the coordinates are *normalized* with respect to v, or v-normalized, if $|p_i|_v \leq 1$ for $0 \leq i \leq n$ and $|p_i| = 1$ for some i. The following lemma is well known, so we omit the proof.

Lemma 1. *Let $P \in \mathbb{P}^n(K)$ and S be sufficiently large such that $\mathcal{O}_S$ is a principal ideal domain. Then there exists coordinates $(p_0 : \ldots : p_n)$ for P which are v-normalized for all $v \notin S$.*

We will also call such coordinates normalized, and context will make it clear whether we refer to a single place v or to all places $v \notin S$. In Chapter 3, we will refer to prime ideals $\mathfrak{p}$ rather than places v. In this case we will use the notation $K_{\mathfrak{p}}, R_{\mathfrak{p}}$, and $\mathbb{F}_{\mathfrak{p}}$ to denote $K_v, \mathcal{O}_v$, and k_v, respectively.

The principle theorems of this dissertation study rational maps with prescribed good reduction. In Chapter 1, we study the Zariski density of points in $\mathcal{M}_2 = \mathcal{M}_2^1$ which have good reduction outside S. In Chapter 2, we study rational maps with prescribed good reduction within the fiber of a K-point under the quotient morphism $\mathrm{Hom}_d^n \to \mathcal{M}_d^n$.

Chapter 1

Good Reduction of Quadratic Maps

1.1 Introduction

In this chaper we consider the following similar but somewhat more geometric question related to a dynamical Shafarevich theorem for rational maps. Rather than a finiteness statement for the set of isomorphism classes of rational maps of degree d having prescribed good reduction, we ask instead whether or not this set is Zariski-dense in the moduli space $\mathcal{M}_d$ parametrizing $\bar{K}$-isomorphism classes of rational maps of degree d. Introduced in the complex-analytic setting by Milnor [12], and further developed geometrically by Silverman [20], $\mathcal{M}_d$ is an affine variety whose definition is given, via geometric invariant theory, as the quotient

$$\mathcal{M}_d = \mathrm{Rat}_d/\mathrm{PGL}_2$$

of the space Rat_d of all rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d, modulo the conjugation action of PGL_2, the automorphism group of $\mathbb{P}^1$. Letting $\langle\cdot\rangle : \mathrm{Rat}_d \to \mathcal{M}_d$ denote the quotient map associated to this action, we propose the following conjecture.

Conjecture 1. *Let K be a number field and let S be a finite set of places of K including the Archimedean places. Then the set*

$$\mathcal{G}_d(K,S) = \left\{ \langle\phi\rangle \in \mathcal{M}_d(K) \,\middle|\, \begin{array}{l} \phi \in \mathrm{Rat}_d(K) \text{ has good} \\ \text{reduction at all } v \in M_K \setminus S \end{array} \right\}$$

is Zariski-dense in $\mathcal{M}_d$.

The first main result of this chapter the proof of this conjecture in the simplest nontrivial setting of quadratic rational maps (the case $d = 2$). The definitions of the required terms are stated precisely in §1.3, and this Zariski-density result is stated as Theorem 2.

The remainder of this chapter is spent showing that, despite our Zariski-density result in the case $d = 2$, if we replace arbitrary quadratic rational maps with objects possessing slightly more dynamical structure, then it is possible to obtain Zariski nondensity results of Shafarevich type for the moduli space $\mathcal{M}_2$.

To explain our motivation in trying to obtain such results, we again consider the analogy with elliptic curves. It is an interesting fact that Shafarevich's finiteness theorem for elliptic curves may fail, in general, for genus-one curves. For example, if S is a finite set of places of $\mathbb{Q}$ containing the Archimedean place as well as all of the places of bad reduction for some rank-zero elliptic curve $E/\mathbb{Q}$, then there are infinitely many non-isomorphic genus-one curves over $\mathbb{Q}$ with good reduction outside S; the argument is explained by Mazur in [11], p. 241. (On the other hand, as he points out, it would follow from the Shafarevich-Tate conjecture that the set of K-isomorphism classes of genus-one curves over K with *everywhere* good reduction is finite.)

Motivated by the elliptic curve analogy, we consider quadratic rational maps with double unramified fixed-point structure. More precisely, we consider the space of triples $\Phi = (\phi, P_1, P_2)$, where $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a quadratic rational map defined over K, and where $P_1, P_2 \in \mathbb{P}^1(K)$ are distinct K-rational unramified fixed points of ϕ. The conjugation action of PGL_2 gives rise to a notion of K-isomorphism between two such triples, and we formulate a natural definition of good reduction for such a triple $\Phi = (\phi, P_1, P_2)$ at a non-Archimedean place v of K which, roughly speaking, requires that (up to K-isomorphism) the reduction $\tilde{\Phi}_v = (\tilde{\phi}_v, \tilde{P}_{1,v}, \tilde{P}_{2,v})$ constitutes a quadratic rational map with double unramified fixed-point structure over the residue field $\mathbb{F}_v$ at v. In §1.4 we give the precise definitions of these terms and we prove the second main result of this chapter, Theorem 3, which shows that among all quadratic rational maps with double unramified fixed-point structure, those having good reduction at all places v outside S comprise a non-Zariski-dense subset of the moduli space $\mathcal{M}_2$. We regard this result as a geometric Shafarevich-type theorem for rational maps.

Finally, with very little extra effort we can also establish a variation on the non-Zariski-density result of Theorem 3, in which maps with unramified 2-cycle structure take the place of maps with unramified double fixed-point structure. This result is stated as Theorem 7.

1.2 Preliminaries

1.2.1 Review of quadratic rational maps

We now fix notation and review basic facts about quadratic rational maps on the projective line; for further details see [17], §2.4, §4.3, §4.6.

An arbitrary quadratic rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\bar{K}$ is given in homogeneous coordinates as

$$\phi(X : Y) = (A(X, Y) : B(X, Y)),$$

where

$$A(X, Y) = a_0X^2 + a_1XY + a_2Y^2$$

$$B(X, Y) = b_0X^2 + b_1XY + b_2Y^2$$

are binary quadratic forms in $\bar{K}[X,Y]$ having no common zeros in $\bar{K}^2 \setminus \{(0,0)\}$. The requirement that $A(X,Y)$ and $B(X,Y)$ share no common zeros in $\bar{K}^2 \setminus \{(0,0)\}$ is equivalent to the nonvanishing of the resultant

$$\mathrm{Res}(A,B) = \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix}$$

associated to the pair (A,B).

The group variety of automorphisms of $\mathbb{P}^1$ is denoted by PGL_2, and each $f \in \mathrm{PGL}_2$ is given in homogeneous coordinates by

$$f(X:Y) = (\alpha X + \beta Y : \gamma X + \delta Y)$$

for some nonsingular matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ with coefficients in $\bar{K}$. Given a quadratic rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, we denote by $\phi^f : \mathbb{P}^1 \to \mathbb{P}^1$ the rational map $\phi^f = f^{-1} \circ \phi \circ f$ defined via conjugation of ϕ by f. Explicitly, if we denote by $(A,B) : \bar{K}^2 \to \bar{K}^2$ the map defined by $(X,Y) \mapsto (A(X,Y), B(X,Y))$, and if we define binary quadratic forms $C(X,Y)$ and $D(X,Y)$ in $\bar{K}[X,Y]$ by the formula

$$(C,D) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \circ (A,B) \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then $\phi^f(X:Y) = (C(X,Y) : D(X,Y))$. The formula

$$\mathrm{Res}(C,D) = (\alpha\delta - \beta\gamma)^2 \mathrm{Res}(A,B), \tag{1.1}$$

which shows the effect of GL_2-conjugation on the resultant, can be verified from direct calculation.

1.2.2 Review of the moduli space $\mathcal{M}_2$

The moduli space $\mathcal{M}_2$ parametrizing isomorphism classes of quadratic rational maps was first studied complex analytically by Milnor [12], who showed that it is isomorphic to the affine plane $\mathbb{A}^2$. A bit later, Silverman [20] used geometric invariant theory to construct $\mathcal{M}_2$ (and more generally the moduli space $\mathcal{M}_d$ of rational maps of degree d) as a scheme over $\mathrm{Spec}(\mathbb{Z})$, and established the isomorphism $\mathcal{M}_2 \simeq \mathbb{A}^2$ in this more geometric context. Since that time, variations and generalizations have been studied by Petsche-Szpiro-Tepper [16], Levy [10], and others. Further references include [17] §4.4 and [19].

We now review the definition and basic properties of the space $\mathcal{M}_2$. The first step is to observe that the set of all quadratic rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\bar{K}$ is parametrized by an affine variety which is commonly denoted by Rat_2. To obtain this variety, note that since the map ϕ is unchanged by scaling its coefficients, one may identify ϕ with the point $(\mathbf{a} : \mathbf{b}) = (a_0 : a_1 : a_2 : b_0 : b_1 : b_2)$ of $\mathbb{P}^5$ defined by its coefficients. In this way, the space Rat_2 of all quadratic rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is identified with the open affine subvariety $\{(\mathbf{a}:\mathbf{b}) \mid \mathrm{Res}(A,B)\neq 0\}$ of $\mathbb{P}^5$.

Let

$$\begin{aligned} \mathrm{PGL}_2 \times \mathrm{Rat}_2 &\to \mathrm{Rat}_2 \\ (f,\phi) &\mapsto \phi^f \end{aligned} \tag{1.2}$$

be the conjugation action of PGL_2 on Rat_2, and let $\mathcal{A} = \Gamma(\mathrm{Rat}_2, \mathcal{O}_{\mathrm{Rat}_2})$ be the coordinate ring of Rat_2. The moduli space $\mathcal{M}_2$ is defined to be the affine variety $\mathrm{Spec}(\mathcal{A}^{\mathrm{PGL}_2})$, where $\mathcal{A}^{\mathrm{PGL}_2}$ is the subring of PGL_2-invariants in $\mathcal{A}$. Using standard facts from geometric invariant theory it can be shown that $\mathcal{M}_2$ is a geometric quotient for the action (1.2), which means roughly that the map

$$\langle\cdot\rangle : \mathrm{Rat}_2 \to \mathcal{M}_2 \tag{1.3}$$

induced by inclusion $\mathcal{A}^{\mathrm{PGL}_2} \subset \mathcal{A}$ possesses many of the nice properties one would expect from the quotient map of a group action in the classical sense. For example, the (geometric) fibers of the map (1.3) are closed, and they are precisely the orbits in Rat_2 with respect to the conjugation action of PGL_2.

To describe Milnor's isomorphism $\mathcal{M}_2 \simeq \mathbb{A}^2$ in detail, recall that each quadratic rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ has three (counting with multiplicity) fixed points $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{P}^1$, and for each fixed point, the multiplier λ_j associated to α_j is the leading coefficient $\lambda_j = \phi'(\alpha_j)$ of the power series expansion of $\phi(z)$ at $z = \alpha_j$ (with respect to some choice of affine coordinate z on $\mathbb{P}^1$).

A standard calculation shows that the multiplier of a fixed point is invariant under PGL_2-conjugation. Since the fixed-point set $\mathrm{Fix}(\phi)$ is naturally an unordered triple, we obtain three scalar-valued PGL_2-invariant functions σ_1 and σ_2 on Rat_2, defined by the first two symmetric functions $\sigma_1(\phi) = \lambda_1 + \lambda_2 + \lambda_3$ and $\sigma_2(\phi) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ in the multipliers of the three fixed points of ϕ. (The third symmetric function, $\sigma_3(\phi) = \lambda_1\lambda_2\lambda_3$, gives only redundant information because of the identity $\sigma_3 - \sigma_1 + 2 = 0$.) Milnor's isomorphism $\mathcal{M}_2 \simeq \mathbb{A}^2$ is defined by

$$\begin{aligned} \sigma : \mathcal{M}_2 &\xrightarrow{\sim} \mathbb{A}^2 \\ \langle\phi\rangle &\mapsto (\sigma_1(\phi), \sigma_2(\phi)). \end{aligned} \tag{1.4}$$

See [12].

1.3 Prescribed good reduction for quadratic rational maps

In this section we show that the quadratic rational maps over K having good reduction at all places outside S comprise a Zariski-dense subset of the moduli space $\mathcal{M}_2$. We first recall the standard definitions of K-isomorphism and good reduction.

Definitions. Two quadratic maps $\phi, \psi \in \mathrm{Rat}_2(K)$ are *K-isomorphic* if $\psi = \phi^f$ for some automorphism $f \in \mathrm{PGL}_2(K)$. Furthermore, $\phi \in \mathrm{Rat}_2(K)$ has *good reduction* at a non-Archimedean place v of K if it is K-isomorphic to some $\psi \in \mathrm{Rat}_2(K)$ such that $\deg(\tilde{\psi}_v) = 2$.

We stress the contrast between the notion of *good reduction*, in which the two rational maps ϕ and ψ are required to be conjugate via a K-rational automorphism $f \in \mathrm{PGL}_2$, with the weaker notion of *potential good reduction*, in which the definition is relaxed to allow automorphisms f defined over $\bar{K}$. To illustrate the difference between the two, consider the elliptic curve setting: an elliptic curve E/K has potential good reduction at all places $v \in M_K \backslash S$ if and only if $j_E \in \mathcal{O}_S$, and so it is trivial that there are infinitely many ($\bar{K}$-isomorphism classes of) such curves. Similarly, in the case of rational maps, it would be straightforward to produce a Zariski-density result for $\bar{K}$-isomorphism classes of rational maps in $\mathcal{M}_2$ with prescribed *potential* good reduction using the fact that $\mathcal{M}_2(\mathbb{Z})$ is Zariski-dense in $\mathcal{M}_2 \simeq \mathbb{A}^2$.

Theorem 2. *Let K be a number field and let S be a finite set of places of K including the Archimedean places. Then the set*

$$\mathcal{G}_2(K,S) = \left\{ \langle\phi\rangle \in \mathcal{M}_2(K) \,\middle|\, \begin{array}{l} \phi \in \mathrm{Rat}_2(K) \textit{ has good} \\ \textit{reduction at all } v \in M_K \setminus S \end{array} \right\}$$

is Zariski-dense in $\mathcal{M}_2$.

Our primary proof of Theorem 2 uses the isomorphism (1.4), as well as a further result of Milnor [12] on quadratic rational maps in critical-point normal form. We also give an alternate proof which holds only when the group of S-units in K is infinite (thus, this alternate proof only fails to apply when K is either $\mathbb{Q}$ or a quadratic imaginary extension of $\mathbb{Q}$, and S consists of the sole Archimedean place of K). While it does not apply in full generality, this secondary proof is sufficiently different from the first proof that it may be of some interest. It is more self-contained, in that it does not rely on special properties of quadratic rational maps in critical-point normal form, and the ideas behind this secondary proof may find wider applicability toward possible generalizations to the higher degree case.

Proof of Theorem 2. We will consider maps $\phi \in \mathrm{Rat}_2$ given in critical-point normal form

$$\phi(X:Y) = (aX^2 + bY^2 : cX^2 + dY^2). \tag{1.5}$$

The critical points are $(1:0)$ and $(0:1)$, and

$$\mathrm{Res}(aX^2 + bY^2, cX^2 + dY^2) = (ad - bc)^2. \tag{1.6}$$

Let $\mathcal{F}$ be the image in the moduli space $\mathcal{M}_2$ of the set of all $\phi \in \mathrm{Rat}_2(\mathbb{Q})$ given in critical-point normal form (1.5) with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. It follows from (1.6) that each such map has good reduction at all non-Archimedean places $v \in M_K$; thus $\mathcal{F} \subset \mathcal{G}_2(K, S)$, and we are reduced to showing that $\mathcal{F}$ is Zariski-dense in $\mathcal{M}_2$.

A direct calculation shows that if ϕ is given in critical-point normal form (1.5) with $ad - bc = 1$, then

$$\sigma_1(\phi) = 8ad - 6$$

$$\sigma_2(\phi) = 8a^2d^2 - 20ad + 4(a^3b + cd^3) + 12;$$

see for example the explicit formula given by Silverman ([17], p. 189), or the calculation for rational maps in critical-point normal form due to Milnor ([12] Corollary C.4).

In view of the isomorphism (1.4), we may identify $\mathcal{M}_2$ with the affine plane $\mathbb{A}^2$ and we may use σ_1 and σ_2 as the two affine coordinates on $\mathcal{M}_2$. Arguing by contradiction, assume on the contrary that the Zariski closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ is not all of $\mathcal{M}_2$. Then $\overline{\mathcal{F}}$ is a finite union of curves and points in $\mathcal{M}_2$. By Bezout's theorem, there exists a positive bound $B = B(\overline{\mathcal{F}}) > 0$ such that, if L is any line in $\mathcal{M}_2$, then either $L \subseteq \overline{\mathcal{F}}$ or $|L \cap \overline{\mathcal{F}}| \leq B$. For each $\alpha \in \bar{K}$, let L_α be the vertical line $\{\sigma_1 = \alpha\}$ in $\mathcal{M}_2$. Then $\overline{\mathcal{F}}$ can contain at most finitely many of these lines, call them $L_{\alpha_1}, \ldots, L_{\alpha_r}$.

We will obtain a contradiction by showing that $\mathcal{F}$ can meet a vertical line L_α at an arbitrarily large number of points, for lines $L_\alpha \notin \{L_{\alpha_1}, \ldots, L_{\alpha_r}\}$. Let N be a positive integer, let p be an arbitrary prime number, and for each $0 \leq n \leq N - 1$ define

$$\phi_{n,N}(X : Y) = (p^n X^2 + Y^2 : (p^{2N} - 1)X^2 + p^{2N-n} Y^2).$$

Since $(p^n)(p^{2N-n}) - (p^{2N}-1)(1) = 1$, we have $\phi_{n,N} \in \mathcal{F}$; denote by $\mathcal{F}(N) = \{\langle \phi_{n,N} \rangle \mid 0 \leq n \leq N-1\}$. Using Milnor's calculation of σ_1 and σ_2 in terms of A and Σ, we have

$$\sigma_1(\langle \phi_{n,N} \rangle) = 8p^{2N} - 6$$

$$\sigma_2(\langle \phi_{n,N} \rangle) = 8p^{4N} - 20p^{2N} + 4(p^{3n} + (p^{2N}-1)p^{6N-3n}) + 12$$

This shows that $\mathcal{F}(N)$ is contained in the line $L_{8p^{2N}-6}$. Further, note that for fixed N, the numbers $p^{3n} + (p^{2N}-1)p^{6N-3n}$ are distinct as n ranges from $0 \leq n \leq N-1$ (for example, because the p-adic absolute value of $p^{3n} + (p^{2N}-1)p^{6N-3n}$ is p^{-3n}). Therefore, the σ_2-coordinates of the N points $\langle \phi_{n,N} \rangle$ are distinct for $0 \leq n \leq N-1$, whereby $|\mathcal{F}(N)| = N$. We have shown that $\mathcal{F}$ meets the line $L_{8p^{2N}-6}$ in at least N points; taking N large enough produces a contradiction, since there are only finitely many N for which $N \leq B$ or $L_{8p^{2N}-6} \in \{L_{\alpha_1}, \ldots, L_{\alpha_r}\}$. $\square$

Alternate proof of Theorem 2. (This proof only holds under the additional assumption that the S-unit group $\mathcal{O}_S^\times$ of K is infinite).

We will consider maps $\phi \in \mathrm{Rat}_2$ given in fixed-point normal form

$$\phi(X:Y) = (X^2 + \lambda_1 XY : \lambda_2 XY + Y^2). \tag{1.7}$$

For rational maps in this form, the fixed points, their multipliers, and the resultant are particularly easy to calculate. The fixed points of ϕ are $(0:1)$,

$(1 : 0)$, and $(1-\lambda_1 : 1-\lambda_2)$, with multipliers λ_1, λ_2, and $\lambda_3 = \frac{2-\lambda_1-\lambda_2}{1-\lambda_1\lambda_2}$, respectively (see [17] §4.6), and

$$\operatorname{Res}(X^2+\lambda_1 XY, \lambda_2 XY + Y^2) = 1 - \lambda_1\lambda_2.$$

For each pair of nonzero elements $\alpha, \beta \in \bar{K}^\times$, define $\phi_{\alpha,\beta} \in \mathrm{Rat}_2$ to be the map given in fixed-point normal form (1.7) with $\lambda_1 = \alpha$ and $\lambda_2 = \frac{1-\beta}{\alpha}$. We obtain a map

$$u : \mathbb{G}_m \times \mathbb{G}_m \to \mathcal{M}_2$$

$$u(\alpha, \beta) = \langle \phi_{\alpha,\beta} \rangle.$$

We first show that u is dominant. For each $\alpha \in \bar{K}^\times$, define $u_\alpha : \mathbb{G}_m \to \mathcal{M}_2$ by $u_\alpha(z) = u(\alpha, z)$, and let $Z_\alpha = \overline{u_\alpha(\mathbb{G}_m)}$ be the Zariski-closure of the image of u_α. In view of the isomorphism (1.4), we may identify $\mathcal{M}_2$ with the affine plane $\mathbb{A}^2$ and we may use σ_1 and σ_2 as the two affine coordinates on $\mathcal{M}_2$. Direct calculations show that $u_1(z) = (3-z, 3-2z)$ and therefore Z_1 is the line $2\sigma_1 - \sigma_2 = 3$ in $\mathcal{M}_2$. Similarly, $u_{-1}(z) = (-3 + z + \frac{4}{z}, 7 - 2(z + \frac{4}{z}))$ and therefore Z_{-1} is the line $2\sigma_1 + \sigma_2 = 1$ in $\mathcal{M}_2$. Now let $Z = \overline{u(\mathcal{O}_S^\times \times \mathcal{O}_S^\times)}$ be the Zariski-closure of the image of u. Then since the torus $\mathbb{G}_m \times \mathbb{G}_m$ is irreducible, Z is irreducible, and since Z contains the two distinct lines Z_1 and Z_{-1}, Z must have dimension 2. Therefore $Z = \mathcal{M}_2$ and u is dominant.

Define $\mathcal{V} = u(\mathcal{O}_S^\times \times \mathcal{O}_S^\times)$ to be the image in $\mathcal{M}_2$ under u of the set of all $\phi_{\alpha,\beta} \in \mathrm{Rat}_2(K)$ for which both α and β are in the S-unit group $\mathcal{O}_S^\times$. The calculation $\mathrm{Res}(X^2 + \alpha XY, (\frac{1-\beta}{\alpha})XY + Y^2) = \beta$ shows that each such map has good reduction at all places $v \in M_K \setminus S$, and therefore $\mathcal{V} \subset \mathcal{G}_2(K, S)$. To complete the proof of the theorem, we only need to show that $\mathcal{V}$ is Zariski-dense in $\mathcal{M}_2$.

Since $\mathcal{O}_S^\times$ is infinite, the subgroup $\mathcal{O}_S^\times \times \mathcal{O}_S^\times$ is Zariski-dense in $\mathbb{G}_m \times \mathbb{G}_m$. Therefore

$$u(\mathbb{G}_m \times \mathbb{G}_m) = u(\overline{\mathcal{O}_S^\times \times \mathcal{O}_S^\times}) \subseteq \overline{u(\mathcal{O}_S^\times \times \mathcal{O}_S^\times)} = \overline{\mathcal{V}}; \tag{1.8}$$

here we have used the fact, which is true of all continuous maps on topological spaces, including morphisms of algebraic varieties, that $f(\overline{X}) \subseteq \overline{f(X)}$. Taking the Zariski-closure of both sides of (1.8) we obtain $\overline{u(\mathbb{G}_m \times \mathbb{G}_m)} \subseteq \overline{\mathcal{V}}$. Since u is dominant, we have $\overline{u(\mathbb{G}_m \times \mathbb{G}_m)} = \mathcal{M}_2$, and therefore $\overline{\mathcal{V}} = \mathcal{M}_2$, completing the proof. □

1.4 Double unramified fixed-point structure

One of the goals of the chapter is to emphasize that dynamical analogues of theorems for elliptic curves over number fields may fail because general rational maps lack the richer structure of elliptic curves. As mentioned in the introduction, when one replaces elliptic curve in the statement of Shafarevich's theorem with genus-one curve, the theorem becomes false. In that setting, the extra structure provided by a marked K-rational point acting as the origin for the elliptic curve has a dramatic influence on the set of K-isomorphism classes of such objects. With the elliptic curve analogy in mind, in this section we consider rational maps equipped with some additional structure arising from fixed points.

Definition. Let $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ be the set of all triples of the form $\Phi = (\phi, P_1, P_2)$, where $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a quadratic rational map defined over K, and where $P_1, P_2 \in \mathbb{P}^1(K)$ are distinct K-rational unramified fixed points of ϕ. We call such a triple Φ a *quadratic rational map with double unramified fixed-point structure over* K; or, when the context is clear, for brevity we may refer to Φ simply as a map.

Remark 1. We have defined $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ only as a set, but observe also that $\mathrm{Rat}_{2,2}^{\mathrm{uf}}$ may be naturally viewed as an open subset of a closed surface in $\mathrm{Rat}_2 \times \mathbb{P}^1 \times \mathbb{P}^1$; thus $\mathrm{Rat}_{2,2}^{\mathrm{uf}}$ is a quasiprojective variety. Further, the map $\mathrm{Rat}_{2,2}^{\mathrm{uf}} \to \mathrm{Rat}_2$ obtained by forgetting the fixed-point structure is dominant; indeed, the set of quadratic rational maps which fail to have three distinct unramified fixed points forms a proper Zariski-closed subset of Rat_2. Finally, given a map $\phi \in \mathrm{Rat}_2(K)$, the three fixed points of ϕ are K'-rational for some extension K'/K of degree at most six. So from the point of view of studying geometric questions concerning generic rational maps, the extra conditions required of a quadratic rational map with double unramified fixed-point structure over K are not terribly restrictive. (Later we will explain our choice of double, rather than single or triple, unramified fixed-point structure.)

Given a map $\Phi = (\phi, P_1, P_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$, and an automorphism $f \in \mathrm{PGL}_2(K)$, observe that $f^{-1}(P_1)$ and $f^{-1}(P_2)$ are distinct unramified fixed points of $\phi^f = f^{-1} \circ \phi \circ f$; we may therefore define $\Phi^f \in \mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$, the conjugate of Φ with respect to f, by

$$\Phi^f = (\phi^f, f^{-1}(P_1), f^{-1}(P_2)).$$

This notion of $\mathrm{PGL}_2(K)$-conjugation on $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ gives rise to the following definitions.

Definitions. Two maps Φ and Ψ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ are *K-isomorphic* if $\Psi = \Phi^f$ for some automorphism $f \in \mathrm{PGL}_2(K)$. A map Φ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ has *good reduction* at a non-Archimedean place v of K if it is K-isomorphic to some $\Psi = (\psi, Q_1, Q_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ such that $\deg(\tilde{\psi}_v) = 2$ and such that $\tilde{Q}_1$ and $\tilde{Q}_2$ are distinct unramified fixed points of $\tilde{\psi}_v$.

We emphasize that this notion of good reduction for a map $\Phi = (\phi, P_1, P_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ is stronger than the standard definition of good reduction for its underlying rational map $\phi \in \mathrm{Rat}_2(K)$; a natural additional condition has been added to ensure that reduction modulo the maximal ideal of $\mathcal{O}_v$ preserves the double unramified fixed-point structure of the triple

$$\tilde{\Phi} = (\tilde{\phi}_v, \tilde{P}_{1,v}, \tilde{P}_{2,v})$$

over the residue field $\mathbb{F}_v$.

Abusing notation slightly, for each $\Phi = (\phi, P_1, P_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$, define $\langle\Phi\rangle = \langle\phi\rangle$. Thus, one may view $\langle\cdot\rangle : \mathrm{Rat}_{2,2}^{\mathrm{uf}}(K) \to \mathcal{M}_2(K)$ as the map which forgets the fixed-point structure of Φ and preserves only the PGL_2-conjugacy class $\langle\phi\rangle$ of its underlying rational map.

The main theorem of this section is the following, which shows that the set of all Φ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ having good reduction outside S comprises a non-Zariski-dense subset of the moduli space $\mathcal{M}_2$.

Theorem 3. *Let K be a number field and let S be a finite set of places of K including the Archimedean places. Then the set*

$$\mathcal{G}_{2,2}^{\mathrm{uf}}(K,S) = \left\{ \langle\Phi\rangle \in \mathcal{M}_2(K) \;\middle|\; \begin{array}{l} \Phi \in \mathrm{Rat}_{2,2}^{\mathrm{uf}}(K) \textit{ has good} \\ \textit{reduction at all } v \in M_K \setminus S \end{array} \right\}$$

is not Zariski-dense in $\mathcal{M}_2$.

We need three preliminary propositions before we can give the proof of Theorem 3. The first states that maps in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ having good reduction at a non-Archimedean place v of K can be represented (up to K-isomorphism) in a certain simple form.

Proposition 4. *Suppose that* $\Phi \in \mathrm{Rat}^{\mathrm{uf}}_{2,2}(K)$ *has good reduction at a non-Archimedean place* v *of* K*. Then* Φ *is* K*-isomorphic to* $\Psi = (\psi, (1:0), (0:1))$ *for some quadratic rational map* $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ *given by*

$$\psi(X:Y) = (X^2 + aXY : bXY + cY^2) \tag{1.9}$$

for $a, b, c \in \mathcal{O}_v^\times$ *such that*

$$\mathrm{Res}(X^2 + aXY, bXY + cY^2)) = c(c - ab) \in \mathcal{O}_v^\times.$$

Proof. According to the definition of good reduction, possibly replacing Φ with some member of its K-isomorphism class, we may assume without loss of generality that $\Phi = (\phi, P_1, P_2)$, where $\deg(\tilde{\phi}_v) = 2$ and where $\tilde{P}_1$ and $\tilde{P}_2$ are distinct unramified fixed points of the reduced map $\tilde{\phi}_v$.

The fact that $\deg(\tilde{\phi}_v) = 2$ means that we may write $\phi(X, Y) = (A(X, Y) : B(X, Y))$ for forms $A(X, Y), B(X, Y)$ in $\mathcal{O}_v[X, Y]$ with $\mathrm{Res}(A, B) \in \mathcal{O}_v^\times$. Set $P_1 = (\alpha_1 : \beta_1)$ for $\alpha_1, \beta_1 \in \mathcal{O}_v$ and at least one of the two α_1, β_1 in the unit group $\mathcal{O}_v^\times$, and set $P_2 = (\alpha_2 : \beta_2)$ subject to the same requirements. Since $\tilde{P}_1 \neq \tilde{P}_2$ in $\mathbb{P}^1(\mathbb{F}_v)$, we have $\tilde{\alpha}_2\tilde{\beta}_1 - \tilde{\alpha}_1\tilde{\beta}_2 \neq 0$ in $\mathbb{F}_v$. In other words $\alpha_2\beta_1 - \alpha_1\beta_2 \in \mathcal{O}_v^\times$, and therefore the matrix $\left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\right)$ is an element of $\mathrm{GL}_2(\mathcal{O}_v)$.

Define $\psi = \phi^f$, where $f \in \mathrm{PGL}_2(K)$ is given by $f(X : Y) = (\alpha_1 X + \alpha_2 Y : \beta_1 X + \beta_2 Y)$. This means that $\psi(X : Y) = (C(X,Y) : D(X,Y))$ where the forms $C(X,Y)$ and $D(X,Y)$ are defined by

$$(C,D) = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}^{-1} \circ (A,B) \circ \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix},$$

Since $\left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_v)$, the formula (1.1) shows that the forms $C(X,Y)$ and $D(X,Y)$ have coefficients in $\mathcal{O}_v$ and resultant $\mathrm{Res}(C,D)$ in $\mathcal{O}_v^\times$. In particular, $\deg(\tilde{\psi}_v) = 2$.

Since $f(1:0) = P_1$ and $f(0:1) = P_2$, it follows that $(1:0)$ and $(0:1)$ are fixed points of ψ, and therefore we have $C(X,Y) = c_0 X^2 + c_1 XY$ and $D(X,Y) = d_1 XY + d_2 Y^2$ for elements $c_0, c_1, d_1, d_2 \in \mathcal{O}_v$, with $\mathrm{Res}(C,D) = c_0 d_2 (c_0 d_2 - c_1 d_1) \in \mathcal{O}_v^\times$. This immediately forces $c_0, d_2 \in \mathcal{O}_v^\times$, since otherwise c_0 or d_2 would be an element of the maximal ideal of $\mathcal{O}_v$, making $c_0 d_2 (c_0 d_2 - c_1 d_1) \in \mathcal{O}_v^\times$ impossible.

Since $\left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_v)$, the automorphism f reduces to an automorphism $\tilde{f} \in \mathrm{PGL}_2(\mathbb{F}_v)$, and $\tilde{\psi}_v = \tilde{\phi}_v^{\tilde{f}}$. Since $\tilde{P}_1$ and $\tilde{P}_2$ are unramified fixed points of $\tilde{\phi}_v$, it follows that $(\tilde{1} : \tilde{0})$ and $(\tilde{0} : \tilde{1})$ are unramified fixed points of $\tilde{\psi}_v$. Standard calculations show that, since $(\tilde{1} : \tilde{0})$ is an unramified point of $\tilde{\psi}_v$, $\tilde{d}_1$ is nonzero in $\mathbb{F}_v$, and since $(\tilde{0} : \tilde{1})$ is an unramified point of $\tilde{\psi}_v$, $\tilde{c}_1$ is nonzero in $\mathbb{F}_v$. Consequently, both c_1 and d_1 are in $\mathcal{O}_v^\times$.

Finally, setting $a = \frac{c_1}{c_0}$, $b = \frac{d_1}{c_0}$, and $c = \frac{d_2}{c_0}$, we obtain a representation for the map ψ in the desired form (1.9), with $a, b, c \in \mathcal{O}_v^\times$ and $c(c - ab) = \frac{d_2}{c_0}(\frac{d_2}{c_0} - \frac{c_1}{c_0}\frac{d_1}{c_0}) = c_0^{-3} d_2(c_0 d_2 - c_1 d_1) \in \mathcal{O}_v^\times$. ☐

Given a map Φ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ having good reduction at all places outside S, Proposition 4 shows that for each $v \in M_K \setminus S$, Φ is K-isomorphic to some map Ψ possessing a particularly simple form which realizes this good reduction at v. A priori, the map Ψ may vary from place to place, but the following proposition shows that, if $\mathcal{O}_S$ is a principal ideal domain, then a global map Ψ can be found satisfying the conclusion of Proposition 4 at every place $v \in M_K \setminus S$.

Proposition 5. *Assume that $\mathcal{O}_S$ is a principal ideal domain. Suppose that $\Phi \in \mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ has good reduction at all places $v \in M_K \setminus S$. Then Φ is K-isomorphic to $\Psi = (\psi, (1 : 0), (0 : 1))$ for some quadratic rational map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ given by*

$$\psi(X : Y) = (X^2 + aXY : bXY + cY^2)$$

for $a, b, c \in \mathcal{O}_S^\times$ such that

$$\mathrm{Res}(X^2 + aXY, bXY + cY^2)) = c(c - ab) \in \mathcal{O}_S^\times.$$

Proof. Replacing $\Phi = (\phi, P_1, P_2)$ with its conjugate by a suitable automorphism in $\mathrm{PGL}_2(K)$ which takes $(1 : 0)$ to P_1 and $(0 : 1)$ to P_2, we may assume without loss of generality that $\Phi = (\phi, (1:0), (0:1))$ for some map ϕ given by

$$\phi(X:Y) = (X^2 + a_0XY : b_0XY + c_0Y^2)$$

for coefficients $a_0, b_0, c_0 \in K$.

Fix a place $v \in M_K \setminus S$. Since Φ has good reduction at v, it follows from Proposition 4 that Φ is K-isomorphic to some map $\Psi_v = (\psi_v, (1:0), (0:1))$, where

$$\psi_v(X:Y) = (X^2 + a_vXY : b_vXY + c_vY^2) \tag{1.10}$$

for $a_v, b_v, c_v \in \mathcal{O}_v^\times$ such that

$$\operatorname{Res}(X^2 + a_vXY, b_vXY + c_vY^2)) = c_v(c_v - a_vb_v) \in \mathcal{O}_v^\times.$$

Let $f_v \in \mathrm{PGL}_2(K)$ be the automorphism such that $\Psi_v = \Phi^{f_v}$. Since both ψ_v and ϕ fix the points $(1:0)$ and $(0:1)$, the automorphism f_v must fix these points as well, so we may write $f_v(X,Y) = (\alpha_vX : Y)$ for some $\alpha_v \in K^\times$. Conjugating ϕ by f_v we obtain

$$\phi^{f_v}(X:Y) = (X^2 + \alpha_v^{-1}a_0XY : b_0XY + \alpha_v^{-1}c_0Y^2). \tag{1.11}$$

Since $\psi_v = \phi^{f_v}$, comparing (1.10) and (1.11) we obtain the three identities

$$a_v = \alpha_v^{-1}a_0$$

$$b_v = b_0$$

$$c_v = \alpha_v^{-1}c_0.$$

Since $\mathcal{O}_S$ is a principal ideal domain, there exists $\alpha \in K^\times$ such that $|\alpha|_v = |\alpha_v|_v$ for each $v \in M_K \setminus S$. Define $\Psi = (\psi, (1:0), (0:1))$ for

$$\psi(X:Y) = (X^2 + aXY : bXY + cY^2)$$

with coefficients given by

$$a = \alpha^{-1}a_0$$

$$b = b_0$$

$$c = \alpha^{-1}c_0.$$

Then $\Psi = \Phi^f$ for the automorphism $f \in \mathrm{PGL}_2(K)$ defined by $f(X:Y) = (\alpha X : Y)$. Furthermore, for each place $v \in M_K \setminus S$ we have $|a|_v = |\alpha^{-1}a_0|_v = |\alpha_v^{-1}a_0|_v = |a_v|_v = 1$, whereby $a \in \mathcal{O}_v^\times$; similar calculations show that b, c, and $c(c-ab)$ are all elements of the unit group $\mathcal{O}_v^\times$ as well. Since these elements are in $\mathcal{O}_v^\times$ for all $v \in M_K \setminus S$, we have $a, b, c, c(c-ab) \in \mathcal{O}_S^\times$. □

Lemma 6. *Given $u \in \bar{K}$, define V_u to be the set of all $\langle\phi\rangle$ in $\mathcal{M}_2$ for $\phi \in \mathrm{Rat}_2$ of the form*

$$\phi(X:Y) = (X^2 + aXY : bXY + cY^2) \tag{1.12}$$

with $\frac{ab}{c} = u$. Then V_u is a proper Zariski-closed subset of $\mathcal{M}_2$.

Proof. Define W_u to be the set of maps $\phi \in \mathrm{Rat}_2$ of the form (1.12) with $\frac{ab}{c} = u$. In the notation of §1.2.1 and §1.2.2, W_u is the intersection of the three hypersurfaces $\{a_2 = 0\}$, $\{b_0 = 0\}$, and $\{a_1b_1 = ua_0b_2\}$ in Rat_2, and thus W_u is Zariski-closed in Rat_2. Since V_u is the image of W_u under the closed quotient map $\langle\cdot\rangle : \mathrm{Rat}_2 \to \mathcal{M}_2$, it follows that V_u is Zariski-closed in $\mathcal{M}_2$.

It remains to show that V_u is a proper subset of $\mathcal{M}_2$; we will prove the stronger statement that V_u and $V_{u'}$ are disjoint whenever $u \neq u'$. For if $V_u \cap V_{u'}$ is nonempty, then there exist maps $\phi \in W_u$ and $\phi' \in W_{u'}$ which are PGL_2-conjugate to one another; say $\phi' = \phi^f$ for $f \in \mathrm{PGL}_2$. In the obvious notation we therefore have $\frac{ab}{c} = u$ and $\frac{a'b'}{c'} = u'$. Since both ϕ and ϕ' fix both $(0 : 1)$ and $(1 : 0)$, the automorphism f must fix these points as well, whereby $f(X : Y) = (\alpha X : Y)$ for some $\alpha \in \bar{K}$. But for maps of the form (1.12), the quantity $\frac{ab}{c}$ is invariant under the action of automorphisms of this form, because conjugating ϕ by f we have

$$\phi^f(X : Y) = (X^2 + \alpha^{-1}aXY : bXY + \alpha^{-1}cY^2)$$

and $\frac{(\alpha^{-1}a)b}{\alpha^{-1}c} = \frac{ab}{c}$. It follows that $u = u'$. This completes the verification that $V_u \cap V_{u'} = \emptyset$ whenever $u \neq u'$, and therefore each V_u is a proper subset of $\mathcal{M}_2$. □

Proof of Theorem 3. As enlarging S proves a stronger statement, first let us increase the size of S so that $\mathcal{O}_S$ is a principal ideal domain.

Each point in $\mathcal{G}_{2,2}^{\mathrm{uf}}(K,S)$ is $\langle\Phi\rangle$ for some $\Phi \in \mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ with good reduction at all places $v \in M_K \setminus S$, and according to Proposition 5 we may assume without loss of generality that each such map takes the form $\Phi = (\phi, (1 : 0), (0 : 1))$ where

$$\phi(X : Y) = (X^2 + aXY : bXY + cY^2)$$

for $a, b, c, c(c-ab) \in \mathcal{O}_S^\times$. It follows that $(\frac{c-ab}{c}, \frac{ab}{c})$ is a solution in $(\mathcal{O}_S^\times)^2$ to the unit equation $x+y=1$. Since there are only finitely many such solutions ([1] §5.1), there exists a finite list of units $u_1, \dots, u_r \in \mathcal{O}_S^\times$ such that

$$\mathcal{G}_{2,2}^{\mathrm{uf}}(K,S) \subseteq V_{u_1} \cup \cdots \cup V_{u_r}, \tag{1.13}$$

where V_u denotes the the set of all $\langle\phi\rangle$ in $\mathcal{M}_2$ for $\phi \in \mathrm{Rat}_2$ of the form $\phi(X : Y) = (X^2 + aXY : bXY + cY^2)$ with $\frac{ab}{c} = u$.

By Lemma 6, each V_u is a proper Zariski-closed subset of $\mathcal{M}_2$, and therefore (1.13) shows that $\mathcal{G}_{2,2}^{\mathrm{uf}}(K,S)$ is not Zariski-dense in $\mathcal{M}_2$. □

Remark 2. Theorem 3 cannot, in general, be improved to a finiteness theorem. For if $\mathcal{O}_S^\times$ is infinite and both u and $1-u$ are S-units, then considering the maps

$$\phi(X : Y) = (X^2 + aXY : a^{-1}uXY + Y^2)$$

as a varies in $\mathcal{O}_S^\times$ produces an infinite subset of $\mathcal{G}_{2,2}^{\mathrm{uf}}(K,S) \cap V_u$.

On the other hand, $\mathcal{G}_{2,2}^{\mathrm{uf}}(\mathbb{Q}, \{\infty\})$ and $\mathcal{G}_{2,2}^{\mathrm{uf}}(\mathbb{Q}(i), \{\infty\})$ are both empty, where $i = \sqrt{-1}$, and where ∞ denotes the sole Archimedean place of these fields, because the rings $\mathbb{Z}$ and $\mathbb{Z}[i]$ have no solutions in units to the unit equation $x + y = 1$. Denoting by ρ a primitive sixth root of unity, the set $\mathcal{G}_{2,2}^{\mathrm{uf}}(\mathbb{Q}(\rho), \{\infty\})$ is finite (by Proposition 5 along with the fact that the ring of integers in $\mathbb{Q}(\rho)$ has finite unit group) and nonempty (consider $\phi(X : Y) = (X^2 + \rho XY : XY + Y^2)$.)

Remark 3. Since a generic quadratic rational map in Rat_2 has three distinct unramified fixed points over $\bar{K}$, it would be reasonable to ask why we consider rational maps with double (rather than single or triple) unramified fixed-point structure.

First, Theorem 3 would be false in general if double unramified fixed-point structure were replaced by *single* unramified fixed-point structure, and a counterexample is given by the same family occurring in the second proof of Theorem 2. Recall that $\mathcal{V}$ is the set of all $\langle\phi\rangle$ in $\mathcal{M}_2$ for rational maps $\phi \in \mathrm{Rat}_2(K)$ taking the form $\phi(X : Y) = (X^2 + \alpha XY : (\frac{1-\beta}{\alpha})XY + Y^2)$ for S-units $\alpha, \beta \in \mathcal{O}_S^\times$. At all places $v \in M_K \setminus S$, the point $(0 : 1)$ reduces to an unramified fixed point of $\tilde{\phi}_v$, but assuming that $\mathcal{O}_S^\times$ is infinite, $\mathcal{V}$ is Zariski-dense in $\mathcal{M}_2$.

One might define the space $\mathrm{Rat}_{3,2}^{\mathrm{uf}}(K)$ of quadratic rational maps with *triple* unramified fixed-point structure over K; but the non-Zariski-density of the image in $\mathcal{M}_2$ of the set of all such maps having prescribed good reduction would follow at once from Theorem 3. Indeed, the map $\mathrm{Rat}_{3,2}^{\mathrm{uf}}(K) \to \mathcal{M}_2(K)$ factors through the map $\mathrm{Rat}_{3,2}^{\mathrm{uf}}(K) \to \mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ which remembers the first two fixed points and forgets the third. In an obvious extension of the notation of Theorem 3, we therefore have $\mathcal{G}_{3,2}^{\mathrm{uf}}(K,S) \subseteq \mathcal{G}_{2,2}^{\mathrm{uf}}(K,S)$.

Remark 4. Although we have not attempted to carry out the necessary details, it may be possible to use geometric invariant theory to define a new moduli space $\mathcal{M}_{2,2}^{\mathrm{uf}} = \mathrm{Rat}_{2,2}^{\mathrm{uf}}/\mathrm{PGL}_2$ as the quotient of the quasiprojective variety $\mathrm{Rat}_{2,2}^{\mathrm{uf}}$ of quadratic rational maps with double unramified fixed-point structure modulo the conjugation action of PGL_2. While such a space may be of interest on its own merits, for our purposes there would be little to gain in such a construction, because a non-Zariski-density result in $\mathcal{M}_{2,2}^{\mathrm{uf}}$ for maps with prescribed good reduction would follow trivially from Theorem 3, using the dominant map $\mathcal{M}_{2,2}^{\mathrm{uf}} \to \mathcal{M}_2$ obtained from forgetting the double unramified fixed-point structure.

1.5 Prescribed good reduction for quadratic maps with unramified 2-cycle structure

With nearly the same proof, a variation on Theorem 3 can be established in which 2-cycle structure is used in place of double fixed-point structure.

Definition. Let $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ be the set of all triples of the form $\Phi = (\phi, P_1, P_2)$, where $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a quadratic rational map defined over K, and where $P_1, P_2 \in \mathbb{P}^1(K)$ are distinct K-rational points which are not ramified points of ϕ and for which $\phi(P_1) = P_2$ and $\phi(P_2) = P_1$. We call such a triple Φ a *quadratic rational map with double unramified* 2*-cycle structure over* K.

Definitions. Two maps Φ and Ψ in $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ are K*-isomorphic* if $\Psi = \Phi^f$ for some automorphism $f \in \mathrm{PGL}_2(K)$, where

$$\Phi^f = (\phi^f, f^{-1}(P_1), f^{-1}(P_2)).$$

A map Φ in $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ has *good reduction* at a non-Archimedean place v of K if it is K-isomorphic to some $\Psi = (\psi, Q_1, Q_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ such that $\deg(\tilde{\psi}_v) = 2$ and such that $\tilde{Q}_1$ and $\tilde{Q}_2$ are distinct points in $\mathbb{P}^1(\mathbb{F}_v)$ which are not ramified points of $\tilde{\psi}_v$ and for which $\tilde{\psi}_v(\tilde{Q}_1) = \tilde{Q}_2$ and $\tilde{\psi}_v(\tilde{Q}_2) = \tilde{Q}_1$.

For each $\Phi = (\phi, P_1, P_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$, define $\langle\Phi\rangle = \langle\phi\rangle$. Thus, as before, the map $\langle\cdot\rangle : \mathrm{Rat}_{2,2}^{\mathrm{uc}}(K) \to \mathcal{M}_2(K)$ forgets the 2-cycle structure of Φ and preserves only the PGL_2-conjugacy class $\langle\phi\rangle$ of its underlying rational map.

Theorem 7. *Let K be a number field and let S be a finite set of places of K including the Archimedean places. Then the set*

$$\mathcal{G}_{2,2}^{\mathrm{uc}}(K,S) = \left\{ \langle\Phi\rangle \in \mathcal{M}_2(K) \,\middle|\, \begin{array}{l} \Phi \in \mathrm{Rat}_{2,2}^{\mathrm{uc}}(K) \textit{ has good} \\ \textit{reduction at all } v \in M_K \setminus S \end{array} \right\}$$

is not Zariski-dense in $\mathcal{M}_2$.

Proof. This proof follows precisely the same strategy as that of Theorem 3, and so we only give a sketch to highlight where this proof differs from the previous one.

Again, without loss of generality we may enlarge S so that $\mathcal{O}_S$ is a principal ideal domain. Each point in $\mathcal{G}_{2,2}^{\mathrm{uc}}(K,S)$ is $\langle\Phi\rangle$ for some $\Phi \in \mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ with good reduction at all places $v \in M_K \setminus S$, and in a similar fashion as in Proposition 5, it may be shown that each such map (up to K-isomorphism) takes the form $\Phi = (\phi, (1:0), (0:1))$ where

$$\phi(X:Y) = (aXY + bY^2 : X^2 + cXY)$$

for $a, b, c, b(b-ac) \in \mathcal{O}_S^\times$. It follows that $(\frac{b-ac}{b}, \frac{ac}{b})$ is a solution in $(\mathcal{O}_S^\times)^2$ to the unit equation $x+y=1$. Since there are only finitely many such solutions ([1] §5.1), there exists a finite list of units $u_1, \ldots, u_r \in \mathcal{O}_S^\times$ such that

$$\mathcal{G}_{2,2}^{\mathrm{uf}}(K,S) \subseteq Z_{u_1} \cup \cdots \cup Z_{u_r},$$

where Z_u denotes the the set of all $\langle\phi\rangle$ in $\mathcal{M}_2$ for $\phi \in \mathrm{Rat}_2$ of the form $\phi(X : Y) = (aXY + bY^2 : X^2 + cXY)$ with $\frac{ac}{b} = u$, and each Z_u is a proper Zariski-closed subset of $\mathcal{M}_2$. $\square$

Chapter 2

Existence of Global Minimal Models

2.1 Definitions and statement of the main results

Let R be a principal ideal domain (PID) with field of fractions K, and let N be a positive integer. In this chapter, our primary objects of study are morphisms $\phi : \mathbb{P}^N \to \mathbb{P}^N$ defined over K. Fixing a choice of homogeneous coordinates $\mathbf{x} = (x_0, \ldots, x_N)$ on $\mathbb{P}^N$, we may write ϕ explicitly as

$$\phi(x_0 : \cdots : x_N) = (\Phi_0(x_0, \ldots, x_N) : \cdots : \Phi_N(x_0, \ldots, x_N)), \tag{2.1}$$

where $\Phi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ is a map defined by an $(N+1)$-

$$\Phi = (\Phi_0, \ldots, \Phi_N)$$

of forms of some common degree $d \geq 1$ in the variables $x_0, x_1, \ldots, x_N$, with the property that

$$\Phi(\mathbf{a}) \neq \mathbf{0} \text{ whenever } \mathbf{a} \in \mathbb{A}^{N+1}(\bar{K}) \setminus \mathbf{0}, \tag{2.2}$$

or equivalently that

$$\operatorname{Res}(\Phi) \neq 0, \tag{2.3}$$

where $\operatorname{Res}(\Phi)$ is the resultant of Φ, a certain homogeneous integral polynomial in the coefficients of the forms Φ_n; see Proposition 15 for a review of the necessary facts about the resultant. We refer to d as the *algebraic degree* of ϕ, and we refer to the map Φ, which is uniquely determined by ϕ up to multiplication by a nonzero scalar in K, as a *homogeneous lift* for ϕ.

Conversely, starting with any map $\Phi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ defined by an $(N+1)$-tuple $\Phi = (\Phi_0, \ldots, \Phi_N)$ of forms of some common degree $d \geq 0$, such that Φ satisfies the nonvanishing condition (2.2), the formula (2.1) gives rise to a morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ of algebraic degree d.

In the study of the dynamical system obtained from iteration of the morphism ϕ, it is generally true that the dynamical properties of ϕ are left unchanged when it is replaced with its conjugate $f \circ \phi \circ f^{-1}$ by an element f of the automorphism group $\mathrm{PGL}_{N+1}(K)$ of $\mathbb{P}^N$ over K. Given a representative $A \in \mathrm{GL}_{N+1}(K)$ for f under the quotient map $\mathrm{GL}_{N+1} \to \mathrm{PGL}_{N+1}$, and given a homgoeneous lift $\Phi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ for ϕ, observe that the map $\Psi = A \circ \Phi \circ A^{-1} : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ is a homogeneous lift for $\psi = f \circ \phi \circ f^{-1}$. It is therefore natural to offer the following loosening of the notion of a homogeneous lift for ϕ.

Definition. Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over K. A *model* for ϕ over K is a map $\Psi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ given by $\Psi = A \circ \Phi \circ A^{-1}$ for some homogeneous lift $\Phi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ of ϕ and some linear automorphism $A \in \mathrm{GL}_{N+1}(K)$ of $\mathbb{A}^{N+1}$.

While $\mathrm{PGL}_{N+1}(K)$-conjugation does not affect purely dynamical properties of morphisms, it does have subtle and unpredictable effects on integrality and divisibility properties in the ring R. For each nonzero prime ideal $\mathfrak{p}$ of R, denote by $K_\mathfrak{p}$ the completion of K with respect to the $\mathfrak{p}$-adic valuation, and let $R_\mathfrak{p}$ be the subring of $\mathfrak{p}$-integral elements of $K_\mathfrak{p}$. Let $\mathbb{F}_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ be the residue field at $\mathfrak{p}$, and denote by $x \mapsto \tilde{x}_\mathfrak{p}$ the surjective reduction map $R_\mathfrak{p} \to \mathbb{F}_\mathfrak{p}$.

Given a model $\Psi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ for a morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ defined over $K_\mathfrak{p}$, we declare that Ψ is *integral* (or $\mathfrak{p}$*-integral*) if each form Ψ_n has coefficients in $R_\mathfrak{p}$. If Ψ is $\mathfrak{p}$-integral, then we may reduce the coefficients modulo $\mathfrak{p}$ and obtain a homogeneous map $\tilde{\Psi}_\mathfrak{p} : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ defined over the residue field $\mathbb{F}_\mathfrak{p}$.

Definition. A morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ defined over $K_\mathfrak{p}$ has *good reduction* if ϕ has a $\mathfrak{p}$-integral model $\Psi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ satisfying either (and therefore both) of the following two equivalent conditions:

- **(a)** the reduced map $\tilde{\Psi}_\mathfrak{p} : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ satisfies $\tilde{\Psi}_\mathfrak{p}(\mathbf{a}) \neq \mathbf{0}$ whenever $\mathbf{a} \in \mathbb{A}^{N+1}(\overline{\mathbb{F}}_\mathfrak{p}) \setminus \mathbf{0}$;
- **(b)** $\operatorname{Res}(\Psi) \in R_\mathfrak{p}^\times$.

According to condition **(a)**, this definition has the following fairly intuitive interpretation: a morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ of algebraic degree $d \geq 1$ defined over $K_\mathfrak{p}$ has good reduction precisely when it is $\mathrm{PGL}_{N+1}(K)$-conjugate to a morphism $\psi : \mathbb{P}^N \to \mathbb{P}^N$ for which reduction modulo $\mathfrak{p}$ gives rise to a morphism $\tilde{\psi}_\mathfrak{p} : \mathbb{P}^N \to \mathbb{P}^N$ of algebraic degree d defined over the residue field $\mathbb{F}_\mathfrak{p}$. The equivalence of conditions **(a)** and **(b)** is a simple conseuquence of basic properties of the resultant, along with the fact that the unit group $R_\mathfrak{p}^\times$ is precisely the set of elements in $R_\mathfrak{p}$ whose image is nonzero under the reduction map $R_\mathfrak{p} \to \mathbb{F}_\mathfrak{p}$.

If $\Psi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ is an arbitrary $\mathfrak{p}$-integral model for ϕ, then

$$\operatorname{ord}_\mathfrak{p}(\operatorname{Res}(\Psi)) \geq 0$$

since $\operatorname{Res}(\Psi)$ is an integral polynomial in the coefficients of Ψ; good reduction at $\mathfrak{p}$ occurs precisely when a $\mathfrak{p}$-integral model Ψ can be found with $\operatorname{ord}_{\mathfrak{p}}(\operatorname{Res}(\Psi)) = 0$. Even in the case of bad reduction, however, one might still ask for a $\mathfrak{p}$-integral model Ψ for ϕ with $\operatorname{ord}_{\mathfrak{p}}(\operatorname{Res}(\Psi))$ as small as possible.

Definition. Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over $K_{\mathfrak{p}}$. A $\mathfrak{p}$-integral model $\Psi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ for ϕ is *minimal* (or $\mathfrak{p}$*-minimal*) if $\operatorname{ord}_{\mathfrak{p}}(\operatorname{Res}(\Psi))$ is minimal among all $\mathfrak{p}$-integral models Ψ for ϕ.

We can now state the main theorem of this chapter. Given a morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ defined over K, and a nonzero prime ideal $\mathfrak{p}$ of R, there always exists a minimal $\mathfrak{p}$-integral model Ψ for ϕ: start with an arbitrary model defined over $K_{\mathfrak{p}}$, scale by a $\mathfrak{p}$-adic uniformizing parameter to obtain a $\mathfrak{p}$-integral model Ψ, and among all such Ψ, select one for which $\operatorname{ord}_{\mathfrak{p}}(\operatorname{Res}(\Psi))$ is minimal. A priori these minimal $\mathfrak{p}$-integral models vary from prime to prime, but it is natural to ask whether one can find a *global minimal model*; that is, a model defined over R which is simultaneously a minimal $\mathfrak{p}$-integral model at all prime ideals $\mathfrak{p}$ of R.

Theorem 8. *Let R be a PID with field of fractions K, and let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over K. Then ϕ has a model $\Psi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$, with coefficients in R, and which is $\mathfrak{p}$-minimal for all nonzero prime ideals $\mathfrak{p}$ of*

R.

An interesting special case of Theorem 8 occurs when the morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ is assumed to have *everywhere* good reduction; that is, when ϕ has good reduction at all nonzero prime ideals $\mathfrak{p}$ of R. While this represents an extremal case of Theorem 8, it is perhaps not as special as it may appear: since any morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ defined over K has good reduction at all except a finite set S of nonzero prime ideals $\mathfrak{p}$ of R, replacing R with the larger PID $R_S = \{r \in K \mid \mathrm{ord}_{\mathfrak{p}}(r) \geq 0 \text{ for all } \mathfrak{p} \notin S\}$, we observe that ϕ has everywhere good reduction over R_S.

Corollary 9. *Let R be a PID with field of fractions K, let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over K, and assume that ϕ has good reduction at all nonzero prime ideals $\mathfrak{p}$ of R. Then ϕ has a model $\Psi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$, with coefficients in R, such that* $\mathrm{Res}(\Psi) \in R^\times$.

In the case $N = 1$, Theorem 8 was proposed by Silverman ([17] pp. 236-237) and proved by Bruin-Molnar [3]; thus our result generalizes this to arbitrary dimension $N \geq 1$. Our proof is not a straightforward generalization the proof by Bruin-Molnar, however. In [3], it is shown that, in order to produce a global minimal model for a rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, one only needs to consider conjugates $f \circ \phi \circ f^{-1}$ of ϕ by f in the group Aff_2 of automorphisms leaving ∞ fixed; i.e. automorphisms taking the form $f(x) = \alpha x + \beta$ in an affine coordinate x. We do not know whether, in the higher dimensional case, a generalization of Aff_2 can be used in a similar fashion leading to a proof of Theorem 8.

Our proof of Theorem 8 relies on the theory of lattices over a PID, and in particular on the action of the adelic general linear group $\mathrm{GL}_n(\mathbb{A}_R)$ on the space of all such lattices of rank n. The main technical lemma of this chapter is a factorization of the group $\mathrm{GL}_n(\mathbb{A}_R)$ as the product of the subgroup $\mathrm{GL}_n(K)$ of principal adeles with the direct product $\mathrm{GL}_n^0(\mathbb{A}_R) = \prod_{\mathfrak{p}} \mathrm{GL}_n(R_{\mathfrak{p}})$. When R is a ring of S-integers in a number field K, this follows from a more general result of Borel [2] on the finiteness of the class number of GL_n. Since we have not been able to find the required material worked out over an arbitrary PID, in this chapter we give a self-contained treatment.

Theorem 8 and Corollary 9 may find arithmetic applications in the setting of a global field K (a number field or a function field with a finite constant field) and a finite subset S of places of K. After possibly replacing S with a suitable larger finite set of places, it is always possible to obtain the situation in which the ring $\mathcal{O}_S$ of S-integers is a PID. Applications of this idea, in slightly different contexts, can be found in the proof of Shafarevich's Theorem for elliptic curves (see [18] §IX.6), as well as an analogue for rational maps due to Petsche [13].

2.2 Global and local lattices over a PID

Throughout this chapter R is a PID with field of fractions K, and $R^\times$ denotes the group of units in R. The set of non-zero prime (and thus maximal) ideals of R will be denoted by M_R. For each $\mathfrak{p} \in M_R$, let $K_\mathfrak{p}$ be the completion of K with respect to the discrete valuation $\mathrm{ord}_\mathfrak{p}(\cdot)$ on K, and let

$$R_\mathfrak{p} = \{a \in K_\mathfrak{p} \mid \mathrm{ord}_\mathfrak{p}(a) \geq 0\}$$

$$R_\mathfrak{p}^\times = \{a \in K_\mathfrak{p} \mid \mathrm{ord}_\mathfrak{p}(a) = 0\}$$

be the subring of $\mathfrak{p}$-integral elements of $K_\mathfrak{p}$, and its unit group, respectively. It is a standard exercise to check the identities

$$\begin{aligned} R &= \{a \in K \mid \mathrm{ord}_\mathfrak{p}(a) \geq 0 \text{ for all } \mathfrak{p} \in M_R\} \\ R^\times &= \{a \in K \mid \mathrm{ord}_\mathfrak{p}(a) = 0 \text{ for all } \mathfrak{p} \in M_R\}. \end{aligned} \tag{2.4}$$

Proposition 10. *Let X be an R-submodule of K^n. Then the following three conditions are equivalent:*

(i) X is free and rank$(X) = n$.

(ii) $aR^n \subseteq X \subseteq bR^n$ for some $a, b \in K^\times$.

(iii) $X = AR^n$ for some $A \in \mathrm{GL}_n(K)$.

Proof. (i) ⇒ (iii): If (i) holds, let A be an $n \times n$ matrix over K whose columns form an R-basis for X. Then $X = AR^n$ and A is nonsingular, hence

$A \in \mathrm{GL}_n(K)$. (If A were singular, then there would be a non-trivial K-linear dependence among the columns of A; multiplying by the product of the denominators of the coefficients of this linear dependence, we would obtain a linear dependence with coefficients in R, in violation of the assumption that the columns of A form an R-basis for X.)

(iii) $\Rightarrow$ (ii): If (iii) holds, let $A \in \mathrm{GL}_n(K)$ such that $X = AR^n$. Let a_{ij} denote the entries of A and let b be the reciprocal of the product of the denominators of the a_{ij} for $1 \leq i, j \leq n$. Then $b^{-1}X = b^{-1}AR^n \subseteq R^n$ since $b^{-1}A$ has entries in R, and therefore $X \subseteq bR^n$. Let b_{ij} denote the entries of A^{-1} and let a be the product of the denominators of the b_{ij} for $1 \leq i, j \leq n$. Then $aR^n \subseteq aA^{-1}X \subseteq X$ since aA^{-1} has entries in R and X is an R-module.

(ii) $\Rightarrow$ (i): Since $x \mapsto ax$ is an isomorphism $R^n \to aR^n$, we see that aR^n is a free R-module of rank n; the same is true of bR^n. Since R is a PID, it follows from Theorem 7.1 of [9] that any R-submodule of bR^n is also free of rank less than or equal to n. Since $X \subseteq bR^n$, X is free and $\mathrm{rank}(X) \leq \mathrm{rank}(bR^n)$. The inequality $\mathrm{rank}(aR^n) \leq \mathrm{rank}(X)$ now follows from the same theorem, as X has been shown to be free. Because aR^n and bR^n are both of rank n, it follows that X has rank n. □

Definition. An *R-lattice* in K^n is a free R-submodule of K^n of rank n.

For each $\mathfrak{p} \in M_R$, the local ring $R_{\mathfrak{p}}$ is itself a PID, and thus Proposition 10 applies to $R_{\mathfrak{p}}$-submodules of $K_{\mathfrak{p}}^n$. In particular, an $R_{\mathfrak{p}}$-lattice in $K_{\mathfrak{p}}^n$ is a free $R_{\mathfrak{p}}$-submodule of $K_{\mathfrak{p}}^n$ of rank n.

If X is an R-lattice in K^n and $\mathfrak{p} \in M_R$ is a nonzero prime ideal of R, there is a natural way to associate to X an $R_{\mathfrak{p}}$-lattice $X_{\mathfrak{p}}$ in $K_{\mathfrak{p}}^n$. By Proposition 3, we may find some $A \in \mathrm{GL}_n(K)$ such that $X = AR^n$, and we define $X_{\mathfrak{p}} = AR_{\mathfrak{p}}^n$. This definition does not depend on the choice of matrix A. For if $X = BR^n$, then $A^{-1}B$ is an isomorphism $R^n \to R^n$, and therefore $A^{-1}B \in \mathrm{GL}_n(R) \subseteq \mathrm{GL}_n(R_{\mathfrak{p}})$. Then $A^{-1}BR_{\mathfrak{p}}^n = R_{\mathfrak{p}}^n$ and therefore $BR_{\mathfrak{p}}^n = AR_{\mathfrak{p}}^n$. The definition of $X_{\mathfrak{p}}$ is equivalent to the $R_{\mathfrak{p}}$-module $X \otimes_R R_{\mathfrak{p}}$ obtained by extension of scalars.

Lemma 11. *Let X be an R-lattice in K^n. Then for every $\mathfrak{p} \in M_R$, $X_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-lattice in $K_{\mathfrak{p}}^n$, and for almost every $\mathfrak{p} \in M_R$, $X_{\mathfrak{p}} = R_{\mathfrak{p}}^n$.*

Proof. Let X=AR^n for $A \in \mathrm{GL}_n(K)$. For any $\mathfrak{p} \in M_R$, we have that $X_{\mathfrak{p}} = AR_{\mathfrak{p}}^n$ and therefore $X_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-lattice in $K_{\mathfrak{p}}^n$ by Proposition 10. Furthermore, $X_{\mathfrak{p}} = R_{\mathfrak{p}}^n$ for all $\mathfrak{p} \in M_R$ except for the finitely many $\mathfrak{p}$ for which $A \notin \mathrm{GL}_n(R_{\mathfrak{p}})$. These primes correspond to the irreducible elements which occur in the denominators of the entries of A or in the numerator of the determinant of A. □

Lemma 12. *Conversely, suppose that* $(X_\mathfrak{p})$ *is a collection of* $R_\mathfrak{p}$*-lattices in* $K_\mathfrak{p}^n$ *for each* $\mathfrak{p} \in M_R$*, such that* $X_\mathfrak{p} = R_\mathfrak{p}^n$ *for almost every* $\mathfrak{p}$*. Then*

$$X' = \{x \in K^n | x \in X_\mathfrak{p} \textit{ for all } \mathfrak{p}\}$$

is an R*-lattice in* K^n*, and* $X'_\mathfrak{p} = X_\mathfrak{p}$ *for each prime* $\mathfrak{p} \in M_R$*.*

Proof. X' is plainly an R-submodule of K^n because $R \subseteq R_\mathfrak{p}$ for all $\mathfrak{p} \in M_R$ and each $X_\mathfrak{p}$ is an $R_\mathfrak{p}$-submodule of $K_\mathfrak{p}^n$. By Proposition 3, to show that X' is free of rank n it is sufficient to show that $aR^n \subseteq X' \subseteq bR^n$ for some $a, b \in K^\times$. As each $X_\mathfrak{p}$ is an $R_\mathfrak{p}$-lattice in $K_\mathfrak{p}^n$, we know by Proposition 3 that a similar chain of inclusions $a_\mathfrak{p} R_\mathfrak{p}^n \subseteq X_\mathfrak{p} \subseteq b_\mathfrak{p} R_\mathfrak{p}^n$ holds for each each prime $\mathfrak{p}$ where $a_\mathfrak{p}, b_\mathfrak{p} \in K_\mathfrak{p}^\times$. By the assumption $X_\mathfrak{p} = R_\mathfrak{p}^n$ for almost every $\mathfrak{p}$, we may assume that $a_\mathfrak{p} = b_\mathfrak{p} = 1$ for almost every $\mathfrak{p}$. Because R is a PID we may assume that both $a_\mathfrak{p}$ and $b_\mathfrak{p}$ are powers of $\mathfrak{p}$-adic uniformizing parameters in R. Let $a = \Pi_\mathfrak{p} a_\mathfrak{p}, b = \Pi_\mathfrak{p} b_\mathfrak{p} \in K^\times$ and it follows that $aR_\mathfrak{p}^n \subseteq X_\mathfrak{p} \subseteq bR_\mathfrak{p}^n$. Using (2.4) we have that $aR^n = \{x \in K^n | x \in aR_\mathfrak{p}^n \text{ for all } \mathfrak{p}\}$ and that $bR^n = \{x \in K^n | x \in bR_\mathfrak{p}^n \text{ for all } \mathfrak{p}\}$. Therefore $aR^n \subseteq X' \subseteq bR^n$ and we conclude X' to be an R-lattice.

Lastly, we show that $X'_\mathfrak{p} = X_\mathfrak{p}$ for all $\mathfrak{p} \in M_R$. The inclusion $X'_\mathfrak{p} \subseteq X_\mathfrak{p}$ follows immediately from the definitions: Proposition 10 provides an element $A \in \mathrm{GL}_n(K)$ such that $X' = AR^n$, and $X'_\mathfrak{p} = AR_\mathfrak{p}^n$. Since $X' \subseteq X_\mathfrak{p}$, the column vectors of A are in $X_{\mathfrak{p}}$, whereby $X'_{\mathfrak{p}} = AR^n_{\mathfrak{p}} \subseteq X_{\mathfrak{p}}$.

To show equality $X'_{\mathfrak{p}} = X_{\mathfrak{p}}$ for all $\mathfrak{p} \in M_R$, suppose there exists some $\mathfrak{p}_0 \in M_R$ with proper inclusion $X'_{\mathfrak{p}_0} \subsetneq X_{\mathfrak{p}_0}$; we will derive a contradiction.

Let $\mathbb{A}^n_R$ be the affine adelic space over R. This space is the restricted direct product of the affine spaces $K^n_{\mathfrak{p}}$ with respect to the subsets $R^n_{\mathfrak{p}}$. Specifically,

$$\mathbb{A}^n_R = \left\{ (a_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in M_R} K^n_{\mathfrak{p}} \,\middle|\, a_{\mathfrak{p}} \in R_{\mathfrak{p}} \text{ for almost all } \mathfrak{p} \right\}.$$

Thus an arbitrary element of $\mathbb{A}^n_R$ is a tuple $(a_{\mathfrak{p}})$, indexed by the primes $\mathfrak{p} \in M_R$, where each $a_{\mathfrak{p}} \in K^n_{\mathfrak{p}}$, and where $a_{\mathfrak{p}} \in R^n_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. The affine adelic space has a topology whose basis consists of sets of the form $\Pi_{\mathfrak{p}} U_{\mathfrak{p}}$, where each $U_{\mathfrak{p}}$ is an open subset of $K^n_{\mathfrak{p}}$ and where $U_{\mathfrak{p}} = R^n_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. Naturally, K^n is a subset of $\mathbb{A}^n_R$ by identifying $a \in K^n$ with the principal adele $(a_{\mathfrak{p}})$, where $a_{\mathfrak{p}} = a$ for all $\mathfrak{p}$.

Define subsets of $\mathbb{A}^n_R$ by $Y' = \Pi_{\mathfrak{p}} X'_{\mathfrak{p}}$ and $Y = \Pi_{\mathfrak{p}} X_{\mathfrak{p}}$. Since we have already shown that $X'_{\mathfrak{p}} \subseteq X_{\mathfrak{p}}$ for all $\mathfrak{p} \in M_R$, and since we have assumed that $X'_{\mathfrak{p}_0} \subsetneq X_{\mathfrak{p}_0}$ for some $\mathfrak{p}_0 \in M_R$, it follows that $Y' \subsetneq Y$. Since an arbitrary $R_{\mathfrak{p}}$-lattice is both open and closed in $K^n_{\mathfrak{p}}$, it follows from the definition of the restricted direct product topology that Y and Y' are both open and closed in $\mathbb{A}^n_R$, and therefore that $Y \setminus Y'$ is a nonempty open subset of $\mathbb{A}^n_R$. It follows from a standard argument that K^n is a dense subset of $\mathbb{A}^n_R$. (When $R = \mathbb{Z}$, this is the most basic form of the weak approximation theorem, the proof of which can be found in Cassels ([4] Ch. II, §14, 15); a direct generalization of this argument holds for an arbitrary PID.) Therefore, there exists $x \in K^n$ whose principal adele (x) is an element of $Y \setminus Y'$. Since $(x) \in Y = \Pi_{\mathfrak{p}} X_{\mathfrak{p}}$, we have $x \in X_{\mathfrak{p}}$ for all $\mathfrak{p}$ and hence by definition, $x \in X'$. It follows that $x \in X'_{\mathfrak{p}}$ for all $\mathfrak{p}$ and consequently $(x) \in \Pi_{\mathfrak{p}} X'_{\mathfrak{p}} = Y'$. This contradiction implies that $X'_{\mathfrak{p}} = X_{\mathfrak{p}}$ for all $\mathfrak{p} \in M_R$. $\square$

2.3 The adelic general linear group over a PID

The adelic general linear group $\mathrm{GL}_n(\mathbb{A}_R)$ associated to R is the restricted direct product of the groups $\mathrm{GL}_n(K_{\mathfrak{p}})$ with respect to the subgroups $\mathrm{GL}_n(R_{\mathfrak{p}})$. More specifically,

$$\mathrm{GL}_n(\mathbb{A}_R) = \left\{ (A_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in M_R} \mathrm{GL}_n(K_{\mathfrak{p}}) \,\middle|\, A_{\mathfrak{p}} \in \mathrm{GL}_n(R_{\mathfrak{p}}) \text{ for almost all } \mathfrak{p} \right\}.$$

The main result of this section shows that the group $\mathrm{GL}_n(\mathbb{A}_R)$ factors into a product of two natural subgroups. First, $\mathrm{GL}_n(K)$ embeds into $\mathrm{GL}_n(\mathbb{A}_R)$ by the identification of each $A \in \mathrm{GL}_n(K)$ with the its *principal* adele $(A_{\mathfrak{p}})$, defined by $A_{\mathfrak{p}} = A$ for all $\mathfrak{p} \in M_R$. The second subgroup of $\mathrm{GL}_n(\mathbb{A}_R)$ is

$$\mathrm{GL}_n^0(\mathbb{A}_R) = \prod_{\mathfrak{p} \in M_R} \mathrm{GL}_n(R_{\mathfrak{p}}),$$

the direct product of the $R_\mathfrak{p}$-integral subgroups $\mathrm{GL}_n(R_\mathfrak{p})$, over all primes $\mathfrak{p} \in M_R$.

Proposition 13. $\mathrm{GL}_n(\mathbb{A}_R) = \mathrm{GL}_n^0(\mathbb{A}_R)\mathrm{GL}_n(K)$.

The following lemma contains most of work toward the proof of Proposition 13.

Lemma 14. *Let $\mathcal{X}_R$ denote the set of R-lattices in K^n. There exists a transitive group action*

$$\mathrm{GL}_n(\mathbb{A}_R) \times \mathcal{X}_R \to \mathcal{X}_R$$

$$(A, X) \mapsto A \cdot X,$$

where $A \cdot X$ is defined to be the R-lattice

$$A \cdot X = \{x \in K^n | x \in A_\mathfrak{p} X_\mathfrak{p} \textit{ for all } \mathfrak{p}\}.$$

Moreover, the stabilizer in $\mathrm{GL}_n(\mathbb{A}_R)$ of the trivial lattice R^n is $\mathrm{GL}_n^0(\mathbb{A}_R)$.

Proof. Let $A, B \in \mathrm{GL}_n(\mathbb{A}_R)$ and $X \in \mathcal{X}_R$. The fact that $A \cdot X$ is an R-lattice in K^n follows from Lemma 5.

Let $I = (I_\mathfrak{p})$ denote the identity adele: $I_\mathfrak{p}$ is the identity matrix in $\mathrm{GL}_n(K_\mathfrak{p})$ for each $\mathfrak{p} \in M_R$. We show that $I \cdot X = X$, or equivalently, that

$$\{x \in K^n | x \in X_\mathfrak{p} \text{ for all } \mathfrak{p}\} = X.$$

First, if $X = R^n$ then the desired identity

$$\{x \in K^n | x \in R_{\mathfrak{p}}^n \text{ for all } \mathfrak{p}\} = R^n$$

follows immediately from (2.4), and thus $I \cdot R^n = R^n$. Now let X be arbitrary. By Proposition 3, $X = AR^n$ for some $A \in \mathrm{GL}_n(K)$, and by definition $X_{\mathfrak{p}} = AR_{\mathfrak{p}}^n$. It follows that

$$\begin{aligned} I \cdot X &= \{x \in K^n | x \in X_{\mathfrak{p}} = AR_{\mathfrak{p}}^n \text{ for all } \mathfrak{p}\} \\ &= \{Ax | x \in K^n, x \in R_{\mathfrak{p}}^n \text{ for all } \mathfrak{p}\} \\ &= AR^n = X. \end{aligned}$$

The equality $A \cdot (B \cdot X) = (AB) \cdot X$ follows from the identity $(B \cdot X)_{\mathfrak{p}} = B_{\mathfrak{p}} X_{\mathfrak{p}}$, which itself is a trivial consequence of Lemma 5. Specifically,

$$\begin{aligned} A \cdot (B \cdot X) &= \{x \in K^n | x \in A_{\mathfrak{p}}(B \cdot X)_{\mathfrak{p}} \text{ for all } \mathfrak{p}\} \\ &= \{x \in K^n | x \in A_{\mathfrak{p}}(B_{\mathfrak{p}} X_{\mathfrak{p}}) \text{ for all } \mathfrak{p}\} \\ &= \{x \in K^n | x \in (AB)_{\mathfrak{p}} X_{\mathfrak{p}} \text{ for all } \mathfrak{p}\} \\ &= (AB) \cdot X. \end{aligned}$$

The transitivity of the action follows from Proposition 3: for any lattice X there is $A \in \mathrm{GL}_n(K)$ such that $X = AR^n$ and considering A as a principal adele it then follows that $X = A \cdot R^n$. Therefore every R-lattice in K^n is in the $\mathrm{GL}_n(\mathbb{A}_R)$-orbit of the trivial lattice.

Finally, we must show that the stabilizer in $\mathrm{GL}_n(\mathbb{A}_R)$ of the trivial lattice R^n is $\mathrm{GL}_n^0(\mathbb{A}_R)$; in other words, that

$$\{A \in \mathrm{GL}_n(\mathbb{A}_R) | A \cdot R^n = R^n\} = \mathrm{GL}_n^0(\mathbb{A}_R).$$

If $A = (A_\mathfrak{p}) \in \mathrm{GL}_n^0(\mathbb{A}_R)$, then $A_\mathfrak{p} \in \mathrm{GL}_n(R_\mathfrak{p})$ for all $\mathfrak{p} \in M_R$, which implies that $A_\mathfrak{p} R_\mathfrak{p}^n = R_\mathfrak{p}^n$. We conclude using (2.4) that

$$\begin{aligned} A \cdot R^n &= \{x \in K^n | x \in A_\mathfrak{p} R_\mathfrak{p}^n \text{ for all } \mathfrak{p}\} \\ &= \{x \in K^n | x \in R_\mathfrak{p}^n \text{ for all } \mathfrak{p}\} \\ &= R^n. \end{aligned}$$

Conversely, suppose $A = (A_\mathfrak{p}) \in \mathrm{GL}_n(\mathbb{A}_R)$ such that $A \cdot R^n = R^n$, which, by definition means that

$$\{x \in K^n | x \in A_\mathfrak{p} R_\mathfrak{p}^n \text{ for all } \mathfrak{p}\} = R^n. \tag{2.5}$$

Let X and Y denote the left-hand side and right-hand side of (2.5), respectively, and fix $\mathfrak{p} \in M_R$. Then trivially $Y_\mathfrak{p} = R_\mathfrak{p}^n$, and Lemma 5 shows that $X_\mathfrak{p} = A_\mathfrak{p} R_\mathfrak{p}^n$. We conclude that $A_\mathfrak{p} R_\mathfrak{p}^n = R_\mathfrak{p}^n$, and this implies that $A \in \mathrm{GL}_n(R_\mathfrak{p})$. [Proof: Let $\{e_i\} \in R_\mathfrak{p}^n$ be the standard basis. Then $A_\mathfrak{p} e_i \in R_\mathfrak{p}^n$ is the i^{th} column of M, showing that $A_\mathfrak{p}$ has coefficients in $R_\mathfrak{p}$. Similarly, $A_\mathfrak{p}^{-1}$ fixes $R_\mathfrak{p}^n$ and therefore $A_\mathfrak{p}^{-1}$ has coefficients in $R_\mathfrak{p}$]. Hence $A_\mathfrak{p} \in \mathrm{GL}_n(R_\mathfrak{p})$ for every prime $\mathfrak{p}$, and so by definition $A \in \mathrm{GL}_n^0(\mathbb{A}_r)$. $\square$

Proof of Proposition 13. Let $A \in \mathrm{GL}_n(\mathbb{A}_R)$ be an arbitrary adele. Let $X = A^{-1} \cdot R^n$ be the lattice obtained by letting A^{-1} act on the trivial lattice. By Proposition 10, $X = BR^n$ for $B \in \mathrm{GL}_n(K)$. Both A^{-1} and B take R^n bijectively onto X, so AB fixes R^n and therefore lies in the stabilizer $\mathrm{GL}_n^0(\mathbb{A}_R)$, say

$AB = C$ for $C \in \mathrm{GL}_n^0(\mathbb{A}_R)$. Therefore $A = CB^{-1} \in \mathrm{GL}_n^0(\mathbb{A}_R)\mathrm{GL}_n(K)$. □

2.4 The existence of global minimal models

In this section we prove the main results of the chapter, Theorem 8 and Corollary 9. First, however, we give a proposition summarizing the relevant properties of the resultant associated to a homogeneous map $\Phi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$.

Proposition 15. *Let* $\Phi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ *be a map defined over a field* K *by an* $(N+1)$*-tuple* $\Phi = (\Phi_0, \ldots, \Phi_N)$ *of forms of some common degree* $d \geq 1$ *in the variables* $x_0, x_1, \ldots, x_N$*, and let* $\mathrm{Res}(\Phi)$ *denote the resultant of* Φ*.*

(i) $\mathrm{Res}(\Phi) = 0$ *if and only if* $\Phi(\mathbf{a}) = \mathbf{0}$ *for some* $\mathbf{a} \in \mathbb{A}^{N+1}(\bar{K}) \setminus \mathbf{0}$.

(ii) If $A \in \mathrm{GL}_{N+1}(K)$ *is a linear automorphism of* $\mathbb{A}^{N+1}$ *defined over* K*, then* $\mathrm{Res}(A \circ \Phi \circ A^{-1}) = \det(A)^{C(N,d)}\mathrm{Res}(\Phi)$ *for some integer* $C(N,d)$ *depending only on* N *and* d*.*

Proof. Part (i) is standard, see [22], §82. Part (ii) follows from [5], Cor. 5. □

Proof of Theorem 8. Let $\Phi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ be an arbitrary homogeneous lift for ϕ. For each $\mathfrak{p} \in M_R$, let $\Phi_{\mathfrak{p}} : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ be a minimal $\mathfrak{p}$-integral model for ϕ; thus $\Phi_{\mathfrak{p}} = A_{\mathfrak{p}} \circ \Phi \circ A_{\mathfrak{p}}^{-1}$ for some $A_{\mathfrak{p}} \in \mathrm{GL}_{N+1}(K)$. If S denotes the finite set of $\mathfrak{p} \in M_R$ for which some coefficient of Φ is not $R_{\mathfrak{p}}$-integral, or for which $\mathrm{Res}(\Phi)$ is not an $R_{\mathfrak{p}}$-unit, then we may take $\Phi_{\mathfrak{p}} = \Phi$ and $A_{\mathfrak{p}} = I$ for all $\mathfrak{p} \notin S$.

By Proposition 13, there exists $A \in \mathrm{GL}_{N+1}(K)$ such that $A_{\mathfrak{p}}A^{-1} \in \mathrm{GL}_{N+1}(R_{\mathfrak{p}})$ for each $\mathfrak{p} \in M_R$. Consider the model $\Psi : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ for ϕ defined by $\Psi = A \circ \Phi \circ A^{-1}$. For each $\mathfrak{p} \in M_R$, we have

$$\Psi = (AA_{\mathfrak{p}}^{-1}) \circ \Phi_{\mathfrak{p}} \circ (AA_{\mathfrak{p}}^{-1})^{-1}. \tag{2.6}$$

Since $AA_{\mathfrak{p}}^{-1} = (A_{\mathfrak{p}}A^{-1})^{-1} \in \mathrm{GL}_{N+1}(R_{\mathfrak{p}})$ and $\Phi_{\mathfrak{p}}$ has coefficients in $R_{\mathfrak{p}}$, it follows from (2.6) that Ψ has coefficients in $R_{\mathfrak{p}}$ as well; since this holds for arbitrary $\mathfrak{p} \in M_R$, it follows from (2.4) that Ψ has coefficients in R. Finally, since $\mathrm{ord}_{\mathfrak{p}}(\det(AA_{\mathfrak{p}}^{-1})) = 0$, it follows from (2.6) and Proposition 15 that

$$\mathrm{ord}_{\mathfrak{p}}(\mathrm{Res}(\Psi)) = \mathrm{ord}_{\mathfrak{p}}(\mathrm{Res}(\Phi_{\mathfrak{p}})),$$

and so Ψ is $\mathfrak{p}$-minimal for each $\mathfrak{p} \in M_R$. □

Proof of Corollary 9. Since ϕ has everywhere good reduction, the model Ψ constructed in Theorem 8 satisfies $\mathrm{ord}_{\mathfrak{p}}(\mathrm{Res}(\Psi)) = 0$ for all nonzero prime ideals $\mathfrak{p}$ of R, and therefore (2.4) implies that $\mathrm{Res}(\Psi) \in R^{\times}$. □

Chapter 3

A Shafarevich Theorem for Twists

3.1 Introduction

In the present chapter we consider a Shafarevich type question originally posed by Silverman in Chapter 3 of [19] regarding the finiteness of rational morphisms on $\mathbb{P}^n$ defined over K, with algebraic degree $d \geq 2$, which have good reduction at all places $v \notin S$, and are twists of a given rational morphism ϕ defined over K.

We say that two rational morphisms $\phi, \psi : \mathbb{P}^n \to \mathbb{P}^n$ defined over K are $\bar{K}$-isomorphic if $\psi = \phi^f$ for $f \in \mathrm{PGL}_{n+1}(\bar{K})$ and K-isomorphic if $\psi = \phi^f$ for $f \in \mathrm{PGL}_{n+1}(K)$. These notions are clearly equivalence relations and we denote the set of rational morphisms which are $\bar{K}$-isomorphic to ϕ by

$$[\phi] = \{\phi^f | f \in \mathrm{PGL}_{n+1}(\bar{K})\},$$

and the set of rational morphisms which are K-isomorphic to ϕ by

$$[\phi]_K = \{\phi^f | f \in \mathrm{PGL}_{n+1}(K)\},$$

We then define the set of twists of ϕ as the set of K-isomorphism classes of rational morphisms ψ defined over K which are $\bar{K}$-isomorphic to ϕ

$$\mathrm{Twist}(\phi/K) = \{[\psi]_K | \psi \text{ is defined over } K \text{ and } [\phi] = [\psi]\}$$

We say that a twist $[\psi]_K \in \mathrm{Twist}(\phi/K)$ has good reduction outside S if there exists a representative $\phi \in [\psi]_K$ of this K-isomorphism class which has good reduction at all places $v \notin S$.

The principle theorem of this chapter is the following.

Theorem 16. *Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be a rational morphism of algebraic degree $d \geq 2$ defined over K and let K be a finite set of places including the Archimedean places. Let*

$$\mathcal{V}(S) = \{[\psi]_K \in \mathrm{Twist}(\phi/K) | [\psi]_K \textit{ has good reduction outside } S\}$$

Then $\mathcal{V}(S)$ is finite.

The theorem is proved by contradiction. Assuming that $\mathcal{V}(S)$ is infinite will give an infinite sequence of elements $f_i \in \mathrm{PGL}_{n+1}(\bar{K})$ which produce infinitely many rational morphisms $\psi_i = \phi^{f_i}$ defined over K, each with good reduction at all $v \notin S$ whose corresponding K-isomorphism classes $[\psi_i]_K$ are distinct.

Fix an integer $M > n+1$ and let $\mathrm{PrePer}(\phi, M)$ denote the set of preperiodic points for ϕ with forward orbit of size less than or equal to M. Then for each $i = 1, 2, \ldots$ the map f_i defines a bijection between $\mathrm{PrePer}(\phi, M)$ and $\mathrm{PrePer}(\psi_i, M)$ by $P \mapsto f(P)$. The premise is the proof is that, after passing to a suitable infinite subsequence and letting M become sufficiently large, the the sets $\mathrm{PrePer}(\psi_i, M)$ can be assumed to all be equal from the finiteness theorem in Section 3.

At this point the sets $\mathrm{PrePer}(\psi_i, M)$ have obtained a significant amount of structure and this structure comes from the following assumptions. First, the good reduction of ϕ and the ψ_i at all places $v \notin S$. Secondly, the ability to enlarge S and thirdly, the ability to enlarge M. By enlarging M we will able to guarantee that $\mathrm{PrePer}(\psi_i, M)$ will have a subset $\mathcal{U}$ of points in general position. Eventually that will allow us to conclude that only finitely many f_i exist. Enlarging S will allow us to assume that the sets $\mathrm{PrePer}(\psi_i, M)$ are linearly S-integral. For a precise definition of linear S-integrality see Definition 3.2. Informally, it guarantees that any set subset of $n+1$ linearly independent points of $\mathbb{P}^n(\bar{K})$ remains linearly independent in $\mathbb{P}^n(k_w)$ for every place w of $\bar{K}$ which lies over a place $v \notin S$.

The previous conditions will allow us to associate a decomposable homogeneous form (see Definition 3.2) to $\mathrm{PrePer}(\psi_i, M)$ which is a polynomial in $\mathcal{O}_S[X_0, \ldots, X_n]$ of fixed degree $N = |\mathrm{PrePer}(\psi_i, M)|$. The group $\mathrm{GL}_{n+1}(\mathcal{O}_S)$ acts naturally on this group by pre-composition and Everste and Győry have shown that there are only finitely equivalence classes of such polynomials with a given discriminant. This theorem will let us conclude that, after passing to an infinite subsequence, for each twist ψ_i we have $\mathrm{PrePer}(\phi, M) = \mathrm{PrePer}(\psi_i, M)$. It then follows that the maps f_i will fix the subset $\mathcal{U}$ in general position and thereby generating the contradiction.

3.2 Linear S-integrality.

Fix a number field K, an algebraic closure $\bar{K}$, and a morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ of algebraic degree at least $d \geq 2$ defined over K. Fix projective coordinates $(\mathbf{x}_0 : \cdots : \mathbf{x}_n)$ on $\mathbb{P}^n$ and let v denote an Archimedean place of K.

Let $\mathcal{V} \subset \mathbb{P}^n(\bar{K})$ be a finite subset $|\mathcal{V}| > n$. Let w be any place of $\bar{K}$ which lies over some $v \notin S$. Let $P_0, \ldots, P_n \in \mathcal{V}$ be a collection of $n+1$ points. We define $\det(P_0, \ldots, P_n)$ to be the determinant of the $n+1$ by $n+1$ matrix of the coordinate of P_i. Such a determine is well-defined up to a scalar multiple of $\bar{K}^\times$.

Definition. Let $P_0 = (p_{00} : \cdots : p_{0n}), \ldots, P_n = (p_{n0} : \cdots : p_{nn}) \in \mathbb{P}^n(\bar{K})$

and w a non-Archimedean place of $\bar{K}$. The *w-adic discriminant* of the P_i is defined as

$$\delta_w(P_0,\ldots,P_n) = \frac{|\det(P_0,\ldots,P_n)|_v}{\max_i(|p_{0i}|_v)\cdots\max_i(|p_{ni}|_v)}$$

The function δ_w of $n+1$ variables on $\mathbb{P}^n(\bar{K})$ is real-valued and uniformly bounded above by 1. When $n = 1$ this definition agrees with the w-adic spherical metric on $\mathbb{P}^1(\bar{K})$.

Lemma 17. *Let $P_0,\ldots,P_n \in \mathbb{P}^n(\bar{K})$ and w be a non-Archimedean place of $\bar{K}$. Then $\delta_w(P_0,\ldots P_n)$ is well defined.*

Proof. Let $\alpha_i \in \bar{K}^\times$ for $i = 0,\ldots,n$ and $(\alpha_i p_{i0} : \ldots : \alpha_i p_{in})$ another choice of coordinates for P_i. Denote $\alpha_i P_i$ for the choice of coordinates above, then it follows from elementary properties of determinants that

$$\det(\alpha_1 P_1,\ldots,\alpha_n P_n) = (\prod_i |\alpha_i|_w)\det(P_0,\ldots,P_n)$$

Hence,

$$\begin{aligned}\delta_w(\alpha_0 P_0,\ldots,\alpha_n P_n) =& \frac{|\det(\alpha_0 P_0,\ldots,\alpha_n P_n)|_v}{\max_i(|\alpha_0 p_{0i}|_v)\cdots\max_i(|\alpha_n p_{ni}|_v)}\\ &= \frac{(\prod_i |\alpha_i|_w|)\det(P_0,\ldots,P_n)|_v}{(\prod_i |\alpha_i|_w)\max_i(|p_{0i}|_v)\cdots\max_i(|p_{ni}|_v)}\\ &= \delta_w(P_0,\ldots,P_n)\end{aligned}$$

□

Lemma 18. *Let* $P_0, \ldots, P_n \in \mathbb{P}^n(\bar{K})$ *and* w *be a non-Archimedean place of* $\bar{K}$*. If* $f \in \mathrm{PGL}_{n+1}(\mathcal{O}_w)$ *then*

$$\delta_w(f(P_0), \ldots, f(P_n)) = \delta_w(P_0, \ldots, P_n).$$

Proof. If $f \in \mathrm{PGL}_{n+1}(\mathcal{O}_w)$, then in terms of the fixed ordered basis on $\mathbb{P}^n$, $f = [L_0 : \cdots : L_n]$ where $L_i = \Sigma f_{i,j} X_j$ is a homogeneous degree 1 polynomial in the variables $X_0, \cdots, X_n$ with coefficients $f_{i,j} \in \mathcal{O}_v$ and $\det(F) \in \mathcal{O}_v^\times$ where F denotes the $(n+1) \times (n+1)$ matrix with entries $f_{i,j}$.

For each P_i let $(p_{0i} : \ldots : p_{ni})$ be w-normalized coordinates. It follows that $\max_j(|p_{ij}|_v) = 1$ for $i = 0, \ldots, n$ and therefore that

$$\delta_w(P_0, \ldots, P_n) = |\det(p_{ij})|_w.$$

Let $f(P_i) = (p'_{0i} : \cdots : p'_{ni})$. Then $p'_{ij} = \Sigma_k f_{ik} p_{kj}$ and it follows that the coordinates p'_{ij} of the $f(P_i)$ are also w-normalized. Hence,

$$\delta_w(f(P_0), \ldots, (P_n)) = |\det(p'_{ij})|_w.$$

The following calculation finishes the lemma:

$$\begin{aligned}\delta_w(f(P_0), \ldots, (P_n)) &= |\det(p'_{ij})|_w \\ &= |\det(F)|_w |\det(p_{ij})|_w \\ &= \delta_w(P_0, \ldots, P_n)\end{aligned}$$

□

Definition. Let S be a finite subsets of places of K including all of the Archimedean places. A finite subset $\mathcal{V} \subset \mathbb{P}^n(\bar{K})$ is *linearly S-integral* if for every collection of $n+1$ linearly independent points $P_0, \ldots, P_n \in \mathcal{V}$ and for every place w of $\bar{K}$ whose restriction to K is not in S, then

$$\delta_w(P_0, \ldots, P_n) = 1$$

The notion of linear S-integrality generalizes the notion of pair-wise S-integrality for points of projective space, which corresponds to linear S-integrality in $\mathbb{P}^1$ and merely requires that two distinct points $P, Q \in \mathbb{P}^1(K)$ reduce to distinct points in $\mathbb{P}^1(k_v)$ for every $v \notin S$. If $n > 1$, then linear S-integrality is strictly stronger than pair-wise S-integrality as it requires that linear independent points of $\mathbb{P}^n(\bar{K})$ not only remain distinct after reduction at each place w, but also that they remain linearly independent.

Definition. Let $N \geq n+1$ be an integer and consider the set $\mathcal{P}(S, N)$ of all subsets $\mathcal{V} \subset \mathbb{P}^n(\bar{K})$ such that the following conditions hold:

1. $\mathcal{V}$ is $\mathrm{Gal}(\bar{K}/K)$-stable.
2. $\mathcal{V}$ is linearly S-integral.
3. $|\mathcal{V}| = N$.
4. $\mathcal{V}$ contains at least one linearly independent $(n+1)$-point subset.

Lemma 19. *There exists an group action*

$$\mathrm{PGL}_{n+1}(\mathcal{O}_S) \times \mathcal{P}(S,N) \to \mathcal{P}(S,N)$$

defined by $(f,\mathcal{V}) \mapsto f(\mathcal{V}) = \{f(P) | P \in \mathcal{V}\}$.

Proof. Let $f, g, h \in \mathrm{PGL}_{n+1}(\mathcal{O}_S)$ and $\mathcal{V} \in \mathcal{P}(S,N)$.

The substance of the lemma consists in showing that the group action is closed. We must show that (1) – (4) of the definition of a linearly S-integral set of size N hold for the set $f(\mathcal{V})$. Requirements (3) and (4) are obviously satisfied since f is an automorphism of $\mathbb{P}^n$. Let $\sigma \in \mathrm{Gal}(\bar{K}/K)$ and $f(P) \in f(\mathcal{V})$. Then, since f is defined over $K = \mathrm{Frac}(\mathcal{O}_S)$ and $\mathcal{V}$ is $\mathrm{Gal}(\bar{K}/K)$-stable we have $\sigma \cdot f(P) = f(\sigma \cdot P) = f(Q) \in f(\mathcal{V})$ for some $Q \in \mathcal{V}$, proving condition (1). For any place w of $\bar{K}$ whose restriction to K is not in S we have that $f \in \mathrm{PGL}_{n+1}(\mathcal{O}_w)$ and it follows for Lemma 18 that

$$\delta_w(f(P_0), \dots, f(P_n)) = \delta_w(P_0, \dots P_n) = 1$$

for every subset of $n+1$ linearly independent points $f(P_0), \dots, f(P_n) \in f(\mathcal{V})$. It follows that (2) holds for $f(\mathcal{V})$.

It is completely trivial that $I \cdot \mathcal{V} = \mathcal{V}$. To show transitivity it suffficies to note that

$$\begin{aligned} f\cdot(g\cdot h)\cdot\mathcal{V} =& \{f(g\circ h)(P)|P\in\mathcal{V}\}\\ =&\{f(g(h(P)))|P\in\mathcal{V}\}\\ =&\{(f\circ g)(h(P))|P\in\mathcal{V}\}\\ =&(f\cdot g)\cdot h\cdot\mathcal{V}\end{aligned}$$

This completes the lemma. $\square$

Because we have fixed an ordered basis $(x_0:\cdots:x_n)$ on $\mathbb{P}^n(\bar{K})$, we can identify $\mathbb{P}^n=\mathbb{P}(V)$, where V is the vector space of $n+1$ tuples $(p_0,\cdots,p_n)$ over $\bar{K}$, and points of $\mathbb{P}^n$ correspond to 1-dimensional linear subspaces of V. Let V^* denote the dual space of V with the dual basis X_i corresponding to the x_i. Then to each point $P\in\mathbb{P}^n(\bar{K})$ we can associate a linear homogeneous form

$$P=(p_1:\cdots:p_n)\in\mathbb{P}^n\mapsto\ell_P=\Sigma_i p_i X_i$$

The linear form ℓ_P depends on the choice of coordinates of P, but if one fixes a non-Archimedean place w of $\bar{K}$ one can use a choice of w-normalized coordinates so that $\ell_P\in\mathcal{O}_w[X_0,\ldots,X_n]$. This choice of ℓ_P is unique up to w-unit.

Definition. A homogeneous form $F\in\mathcal{O}_S[X_0,\cdots,X_n]$ is called a *decomposable form* if it can be factored over its splitting field as $F=\lambda\ell_1^{k_1}\cdots\ell_t^{k_t}$ for $\lambda\in$

K^*, $\ell_1, \cdots \ell_2$ are pair-wise non-proportional homogeneous linear polynomials over $\bar{K}$ and $k_1, \cdots, k_t$ are positive integers such that $k_1 + \cdots + k_t = \deg(F)$.

This type of form is studied by Evertse and Győry in [6]. Each decomposable form has an associated discriminant, which is a fractional ideal of $\mathcal{O}_S$. For the remainder of this section by ideal, we mean fractional ideal.

Definition. Suppose $F(X_0, ..., X_n)$ is a decomposable form and

$$F = \lambda \ell_1^{k_1} \cdots \ell_t^{k_t}$$

in the splitting field L of F. Then the *discriminant* of F, denoted D_F, an ideal of $\mathcal{O}_S$ defined as follows

$$D_F = \prod_{\mathcal{I}(F)} \left(\frac{\det(\ell_{i_0}, \cdots, \ell_{i_n})}{(\ell_{i_0}) \cdots (\ell_{i_n})} \right)^2 \tag{3.1}$$

where (ℓ_i) denotes the ideal generated by the coefficients of ℓ_i and $\mathcal{I}(F)$ is the collection of L-linearly independent subsets $\{\ell_{i_0}, \dots, \ell_{i_n}\}$ of $\{\ell_0, \dots, \ell_t\}$.

Remark 5. For each linear form ℓ_i, the ideal (ℓ_i) is actually an ideal of the integral closure of $\mathcal{O}_S$ in the splitting field L, but D_F is an ideal of $\mathcal{O}_S$. This follows from D_F being invariant under $\mathrm{Gal}(L/K)$.

Remark 6. If $F = \lambda \ell_1^{k_1} \cdots \ell_t^{k_t}$ is a decomposable form, then the scalar λ is not used in the definition of the discriminant of F. In particular, if $\gamma \in K^\times$, then $D_F = D_{\gamma F}$.

To each subset $\mathcal{V} \in \mathcal{P}(S, N)$ we can associate a decomposable homogeneous form $F_{\mathcal{V}} \in \mathcal{O}_S[X_0, \ldots, X_n]$ in the following manner. Let L denote the splitting field of the set $\mathcal{V}$. For each point $P \in \mathcal{V}$ Let ℓ_P be the associated homogeneous form of degree 1 for a choice of L-rational coordinates. The form ℓ_P is well defined upto multiplication by a non-zero scalar of L. Define

$$F_{\mathcal{V}} = \prod_{P \in \mathcal{V}} \ell_P$$

As $\mathcal{V}$ is $\mathrm{Gal}(\bar{K}/K)$-stable, it follows that $F_{\mathcal{V}}$ is a decomposable form of $K[X_0, \ldots, X_n]$ of degree $|\mathcal{V}|$ and is well defined upto multiplication by a non-zero scalar of K. It follows that $D_{F_{\mathcal{V}}}$ is a fractional ideal of $\mathcal{O}_S$. By the above remark, the discriminant ideal is unchanged if we multiply $F_{\mathcal{V}}$ by a scalar of $\gamma \in K^{\times}$, so we may choose some γ such that $F_{\mathcal{V}} \in \mathcal{O}_S[X_0, \ldots X_n]$.

Proposition 20. *Let $\mathcal{V} \in \mathcal{P}(S, N)$ and let $F_{\mathcal{V}}$ be the associated decomposable form. Then $D_{V_{\mathcal{V}}} = \mathcal{O}_S$.*

Proof. As before, let L be the splitting field for $\mathcal{V}$ and T be the places of L lying over S. It suffices to show that for any collection of $n+1$ linearly independent points $P_0, \ldots, P_n \in \mathbb{P}^n$ that $(\det(P_0, \ldots, P_n)) = (\ell_{P_0}) \cdots (\ell_{P_n})$ as ideals of $\mathcal{O}_T$. Without loss of generality, we may assume that all of the forms have coefficients in $\mathcal{O}_T$.

As $\mathcal{O}_T$ is a Dedekind domain, it follows from unique factorization into prime ideals that

$$\begin{aligned}(\det(P_0,\dots,P_n)) =& \prod_{w\in M_L\setminus T}\mathfrak{p}_v^{a_v}\\ (\ell_{P_0})\cdots(\ell_{P_n}) =& \prod_{w\in M_L\setminus T}\mathfrak{p}_v^{b_v}\end{aligned}$$

for nonnegative integers a_v, b_v. It suffices to show that $a_v = b_v$ for every $v \in M_L \setminus T$. Fix such a v. As $\mathcal{V}$ is assumed to the linearly S-integral, it follows the fact that $\delta_w(P_0,\dots,P_n) = 1$ that

$$\begin{aligned}a_v =& \mathrm{ord}_v(\det(P_0,\dots,P_n))\\ =& \mathrm{ord}_v(\det(p_{ij}))\\ =& \sum_{0\le i\le n}\min_{0\le j\le n}(\mathrm{ord}_v(p_{ij}))\\ =& \sum_{0\le i\le n}\mathrm{ord}_v(\ell_{P_i}) = \mathrm{ord}_v((\ell_{P_0})\cdots(\ell_{P_n})) = b_v\end{aligned}$$

□

Definition. Let F and G be two decomposable forms in $n+1$ variables of degree d. The forms F and G are *weakly $\mathcal{O}_S$-equivalent* if

$$F(X_0,\dots,X_n) = \lambda G(A(X_0,\dots,X_n))$$

for some $\lambda \in K^\times$ and some $A \in \mathrm{GL}_{n+1}(\mathcal{O}_S)$.

Theorem 21. *The group action of $\mathrm{PGL}_{n+1}(\mathcal{O}_S)$ on $\mathcal{P}(S,N)$ has only finitely many orbits.*

Proof. This theorem is a reformulation of Corollary 2 by Everste and Győry in [6]. Their corollary states that there are only finitely many weak $\mathcal{O}_S$-equivalence classes of decomposable forms in $K[X_0, \ldots, X_n]$ of fixed degree N and given discriminant ideal D.

To conclude the proof of this theorem it suffices to show that if $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{P}(S, N)$ and if the forms $F_{\mathcal{V}_1}$ and $F_{\mathcal{V}_2}$ are weakly $\mathcal{O}_S$-equivalent, then the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are in the same $\mathrm{PGL}_{n+1}(\mathcal{O}_S)$-orbit. Let

$$\begin{aligned}\mathcal{V}_1 =& \{P_1, \ldots, P_N\}\\ \mathcal{V}_2 =& \{Q_1, \ldots, Q_N\}.\end{aligned}$$

Let F_1 and F_2 denote $F_{\mathcal{V}_1}$ and $F_{\mathcal{V}_2}$, respectively. If F_1 and F_2 are weakly $\mathcal{O}_S$-equivalent. Then there exists $\lambda \in K^\times$ and $A \in \mathrm{GL}_{n+1}(\mathcal{O}_S)$ such that

$$F_1(X_0, \ldots, X_n) = \lambda F_2(A(X_0, \ldots, X_n))$$

It follows that

$$\prod_{1 \le i \le N} \ell_{P_i}(X_0, \ldots, X_n) = \lambda \prod_{1 \le i \le N} \ell_{Q_i}(A((X_0, \ldots, X_n))$$

and that after reordering

$$\ell_{P_i}(X_0, \ldots, X_n) = \lambda_i \ell_{Q_i}(A(X_0, \ldots, X_n))$$

for scalars $\lambda_i \in K^\times$.

Equating coefficients gives that

$$(p_{0i}, \ldots, p_{ni}) = \lambda_i A^t (q_{0i}, \ldots, q_{ni})$$

where $A^t \in \mathrm{GL}_{n+1}(\mathcal{O}_S)$ is the transpose of A. Let $a \in \mathrm{PGL}_{n+1}(\mathcal{O}_S)$ be the corresponding projective linear transformation to A^t. Then $P_i = a(Q_i)$ and therefore $\mathcal{V}_1 = a(\mathcal{V}_2)$. □

3.3 Main Theorem

Let $M \geq 1$ and $\mathrm{PrePer}(\phi, M)$ denote the set of all points which are ϕ-pre-periodic and whose forward orbit has size of at most M. It is known from the theory of canonical heights that the set $\mathrm{PrePer}(\phi, M)$ is finite (see [17]). Let $N = |\mathrm{PrePer}(\phi, M)|$. Every rational morphism of algebraic degree at least 2 has infinitely many pre-periodic points, so by increasing M we may assume that $N \geq n+2$ and moreover, by Fakhruddin's result on the Zariski density of pre-periodic points (see Theorem 5.1 of [7]) we may assume that there is a subset of $\mathrm{PrePer}(\phi, M)$ consisting of $n+2$ points which lie in general position.

Definition. A subset $\mathcal{V} \subset \mathbb{P}^n(\bar{K})$ with $|\mathcal{V}| \geq n+1$ is in *general position* if no $(n+1)$-point subset lies in a hyperplane.

Lemma 22. *Let $\mathcal{V}, \mathcal{W} \subset \mathbb{P}^n(\bar{K})$ be finite , and assume that $\mathcal{V}$ has a subset $\mathcal{V}_0$ in general position with $|\mathcal{V}_0| = n+2$. Then there exist only finitely many automorphisms $f \in \mathrm{PGL}_{n+1}(\bar{K})$ such that $f(\mathcal{V}) = \mathcal{W}$.*

Proof. Suppose the contrary and that $f_1, f_2, f_3, \ldots \in \mathrm{PGL}_{n+1}(\bar{K})$ is an infinite sequence of distinct automorphisms such that $f_i(\mathcal{V}) = \mathcal{W}$. Since $\mathcal{W}$ is finite, it has only finitely many subsets, and we may assume that there exists a subset $\mathcal{W}_0 \subset \mathcal{W}$ in general position with $|\mathcal{W}_0| = n+2$ and, perhaps after passing to an infinite subsequence, that $f_i(\mathcal{V}_0) = \mathcal{W}_0$ for all i. Choose $g \in \mathrm{PGL}_{n+1}(\bar{K})$ such that $g(\mathcal{W}_0) = \mathcal{V}_0$. Then the compositions $g \circ f_i$ form and infinite sequence of distinct automorphisms in $\mathrm{PGL}_{n+1}(\bar{K})$ such that $g \circ f_i(\mathcal{V}_0) = \mathcal{V}_0$. This gives a contradiction as there are only $(n+2)!$ such automorphisms. □

Definition. An *$\mathcal{O}_S$-model* for a rational morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ defined over K is a homogeneous lift $\Psi : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ such that $\Psi = A^{-1} \circ \Phi \circ A$ for $A \in \mathrm{GL}_{n+1}(K)$ and Ψ is defined over $\mathcal{O}_S$.

Proposition 23. *Assume that $\mathcal{O}_S$ in a PID. Let $\phi, \psi : \mathbb{P}^n \to \mathbb{P}^n$ be rational morphisms defined over K of algebraic degree d, both having good reduction at all places v of K outside S. Assume that $[\psi]_K \in \mathrm{Twist}(\phi/K)$.*

(a) There exist rational morphisms $\phi_0 \in [\phi]_K$ and $\psi_0 \in [\psi]_K$, and homoge-

neous lifts $\Phi, \Psi : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ *of* ϕ_0 *and* ψ_0*, respectively, such that* Φ, Ψ *have coefficients in* $\mathcal{O}_S$ *and resultants* $\mathrm{Res}(\Phi), \mathrm{Res}(\Psi) \in \mathcal{O}_S^{\times}$*.*

(b) There exists $A \in GL_{n+1}(\overline{\mathcal{O}}_S)$ *such that* $\Psi = \Phi^A$*.*

(c) For each integer $M \geq 1$*, we have*

$$\mathrm{PrePer}(\psi_0, M) = f(\mathrm{PrePer}(\phi_0, M)),$$

where $f : \mathbb{P}^n \to \mathbb{P}^n$ *is the automorphism associated to* A*. Moreover,* $\mathrm{PrePer}(\phi_0, M)$ *is linearly* S*-integral if and only if* $\mathrm{PrePer}(\psi_0, M)$ *is linearly* S*-integral.*

Proof. Part (a) follows from the main theorem of [15]. The existence of $A \in \mathrm{GL}_{n+1}(\bar{K})$ follows immediately from Φ, Ψ being lifts of twists. That $A \in \mathrm{GL}_{n+1}(\overline{\mathcal{O}}_S)$ follows from Lemma 6 in [16]. For (c), that $\mathrm{PrePer}(\phi_0, M) = f(\mathrm{PrePer}(\psi_0, M))$ is immediate. As $A \in \mathrm{GL}_{n+1}(\overline{\mathcal{O}}_S)$ it follows that $f \in \mathrm{PGL}_{n+1}(\overline{\mathcal{O}}_S)$, in particular $f \in \mathrm{PGL}_{n+1}(\mathcal{O}_w)$ for any place w of $\bar{K}$ over a place of K not in S. It follows from Lemma 18 that for any collection of $n+1$ points $P_0, \ldots, P_n$ that

$$\delta_w(P_0, \ldots, P_n) = \delta_w(f(P_0), \ldots, f(P_n))$$

and hence $\mathrm{PrePer}(\phi_0, M)$ is linearly S-integral if and only if $\mathrm{PrePer}(\psi_0, M)$ is linearly S-integral. □

We are now ready to prove the main theorem.

Theorem 24. *Let* $\phi : \mathbb{P}^n \to \mathbb{P}^n$ *be a rational morphism of algebraic degree* $d > 1$ *defined over* K *and* $\mathrm{Twist}(\phi/K)$ *the set of* K*-twists. Then there are only finitely many twists* $[\psi]_K \in \mathrm{Twist}(\phi/K)$ *which have good reduction at all places* $v \notin S$*.*

Proof. If no twists of ϕ have good reduction outside S then there is nothing to prove. Therefore, assume that at least one such twist exists, and since being twists is an equivalence relation, without loss of generality assume that ϕ has good reduction outside S.

Assume contrary to the theorem and let

$$[\psi_1]_K, [\psi_2]_K, [\psi_3]_K, \ldots \tag{3.2}$$

be an infinite sequence of distinct twists which have good reduction outside S.

We may assume that by Fakrhuddin's result on Zariski density of preperiodic points (see [7]) that we can increase M so that the set $\mathrm{PrePer}(\phi, M)$ contains a set of $n+2$ points of $\mathbb{P}^n$ in general position. Because ϕ is defined over K the set $\mathrm{PrePer}(\phi, M)$ is $\mathrm{Gal}(\bar{K}/K)$-stable. There are only finitely many combinations of $n+1$ points of $\mathrm{PrePer}(\phi, M)$ which are linearly independent and let $D_0, \ldots D_t$ be their determinants. There are only finitely many places v of K such that $v(D_i) > 0$ for any $i = 0, \dots, t$. Enlarge S by these places. $\mathrm{PrePer}(\phi, M)$ is now linearly S-integral and further enlarging S does not change this condition. It follows that

$$\mathrm{PrePer}(\phi, M) \in \mathcal{P}(S, N)$$

where $N = |\mathrm{PrePer}(\phi, M)|$. Finally, we may further enlarge S until $\mathcal{O}_S$ is a PID and by Proposition 23 that $\mathrm{PrePer}(\psi_i, M) \in \mathcal{P}(S, N)$ for each $i = 1, 2, \dots$, perhaps after replacing ψ_i with some K-isomorphic map within $[\psi_i]_K$.

By Theorem 21, we may assume, after passing to a subsequence, that $\mathrm{PrePer}(\phi, M), \mathrm{PrePer}(\psi_i, M)$ lie in the same $\mathrm{PGL}_{n+1}(\mathcal{O}_S)$-equivalence class for all $i \geq 1$. It follows that there exists a sequence of linear transformations

$$g_i \in \mathrm{PGL}_{n+1}(\mathcal{O}_S)$$

such that

$$\mathrm{PrePer}(\psi_i, M) = g_i(\mathrm{PrePer}(\phi, M))$$

and therefore that

$$\mathrm{PrePer}(\psi_i^{g_i}, M) = g_i^{-1}(\mathrm{PrePer}(\psi, M)) = \mathrm{PrePer}(\phi, M)$$

As $\psi_i^{g_i}$ also has good reduction at all $v \notin S$, it suffices to replace ψ_i with $\psi_i^{g_i}$ and assume that $\mathrm{PrePer}(\psi_i, M) = \mathrm{PrePer}(\phi, M)$ for all $i = 1, 2, \dots$.

Let $f_i \in \mathrm{PGL}_{n+1}(\bar{K})$ be such that $\psi_i = \phi^{f_i}$. As the rational morphisms ψ_i are assumed to be distinct, so must the f_i be distinct, and it follows that f_i gives a bijection

$$f_i : \mathrm{PrePer}(\phi, M) \to \mathrm{PrePer}(\phi, M) \tag{3.3}$$

As $\mathrm{PrePer}(\phi, M)$ is finite and contains a subset of $n+2$ points in general position, it follows from Lemma 22 that only finitely many f_i can exist and therefore gives the necessary contradiction. □

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