

A TRANSFORMATIONAL APPROACH TO INVERSIONAL RELATIONS

by

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Abstract

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Inversion has been explored as an essential device in post-tonal music and discussed in the relevant literature. In particular, many music theorists have demonstrated that inversional symmetry plays a significant role in the music of Bartók, which often includes inversional relations on the musical surface. In many other musical works, however, inversion, or symmetrical inversion, is often ambiguous and not immediately apparent; thus its role is easily overlooked or underestimated. This dissertation argues that inversion may play an important role in pitch organization within a piece or a passage of post-tonal music. Significantly, since inversional relations can more effectively be analyzed by using a transformational approach, at both foreground and background levels, the bulk of this dissertation is thus based in such a transformational approach.

Chapter 1 outlines many different methods for defining and illustrating pitch and pitch-class inversion as provided in the analytic literature. Chapter 2 examines symmetrical inversion as it appears in Klumpenhower networks which transform into each other among twelve index-zones. This chapter also introduces new axial

isographies for tetrachords. Chapter 3 explores inversive relations between pitch-class sets of different sizes, i.e., a trichord and a tetrachord, which are often the important groupings in post-tonal music. Chapter 4 presents specific aspects of symmetrical inversion suggested in Perle's theory of twelve-tone tonality and in his music.

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Chapter 1

Inversional Relations in Post-tonal Music

Inversion, or symmetrical inversion, is widely recognized as a fundamental technique in twentieth-century music. In the discussion of inversion, a number of theorists have suggested that inversion plays a major role in relating, or grouping, of pitches in atonal as well as twelve-tone music.¹ Among all of these investigations, Schoenberg's theories connecting inversion and its application to his twelve-tone system have been crucial both in his music and in the music of other twentieth-century composers. Schoenberg described the concept of inversion precisely:

Symmetry is one of the simplest principles; to the right of the axis there is the same thing (in equal distances, in equal amounts, etc.) as to the left of the axis. Inversion and the principal of mirror and retrograde are basically the same.²

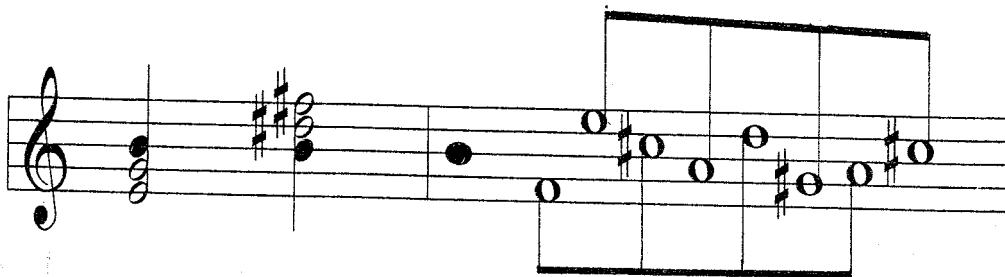
¹ For general studies of inversion in twentieth-century music, see Lora L. Gingerich, "Explaining Inversion in Twentieth-Century Theory," *Journal of Music Theory Pedagogy* 3/2 (1989), 233-243; Wolfgang Scherzinger, "Anton Webern and the Concept of Symmetrical Inversion," *Repercussions* (1997), 109-114; David W. Bernstein, "Symmetry and Symmetrical Inversion in Turn-of-the-Century Theory and Practice," *Music Theory and the Exploration of the Past*, edited by Christopher Hatch and David W. Bernstein (University of Chicago Press, 1993), 377-407.

² Arnold Schoenberg, *The Musical Idea and the Logic, Technique, and Art of Its Presentation*, edited and translated by Patricia Carpenter and Severine Neff (New York: Columbia University Press, 1995), 296-297; *Symmetrie ist eines der einfachsten Prinzipien: rechts von der achse befindet sich dasselbe (in gleichen Abständen, in gleichen Maben etc) wie links von der Aches.*

These descriptions, however, induce confusion or misunderstanding that arises from mixing together the terms “symmetry” and “inversion.”³ In a number of theoretical writings, inversion is only partially dealt with within the larger domain of symmetry, or even considered only as a limited technique for certain isolated sounds.⁴

In general, twentieth-century theoretical literature introduces two types of inversion, namely, pitch inversion and pitch-class inversion (hereafter referred to as pc inversion). The center of a pitch inversion is a single pitch, or a pair of pitches forming a half-step, in a specific register. Example 1.1 shows the pitch inversion of a chord, or a melody, around the given center (B_4) as the related pitches which lie an equal distance from the center, but in opposite directions.

Example 1.1 Pitch inversion of a chord and a melody around the given center, B_4



³ Note that in the broad sense, symmetry mapping into itself includes both transpositionally and inversionally symmetrical structure.

⁴ Michael Cherlin interpreted inversion in terms of mirror symmetry in “Dramaturgy and Mirror Imagery in Schoenberg’s *Moses Und Aron*: Two Paradigmatic Interval Palindromes,” *Perspectives of New Music* 29/2 (1991), 50-71.

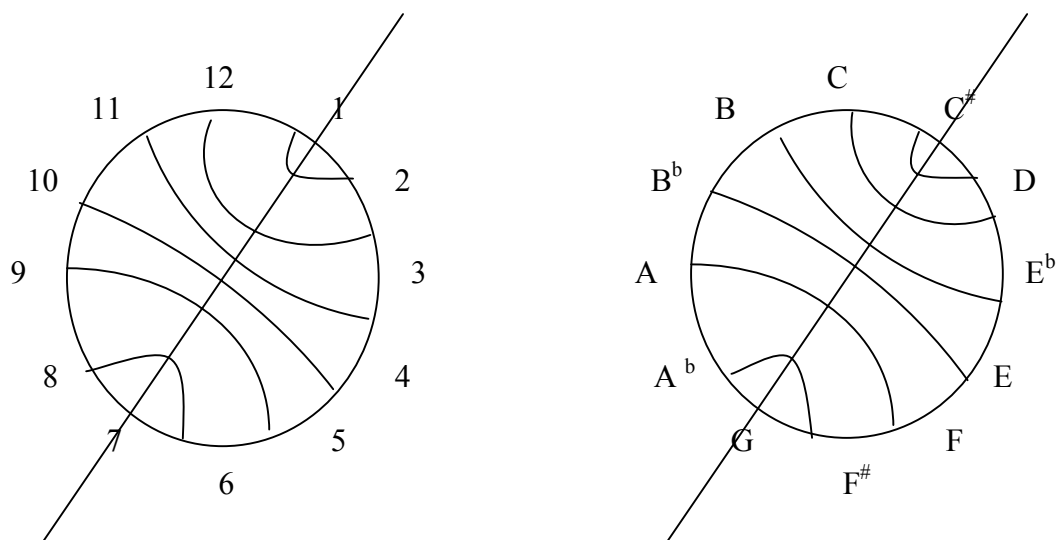
Here, the pitch inversion of a chord, in which the lower tones symmetrically reflect the higher, is strikingly similar to the concept of the dualistic harmonic theories, a move from major to minor chord or vice versa.⁵ The inversion of the melody in Example 1.1 is also a related melody whose pitches lie a certain interval above or below a specific inversional center, B₄. Rather than having a single center of inversion in a particular register, a pc inversion will have an axis of symmetry with no specific register, and is represented by two pitch-classes or two pairs of pitch-classes separated by a tritone.⁶ A convenient visual model for explaining pc inversion is a circular diagram resembling a clock face, as shown in Example 1.2.⁷

⁵ It is what Riemann termed *Wechselwirkung*, a discussion of Riemann's theories can be found in William C. Mickelsen, *Hugo Riemann's Theory of Harmony, with The History of Harmonic Theory, Book III, by Hugo Riemann*, translated and edited by William C. Mickelsen (Lincoln, 1977). Riemann's theories, however, were not typical and only now have been revived with neo-Riemannian theories.

⁶ Lora L. Gingerich, "Explaining Inversion," 236.

⁷ David Lewin illustrates pitch-class inversion in a clock face in "Inversional Balance as an Organizing Force in Schoenberg's Music and Thought," *Perspectives of New Music* 6/2 (1968), 1-21.

Example 1.2 Pitch-class inversion



The system is now mod 12. 12 o'clock and 2 o'clock are equidistant from 1 o'clock by one hour and from 7 o'clock by five hours. That is, the distance C-D has the midpoint C[#] (and G) and this midpoint is the inversional axis that maps C onto D.

In essence, the criterion for making a distinction between pitch inversion and pc inversion is consideration of registral placement. If in a given context only pitch inversion is dealt with within a register, the other will not be properly defined. In fact, the decision whether to give priority to pitch inversion or pc inversion is controversial.

Jonathan Bernard, for instance, prefers pitch inversion and maintains that (inversional) symmetry should be represented in literal, registral terms:

We have long been accustomed to the concepts of inversional and octave equivalence in tonal music, and for the most part we have retained these concepts in analysis of the post-tonal repertoire. Nevertheless, in a symmetrical context there is no reason why the metaphorical space of the modulo 12 pitch-class system should be equated with real space.⁸

⁸ Jonathan Bernard employed pitch inversion but rejected pitch-class inversion as an analytic concept in "Space and Symmetry in Bartók," *Journal of Music Theory* 30/2 (1986), 185-201.

Bernard's assertion suggests the justification for pitch inversion, that it is an extension of traditional tonality.

Tom Demske also insists that considering the registral centers of pitch inversion can lead to helpful insights into atonal music:

The registral disposition of pitches is often an important structural feature of music in the atonal style, as has been amply demonstrated in various studies by David Lewin and others. For example, while Lewin's theory of inversional balance does embrace pitch-class inversion, it is most compelling when it is supported in analytical contexts by registral phenomena, specially, pitch inversion.⁹

These discussions show that pitch inversion has significance for pitch relations as well as defining symmetrical structures in the works of Bartók, Schoenberg and Webern. The authors' assertions in the above quotations, however, also suggest some questions: Why should inversion be considered to only be in literal pitch space instead of in pitch-class space (mod 12), when we consider the context as being determined by a modulo-12 pitch class system? Why may we make an exception by admitting a single external element, register, rather than timbre, rhythm, dynamics or some other element? And, also, if we accept only pitch inversion, then how can we realize the centers in asymmetrical layouts?

In answer to the first question, Perle explained inversion in the chromatic scale as defining a pitch-class domain:

The placement of a collection of functionally differentiated notes whose scalar ordering produces a series of whole steps and half steps, by a collection of functionally undifferentiated notes whose scalar ordering produces a series of half step only, completely transforms the meaning of inversion, which now

⁹ Tom Demske, "Registral Centers of Balance in Atonal Works by Schoenberg and Webern," *In Theory Only* 9/23 (1986), 60.

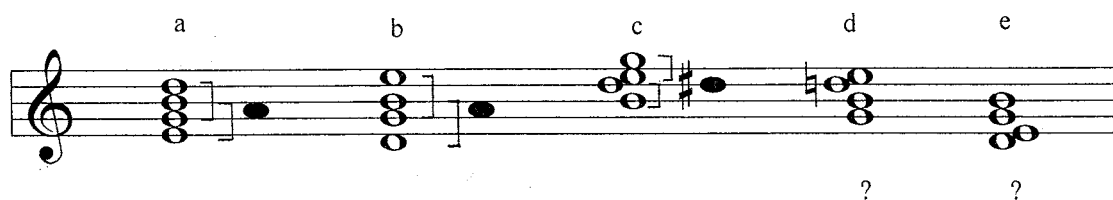
becomes a precompositional means of symmetrically partitioning the tone material...¹⁰

Babbitt responds to the second question by defining a new concept of inversion for twelve-tone music:

Inversion, in the traditional sense, implies inversion of contour. In the twelve-tone system inversion, like transpositions, cannot be characterized in terms of registral consideration, but merely as that permutation of pitch numbers, which results from the substitution of complementary pitch numbers in S...¹¹

Such theorists preferred to make these claims about pc inversion without regard to register, even when pitch inversion can be realized. To address the last question, it will be worth examining all possible registral arrangements of a pc-set (here a minor-seventh chord) as shown in Example 1.3.

Example 1.3 Possible registral arrangements of a minor seventh chord



We can observe that not every realization of the tetrachord deploys its members symmetrically about a pitch-axis; thus the registral centers cannot be determined simply

¹⁰ George Perle, *Twelve-Tone Tonality* 2nd ed. (Berkeley: University of California Press, 1980), 5.

¹¹ Milton Babbitt, "Twelve-Tone Invariants as Compositional Determinants," *The Musical Quarterly* 46/2 (1960), 253.

in such asymmetrical configurations as d and e. This example also demonstrates that a pc inversion operator has the potential to be realized as different pitch inversions. On the other hand, the realization of pitch inversion occurs only within its registral boundary. Therefore inversional centers, which depend on registral arrangement, cannot be applied broadly, in spite of the fact that some advocates claim pitch inversion is a more useful means for characterizing certain aspects of particular composer's works.¹²

The concept of pc inversion began to take shape in the early twentieth century in writings by Schröder, Ziehn, and Capellen, descendants of dualistic harmonic theories. Among them, Schröder's treatise *Die symmetrische Umkehrung in der Musik* (1902) remains the most extensive work written on the subject.¹³ Indeed, in his study Schröder contributed to a conceptual preparation for the notion of inversion in twelve-tone theory. His investigation of inverted and then transposed chromatic scales, which may be described as being located around an axis of inversion, is suggestive of contemporary inversionally related twelve-tone rows.¹⁴ However, the concept of inversion seems to be restricted for him to diatonic lines since he claims that inversion cannot be effective in chromatic scales:

¹² Pitch inversion taking symmetrical significance is discussed in David Lewin, "Inversional Balance,"; Bruce Archibald, "Some Thoughts on Symmetry in Early Webern, Op.5, no.2," *Perspectives of new Music* 10/2 (1972), 159-63

¹³ Hermann Schröder, *Die symmetrische Umkehrung in der Musik* (Leipzig: Publikationen der International Musikgesellschaft, 1902)

¹⁴ David W. Bernstein discusses Schröder's theory in "Symmetry and Symmetrical Inversion in Turn-of-the-Century Theory and Practice," in *Music Theory and the Exploration of the Past*, edited by Christopher Hatch and David W. Berstein (Chicago: University of Chicago Press, 1993), 395-399.

The inversion of diatonic tone-rows yields an absolutely effective contrast in identity and character; major becomes minor and vice versa. On the other hand, with the inversion of a chromatic scale, little of contrast in identity and character, that is, of major and minor, is recognizable--a proof that symmetrical inversion is notably more effective in the diatonic than it is in the chromatic.¹⁵

His view of inversion did not yet extend beyond diatonic limits. The full development of the concept of pc inversion as it is understood in contemporary music theory did not take place until the mid-twentieth century. Then its exponents attempted to account for inversion in a twelve-tone system within a twelve-tone context, just as theorists of tonal harmony examined symmetrical inversion within a tonal context.

I. Theoretical studies on pitch-class inversion

Since the mid-twentieth century, as pitch-class set theories have undergone great refinement through using modular arithmetic to govern the twelve-tone system, inversion has also been subject to greater mathematical formulation. The significant theoretical writings from the 1960s by Milton Babbitt, Allen Forte, David Lewin, George Perle, and others contributed not only to establishing the concept of inversion but also to demonstrating its functional expansion through the comprehensive set-theoretic relations of invariance and literal and abstract inclusion, as well as through the organization of equivalence classes into larger complexes, all related under inversion. Each of the authors, however, interprets inversion and its applicability from a different viewpoint. The literature reflecting and offering the various definitions, notation, and role of inversion and its relations deserve to be examined carefully in order to grasp the

¹⁵ Hermann Schröder, *Die symmetrische Umkehrung in der Music*, 6.

nature of inversion itself as well as its role in relating pc sets. Let us consider these meaningful mid-century achievements in inversion theory in more detail.

1-1. Milton Babbitt

The main groundwork in modern theory for pc inversion was first established by Babbitt's well-known articles.¹⁶ His investigations embodied inversion through serial operations and the modeling of a system of inversion by means of arithmetic expression. Babbitt defined an index number as the sum (mod 12) of the pitch classes derived from inversionally-related sets.¹⁷ Babbitt considered inversion in connection with transposition as an indivisible operation.

The importance of transpositional complementation alone would serve to suggest the systematic operation of inversion (designated; I), which is definable as complementation (mod 12) of each pitch number of S, as opposed to complementation of the transposition applied to all pitch numbers of S.¹⁸

In addition, Babbitt divided the twelve inversionally related set pairs into two groups, depending on the even or odd index number, and also explored the characteristics of even and odd indices.

The cyclic representation of the pitch class permutations effected by IT shows that all even t's (transposition's) produce similar permutations of six cycles of

¹⁶ Milton Babbitt, "Some Aspects of Twelve-Tone Composition," *The Score and I.M.A. Magazine* XII (1955), 53-61; idem, "Twelve-Tone Invariants as Compositional Determinants," *Musical Quarterly*, 46/2 (1960), 246-259; idem, "Set Structure as a Compositional Determinant," *Journal of Music Theory*, 5 (1961), 72-94.

¹⁷ Milton Babbitt, "Twelve-Tone Rhythmic Structure and the Electronic Medium," *Perspectives of New Music* 1/1 (1962), 57.

¹⁸ Milton Babbitt, "Twelve-Tone Invariants," 252.

two elements each, while odd t's produce similar permutations of five two-element cycles and two unit cycles.¹⁹

Babbitt proposed, however, that his ultimate purpose in examining pitch or intervallic contents and operations upon them is to effect combinatorial matching. The hexachord inversive combinatoriality, for instance, presents a set of twelve pitch classes consisting of two hexachords in which the content of one is an inversion of the pitch classes of the other.²⁰ The inversive operation serves to associate pitch-classes as well as collections of an equivalent size.

Interestingly, Babbitt regarded the index numbers resulting from adding pc integers as merely theoretical terms, since there are difficulties in hearing intervals as sums of pitch numbers.

What could be meant or designated by a sum of pitch numbers? The difference between such numbers defines what has always been termed an interval... And yet, in some musically important sense, these two concepts would seem to require differentiation at some level. Surely interval is an "observation concept"; does the other concept require categorization as "theoretical," in the usual sense of the term, since it is not apparently translatable made of this terminological differentiation, this later concept, for all of its hierarchical implications, will be formable only on theoretical terms.²¹

1-2. David Lewin

Lewin described inversion along with transposition in terms of "interval content" since he assumes that:

¹⁹ Milton Babbitt, "Twelve-Tone Invariants," 254.

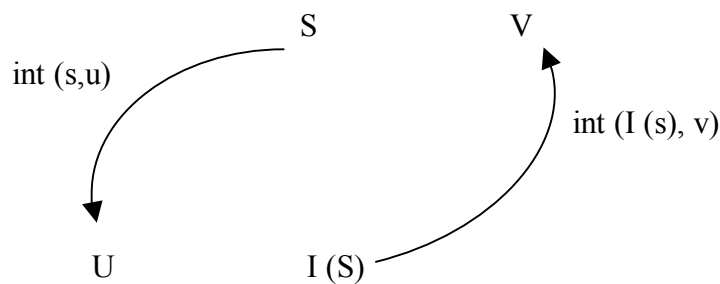
²⁰ Milton Babbitt, "Set Structure as a Compositional Determinant," *Journal of Music Theory* 5 (1961), 73-94.

²¹ Milton Babbitt, "The Structure and Function of Music Theory: I," *College Music Symposium* 5 (1965), 56.

...While the pcs themselves do not exhibit algebraic behavior, the intervals do. Specifically, the intervals take on values on the additive group of residues modulo 12, and the intervals compose with one another, and have complements, following the arithmetic of that group.²²

For inversional operation, therefore, he maintained that any element s and its inversion $I(s)$ are balanced about the given u and v in a certain intervallic proportion as shown in Example 1.4.²³

Example 1.4 Interval s and its inversion $I(s)$ as balanced about the given u and v as suggested by Lewin



An interval (i) between pcs can be inverted to interval- I , such that there is also an identical distance between pcs for the mod12 complement of the interval.²⁴ That is, he considered the inversion as an interval-reversing operation, while transposition is

²² David Lewin, "A Label-free Development for Twelve Pitch-Class Systems," *Journal of Music Theory* 21/1 (1977), 33.

²³ The Example is provided in his *Generalized musical Intervals and Transformations* (New Haven: Yale University Press, 1987), 51.

²⁴ In his interval function, he expressed his "invertible" interval function as $\text{IFUNC}(Y, X)(i) = \text{IFUNC}(X, Y)(i^{-1})$ in *Generalized musical Intervals and Transformations*, 99.

interval-preserving.²⁵ Since inversion is defined as an operation on intervals, his notation for the inversion became more analytically precise than that of conventional notation. The construction used in Lewin's earlier writing, " I^{uv} ", was later changed to " I_y^u " meaning the inversion around the pair u, v .²⁶

Lewin, however, claimed that certain pc sets should be distinguished from their inverted forms as functionally dissimilar. For instance, an interval-pair (3,4), corresponding to (C, E^b, G), is different from its inverted form (4,3) corresponding to (C, E, G); the interval-pairs should not be confused with intervals themselves.²⁷ In addition, in his writing on inversional balance, Lewin claimed that direct inversional relations between pitch classes could be perceived without the aid of the ubiquitous mirror texturing in Schoenberg's music.²⁸

1-3. Allen Forte

The Structure of Atonal Music, Forte's pioneering study of set-theoretic relations, begins with illustrations of transpositional and inversional relationships in a variety of atonal works, and then introduces his "interval vectors," which measure the total

²⁵ David Lewin, "On Generalized Intervals and Transformations," *Journal of Music Theory* 24/2 (1980), 245.

²⁶ Lewin used " I^{uv} " throughout his "A Label-Free Development" and changed in his *Generalized Musical Intervals and Transformations*.

²⁷ David Lewin, "Forte's Interval Vector, My Interval Function, and Regner's Common-note Function," *Journal of Music Theory* 21 (1977), 211.

²⁸ Lewin, "Forte's Interval Vector, My Interval Function," 235.

intervallic content of a given pitch class set and its inversion.²⁹ He defined an inversional operation as the complementation of a specific index number, zero (I_0), and the other index numbers (I_{1-11}) which are produced by the subsequent transposition. In this scheme, inversion seems less functional, requiring assistance:

It is important to observe that whereas t (transposition) does not imply inversion, inversion always implies subsequent t , even if it is the trivial case $t=0$...³⁰

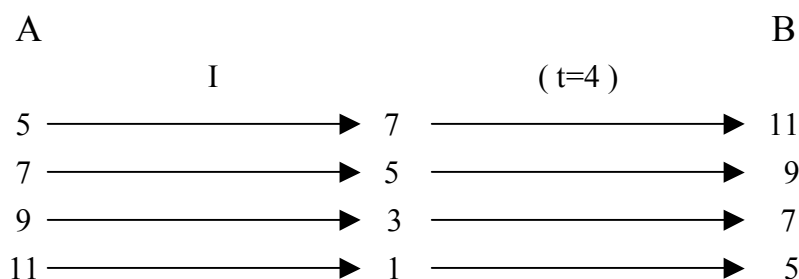
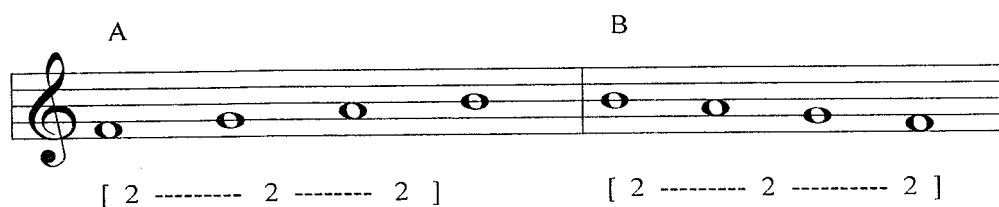
Most importantly, Forte formalized the concepts of equivalence and inclusion between pc sets through the inversional operation. He suggested that, under inversion, a pitch class set is interpreted as equivalent to another when there is ordered inversion. Here, Forte pointed out that the double mapping (IT) preserves the sequence of intervals (2-2-2) formed by adjacent pitch classes as shown in Example 1.5.³¹

²⁹ Forte's 220 pitch-class sets cannot be distinguished from their inversions and it is criticized by David Lewin, "Forte's Interval Vector," and Eric Regener, "On Allen Forte's Theory of Chords," *Perspectives of new Music* 13 (1974), 194-212.

³⁰ Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press), 9.

³¹ Allen Forte, *The Structure of Atonal Music*, 62.

Example 1.5 Double mapping of inversion and transposition present preserves the sequence of intervals (2-2-2)



He laid more stress on transposition than on inversion itself and so proposed, in his explanation of complement relations, that:

Since we know that interval-content is preserved under the I-mapping, we have only to determine the requirements (on K), the transposition number for the subsequent T-mapping.³²

His reply to Clough's criticism again clarified his concept of inversion, that is, that inversion is regarded as trivial:

He wishes to convince us that inversion somehow has priority over other transformations. I see no reason to believe that inversion as a general operation has any "inherent" or "natural" properties. It too is a mathematical abstraction, one which may be relevant to a partial musical situation, or again may not be of interest.³³

³² Allen Forte, "A Theory of Set-Complexes for Music," *Journal of Music Theory* 8/2 (1964), 155.

³³ Allen Forte, "The Domain and Relations of Set-Complex Theory," *Journal of Music Theory* 9/1 (1965), 175; John Clough criticized Forte's theory of set-complexes in "Pitch-Set Equivalence and Inclusion," *Journal of Music Theory* 9/1 (1965), 163-172.

1-4. Robert Morris

Morris proposed the following basic definition of inversion in post-tonal music:

The operation of inversion in atonal music theory is not to be confused with the concept of “chord inversion” from traditional figured bass theory.... Rather, inversion means the operation of finding the inverse of a pitch or pitch class. If x is a pitch class or pitch, its inverse is x' , such that the following holds: $x+x'=0$.³⁴

He classified inversions into two types based on the type of space assigned, either p-space or pc-space, and insisted that each type should be understood to be a wholly different object.³⁵ He defined the inversional center in p-space, mapping a pitch x into $-x$ as its inverse and vice versa, with arithmetic ($x+(-x)=0$); he also supplied the reason why this notion of p-space inversion must be incoherent in pc-space:

Let us define the center of inversion for T_nI_s in pc-space as we would in p-space; n divided by 2. However, if n is odd, $n/2$ does not exist (is not defined in pc-space). If n is even, then there are two centers of inversion: these are $n/2$ and $(n/2) + 6$ This means that the “axis of inversion” will not be unique: two axes a tritone apart have been defined. Therefore, we throw out the idea of inversional centers in pc-space.³⁶

That is, he repudiated the “center of inversion” in pc-space owing to his assumption that it is useless to determine the inversional centers, but did not reject pc inversion itself. On the other hand, he suggested that there is an exceptional case for translating the inversion from pc- to pitch-space, that is, when pitch-class segments

³⁴ Robert Morris, *Class Notes for Atonal Music Theory* (Hanover, New Hampshire: Frog Peak Music, 1991), 13.

³⁵ Robert Morris defines various musical spaces, e.g., contour space, pitch space, and pc-space in his *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven: Yale University Press, 1987); and Morris, *Class Notes for Atonal Music Theory* (Hanover, New Hampshire: Frog Peak Music, 1991).

³⁶ Robert Morris, *Class Notes for Atonal Music Theory*, 26.

remain invariant under RT_nI . Such pc segments as $\{0,5,9,1,6\}$, for instance, which is invariant under RT_6I , can be interpreted in p-space as “mirror chords”, $\{0,5,9,13,18\}$.³⁷ In his studies, while he suggests that there are some ideas associated with inversion and its applications in pc-space, he allows for an inversive center or axis of symmetry as a workable concept only in p-space and explains that:

For the composite operator T_nI it is the pitch $n/2$. The “n” is equal to the sum of the two pitches that are inversive related by T_nI . If n is an odd number, the inversive center is a quarter-tone between two pitches.³⁸

1-5. Elliott Antokoletz

Antokoletz attempted to demonstrate that “an all-encompassing system of pitch relations” in Bartók’s music is based on symmetrical inversion as he proposed that:

In contrast to the traditional tonal system...pitch relations in Bartók’s music are primarily based on the principle of equal subdivisions of the octave into the total complex of interval cycles. The fundamental concept underlying this equal division system is that of symmetry.³⁹

In the theoretic kernel of the book, the fourth chapter (“Basic principles of symmetrical pitch construction”), Antokoletz connected a “principle” of structure with the central concepts of inversion and the interval cycle. He first introduced the various

³⁷ He also explained that some pc-space invariants, e.g., (0127) under invariant under T_2I require octave doubling (that is, 70127) in order to be illustrated in p-space since these contain two singleton cycles for the operator yielding the invariance; Robert Morris, *Class Notes for Atonal Music Theory* (Hanover, New Hampshire: Frog Peak Music, 1991), 31-32.

³⁸ Robert Morris, *Class Notes*, 15.

³⁹ Antokoletz explains in a footnote that by symmetry he means only inversive symmetry in his *The Music of Béla Bartók* (Berkeley and Los Angeles: University of California Press, 1984), xii.

interval cycles combining each interval and its complement.⁴⁰ Each pair of complementary intervals contains two intervallic differences that add up to an octave. In the pairs of complementary intervals other than that of the perfect fourth and fifth, the smaller interval of each pair generates a cycle that subdivides one octave symmetrically.⁴¹ Antokoletz then defined the properties of inversional pitch structures as follows:

Any collection of two notes is symmetrical, since the two notes are equidistant from an imaginary axis. If we join a second dyad to the first, with the two notes of the second dyad equidistant from the same axis of symmetry, a four-note symmetry results.⁴²

His inversional four-note cell consists of two dyads forming two intervals with equal sums, termed a “sum couple”. He identified three such cells from Bartók’s Fourth Quartet, labeled X, Y, and Z. As Example 1.6 illustrates, in contrast with the first chromatic tetrachord of “X” and the second whole-tone tetrachord of “Y”, the third, “Z”, is special since it contains double tritones and thus produces two sum couples. Antokoletz made much of the “Z” cell since he noted, “the pivotal functions of [Z cells] are employed by Bartók as a means of exchanging one axis of symmetry for the other.”⁴³

⁴⁰ He provides six cycles, those built from the perfect fourths or fifths, major seconds, minor thirds, major thirds, and tritone. He uses these interval names rather than measuring the intervals in semitones since he gives emphasis to modality in Bartók’s music.

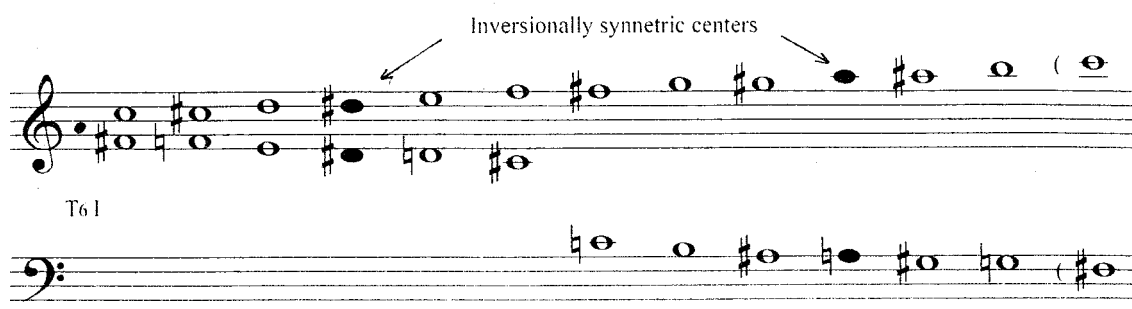
⁴¹ Elliott Antokoletz, *The Music of Béla Bartók*, 68.

⁴² *Ibid.*, 69.

⁴³ *Ibid.*, 72.

different notation for pitches and for pitch classes: pitch inversion (affixed as $T_n^p I(x)$) preserves contours with pitch intervals in opposite directions, while pitch-class inversion (notated as $T_n I(x)$) maintains the absolute value of intervals producing complementary (mod 12) intervals. With respect to the inversional center, he also pointed out the way applications of theory differ for ordered and unordered sets. In the modular space of a $T_n I$ -related pc, the center of inversion is one-half the index. That is, in each of the twelve inversions, there are always two centers of inversion, which are equal to one-half the index and one-half the index plus six. As shown in Example 1.7, the characteristic “wedge” shape presents pc unisons at $D^\sharp$ and A, that is, the two centers for $T_6 I$. On the other hand, for $T_n^p I$ in registral space, this inversional center is a place in pitch exactly at the halfway point.

Example 1.7 Center of inversion, A, is half way between each pair of $T_6 I$ related pitches; there are pc unisons at $D^\sharp$ and A



Rahn also suggested that as a pc-set for inversion can be formed by combining two inversionally related subsets, so it can also be disunited into two or more inversionally related subsets.

The union of any two inversionally related sets is an inversionally symmetrical set. Conversely, an interval symmetrical set may always be “disunited” into at least one pair of inversionally related subsets.⁴⁶

His partitioning subsets, produced by disuniting a larger set, form a connection between sets of different sizes and types under the inversion operation.

For generating inversionally symmetrical pc sets, his examples permitted an octave doubling of one pc to be displayed as symmetric chords when a set cannot be mapped into itself under inversion. For example, the set inversionally symmetrical to {0,1,2,7} is {7,0,1,2,7}, which shows a symmetrical arrangement of pitches centering around pc1. In this instance, his distinction between pitch and pitch-class inversion is somewhat obscure although he makes a clear notational pitch/pc distinction.

1-7. George Perle

In his twelve-tone tonal system, Perle employed inversion as a significant operator at all levels of the system, from the deviation of the basic cell to the formation of the largest structural features.⁴⁷ He proposed that the inversional structure is naturally implied by the semitone scale, which divides the octave by equal intervals as a self-invertible subscale.⁴⁸ Perle inferred the properties of inversion from the paired semitonal scales of opposite aspect, called a sum scale:

⁴⁶ John Rahn, *Basic Atonal Theory* (New York: Longman Inc. 1980), 93.

⁴⁷ George Perle, *Twelve-Tone Tonality* (Berkeley: University of California Press, 1977).

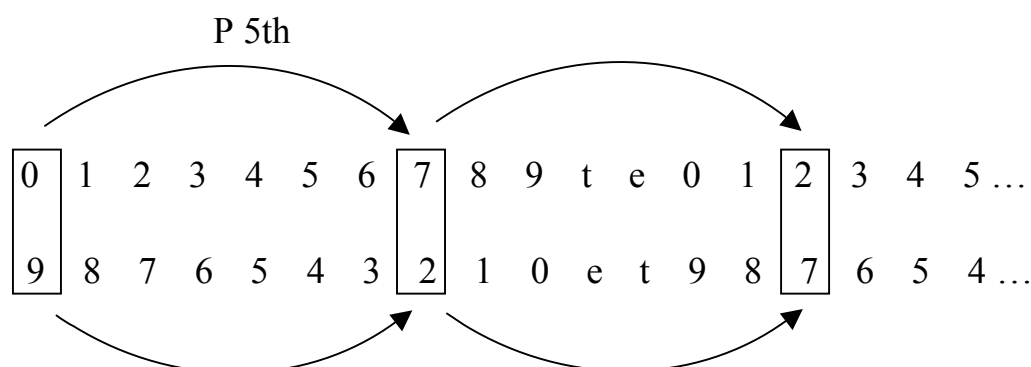
⁴⁸ There are five types of uni-intervallic divisions of the octave; twelve semitone scales, two whole-tone scales, three diminished seventh chords, four augmented triads, and six tritones in George Perle, “Symmetrical Formations in the String Quartets of Béla Bartók,” *The Music Review* 16 (1955), 301.

[Similarly,] the pitch-class relations between any pair of inversionally related sets are reducible to the pitch-class relations between two inversionally related representations of the semitonal scale. For example, the relations between corresponding pitch classes of the P_2 and I_7 sets [illustrated above], or of any other pair of sets of which one is the inversion of the other at sum of complementation 9....⁴⁹

His basic collection, a cyclic set, is also derived from the inversional alignment of semitonal scales by reading successive columns an interval apart consistently from top to bottom as shown in Example 1.8.

Example 1.8 Cyclic set of interval 7 cycles in an inversional alignment

(Sum 9)



Cyclic set : 0 9 7 2 2 7 9 0

That is, in this cyclic set, alternate elements are members of inversionally complementary cycles of interval 7. In addition to his tonal system, Perle attempted to

⁴⁹ George Perle, *Twelve-Tone Tonality* (Berkeley: University of California Press, 1977), 5.

demonstrate the significance of inversion in Bartók's and Berg's music.⁵⁰ In his analyses, as well as his theoretical writings, the sum-equivalence of inversion plays a major role since he considered inversion as the basis of the harmonic organization characteristic of twelve-tone music. As he notes, "Inversional complementation in post-diatonic music generates the harmonic context and is, by definition, literal."⁵¹ On the other hand, in earlier writings, Perle claimed that the role of inversional complementation could not be overestimated, and so asserted that:

It must be observed that Bartók's symmetrical formations are only an incidental aspect of his total compositional means. Even in those few works where they perform a significant structural role they do not ultimately define the context...⁵²

1-8. Brian Alegant

Alegant demonstrated the analytic applicability of pitch-class inversion through the examination of characteristics of even and odd index numbers. On the basis of Babbitt's formulations, he suggested that the inversion operator divides pitch-class sets into two partitions, each of which reveals an aurally discernible salient feature:

While even index numbers produce five pc pairs and two singletons and feature even sums and differences between their corresponding pcs, odd index numbers

⁵⁰ George Perle, "Symmetrical Formations in the String Quartets of Béla Bartók," *The Music Review* 16 (1955), 300-312; "Berg's Master Array of the Interval Cycles," *Musical Quarterly* 63/1 (1977), 1-30.

⁵¹ George Perle, *Twelve-Tone Tonality* (Berkeley: University of California Press, 1977), 3.

⁵² George Perle, "Symmetrical Formations in the String Quartets of Béla Bartók," *The Music Review* 16 (1955), 312.

feature six pc pairs, and odd index sums and differences. These are some reasons why passages based on even and odd index numbers have different “sounds.”⁵³

He assumes that the nature of inversive settings is realized by comparing and contrasting the two index numbers. He did not deal with pitches, but with pitch classes, and employed only the inversion operation in the pc sets.

In his analyses, he divides the universe of set-classes into four distinct categories: even, odd, collections in both, and collections in neither. In other words, we can combine any odd index numbers to build the odd set-class inventory, while even index numbers combine to generate the even inventory, which is displayed in Example 1.9.⁵⁴

⁵³ Brian Alegant, “When Even becomes Odd: A Partitional Approach to Inversion,” *Journal of Music Theory* 43/2 (1999), 195.

⁵⁴ This example is Alegant’s Figure 2 and 3 in “When Even becomes Odd,” 196-197.

Example 1.9 odd and even inventories provided by Alegant

(Odd inventory)

Dyads: 2-; 1 1 3 3 5 5

Tetrachords: 4-; 1 1 7 7 9 9 10 10 17 17 20 20 23 23 28

Hexachords: 6-; 1 1 6 6 8 8 13 13 20 20 29 29 32 32 38 38 42 42 50 50

4-1 [0123] 4-5 [0145] 4-9 [0167] 4-10 [0235] 4-17 [0347] 4-20 [0158]

4-23 [0257] 4-28 [0369] 6-1 [012345] 6-8 [023457] 6-20 [014589] 6-32 [024579]

(Even inventory)

Singletons: 1-; 1 1 dyads: 2-; 2 2 4 4 6 (this is produced by the single tons)

Dyads: 2-; 6

Trichords: 3-; 1 1 6 6 9 9 10 10 12 12

Tetrachords: 4-; 3 3 6 6 8 8 21 21 24 24 25 25 26 26 28

Pentachords: 5-; 1 1 8 8 12 12 15 15 17 17 22 22 33 33 34 34 35 35 37 37

Hexachords: 6-; 4 4 7 7 23 23 26 26 28 28 35 35 37 37 45 45 48 48 49 49

3-6 [024] 3-9 [027] 3-10 [036]

3-12 [048] 4-3 [0134] 4-6 [0127] 4-8 [0156] 4-21 [0246] 4-24 [0248]

4-25 [0268] 4-26 [0358] 4-28 [0369] 5-1 [01234] 5-35 [02479] 6-35 [02468a]

In this, where the set-classes 4-28 and 8-28 belong to both inventories, some set-classes belong to neither inventory. These include set-classes 3-3, 4-13, 5-3, and so

on which do not fall into odd or even inventory. He then defines set-classes as being inside or outside the governing inventory of either even or odd. In his examples, set-classes inside an inventory are realized as structural ones that can aurally be distinguished from set-classes outside the inventory which are ornaments.

In addition, there are other approaches to pc inversion besides those selected above. A number of articles--including those of Bruce Archibald, Robert Clifford, Wallace Berry, and Leo Treitler--also explored the nature and properties of inversion through analyzing twentieth-century repertoires.⁵⁵ In fact to what extent the inversion in pc space defines pitch and interval relations has been a controversial issue for many years.⁵⁶ A number of theorists minimize the role of inversion as a subsidiary device or admit pc inversion only in part. As we observed, theorists such as Babbitt and Morris recognize defining pc inversion just as arithmetic, though they do claim its broad applicability to post-tonal music. Forte also tends to view inversion as less determining and functional than transposition while at the same time he demonstrates the significance of inversion in associating pitches as well as collections.⁵⁷ These authors,

⁵⁵ Wallace Berry, "Symmetrical Interval Sets and Derivative Pitch Materials in Bartók's String Quartet No.3," *Perspectives of New Music* (1979-80), 287-380.; Robert Clifford, "Multi-Level Symmetrics in Webern's Op.11, No.1," *Perspectives of New Music* 40/1 (2002), 198-215.; Leo Treitler, "Harmonic Procedures in the Forth Quartet of Béla Bartók," *Journal of Music Theory* 3/2 (1959), 292-298.; Bruce Archibald, "Some Thoughts on Symmetry in Early Webern: Op.5, No.2," *Perspectives of New Music* 10/2 (1972), 159-163.

⁵⁶ Many theorists instead encourage the use of pitch inversion alone or look for some other device to substitute for inversion; Jonathan Bernard, "Space and Symmetry," 185-201; Tom Demske, "Registral Centers of Balance," 60-76.

⁵⁷ Allen Forte, "The Domain and Relations of Set-Complex Theory," *Journal of Music Theory* 9/1 (1965), 175.

nevertheless, established a theoretical basis for inversion that has been essential to the identification and systematization of inversional structure. Perle and Antokoletz, by contrast, fully recognize inversion as essential and central for describing the work of Bartók and other twentieth-century composers. In addition, Alegant shows how to use pitch-class inversion as a means for discerning the difference between structural pitches and embellishing ones in atonal and twelve-tone music.

In recent studies, the transformational approaches such as those of David Lewin, Henry Klumpenhouwer, Philip Lambert and Philip Stoecker, have examined the inversional operation in great depth by means of Klumpenhouwer networks and the index zone. This will be discussed in the following chapter.⁵⁸ For these theorists and in network relations, however, the inversional operation seems to perform a less significant role since such isographic relations as strongly and positively can easily be identified by the same size of an interval or intervals. Although Stoecker has recently attempted to demonstrate the significant role of inversion in isographic relations through introducing axial isography, his axial isographic relations are limited to trichords only. Indeed in both modern set-theoretic and in transformational investigations, the inversional operation has an effect mainly on relationships or transformations of pitch-class sets of the same cardinality.

⁵⁸ Their representative works are David Lewin, "Klumpenhouwer Networks and Some Isographies That Involve Them," *Music Theory Spectrum* 12/1 (1990), 83-120; Henry Klumpenhouwer, "A Generalized Model of Voice-Leading for Atonal Music," Ph.D. diss., Harvard University, 1991. Phillip Lambert, "On Contextual Transformations," *Perspectives of New Music* 38/1 (2000), 45-76.; Philip Stoecker, "Klumpenhouwer Networks, Trichords, and Axial Isography," *Music Theory Spectrum* 24/2 (2002), 231-245.

The foregoing discussion, therefore, is intended not only to examine how broadly inversion can be applied as the most basic way of combining, disuniting, transforming, and relating pc sets but also to demonstrate the significant structural role of inversion in isographic relations through interpreting Klumpenhouwer networks in a new context, that of inversional regions or index zones. The process of the realization of isographies in inversional regions will also suggest new types of isographies including axial isography for tetrachords. Especially, the role of the inversional relation between sets of different sizes, such as a superset and one of its subsets, will be examined by organizing and interpreting the network relations in twelve inversional regions. The ultimate goal of this examination is to show, through the analysis of a variety of post-tonal compositions, how inversional relations play a significant role as structural determinants and also how inversional background design can be projected in the foreground musical structure.

The following text is divided into three distinct sections. The first, Chapter 2, will discuss the inversional operation in Klumpenhouwer networks, reinterpreting inversion in terms of inversional regions, or index zones, and will also evaluate the ways in which these index zones realize inversional designs. It also will introduce new analytic devices including axial isographies for tetrachords. The second section, Chapter 3, will examine a methodology for determining the relationships between pc sets of different sizes, mainly a trichord and a tetrachord, using a transformational approach, and, in this sense, the process will reevaluate inversion as an essential device providing structural unity through grouping pitch elements together. The last chapter, Chapter 4, will describe Perle's twelve-tone tonality by means of a transformational approach for

the sake of observing both general and specific aspects of inversion in the relations among the larger sets, e.g. hexachords and their subsets.

Chapter 2

Klumpenhouwer Networks and Axial Isography for Tetrachords

A number of transformational approaches have shown how Klumpenhouwer networks can be used for interpreting either pitch-class sets or the interrelation among several interpreted pitch-class sets.⁵⁹ While developing the notion of Klumpenhouwer networks (or K-nets), Klumpenhouwer has allowed for both transpositional and inversional operations within a network in order to interpret more chords of different set classes with isographic networks, while Lewin has demonstrated the analytic utility of K-nets for understanding the isographic relations among networks.⁶⁰ Recently, Lewin

⁵⁹ Edward Gollin, "Some Unusual Transformations in Bartok's 'Minor Seconds, Major Sevenths'," *Intégral* 12 (1998), 25-51; Shaugn O'Donnell, "Transformational Voice-Leading in Atonal Music," Ph.D. diss., City University of New York, 1997; idem, "Klumpenhouwer Networks, Isography, and the Molecular Metaphor," *Intégral* 12 (1998), 53-80; Philip Lambert, "Isographies and Some Klumpenhouwer Networks They Involve," *Music Theory Spectrum* 24/2 (2002), 165-195.

⁶⁰ Klumpenhouwer expands Lewin's network model involving transpositional relations only. Lewin and Klumpenhouwer provide their analyses of K-nets in David Lewin, "Klumpenhouwer Networks and Some Isographies That Involve Them," *Music Theory Spectrum* 12/1 (1990), 83-120; idem, "A Tutorial on Klumpenhouwer Networks, Using the Chorale in Schoenberg's Opus 11, No.2," *Journal of Music Theory* 38/1 (1994), 79-101; Henry Klumpenhouwer, "A Generalized Model of Voice-Leading for Atonal Music," Ph.D.diss., Harvard University, 1991; idem, "Aspects of Row Structure and Harmony in Martino's Impromptu Number 6," *Perspectives of New Music* 29/2 (1991), 318-54; idem, "An Instance of Parapraxis in the Gavotte of Schoenberg's Opus 25," *Journal of Music Theory* 38/2 (1994), 217-48; idem, "Network Analysis and Webern's Opus 27/III," *Tijdschrift voor Muziektheorie* 3/1 (1998), 24-37; idem, "The Inner and Outer Automorphisms of Pitch-Class Inversion and Transposition: Some Implications for Analysis with Klumpenhouwer Networks," *Intégral* 12 (1998), 81-93.

and other theorists have also suggested additional uses of K-nets.⁶¹ In order to extend the analytic reach of K-nets, Lewin defines three types of isography: “strong,” “positive,” and “negative.” These can be characterized by comparing the T_n and I_n labels among various network interpretations. In these isographic relations, the transpositional operations appear to be more significant since only one or more common interval classes are required to produce the isographies.⁶² For instance, any pair of trichords containing the same ic2 has either a strong or a positive isography by yielding the same I_n numbers or equivalent difference between the corresponding I_n numbers. In addition, for 2+1 and 2+2 K-nets, negative isography can also be produced by simply reversing the positions of the t-related pitches.⁶³

In such conventional isographies, the realization of the inversive operations is not always made explicit since we can easily realize some other operation as a relevant analytic tool and regard inversions as of secondary importance. Stoecker, however, recently introduced an additional type of isography, which gives more emphasis to

⁶¹ Some articles suggest several points of intersection between K-nets and Perle’s theoretical constructs. It will be discussed more in detail in the last chapter of this dissertation. See Perle, “Communication,” *Music Theory Spectrum* 15/2 (1993), 300-303; Robert Morris, “Voice-Leading Spaces,” *Music Theory Spectrum* 20/2 (1998), 191-193; David Lewin, “Thoughts on Klumpenhouwer networks and Perle-Lansky Cycles,” *Music Theory Spectrum* 24/2 (2002), 196-230; David Headlam, “Perle’s Cyclic Sets and Klumpenhouwer Networks: A Response,” *Music Theory Spectrum* 24/2 (2002), 246-256; Gretchen C. Foley, “Arrays and K-Nets: Transformational Relationships within Perle’s Twelve-Tone Tonality,” *Indiana Theory Review* 23 (2002), 69-97.

⁶² In a recent article, Michael Buchler criticizes K-nets including both T_n and I_n arrows and instead suggests T-nets comprising only transpositional relations in his “Reconsidering Klumpenhouwer Networks,” *Music Theory Online* 13/2 (2007).

⁶³ Lewin, “Tutorial,” 88-89; Lambert, “Isographies and Some Klumpenhouwer Networks,” 70.

inversion operations than to transpositions, although his axial isography is limited only to trichords.⁶⁴ Because of this limitation there remains a need to discuss such axial isography among larger sets, especially of tetrachords, in order to explore more ways that the inversional operations may be used to find meaningful isographic relations among K-nets.

This chapter will show how the inversional operation plays a major role in certain isographic relations and how these isographies can provide insight into the nature of inversion as reflected in inversionally designed works. In addition, this chapter will provide the analyses of three pieces in which inversion operations are prominent features of the compositional surface while at the same time affecting deeper structure. The chapter will focus primarily on tetrachords.

The first part of the chapter is an examination of Lambert's "index zone," which will then be used to show how both conventional and new isographies can be implemented in the new context of inversional areas. In the second part of the chapter, three analyses, of sections of Schoenberg's *Drei Klavierstücke*, op.11, no.2, Scriabin's *Vers La Flamme*, op.72, and a short passage of Webern's *Streichquartett* 1905, will each introduce a new type of axial isography. This part will also explore how the tetrachordal axial isographies can be applied to realize the contextual nature of inversion.

⁶⁴ Philip Stoecker, "Klumpenhauer Networks, Trichords, and Axial Isography," *Music Theory Spectrum* 24/2 (2002), 231-245; idem, "Studies in Post-Tonal Symmetry: A Transformational Approach," Ph.D. diss., City University of New York, 2003, especially chapter 2.

2-1. K-nets in inversional zones

In his analysis of Webern's Three Pieces for Violoncello and Piano, op.11, Philip Lambert defines the "index zone," in which a specific index number is pervasive, in order to show how a particular inversional operation dominates the transformational pitch and pitch-class relations within a single zone and also to show how a progression of different index zones determines the structural background of the piece.⁶⁵ Having suggested several ways to establish index zones in pitch space, Lambert explores particular inversional relationships among pitch classes that sum to the same index number yet do not participate in a display of pitch symmetry. In his analysis, Lambert observes especially the transformational path of a single inversional tetrachord, a member of set-class (0145), and he emphasizes the tetrachords that involve inversional pairings with the same index number by asserting that "dyads from that constellation may cooperate to present at least four pitches surrounding a single p-space axis, may subsequently bring together at least four other pitches surrounding an ic-6-related axis, or may simply be paired as pc dyads without supporting p-space symmetry."⁶⁶ For his index zones, he basically adopts Lewin's view of inversion that particular inversional

⁶⁵ Philip Lambert, "On Contextual Transformations," *Perspectives of New Music* 38/1 (2000), 45-76.

⁶⁶ Philip Lambert, "On Contextual Transformations," 67.

relations strongly suggest that certain notes are attracted to their inversional partners.⁶⁷

Within an index zone, the inversion operator indeed has great ability to combine pitch classes forming a new combination under the same number. I will use his term “index zone” in the following discussion, in which at least one pair of dyads or dyadic pairings project the same index number, but I will limit the discussion to pc inversion only.

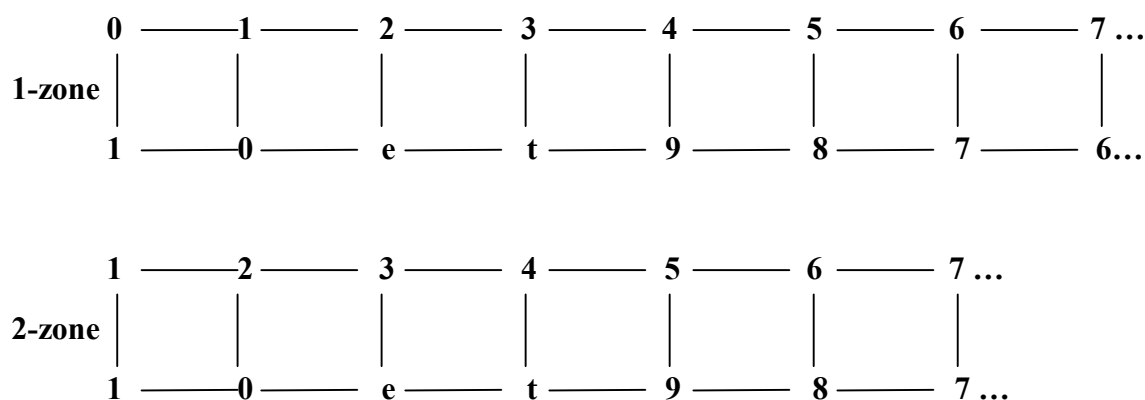
In addition, as Lambert shows that the (0145) tetrachord, which involves inversional pairings about the same axis, is of particular importance in index zones, any inversional tetrachord in fact draws our attention to a specific inversional operation by including a dyadic pair which sums to the same index. Antokoletz refers to such inversional tetrachords as “sum couples” or “sum of complementation” and demonstrates the significance of inversional four-note cells as structural generators producing axes in his analyses of Bartók’s music.⁶⁸ I will also emphasize inversional tetrachords in the transformations of K-nets within index zones, since an inversional tetrachord can easily establish a zone by presenting a pair of sum dyads which have the same index number, especially among many hybrid tetrachords with different I_n numbers. Moreover, certain axial isographies can only be produced by a sum couple and either another tetrachord or another trichord. This will be discussed in the following analyses and in the following chapter.

⁶⁷ Lambert briefly mentions Lewin’s examination of inversional symmetry in “On Contextual Transformations,” 60-61. See Lewin’s discussion of inversion in his *Musical Form and Transformation: 4 Analytic Essays* (New Haven: Yale University Press, 1993), 88; idem, “Inversional Balance as an Organizing Force in Schoenberg’s Music and Thought,” *Perspectives of New Music* 6 (1967), 1-21.

⁶⁸ Antokoletz, *The Music of Béla Bartók*, 70.

For now let me examine how “strong” and “positive” isography for tetrachords can be interpreted in inversive zones. In pitch-class space, twelve inversive axes (I_0 - I_6) yield twelve inversive zones. Any pair of dyads summing to the same index number may comprise a tetrachordal K-net in a zone. Drawing quadrangles on the twelve index zones can be used to show the isographies between tetrachordal K-nets. As illustrated in Example 2.1, an odd index number, for instance in the 1-zone, produces seven members with the same quadrangular shape and size while an even index number, for instance in the 2-zone, yields six members with the same shape and size.⁶⁹ In other words, an index zone can yield six or seven members which can then be called a K-class.⁷⁰

Example 2.1 Seven or six members with the same quadrangular shape within an index-zone

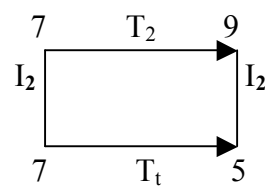
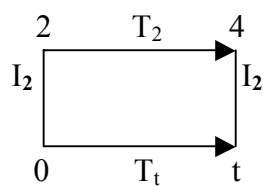
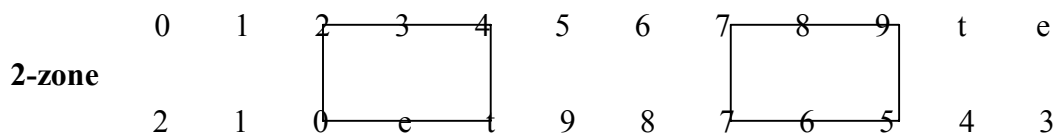


⁶⁹ In each zone of Example 2.1, the pitch-classes which follow yield the same members as the ones shown in this example.

⁷⁰ O'Donnell introduces this group as a Klumpenhower class in “Transformational Voice-Leading in Atonal Music,” 38-39. Lambert explains the term in “Isographies and Some Klumpenhower Networks,” 169-70.

To perform spatial transformations of the K-nets, a pair of dyads can be combined with another dyadic pair summing either to the same or to different index numbers. For strong isography, any one quadrangle can be associated with another of the same size and shape within a zone. That is, in a zone, all quadrangles of the same size have strong isography. A strong isography between tetrachordal K-nets, labeled as $\langle T_0 \rangle$, thus represents the movement of quadrangles within a zone.⁷¹ Example 2.2 shows that a pair of quadrangles of the same size within a 2-zone produces strong isography; the network interpretations contain the same I_2 arrow and the same T_2/T_t arrows.

Example 2.2 Strong isography between tetrachords within 2-zone

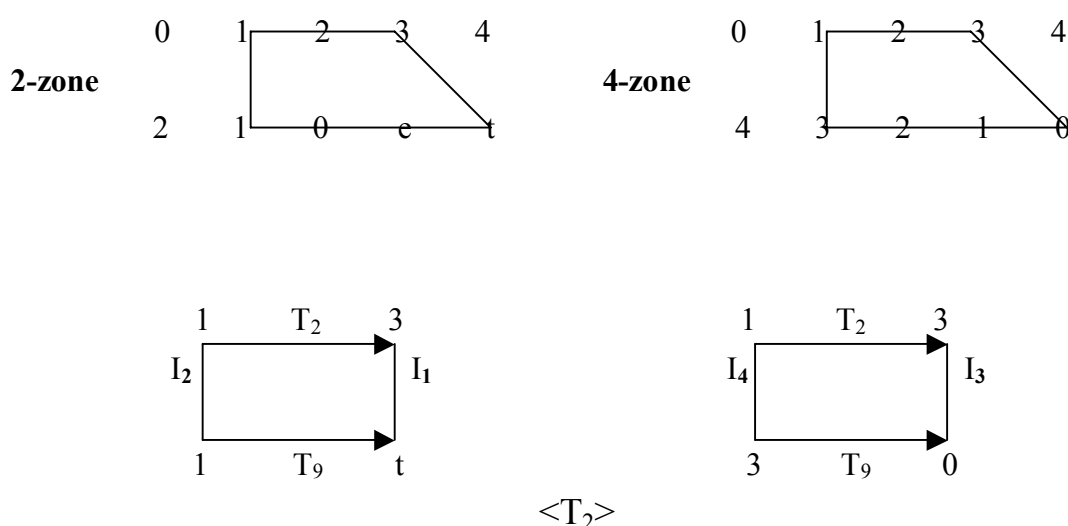


$\langle T_0 \rangle$

⁷¹ Lewin defines $\langle T_n \rangle$, “hyper T-n,” as an operation that maps one graph onto another, with n equal to the difference between corresponding I-numbers in his “A Tutorial” and this notation is primarily presented by Klumpenhouwer, “A Generalized Model.” Lewin also originally represents strong isography as $F\langle 1,0 \rangle$ and positive isography as $F\langle 1,j \rangle$; j is equal to the difference between corresponding I-numbers in his ‘Klumpenhouwer Networks,’ 89-90.

In addition, a quadrangle within a zone can move to another quadrangle of the same size in a different zone. As Example 2.3 shows, the quadrangles of the same shape and size but derived from different zones have positive isography; the network interpretations both involve the same T_2 and T_9 arrows, but each contain different I_2/I_1 and I_4/I_3 arrows.

Example 2.3 Positive isography between tetrachords drawn from different index-zones

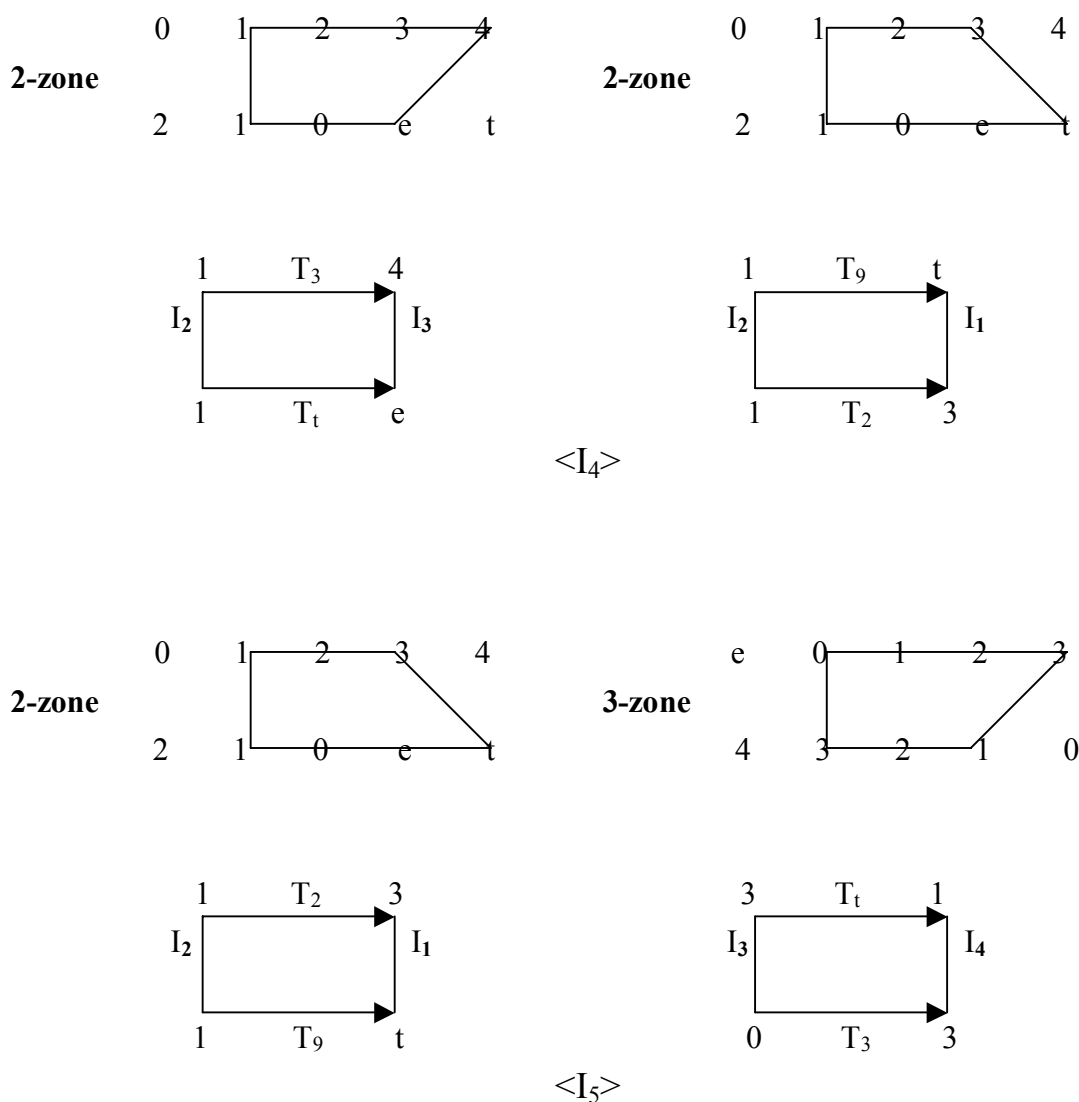


In this, the notation $\langle T_2 \rangle$ represents the difference between their inversions as being two semitones apart. This positive isography can be demonstrated by moving the quadrangles of the same shape and size between index zones while their movement within a single zone yields strong isography. Lambert terms all K-nets which can be related to each other by positive and strong isographies as a “family.”⁷²

⁷² Lambert, “Isographies and Some Klumpenhouwer Networks,” 170.

On the other hand, a quadrangle such as the one in Example 2.2 or 2.3 can be flipped upside down either within a zone or in a different zone. Example 2.4 is an illustration of the way in which a K-net's negative isography can also be produced within a zone as well as between different zones by the flipping of quadrangles; in the network interpretations, their T_n numbers sum to 0 (mod 12), while their I_n numbers sum to 4 and 5 respectively.

Example 2.4 Negative isography for tetrachords within a zone and between index-zones



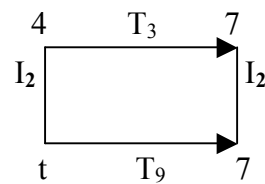
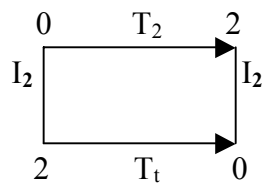
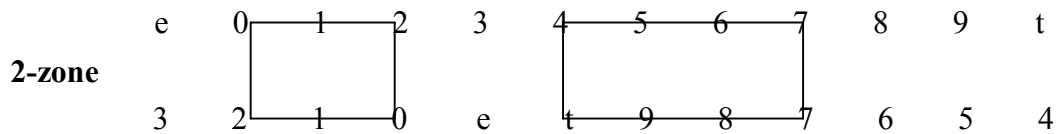
The negative isographies in this example are thus represented by $\langle I_4 \rangle$ and $\langle I_5 \rangle$.⁷³ As we examined above, any given quadrangle can move to another of the same

⁷³ A number of negatively isographic relations omitted here can also be generated by additional flips and swaps of quadrangles. Lambert also asserts that “negative isography simply reverses the position of the I_n arrows, without affecting the pitch-class content of the set” in his “Isographies and Some Klumpenhouwer Networks,” 170.

size within a zone or to a different zone, with or without flipping, to have strong, positive, or negative isography.

The axial isography for trichords introduced by Stoecker, which incorporates different T_n arrows but shares the same I_n arrow, can also be implemented as the realization of transformation within a zone.⁷⁴ Unlike the axial isography for trichords, there are three possibilities for the realization of axial isography for tetrachords. Just as in Stoecker's trichordal axial isography, the first type of axial isography for tetrachords allows different T_n arrows. As Example 2.5 shows, a quadrangle can be associated with another quadrangle of a different size but the same shape in the same 2-zone.

Example 2.5 First type of axial isographic relationship between tetrachords within 2-zone



The network interpretations thus have the identical I_2 arrow whereas the analogous T_n arrows differ. Instead of simply presenting the operations in the abstract,

⁷⁴ The trichordal K-nets will be explored in the following chapter of this dissertation.

let us look at an analytic example which will illustrate how the new methodology can be applied and can work together with other operations in a particular musical context.

2-2. Axial isography I: Arnold Schoenberg's op.11, no.2

Example 2.6 displays an annotated score of a short passage, measures 4 to 9, from Schoenberg's op.11, no.2.⁷⁵

Example 2.6 Schoenberg's op.11, no.2, right-hand part in mm. 4-9

The image shows a musical score for the right-hand part of Arnold Schoenberg's op.11, no.2, measures 4 to 9. The score is annotated with nine symmetrical tetrachords labeled C1 through C9. C1 and C2 are in measures 4 and 5, C3 and C4 in measure 6, C5, C6, C7, and C8 in measure 7, and C9 in measure 8. The score includes dynamic markings (rit., p, f) and articulation (accents).

As shown in Example 2.7, the passage presents nine symmetrical tetrachords. These tetrachords, labeled C₁ through C₉, represent five different set classes. It is not

⁷⁵ Lewin presents extensive analysis of this piece, especially the first three measures and the “chorale” passage measures from 9 to 12, in “A Tutorial,” 79-101.

difficult to capture the inversional gesture in this passage, since the outer voices of the tetrachords present consecutive paired dyads involving complementary T_n values.

Example 2.7 Tetrachords C_1 - C_9 from Schoenberg's op.11, no.2, mm. 4-9

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
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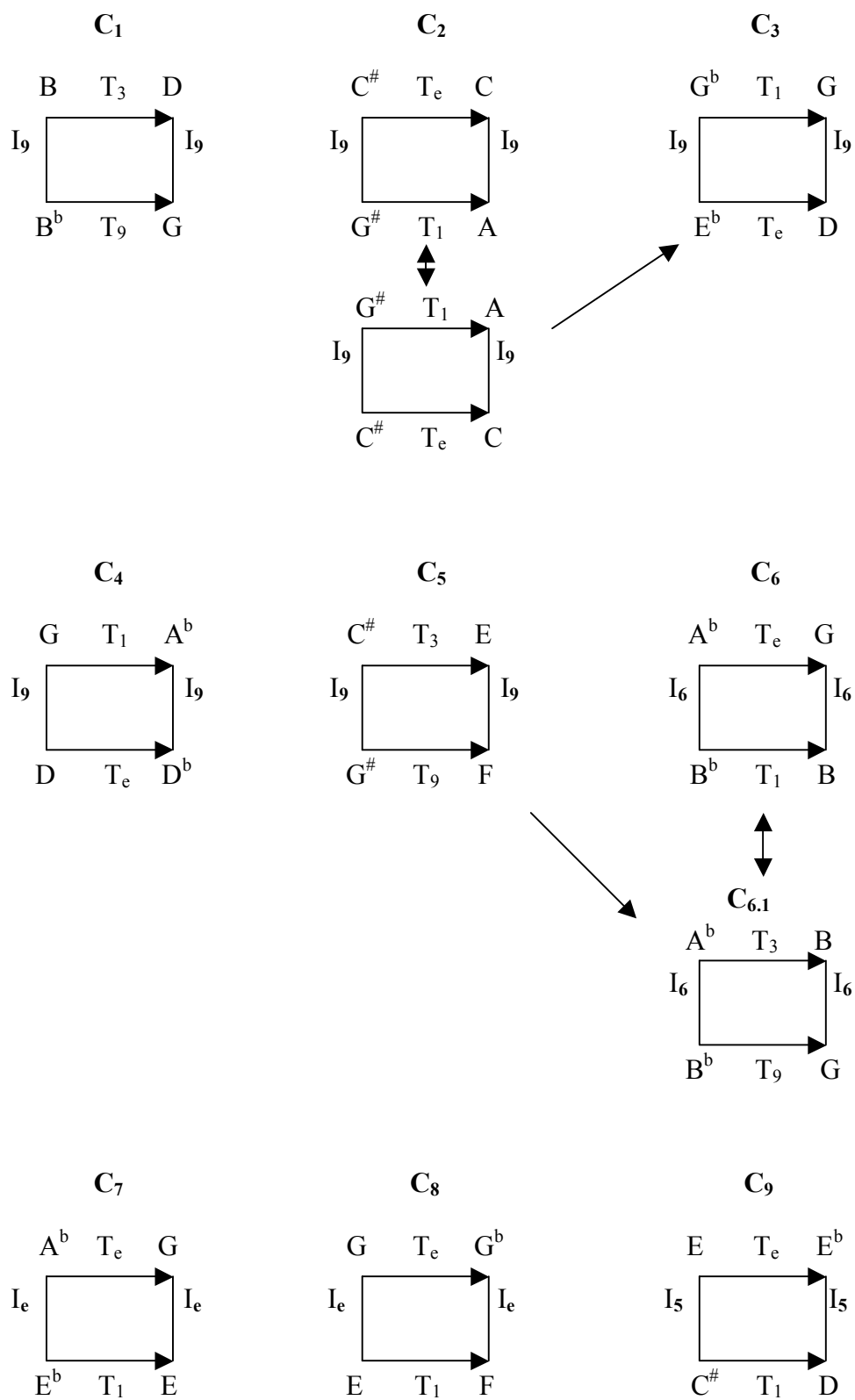


4-1 /	4-7	4-7	4-9	4-17	4-3	4-7	4-1	4-1
(0347)	(0145)		(0167)		(0134)		(0123)	

Example 2.8 interprets this passage as nine tetrachordal K-nets. All network interpretations correspond to the musical surface.⁷⁶ That is, the K-nets maintain registral integrity and model the tetrachords, each featuring two identically related inversions with complementary T_n arrows.

⁷⁶ In this example, The K-nets of C_2 and C_{10} are reversed without changing pitches and inversional relations. Specifically for inversional tetrachords, the negative forms do not affect the transformational path among zones.

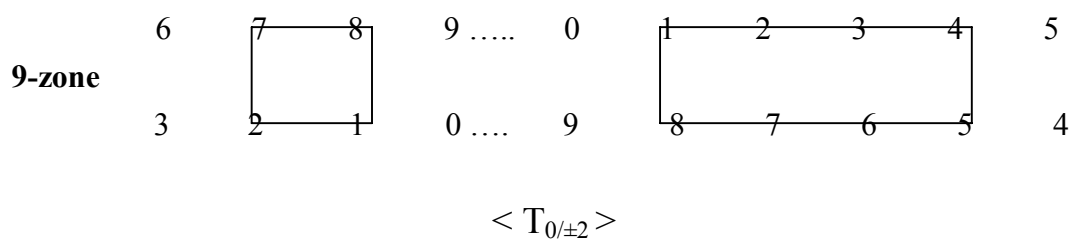
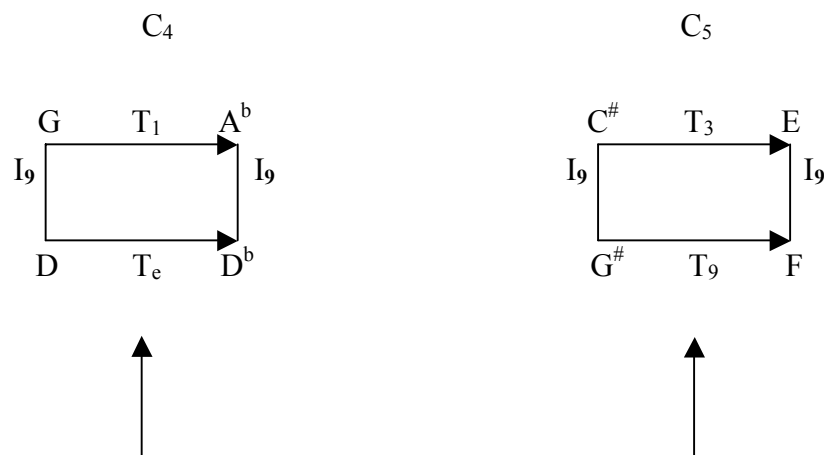
Example 2.8 Network interpretations of $C_1 - C_9$ in Schoenberg's op.11, no.2, mm. 4-9



Each network interpretation of an inversional tetrachord necessarily yields one I_n arrow: C_1 to C_5 have identical I_9 arrows while the remaining chords involve I_6 , I_e and I_5 arrows. The network interpretations of C_2 and C_3 , each a member of set class 4-7(0145), and C_4 , a member of 4-9(0167), are all identical, thus yielding a strongly isographic relation. However, the network interpretations for C_1 and C_5 , each a member of 4-17(0347), cannot be related to the adjacent tetrachords either by strongly or by positively isographic relations since the tetrachords do not share the same interval classes. Significantly, the networks instead share the same I_n arrows with the neighboring tetrachords. In this case, the concept of Stoecker's axial isography for trichords is applied to tetrachords; the tetrachordal K-nets under the axial isography draw our attention to the invariant inversional relations more than to the varying interval classes.⁷⁷ Example 2.9 demonstrates that in the progression from C_1 to C_2 , it is easy to focus on the successive I_9 dyads since the sum dyads of the outer voices move away from each other by complementary T_n arrows even though the third dyad is placed an octave higher. In addition, the C_4 to C_5 motion exhibits the same consecutive I_9 -dyads though the C_5 dyads are placed also an octave higher.

⁷⁷ Stoecker, "Studies in Post-Tonal Symmetry," 31-32.

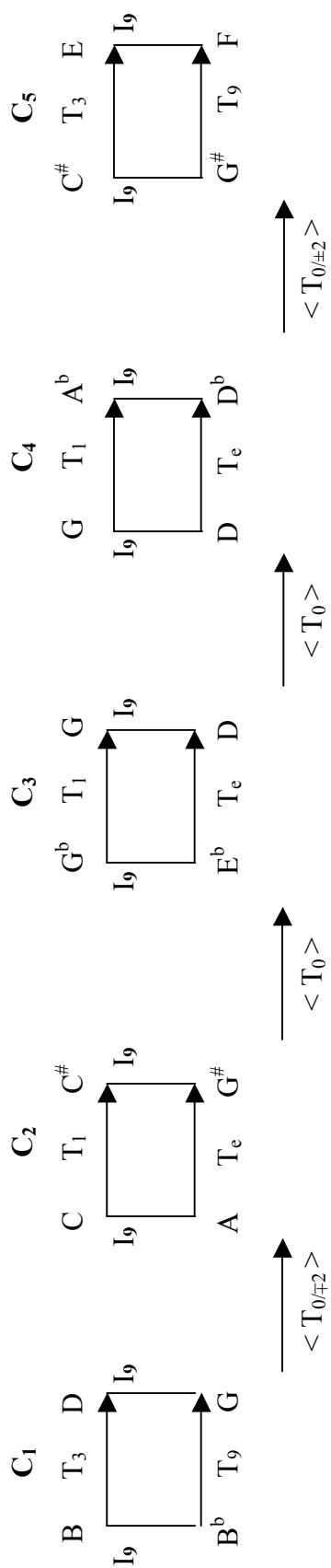
Example 2.9 First type of axial isography: network interpretations of tetrachords have identical I_n arrows while their T_n arrows change



Here, the network interpretations share identical I_n arrows while their T_n arrows change. We can define the tetrachordal K-nets that share identical I_n arrows but involve different T_n arrows as axially isographic for tetrachords. Example 2.9 also shows that quadrangles of different sizes can associate with each other within a zone since the quadrangles are still controlled by the same inversional operator, even though their sizes differ. This type of axial isography for tetrachords can be labeled $\langle T_{0/\pm n} \rangle$, where T_0 represents the consistent I_n arrow, namely the projection within a zone, and $\pm n$ indicates the difference between complementary T_n numbers. Therefore, in Example 2.9, the movement within a zone in which the T_n arrows change from $T_{\pm 1}$ of C_4 to $T_{\pm 3}$ of C_5 is expressed by the notation $\langle T_{0/\pm 2} \rangle$, which is equivalent to difference of the sizes of the rectangles.⁷⁸ Similarly, the operation $\langle T_{0/\mp 2} \rangle$ can be yielded by mapping the K-net of C_1 to C_2 as displayed in Example 2.10 shows.

⁷⁸ For the first axial isography, I prefer the notation $\pm$ or $\mp$ to emphasize the complementary $T_{\pm}$ values tracing the wedge path to the T_{+n} and the T_{-n} value.

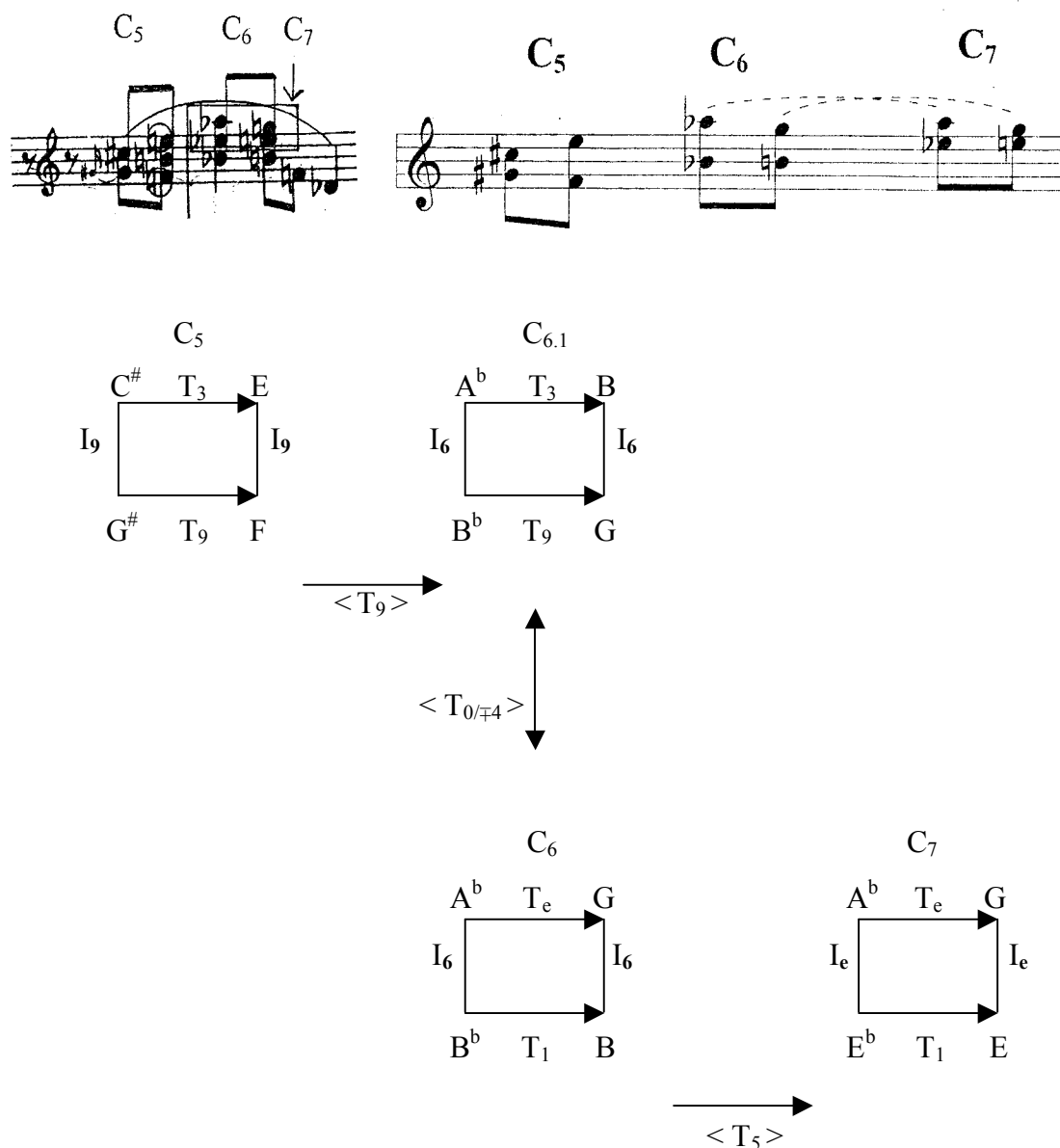
Example 2.10 Strong and first type of axial isography in Schoenberg's op.11, no.2, mm. 4-7



Here, in the progression from C_1 to C_5 , we observe that of all the tetrachords only the I_9 arrows are the same while the inversional wedge centering on the inversional axis is reflected by $\langle T_{0/\mp 2} \rangle$, $\langle T_0 \rangle$, and $\langle T_{0/\pm 2} \rangle$. That is, the strong and the axial isographies produced in progression C_1 to C_5 cooperate with each other to extend the given inversional wedge by the first inversional tetrachord.

After maintaining the wedging motion in the 9-zone, the network interpretations for C_5 and C_6 share neither $T_{\pm n}$ arrows nor I_n arrows. Example 2.11, however, illustrates that the network interpretation of $C_{6,1}$ can also be articulated as a dyad of complementary interval classes, ic 3 and ic 9.

Example 2.11 Positive and axial isography of network interpretations of C_5 - C_7 in Schoenberg's op.11, no.2, mm. 7-8

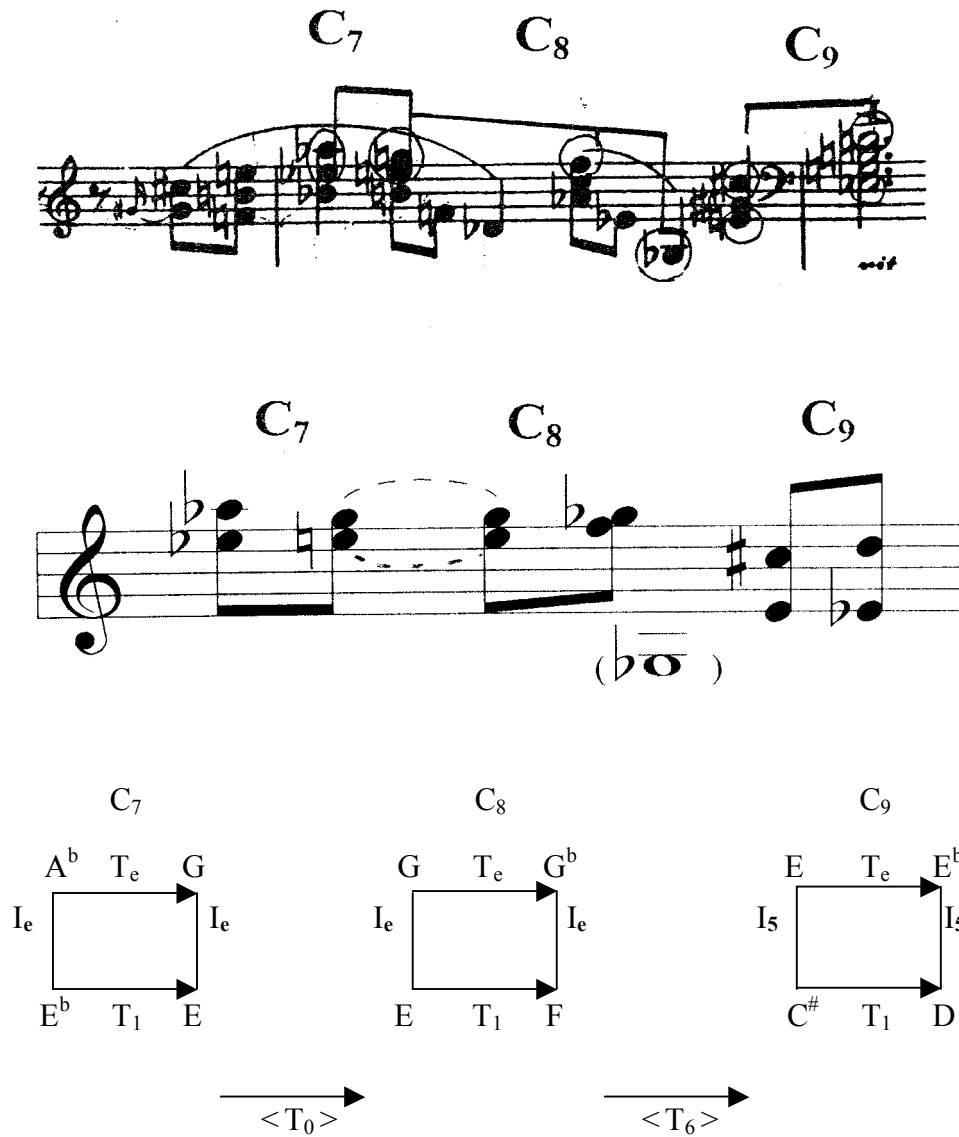


That is, the network of C_6 can bear other T_n numbers through the exchange of t-related dyadic partners while still having the same inversive relations exposed in musical surface. Note that C_6 involves two tetrachordal K-nets, C_6 and $C_{6.1}$, as displayed in Example 2.11, that contain the same I_6 arrows, each having $T_{\pm 1}$ or $T_{\pm 3}$ arrows. Thus,

the network interpretations of C_5 and $C_{6.1}$ are positively isographic with each other since both contain $T_{\pm 3}$ arrows while showing a consistent difference in the corresponding I_n arrows. In the progression from C_5 to $C_{6.1}$, the hyper- T_n number, $\langle T_9 \rangle$, indicates by how much objects of transformations move apart between index zones. Also, the network interpretations of C_6 and $C_{6.1}$ show axial isography at $\langle T_{0/\mp 4} \rangle$. Meanwhile, tetrachord C_6 is drawn between the outer voices of the right-hand part while C_7 is drawn between the upper and middle voices, as shown in Example 2.11. That is, in the inversive tetrachord C_7 , the symmetrical motion between the outer voice A^b_5 - G_5 and the inner voice E^b_5 and E_5 provides an inversive operation I_e . In the progression from C_6 to C_7 , the shift from 6-zone to 11-zone is represented by $\langle T_5 \rangle$.

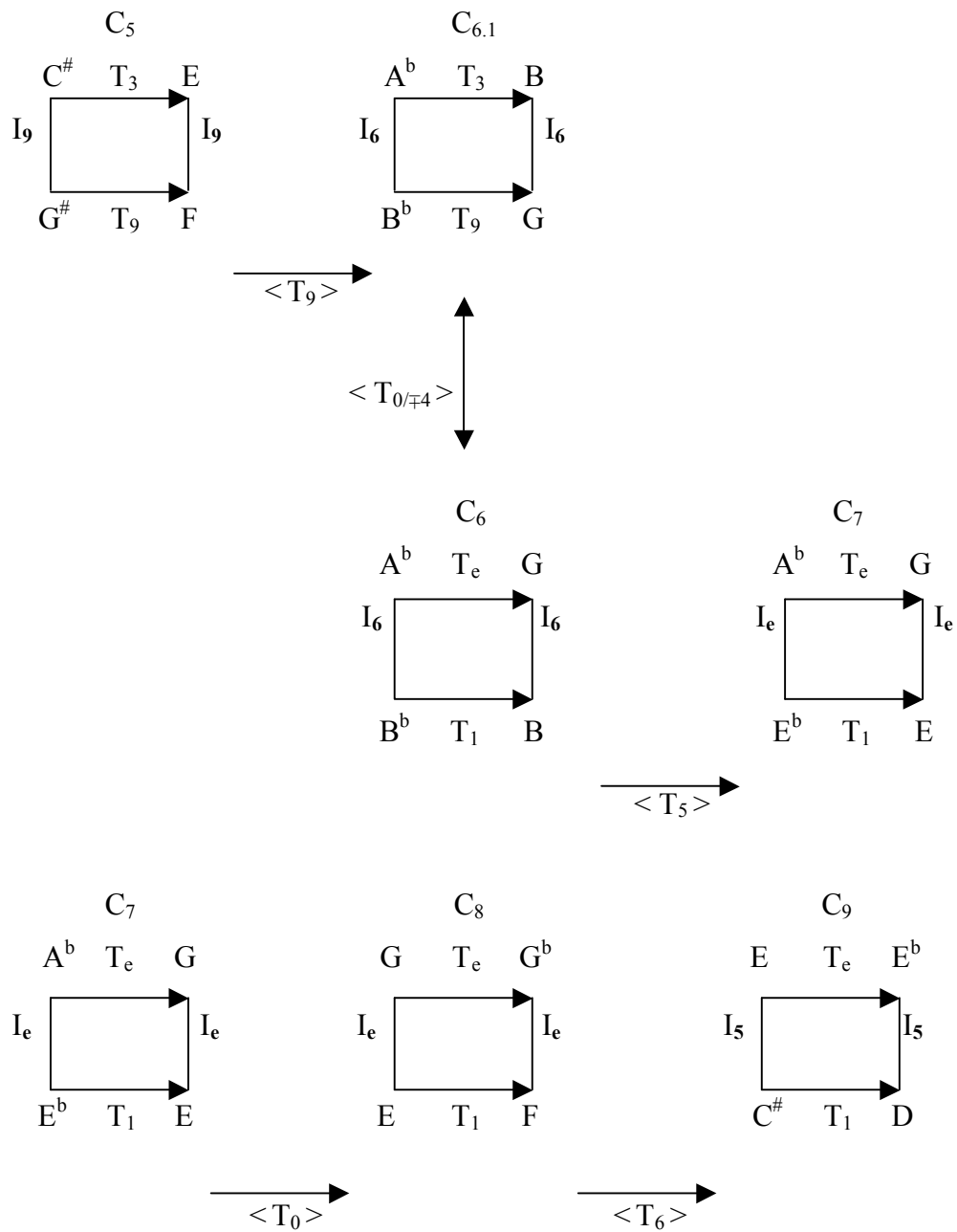
In addition, C_8 presents the same inversive operation, I_e , between the outer voices G_5 - G^b_3 (this G^b being lowered two octaves) and the inner voice E_5 - F_5 and thus the network interpretations of C_7 and C_8 show strong isography as Example 2.12 illustrates.

Example 2.12 Strong and positive isography of network interpretations of C_7 - C_9 in Schoenberg's op.11, no.2, mm. 8-9



We can also see in this example that the K-nets of C_8 and C_9 that contain the same T_n arrows while having different I_n arrows are positively isographic with each other. Example 2.13 provides a transformational path from C_5 to C_9 using strong, positive and axial isographies and notates the series of transformations as $\langle T_9 \rangle$, $\langle T_{0/\mp 4} \rangle$, $\langle T_5 \rangle$, $\langle T_0 \rangle$ and $\langle T_6 \rangle$.

Example 2.13 Network interpretations of C_5 - C_9 in Schoenberg's op.11, no.2, mm. 6-9

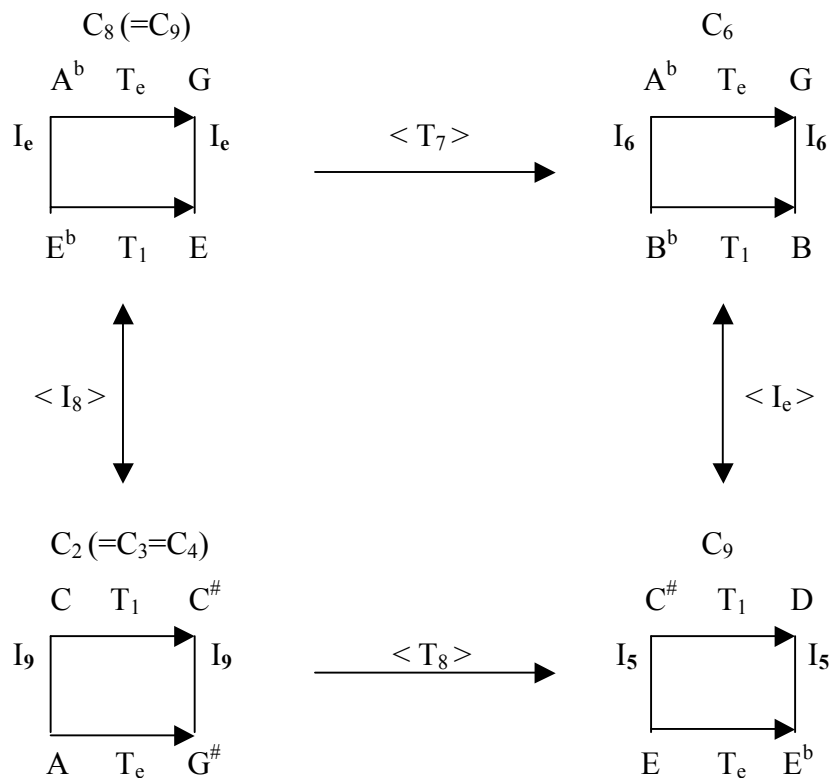


Finally, in this entire passage there are four network interpretations that can be paired to be positively isographic; the networks for C_2 ($=C_3, C_4$), C_6, C_7 ($=C_8$) and C_9 share the same T_n arrows but each has different I_n arrows. Example 2.14 arranges these

K-nets in a higher-level network designated as H, whose model has new $\langle T_n \rangle / \langle I_n \rangle$ relations differing from those of the above tetrachords.

Example 2.14 Schoenberg's op.11, no.2, mm. 4-9, supernetwork of tetrachordal K-nets

H



Since the network recursion containing the new $\langle T_n \rangle$ and $\langle I_n \rangle$ arrows cannot be related to any K-net of the given chord, this example thus does not support Lewin's suggestion that "one could conceive of the higher-level-network as prolonging the lower-level one, particularly when the given chord is part of the given progression."⁷⁹

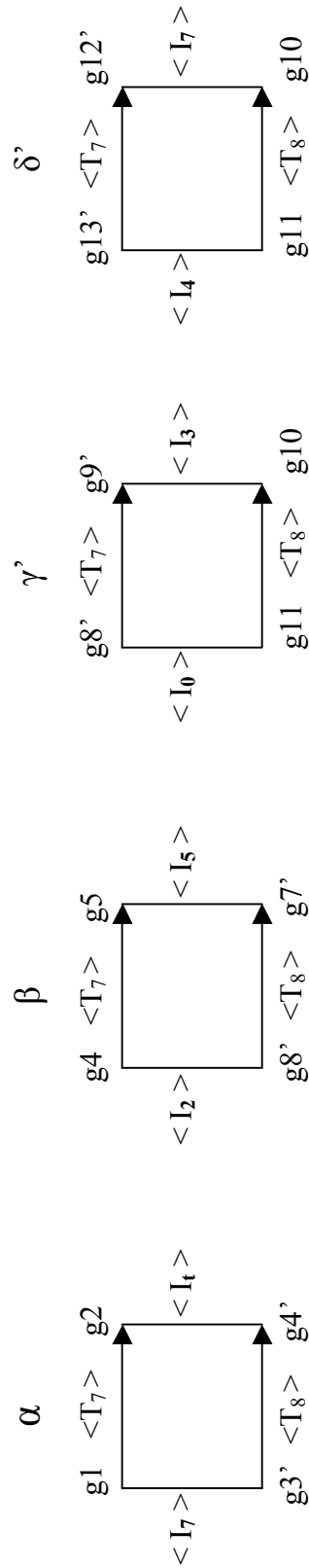
The higher-level network H, however, can be integrated into both the higher- and lower-level networks of the chorale passage which follows. Example 2.15 reproduces Lewin's

⁷⁹ Lewin, "Klumpenhouwer Networks," 115.

higher-level graphs which are obtained by interpreting all thirteen chords (g_1 - g_{13}) of this chorale passage.⁸⁰ In comparing the higher-level network H to Lewin's network, we see that the H and recursive networks share the same $\langle T_n \rangle$ arrows and thus that they are isographic with each other. Therefore, the passage examined first and the following chorale share a larger structural trait, the higher-level K-structure, making into a single whole. In this analysis, axial isographies are irrelevant to the recursive K-net structures but serve to describe how a zone can be extended by embracing differing T_n arrows.

⁸⁰ The retrogrades of Lewin's gamma and delta networks are provided in this example. Lewin also demonstrates that the first three measures can be related to the opening of the chorale passage by K-nets. He constructs the higher-level-graphs gamma and delta in a different ordering from others to provide recursion at a higher level and also provides the higher-level-graphs for the $\langle\langle T_n \rangle\rangle$ and $\langle\langle I_n \rangle\rangle$ relations at a higher structural level in "A Tutorial," 90, 92.

Example 2.15 Schoenberg's op.11, no.2, mm. 10-13; supernetwork of graphs as interpreted by Lewin



2-3. Axial isography II: Alexander Scriabin's "Vers La Flamme," op.72 (1914)

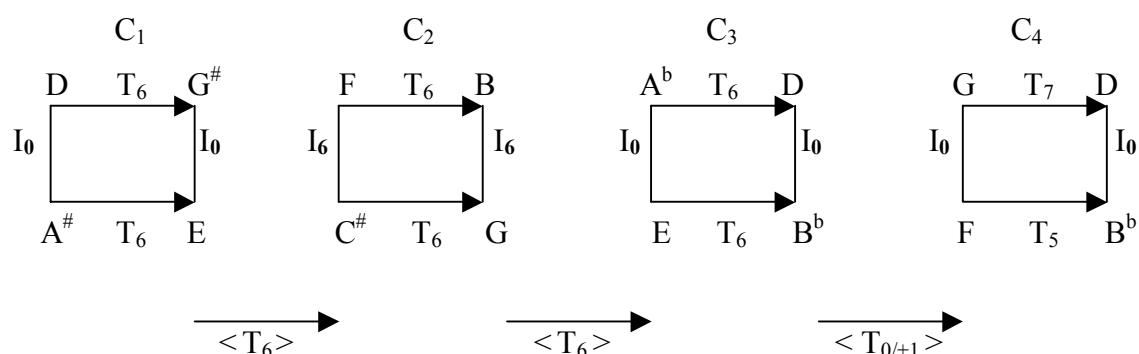
Example 2.16 presents the first nineteen measures from Scriabin's "Vers La Flamme," op.72. Symmetrical inversion is certainly an important feature of this piece as the upper part alone as well as the upper and lower parts together show abundant inversive gestures of various four-note pc collections. In this analysis, I will introduce a new axial isography for tetrachords and will show how it can be used to unify diverse pc collections and model inversive motions reflected in the musical surface in both surface- and deeper-level K-nets. Let us begin with the five tetrachords presented in right- and left-hand together.

Example 2.16 Scriabin's "Vers La Flamme," op.72, mm.1-19

The image displays three systems of musical notation for the first 19 measures of Scriabin's "Vers La Flamme," op.72. Each system consists of a piano (left) and right-hand (right) staff. The first system (measures 1-4) is marked *pp* *sombre* and includes the annotation C_1 4-25(0268) above the right-hand staff, with a boxed-in tetrachord in the right hand. The second system (measures 5-8) is marked *pp* and includes the annotation C_2 4-25(0268) above the right-hand staff, with a boxed-in tetrachord in the right hand. The third system (measures 9-19) is marked *p* and *mp*, and includes the annotation C_3 4-25(0268) above the right-hand staff (with a boxed-in tetrachord), and C_4 C_5 4-26(0358) 3-11(037) above the right-hand staff. The notation includes various accidentals, dynamics, and articulation marks.

In this example, I have boxed and labeled the chords as C_1 through C_5 . C_1 , C_2 and C_3 are members of set-class 4-25 (0268), while C_4 and C_5 are members of set class 4-26 (0358) and 3-11 (037) respectively. Each of the tetrachords has a pair of tritones, one in the right hand and one in the left hand. In Example 2.16, we can also hear the operations in pitch space inasmuch as C_1 moves up three semitones to C_2 and C_2 then moves up three more semitones to C_3 . As Example 2.17 shows, the network interpretation of C_1 is positively isographic to the network C_2 at $\langle T_6 \rangle$, and again, C_2 is positively isographic to the network C_3 at $\langle T_6 \rangle$.

Example 2.17 Positive and axial isography of network interpretations of C_1 - C_4



Here, the networks of C_3 and C_4 also share identical I_0 arrows while the $T_{\pm n}$ arrows change into one another by adding and subtracting 1; thus the graphs demonstrate the first type of axial isography at $\langle T_{0/\pm 1} \rangle$. The network interpretation for C_4 , a member of set class 4-26 (0358), however, cannot be isographic with C_5 , a member of 3-11 (037), since the K-net does not share the same I_0 arrows or the same interval classes, as shown in Example 2.18.

Example 2.18 Transformations of tetrachord C_4 to C_5 in Scriabin's op.72, mm.18-19,

C_4

C_5

C_4

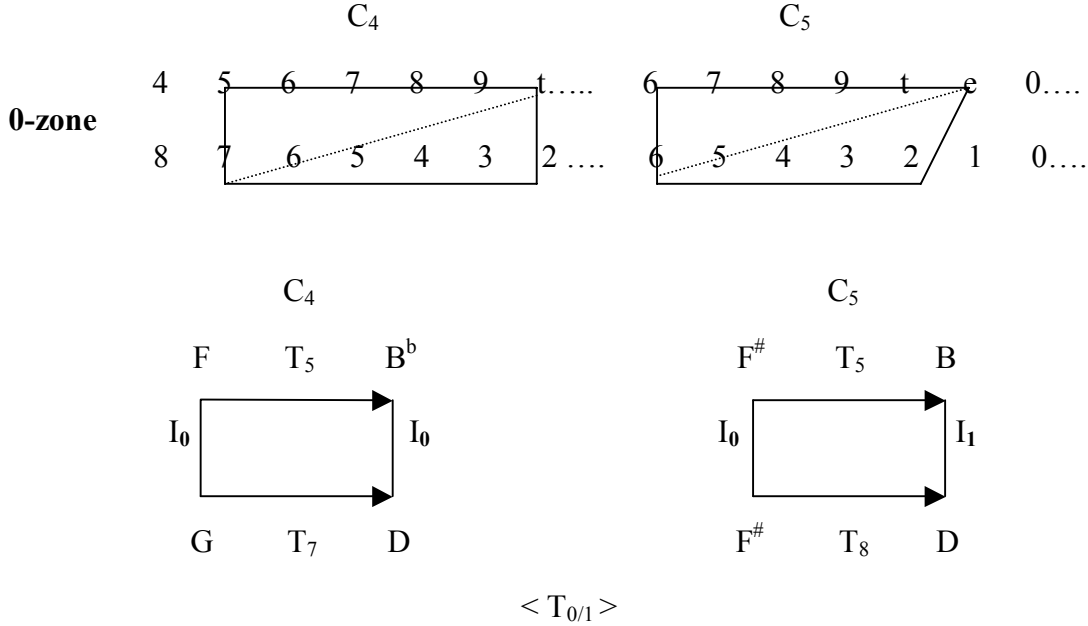
C_5

Note that the networks for C_4 and C_5 share identical I_n arrow and one T_n arrow.

We can hear the inversive motion between the upper and lower part but one pitch is sustained. To provide symmetrical motion or to produce strong isography in C_4 to C_5 , G_4 and F_3 in C_4 would have to move inward by a semitone to $F^\#_4$ and $F^\#_3$ in C_5 , and D_4 and $B^\flat_2$ in C_4 would also move inward to $D^\flat_4$ and B_2 . In C_4 to C_5 , $B^\flat_2$ in C_4 moves inward, but D_4 stays the same. That is, the pitch D_4 , as it deviates from the wedging motion, yields different I_n and T_n arrows in C_5 . Example 2.19 illustrates that within a zone, the tetrachords C_4 and C_5 can emerge as two quadrangles of different sizes involving an invariant triangle.⁸¹

⁸¹ The relation with the subset of trichords will be discussed more in detail in Chapter 3.

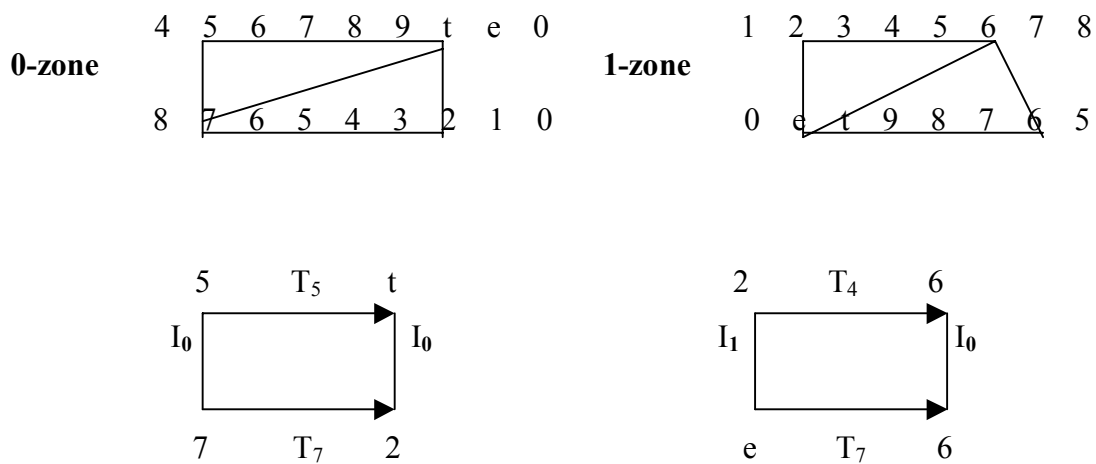
Example 2.19 Second type of axial isography within a zone; network interpretations share only one I_n arrow and one T_n arrow



The K-nets of C_4 and C_5 that share single I_n and T_n arrows while the other I_n and T_n arrows vary, define the second type of axial isography. Compared with the strong isography, the second axial isography, which shares only one I_n arrow, is less likely to belong to one particular zone; however, it still shares a triadic subset which traces the same inversional wedge. Example 2.19 also shows that the hyper T_n number reflects the degree to which the motion between C_4 and C_5 is inconsistent with strong isography. Any operation such as that in C_4 and C_5 which maps the K-nets with only one I_n arrow and one T_n arrow is labeled $\langle T_{0/n} \rangle$. The left numerical subscript, 0, represents the shift within the same zone while the right subscript indicates the difference between the I_n numbers, which is necessarily equivalent to the difference between T_n numbers. Therefore, the network of C_4 maps onto the network of C_5 at $\langle T_{0/1} \rangle$.

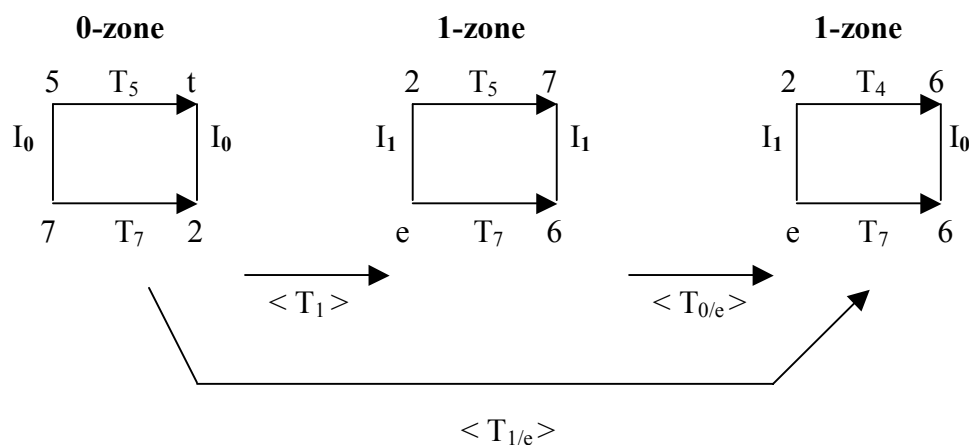
On the other hand, there is another way to discover the second axial isography. For example, C_4 and C_5 in Example 2.18 may be considered as K-nets derived from different zones. As we see in Example 2.20, although the quadrangles derive from different zones, these still have an invariant triangle.

Example 2.20 Second type of axial isography between inversive zones



That is, the network interpretations sharing one I_n and T_n arrow demonstrate the second axial isography, but in this case the shared I_n arrow is placed on the right side. To map the first K-net onto the second, each belonging to a different zone, the first K-net, in the given 0-zone, can shift to the 1-zone by $\langle T_1 \rangle$ and then it maps onto second as $\langle T_{0/e} \rangle$ as shown in Example 2.21.

Example 2.21 Second type of axial isography between inversive zones



The operation can be labeled as $\langle T_{1/e} \rangle$; the left subscript represents the difference between the varying I_n arrows, in this instance 1, and the right subscript indicates the difference between the T_n arrows, of 11.⁸² The axial isography that relates the K-nets which share single identical T_n and I_n arrows, especially in their right sides, can thus be labeled $\langle T_{n/-n} \rangle$.⁸³

Of the two possible isographic interpretations that can be defined as type 2 axial isography, I would like to choose the first interpretation, shown in Example 2.18, for two reasons. First, the first interpretation more clearly retains the inversive gesture reflected in the musical surface. Example 2.18 shows the wedge motion between the treble-clef and the bass-clef, consistent with other chords but with a deviation in the alto part. As we see in the relationships illustrated in Example 2.18, the type 2 axial

⁸² One might obtain this axial isography within a zone by flipping the first quadrangle; however, the exchange of I_n numbers by such flipping is not possible since it produces negative isography in another zone.

⁸³ The notation is the same as Stoecker's notation for trichordal axial isography. He interprets axial isography as a variant of positive isography. See Stoecker, "Studies in Post-Tonal Symmetry," 30-31.

isography accepts the asymmetric wedging motion within the 0-zone. Second, for the type of axial isography, we should consider the context of a musical passage before making an interpretive decision. In the passage in Example 2.17, each of the chords C_1 , C_3 , and C_4 constitutes the 0-zone and the passage which follows it also defines the same zone.⁸⁴ These preceding chords, as well as the following chords, which have the same I_0 values, lead us to focus on the I_0 value of C_5 rather than its I_1 value. Thus, C_5 can be interpreted as being in the 0-zone although C_5 has a different I_n arrow.

We can now examine how the inversional and the near-inversional tetrachords in the right-hand part can be associated with each other. The right-hand part comprises seven tetrachords, boxed in Example 2.22.

⁸⁴ In a small-scale passage a zone can be defined by a single tetrachordal K-net with the same I_n values while in some larger-scale sense, an inversional tetrachord with the same I_n values can be regarded as an embellishment rather than as part of the structural progressions.

Example 2.22 Scriabin's "Vers La Flamme," op.72, mm.1-19

34

C_6 C_7 C_8

C_9 C_{10} C_{11}

C_{12}

Example 2.23 labels these tetrachords C_6 through C_{12} and displays their set-class content below the music.

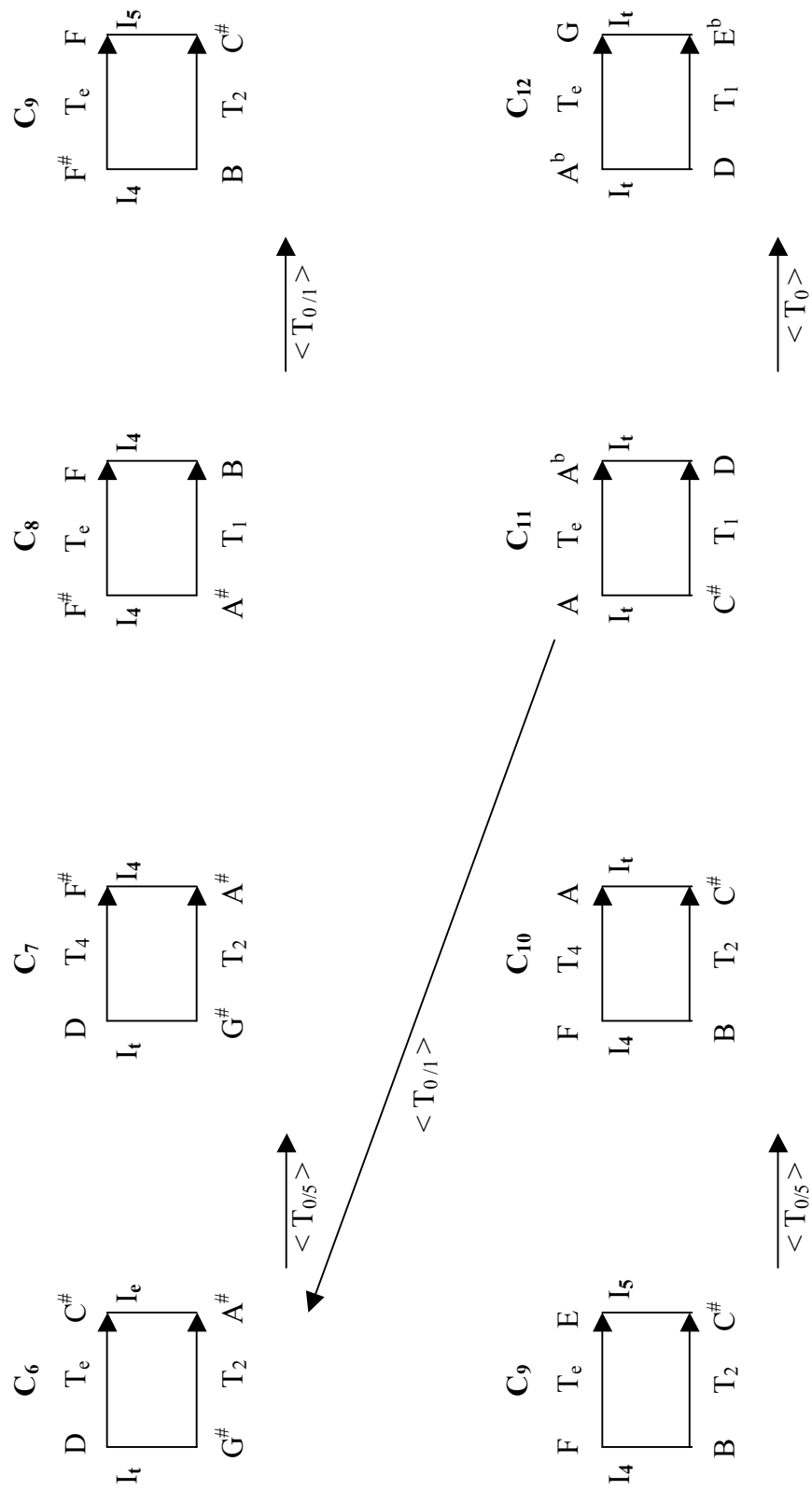
Example 2.23 Scriabin's "Vers La Flamme," op.72, mm.1-19

C_6 C_7 C_8 C_9 C_{10} C_{11} C_{12}

4-Z15 4-24 4-8 4-Z15 4-24 4-8 4-8
(0146) (0248) (0156)

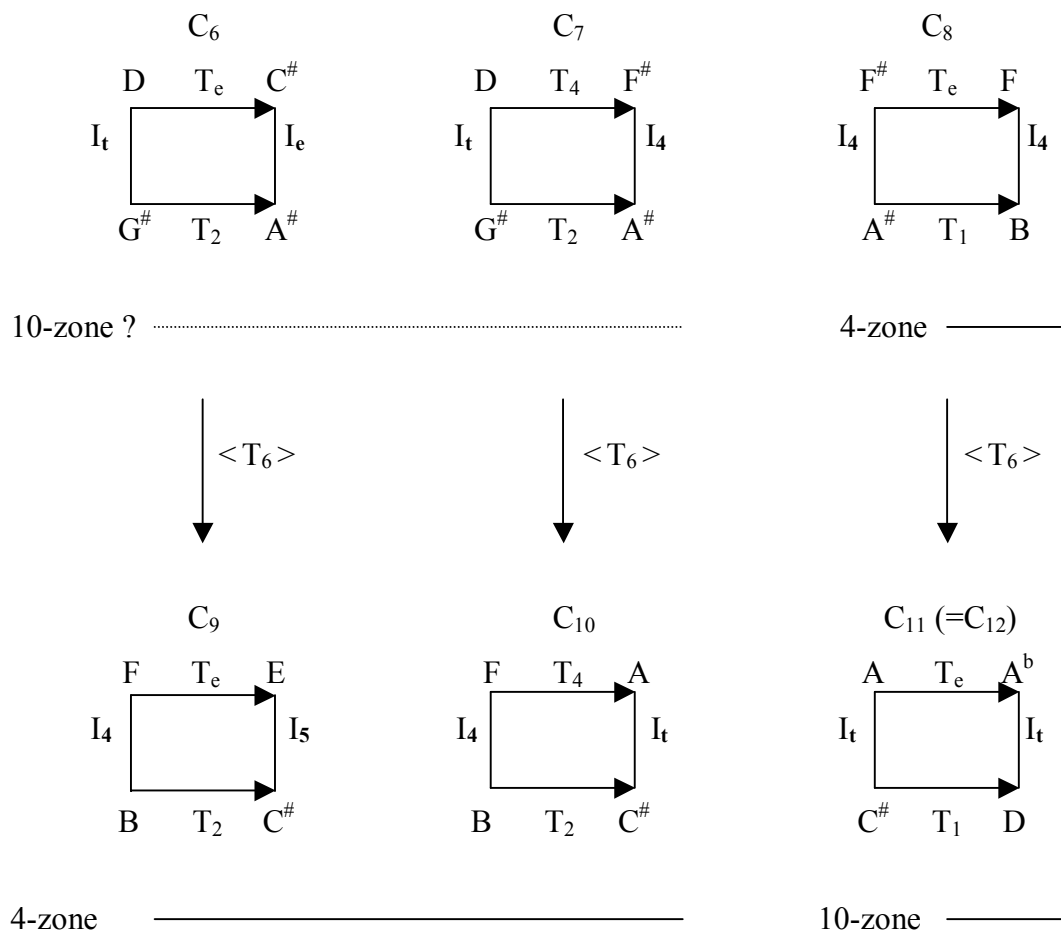
C_6 and C_9 are members of set class 4-Z15 (0146), while C_8 , C_{11} , and C_{12} are members of set class 4-8 (0156). In addition, C_7 and C_{10} are members of set class 4-24 (0248). Example 2.24 provides network interpretations for C_6 to C_{12} , producing strong and axial isographies and also corresponding to the musical surface.

Example 2.24 Strong and axial isography in Scriabin's op.72, mm.1-19



In this, we find that C_6 and C_7 have three identical pitch classes and thus their network interpretations sharing I_t and T_2 arrows are the type 2 axial isographic, denoted by $\langle T_{0/5} \rangle$, and that the same relation obtains between the networks of C_9 and C_{10} .⁸⁵ Likewise, C_{11} and C_6 , sharing I_t and T_e arrows, also produce the type 2 axial isography and that same relation, denoted by $\langle T_{0/1} \rangle$, obtains between the networks of C_8 and C_9 . Example 2.25 illustrates how the inversive and near-inversive tetrachords which relate according to this second axial isography establish index zones.

Example 2.25 Positive isography in Scriabin's op.72, mm.1-19

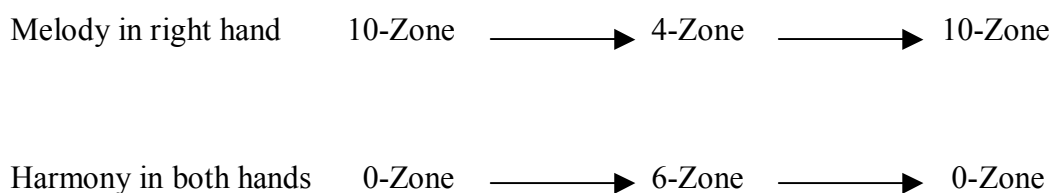


⁸⁵ These chords provide the strong R_p relation, sharing all the same pitch classes except only one note.

The 4-zone established by the inversive tetrachord C_8 is extended by near-inversive tetrachords, C_9 and C_{10} . Also, the inversive tetrachords C_{11} and C_{12} confirm the 10-zone.

Note that the tetrachords, C_6 and C_7 , cannot be interpreted as belonging to one zone while having two different I_n values. The chords, however, may be interpreted as being in the 10-zone. In other words, the K-nets of C_6 , C_7 , C_{11} , and C_{12} , while all establishing the 10-zone, split in both ends of this section in order to provide the same transformational paths among zones as the tetrachords in both parts, which is suggested in Example 2.17. Therefore, Example 2.26 shows that the right-hand part alone and both parts together provide the same transformational path between zones as notated by the hyper- T_n arrows, $\langle T_6 \rangle$.

Example 2.26 Transformational path between inversive zones in Scriabin's op.72, mm.1-19

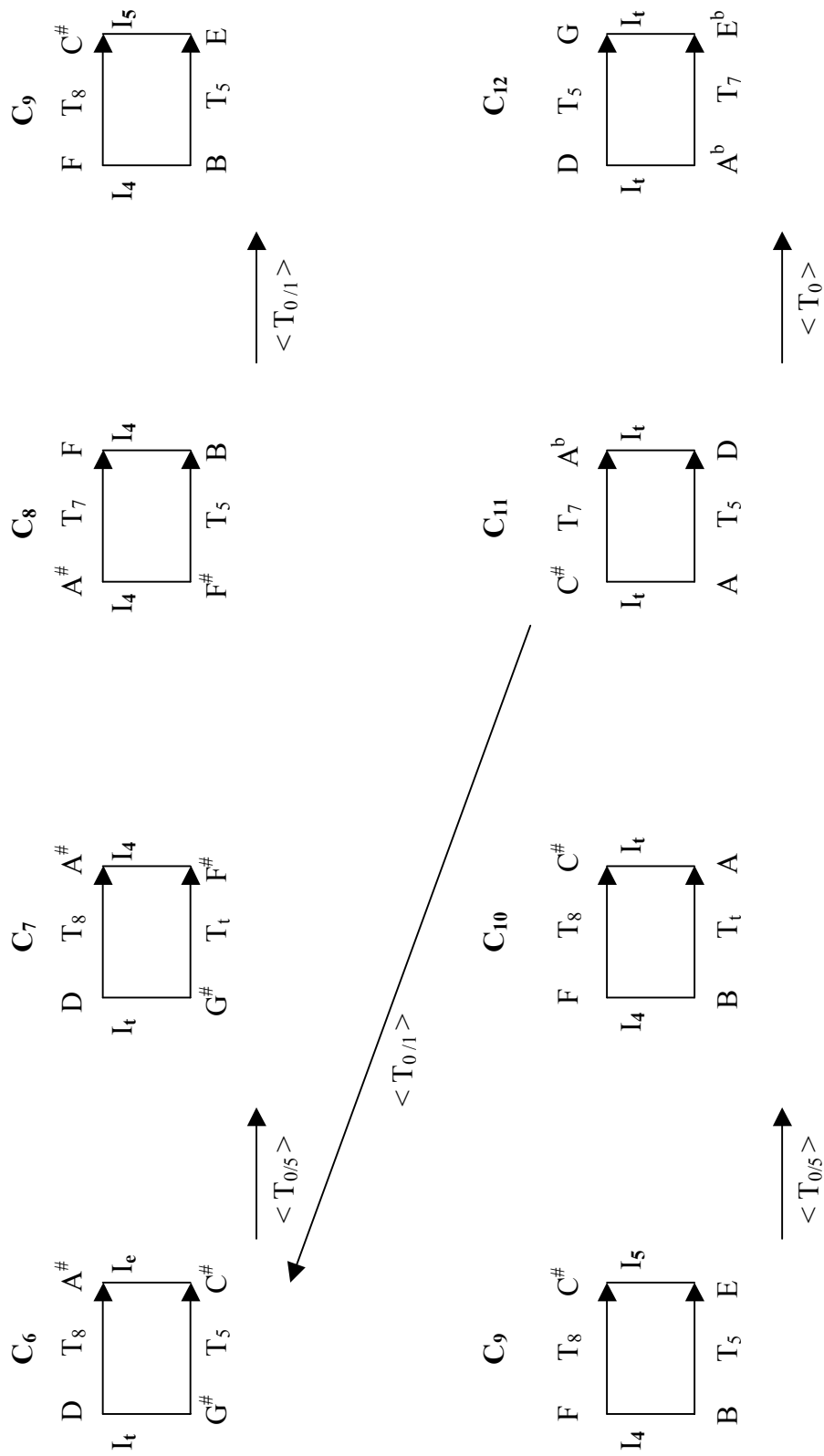


The type 2 axial isography, which has been examined above, embraces relationships either between inversive and near-inversive tetrachords or just between near-inversive tetrachords. In the former case, a zone can be defined and extended by this type 2 axial isography. In the latter case, however, the near-inversive tetrachords which share only an I_n and T_n arrow can be realized in a zone, but it is not necessary to do so. The judgment that the tetrachords belong to the same zone depend on an

assessment of the musical context through which we can identify the zone in which the tetrachords are found. For instance, the near tetrachords, C_9 and C_{10} , in Example 2.24, can be realized in the 4-zone that was previously established by the inversional tetrachord, C_8 .

On the other hand, the tetrachords C_6 through C_{12} can each be articulated as a dyad of complementary interval classes, $ic5$ and $ic7$, or a dyad of $ic8$ plus either a dyad of $ic10$ or $ic5$. That is, the tetrachords can bear other T_n numbers through the exchange of transpositionally related dyadic partners while still having the same inversional relations exposed in musical surface. We can see the networks for C_6 to C_{12} in Example 2.27 which contains T_5/T_7 , T_8/T_5 , or T_8/T_t arrows whereas the networks have the same I_n arrows as in Example 2.24.

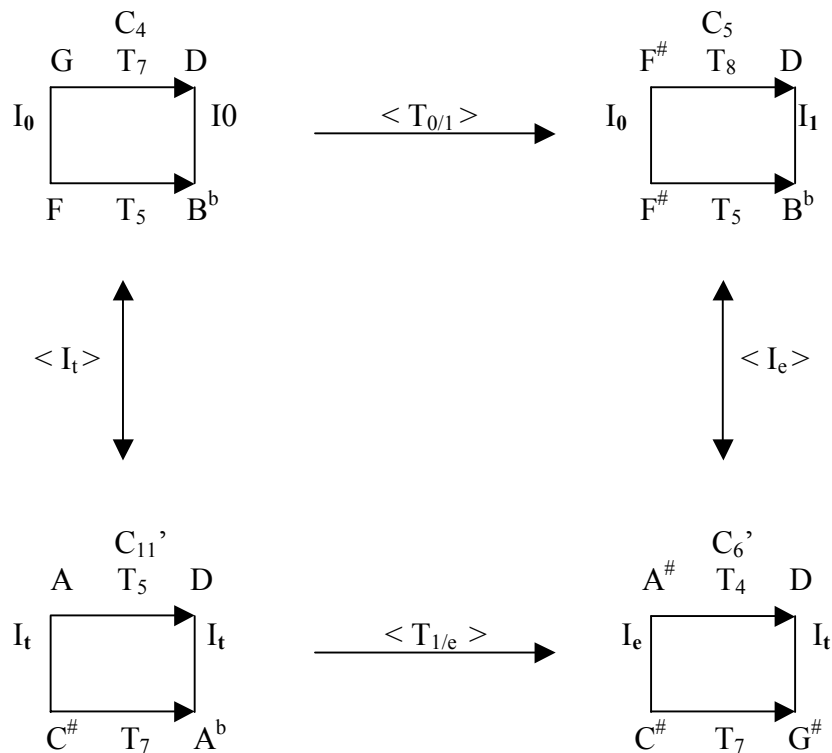
Example 2.27 Strong and axial isography in Scriabin's op.72, mm.1-19



Although the reinterpretation of the tetrachords results in the same isographic relations, it is worth considering the networks since an interesting higher-level structure emerges in connection with the last two chords of both parts. That is, a possible search for higher-level networks is seen in Example 2.28 which arranges the networks of C_4 , C_5 , C_6' and C_{11}' into a higher-level network designated by S.

Example 2.28 Scriabin's op.72, mm. 1-19, supernetwork of tetrachordal K-nets drawn from 10- and 0-zones

S

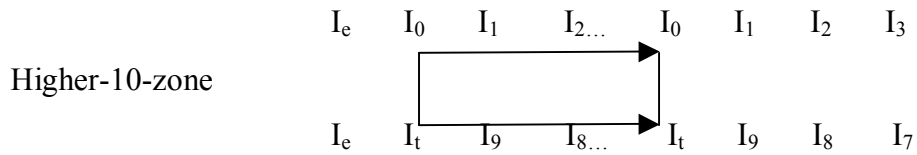
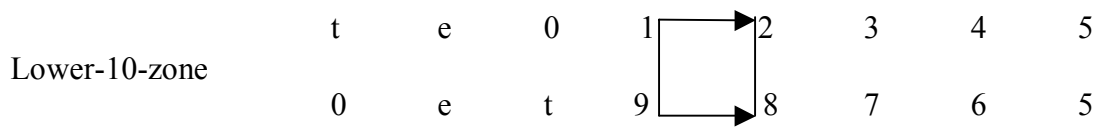


Here, the network interpretations of C_6 and C_{11} have been replaced by the negative forms of C_6' and C_{11}' . In both hands, only the last two chords, C_4 and C_5 , present type 2 axial isography within the 0-zone and the first and the last chords of the right-hand part, C_6 and C_{11} , also display the same relationship within the 10-zone. The

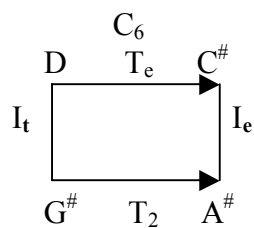
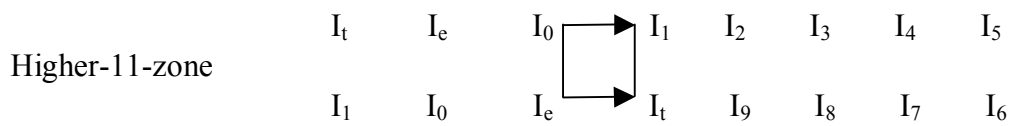
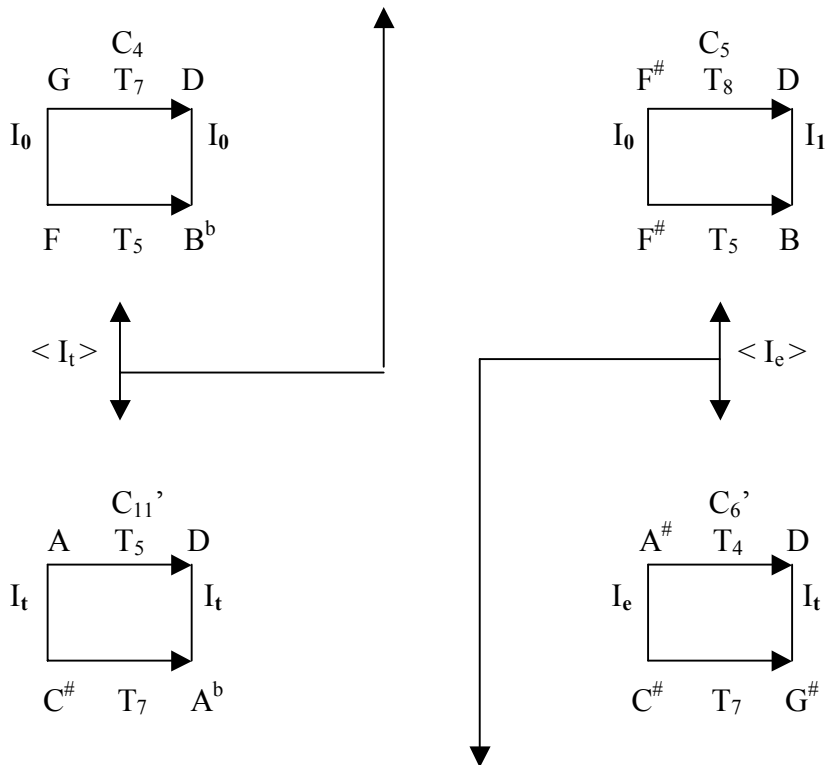
K-nets of the four chords are quite evident since these represent the significant index-zones assigned to the beginning and the end of the passage. On the other hand, the higher-level axis of inversion, which is implicit, may pose some problems for obtaining an axial isography between the different levels. In discussions of K-nets, the hyper $\langle I_n \rangle$ has been a controversial issue since $\langle I_n \rangle$ indicates the sum of two inversional axes which is quite abstract from the point of view of the musical surface.⁸⁶ As we examined above, the T_n and I_n arrows operate among pitch classes within a zone while the $\langle T_n \rangle$ and $\langle I_n \rangle$ arrows operate among zones. Indeed, the $\langle I_n \rangle$ operation may be difficult to realize between index zones while $\langle T_n \rangle$ simply indicates the distance between the zones. Example 2.29 shows that as a zone can be established by a pair of sum dyads, a couple of paired sum dyads can be realized in a higher-level zone.

⁸⁶ The discussion about conceptional and structural distinctions between a lower-level K-net and a higher-level network appears in Klumpenhouwer, "The Inner and Outer Automorphisms," 81-93 and Lewin, "Klumpenhouwer Networks," 114-115.

Example 2.29 Inversional relations in the higher-level and the lower-level zones



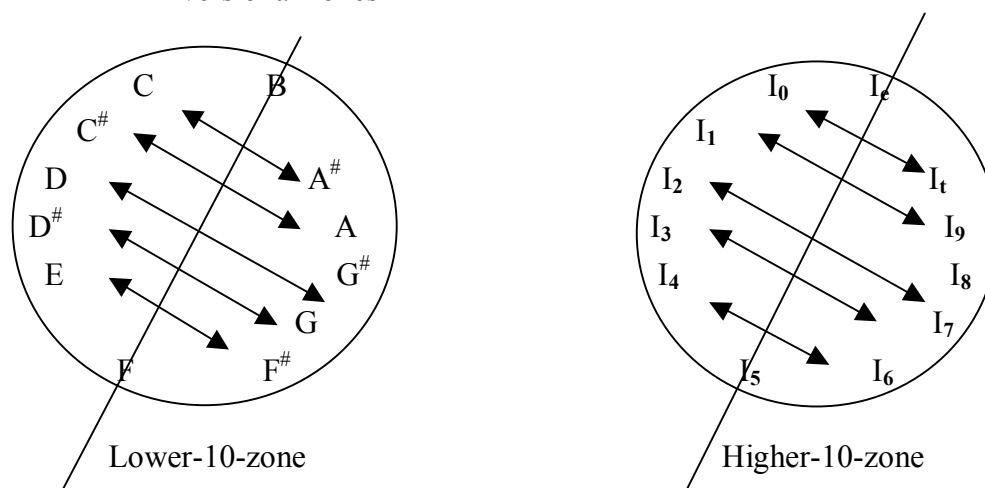
S'



Example 2.29 also shows a tetrachord having I_0 arrows and another tetrachord consisting of I_t arrows combining to establish a higher-level inversional zone. That is, each of the I_0 dyads of C_4 is paired with one of the I_t dyads of C_{11} vertically within a new higher-level 10-zone. In a similar fashion, the I_0 and I_t arrows of C_5 are combined with the I_e and I_t arrows of C_6' in the higher-level 11 zone; that is, a quadrangle in the second-level zone consists of four I_n operations. Here we should remember that in the higher-level zone the members are not pitch classes, but are inversions themselves and thus the horizontal dimension is no longer meaningful. In addition, in the higher-level inversional space, twelve I_n members can be arranged around the higher-level twelve inversional axes and thus there are twelve inversional higher-level zones.

Although the numerical subscripts in I_n and $\langle I_n \rangle$ have completely different meanings, in that one indicates the sum of two pitch classes while the other signifies the sum of two axes, the same subscripts yield the same inversional structure with a pair of antipodal centers in the same positions. Example 2.30 shows the lower and higher “clocks” which display the same inversional structure with the midpoints (B and F in the lower clock and I_e and I_5 in the higher clock) in the same positions.

Example 2.30 Structural similarity between higher-level and lower-level inversional zones



Consequently, the matching $\langle I_n \rangle / I_n$ numbers are meaningful since in both the higher-level and the lower-level zones, all twelve members are arranged by the same inversive generator and with the same structural layout. For the $\langle I_n \rangle$ arrows, Klumpenhouwer suggests that $\langle I_n \rangle$ should not be conceived as an operation on pitch class I_n because of notational similarity, but rather because of structural similarity.⁸⁷ Returning to Example 2.29, S' present the higher-level structures that reproduce the same inversive structures as the given lower-level chords, C_6 , as a prolongation.

2-4 Axial isography III: Anton Webern's String Quartet, (1905)

Example 2.31 excerpts a short passage, mm. 8-12, from Webern's String Quartet (1905).

Example 2.31 Webern's String Quartet, mm.8-12

$C_1 \ C_2 \ C_3 \ C_4 \ C_5 \ C_6 \ C_7 \ C_8 \ C_9$

4-25 - - 3-12 - - 4-27 3-5 4-25
(0268) (048) (0258) (016) (0268)

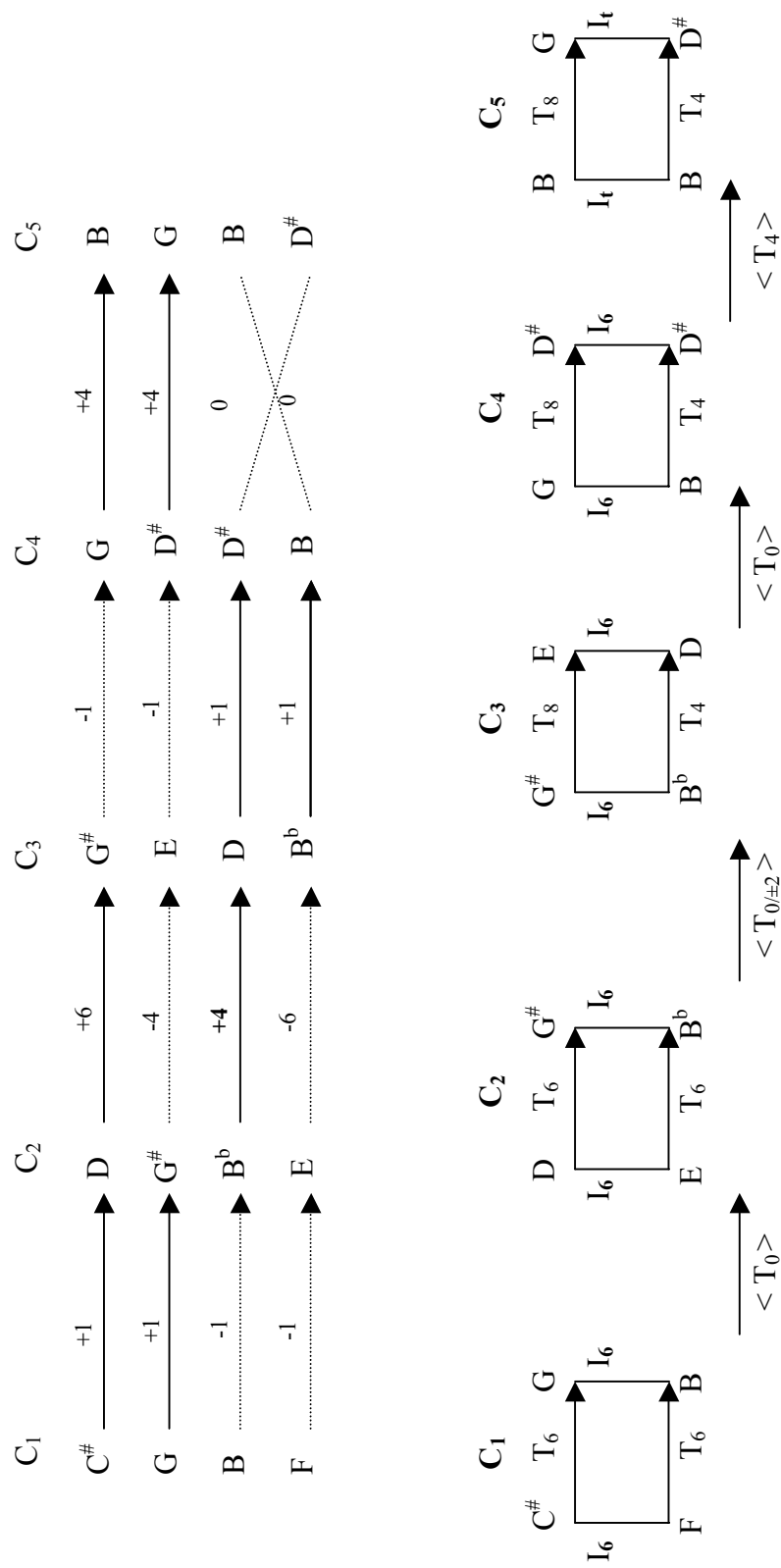
⁸⁷ Klumpenhouwer, "Inner and Outer Automorphisms," 90.

As the harmonic structure in four voices presents consecutive wedging motion mostly between the upper dyads and the lower dyads, inversional symmetry is a prominent feature of this passage. In this analysis, we shall see how the axial isographies examined above can be applied to bring inversional voices to light and how a third type of axial isography can be used together with the other isographies to reveal a background inversional design.

As Example 2.31 shows, four different set-classes, 4-25 (0268), 4-27 (0258), 3-5 (016), and 3-12 (048), are represented by all nine tetrachords, labeled C_1 to C_9 . Example 2.32 lays out the network interpretations of the first five tetrachords and their voice-leading transformations.⁸⁸

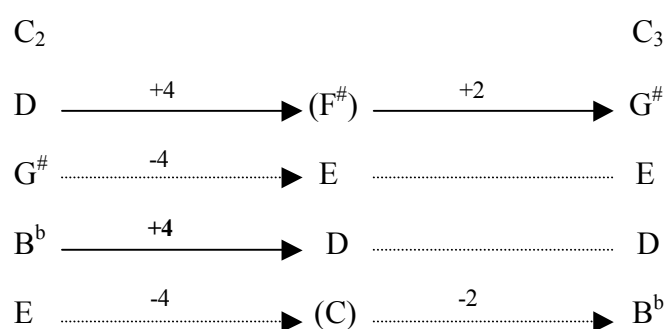
⁸⁸ In order to underscore the inversional motion in transformational voice leading, after this I will use the notation $+n/-n$ for the inversional voices in the examples.

Example 2.32 Transformations and network interpretations of tetrachords C_1 to C_5 in Webern's String Quartet, mm. 8-10



In this example, the straight lines indicate ascending motions while the dotted lines indicate descending motions. Also, these dual transformations bring out the symmetrical foreground voices reinforced by registral lines.⁸⁹ We observe in this example that axial isographies can embrace the inversive voices over a wider range. The networks of C_1 and C_2 are strongly isographic presenting the wedging motion in the 6-zone. Since the K-nets of C_2 and C_3 share the same I_6 arrows while their T_n arrows change, the networks of C_2 and C_3 are the type 1 axial isography. For now, let us focus our attention on a transformational modeling of the first axial isography in order to examine how a mirrored voice leading is redistributed to produce such a relationship.⁹⁰ Example 2.33 illustrates that in C_2 to C_3 the inversive motion by four semitones, the D-G[#] dyad moving up to F[#]-E and the B^b-E moving down to D-C, takes the additional step of moving the F[#] up by two more semitones and the C down by two more semitones.

Example 2.33 Transformations of tetrachords C_2 to C_3 in Webern's String Quartet, m.9

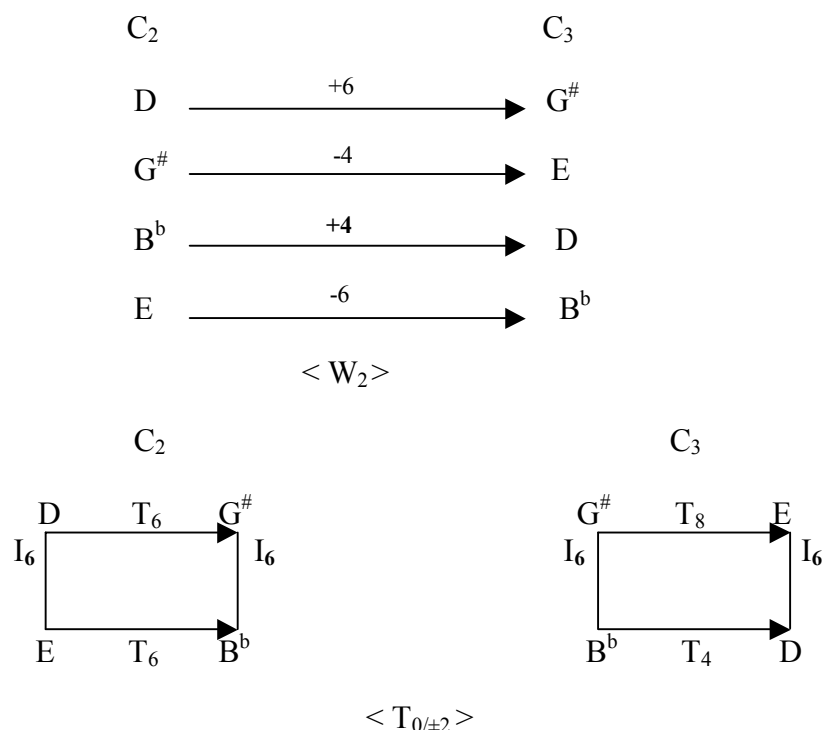


⁸⁹ O'Donnell introduces dual inversions and transpositions as transformations utilizing two simultaneous transpositions or inversions in "Transformational Voice-Leading," 48-68.

⁹⁰ Klumpenhouwer discusses the relations of transformational mappings to isography in "An Instance of Parapraxis," and "A Generalized Model." Additional interpretations of K-nets as transformational voices are in Lambert, "Isographies," 167-168; Gollin, "Some Unusual Transformations," 29-35; O'Donnell, "Transformational Voice-Leading."

That is, each pitch class in one I-related dyad of C_2 is transposed by ± 4 to obtain the corresponding pitch classes in C_3 while each pitch in the other I-related dyad of C_2 is transposed by ± 6 . O'Donnell introduces this type of transformational voice as $\langle W_n \rangle$, in which the I_n numbers are identical while the T_n values show a consistent difference.⁹¹ Accordingly, Example 2.34 illustrates that the $\langle W_2 \rangle$ transformations, in which the subscript 2 represents the sum of two $T_{\pm}$ values, can be identified with the type 1 axial isography at $\langle T_{0/\pm 2} \rangle$. Here, the K-nets of C_2 and C_3 also reflect the wedging motion of the musical surface.

Example 2.34 First type of axial isography between network interpretations of C_2 and C_3 in Webern's String Quartet, m.9



⁹¹ O'Donnell defines $\langle W_n \rangle$ as a composite transformation expressing the sum of the two wedges in a dual wedge in "Transformational Voice-Leading," 162-163.

We also see that C_3 and C_4 are strongly isographic with an inversional wedging motion as Example 2.32 displays. Therefore, the progression C_1 to C_4 shows the way in which this wedging motion within the 6-zone is determined and extended by strongly and axially isographic relations. C_4 and C_5 , however, break the wedging motion and produce a different inversional wedge. In C_4 to C_5 , the upper dyad moves up four interval classes while the lower dyad stays the same. Thus, K-nets of C_4 and C_5 are positively isographic at $\langle T_4 \rangle$.

In C_5 to C_6 , T_7 arrows transform the upper dyad while the T_{-1} arrows transform the lower dyad. In Example 2.35 the motion is interpreted as positively isographic at $\langle T_6 \rangle$.

Example 2.35 Transformations and network interpretations of tetrachords $C_5 - C_9$ in Webern's String Quartet, m.10-12

Diagram illustrating transformations between tetrachords C_6 and C_7 on a musical staff. The top staff (treble clef) shows a transformation from C_6 to C_7 with intervals $+2$ and $+4$. The bottom staff (bass clef) shows transformations from C_6 to C_7 with intervals -4 and -2 .

Diagram illustrating transformations between tetrachords C_5 , C_6 , C_7 , C_8 , and C_9 . The transformations are as follows:

- C_5 to C_6 : $B \xrightarrow{+7} D$, $G \xrightarrow{+4} F^\#$, $B \xrightarrow{-1} A^\#$, $D^\# \xrightarrow{-1} D$
- C_6 to C_7 : $D \xrightarrow{+1} D^\#$, $F^\# \xrightarrow{+4} A^\#$, $A^\# \xrightarrow{+4} F^\#$, $D \xrightarrow{-2} C$
- C_7 to C_8 : $D^\# \xrightarrow{+6} E$, $A^\# \xrightarrow{-1} F$, $F^\# \xrightarrow{-1} F$, $C \xrightarrow{-1} B$
- C_8 to C_9 : $E \xrightarrow{+1} F$, $A \xrightarrow{+2} B$, $F \xrightarrow{-2} A$, $B \xrightarrow{-2} D^\#$

Some transformations are indicated by dashed lines or crossed out, suggesting they are not part of the primary network.

Diagram illustrating transformations between tetrachords C_5 , C_6 , C_7 , C_8 , and C_9 using a network of transformations. Each tetrachord is represented by a rectangle with vertices labeled with notes and transformations:

- C_5 : B (top-left), T_8 (top-right), G (bottom-right), T_4 (bottom-left), $D^\#$ (bottom-right), I_t (top-left), I_t (top-right), I_t (bottom-left), I_t (bottom-right)
- C_6 : $F^\#$ (top-left), T_8 (top-right), D (bottom-right), T_4 (bottom-left), $A^\#$ (bottom-right), I_4 (top-left), I_4 (top-right), I_4 (bottom-left), I_4 (bottom-right)
- C_7 : $A^\#$ (top-left), T_5 (top-right), $D^\#$ (bottom-right), T_6 (bottom-left), $F^\#$ (bottom-right), I_4 (top-left), I_3 (top-right), I_3 (bottom-left), I_3 (bottom-right)
- C_8 : E (top-left), T_5 (top-right), A (bottom-right), T_6 (bottom-left), F (bottom-right), I_9 (top-left), I_8 (top-right), I_8 (bottom-left), I_8 (bottom-right)
- C_9 : $D^\#$ (top-left), T_6 (top-right), A (bottom-right), T_6 (bottom-left), F (bottom-right), I_8 (top-left), I_8 (top-right), I_8 (bottom-left), I_8 (bottom-right)

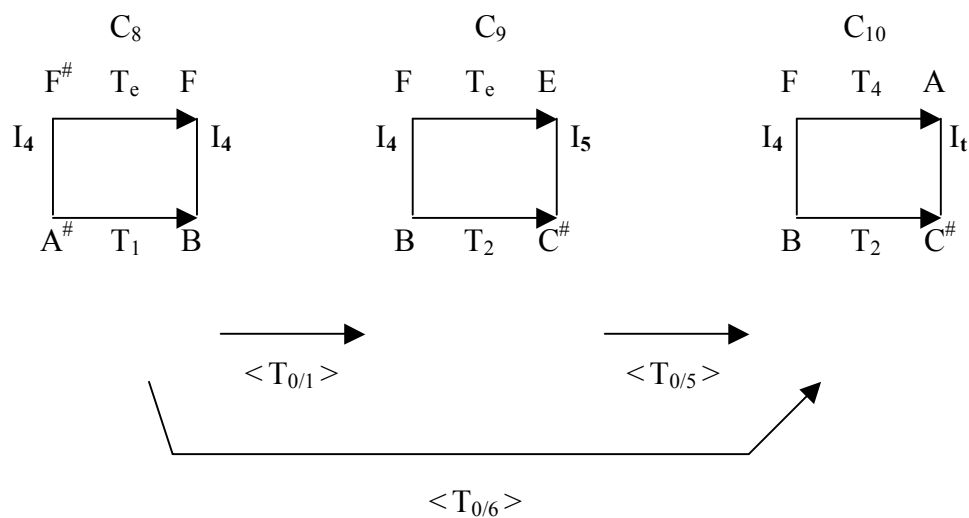
Transformations between tetrachords are indicated by arrows:

- $C_5 \rightarrow C_6$: $\langle T_6 \rangle$
- $C_6 \rightarrow C_7$: $\langle T_{0/e} \rangle$
- $C_7 \rightarrow C_8$: $\langle T_5 \rangle$
- $C_8 \rightarrow C_9$: $\langle T_{e/1} \rangle$
- $C_9 \rightarrow C_8$: $\langle T_{e/1} \rangle$
- $C_9 \rightarrow C_7$: $\langle T_5 \rangle$

Notice, however, that the voices from C_6 to C_7 are highly diffused and that the network interpretation of C_6 cannot be isographic with the network of C_7 since the K-nets do not share the same T_n arrows with positive or strong isography or the same I_n arrows with axial isography. There is, however, a single I_n arrow shared between two of the K-nets; the shared inversion is audible here with the exchange of pitch classes $F^\#$ and $A^\#$. The K-nets that have a single I_n arrow shared between identical pitch classes and between inversive and near-inversive tetrachords exemplify the type 3 axial isography; the operation is labeled $\langle T_{0/n} \rangle$, which is the same notation as that of the type 2 axial isography. It is important to remember that the type 3 axial isography will occur under the condition that the precedent tetrachord must be inversive and also have one I_n arrow shared between identical pitch classes since it validates its transformations within a zone and also enables us to perceive the associated wedging motion. In the notation $\langle T_{0/n} \rangle$ used for the type 3 axial isography, the left subscript 0 indicates a shift within the same zone and the right subscript n represents the difference between the varying I_n numbers, which is also equivalent to the difference between the sums of the T_n numbers. In Example 2.35, then, $\langle T_{0/e} \rangle$ is the relevant operation to map the K-net of C_6 to C_7 since the mapping is produced within the same 4-zone and the difference between the varying I_n numbers is 11. It is also possible to consider the type 3 axial isography as a variant of the type 1 axial isography. That is, if the networks of C_6 and C_7 have the type 1 axial isography, as in Example 2.35, the $D_6^\#$ in the first violin part would have moved to E_6 , in a wedging motion with D_3 and C_3 in the cello part. The middle parts exhibit outward wedging motion indicated by the complementary T_n arrow.

Example 2.36 reproduces Example 2.24, from the analysis of Scriabin's *Vers La Flamme*, and displays the transformations $\langle T_{0/1} \rangle$ and $\langle T_{0/5} \rangle$ produced by the type 2 axial isography.

Example 2.36 Type 2 and 3 axial isography among network interpretations of C_8 - C_{10} in Scriabin's op.72, m.5-9

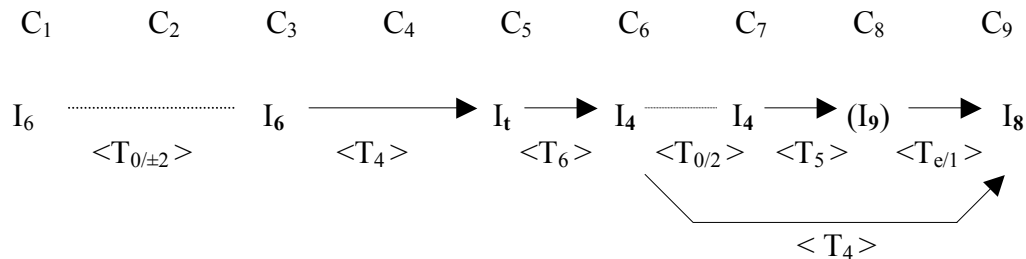


Moreover, the networks of C_8 and C_{10} are the type 3 axial isography, at $\langle T_{0/6} \rangle$. The network interpretations of an inversive C_8 and a near-inversive C_{10} share a single I_4 arrow with two same pitch classes, F and B , thus satisfying the conditions of the type 3 axial isography. Finally, the networks C_8 , C_9 , and C_{10} are all axially isographic with each other.

Returning to Example 2.35, we can interpret C_7 to C_8 as T_{+6} and T_{-1} since the upper dyad moves up a tritone and the lower dyad moves down a half step. The skewing of the wedge is represented by $\langle T_5 \rangle$. However, the motion of C_8 to C_9 displays $\pm 2/+1$ voices. As Example 2.35 shows, the ± 2 in three voices produces the type 2 axial isography involving an invariant trichord subset. We can see that the network

interpretations of C_8 and C_9 share only one T_6 arrow and one I_8 arrow. Since the shared I_8 arrows are on their right sides, the type 2 axial isography can be labeled as $\langle T_{e/1} \rangle$. Finally, Example 2.37 notates the series of transformations $\langle T_{0/\pm 2} \rangle$, $\langle T_4 \rangle$, $\langle T_6 \rangle$, and $\langle T_4 \rangle$ that indicate how the K-nets either stay in one zone or move among zones.

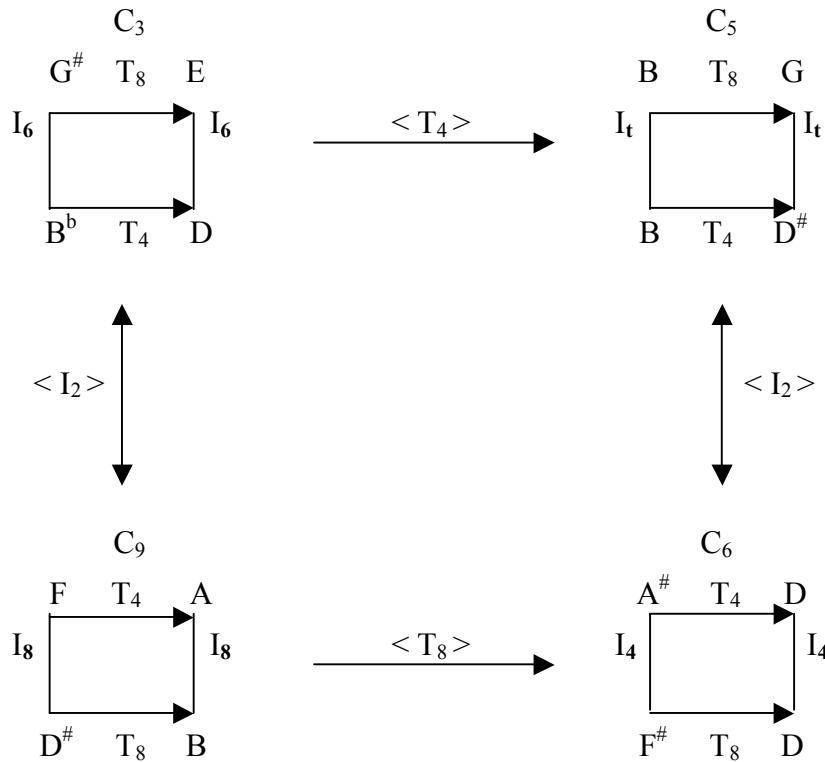
Example 2.37 Transformational path among inversive zones in Webern's String Quartet, mm.8-12



Here, there can be no 9-zone since a dyad cannot establish a zone by itself. As we examined above, axial isographic relations can maintain or extend a zone which has already been established and they can also act as pivots to link different zones.

Significantly, a higher-level network in Example 2.38 can be seen to incorporate the relationships examined earlier in Examples 2.32 and 2.35. The last chord C_9 is now reproduced by the exchange of t-related partners generating the same T_4/T_8 arrows with other chords while still preserve the same I_8 arrows.

Example 2.38 Webern's String Quartet, mm. 8-12, supernetwork of tetrachordal K-nets C_3 , C_5 , C_9 , and C_6 ; positive isography with each tetrachord



Of the nine tetrachords in this passage, the four K-nets of C_3 , C_5 , C_6 and C_9 , which are displayed in the higher-level network, can be considered as representatives of each index-zone since the tetrachords contain a single I_n arrow and share the same complementary T_n arrows which can be related to each other. That is, the tetrachords direct our attention toward the analytic observations which both associate different zones and also provide their recursive combinations at a deeper level. The angle-bracket operations $\langle T_4 \rangle$ and $\langle T_8 \rangle$ that label the arrow from C_3 to C_5 and from C_9 to C_6 and thus the higher-level network is positively isographic to the networks of the lower level. In this short passage, axial isography not only serves to depict an inversive gesture

between inversional and near-inversional tetrachords but also work together with strong isography to support an inversional zone.

I have examined the spatial transformations of the K-nets among index zones since it may be a way of interpreting the traditional and new isographies to reveal inversional structure either inside or outside of a network. While positive isography relates K-nets derived from different zones, strong and axial isographies combine K-nets within a zone. Even more importantly, a zone can be established by an inversional tetrachord or by a number of inversional or near-inversional tetrachords, depending on the musical context. For instance, between two phrases, each consisting of a number of tetrachords which establish a given zone, an inversional tetrachord constructing yet another zone can be considered as an embellishment. Also, in a higher-level structure considering the musical context, it is possible to select only some of the chords as structural ones. I also have proposed three types of axial isography for tetrachords as a method for analyzing inversionally based works. The table below compares all types of isographies for tetrachords.

Type	I_n arrows	T_n arrows	Notation
Strong	Share two I_n arrows	Share two T_n arrows	$\langle T_0 \rangle$
Positive		Share two T_n arrows	$\langle T_n \rangle$
Axial Type I Type II Type III	I Share two I_n arrows		$\langle T_{0/\pm n} \rangle$
	II Share one I_n arrow	Share one T_n arrow	$\langle T_{0/n} \rangle$ or $\langle T_{n/-n} \rangle$
	III Share one I_n arrow		$\langle T_{0/n} \rangle$

Axial isography associates diverse tetrachords even those which involve different T_n arrows, going beyond the limits of traditional isography. Axial isography also functions as a pivot connecting the K-nets which arise from different zones by sharing either a single or both I_n arrows. By using traditional and these types of axial isography, it will be meaningful either to trace an orderly or an interesting transformational path between different zones, or to construct a network or networks of structural chords.

It should also be noted that my analysis is merely a way of bringing out many possible readings of the same passage of music. In the above analyses, some of the axial isographies can also be interpreted as some other isographic relation and, in essence, all chords can be identified by a number of possible network interpretations. However, most of the K-nets in my analysis retain the same inversional relations exposed in musical surface, that is, preserve the inversional surface characters of the pieces although the network interpretations need not necessarily conform to the musical surface, as Lewin shows.⁹² One of the main reasons for choosing these models is to demonstrate how K-nets, especially those with axially isographic relations, can be useful for reflecting foreground inversional operations and also for revealing background inversional structure at a deeper level.

Meanwhile, there is no doubt that a number of pieces and sections evidence transpositional operations as meaningful musical features. In his recent article, Buchler wrote that “network-internal I_n values are unnecessary to determine whether two K-nets are isographic” and suggested a T-net in which all nodes are only transpositionally

⁹² Lewin, “Klumpenhouwer Networks,” 114.

connected.⁹³ His assertion seems to be based on the assumption that inversional operations are only a byproduct of transposition and can be brought to light through intervallic analysis.⁹⁴ However, it must not be overlooked that a number of pieces and sections employ inversional operations as generators of that piece or section. In the pieces with very compelling transposition or inversion, the possibility of there being some other property should not be disregarded since any transpositional or inversional relation can indicate another property and the result exists whether it is perceptible or not. I also found that in many interesting pieces or sections, transpositional and inversional relations interact and inform each other rather than there being only one property which is in control throughout. This is one of the reasons why the K-nets having both I_n and T_n values is a valuable analytic tool. Certainly, it will be more fruitful to use such axial isography to identify inversional features exposed in the musical surface thus avoiding analytic promiscuity.⁹⁵ For axial isography, Stoecker asserts that “it does not push our analytic tool to the point of promiscuity, but it does allow us additional opportunities to explore and refine our intuitions of a passage of music.”⁹⁶ Axial isography is indeed capable not only of extending the K-nets’ territory by focusing inversional operation but also of suggesting more possibilities in musical cognition.

⁹³ Michael Buchler, “Reconsidering Klumpenhouwer Networks,” *Music Theory Online* 13/2 (2007), 6.

⁹⁴ Michael Buchler, “Reconsidering Klumpenhouwer Networks,” 9.

⁹⁵ O’Donnell refers to the “exclusivity - promiscuity continuum” in “Transformational Voice-Leading,” 62-63 and in O’Donnell, “Klumpenhouwer Networks,” 61-62.

⁹⁶ Stoecker, “Studies in Post-tonal Symmetry,” 39.

Chapter 3

Transformations between a Trichord and a Tetrachord

Since using K-nets has been suggested as an analytic method that includes both inversional and transpositional operations, considerable efforts have gone into the demonstration or evaluation of their value in the analytical literature.⁹⁷ These writings provide many instances which show the relative merits and strengths of K-nets and their ability to associate chords from different set-classes with the additional potential for replication at different structural levels. In contrast, some authors raise issues about the network interpretations or hierarchical structures that seem to have no relevance to the musical surface. In a recent article, Buchler points out one of the reasons that the K-nets and the recursive K-net structures seem arbitrary or extraordinary:

K-nets of size q must be grouped into q -sized hyper-networks for recursion to be drawn. Extended outward, q -sized K-nets must form into hyper-networks of size qn , where n is an integer that reflects the recursive level. There are not many situations where q -note chords cluster into q -chord progressions (and q^3 -progressions of progressions, and so on). . . .⁹⁸

In fact, the K-nets and their recursive K-nets treated in a number of analyses are limited in the sizes of their networks. That is, any single network relationship permits only sets of the same cardinality, or more precisely, networks of the same nodes. In practice, however, the network relationships that have been defined in the earlier writings are not

⁹⁷ In a recent article, Michael Buchler criticizes K-nets and their analytical deployment in his "Reconsidering Klumpenhouwer Networks," *Music Theory Online* 13/2 (2007). Also, see footnotes 1 and 2 of Chapter 2.

⁹⁸ Buchler, "Reconsidering Klumpenhouwer Networks," 18.

more broadly applicable since we can easily see a number of pieces or passages which contain sets of different cardinalities, especially by including three- and four-note motivic elements. Hence, in these pieces, network relationships which only permit the same nodes are not able to fully explicate the entire piece or passage. In addition, formatting hyper-networks of the same size in a recursive way may also be incompatible with the music itself in many cases.

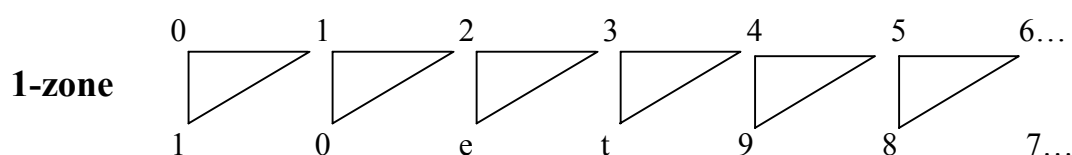
In this chapter, I will examine a method for associating K-nets of different sizes, especially trichords and tetrachords, by paying attention to the spatial transformations of K-nets among index zones. The first part discusses what possibilities the consideration of tetrachords creates for thinking about the trichordal subsets and how the trichordal K-nets can be related to tetrachordal K-nets. This first part introduces three types of transformations between a trichord and a tetrachord. The second part of the chapter demonstrates how possible transformations between a trichordal K-net and a tetrachordal K-net may be profitably employed in the analysis of the first section from Bartók's *Dance Suite*, op.14 (1916), the first pieces from Schoenberg's *Das Buch der hängenden Gärten*, op.15, and Eugene Cines's "Abbreviations" for solo piano (1974).

3-1. Transformations between a trichordal K-net and a tetrachordal K-net.

First let us consider the possible transformations within an index zone that may be yielded by the relations among a trichord and the trichordal subsets of a tetrachord. As we examined in the previous chapter, for a trichord a K-net can be viewed as a contiguous triangle, embedded within an index zone, which is produced by associating an inversive dyad with any single note, whereas a tetrachordal K-net is produced by

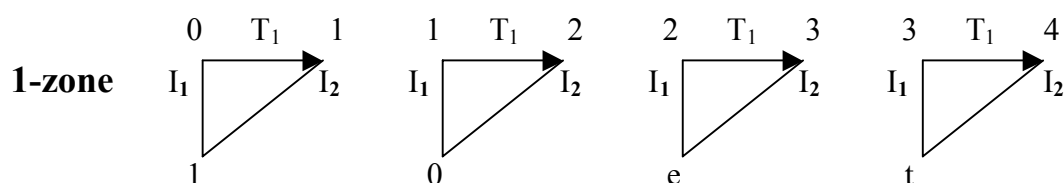
combining an inversive dyad with another dyad within the same zone. Let us now look at the way in which trichordal K-nets may occur within an index-zone. Example 3.1 shows how the trichords within an index zone, for instance within the 1-zone, with the same transpositional relations necessarily include the same inversive relations.

Example 3.1 Trichords within an index-zone



Example 3.2 thus interprets as K-nets those triangles with the same T_n and I_n operations demonstrating strong isography. That is, the triangles within the 1-zone contain identically positioned T_1 , I_1 , and I_2 arrows.⁹⁹

Example 3.2 Strong isography among trichords within the 1-zone

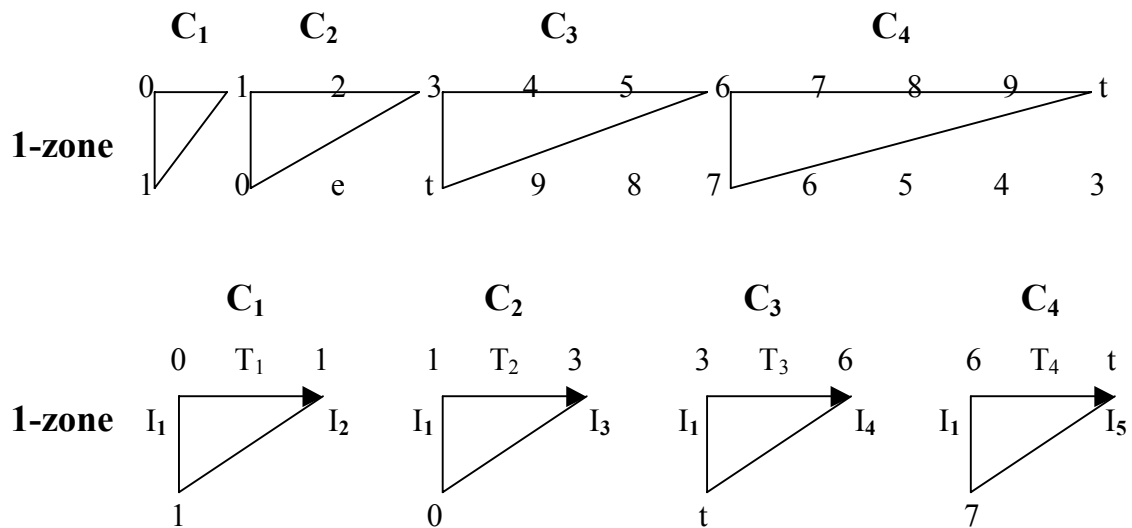


In a similar manner, it is also possible for a trichord to transform into other trichords containing different T_n relations within an index zone. Example 3.3 gives an

⁹⁹ It is also possible to represent negative isography in an index zone since the reverse position of any triangle ($I_1 \rightarrow I_2 < I_3 > I_2 \rightarrow I_1$) simply exhibits the reverse position of the two I -subscripts with a T_n arrow replacing T_{12-n} .

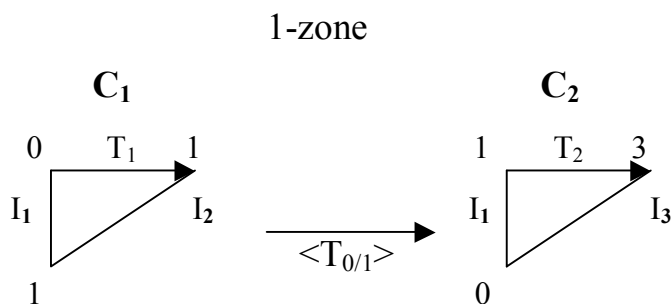
instance in which, within the 1-zone, each of the T_n -relations yields different inversional relations, which all, however, involve the same I_n arrow on the left side.

Example 3.3 Axial isography among trichords within an index-zone



The axial isography for trichords sharing a single I_n arrow will be labeled $\langle T_{0/n} \rangle$; the left numerical subscript, 0, represents their shared I_n arrows within an index zone and the right numerical subscript indicates the difference between T_n arrows which will also yield the same difference between I_n arrows. Example 3.4 presents one possible axial isography of the network interpretations of the triangles which share I_1 arrow within the 1-zone while the difference between T_n arrows is 1.

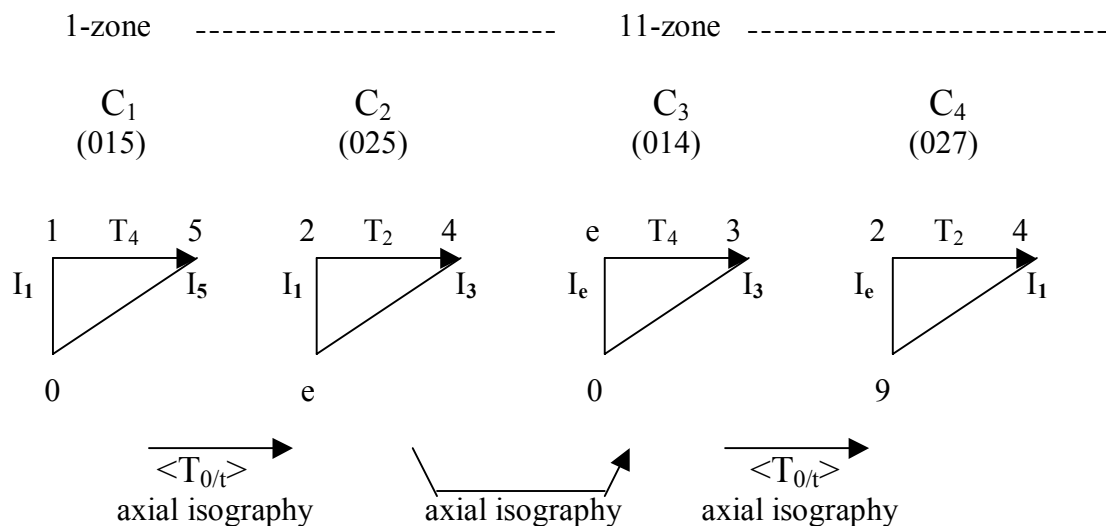
Example 3.4 Axial isography between trichords in the 1-zone



Thus, Example 3.4 demonstrates that under axial isography the network interpretation of C_1 maps onto the K-net that models C_2 at $\langle T_{0/1} \rangle$.

On the other hand, there is another way to represent axial isography between trichordal K-nets. Example 3.5 shows that four trichordal K-nets derived from different index zones interpret four different set-classes, C_1 - C_4 .

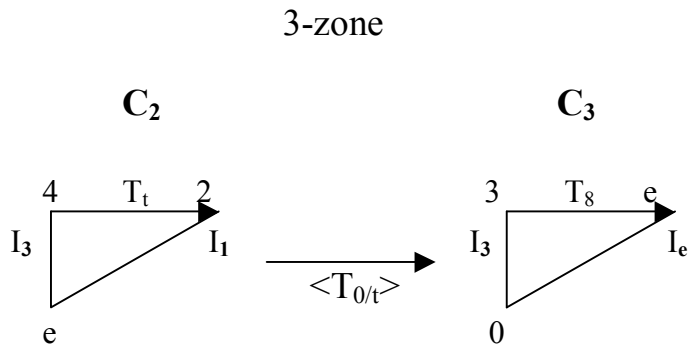
Example 3.5 Axial isography between index-zones



Note that each pair of network interpretations, C_1 and C_2 and C_3 and C_4 , is axially isographic at $\langle T_{0/t} \rangle$; C_1 has an identical I_1 arrow in common with C_2 within the 1-zone and C_3 shares the identical I_e arrow with C_4 within the 11-zone. Notice also that the network interpretation for C_2 is axially isographic with C_3 since the K-nets share identical I_3 arrows. In view of the relationships shown in Example 3.6 and 3.7, we now

have two different K-net interpretations of the same chords. The first possibility displayed in Example 3.6 interprets the chords, C_2 and C_3 , as placed in the same zone.

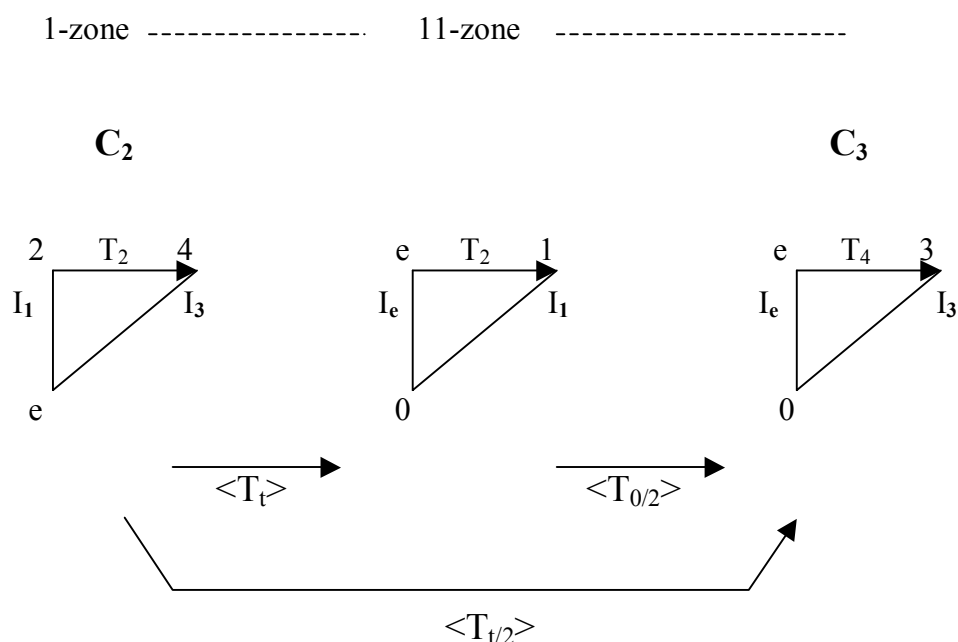
Example 3.6 Axial isography in the 3-zone



That is, the network interpretations of C_2 and C_3 in this example can be shown to be axially isographic at $\langle T_{0/t} \rangle$ within the 3-zone.

The second possible interpretation of C_2 and C_3 , both containing an I_3 arrow, can also be interpreted in each of the different zones as shown in Example 3.7.

Example 3.7 Axial isography between the 1-zone and 11-zones



That is, in this view, only the I_3 arrows shared between C_2 and C_3 are capable of pivoting from one zone to another. To map the network interpretation of C_2 onto C_3 , as indicated in Example 3.7, the network of C_2 in the given 1-zone can shift to the 11-zone by $\langle T_t \rangle$ and then maps onto C_3 as $\langle T_{0/2} \rangle$. The operation can thus be labeled as $\langle T_{t/2} \rangle$; the left subscript represents the difference between the varying I_n arrows, especially on their left sides, and the right subscript indicates the difference between the T_n arrows. Such an axial isography that relates K-nets belonging to different zones but sharing a single identical I_n arrow can thus be labeled $\langle T_{n/-n} \rangle$.¹⁰⁰ Let us look at these conditions, which suggest two possibilities for interpreting trichords in different index-

¹⁰⁰ It is the same notation as the type 2 axial isography between tetrachordal networks in which each belongs to a different index zone while sharing a trichordal subset. Stoecker also defined a type of “axial isography” and labeled it $\langle T_{n/-n} \rangle$ in which two numbers within the angle brackets represent the difference between I_n and T_n arrows and these are always complements of each other. See his “Klumpenhauer Networks, Trichords, and Axial Isography,” 235-240.

zones, in the trichordal K-nets at work in one of the examples, the third part of Werner Heider's "*Kleinwelt*" for piano (1992). Example 3.8 presents the last passage of the third part from the work.

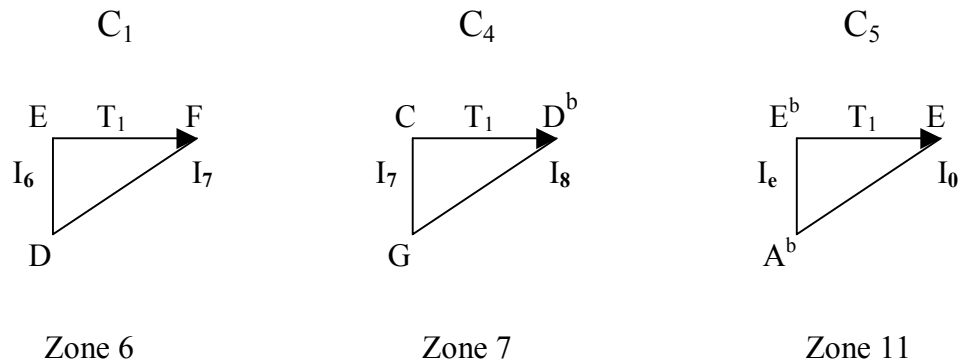
Example 3.8 Werner Heider's "*Kleinwelt*," right-hand part in part 3

C_1	C_2	C_3	C_1	$(C_2 \ C_1 \ C_3)$	C_4
3-2	4-11	4-23			3-5
(013)	(0135)	(0257)			(016)

C_5	$(C_3 \ C_2 \ C_1)$	C_6
3-4		3-1
(015)		(012)

Here, the right-hand part of this passage includes six different chords, which contain members of six different set-classes and do not all share the same cardinality. The three circled trichords provided in Example 3.8, C_1 , C_4 , and C_5 can be interpreted as K-nets each derived from different index-zones as shown in Example 3.9.

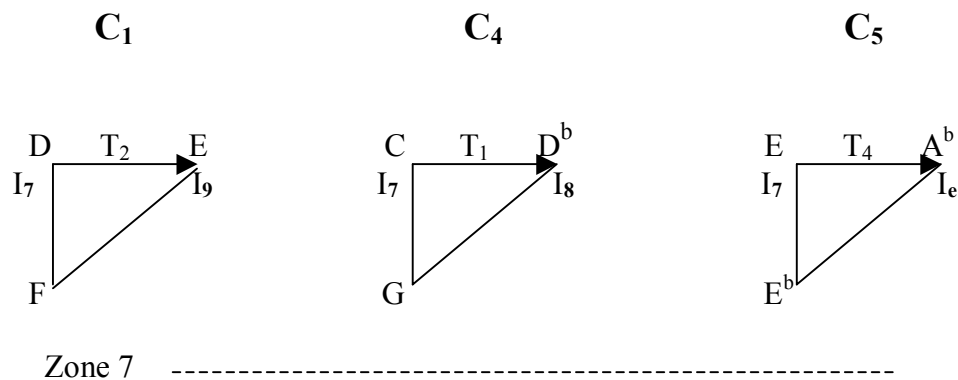
Example 3.9 An interpretation of trichords C_1 , C_4 , and C_5 in different index-zones



Since each of three trichordal K-nets has different I_n arrows while all have identical T_1 arrows, we can see that the graphs are all positively isographic one with another.

On the other hand, the identical chords can also be interpreted as K-nets in the same index-zone as displayed in Example 3.10.

Example 3.10 An interpretation of trichords C_1 , C_4 , and C_5 in the same index-zone

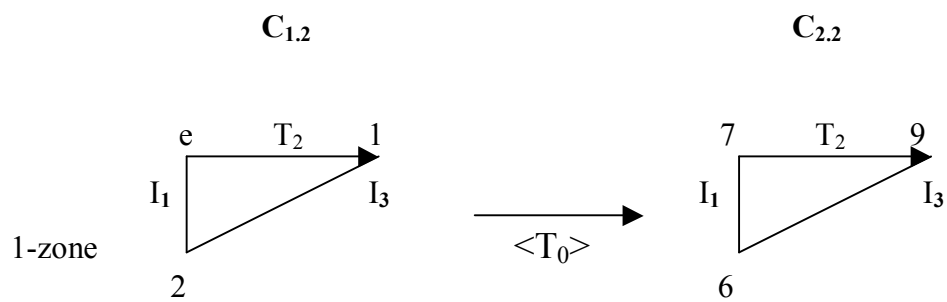
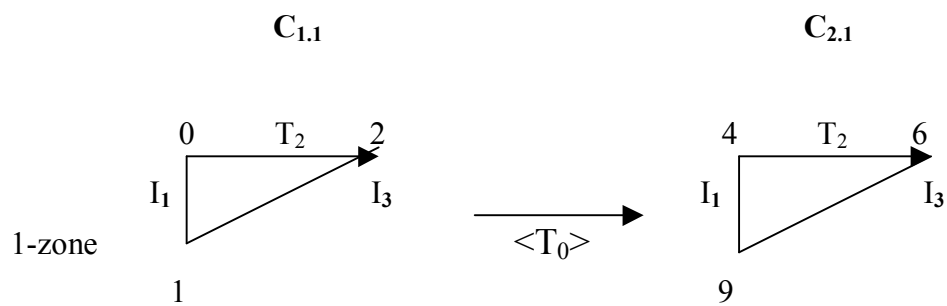
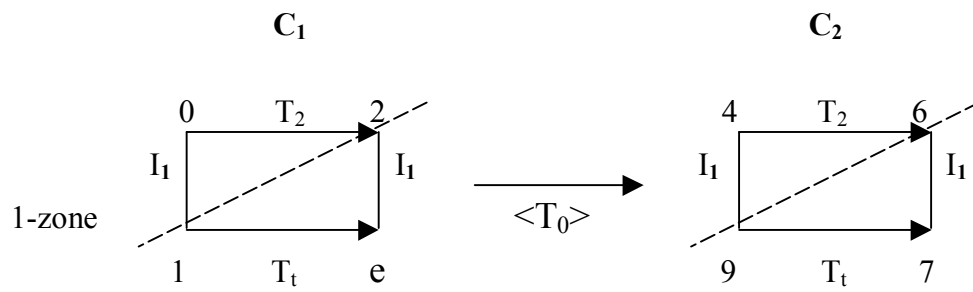
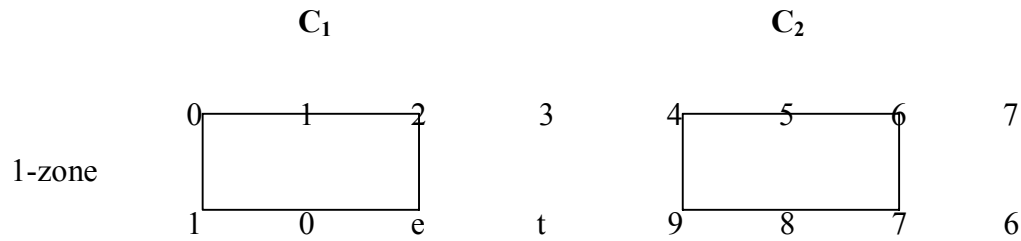


In this, all trichordal K-nets share an I_7 arrow while each contains different T_n and I_n arrows, thus demonstrating axial isography within the 7-zone. As we have seen in the examples above, we may often be faced with having to decide between different

readings for the same trichords. In effect, it is not easy to decide whether Heider's passage in Example 3.8 is better regarded as different zones or as a single zone. The trichordal K-nets indeed are less determined when being generated either by the same index-zone or by different index-zones since the trichordal K-nets can share at most a single I_n arrow showing which zone they belong to. In order to clarify to which zone the trichordal K-nets belong, we can rely on neighboring tetrachordal K-nets which can share with them a pair of identical I_n arrows, indicating their places.

For tetrachords, we can draw a strong isography and three types of axial isography through the spatial transformations within an index-zone as examined in the previous chapter. Once we have transformed one quadrangle of a tetrachord into another, we can now consider each quadrangle of the tetrachords to be a complex of two trichordal subsets. For instance, two quadrangles of the same size and same shape within an index-zone are presented in Example 3.11.

Example 3.11 Strong isography between tetrachords and between their trichordal subsets.

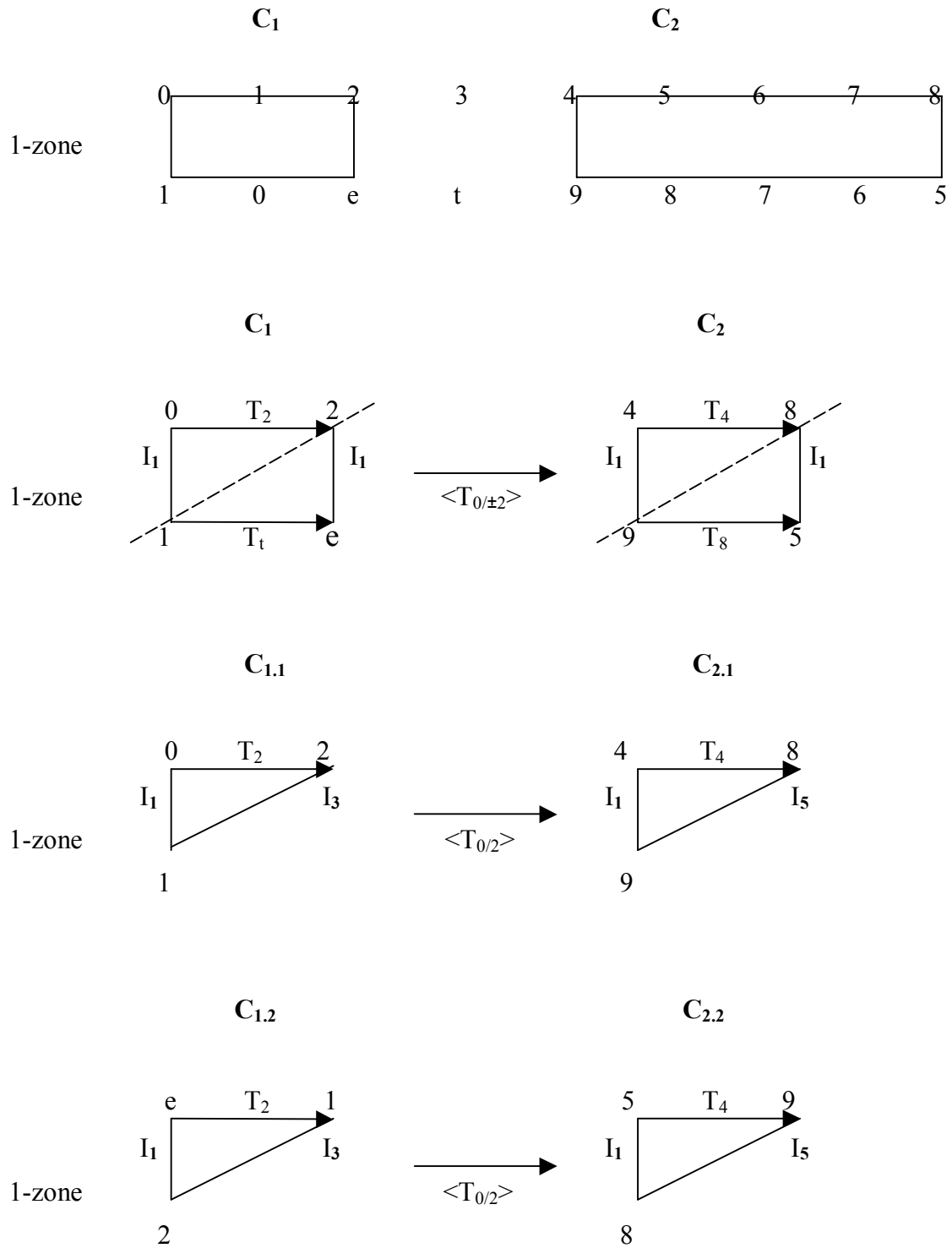


As seen here, if two tetrachords are strongly isographic with one another, their trichordal subsets will in turn be related by the same relation.¹⁰¹ In such a case, two pairs of trichordal subsets will always be strongly isographic to each other.

In Example 3.12 there are two quadrangles within an index-zone though of different size and shape.

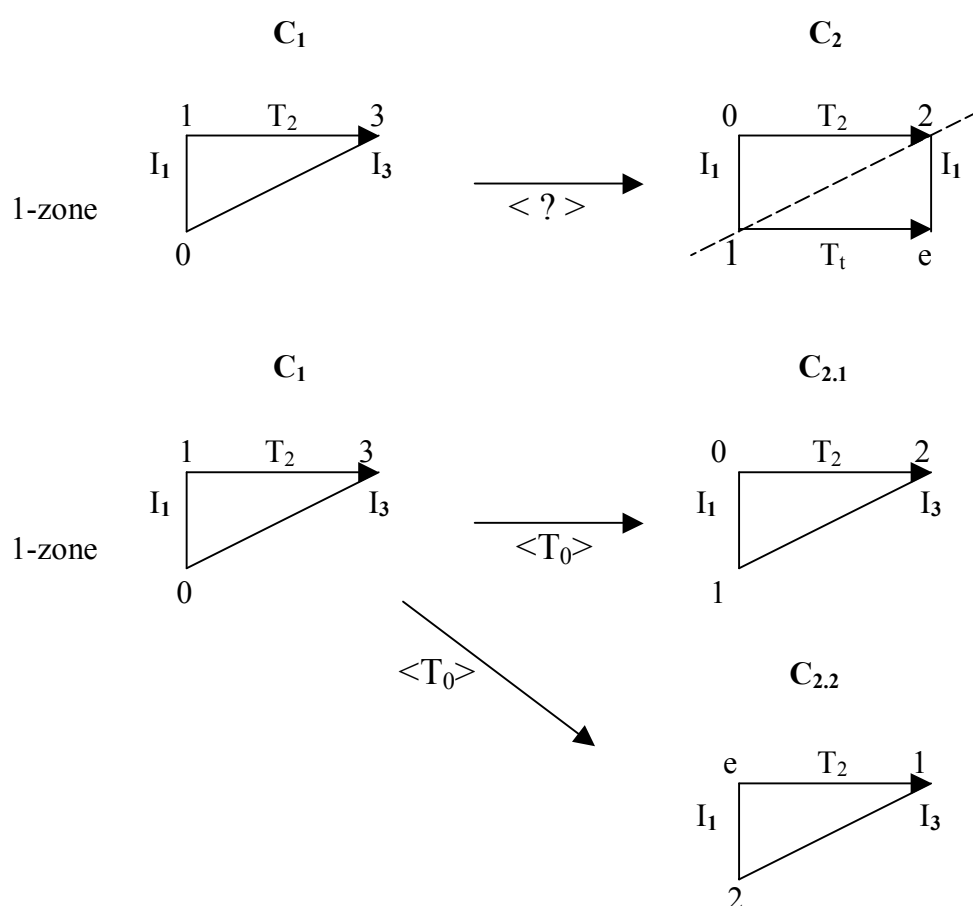
¹⁰¹ We can also attain equivalent results by drawing a diagonally opposite line from the northwest to the southeast.

Example 3.12 Axial isography between tetrachords and between their trichordal subsets.



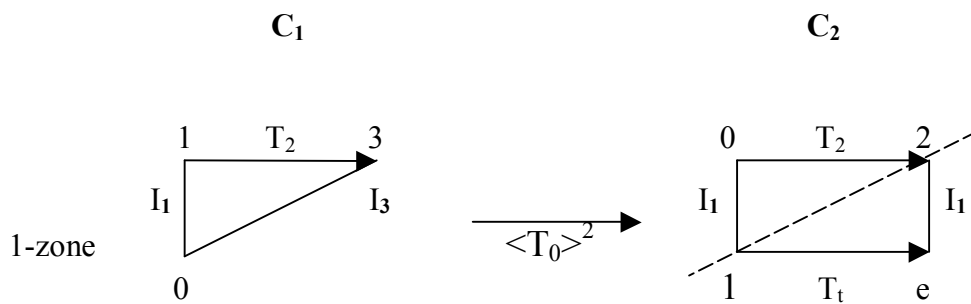
Note that the tetrachords, which have the first type of axial isography at $\langle T_{0/\pm 2} \rangle$, have subsets, which also show axially isographic relations with each other at $\langle T_{0/2} \rangle$. Both strong and type 1 axially isographic relationships between tetrachords assure the same relationships between their trichordal subsets as shown in Examples 3.11 and 3.12. Viewing in this way provides a useful means of describing the relations between a trichord and a tetrachord now involving a couple of trichordal subsets. Example 3.13(a) provides an example of strongly isographic relationships between a trichord and a tetrachord laid out in a simple way.

Example 3.13(a) Strong isography between a tetrachord and a trichord within an index-zone



It is not possible in this example to associate the trichordal K-net of C_1 with the tetrachordal K-net of C_2 , however. Instead, since the tetrachordal K-net for C_2 consists of a pair of trichordal subsets containing the same I_n and T_n arrows, we can see that the trichordal network, C_1 , and the networks of trichordal subsets, $C_{2,1}$ and $C_{2,2}$, share identical I_n and T_n arrows within the 1-zone. That is, the trichordal K-net of C_1 is strongly isographic to each network of trichordal subsets, $C_{2,1}$ and $C_{2,2}$, represented by $\langle T_0 \rangle$. As a result, the trichordal K-net of C_1 and the tetrachordal K-net of C_2 can be related to each other despite their different number of nodes. Such transformations between tetrachordal and trichordal K-nets which share the same I_n and T_n arrows within an index-zone can be defined as examples of *strongly reductive isography*. In Example 3.13(b), the diagram labeled $\langle T_0 \rangle^2$ represents the mapping of the same trichord, C_1 , into both halves of the tetrachord, $C_{2,1}$ and $C_{2,2}$, with the same $\langle T_0 \rangle$ relations.

Example 3.13(b) Strongly reductive isography between a tetrachord and a trichord



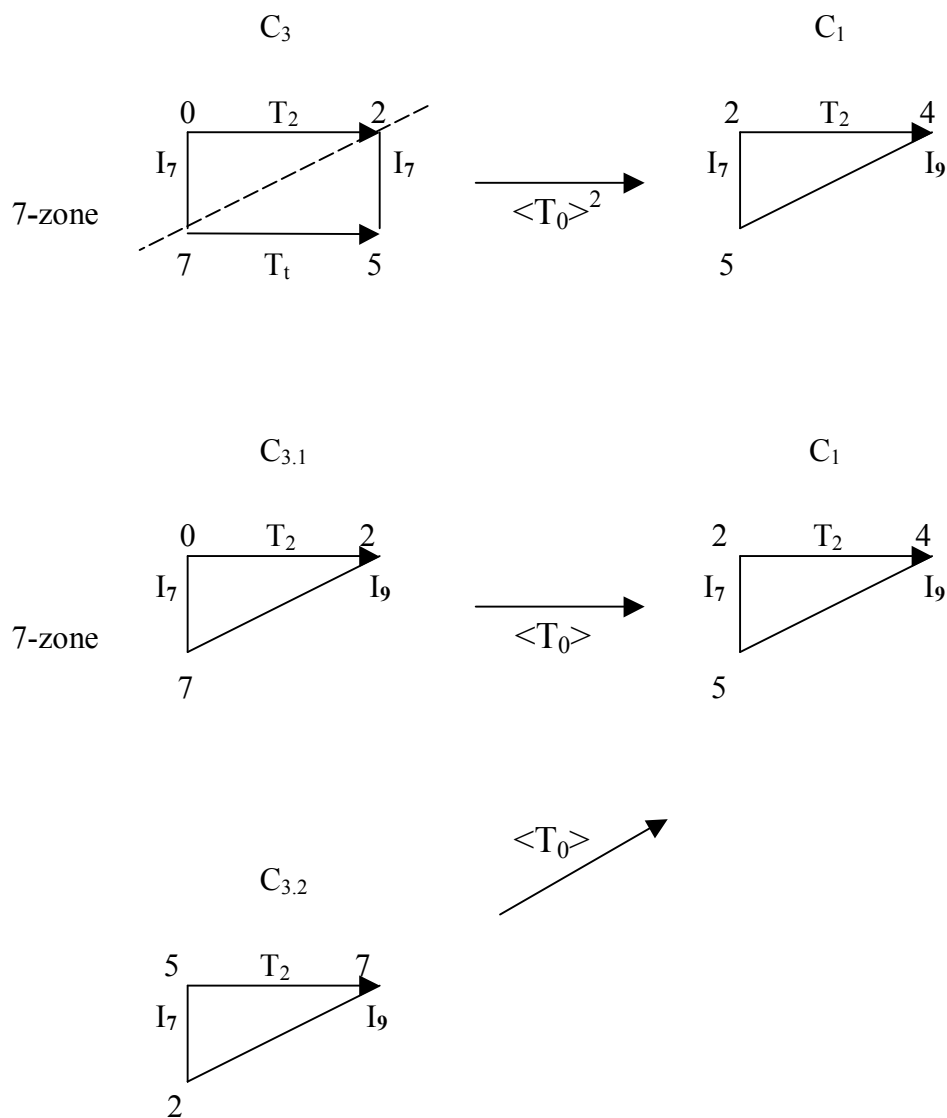
Now, we can revisit the Heider passage in order to give a more concrete explanation. Example 3.14 returns to the passage from Example 3.8 and includes four different chords which present isographic relationships of this sort and whose graphs are

labeled $\langle T_0 \rangle^2$. In this progression, we can hear the upper outer voice as an inversion of the lower voice, though the B in the second chord deviates slightly. As shown in Example 3.14(a), each chord has at least one I_7 arrow which occurs in pitch space, making it particularly noticeable.

Example 3.14(a) Heider's "*Kleinwelt*," last passage of part 3

Note that the K-nets of C_3 and C_1 in Example 3.14(b) especially share identical I_7 and T_2 arrows within the same 7-zone although the network interpretations have different number of nodes. In other words, the network interpretation of the trichord, C_1 , duplicates exactly the two trichordal subsets of the tetrachord, $C_{3.1}$ and $C_{3.2}$, and thus the corresponding pairs of networks are strongly isographic as shown in Example 3.14(b).

Example 3.14(b) Strongly reductive isography between the tetrachord of C_3 and the trichord of C_1



Therefore, under strongly reductive isography, the tetrachordal K-net that models C_3 maps onto the K-net that models C_1 at $\langle T_0 \rangle^2$.

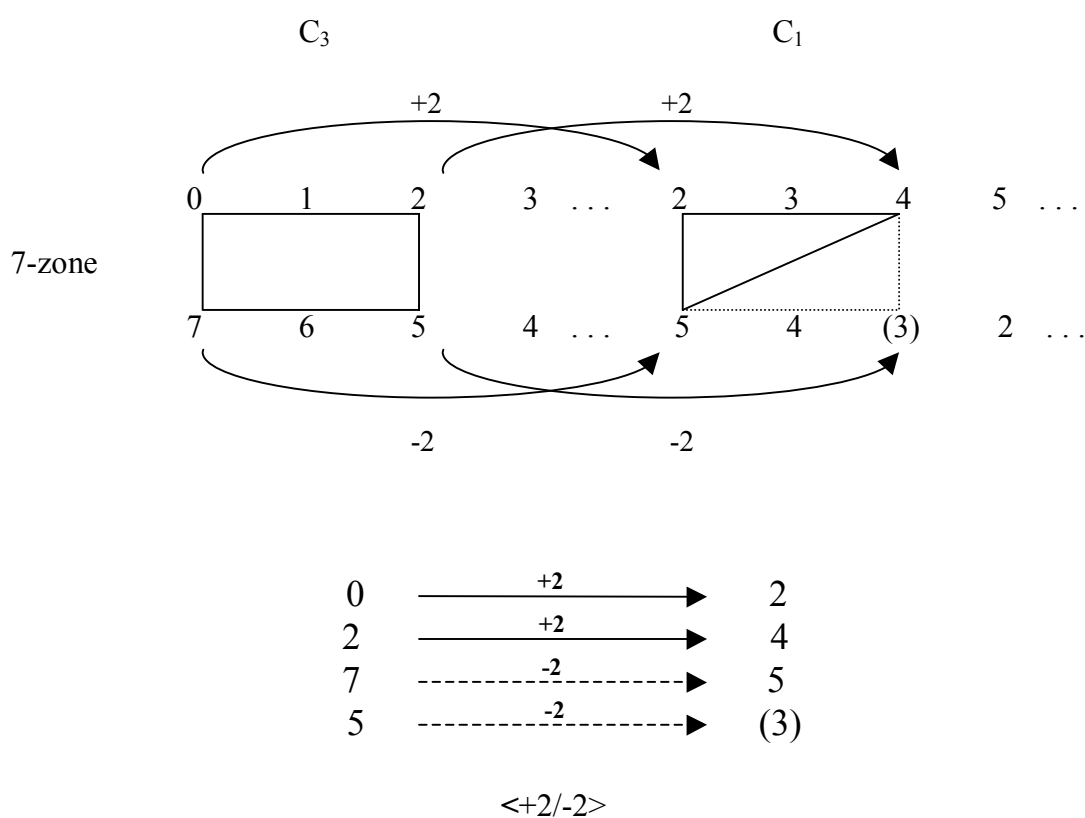
Let us take a moment to explore how the relation between a tetrachord and a trichord can be portrayed by transformational voice-leading. When the mappings between any pair of a trichord and a tetrachord involve every note, there will be thirty-

six possible mappings by allowing splits or fusing of any of the notes of either the smaller set or the larger one.¹⁰² It is, however, important to remember that a tetrachord and a trichord map to each other within the special condition of the index-zones. Above all, generating voice leadings in the index-zones will be much simpler. Within an index-zone, if two sets of the same cardinality have at least one common I_n arrow under axial isography, then one set can move to the other whatever their shape. In this situation, all the notes of both sets must occur in the voice leading. In the same way, when two sets, especially a tetrachord and a trichord of different cardinalities, have a common trichordal subset under strongly reductive isography, the tetrachord can also move to the trichord without regard to size, but one note of the tetrachord will be left out. Then the question can be raised: Can we be sure that the odd note of the tetrachord will be left out of the mapping to the trichord? The answer is no; indeed, the destination of the odd note is simply omitted. We have seen how a trichord can be embedded within an index-zone in which inversional dyads are significant determinants. That is, a trichord, which is produced by associating an inversional dyad with any single other note, can be regarded as insufficient and thus undetermined, seeking to be perfect. In other words, a trichord in an index-zone may be realized as a defective model of inversional tetrachords seeking an inversional partner. Therefore, just as any quadrangle maps to another of the same size for strong isography, a quadrangle can map to a trichord for strongly reductive isography by adding an imaginary note. We may have two possible voice leadings between a quadrangle and a triangle within an index-zone. First, a quadrangle

¹⁰² Straus discusses how pitch-class sets of different cardinality can generate voice leadings and shows thirty-six possible mappings between a trichord and a tetrachord in Example 5, Joseph N. Straus, "Uniformity, Balance, and Smoothness in Atonal Voice Leading," *Music Theory Spectrum* 25 (2003), 313-314.

may move to a triangle without flipping. Example 3.14(c) lays out the voice leading transformation between the tetrachord of C_3 and the trichord of C_1 derived from the Heider passage in Example 3.14(a).

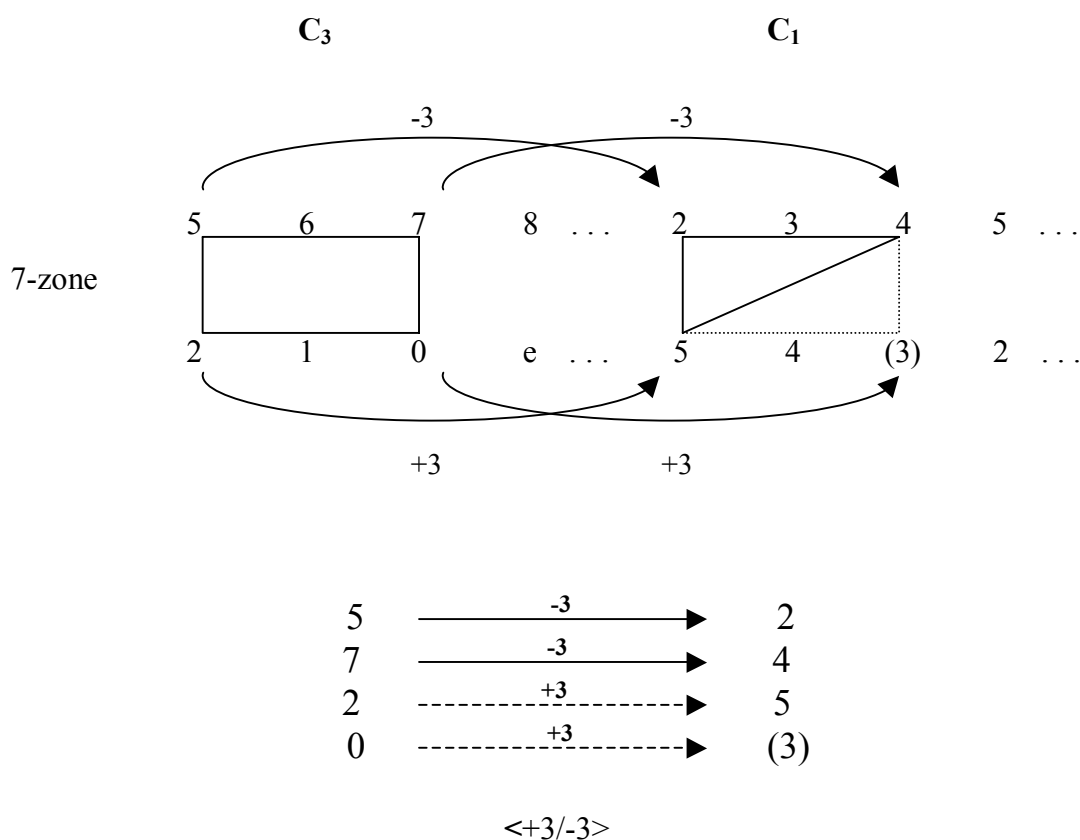
Example 3.14(c) First possible voice leading between the tetrachord of C_3 and the trichord of C_1



Example 3.14(c) has each note of C_3 connected to one and only one note of C_1 including an imaginary note. We can see more clearly the wedging motion within the 7-zone by adding the missing E^b to the trichord of C_1 . The quadrangle in the second diagram flips diagonally both southeast- to -northwest and southwest- to -northeast and may move to a triangle. This is brought out in Example 3.14(d), where each note of the tetrachord,

C_3 , follows the individual pitch-class mappings enacted by the wedging motion within the 7-zone.

Example 3.14(d) Second possible voice leading between the tetrachord of C_3 and the trichord of C_1

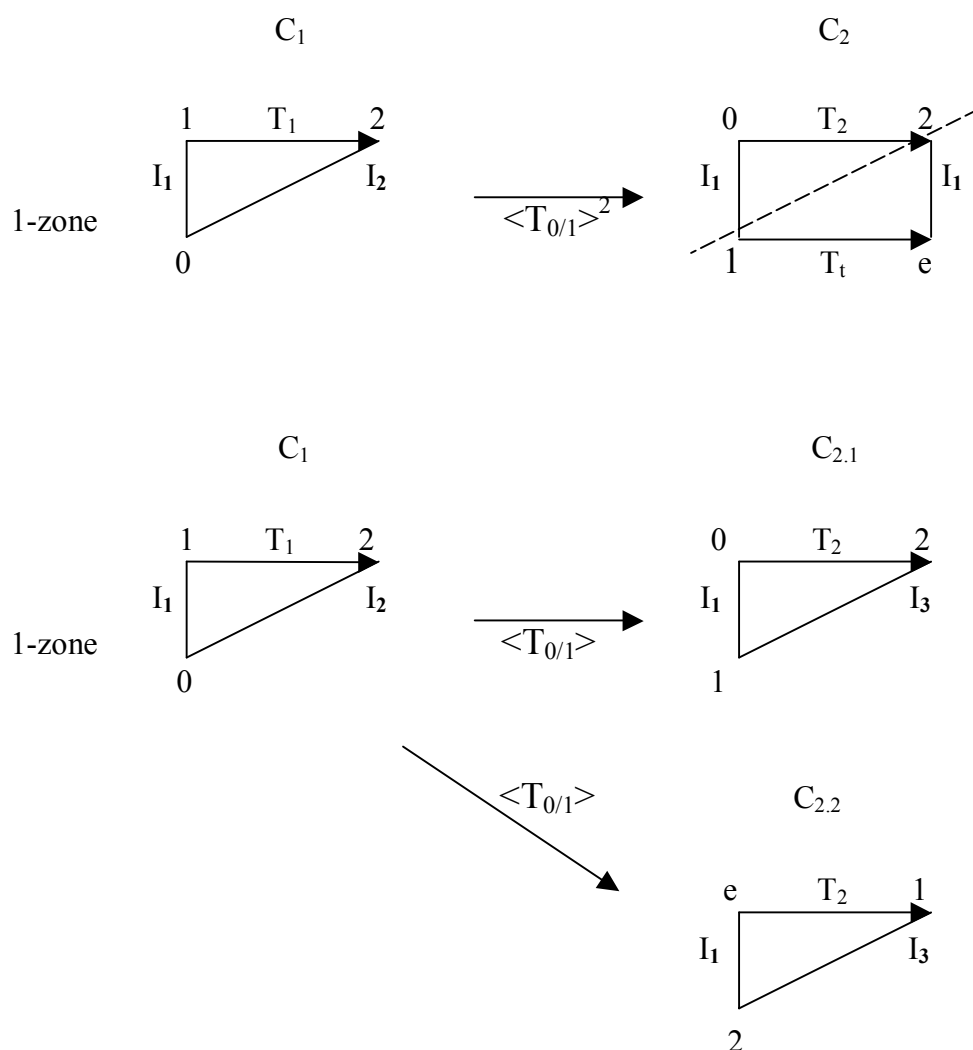


These dual transformations, either T_{+3}/T_{-3} or T_{+2}/T_{-2} , shown in Examples 3.14(c) and (d) suggest hearing the symmetry of the progression as two dyads wedged apart by the same interval-classes, though we can only literally hear three of the four voices. The two possible voice leadings of the progression C_3 - C_1 shown in Examples 3.14(c) and (d) also embody two possible mappings among the trichordal subsets of C_3 and the trichord of C_1 , $C_{3.1}$ - C_1 and $C_{3.2}$ - C_1 , displayed in Example 3.14(b). In particular, the

first interpretation of the voice leading shown in Example 3.14(c) seems more plausible since it is realized by registral lines in the Heider passage.

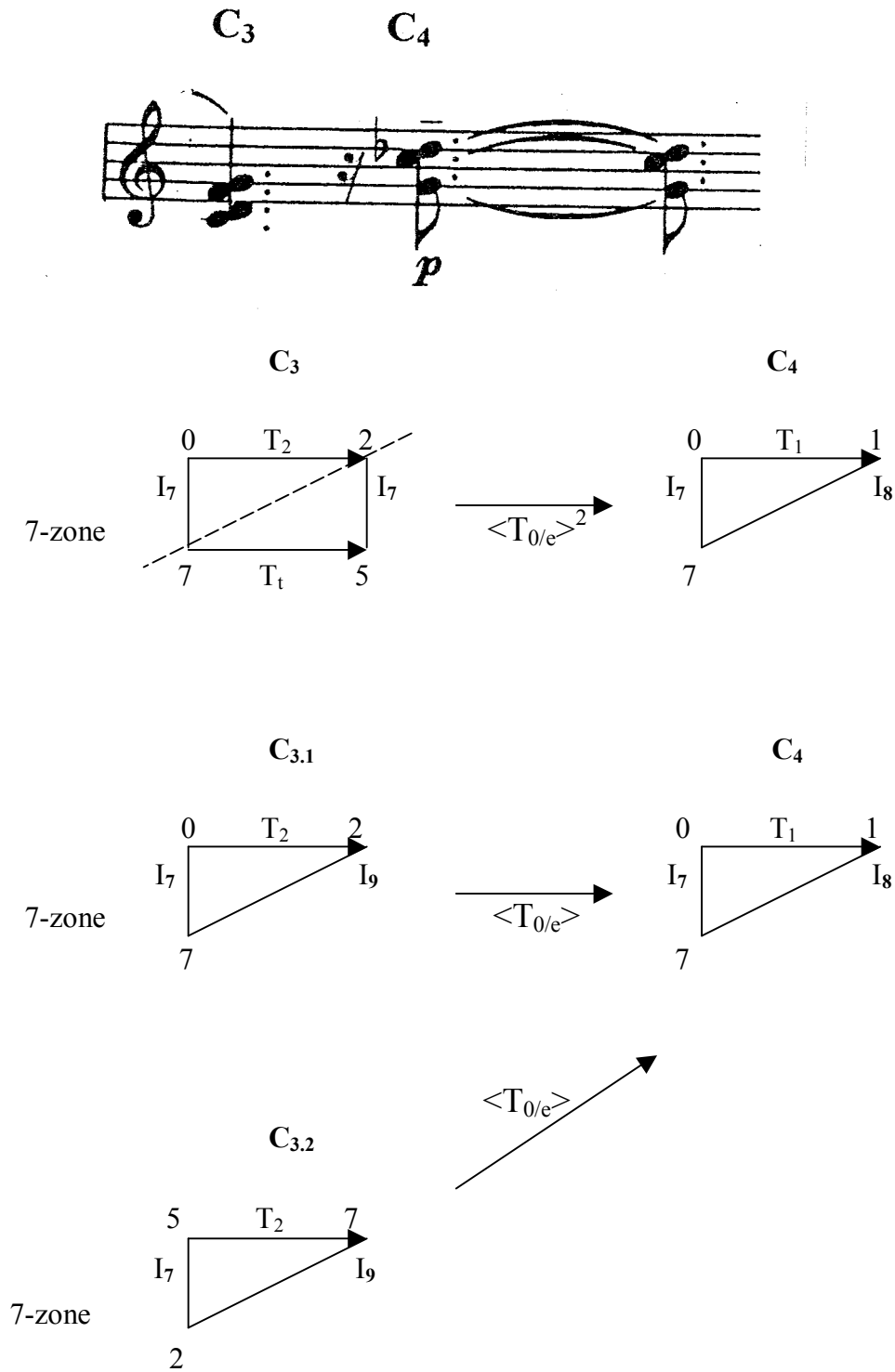
We can also examine a possible transformation between a trichord and a tetrachord within a zone. Example 3.15 shows how the triangle C_1 can be related to the quadrangle C_2 despite their different size and shape.

Example 3.15 Type 1 axially reductive isography between a tetrachord and a trichord



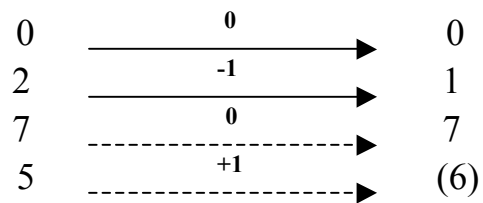
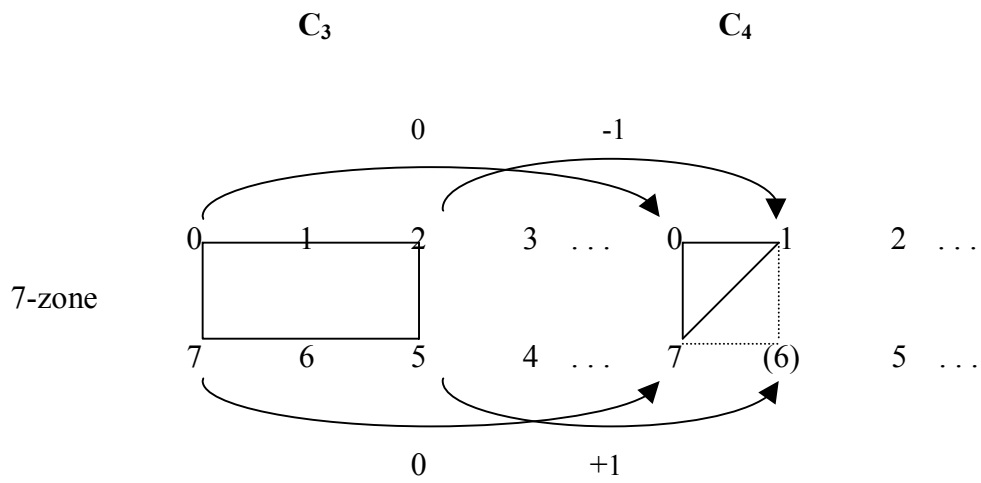
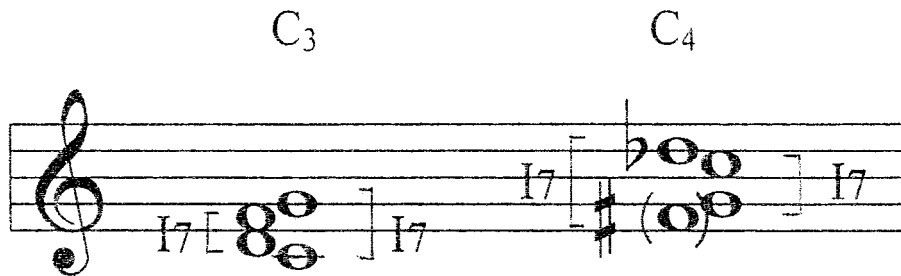
Example 3.15 illustrates that the trichord, C_1 , and the trichordal subsets of C_2 , $C_{2.1}$ and $C_{2.2}$, all have identical I_1 arrows within the same zone while having different T_n arrows: under axial isography, the K-net of C_1 maps onto both K-nets of $C_{2.1}$ and $C_{2.2}$ as $\langle T_{0/1} \rangle$ relations. That is, the transformation between the trichordal and tetrachordal K-nets is labeled $\langle T_{0/1} \rangle^2$. The left numerical subscript, 0, indicates the identical I_n arrows within the same zone while the right numerical subscript, 1, equals the difference between analogous T_n arrows. K-nets of three and four nodes that have different T_n arrows within an index-zone demonstrate *type 1 axially reductive isography*. A return to Heider's passage provides an instance of using this type of isography. In Example 3.16(a) the network interpretations of C_3 and C_4 neither share the same configuration of nodes and arrows nor have identical T_n arrows.

Example 3.16(a) Heider's "*Kleinwelt*," last passage of part 3: type 1 axially reductive isography between the tetrachord of C_3 and the trichord of C_4



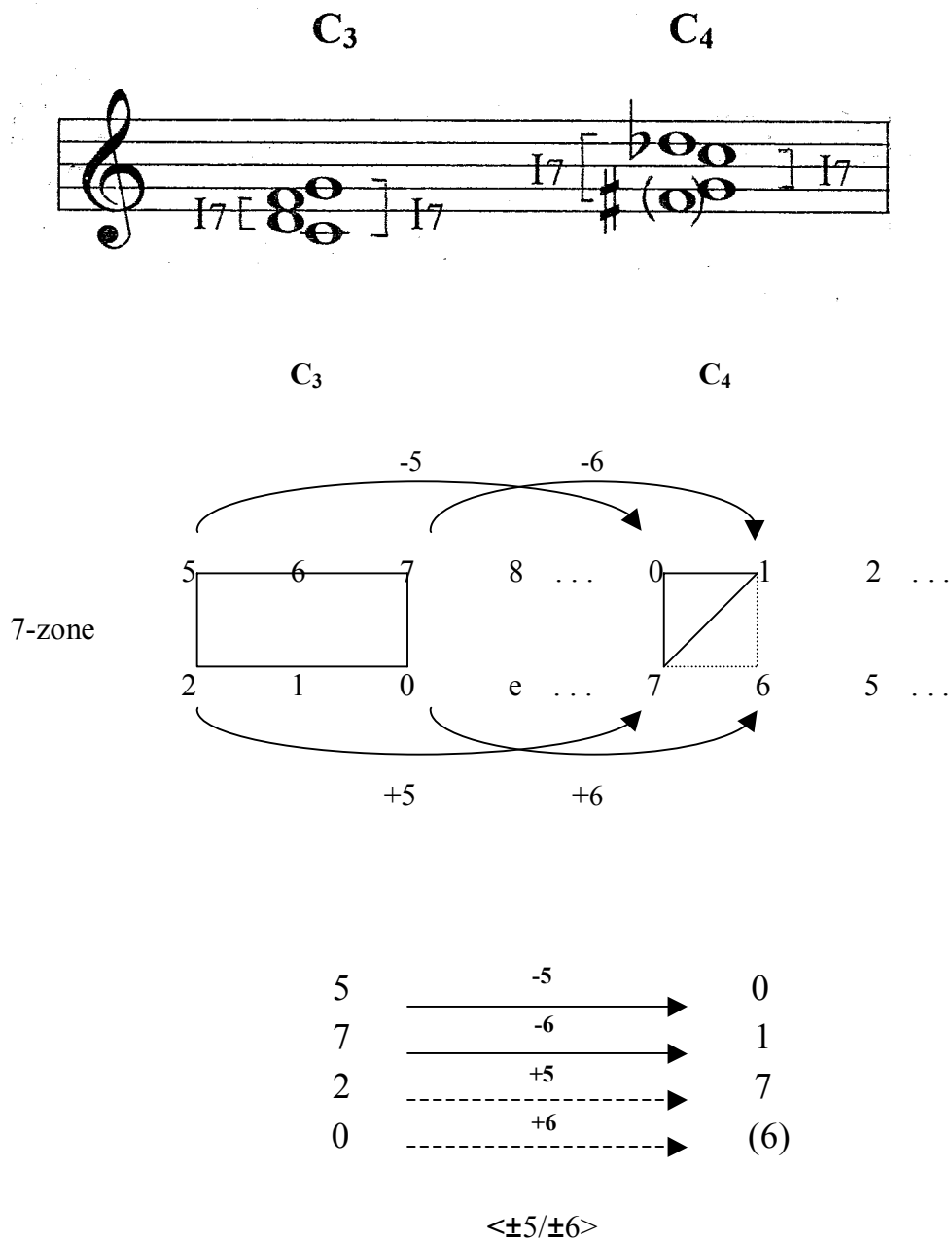
Notice, however, that both K-nets have identical I_7 arrows within the 7-zone. The surface arrangement of the chords clearly exhibits the symmetrical inversion I_7 , by including the I_7 -dyad (C, G). That is, Example 3.16(a) illustrates that the K-net of C_3 can transform into the K-net of C_4 by embracing a pair of equivalent trichordal subsets within the 7-zone; the axially isographic relations among their trichordal K-nets are represented by $\langle T_{0/e} \rangle^2$. There are also two possible voice leadings produced by the type-1 axially reductive isography between C_3 and C_4 , as displayed in Examples 3.16(b) and (c). The dual transformations of the first interpretation shown in Example 3.16(b) especially bring out the symmetrical foreground voices reinforced by registral lines. That is, the C-G dyad stays while the D-F dyad in C_3 descends or ascends a semitone to the D^b - $F^\sharp$ dyad in C_4 , though the $F^\sharp$ is not present in the trichord C_4 .

Example 3.16(b) First possible voice leading of the type 1 axially reductive isography between the tetrachord of C_3 and the trichord of C_4



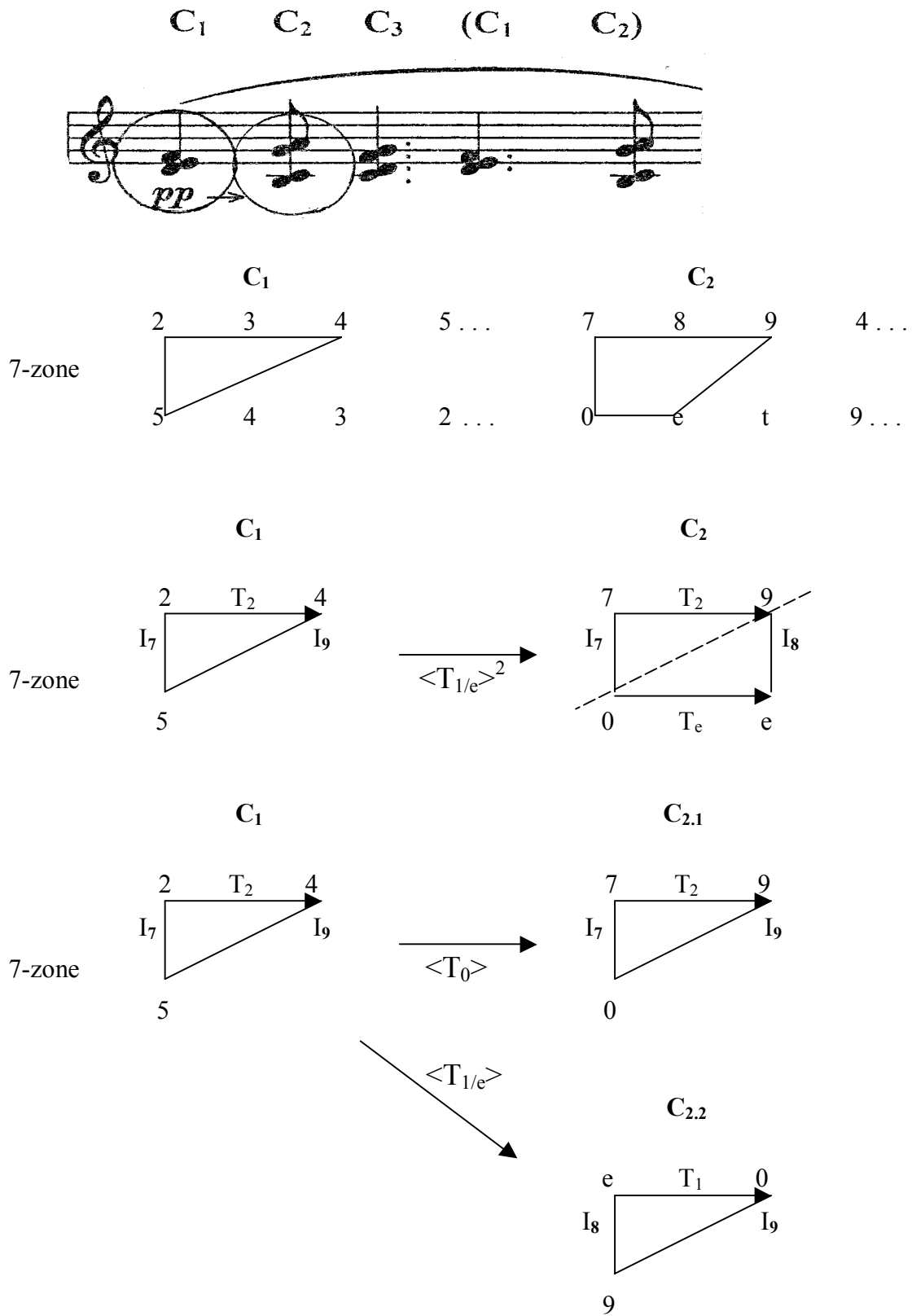
$\langle 0/\pm 1 \rangle$

Example 3.16(c) Second possible voice leading of the type 1 axially reductive isography between the tetrachord of C_3 and the trichord of C_4



For another possible transformation between a trichord and a tetrachord within an index zone, Example 3.17(a) revisits the Heider passage from Example 3.8.

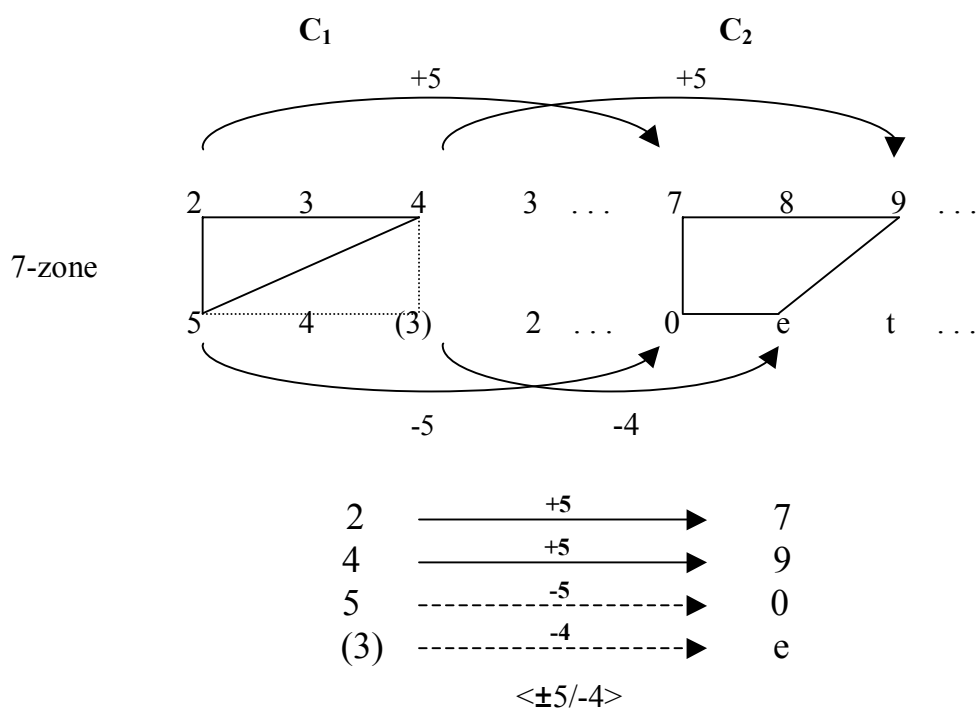
Example 3.17(a) Type 2 axially reductive isography between the tetrachord of C_2 and the trichord of C_1 within an index-zone



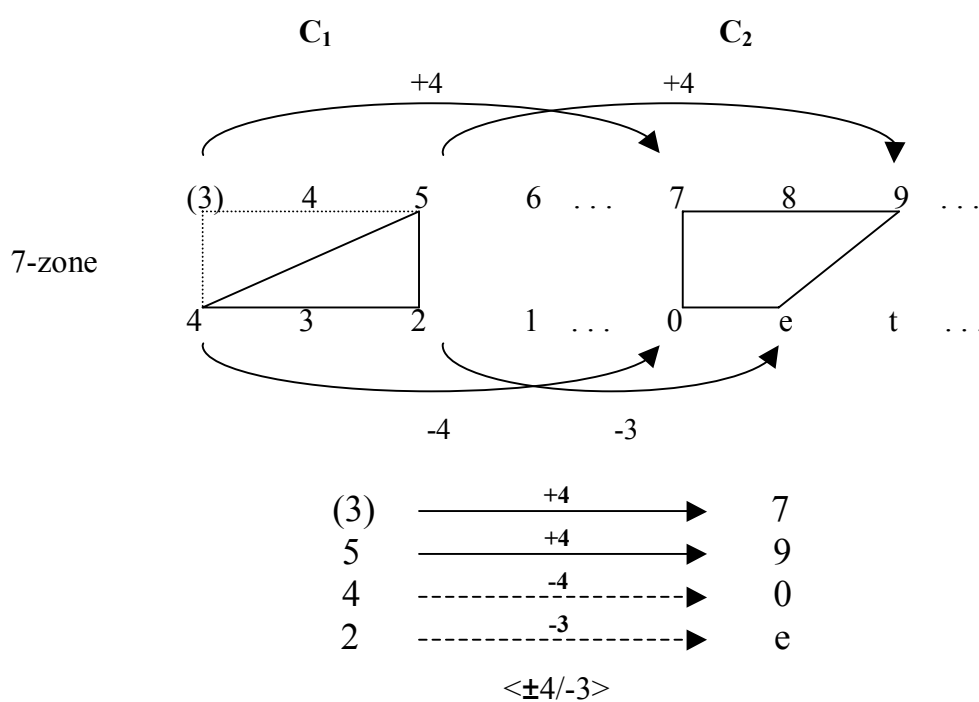
In this, we note that the trichordal network of C_1 and the tetrachordal network of C_2 share single identical I_7 and T_2 arrows. That is, the K-nets of C_1 and C_2 share the trichordal network of C_1 itself as an invariant within the 7-zone. In Example 3.17(a), the K-net of C_1 is strongly isographic with the trichordal subset of $C_{2,1}$ at $\langle T_0 \rangle$ while the K-nets for C_1 and another trichordal subset of $C_{2,2}$ share the same I_9 arrows in their right sides and are axially isographic with each other at $\langle T_{1/e} \rangle$. Even though the transformation does not assure the same isographic relations among the trichordal K-nets, the possible relations of trichordal K-nets suggest another transformation between the trichord C_1 and the tetrachord C_2 . Such a transformation, shown in Example 3.17(a), can be defined as a *type 2 axially reductive isography* and labeled with $\langle T_{1/e} \rangle^2$.¹⁰³ The left subscript, 1, indicates the difference between the I_7 arrow in the left sides of the trichord C_1 and the I_8 arrow in the same side of the tetrachord C_2 and the two numbers within the angle brackets will always be complements of each other. We also see in Example 3.17(b) and (c) that under type 2 axially reductive isography two possible voice leadings display T_{+5}/T_{-4} and T_{+4}/T_{-3} voices.

¹⁰³ $\langle T_0 \rangle * \langle T_{1/e} \rangle = \langle T_{1/e} \rangle^2$

Example 3.17(b) First possible voice leading of the type 1 axially reductive isography between the trichord of C_1 and the tetrachord of C_2



Example 3.17(c) Second possible voice leading of the type 1 axially reductive isography between the trichord of C_1 and the tetrachord of C_2



In both cases, it is not easy to hear the wedging ± 5 or ± 4 in three voices. In particular, the second interpretation of voice leading shown in Example 3.17(c) is perceptually ambiguous since one voice which occurs in the wedge motion is omitted. Nevertheless, the second voice leading is preferable because of its realization in pitch space. The type 2 axially reductive isography between C_1 and C_2 , while not easily audible, creates more tension with the strong reductive isography surrounding them in this passage.

So far, we have examined three possible transformations between trichordal and tetrachordal K-nets; strong, type 1 and type 2 axially reductive isographies all of which may be yielded within the same zone. We can finally see in Examples 3.18 and 3.19 how these transformations among trichords and tetrachords can be applied in the first and in the last passage of the third part of Heider's "*Kleinwelt*". As we have seen above, Example 3.18 interprets the five circled chords of the last passage shown in Example 3.8, C_1 to C_5 , as trichordal or tetrachordal K-nets.

Example 3.18 Heider's "*Kleinwelt*," last passage of part 3

Zone 7

In this progression, it is not sufficient to draw network isographies since the five chords have different numbers of notes. The surface arrangements of the chords, however, clearly exhibit a symmetrical inversion. We can see that the five K-nets have at least a single I_7 arrow in common within the 7-zone. In particular, the network interpretation of the central tetrachord, C_3 , which contains a pair of identical I_7 arrows, shows us their places in a clear way. The network interpretation of the trichord C_1 can be associated with the tetrachords which follow it by the operation which maps C_1 with at least one identical I_n arrow onto the trichordal subsets of C_2 and C_3 . In the progression C_1 - C_2 , the K-net of C_2 consists of two trichordal subsets, the trichordal K-net of C_1 itself and another which is axially isographic with the K-net of C_1 at $\langle T_{1/e} \rangle$, as previously shown in Example 3.17(a). This type 2 axially reductive isography between the K-nets of C_1 and C_2 is denoted by $\langle T_{1/e} \rangle^2$. In addition, the K-net of C_1 can transform into the K-net of C_3 by embracing a pair of equivalent trichordal subsets within the 7-zone; the strongly isographic relations among their trichordal K-nets can be reflected by the notation $\langle T_0 \rangle^2$. Note also that the tetrachordal K-net of C_2 is type 2 axially isographic with the tetrachordal K-net of C_3 since both contain a subset of a trichord, expressed in this example by the sharing of I_7 and T_2 arrows. Notice that there are two pairs of axially isographic networks, the tetrachordal K-nets of C_2 and C_3 and the trichordal K-nets of C_4 and C_5 , and that these pairs are connected by axially reductive isography at $\langle T_{0/e} \rangle^2$ as shown in Example 3.18. Accordingly, Example 3.18 generates a transformational path from C_1 to C_5 presenting axially reductive isographies between K-nets which have different number of nodes and also axial isographies between K-nets which share the

same nodes. The five chords, C_1 - C_5 , ultimately extend the 7-zone beyond their respective nodes.

Example 3.19 excerpts the first passage of the same part, including four different chords.

Example 3.19 Heider's "*Kleinwelt*," first passage of part 3

C_1	C_2	C_3	C_7		$(C_2$	C_3	C_1	$C_2)$
3-2	4-11	4-23	3-2					
(013)	(0135)	(0257)						

The first three chords in this example, C_1 - C_3 , are identical to the first chords of the previous passage, shown in Example 3.18. In the progression C_3 to C_7 of the first passage, we can see yet another transformation between a trichordal K-net and a tetrachordal K-net capable of shifting from one zone to another. Example 3.20 interprets the trichord of C_7 as a K-net containing a T_2 arrow within the 11-zone.

Example 3.20 An interpretation of C_7 in the 11-zone

C_3 C_7

C_3

7-zone

012

765

C_7

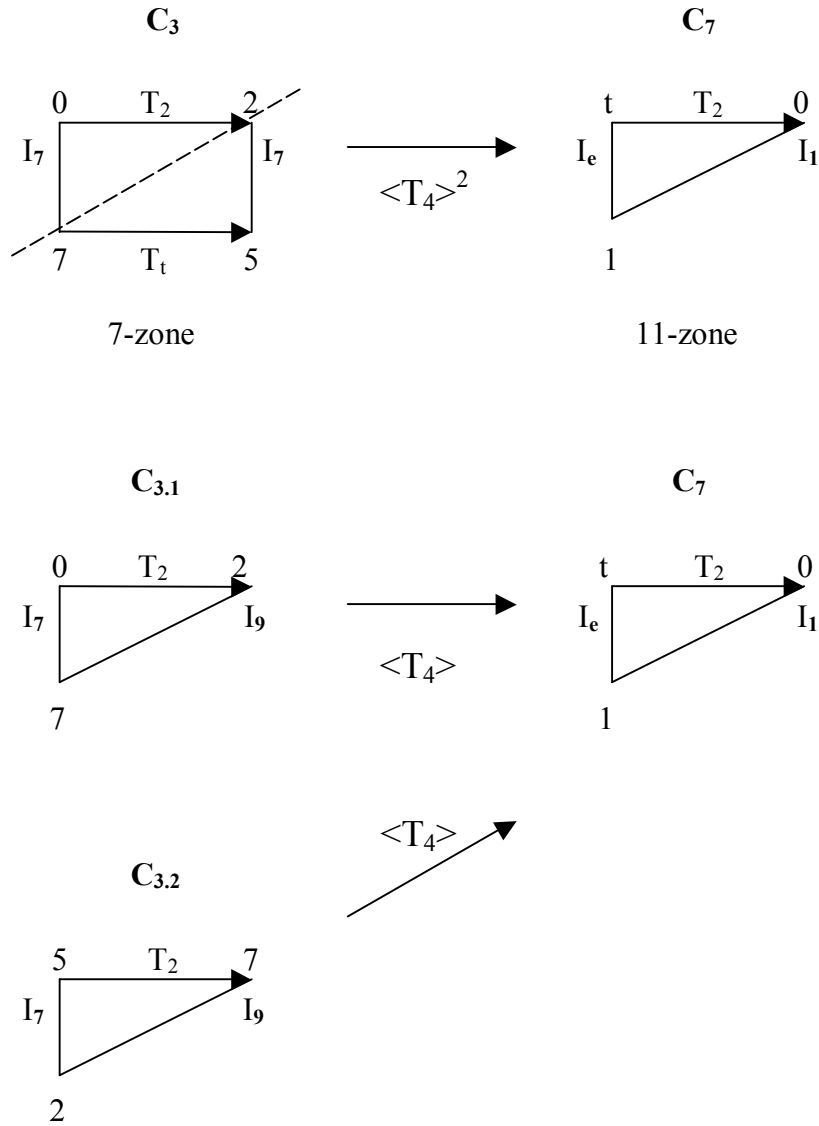
11-zone

te0

10e

We can see in Example 3.21 that the tetrachordal K-net of C_3 can transform to the trichordal K-net of C_7 although they are in different zones.

Example 3.21 First network interpretation of C_3 and C_7 in Heider's passage;
positively reductive isography at $\langle T_4 \rangle^2$

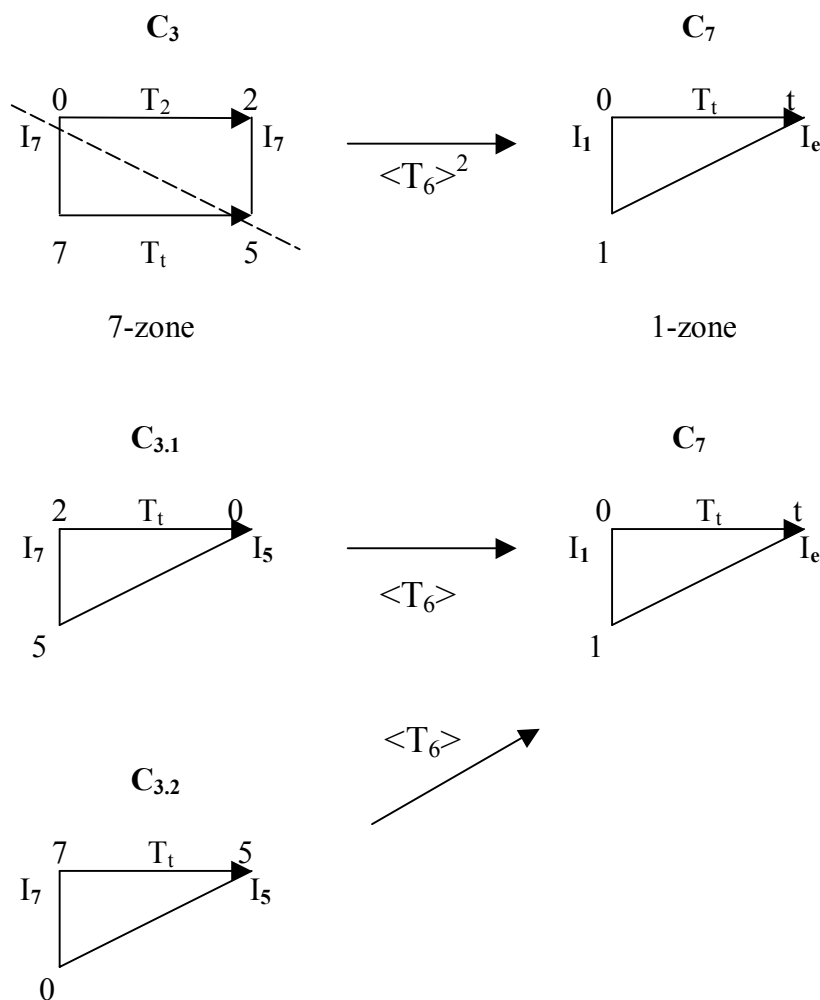


Note that both K-nets for the trichordal subsets of C_3 are positively isographic with the trichordal K-net of C_7 where the $\langle T_4 \rangle$ operations are indicated. Thus, the transformation between a trichordal K-net in an index-zone and a tetrachordal K-net in another index-zone with the sharing of the same T_n arrows among the trichord and the trichordal subsets, such as C_3 - C_7 shown in Example 3.21, can be defined as a *positively reductive*

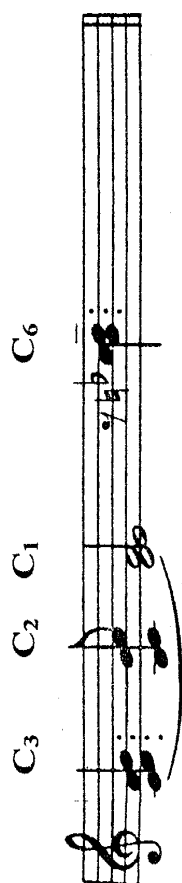
isography and labeled $\langle T_n \rangle^2$. That is, the tetrachordal K-net of C_3 transforms to the trichordal K-net of C_7 by $\langle T_4 \rangle^2$. For such a positively reductive isography, it must be emphasized that a tetrachord must be an inversional tetrachord, retaining the same I_n arrows with complementary T_n arrows, in order to have the same positively isographic relations between a pair of equivalent trichordal subsets and a trichord.

Meanwhile, Example 3.22 suggests another interpretation of the progression C_3 - C_7 in which the trichordal K-net of C_7 is placed in the 1-zone and contains a T_t arrow.

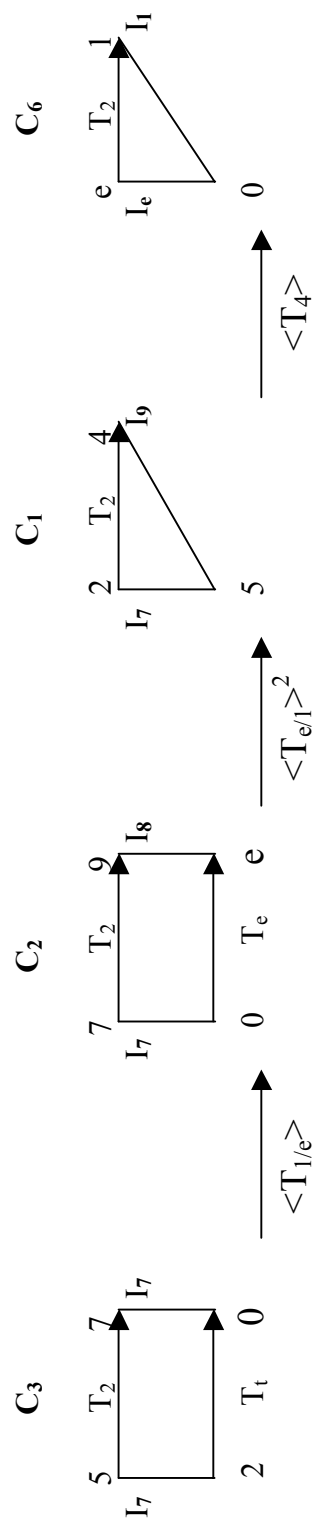
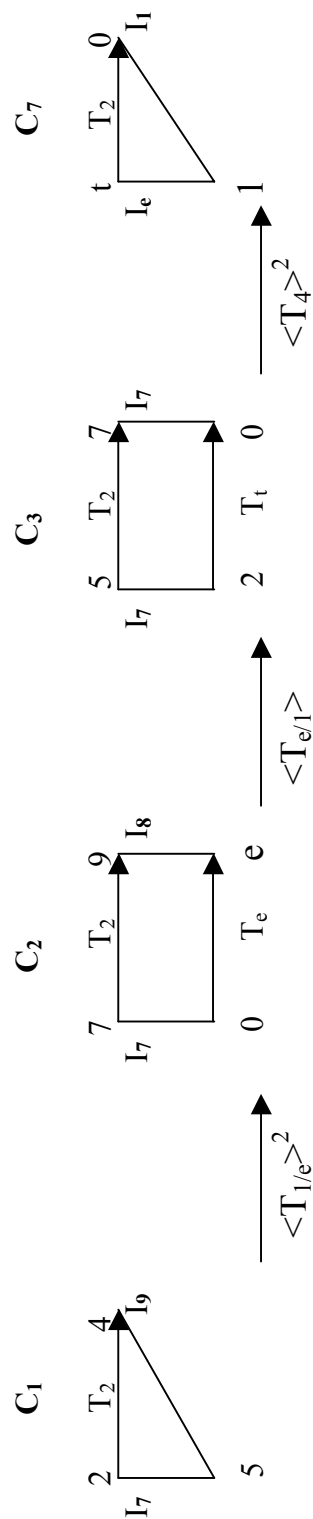
Example 3.22 Second network interpretation of C_3 and C_7 in Heider's passage; positively reductive isography at $\langle T_6 \rangle^2$



The example reveals the same $\langle T_6 \rangle$ relations both between the trichordal subset $C_{3,1}$ and the trichord C_7 and between another subset $C_{3,2}$ and C_7 . We thus see a $\langle T_6 \rangle^2$ transformation in this C_3 - C_7 progression. Comparing the two interpretations shown in Example 3.21 and 3.22, it is significant that the inversional zones, to which the trichordal and tetrachordal K-nets belong, need to be made more clear by taking their musical environments into consideration. For this passage in “*Kleinwelt*,” we can assume in both interpretations that the tetrachordal K-net of C_3 is to be placed in the 7-zone in relation with the previous chords while the index-zone of the isolated trichord, C_7 , can not easily be determined. However, the first interpretation, suggested in Example 3.21, would be made more persuasive by noticing the progression which follows and that it traces the same transformational path between index-zones. As an example, Example 3.23(a) and (b) copy from Example 3.8 and Example 3.19 the first and last four chords of the same part of “*Kleinwelt*.”

Example 3.23(a) Heider's "*Kleinwelt*," last four chords of part 3

Last four chords in part 3

**Example 3.23(b)** Heider's "*Kleinwelt*," first four chords of part 3

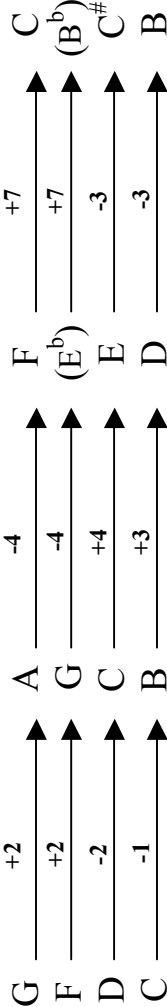
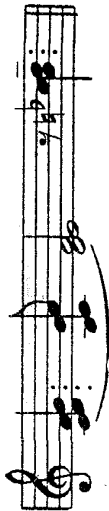
The temporal order of the first three chords in both progressions is now reversed.

Among the final four chords in Example 3.23(a), the trichordal K-net of C_1 is positively isographic with the network of a trichord, C_6 at $\langle T_4 \rangle$. As we see in both Examples 3.23(a) and (b), the first and the last four chords from this passage provide the same transformational path as a couplet, especially in their making an identical shift from zone 7 to 11, although the operations labeled $\langle T_n \rangle$ or $\langle T_{n/-n} \rangle$ replace the transformations labeled $\langle T_n \rangle^2$ or $\langle T_{n/-n} \rangle^2$ and vice versa. All chords in the example include a shared triangle of the same shape and size within the 7- or 11-zone. Thus, C_7 in the first four chords would necessarily be interpreted as a trichordal K-net which shares identical T_2 arrows with the previous chords in the 11-zone as suggested in the first interpretation of Example 3.21. Example 3.24 lays out voice leadings of the first and the last four chords given in Examples 3.23(a) and (b), in which the mappings coincide with the registral lines.

Example 3.24 Voice leadings of the first and the last four chords from Heider’s “*Kleinwelt*,” part 3

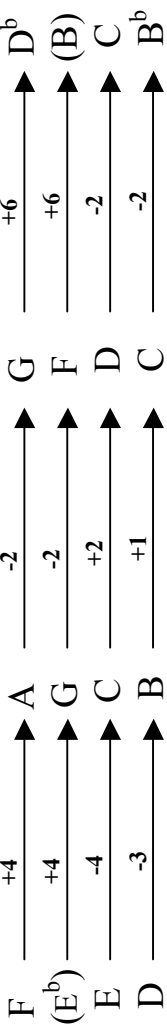
Last four chords in part 3

C₃ C₂ C₁ C₆



First four chords in part 3

C₁ C₂ C₃ C₇



In both voice-leading transformations, the motion from the first chord to the third chord display $\pm n/+m$ or $\pm n/-m$ voices. Note that the $\pm n$ in three voices produces type 2 axially reductive isography involving a trichordal subset of the same shape as an invariant within the same zone. That is, type 2 axially reductive isographies between a tetrachord and a trichord embrace wedging motion over a wider range. However, the voices from the penultimate chord to the last in both progressions break the wedging motion and produce a different inversional wedge; the upper and the lower dyads move up or down by different interval classes. The skewing of the wedge is represented by $\langle T_4 \rangle$ or $\langle T_4 \rangle^2$. Although both transformations may be less apparent to the ear because the second highest voices exclude some notes, it can not be denied that in both pitch and pitch-class space inversional motions are important aspects of this passage and that the notes which can only be imagined are implied in the music.

The problem of making choices among multiple K-net interpretations of a pc set and among many possible K-net isographies relating pc sets has been discussed in several places in the literature.¹⁰⁴ The axial isographies between tetrachords introduced in the previous chapter and also the transformations between three- and four-nodes examined above might allow a far wider range of possible interpretations of pc sets and visualization of a much broader network of relationships. The greater number of potential interpretations or greater width of network relationships, however, also suggests additional analytic possibilities for realizing or understanding significant musical features, especially inversional relations beyond the limits of traditional

¹⁰⁴ Lambert, "Isographies, and Some Klumpenhouwer Networks," 167. O'Donnell considers the problem as "promiscuity" in his "Klumpenhouwer Networks," 60-62. Buchler, "Reconsidering Klumpenhouwer Networks," 11-15.

isographies which highlight only one or two transpositional relations between the same nodes. In particular, in a number of pieces or sections which center around competing inversional axes, this range of possible transformations can help us grasp inversional properties without regard to the number of nodes. Even more significantly, when the inversional relations are evident on the surface, we can discern the salient features of inversional relations more effectively by means of the axial isographies and the axial reductive isographies between different nodes. That is, these isographies interpret chords as objects which shift among inversional zones and thus eventually reveal significant inversional properties through producing transformational paths among inversional axes. In these pieces or sections which have inversional properties as significant features, the axial isographies and the axial reductive isographies between trichordal and tetrachordal K-nets can help us easily realize both the inversional properties that are presented as surface events and also those concealed as background design. The following three analyses examine the ways in which the reductive isographies between trichords and tetrachords, in cooperation with other network isographies, can be used to illuminate the significant inversional relations in pieces or passages having specific wedging motions as a significant feature.

3-2. Bartók's *Dance Suite*, op.14 (1916)

Example 3.25, from the opening theme of Bartók's *Dance Suite*, op.14 (1916), is an example of a texture in which inversional relations are employed as a significant device for associating trichords and tetrachords.¹⁰⁵

¹⁰⁵ The *Dance Suite* was written originally as a work for orchestra in 1873; the example gives Bartók's transcription of the score for piano. Also, according to Bartók, the opening theme of this piece is his first "chromatic" melody. See the *Essays*, Edited by Benjamin Suchoff, New York: St. Martin's Press, 1976, 379.

Example 3.25 Bartók's Dance Suite, Op.14, mm.1-47

Moderato, $\text{♩} = 92$ 5

Piano

Measures 1-10: Moderato, $\text{♩} = 92$. Dynamics: *gliss.*, *pp*, *p*, *sf*, *mf*, *dim.*, *martellato*.

Measures 15-20: *p*.

Measures 25-30: *mf*, *f*.

Meno mosso, $\text{♩} = 84$ 35

Measures 31-35: *dim.*, *p*, *accel.*.

Measures 40: *al*, *cresc. molto*.

Sostenuto, $\text{♩} = 84$ 45

Measures 45-47: *f*, *sf*, *dim.*, *accelerando*, *al*.

As Bartók's music often employs an abundance of inversive tetrachords, the right-hand part of this section also provides inversionally symmetric four-note pitch sets; there are only two exceptions, 4-4 (0125) and 4-13 (0136).¹⁰⁶ Example 3.26 interprets the first inversive tetrachord, a member of set class 4-1 (0123), as two possible K-nets, containing either complementary T_n arrows with I_7 arrows or T_1/T_3 arrows with I_5/I_9 arrows, which are coincident with the registral and the temporal order.

Example 3.26 Two possible network interpretations of the first tetrachord in Bartók's Dance Suite, Op. 14

C_1
4-1(0123)

Moderato, $\text{♩} = 92$

7-zone

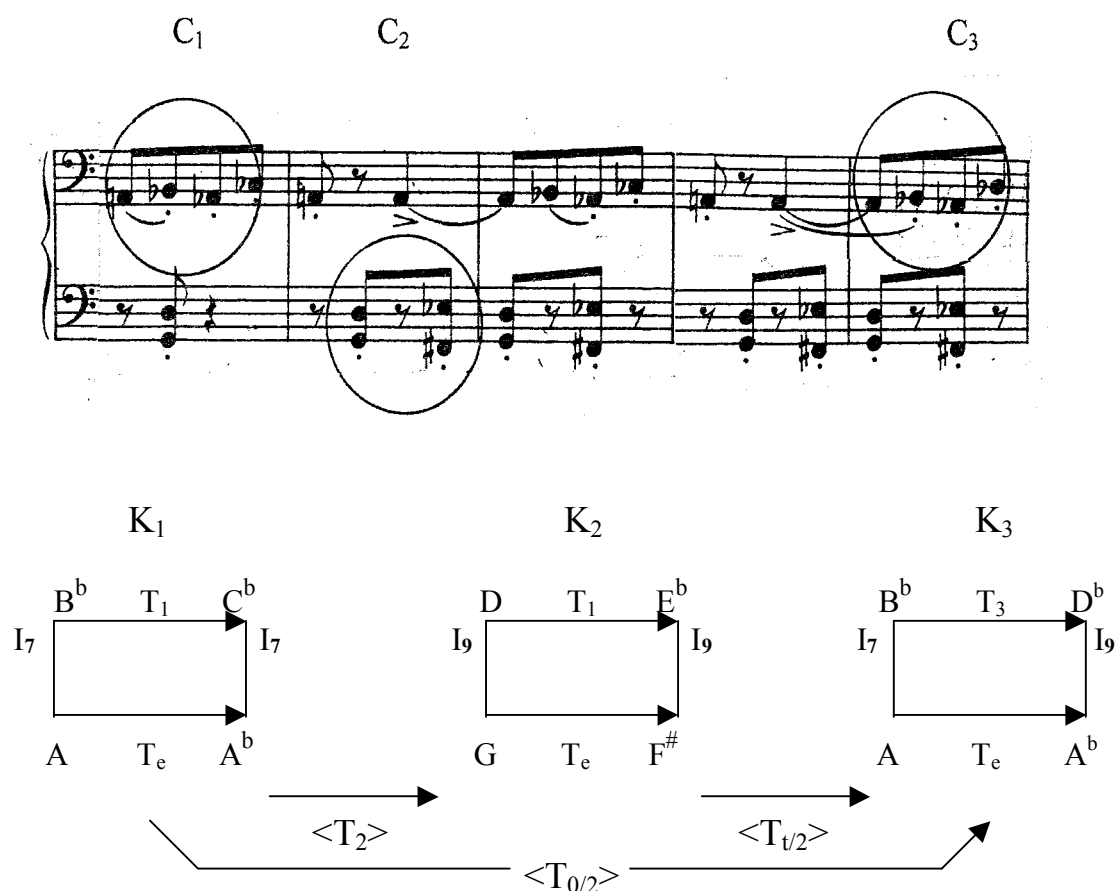
5-zone

Between the two network structures perceptible in the musical surface, the first interpretation seems more plausible since it brings out a specific inversive relation, I_7 , primarily focusing on the inversive properties which have significant force in this section. Thus, these opening two inversive tetrachords, C_1 in the right-hand part and

¹⁰⁶ Antokoletz called such inversive tetrachords as 4-1 (0123), 4-21 (0246), 4-9 (0167) “X”, “Y”, and “Z” cells in his *The Music of Béla Bartók*, 69-71. Lendvai also calls those such as 4-3 (0134) and 4-10 (0235) “gamma” chords in *The Workshop of Bartók and Kodály*, (Musica Budapest, 1983), 353.

another inversiveal chord C_2 in the left-hand part, are interpreted as K-nets, K_1 and K_2 , each of which contains a single I_n arrow with complementary T_n arrows as shown in Example 3.27.

Example 3.27 Positive and axial isography between tetrachords in Bartók's Op.14, mm.3-7

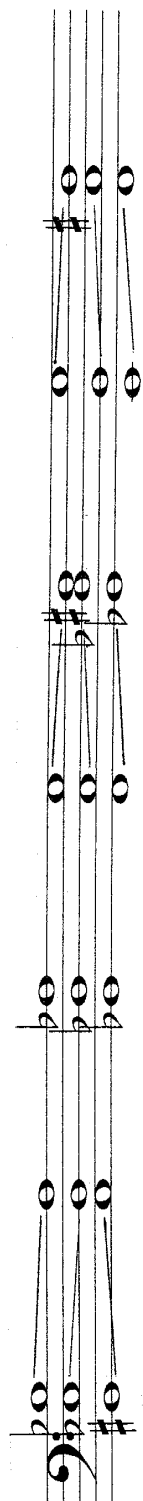


The network interpretation of C_3 continues this pattern but retains I_7/I_9 arrows with T_3/T_e arrows. Note that the network interpretation of C_1 is positively isographic with the network of C_2 at $\langle T_2 \rangle$ and the network of C_2 has type 2 axial isography sharing a trichordal subset with the network of C_3 at $\langle T_{t/2} \rangle$. In addition, K_1 has type 1 axial isography with K_3 at $\langle T_{0/2} \rangle$. On the other hand, from m.8, the left-hand part presents

contiguous homorhythmic trichords, C₄- C₁₀, sharing ic 5 as an ostinato figure. In the musical surface, we can perceive the wedging motion between each pair of trichords, especially C₄ and C₅, C₇ and C₈, and C₉ and C₁₀, by listening for the parallel T₅s of a dyad with the contrary motion of another note as shown in Example 3.28(a).

Example 3.28(a) Bartók, Op. 14, left-hand part in mm. 9-45

m. 8-12	m. 8-12	m. 13	m. 14-22	m. 14-22	m. 24-44	m. 24-44
C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
3-9(027)	2-5(05)	3-9(027)	3-9(027)	2-5(05)	3-11(037)	3-5(016)



That is, we can see in the example that the trichords in the left-hand part all contain ic 5 dyads and also that they are members of five different set-classes which are strongly and positively isographic. Example 3.28(b) models the seven trichords of the left-hand part as isographic K-nets, K_4 - K_{10} .

In the progression K_6 - K_7 , the network interpretation does not coincide with registral lines, but in the subsequent trichords, K_7 - K_{10} , noting the consistent ic 5 dyads between the lower two notes makes this interpretation more compelling since the specific wedge motions from C_7 occur between the upper two notes. That is, K_6 is a link which not only connects contiguous networks belonging to different inversional zones but also alters the registral disposition of the specific wedge motions from the lower two notes to the upper two notes. In Example 3.28(b), the progression as a whole involves the shift from the 6-zone to the 7-zone and then back to the 6-zone.

On the other hand, the right-hand part in mm.14-44 is conspicuously comprised of six tetrachords and two trichords as shown in Example 3.29(a).

Example 3.29(a) Bartók's op.14, right-hand part in mm.15-44

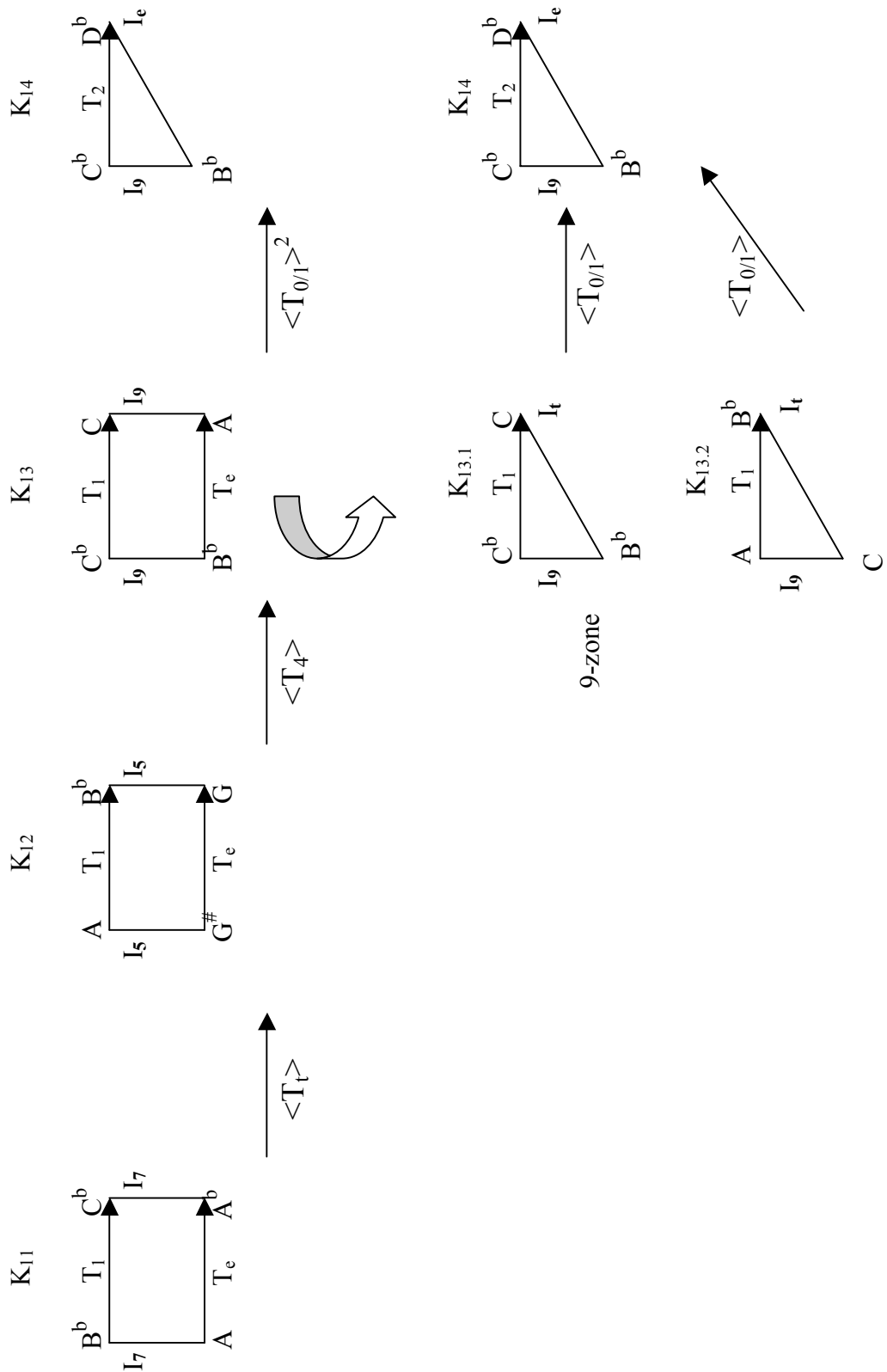
m.15 $C_{11}(=C_1), 4-1(0123)$ T_1 T_e
 m.24-25 $C_{12}, 4-1(0123)$ T_1 T_e
 m.36 $C_{14}, 3-2(013)$ T_2 T_2
 m.37-38 $C_{15}, 3-1(012)$ T_t T_t
 m.42-44 $C_{16}, 4-10(0235)$ T_t T_2
 m.44 $C_{17}, 4-3(0134)$ $C_{18}, 4-26(0358)$ T_{31}

Example 3.29(b) Bartók, op.14, transpositional property in the right-hand part of mm.15-44

C_{11} C_{13} T_1 C_{17} T_4 T_3 T_1

In the progression C_{16} - C_{18} , each tetrachord contains members of different set-classes while the tetrachords C_{11} , C_{12} and C_{13} are members of the same set-class 4-1 (0123). Among the eight chords, C_{11} - C_{18} , the six inversional tetrachords can be articulated into complementary T_n arrows yielding single I_n arrows while the two trichords have different T_n arrows. The voice leading implied by the isographies coincides with registral lines although some voices contrast in metric placement. For this passage, one may claim that a pair of dyads in m.15 simply transposes that of m.33 upward by a semitone and then transposes C_{17} of m.44 upward again by a minor and a major third as displayed in Example 3.29(b). However, we can also hear the regularly increasing transpositional property from the semitone of m.15 to the minor third of m.44, each moving around different axes as shown in Example 3.29(a). The following examples will show how this entire passage can be structured about some inversional wedges. We can see in Example 3.30 that K_{11} , K_{12} , and K_{13} all contain identical $T_{\pm 1}$ arrows while the analogous I_n arrows vary; thus they can be related by positive isography.

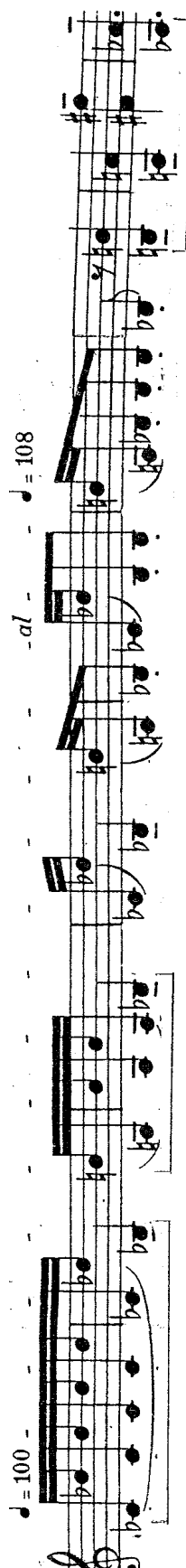
Example 3.30 Positive and axially reductive isography of network interpretations of C_{11} to C_{14} in mm.15-36



Notice that the network interpretations of C_{13} and C_{14} retain the same inversional symmetry around I_9 despite their different nodes. The tetrachordal K-net of C_{13} can thus have axially reductive isography with the subsequent trichordal K-net of C_{14} since both trichordal subsets of K_{13} and trichordal K_{14} are axially isographic. That is, as Example 3.30 displays, the trichordal subsets of K_{13} which share identical graphs can map by means of the shared I_9 arrow onto the trichordal K-net, K_{14} , by the same relationship, $\langle T_{0/1} \rangle$. Thus, the transformation between tetrachordal K_{13} and trichordal K_{14} made up of differing T_n arrows within the same zone is reflected by $\langle T_{0/1} \rangle^2$. In the following progression, K_{14} - K_{15} , the two trichords, members of set classes 3-1 (012) and 3-2 (013), can be interpreted in multiple ways. Example 3.31, however, shows that in temporal order the trichordal K-nets, K_{14} and K_{15} , show axial isography by sharing an I_9 arrow connecting the same B (C^b) and B^b .

Example 3.31 Axial and positive reductive isography in Bartók's op.14, right-hand part in mm.36-44

$\text{♩} = 100$



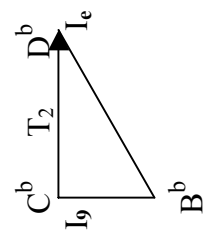
$\text{♩} = 108$

C_{14}

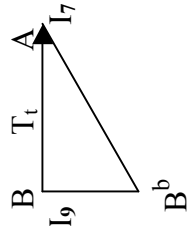
C_{15}

C_{16}

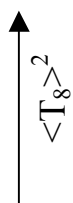
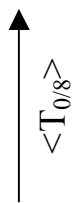
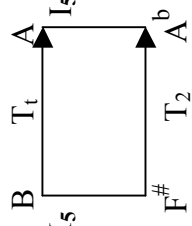
C_{14}
 K_{14}



C_{15}
 K_{15}



C_{16}
 K_{16}



Zone 9

The eight different semitones between the T_n arrows are reflected by $\langle T_{0/8} \rangle$. In addition, the progression C_{15} - C_{16} provides another transformation between different nodes. The tetrachordal and trichordal K-nets are positively transformed into each other. That is, the trichordal subsets of K_{16} and the trichordal K_{15} share identical T_t arrows and all the I-numbers of the trichordal subsets of K_{16} increase by 8 over the I-numbers of the trichordal K_{15} . Since the trichordal K_{15} is positively isographic with the trichordal subsets of K_{16} by means of $\langle T_8 \rangle$ relations, we can derive the transformation $\langle T_8 \rangle^2$ as the pertinent mapping of K_{15} to K_{16} . Example 3.32 interprets the last three tetrachords of the right-hand part as the K-nets, K_{16} to K_{18} .

Example 3.32 Positive and strong isography in Bartók's op.14, right-hand part in mm.42-44

Sostenuto, $\text{♩} = 84$

K_{16}
 K_{17}
 K_{18}

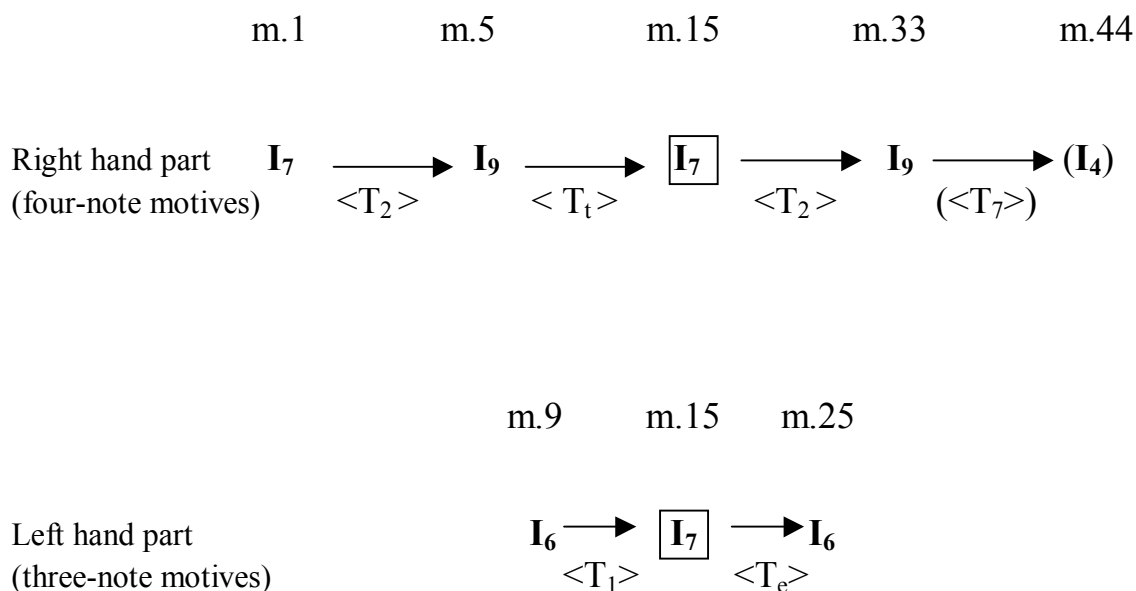
$\xrightarrow{\langle T_e \rangle}$
 $\xrightarrow{\langle T_0 \rangle}$

Zone 4 -----

The tetrachords share $T_{\pm 3}$ arrows and the network K_{16} shown in Example 3.31 now contains $T_{\pm 3}$ arrows through the exchange of t-related dyadic partners while still having the same inversive relations. Note that networks K_{16} and K_{17} are positively isographic at $\langle T_e \rangle$ and network K_{17} has a strong isography with the network K_{18} . As we see in Example 3.31 and Example 3.32, the tetrachordal K_{16} here functions as a link connecting the adjacent K-nets, K_{15} and K_{17} , belonging to different zones as well as having different numbers of nodes.

Example 3.33 shows the overall transformational paths of both right- and left-hand parts, and the right-hand part of this section includes the strongly suggested four-note motive of C_2 by way of exception though it is presented in the left-hand part.

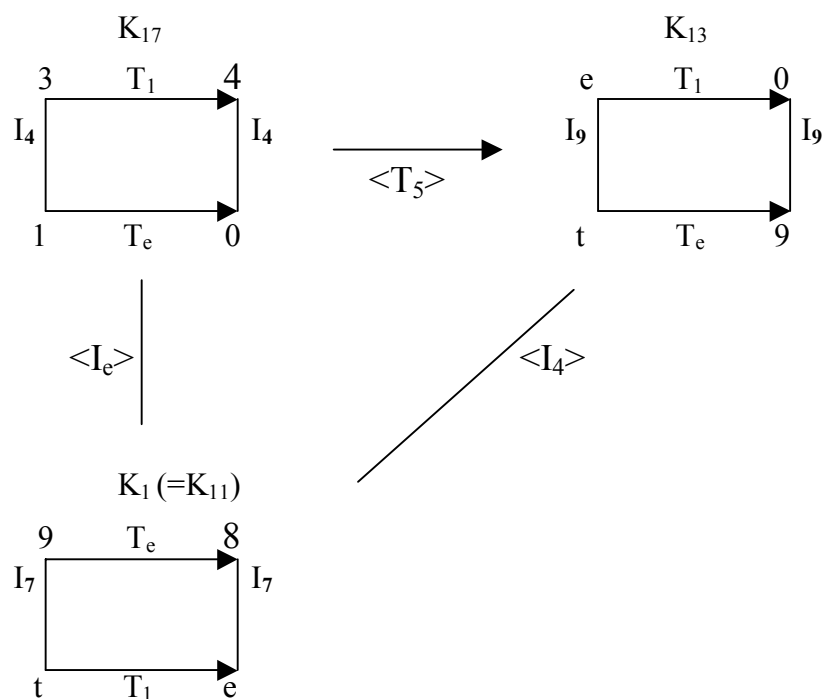
Example 3.33 Summary of the index-zones in Bartók's op.14, mm.1-44



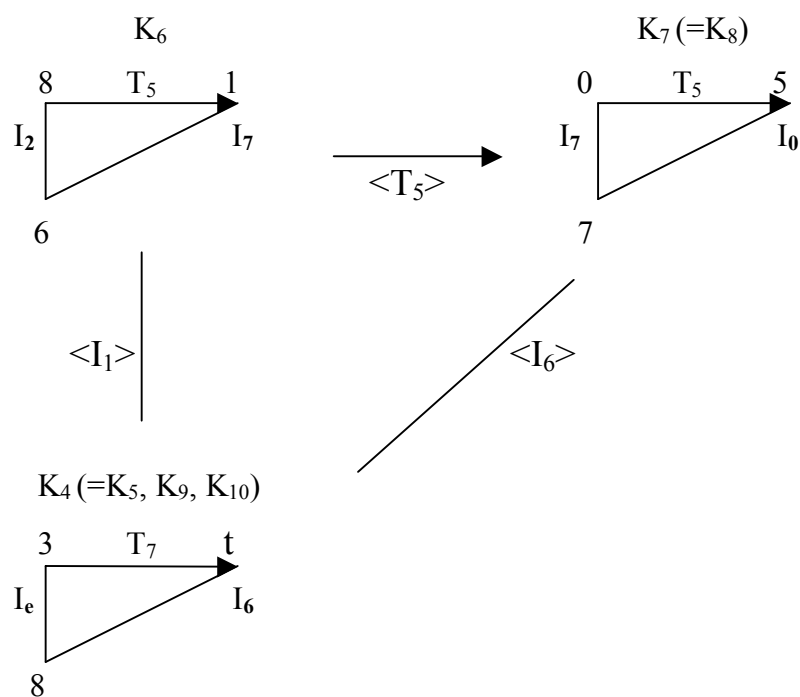
As we found in the previous examples, the index zones, 7, 9, and 4, are extended by the repetition of inversional tetrachords or by axially reductive isographies and axial isographies within the same zone. Thus, we can see the transformations $\langle T_2 \rangle$, $\langle T_4 \rangle$, and $\langle T_7 \rangle$ among index zones in Example 3.33. In addition, it is interesting in the example that the transformational paths in the parts for both hands present the same inversional axis for zone 7 in the center. In other words, the tetrachords of the right hand and the trichords of the left hand can be transformed into each other within the same zone by emphasizing the central 7-zone, which is defined by the initial tetrachordal motive. There would be a perfect symmetrical disposition in both parts, if the final tetrachords in the right-hand part established the 7-zone instead of the 4-zone. The last 4-zone, which does not fit into the overall transformational progression with its symmetric layout, however, suggests a possible relation between tetrachordal and trichordal aggregates. That is, Example 3.34 organizes each tetrachordal K-net from the 4-, 7-, and 9-zones within a node of a 2+1 super-network labeled S_1 .

Example 3.34 Bartók's Dance Suite, op.14, mm.1-44; recursion of the K-nets

Right-hand part $\boxed{S_1}$



Left-hand part $\boxed{S_2}$



Here, each tetrachordal K-net has complementary $T_{\pm 1}$ arrows and a single I_n arrow. This example thus illustrates that the super-network S_1 comprised of the $\langle T_5 \rangle$ relation is positively isographic to the individual trichordal K-nets of the left-hand part as well as the super-network S_2 , incorporating all the trichords of the left-hand part. Consequently, the transformations between a trichordal K-net and a tetrachordal K-net in this analysis make possible the generating of a transformational path despite the employment of different nodes for the network interpretations. In addition, these ultimately serve to unify this section as a whole by providing larger scale structures which incorporate all the aggregates of three- and four-note motives.

3-3. Arnold Schoenberg's *Das Buch der hängenden Gärten*, op.15, no.1.

We can take a look at another excerpt, from Schoenberg's *Das Buch der hängenden Gärten*, op.15, no.1, to examine the analytic utility of using strongly, positively, and axially reductive isographies between tetrachords and trichords. Furthermore, I will demonstrate how K-nets, used with special regard for these reductive isographies can reveal a background inversional design by tracing the transformational paths among index-zones. Note that throughout the entire piece, inversional properties are employed as a significant compositional strategy. The piece consists of three formal sections in an A-B-A' structure (9+6+8) and provides a predominantly two-voice texture although the opening eight measures provide an organization which is not clearly stratified into two contrasting textures. In this piece,

the left-hand part presents a prevalent inversive relation primarily among tetrachords while the right-hand part suggests a specific inversive relation only among trichords.

An annotated score of the first section, mm.1-8, is shown in Example 3.35.

Example 3.35 Arnold Schoenberg's *Das Buch der hängenden Gärten*, op.15, no.1, mm.1-8

Right-hand part

Left-hand part

S_1 S_2

b_1 b_2

4-2 (0124) 3-3 (014)

S_3

Right-hand part

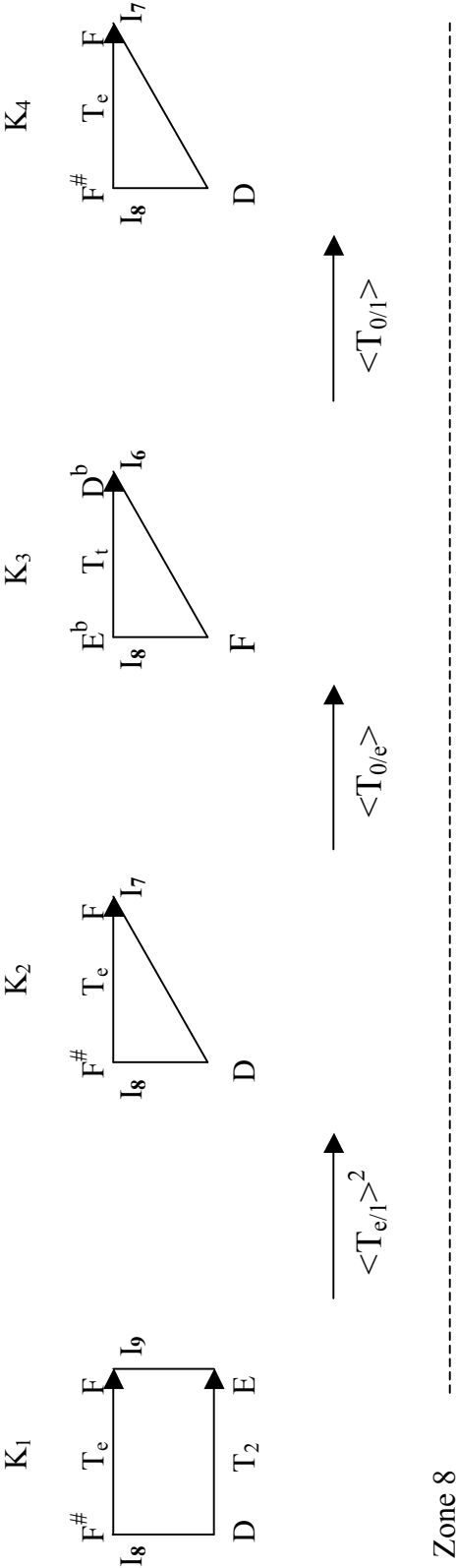
Left-hand part

b_3 b_4

3-6 (024) 3-3 (014)

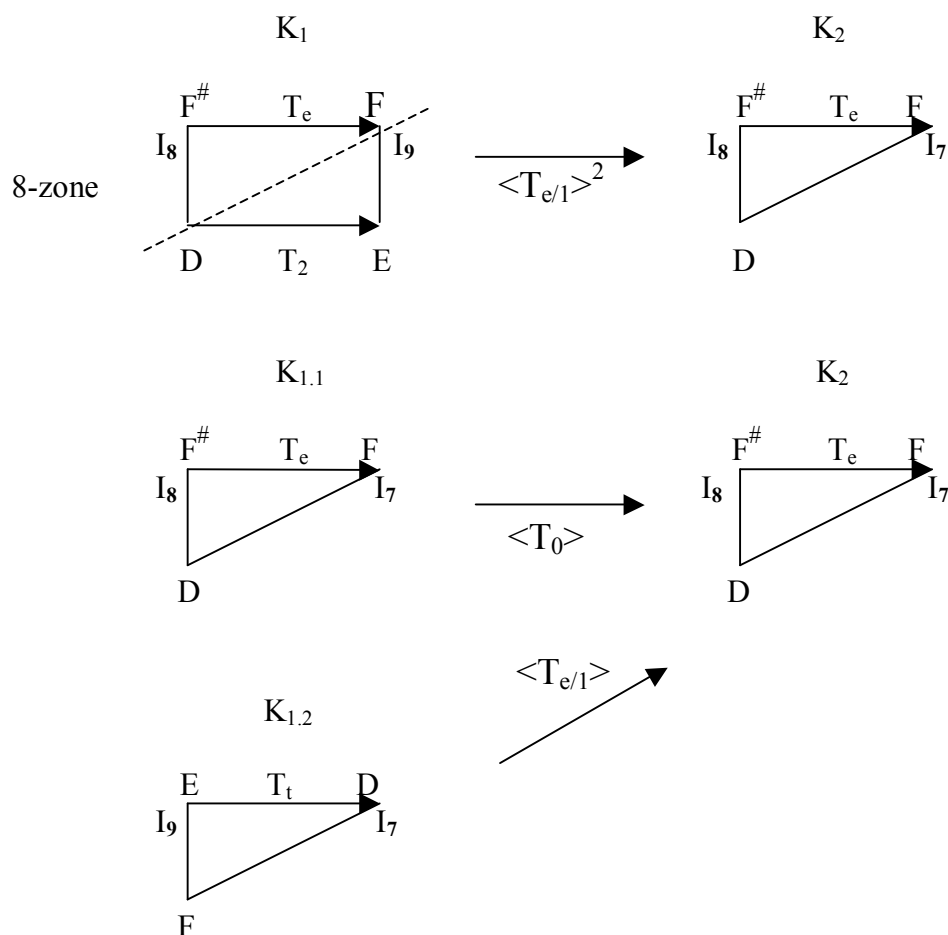
The piano accompaniment can be stratified into two textures: in each of four phrases, the first three- or four-notes alternate as a strand of the left-hand part with the last three-notes as a strand of the right-hand part. The left-hand part involves four chords, labeled b_1 - b_4 in Example 3.35, which are members of three different set-classes: 4-2 (0124), 3-3 (014), and 3-6 (024). Example 3.36 interprets the four chords as K-nets, K_1 - K_4 .

Example 3.36 Axial and axial reductive isography among the K-nets of the left-hand part in mm.1-8



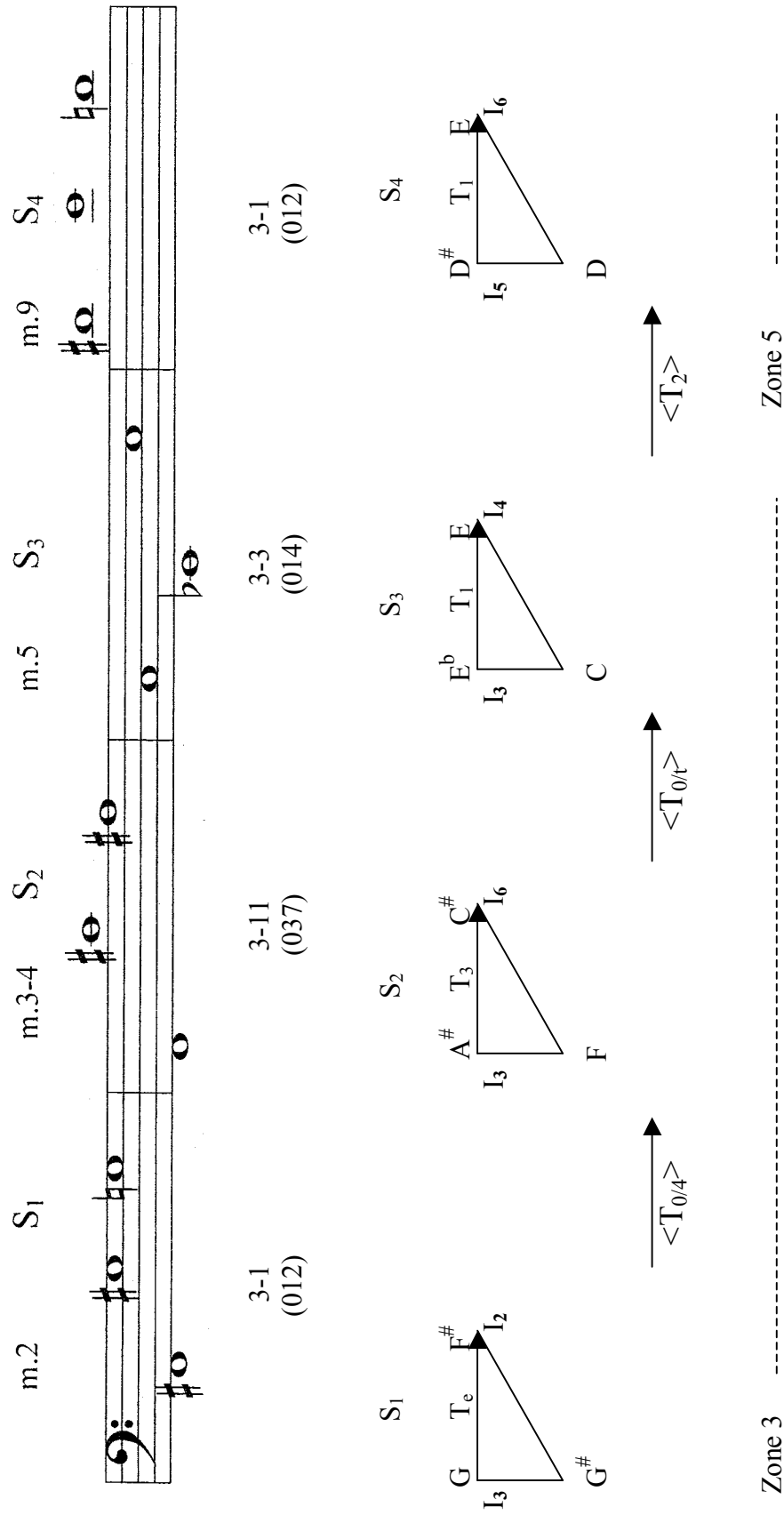
The musical surface supports this interpretation by its temporal order. That is, in each chord of b_1 - b_4 , the first two notes produce the I_8 arrows while the first and third notes generate the T_n arrows, either T_e or T_t . The tetrachordal K_1 and trichordal K_2 share identical I_8 and T_e arrows. We can see in Example 3.37 that the trichordal subsets of b_1 and trichord b_2 have strong and axial isography and thus can be represented by $\langle T_{e/1} \rangle^2$ that indicates the transformation between tetrachordal K_1 and trichordal K_2 sharing K_2 itself as an invariant while another trichordal subset of K_1 and trichordal K_2 are related by $\langle T_{e/1} \rangle$.

Example 3.37 Axially reductive isography; the trichordal subsets of K_1 are strong and axial isographic with trichordal K_2



In Example 3.36, the following network interpretations of trichords, K_2 - K_4 , are axially isographic with each other sharing I_8 arrows. In contrast, the right-hand part presents four trichords, labeled S_1 - S_4 in Example 3.38. Below the music, each trichord is interpreted as a K-net.

Example 3.38 Axial and positive isography among K-nets S_1 - S_4 of the right-hand part in mm.2-9



Since no shared interval class exists for the first two chords, members of set classes 3-1 (012) and 3-11 (037), the network interpretations can not be traditionally isographic. There is, however, an inversive motion; the $G^\sharp$ -G dyad wedges out the I_3 dyad, F- $A^\sharp$. Here, it does not matter which transpositional relation is chosen for the network interpretations since any interpretative choice does not affect the overall transformational path. In any case, they can be axially isographic sharing identical I_3 arrows even though the network interpretations do not correspond to the musical surface. The network interpretations of S_3 and S_4 share a T_1 arrow drawn between the same pitch classes, $D^\sharp$ (E^b) and E, though differently articulated in the musical surface. Thus, the K-nets S_3 and S_4 are positively isography at $\langle T_2 \rangle$. In fact, the reasons for these interpretations are twofold. Only these network interpretations can include all four trichords by suggesting the I_3 relation as a prevailing inversive axis that is also prominent in the subsequent sections. This interpretation of the first section may also anticipate the transformational pathway of the section which follows it by providing identical shifts among index-zones.

We can now see how the next section traces the same transformational pathway among inversive zones as does the first section. Example 3.39 and 3.40 include the music and relevant K-nets that interpret the tetrachords and trichords of the right-hand part in mm.11-15. It is interesting that the right-hand part is again divided into two strands in mm.11-12: a strand involving only trichordal K-nets centering the 3-zone and another embracing trichords and tetrachords mainly in the 5-zone as indicated in Example 3.39. Beginning with the upper clef of mm.11-13, five different chords, labeled s_5 - s_9 , are interpreted as K-nets in Example 3.40(a).

Example 3.39 Schoenberg, op. 15, no. 1; two strands in mm. 11-13

m. 11 I_3 (trichord) ——— $\{I_5$ (tetrachord) ———

rit. - - - *flüchtig*

pp espress.

pp

I_3 (trichord) ———

m. 13

I_5 (tetrachord) } I_3 (trichord) ———

pp

Left-hand part

Example 3.40(a) Schoenberg, op.15, no.1, the first strand of the right-hand part in mm.11-13

rit. - - - flüchtig

pp espress.

S₅

B T₁ C

I₃ I₄

E

S₆

E^b T₁ E

I₅ I₆

D

S₇

C[#] T₁ D

I₅ I_e

E T₅ A

S₈

F T₁ F[#]

I₀ I₆

G T₅ C

S₉

C[#] T₁ D

I₅ I_e

E T₅ A

→ <T₂>

→ <T_{6/6}>²

→ <T₇>

→ <T₅>

Zone 3 ----- Zone 5 -----

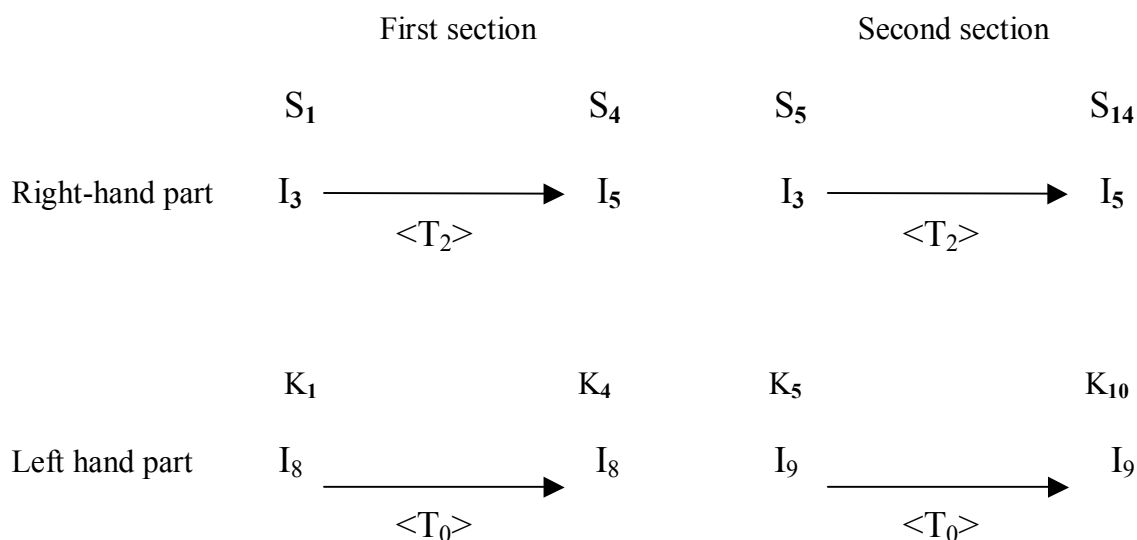
On the musical surface, these chords do not seem to emphasize any specific inversive axis but instead stress particular transpositional relations, T_1 or T_5 . Example 3.40(a) illustrates that the K-nets for S_5 and S_6 share a T_1 arrow and thus exhibit the $\langle T_2 \rangle$ operation. Here the trichordal S_6 is interpreted as having the same graph as S_4 suggested in Example 3.38 since the trichord S_4 recurs as S_6 in a higher register. The tetrachords which follow, S_7 - S_9 , can be articulated into a dyad of ic 1 plus a dyad of ic 5. The transformations $\langle T_7 \rangle$ and $\langle T_5 \rangle$ are next, transforming S_7 into the positively isographic networks S_8 and S_9 . That is, S_7 reappears as S_9 but in a different temporal order. Note that in the progression S_6 - S_7 the transformation between trichordal S_6 and tetrachordal S_7 is denoted by $\langle T_{6/6} \rangle^2$ since the network interpretation of S_7 within the 5-zone not only involves the trichordal S_6 itself as a subset but also has another trichordal subset containing a new I_e arrow. The left numerical subscript within the angle brackets, 6, represents the difference between the I_5 arrow of S_6 and the I_e arrow of S_7 while the right numerical subscript is always a complement of the left one. On the other hand, we can consider the K-nets from S_6 to S_9 , which extend the 5-zone, to be embellishing chords in mm.11-13. That is, as previously shown in Example 3.39, K-nets S_6 - S_9 are momentarily suggested between trichordal aggregates, which support only the prevalent inversive relation, I_3 . As the second strand, the trichordal aggregates in mm.11-14 present four different set classes represented by five different trichords, $S_{10}(=S_5)$ - S_{14} as displayed in Example 3.40(b). We can see in this example that the networks, S_{10} - S_{13} , have axial and strong isographies within the 3-zone. Unlike the first strand shown in Example 3.40(a), we can easily hear the I_3 dyads since they are realized by the mirrored voice-leading motion of the outer voices. Notice that the final chord S_{14} is positively

isographic with S_{13} at the $\langle T_2 \rangle$ operation, which now emerges as the significant transformation already exhibited in the right-hand part of the first section.

In another interpretation, the left-hand part of this section begins with an inversional tetrachord containing an I_9 arrow with the complementary $T_{\pm 2}$ arrows in m. 10; then it resumes with the same tetrachordal K-net which involves I_9 and $T_{\pm 2}$ arrows in m. 13 after a two-measure interruption of the right-hand part. Example 3.41 shows the six tetrachords of the left-hand part, K_5 - K_{10} , as members of the same set class 4-23 (0257).

We can see in this example that each tetrachordal K-net has identical $T_{\pm 2}$ arrows and a single I_n arrow. The example thus notates the series of transformations $\langle T_8 \rangle$, $\langle T_6 \rangle$, $\langle T_2 \rangle$, $\langle T_4 \rangle$ and $\langle T_4 \rangle$ under positive isographies. That is, the left-hand part in this section establishes the 9-zone with the first two chords and returns to the same zone in the final tetrachord. Example 3.42 illustrates that the two strands of the second section eventually provide transformations among index zones identical with those of the first section, precisely the $\langle T_2 \rangle$ relation in the right-hand part and the $\langle T_0 \rangle$ relation in the left-hand part.

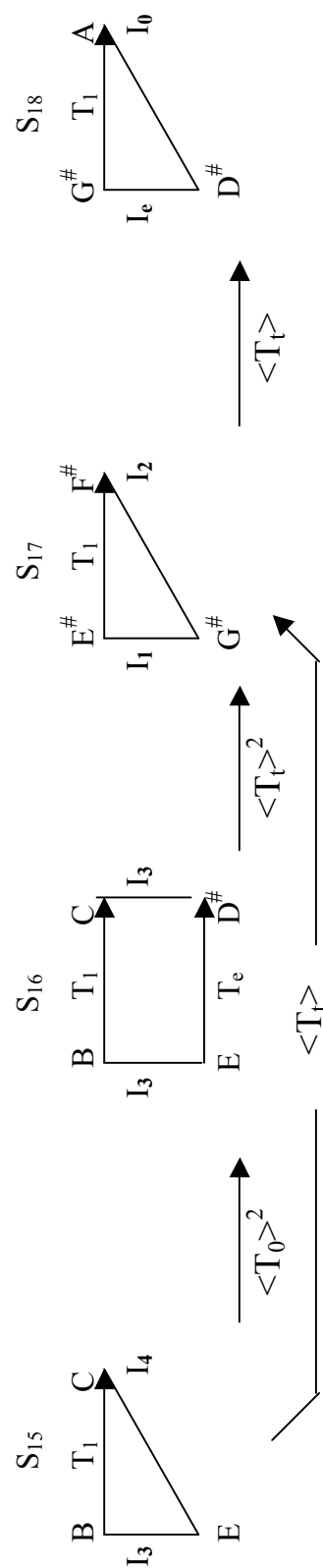
Example 3.42 Summary of the index-zones in Schoenberg op.15, no.1, the first and second sections.



The final section of this piece, mm.17-23, preserves the prevailing inversive relations of the previous section, I_3 for the right-hand part and I_9 for the left-hand part. For the right-hand part, Example 3.43 interprets the four chords of the right-hand part as K-nets, S_{15} - S_{18} .

Example 3.43 Positive, positively reductive and strongly reductive isography among K-nets S_{15} - b_{18} of the right-hand part in mm. 17-23

The musical score consists of two staves. The upper staff is labeled with S_{15} 3-3 (014) and S_{16} 4-7 (0145). The lower staff is labeled with S_{17} 3-2 (013) and S_{18} 3-5 (016). The score includes various musical notations such as notes, rests, and dynamic markings like sf and p .



This example also illustrates that the two strands of the right- and the left-hand parts are reversed in register in the final three measures. In Example 3.43 one can easily hear the strongly reductive isography between S_{15} and S_{16} since the trichordal S_{15} transforms into the tetrachordal S_{16} by simply adding the $D^{\#}_4$ to supplement another I_3 arrow. The $\langle T_0 \rangle^2$ transformation between trichordal S_{15} and tetrachordal S_{16} shows that the trichordal S_{15} and the trichordal subsets of S_{16} can have the same graphs within the 3-zone demonstrating strong isography. The inversional tetrachordal S_{16} , which retains $T_{\pm 1}$ arrows within the 3-zone, can also transform into the trichordal S_{17} retaining a T_1 arrow in another zone though the T_1 arrow is not apparent in the musical surface. That is, for the network S_{17} the highest notes of this piece, alternating $G^{\#}_5$ and $E^{\#}_4$, are realized as an inversional relation, I_1 , rather than as an unfamiliar T_3 to other adjacent chords. We can thus derive the transformation $\langle T_t \rangle^2$ from the progression S_{16} - S_{17} and also $\langle T_t \rangle$ as the pertinent mapping of S_{17} - S_{18} . As a result, the right-hand part in this section finally reveals two $\langle T_t \rangle$ relations among three trichordal K-nets each containing a T_1 arrow as shown in Example 3.43.

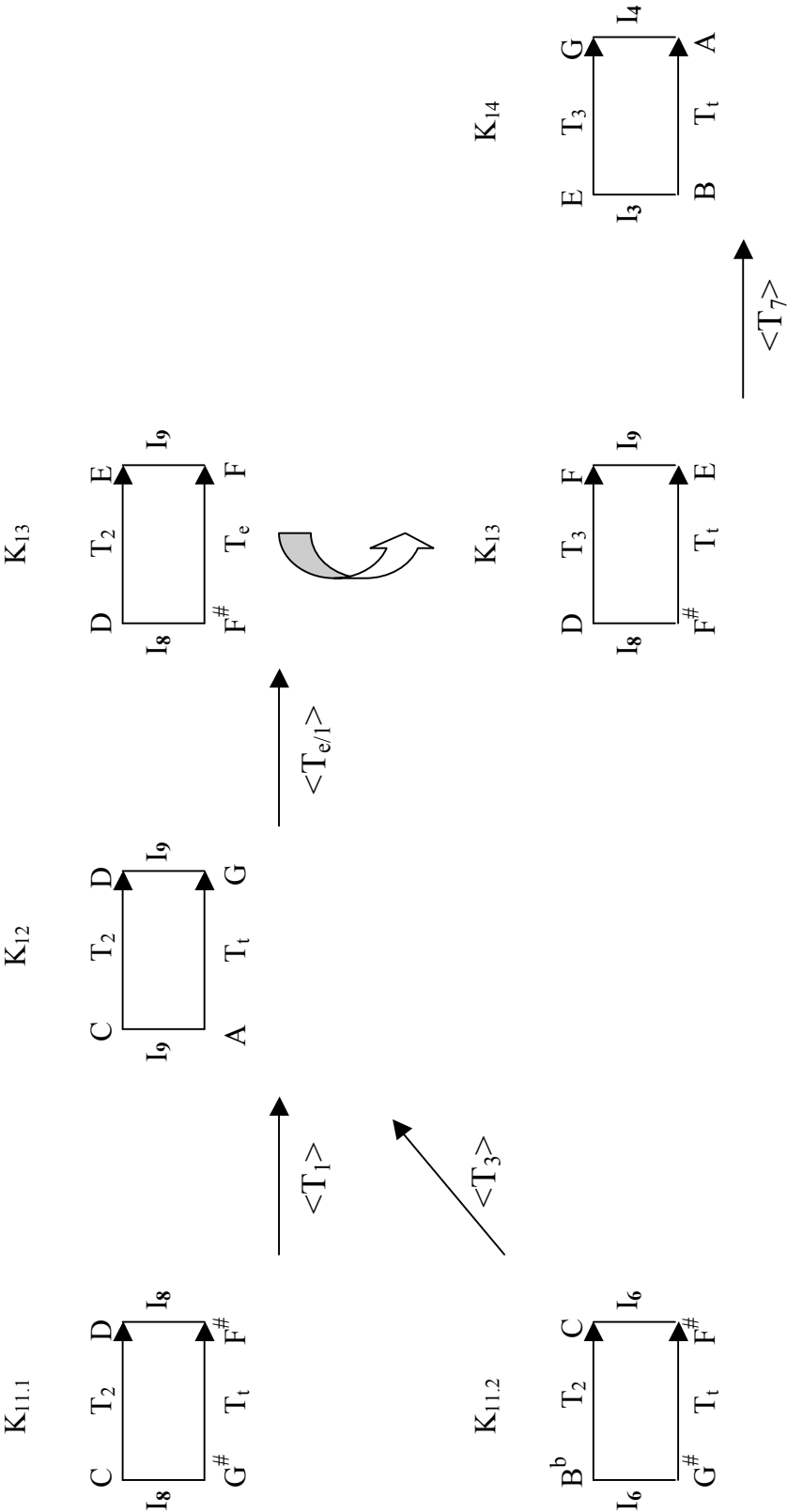
The left-hand part, however, presents four chords labeled b_{11} - b_{14} as Example 3.44(a) displays.

Example 3.44(a) Schoenberg, op.15, no.1, left-hand part in mm. 17-23

b_{11}
 δ
flüchtig
f
 b_{12}

b_{14}
sf
p
 b_{13}

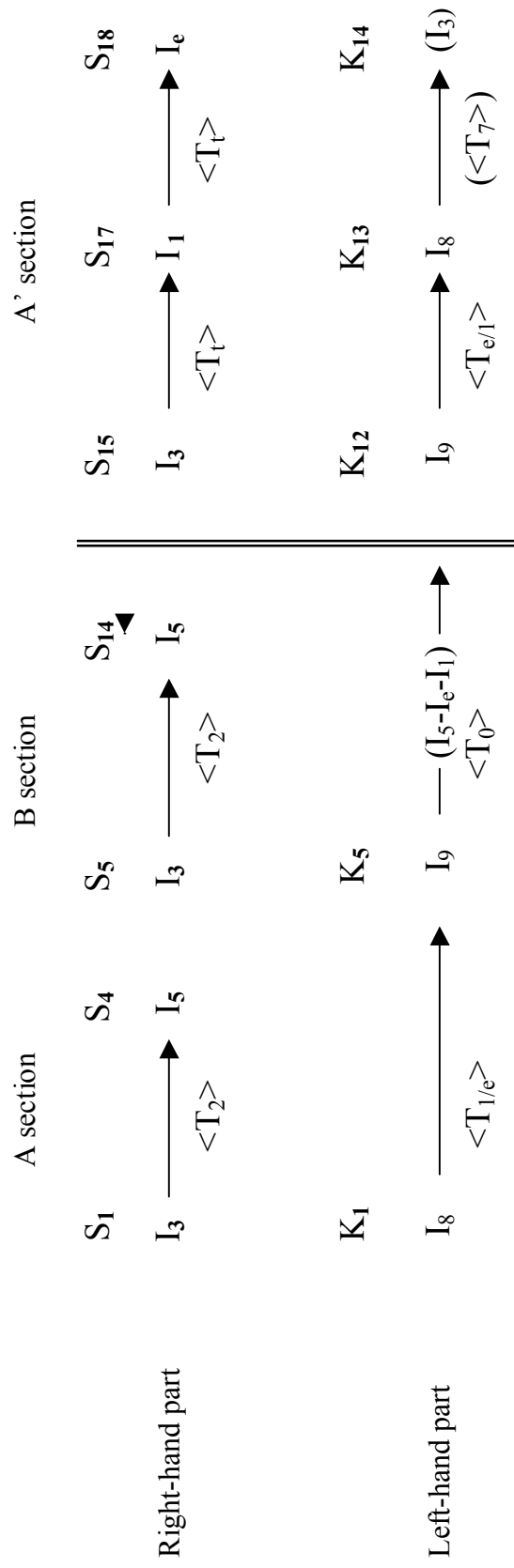
Example 3.44(b) Positive and axial isography among K-nets S_{15} - b_{18} of the left-hand part in mm. 17-23



The first pentachord b_{11} , a member of set class 5-33 (02468), can yield two tetrachordal K-nets retaining complementary $T_{\pm 2}$ arrows. Example 3.44(b) shows the tetrachordal K-nets for b_{11} , $K_{11.1}$ and $K_{11.2}$, each belonging to the 8-zone and 6-zone. In fact, it does not matter which interpretation is chosen between $K_{11.1}$ and $K_{11.2}$ since the chord can be considered an embellishing gesture which introduces the subsequent prominent tetrachords which already appeared in the first section. That is, any interpretative choice for b_{11} does not affect the overall transformational path. Example 3.44 also illustrates that the K-net K_{12} is the type 2 axially isographic with K_{13} since both tetrachordal K-nets share a trichordal subset containing T_2 and I_9 arrows. We can also see the positive isography of K_{13} and K_{14} represented by $\langle T_7 \rangle$ by providing the same T_n arrows with varying analogous I_n arrows. Here, K_{13} is reinterpreted as a K-net still having the same I_n arrows while also having different T_n arrows through an exchange of transpositional partners.

Finally, Example 3.45 displays a summary of the K-net progressions and an overall transformational path among index zones for both hand parts, providing a symmetrical layout.

Example 3.45 Overall transformational paths among index-zones in Schoenberg's op.15, no.1

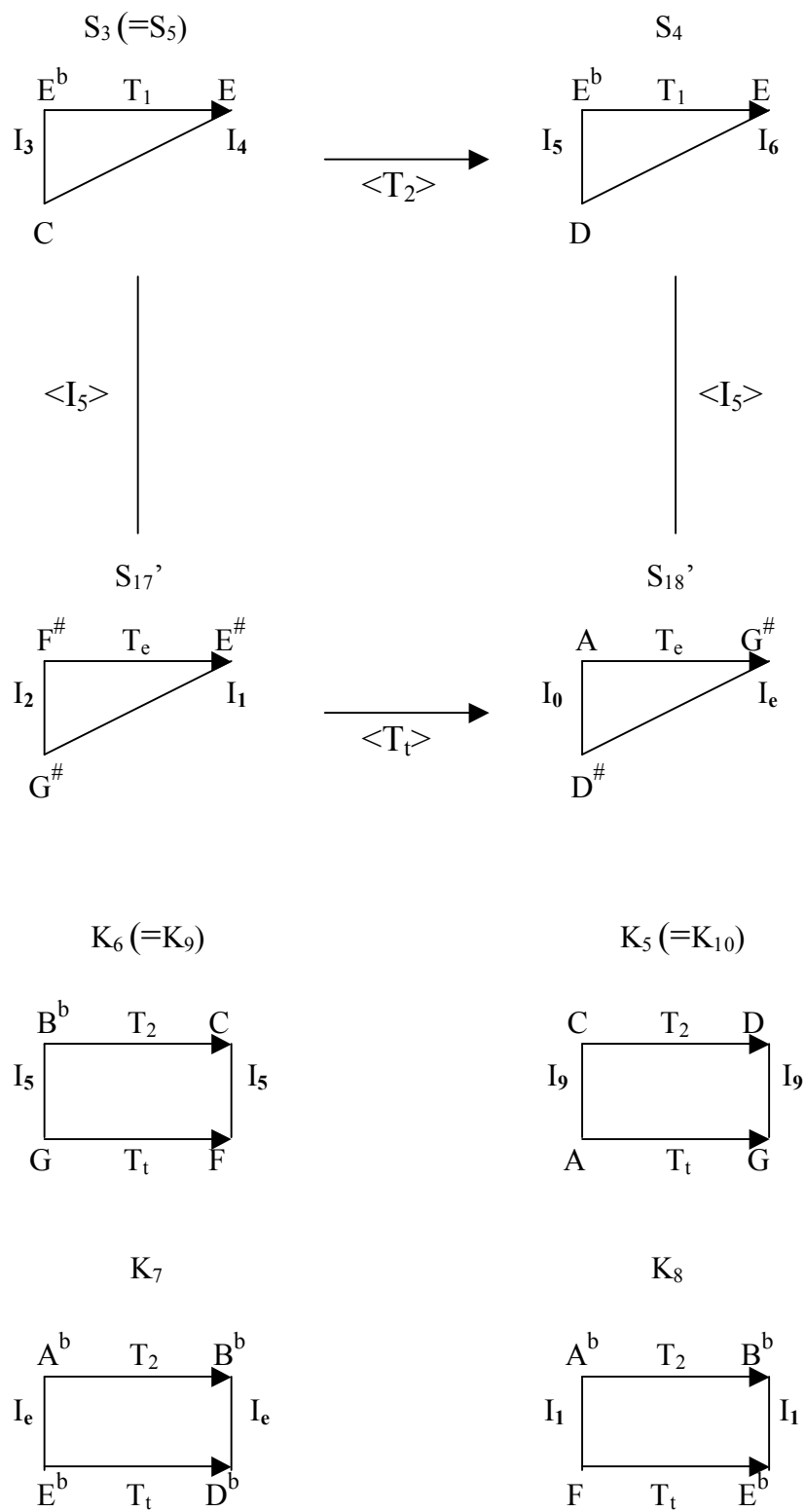


In other words, the $\langle T_2 \rangle$ relations of the right-hand part featured in the first and second sections contrast with the complementary $\langle T_t \rangle$ relations presented in the last section.

Likewise, the $\langle T_{1/e} \rangle$ relation of the left-hand part presented between tetrachordal K-nets of the first and second sections contrasts with the $\langle T_{e/1} \rangle$ of the last section. In particular, it is possible to arrange the four trichordal K-nets of the right-hand part suggested in Example 3.45 into a tetrachordal supernetwork. Example 3.46 illustrates that the tetrachordal supernetwork S retains a $\langle T_2 \rangle$ operation between the first two trichordal K-nets and a $\langle T_t \rangle$ operation between the last two trichordal K-nets shown in Example 3.45.

Example 3.46 Schoenberg's op.15, no.1; recursion of the K-nets of the right-hand part

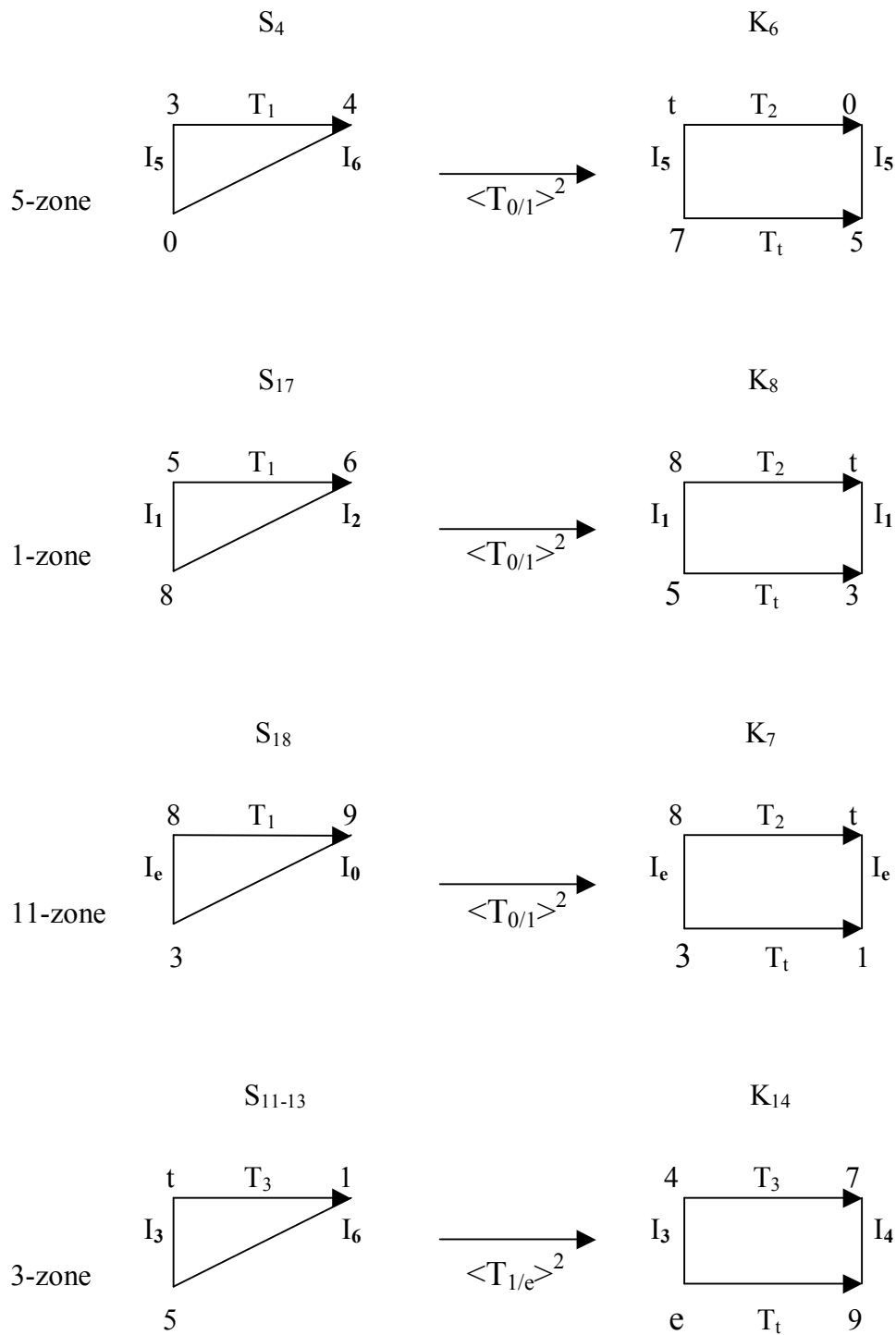
S



That is, S consists of all of the prominent trichordal K -nets, each representing the four index-zones suggested in the right-hand part. The supernetwork S is also strongly or positively isographic to other individual K -nets of the inversive tetrachords, especially those presented in the left-hand part of the middle section.

In addition, Example 3.47 illustrates that all of the trichordal K -nets of the right-hand part, each derived from a different zone, can transform to the tetrachordal K -nets of the left-hand part presented in the middle section.

Example 3.47 Schoenberg's op.15, no.1; axially reductive isography between trichordal K-net of the right-hand part and tetrachordal K-net of the left-hand part



We can see in the example that each transformation between K-nets of different numbers of nodes accomplishes the same $\langle T_{0/1} \rangle^2$ relation within the same index-zone except for the last transformation between S_{11} ($=S_{12}-S_{13}$) and K_{14} . The tetrachordal K_{14} contains the trichordal K-net S_{11} itself within the 3-zone while K_{14} has a different I_4 arrow and thus we can derive $\langle T_{1/e} \rangle^2$ label for this transformation. The reductive isographies between trichordal and tetrachordal K-nets in Example 3.47 show us the role of the embellishing tetrachords of the left-hand part suggested in Example 3.41 that repeat or anticipate the inversive properties of the trichords presented in the right-hand part. In addition, those reductive isographies illustrate the reason why the last tetrachordal K-net, K_{14} , is provided in the left-hand part in Example 3.45 though it breaks the balanced layout. In other words, all tetrachordal K-nets retaining complementary T_n arrows in the left-hand part have associations with all the trichordal K-nets of the right-hand part. In this piece, although inversive motions may not be always perceptible on the musical surface, symmetrical inversion plays a significant role as we have examined above. Especially, the possible transformations between trichords and tetrachords were used in this piece to highlight a single inversive axis, and they thus help us to easily grasp an overview of the inversive properties provided throughout the entire piece. In addition, the symmetrical layout revealed by the transformational paths among index-zones contributes to coherence in this piece.

3-4. Eugene Cines's "Abbreviations" for solo piano, no.1 (1974).

An analysis of the first movement of Eugene Cines's "Abbreviations" for solo piano, written in 1974, also examines how we can use K-nets to show large-scale

unifying symmetrical structures in three- and four-note groupings, especially, the transformations between the K-nets with different numbers of nodes. In particular, the analysis of this piece will show how the first tetrachord contributes as a motivic gesture to the large-scale organization which unifies all the trichords and tetrachords presented in the entire piece and, especially, how the inversional property of the four-note motive is composed-out as material for development.

An annotated score of Cines's work is shown in Example 3.48.

Example 3.48 Eugene Cines's "Abbreviations" for solo piano, no.1

A section

A' section

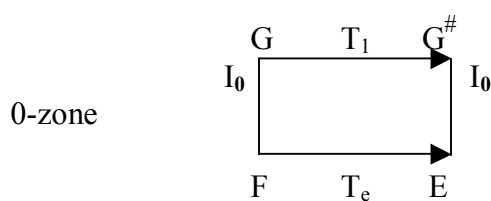
This piece can be divided into two sections, A and A', by the silence and meter change in m.10. The opening three measures introduce an inversive tetrachord as a motive, a member of set class 4-3 (0134). Example 3.49 parses the first tetrachord, labeled C_1 , into T_1 and T_e dyads with a single I_0 relation and this interpretation receives a registral emphasis in the musical surface.

Example 3.49 Eugene Cines's "Abbreviations," network interpretation of the first chord in mm.1-2



C_1

4-3 (0134)



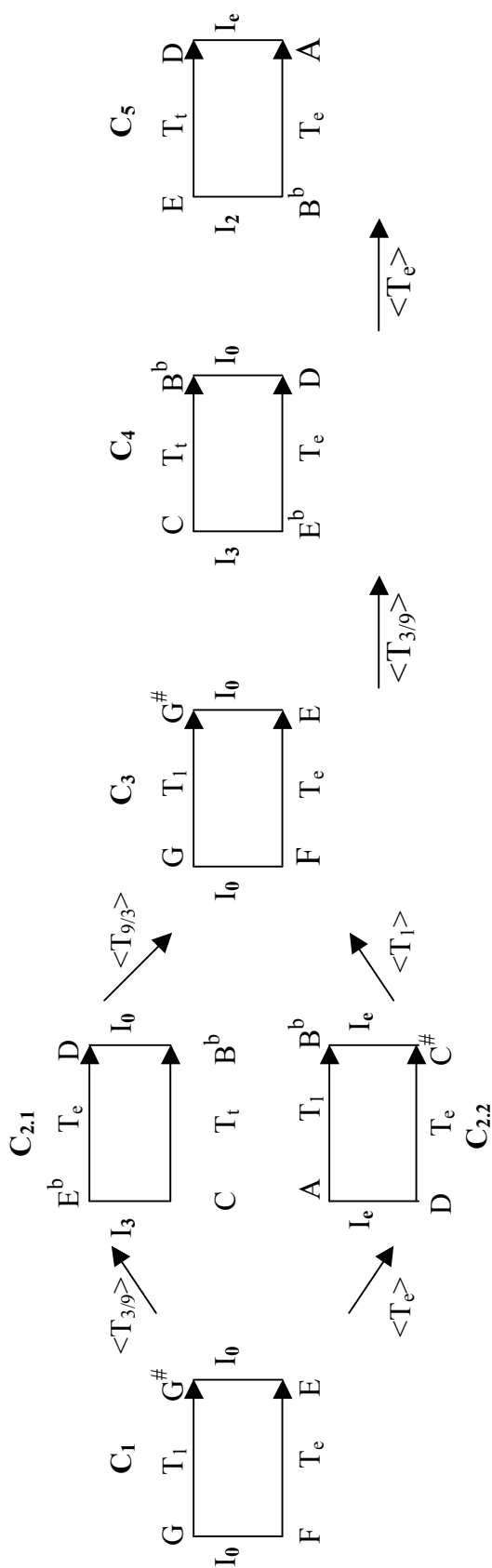
The other tetrachords which are arranged just like C_1 can be interpreted in a similar manner, the melody producing T_n arrows while the harmony produces I_n arrows.

Example 3.50 interprets the next two tetrachords, $C_{2.1}$ and $C_{2.2}$, each segmented both horizontally and vertically, by sharing two notes (Bb-D).

Example 3.50 Positive and axial isography among the K-nets C_1 - C_5 of the right-hand part in mm. 3-9

C_1 $C_{2.1}$ C_3 C_4 C_5

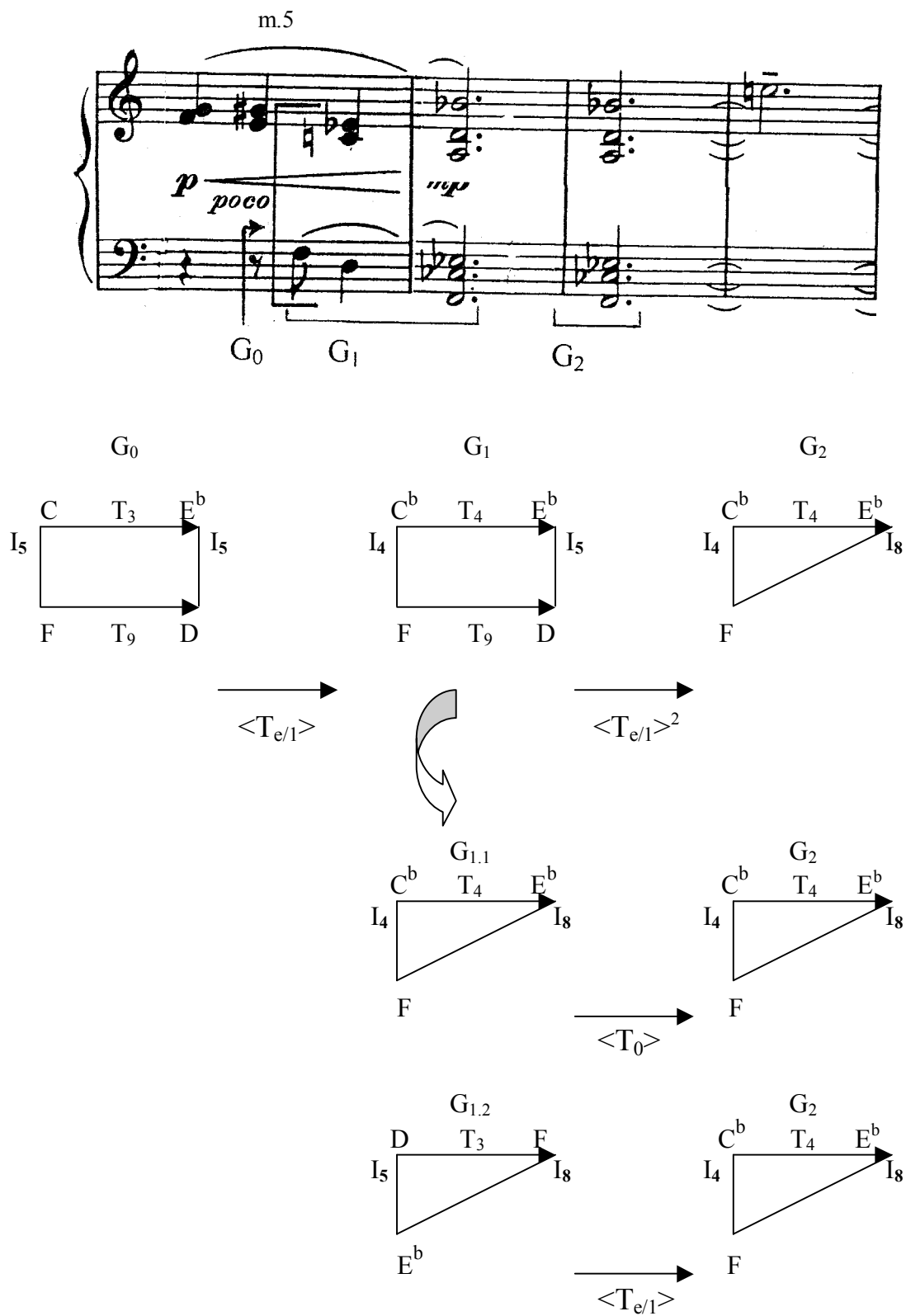
$C_{2.2}$



In this, the chord $C_{2.1}$ is articulated into I_3 and I_0 dyads produced by the vertical harmony while $C_{2.2}$ is interpreted as an inversive tetrachord involving a pair of I_e dyads. That is, the tetrachords $C_{2.1}$ and $C_{2.2}$ diverge from C_1 and then converge on C_3 , so that the progression $C_1 - C_3$ of the right-hand part show two transformational paths, $\langle\langle T_{3/9} \rangle\rangle$, $\langle T_{9/3} \rangle$ and $\langle\langle T_e \rangle\rangle$, $\langle T_1 \rangle$. It is clear that, in either way, the 0-zone is extended in the progression $C_1 - C_3$ of the right-hand part by the inversive tetrachords C_1 and C_3 , which contain only the same I_0 arrows. We can regard the middle chords, $C_{2.1}$ and $C_{2.2}$, as a passing motion. The subsequent tetrachords C_4 and C_5 can be articulated into a dyad of ic 1 plus a dyad of ic 2. We can see in Example 3.50 the $\langle T_e \rangle$ arrow, transforming C_4 into the positively isographic network C_5 . Therefore, the right-hand part of the first section largely extends the 0-zone and then shifts to the 2-zone.

In contrast, in the left-hand part the first section begins with the tetrachord labeled G_1 in m. 5. The tetrachord G_1 can be articulated into a dyad of ic 9 plus a dyad of ic 4 and thus the network interpretation of G_1 in Example 3.51 contains T_9 and T_4 arrows along with I_4 and I_5 arrows.

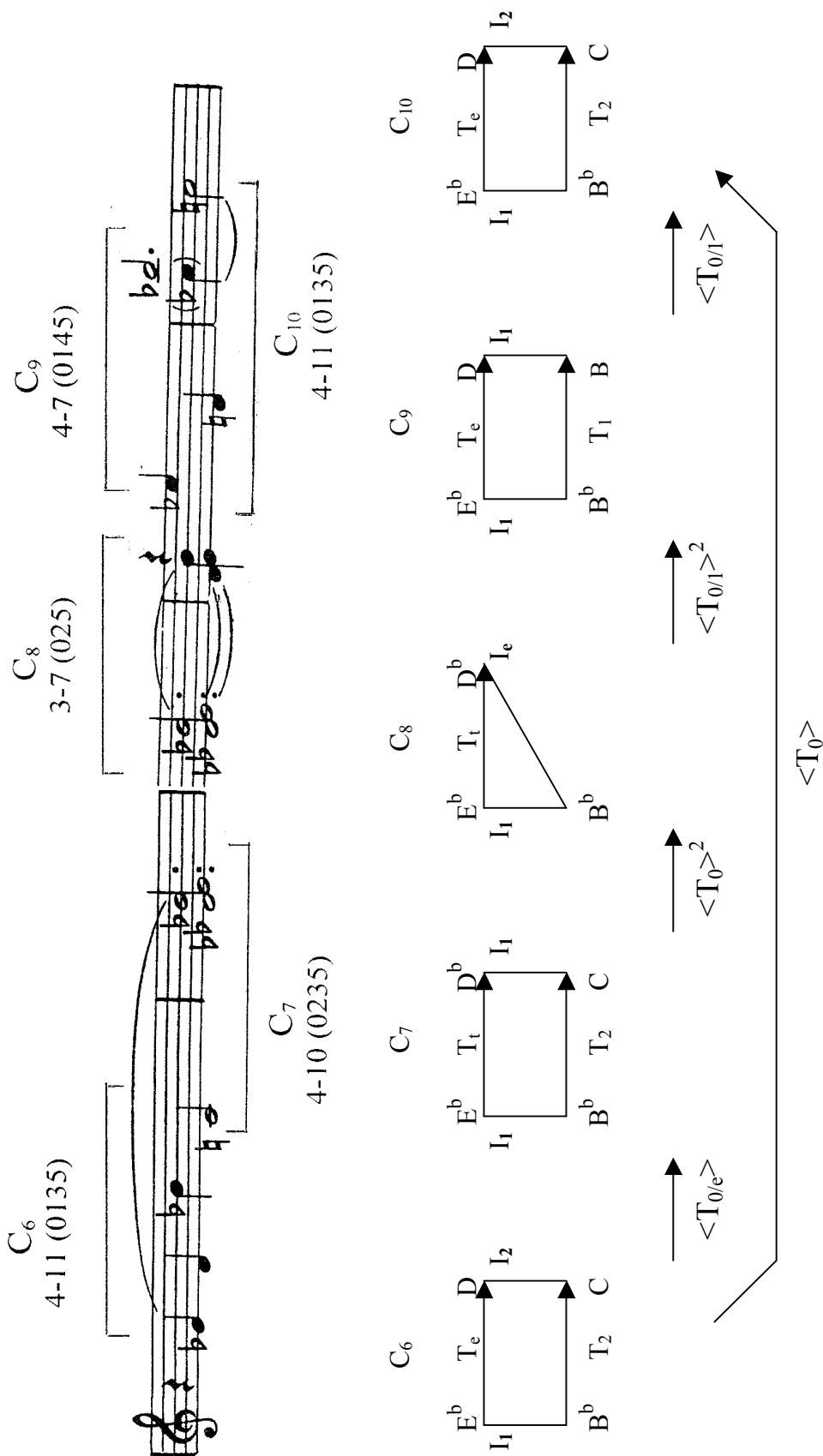
Example 3.51 Axial and axially reductive isography among the K-nets G_0 - G_2 of the left-hand part in mm.5-7



The reason for this particular interpretation for G_1 , among multiple possible interpretations, is that the vertically segmented tetrachord, G_0 , which emerges from the right-hand part, suggests I_5 arrows in Example 3.51 and can also be related to the K-net of G_1 by the sharing of a trichordal subset, namely the type 2 axially isographic network. We can also see in Example 3.51 that under axially reductive isography the tetrachordal K-net of G_1 and its trichordal subset, G_2 , presented in mm.7-9, generate a $\langle T_{e/1} \rangle^2$ transformation within the 4-zone since one trichordal subset of G_1 is strongly isographic with G_2 at $\langle T_0 \rangle$ while another subset of G_1 is axially isographic with G_2 at $\langle T_{e/1} \rangle$.

In the second section, the right-hand part in mm.10-15 presents five different chords, C_6 - C_{10} , representing four different set-classes. Example 3.52 models the five chords, C_6 - C_{10} , as isographic K-nets.

Example 3.52 Cines’s “Abbreviations,” axial isography among the K-nets C₆–C₉ of the right-hand part in mm.10-15

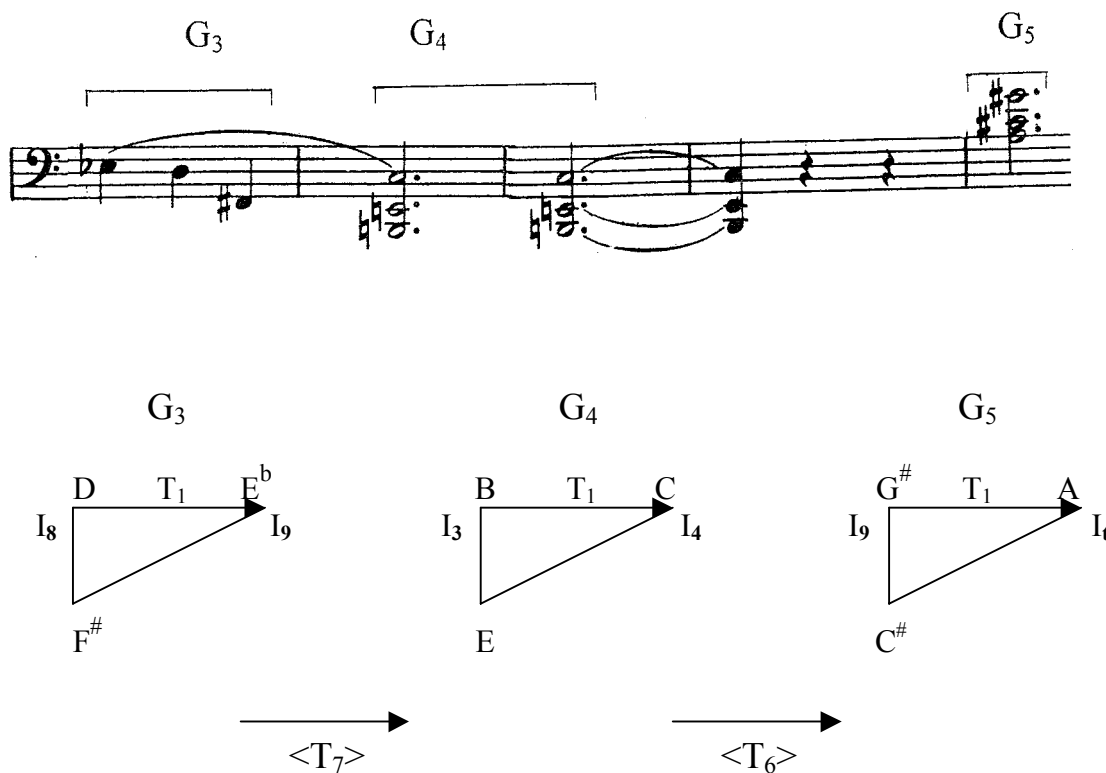


Zone 1

In interpreting C_6 , the temporal order in musical surface emphasizes the T_e and T_2 arrows produced by the melodic motion E^b -D and B^b -C although the same notes are interpreted as a different K-net in C_2 of the first section. Note that the tetrachords, C_6 and C_7 , share the pitch-class set $\{E^b, B^b, C\}$ and thus gives rise to type 2 axial isography, sharing a trichordal subset. In addition, C_7 is strongly reductive isography with its trichordal subset, C_8 , by $\langle T_0 \rangle^2$ and C_8 is again axially reductive isography with C_9 by $\langle T_{0/1} \rangle^2$. We can also see that the tetrachordal K-net of C_9 is type 2 axially isographic with the tetrachordal network of C_{10} at $\langle T_{0/1} \rangle$ since they share a trichordal subset. In the progression C_6 - C_{10} , each tetrachordal segment of C_7 and C_9 is necessary as a pivot chord to associate the preceding tetrachordal K-net with the trichordal K-net which follows it and vice versa. In fact, we can hear in this example that the first tetrachord C_6 is prolonged by the tetrachord C_{10} but is placed an octave higher. Thus, the right-hand part of the second section largely extends the 1-zone.

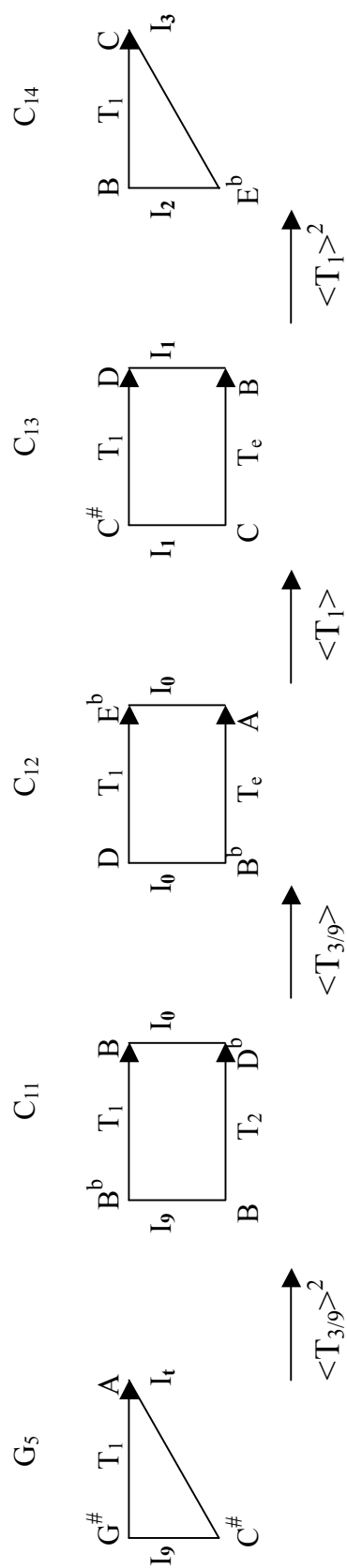
In the second section, the left-hand part provides three trichords in mm.10-14. The network interpretations of G_3 - G_5 model each of the trichords as displayed in Example 3.53.

Example 3.53 Positive isography among the K-nets G_3 - G_5 of the left-hand part in mm.10-14



These trichordal set-classes sharing invariant ic 1s are positively isographic with each other. Example 3.53 notates the transformations $\langle T_7 \rangle$ and $\langle T_6 \rangle$ under the mappings suggested by analogous nodes. In m.16, the both hands, however, converge into a single strand by presenting the same inversive axis between trichordal G_5 in the left-hand part and tetrachordal C_{11} . That is, Example 3.54 illustrates that the trichord G_5 and the tetrachord C_{11} can be interpreted as K-nets sharing T_1 arrows and I_9 arrows.

Example 3.54 Cines's "Abbreviations," network interpretations of C_{11} - C_{14} in mm. 16-21

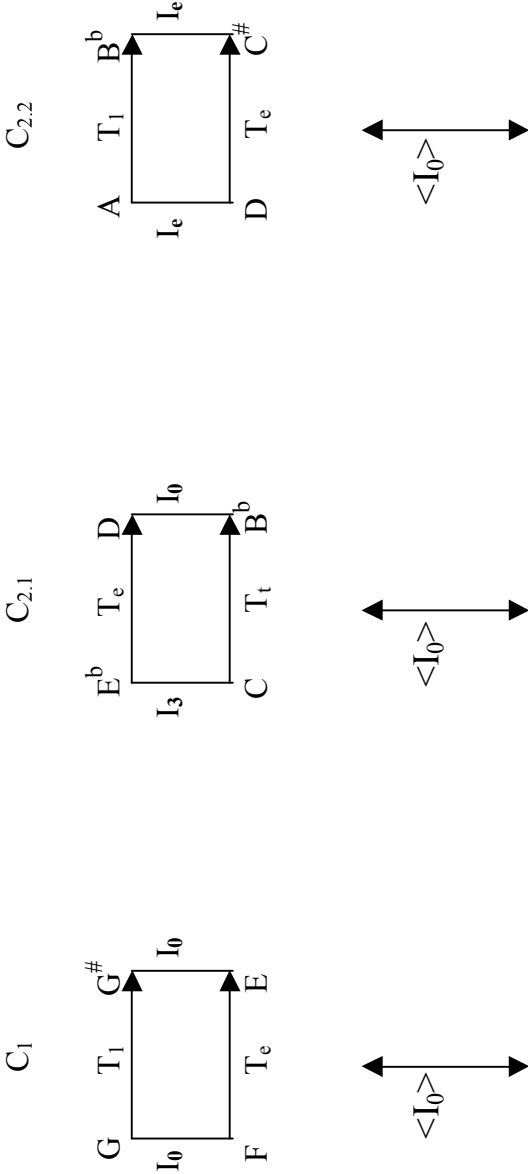


We can see that the network interpretation of G_5 has axially reductive isography with the network of C_{11} , represented by $\langle T_{3/9} \rangle^2$. In addition, the subsequent tetrachords and trichords can be articulated into a pair of dyads of $ic1$ or a single dyad of $ic1$. Note that these chords also provide the series of transformations $\langle T_{3/9} \rangle$, $\langle T_1 \rangle$, and $\langle T_1 \rangle^2$ between their trichordal and tetrachordal K-nets.

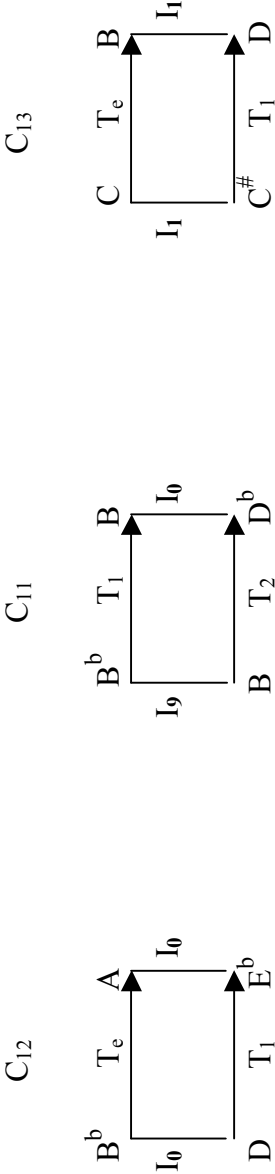
In summary, while the right-hand part in each section provides an extension for a different index-zone, I_0 or I_1 , as shown in Examples 3.50 and 3.52, the first and the last three tetrachords of this piece provide a balanced structure in the transformational pathways between their inversional zones. Example 3.55 illustrates that the opening three tetrachords of the first section can be paired with the final three tetrachords of the second section to highlight a specific inversional relation, I_0 , which is suggested in the first four-note motive, C_1 .

Example 3.55 Inversional relations between each pair of the first and the last three tetrachords

First three tetrachords



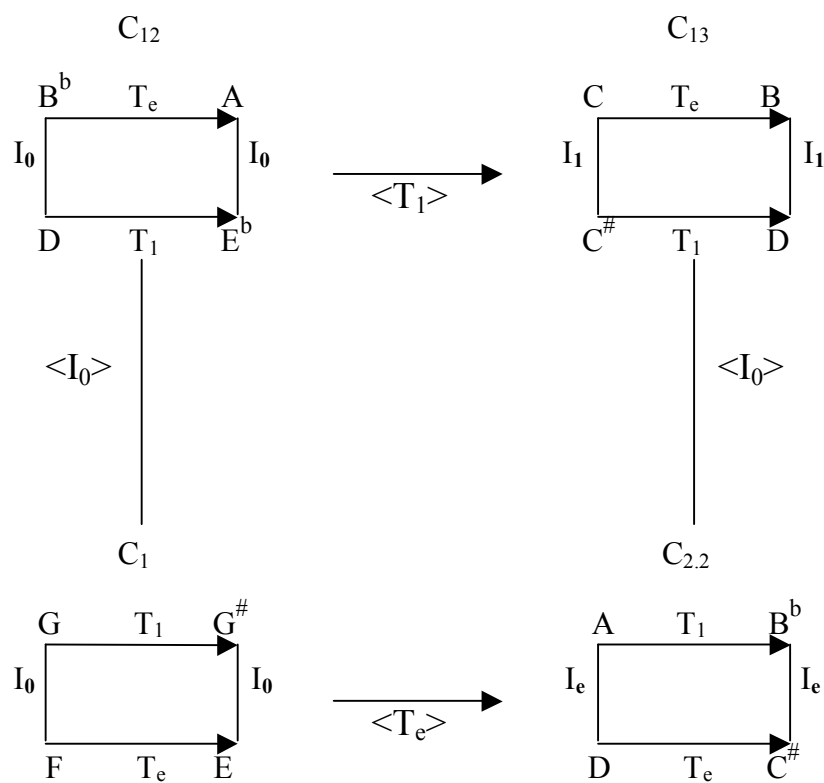
Final three tetrachords



Other tetrachords in the right-hand part, however, simply serve to either connect the different index-zones of two sections or extend an index-zone within each section. Significantly, it is possible to arrange the four tetrachordal graphs of C_1 , $C_{2.2}$, C_{11} , and C_{12} , suggested in Example 3.55, into a higher-level network isography labeled S_1 . As Example 3.56 shows, the super-network S_1 is strongly isographic to the network of the first tetrachord, C_1 .

Example 3.56 Cines's "Abbreviations," supernetwork S_1 of tetrachordal K-nets C_1 , $C_{2.2}$, C_{12} , and C_{13} ; strong isography with the first tetrachord C_1

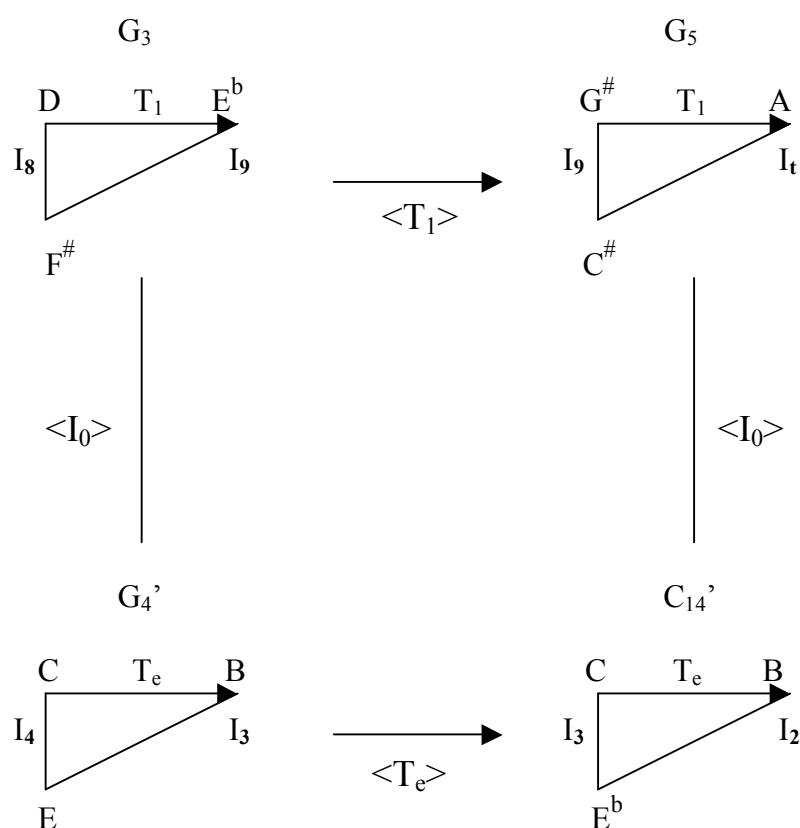
S_1



Furthermore, it is also possible to arrange all four trichordal K-nets which retain T_1 arrows presented in the left-hand part into a tetrachordal super-network, labeled S_2 , as shown in Example 3.57.

Example 3.57 Cines's "Abbreviations," supernetwork S_2 of trichordal K-nets G_3 , G_5 , G_4 , and G_{14} ; strong isography with the first tetrachord C_1

S_2



The four aggregates of trichords, S_2 , containing the $\langle T_1 \rangle$, $\langle T_e \rangle$ and $\langle I_0 \rangle$ arrows is also a strongly isographic to the network of the first four-note motive as well as the supernetwork S_1 of the tetrachord aggregates. That is, both the higher-level networks of trichord and tetrachord aggregates embody the motive of the first tetrachord so that the recursive structures unify all the K-nets of different numbers of nodes as a whole.

In this chapter, we have seen the transformations capable of associating trichordal K-nets with tetrachordal K-nets, helping us to better understand the nature of inversions of the pieces, employing three- and four-note aggregates as significant components. We have also disclosed how these transformations between different sizes of networks can express more effectively the inversional properties in both the cases of being emphasized and of being ambiguously suggested in the musical surface, going beyond the limits of the traditional isographies which permit only the same number of nodes. In the analyses of the three musical excerpts we have examined above, the reductive isographies between different K-net nodes lead us to the possibility of grasping the inversional properties exposed in the surface of the music and also provide strong evidence for employing inversional properties as a background design through the revealing of symmetrical structures between sections or phrases. In other words, in this type of analysis, the reductive isographies which relate segments of three- and four-notes within a zone or between zones and other isographies which relate the same number of notes together produce prominent transformational pathways among index-zones. In many cases, strongly reductive and strong isographies ensure or extend a zone while positively reductive and positive isographies make us aware of a shift to a new zone either for a moment or for longer. Axially reductive and axial isographies, however, may prove to be more useful since they serve either to extend a zone or to introduce a new zone, or may in fact perform both roles at the same time. Furthermore, the recursive structures within different levels also point to possible relations between the three- and four-note aggregates. That is, the hyper-networks which incorporate aggregates of a different number of notes relate all the different sizes of networks at a

higher-level and thus achieve coherence for the work as a whole. Of course, the long-range transformational progressions among index-zones and the deeper-level structures may be rejected as inadequate methods on the basis that these background structures cannot easily be discerned in hearing the musical works. The analyses, however, not only prove these methods to be useful but indeed show them to be necessary for taking a comprehensive approach to compositions which employ symmetrical inversions as significant forces.

Chapter 4

Klumpenhouwer Networks and Perle's Cyclic Arrays

In a number of his theoretical writings, and in corresponding compositions, George Perle introduces his compositional method, which he calls “twelve-tone tonality,” and illustrates how his theory can be applied or realized both in his own works and in the works of others.¹⁰⁷ He devises his unique system of twelve-tone tonality by utilizing interactions between interval cycles and inversional symmetry as basic components.¹⁰⁸ In view of the fact that these fundamental elements may correspond to the transpositional and inversional relations of K-nets, both Perle himself and other writers as well suggest that there are affinities between Perle's compositional

¹⁰⁷ Perle's earlier publications on twelve-tone tonality were “Evolution of the Tone Row: the Twelve-Tone Modal System,” *The Music Review* 2 (1941), 273-287; “Twelve-Tone Tonality,” *Monthly Musical Record* 73 (1943), 175-179; “An Approach to Simultaneity in Twelve-Tone Music,” *Perspectives of New Music* 3/1 (1964), 91-101; and the book *Twelve-Tone Tonality* (Berkeley: University of California Press, 1977), revised in *Twelve-Tone Tonality* 2ed (Berkeley: University of California Press, 1996). Perle's later articles and books on his theory of twelve-tone tonality include “Berg's Master Array of the Interval Cycles,” *Musical Quarterly* 63 (1977), 1-30; “Symmetry, the Twelve-Tone Scale, and Tonality,” *Contemporary Music Review* 6/2 (1992), 81-96; “Communication,” *Music Theory Spectrum* 15/2 (1993), 300-303; *The Listening Composer* (Berkeley: University of California Press, 1990), Ch. 5; and *The Right Notes: Selected Essays by George Perle on Twentieth-Century Music* (Stuyvesant, NY: Pendragon Press, 1995).

¹⁰⁸ Both a full and a brief description of Perle's system of twelve-tone tonality are given by Thomas Carrabr , “Twelve-Tone Tonality and the Music of George Perle,” Ph.D. diss., City University of New York, 1993; Steven L. Rosenhaus, “Harmonic Motion in George Perle's Wind Quartet No.4,” Ph.D. diss., New York University, 1995; David Pitt, “What is Tonality?,” *International Journal of Musicology* 4 (1996), 291-300; and David Headlam, “Tonality and Twelve-Tone Tonality: The Recent Music of George Perle,” *International Journal of Musicology* 4 (1996), 301-333.

methods and K-nets.¹⁰⁹ Recognizing the structural similarity, Perle remarks in his “Communications”:

In Twelve-Tone Tonality (p.34) I say that each segment of the cyclic set “comprises all the criteria that define the set- the alternate sums formed by adjacent pcs and the cyclic interval expressed in the difference between these sums.” A pitch-class collection analyzed in this way is what Klumpenhouwer calls a “network.”¹¹⁰

A similar view had previously been proposed by Lewin, who commented that “I suspect isographies of the networks, and networks-of-networks, could reveal aspects of large-scale structuring in Perle’s music beyond the extent of the formal analyses which himself presents.”¹¹¹ In his later writing, Lewin lays out possible transformations of K-nets within Perle-cycles to demonstrate their structural connections.¹¹² Although many authors besides those mentioned above have demonstrated the structural similarities between K-nets and Perle’s compositional system, their discussions have primarily been limited to trichordal pc segments.¹¹³ Foley, however, attempts to represent the principal

¹⁰⁹ George Perle, “Communications,” *Music Theory Spectrum* 15/2 (1993), 300-303; Philip Lambert, “Isographies and Some Klumpenhouwer Networks They Involve,” *Music Theory Spectrum* 24/2 (2002), 165-195; David Lewin, “Thoughts on Klumpenhouwer networks and Perle-Lansky Cycles,” *Music Theory Spectrum* 24/2 (2002), 196-230; Dave Headlam, “Perle’s Cyclic Sets and Klumpenhouwer Networks: Response,” *Music Theory Spectrum* 24/2 (2002), 246-257; Gretchen C. Foley, “Arrays and K-nets: Transformational Relationships within Perle’s Twelve-Tone Tonality,” *Indiana Theory Review* 23 (2002), 69-97; Michael Callahan, “Measuring K-net Distance: Parallels between Perle and Lewin, and a Generalized Representation of Sum- and -Difference space,” paper presented at the Annual Meeting of the Society for Music Theory (Los Angeles, 2006).

¹¹⁰ Perle, “Communications,” 300.

¹¹¹ Lewin, “Klumpenhouwer Networks,” 114.

¹¹² Lewin, “Thoughts on Klumpenhouwer Networks,” 196-230.

¹¹³ Lambert shows that only some limited larger pc collections can be represented in an isographic relationship in his “Isographies,” 187.

collection of six pitch-class sets, termed “axis-dyad chords” in twelve-tone tonality, as K-nets, and suggests:

In his theory of twelve-tone tonality, Perle expounds on many other increasingly abstract relationships among arrays, in terms of membership in his unique concepts of modes, keys, synoptic arrays, and tonalities. It is my belief that K-nets will continue to be an invaluable tool in revealing the transformational relationships underlying these concepts....¹¹⁴

For Foley, however, the network interpretations of Perle’s axis-dyad chords, which produce isographic relations, include only those cyclic intervals which are identical, either within an array or between arrays. I intend to expand the discussions by examining how K-nets can efficiently construe or represent Perle’s compositional methods, especially focusing on his cyclic arrays. That is, I aim to show how the K-nets of axis-dyad chords can be used to interpret the diverse harmonies within an array or between arrays, highlighting their symmetrical structural design revealed through analyses of Perle’s compositions.

This chapter is divided into three sections. The first part briefly reviews how trichordal segments within Perle’s cyclic sets can be represented as K-nets. The second part focuses on Perle’s axis-dyad chords and shows how the K-nets of these chords can clarify the abstract relationships between cyclic arrays. Finally, this chapter will provide an analysis of Perle’s Etude No.5 from *Six Etudes for Piano* exemplifying a symmetrical structure articulated by isographic relationships.

¹¹⁴ Foley, “Arrays and K-nets,” 94.

4-I. K-nets and trichordal segments in cyclic sets.

Perle's fundamental compositional material, called a "cyclic set," is comprised of interwoven ascending and descending cycles.¹¹⁵ Example 4.1 shows a cyclic set formed by two interlocking inversionally related interval 1 cycles.

Example 4.1 Cyclic interval 1 of cyclic set 0,1

The musical notation shows a single staff with 14 notes. Open noteheads represent the ascending interval cycle, and filled-in noteheads represent the descending interval cycle. The notes are: C4 (open), C#4 (filled), D4 (open), D#4 (filled), E4 (open), E#4 (filled), F4 (open), F#4 (filled), G4 (open), G#4 (filled), A4 (open), A#4 (filled), B4 (open), and B#4 (filled).

Below the staff, the interval cycle is shown as a sequence of 14 numbers: 0, 0, 1, e, 2, t, 3, 9, 4, 8, 5, 7, 6, 6, 7. The first two numbers (0, 0) are grouped under C₁. The next three numbers (1, e, 2) are grouped under C₂. The next three numbers (t, 3, 9) are grouped under C₃. The remaining six numbers (4, 8, 5, 7, 6, 6, 7) are not grouped.

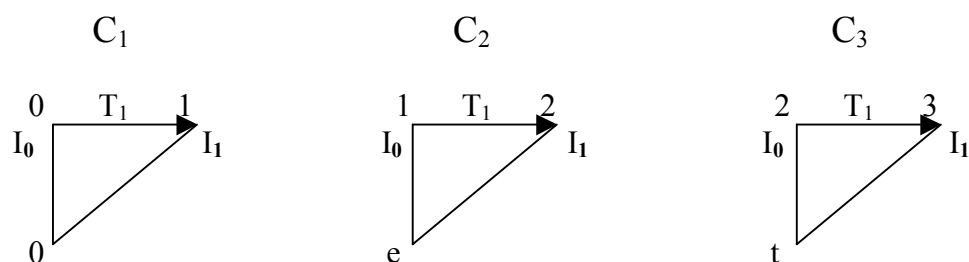
Below the interval cycle, the label "0,1" is followed by the same sequence of numbers: 0, 0, 1, e, 2, t, 3, 9, 4, 8, 5, 7, 6, 6, 7.

The ascending ordered pc interval 1 cycle appears in open noteheads and the descending interval cycle appears in filled-in noteheads. Since a cyclic set produces two sums and the same interval, any trichordal segment can represent the cyclic set which it belongs to as the basic unit. In the cyclic set of Example 4.1, the label "0,1" not only indicates two repeating sums, named *tonic sums*, but also reveals the interval cycle, in this case 1, which may be determined by subtracting the first tonic sum from the second. Any trichordal segment of a cyclic set can thus be interpreted as a trichordal K-net involving

¹¹⁵ For his interval cycles, Perle initially employed only cycles of intervals 1 and 5, but he later allows other interval cycles. Because ics 1 and 5 exhaust all twelve members of pitch classes within a cycle, Lewin defines interlocking ascending and descending interval 1 and 5 cycles as "particularly useful" in his "Thoughts on Klumpenhouwer Networks," 217.

two I_n and one T_n arrows. In other words, the contiguous trichordal segments within the cyclic set of Example 4.1 can be interpreted as K-nets that are all related by strong isography with the same configuration of nodes and arrows as displayed in Example 4.2.

Example 4.2 Strongly isographic K-nets of trichordal segments from cyclic set 0,1



On the other hand, shifting any interval cycle of the cyclic set produces K-nets related by positive isography. We can obtain twelve possibilities for transposing a cyclic set by shifting either the ascending or the descending cycle. For instance, transposing the descending cycle in Example 4.1 down by two semitones while leaving the ascending one as it produces new tonic sums, 10,11, as shown in Example 4.3.

Example 4.3 Cyclic interval 1 of cyclic set 10, 11

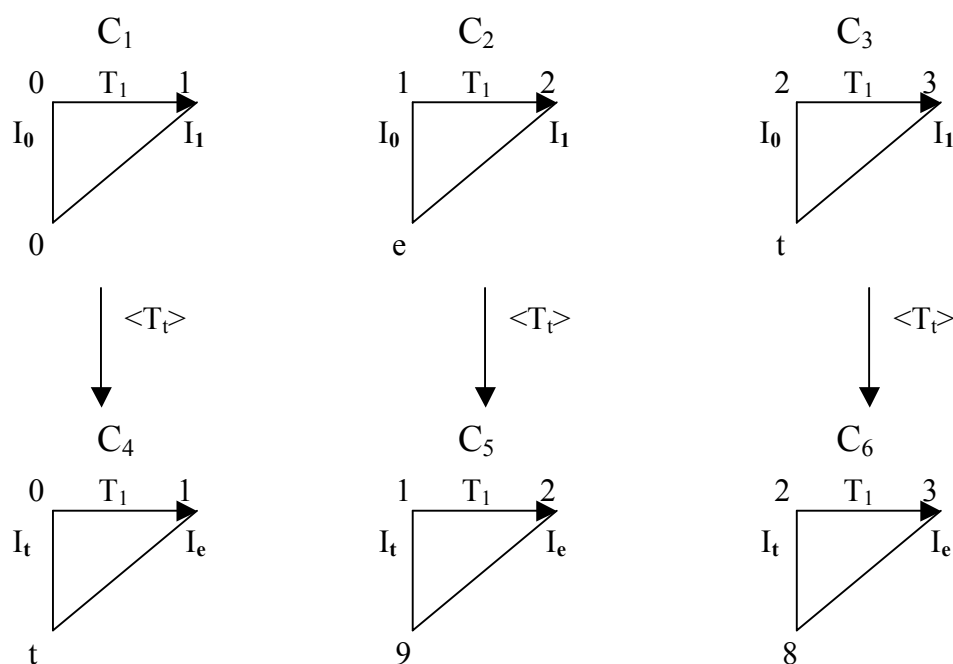
0 t 1 9 2 8 3 7 4 6 5 5 6 4 7

10,11: 0 t 1 9 2 8 3 7 4 6 5 5 6 4 7

C₄ C₅ C₆

The K-nets of all the trichordal segments from this new cyclic set exhibit strong isographies within themselves, but positive isographies with those from Example 4.2. That is, both groups from Example 4.1 and 4.3 retain the same interval cycles, presenting the same T_1 arrows in their K-nets.¹¹⁶ Example 4.4 shows that the K-nets derived from each of the cyclic sets, “0,1” and “10,11,” contain identical T_1 arrows while the analogous I_n arrows vary by 10.

Example 4.4 Positively isographic K-nets among trichordal segments from cyclic set 0,1 and those from cyclic set 10, 11



¹¹⁶ Lambert defines all members of the same K-class, which is derived from all cyclic sets sharing the same interval cycles, as a *family* in his “Isographies,” 185. Callahan also terms a group of 144 trichordal K-nets that can be related to each other by some positive isography as a *positive isography class* and a group of 12 trichordal K-nets that share strong isography with each other as a *strong isography class*, in his recent paper “Measuring K-net Distance.”

Here, the label $\langle T_i \rangle$ represents the difference between the analogous I_n arrows and also indicates the difference between the corresponding tonic sums of two cyclic sets.

Note that the K-nets in Perle's cyclic sets differ from the spatial transformations of K-nets in index zones, as they share only shape and size, providing those specific T_n arrows.¹¹⁷ Therefore, Stoecker's "axial isography" containing different T_n arrows does not occur in a cyclic set. Perle's cognate sets with different interval cycles, however, can engage the trichordal K-nets in an axial isography.¹¹⁸ In Example 4.5, each of the cyclic sets contains different cyclic intervals while at the same time sharing a common sum.

Example 4.5 Cyclic interval 1 of cyclic set 1, 2 and cyclic interval 2 of cyclic set 1,3

1,2 : $\boxed{0 \quad 1 \quad I} \mid 0 \quad 2 \quad e \quad 3 \quad t \quad 4 \quad 9 \quad 5 \quad 8 \quad 6 \quad 7 \quad 7$
 C_1

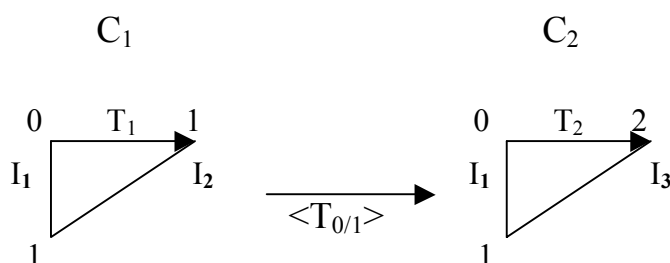
1,3 : $\boxed{0 \quad 1 \quad 2} \mid e \quad 4 \quad 9 \quad 6 \quad 7 \quad 8 \quad 5 \quad t \quad 3 \quad 0 \quad 1 \quad 2$
 C_2

The trichordal segments from each cyclic set, C_1 and C_2 , can be interpreted as K-nets related by axial isography as given in Example 4.6.

¹¹⁷ In the previous chapters, 2 and 3, I examined the spatial transformations of trichordal and tetrachordal K-nets in inversional zones, both those which are identical as well as those of different sizes and shapes.

¹¹⁸ Headlam discusses the structural similarities between axial isography and Perle's cognate sets in his "Perle's Cyclic Sets," 249.

Example 4.6 Axially isographic K-nets between trichordal segments from cyclic set 1,2 and from cyclic set 1, 3



Here, the K-nets contain different T_n and I_n arrows while maintaining identical I_1 arrows. Thus, the relationship is denoted by $\langle T_{0/1} \rangle$.¹¹⁹ Therefore, any trichordal segments of Perle's cyclic sets which share the same interval cycles or any cognate sets which share a single sum can be interpreted as K-nets related by strong, positive, and axial isography.

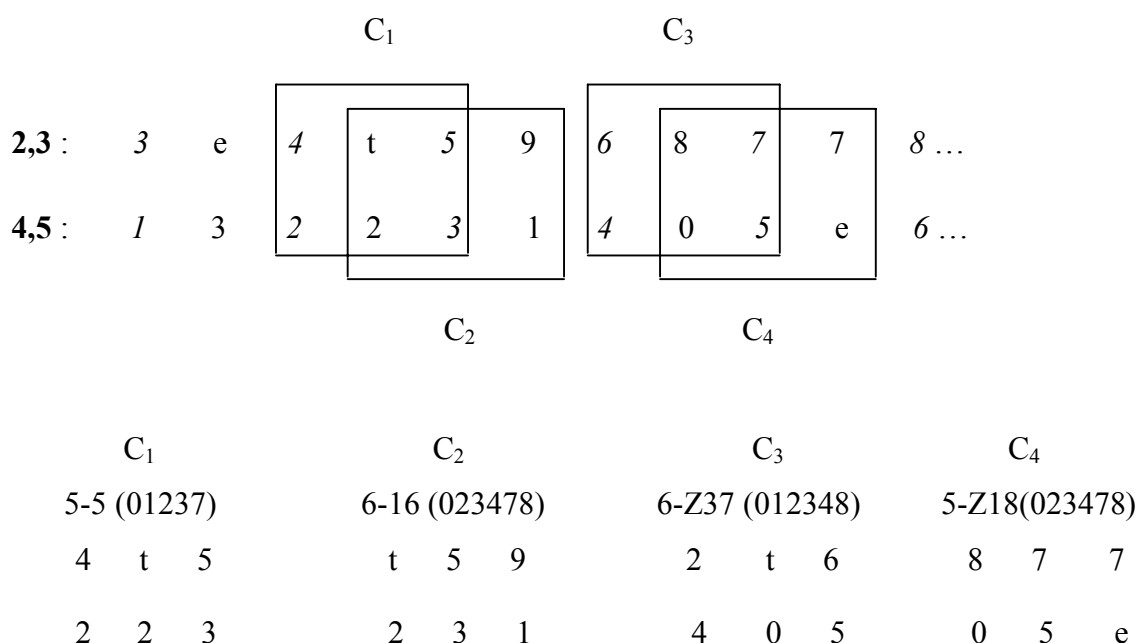
4-II. K-nets and axis-dyad chords in cyclic arrays.

To generate larger pitch-class collections, Perle vertically aligns two cyclic sets into an “array” as shown in Example 4.7. That is, the two- or three-note segments from each of the cyclic sets can be combined to generate larger chords, tetrachords termed “sum tetrachords” and hexachords termed “axis-dyad chords,” which is the main unit in

¹¹⁹ In chapter 2 I used Stoecker's axial isography, labeled $\langle T_{n/-n} \rangle$ as $\langle T_{0/n} \rangle$, for indicating the spatial transformation within an index zone.

Perle's system.¹²⁰ Example 4.7 shows four axis-dyad chords, C₁-C₄. Each axis-dyad chord is comprised of a middle dyad, called the "axis notes," with the four pitch classes surrounding the axis notes referred to as "neighbor notes."¹²¹

Example 4.7 Axis-dyad chords drawn from the same array 2,3/4,5



The array 2,3/4,5 in Example 4.7 retains the same interval 1 cycles in both the upper and the lower cyclic sets. In this example, the axis-dyad chords, C₁-C₄, are of different cardinalities and contain members of four set classes: the first and last chords, C₁ and C₄, are members of sc 5-5 (01237) and 5-Z18 (01457), and C₂ and C₃ are members of sc 6-16 (023467) and 6-Z37 (012348). Within the array 2,3/4,5, each of the chords C₃ and C₄ generated from C₁ and C₂ by moving both the upper and lower cyclic sets two places to the right, however, need to be flipped horizontally (from right to left) to show the

¹²⁰ Perle, *Twelve-Tone Tonality*, 2ed., 29.

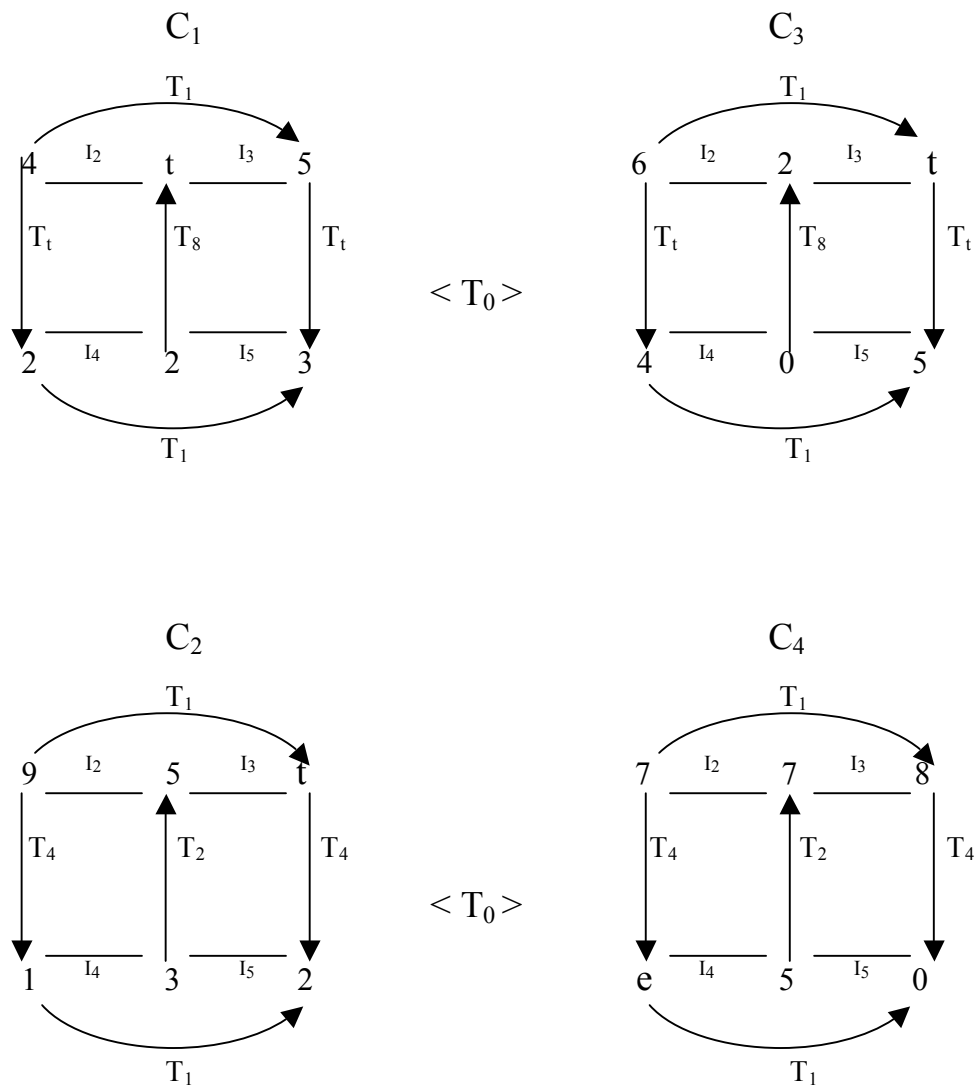
¹²¹ Perle, *Twelve-Tone Tonality*, 2ed., 28.

cycles of T_1 . There are two ways we can interpret these axis-dyad chords as K-nets.

First, any hexachordal K-nets for an axis-dyad chord can be related by transposition.

That is, the hexachordal K-nets can be parsed as a shared tetrachord sc and a shared dyad as shown in Example 4.8.

Example 4.8 Network interpretations of axis-dyad chords are transpositionally related



Foley proposed this type of isography, which provides isographic relationships at two levels, referring to it as “composite isography.”¹²² In her composite isography, a tetrachord and a dyad are combined into the K-net of each axis-dyad chord in a cyclic array. In this case, the internal T_n arrows can describe the moves between axis-dyad chords more simply, especially when their array produces two alternating vertical intervals, since all the K-nets of the axis-dyad chords drawn from a single array share identical graphs. We can see in Example 4.8 two pairs of strongly isographic relations between axis-dyad chords, C_1 - C_3 and C_2 - C_4 . That is, in each pair, the first axis dyad chord moves to the second by shifting both the upper and lower cyclic sets two places to the right, and thus the chords share the same nodes and arrows.¹²³ This method may, however, consistently be applied to only limited axis-dyad chords drawn from the same difference alignment of the array.¹²⁴ In other words, the K-nets of axis-dyad chords with internal T_n arrows cannot include all of the kinds of cyclic arrays, such as when an array is generated by different cyclic intervals. In the case of Example 4.9(a), the array 10,0/9,10 produces different 2 and 1 cycles.

¹²² Foley, “Arrays and K-nets,” 77.

¹²³ Foley defines the process, which exhibits the moves of both the upper and the lower trichordal segments by the same amount in the same direction, as *parallel shifting* and illustrates the moves should be by even numbers to maintain the same difference alignment. See “Arrays and K-nets,” 74 and 77.

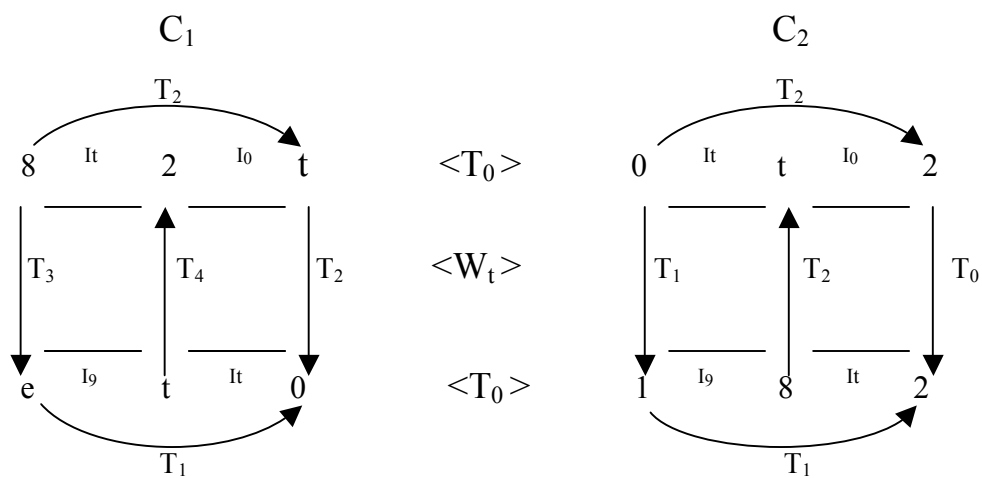
¹²⁴ Perle provides two types of alignments, difference and sum alignments. While the corresponding ascending and descending cycles are vertically aligned in a difference alignment, in sum alignment the ascending cycle of a cyclic set is aligned with the descending cycle of another set. See Perle, *Twelve-Tone Tonality*, 2ed., chapter 9 and 12.

Example 4.9(a) Axis-dyad chords C_1 and C_2 drawn from array 10,0/9,10

			C_1		C_2		
10,0 :	6	4	8 2 t	0	0 t 2	8	4 ...
9,10 :	t	e	e t 0	9	1 8 2	7	3 ...

Despite its having a difference alignment, the array does not provide two alternate vertical intervals between cyclic sets. In such arrays, the K-nets of the axis-dyad chords within the array which involve internal T_n arrows cannot coherently be shown to have any particular isographic relation. That is, in Example 4.9(b) the axis-dyad chords, which are generated by parallel shifting of the array's cyclic sets, are not strongly isographic.

Example 4.9(b) Composite isographic relationship between axis-dyad chords C_1 and C_2 drawn from array 10,0/9,10 whose component cyclic sets comprise differing intervals



While the chords share the same T_n and I_n arrows between the corresponding upper and lower trichordal segments, they show a consistent difference of 10 between their internal T_n values. Foley labels the internal segment relationships between C_1 and C_2 , shown in Example 4.9(b), as $\langle W_t \rangle$, which represents the difference between the corresponding internal T_n values.¹²⁵ In other words, the internal $\langle W_n \rangle$ value in her composite isography is generated by the difference in the transpositional values of the upper and lower chordal segments.¹²⁶ Second, the trichordal K-nets can be related by inversion rather than transposition although the array has recurrent vertical intervals. In fact, the K-nets of axis-dyad chords as I introduced here can invariably be related to each other by applying the concept of axial isography. Moreover, these K-nets involving internal I_n arrows may more effectively and more broadly interpret both the symmetrical motions between axis-dyad chords within an array and the various relationships between cyclic arrays. Perle also suggests the possibility of interpreting using inversional arrows even within an array generated from different scales: “We have seen that the compositional realization of an array may often be equally well interpreted in terms of either sum or difference scales.”¹²⁷

The following examples will show how the K-nets of axis-dyad chords which contain internal I_n arrows may interpret diverse harmonies drawn from either one array or from different arrays. As we see in Example 4.10(a), since each of the upper and lower trichordal segments which belong to the same cyclic set, the K-nets share

¹²⁵ Foley, “Arrays and K-nets,” 82.

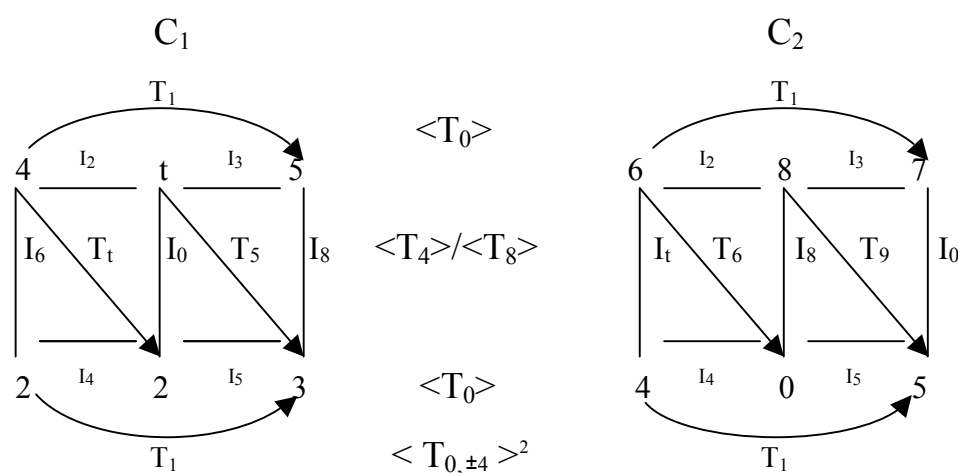
¹²⁶ O’Donnell has proposed a new type of transformation, $\langle W_n \rangle$, in which the I_n values are identical while the T_n values change by equal amounts. O’Donnell, “Transformational Voice Leading,” 162-163.

¹²⁷ Perle, *Twelve-Tone Tonality*, 2ed., 109.

identical T_n and I_n arrows between their corresponding upper and lower segments. The axis-dyad chords can be parsed as 3+3, though it is not upper and lower trichords, but rather two overlapping triangles.

Example 4.10(a) Network interpretations of axis-dyad chords in which the upper and lower trichordal segments are inversionally related

			C_1		C_2		
2,3 :	3	e	4 t 5	9	6 8 7	7	8 ...
4,5 :	1	3	2 2 3	1	4 0 5	e	6 ...



The inner I_n arrows between the trichordal K-nets, however, present complementary values, denoted by $\langle T_n \rangle / \langle T_{-n} \rangle$. For example, the internal $I_6 / I_0 / I_8$ relations (hereafter the internal $I_p / I_q / I_r$ relations will be referred to as $p/q/r$) of the network interpretation of C_1 increase or decrease by 4 over the corresponding I_n values of C_2 . Thus, this example

displays differences of 4/8 between the internal I_n arrows.¹²⁸ For these chords, I propose a new Klumpenhower Transformation $\langle T_{0,\pm n} \rangle^2$ to represent both these relations together, $\langle T_0 \rangle$ between their trichordal segments and $\langle T_n \rangle / \langle T_{-n} \rangle$ between the corresponding internal I_n arrows. In Example 4.10(a), the K-net of C_1 maps onto the K-net of C_2 at $\langle T_{0,\pm 4} \rangle^2$. That is, the left subscript prior to the comma, 0, represents the identical T_n/I_n arrows between the upper and lower trichordal segments and the right numerical subscript after comma, ± 4 , indicates the differences between their internal I_n values, 6/0/8 and t/8/0, and also the same differences between their internal T_n values, drawn diagonally as t/5 and 6/9. Due to the symmetrical properties inherent within an array, caused by inversionally related interval cycles within each of its cyclic sets, all axis-dyad chords from the same array will always provide the same $\langle T_{0,\pm n} \rangle^2$ relations: $\langle T_0 \rangle$ in the outer segment relations and $\langle T_{\pm n} \rangle$ in the internal segment relations.¹²⁹ I will define this type of K-net, which combines such $\langle T_0 \rangle$ and $\langle T_{\pm n} \rangle$ relationships, as *strongly composite isography*. In fact, this type of isography may also be realized as a type of axial isography since the outer I_n arrows stay the same while the internal I_n arrows vary.

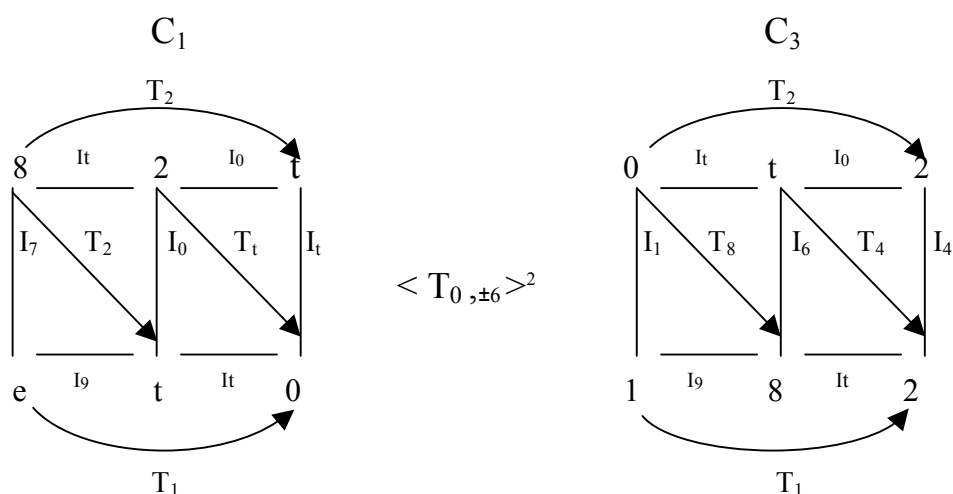
Example 4.10(b) also interprets the C_1 and C_2 chords as K-nets retaining their internal I_n arrows.

¹²⁸ Example 4.10(a) also shows differences of 4/8 between the internal T_n arrows.

¹²⁹ Stoecker demonstrates that within an array axis-dyad chords are internally symmetrical, mapping onto themselves, and are also inversionally related to each other. That is, a chord drawn from the array 0,1/7,8, for instance, 485/e80, can map to itself by I_4 (4+0, 8+8, 5+e=4) and can be inversionally related to another axis-dyad chord in the array. See Stoecker, "Studies in Post-Tonal Symmetry," 99-100.

Example 4.10(b) Network interpretations of axis-dyad chords C_1 and C_2 drawn from array 10,0/9,10 whose component cyclic sets comprise differing intervals

	C_1					C_2				
10,0 :	6	4	8	2	t	0	0	t	2	8 4 ...
9,10 :	t	e	e	t	0	9	1	8	2	7 3 ...



In this array of different cycles, 10,0/9,10, the C_1 chord moves to the C_2 chord by shifting both the upper and lower trichordal segments two places to the right. Each note in the upper and lower segments of C_1 moves up or down four or two semitones.

Example 4.10(b) illustrates that the K-net of C_1 is related by strongly composite isography, $\langle T_{0,\pm 6} \rangle^2$, to the K-net of C_2 ; the right numerical subscript, ± 6 , indicates the differences between internal I_n arrows, 7/0/ t and 1/6/4.

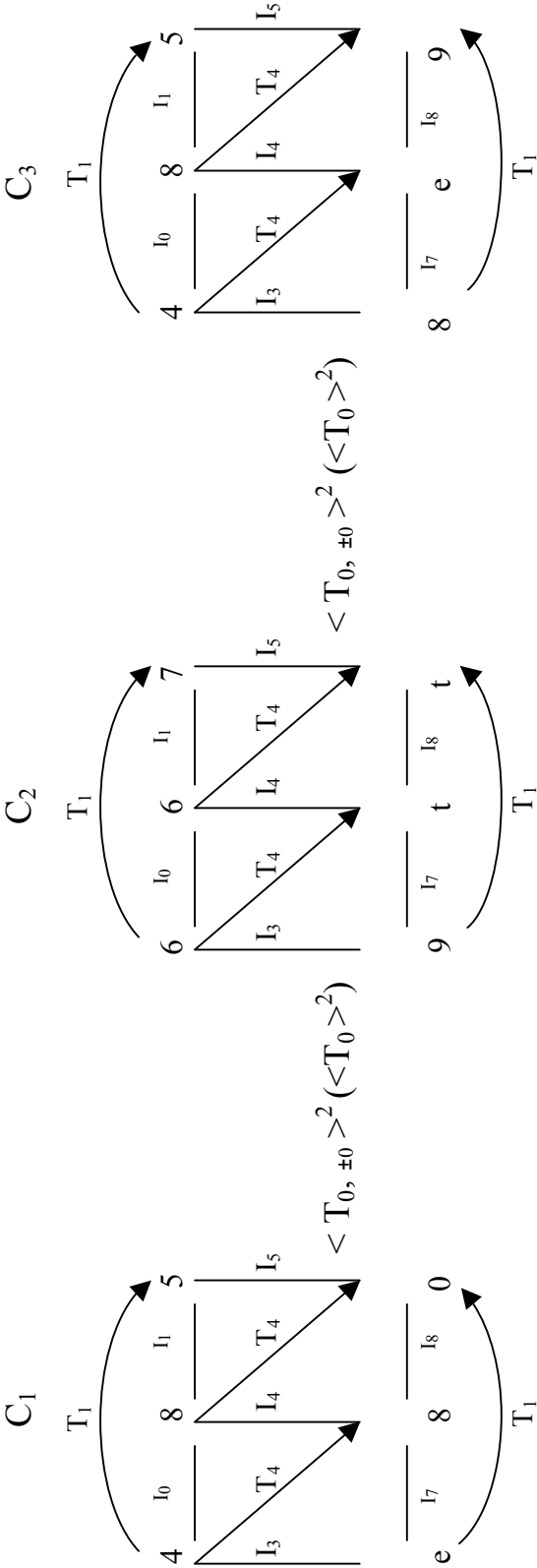
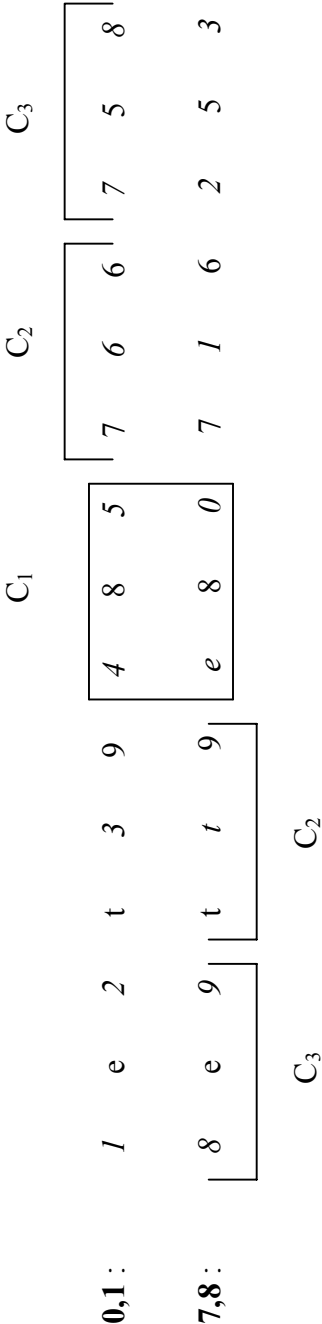
The $\langle T_{0,\pm n} \rangle^2$ relation thus not only represents the way in which the corresponding upper and lower trichordal segments of the axis-dyad chords are symmetrically related to each other in the same cyclic array but also indicates more precisely the degree to which the upper and lower trichords are symmetrically aligned within an array. We can see in Example 4.11 that there is another way to generate axis-dyad chords yielding $\langle T_{0,\pm n} \rangle^2$ relationships within an array by moving either the upper or the lower trichordal segment.¹³⁰

¹³⁰ In his “Thoughts on Klumpenhower Networks,” 196, Lewin labels such shifting of a single cyclic set as “SLIDE (n)” where n is equal to the number of places either the upper or lower cyclic set has been slid.

In the array 0,1/7,8, the upper cyclic set is fixed while the lower segments shift two units to the right or to the left. This example illustrates that the network interpretations of these chords have identical outer T_n and I_n arrows. In their internal relations, however, the differences between the analogous I_n arrows are $+1/-1$ from C_1 to C_2 and from C_2 to C_3 . Thus, we can see that the network interpretations of the axis-dyad chords have strongly composite isography with each other, represented by $\langle T_{0,\pm 1} \rangle^2$.

Example 4.12 presents another possible strong composite isography generated by symmetrical shifts between the upper and lower trichordal segments, yielding a traditional strong isography.

Example 4.12 Strongly composite isographic relations among axis-dyad chords from symmetrical shifting of cyclic sets
(array 0,1/7,8 comprising same interval 1cycles)



This type of strongly composite isography, however, is limited to the upper and lower trichordal segments which belong to the same trichordal interval cycle. In this example C_2 and C_3 are generated from C_1 by moving the upper trichordal segment three or six units to the right, and the lower segment three or six units to the left. The network interpretations of these axis-dyad chords all contain the same outer and internal I_n arrows. Both the symmetrical motion between the corresponding upper trichords and the lower trichords and the symmetrical shifts between the upper and lower segments is reflected by $\langle T_0 \rangle$ and $\langle T_{\pm 0} \rangle$, providing an example of strongly composite isography at $\langle T_{0,\pm 0} \rangle^2$. For the strongly composite isography, the numerical subscript after the comma, $\pm n$, can occasionally be omitted for convenience and thus simply be labeled $\langle T_0 \rangle^2$.

Example 4.13 provides an example of the use of this type of composite isography from Perle's music, a short passage of six chords in measures 62-63 from the third movement of his *Seventh Quartet* (1973).

Example 4.13 Perle, *Seventh Quartet*, III (1973), mm.62-63



	C_1	C_2	C_3	C_4	C_5	C_6
0,8:	e 1 7	5 7 1	t 2 6	7 5 3	7 5 3	5 7 1
5,6:	4 1 5	t 7 e	3 2 4	7 t 8	6 e 7	t 7 e

Below the music we can see Perle's realization of the axis-dyad chords within the array, 0,8/5,6. I have labeled six axis-dyad chords provided in this passage as C_1 to C_6 .¹³¹ The cyclic array for this passage, which is a combination of two kinds of cycles, involves an interval 8 cycle and an interval 1 cycle. Example 4.14 illustrates the way in which the upper and lower trichordal segments of the first six chords are drawn from each kind of cyclic set.

Example 4.14 Sources of Upper and lower trichordal segments of chords C_1 - C_6 from array 0,8/5,6

	C_1	C_2	C_3	C_4	C_5	C_6
0,8:	e 1 7	5 7 1	t 2 6	7 5 3	7 5 3	5 7 1
5,6:	4 1 5	t 7 e	3 2 4	7 t 8	6 e 7	t 7 e

5,6 : $\boxed{3 \ 2 \ 4} \boxed{1 \ 5} \ 0 \ \boxed{6 \ e \ 7} \boxed{t \ 8} \ 9 \ 9 \ 8 \ \boxed{t \ 7 \ e} \ 6 \ 0$
 $\qquad\qquad C_3 \qquad C_1 \qquad\qquad C_5 \qquad C_4 \qquad\qquad C_2 (=C_6)$

0,8(i): $\boxed{5 \ 7 \ 1} \boxed{e} \ 9 \ \boxed{3 \ 5 \ 7} \ 1$
 $\qquad\qquad C_1 \qquad\qquad C_2 (=C_6) \qquad\qquad C_4 (=C_5)$

0,8(ii) : $6 \ 2 \ t \ \boxed{t \ 2 \ 6} \ 6 \ 2$
 $\qquad\qquad\qquad\qquad\qquad\qquad C_3$

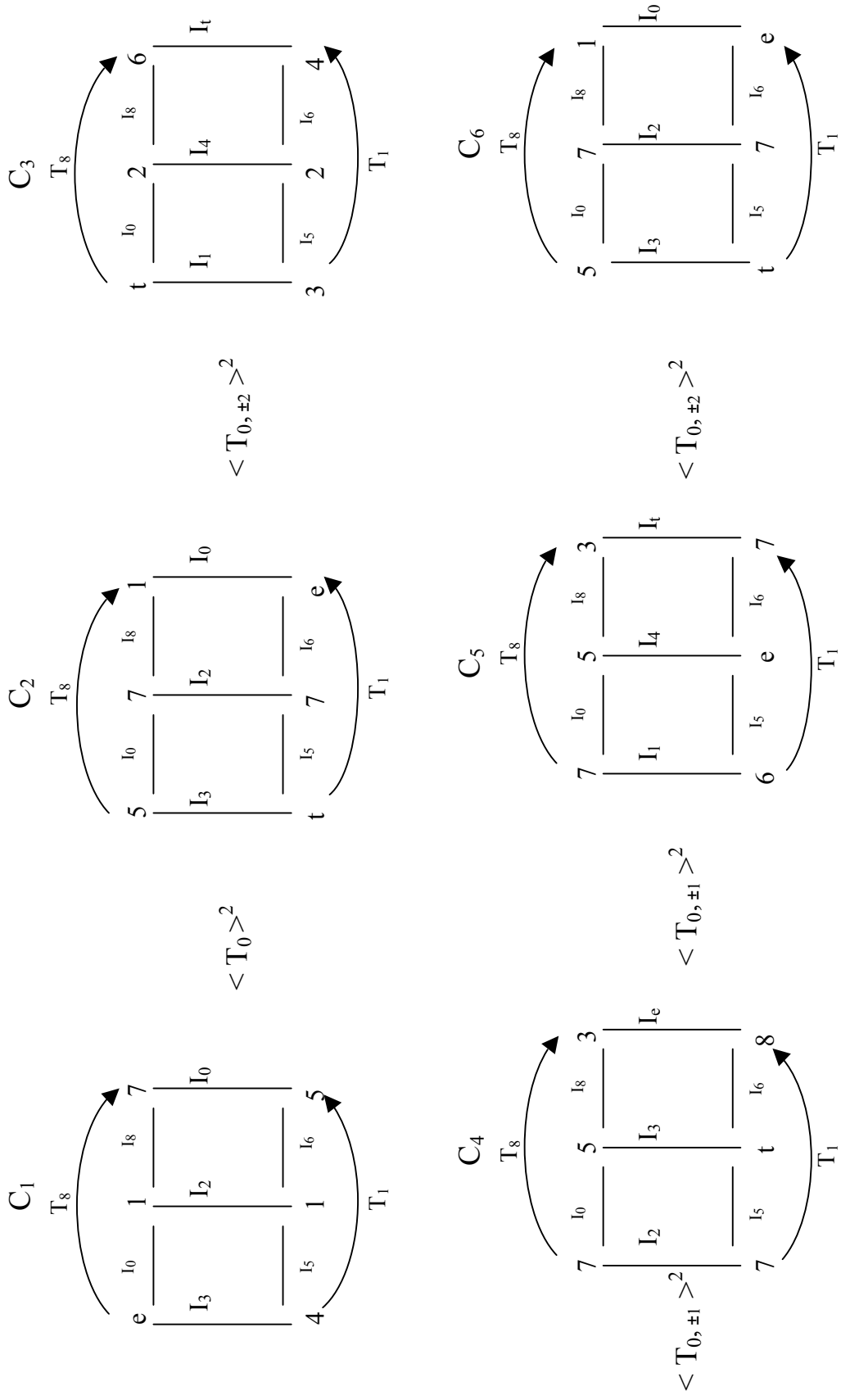
0,8(iii) : $0 \ 0 \ 8 \ 4 \ 4 \ 8 \ 0$

¹³¹ This example is a part of Perle's Example 63 in *Twelve-Tone Tonality*, 2ed., 133.

We can obtain all the lower trichordal segments by shifting within the cyclic set, 5,6. On the other hand, since the 8 cycle does not complete the entire aggregate of twelve pitch classes, the upper cyclic set, 0,8, must be partitioned into three segments. Notice that it is rather difficult to explain how each of the upper and lower segments are paired to generate these chords and also how the axis-dyad chords are interrelated within the same array.¹³² Example 4.15(a) interprets six axis-dyad chords as K-nets.

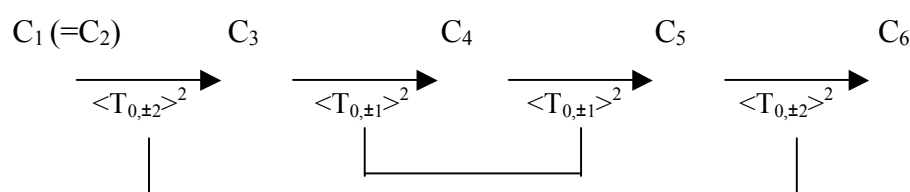
¹³² Foley addresses the difficulty explaining how the axis-dyad chords arise from different cyclic partitions and different alignments of an array in her “Arrays and K-nets,” 89-90.

Example 4.15(a) Strongly composite isographic relationships between successive chords C_1 - C_6



Since all six chords are drawn from the same array, 0,8/5,6, all the network interpretations present identical $\langle T_0 \rangle$ relations between the outer I_n arrows. We can also see in Example 4.13 that in the C_1 - C_3 progression, the symmetrical motion between axis-dyad notes and neighbor notes are audible in pitch space while in the subsequent chords the symmetry occurs only in pitch-class space. Note that in Example 4.15(a) the corresponding internal I_n arrows are either the same or varied by $\pm n$. The first two chords, C_1 and C_2 , share a ± 6 transpositional relationship between the pcs of their corresponding upper and lower trichordal segments. That is, the sum of the transpositional values (± 6 and ± 6) of the upper and lower segments yield the same internal I_n values and thus generate the $\langle T_0 \rangle^2$ relationship between C_1 and C_2 . On the other hand, the chords C_4 and C_5 hold the upper segment invariant while the pcs of the lower segment are transposed by ± 1 . The numerical subscript after the comma in $\langle T_{0,\pm n} \rangle^2$, in this case ± 1 , reflects the sum of the transpositional values of the upper and lower chordal segments. All network interpretations of the axis-dyad chords, C_1 - C_6 , thus show strongly composite isography, labeled $\langle T_{0,\pm n} \rangle^2$, where the right subscript, $\pm n$, equals the complementary difference between analogous internal I_n arrows. Example 4.15(b) notates the series of transformations $\langle T_{0,\pm 2} \rangle^2$, $\langle T_{0,\pm 1} \rangle^2$, $\langle T_{0,\pm 1} \rangle^2$, and $\langle T_{0,\pm 2} \rangle^2$.

Example 4.15(b) Symmetrical layout of the same transformational relationship in succession of chords C_1 - C_6

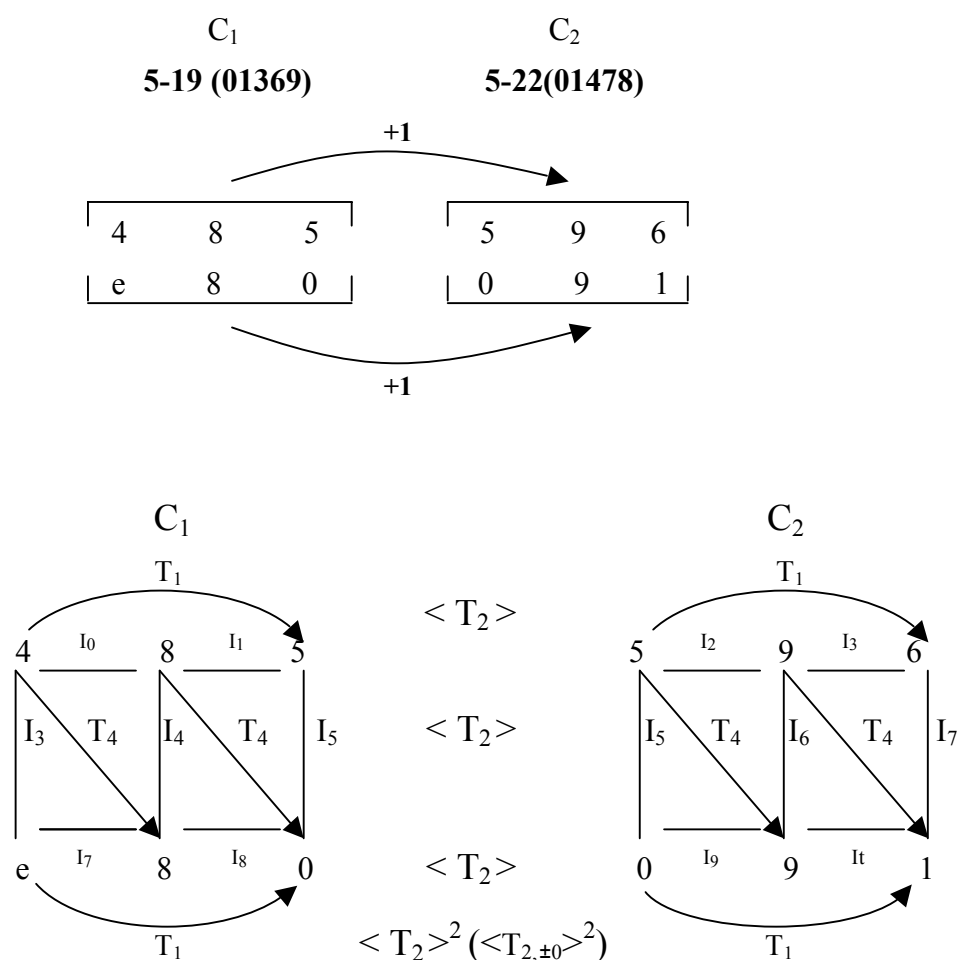


There are also two pairs of network interpretations of C_1 - C_6 and C_3 - C_5 which have exactly the same outer and internal I_n arrows, as expressed by $\langle T_0 \rangle^2$ (or $\langle T_{0,\pm 0} \rangle^2$).

Example 4.15(b) thus finally shows the symmetrical display of the same transformational relationships in the succession of the chords.

As I have examined above, any pair of axis-dyad chords derived from the same array can be associated by using strongly composite isography. More significantly, different examples of Perle's music provide many different cyclic arrays rather than relying on only one array. Therefore, modulating from one array to another is a significant process in the compositional realizations in Perle's music. Example 4.16 illustrates that all pitch-classes of the axis-dyad chord C_1 have been transposed by the same value, T_1 , in C_2 .

Example 4.16 Positively composite isography between axis-dyad chords drawn from arrays 0,1/7,8 and 2,3/9,10



Both chords share the same interval system while the corresponding tonic sums exhibit a consistent difference of 2. The network interpretations of C_1 and C_2 shown in Example 4.16 thus display the same difference of 2 between the outer I_n arrows and also between the corresponding internal I_n arrows while the internal and outer T_n arrows are now identical. Note that the network interpretations of C_1 , a member of set class 5-19 (01369), and C_2 , a member of set class 5-22 (01478), have identical T_n arrows while analogous I_n arrows vary by equal amounts, illustrating *positively composite isography*.

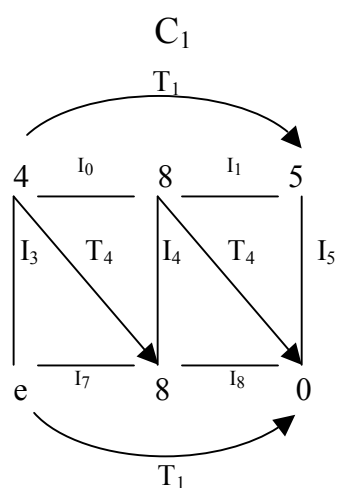
I will use $\langle T_{n, \pm n} \rangle^2$ ($n=n$ or $n \neq n$) for this type of isography, which can also be simply represented by a $\langle T_n \rangle^2$ relation.

Example 4.17 shows axis-dyad chords each drawn from different arrays, 0,1/7,8 and 2,3/9,10, in which all pitch classes of the axis-dyad chord C_1 are transposed by T_1 in C_2 .

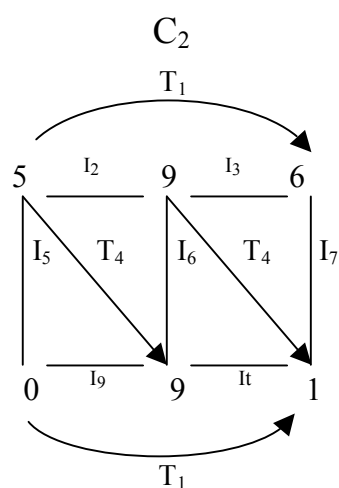
Example 4.17 Positively composite isography between axis-dyad chords of arrays
 0,1/7,8 and 2,3/9,10 drawn from symmetrical shifting of cyclic sets

					C ₁							
0,1:	2	t	3	9	4	8	5	7	6	6	7	5
7,8:	9	t	t	9	e	8	0	7	1	6	2	5

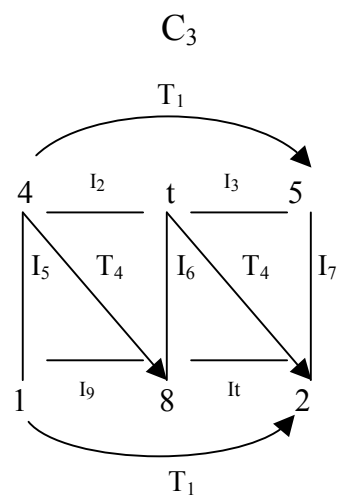
					C_2							
0,1:	3	e	4	t	5	9	6	8	7	7	8	6
7,8:	t	e	e	t	0	9	1	8	2	7	3	6
								C_3				



$\langle T_2 \rangle^2$



$\langle T_2 \rangle^2$



The network interpretations of C_1 and C_2 are represented by $\langle T_{2,\pm 0} \rangle^2$ (or $\langle T_2 \rangle^2$), since the outer and internal I_n arrows present the same difference of 2. Within the array 2,3/9,10 shown in Example 4.17, C_3 is generated from C_2 by symmetrical shifts between the upper and lower trichordal segments; the upper trichordal segment moves two units to the left and the lower segment two units to the right. Both network interpretations of these chords, C_2 and C_3 , contain the same outer and internal I_n and T_n arrows. Thus, in the C_1 - C_3 progression the symmetrical shifts between the upper and lower segments after transposing $\langle T_2 \rangle^2$ to the new array 2,3/9,10 are reflected by $\langle T_2 \rangle$ and $\langle T_{\pm 0} \rangle$, providing an example of positively composite isography at $\langle T_{2,\pm 0} \rangle^2$ (or $\langle T_2 \rangle^2$).

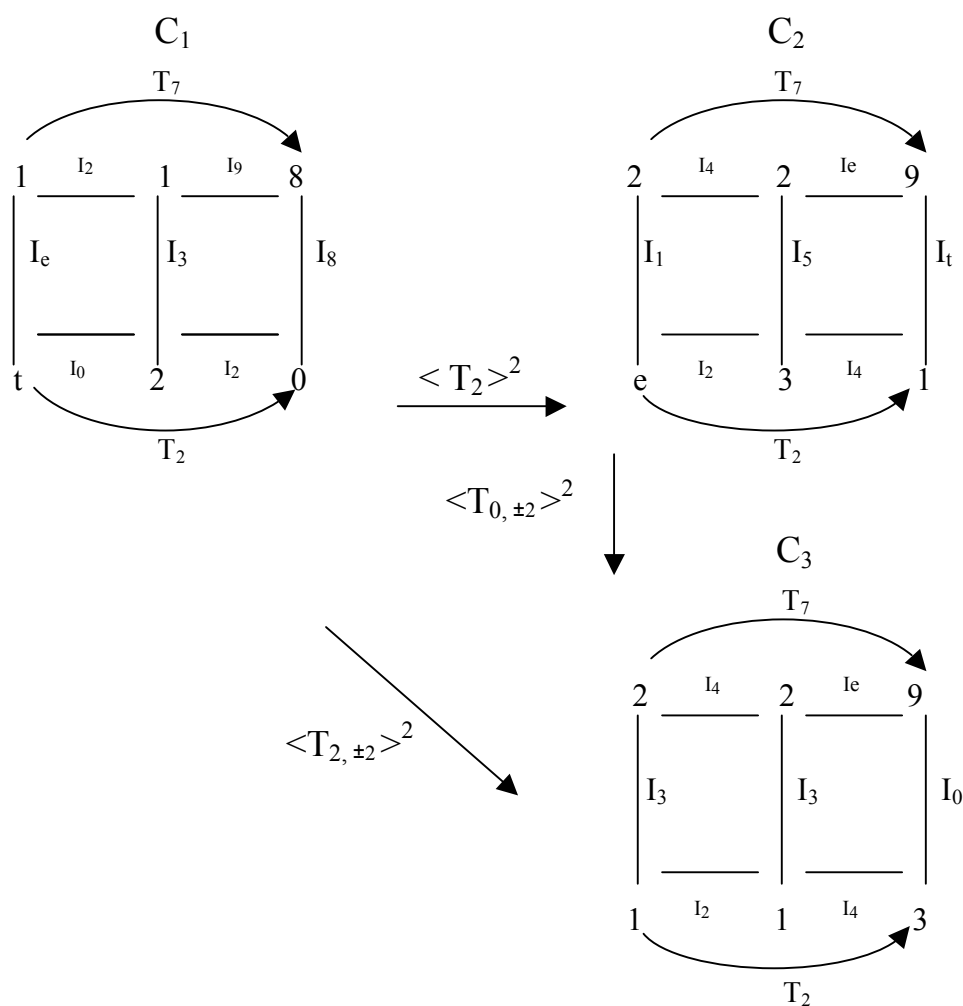
We can also see in Example 4.18(a) that the arrays 2,9/0,2 and 4,e/2,4, both comprising the same cyclic intervals 7 and 2, are related by T_2 .

Example 4.18(a) Positively composite isography between axis-dyad chords of arrays 2,9/0,2 and 4,11/2,4 drawn from asymmetrical shifting of cyclic sets

					C_1								
2,9:	<i>e</i>	3	6	8	$\begin{bmatrix} 1 & 1 & 8 \\ t & 2 & 0 \end{bmatrix}$	6	3	<i>e</i>	<i>t</i>	4			
0,2:	6	6	8	4		0	2	<i>t</i>	4	8			

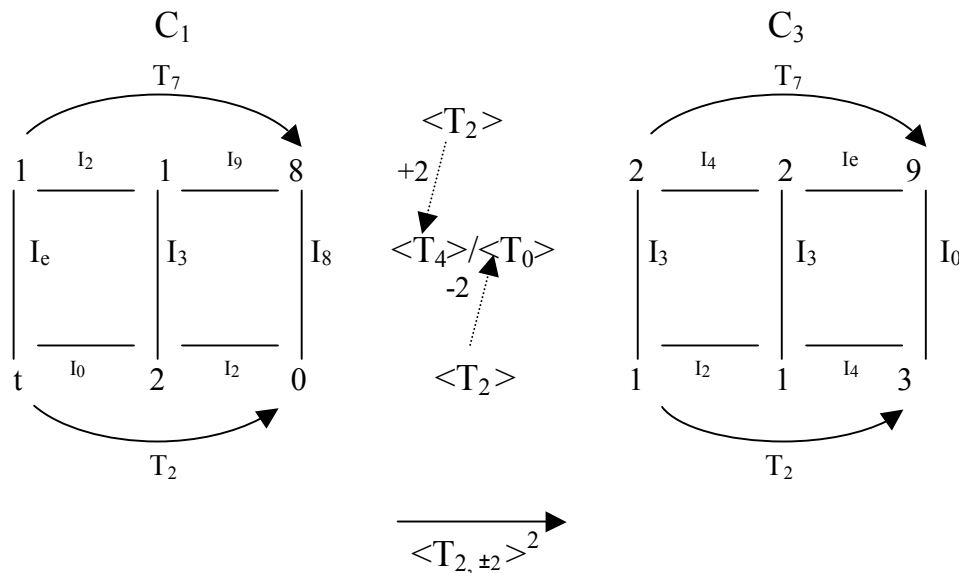
					C_2								
4,e:	0	4	7	9	$\begin{bmatrix} 2 & 2 & 9 \\ e & 3 & 1 \end{bmatrix}$	7	4	0	<i>e</i>	5			
2,4:	7	7	9	5		1	3	<i>e</i>	5	9			

C_3



That is, both chords C_1 and C_2 share the same T_n arrows while their outer and internal I_n arrows display a consistent difference of 2. The inner I_n arrows between C_1 and C_3 , however, do not present the same difference as their outer I_n arrows. Since C_3 is generated from C_1 by transposing to C_2 in the new array and then shifting only the lower segment two units to the right, C_1 and C_3 are positively composite isographic at $\langle T_{2,\pm 2} \rangle^2$, which is the sum of $\langle T_{2,\pm 0} \rangle^2$ and $\langle T_{0,\pm 2} \rangle^2$. We can also obtain the resultant right numerical subscript after the comma, ± 2 , within the angle brackets, by subtracting the differences of 2/2 between the outer I_n arrows from the differences of 4/0 between the internal I_n arrows as displayed in Example 4.18(b).

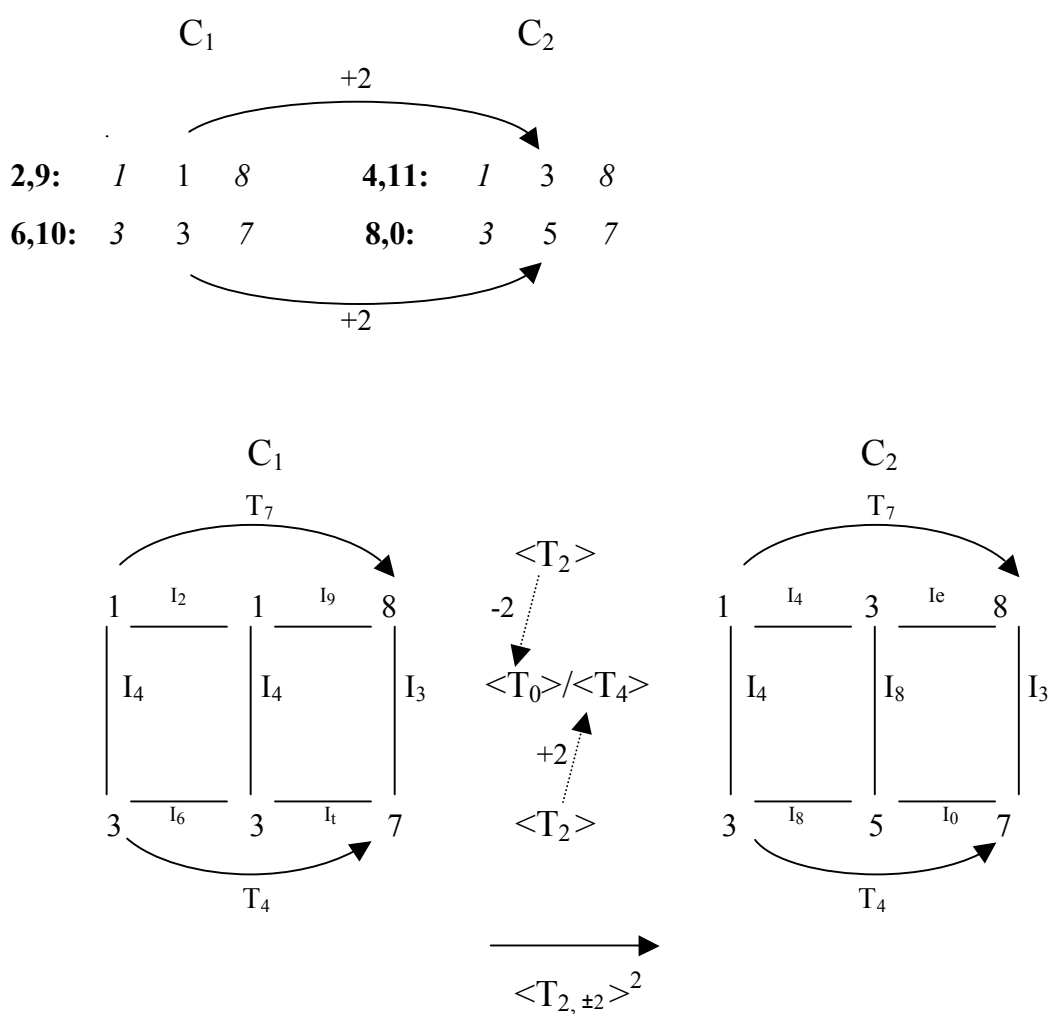
Example 4.18(b) Positively composite isography between axis-dyad chords of arrays 2,9/0,2 and 4,11/2,4



Just as in traditional positive isography, $\langle T_{n, \pm n} \rangle^2$ tells us by how much the axis notes and the neighbor notes of each trichordal segment do not move in a symmetrical way and also by how much the upper and lower trichords are symmetrically aligned after transposing to a new array.

Example 4.19 also shows an instance of positively composite isography generated by an asymmetrical transpositional relationship between chords from different arrays, providing a $\langle T_{n, \pm n} \rangle^2$ relationship.

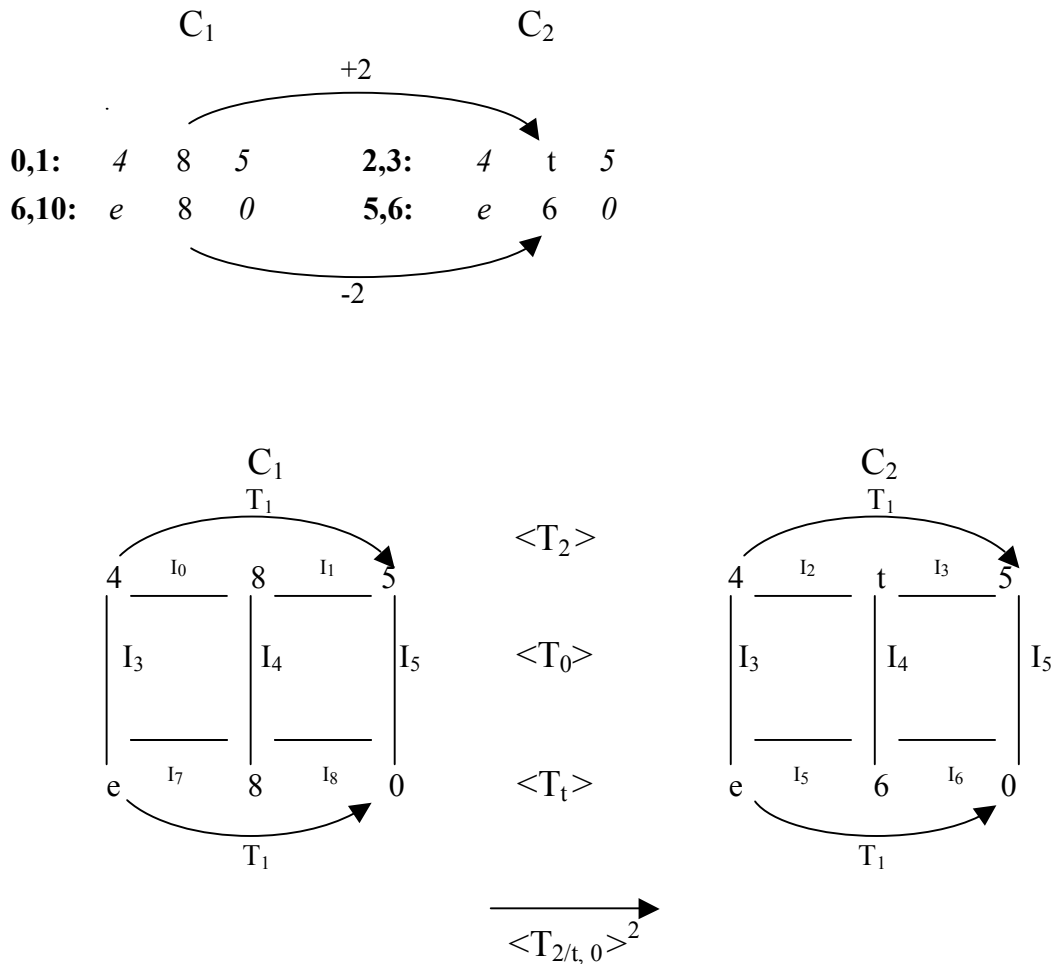
Example 4.19 Positively composite isography between axis-dyad chords in an asymmetrical transpositional relationship from arrays 2,9/6,10 and 4,11/8,0



The moves which hold the neighbor-notes of the lower and upper segments while increasing the axis-note dyad by 2 will produce the $\langle T_2, \pm 2 \rangle^2$ relationship. In this example, the left number within the angle brackets, 2, represents the difference between the outer I_n arrows and the right number after the comma, ± 2 , indicates the differences between the outer and internal segment relationships, $\langle T_2 \rangle / \langle T_2 \rangle$ and $\langle T_0 \rangle / \langle T_4 \rangle$.

Now we can see other possible modulations frequently used in Perle's music, which may broadly fall into another type of composite isography. Among the many possibilities, I will examine two ways in which the axis-dyad chords between the upper and lower trichordal segments are aligned by the same I_n value, yielding internal $\langle I_0 \rangle$, while their corresponding upper and lower trichordal segments are related by positive and axial isographies. Example 4.20 shows the first instance of this type of composite isography in that an integer, in this case of 2, is added to the upper axis note of C_2 while the same integer is subtracted from the lower axis note of C_2 .

Example 4.20 Composite isographic relationship between axis-dyad chords from arrays 0,1/7,8 and 2,3/5,6 comprising the same internal I_n arrows



This produces a new array, 2,3/5,6, in C_2 . The network interpretations of C_1 and C_2 in Example 4.20 show complementary differences between the outer I_n arrows, as denoted by $\langle T_2 \rangle$ and $\langle T_t \rangle$, while their internal I_n arrows are all the same. Thus these axis-dyad chords do not share the same secondary difference patterns (7/0/7 in the array 0,1/7,8 and 7/8/7 in the array 2,3/5,6). Note that the K-nets of C_1 and C_2 may be considered as having a type of axial isography since the internal I_n arrows are the same while the outer

I_n arrows vary. The K-nets of C_1 and C_2 in Example 4.20 can be labeled $\langle T_{2/t,0} \rangle^2$. The left and right subscripts prior to the comma indicate the difference between the I_n arrows of the upper segments and the difference between the I_n arrows of the lower segments respectively. The middle $\langle T_0 \rangle$ relation between the corresponding internal I_n arrows will always be represented by the numerical subscript 0 after comma without $\pm$ sign.

Let us look at one of many examples using this type of composite isography, a small fragment of “Etude No.6” from Perle’s *Six New Etudes for Piano*. Example 4.21(a) shows the two subject heads provided in m.1 and m.12.

Example 4.21(a) Perle, *Six New Etudes for Piano*, No.6, m.1 and m.12.

m.1

$\text{♩} = 152$
sempre legato
ppp

C_1 C_2 C_3 C_4

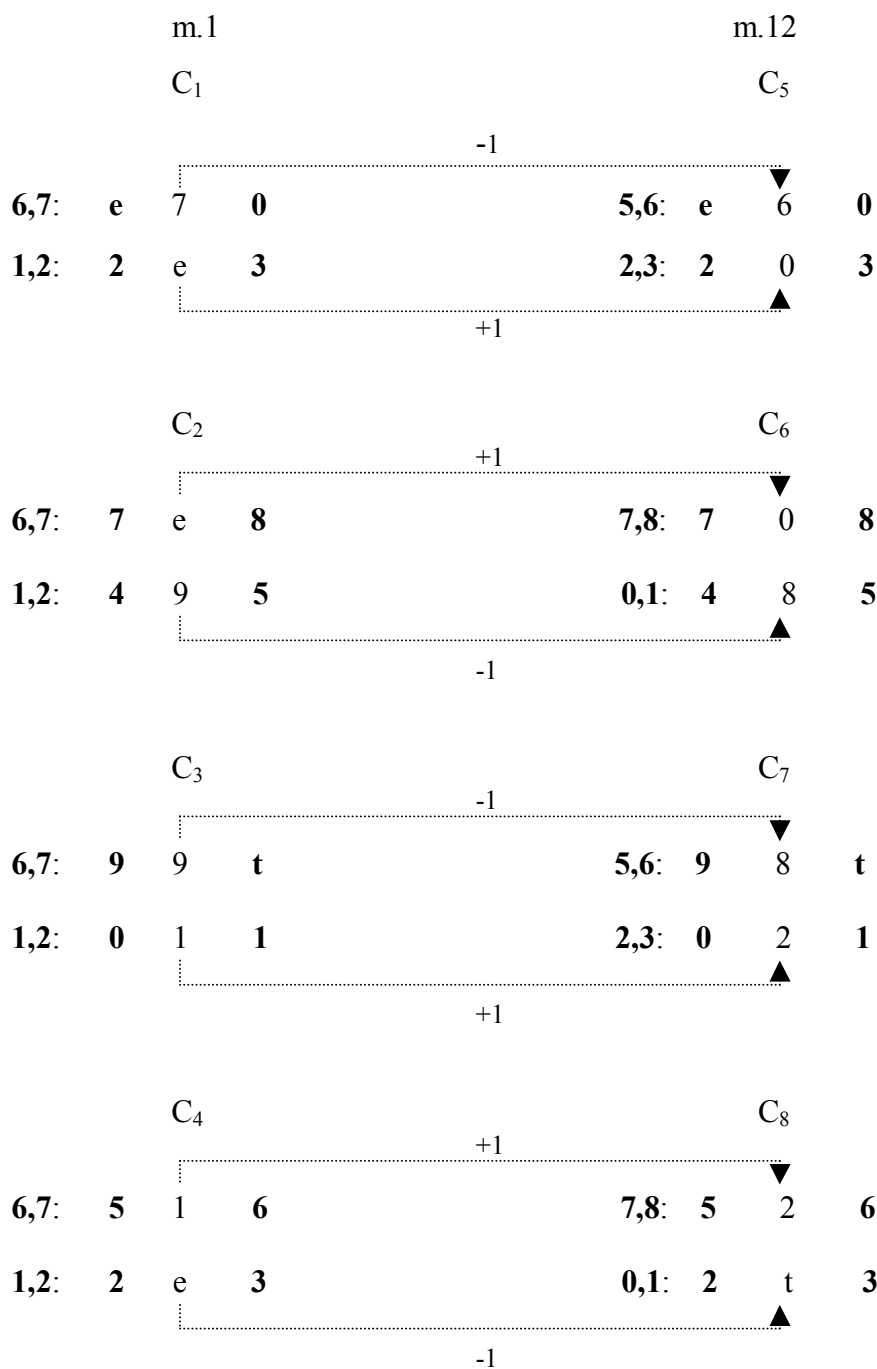
C_5 C_6 C_7 C_8

m.12

Although the treble part of m.1 is unchanged in m.12, the pitches of the bass part in m.1 move either down or up by a semitone in m.12. Example 4.21(b) illustrates how the

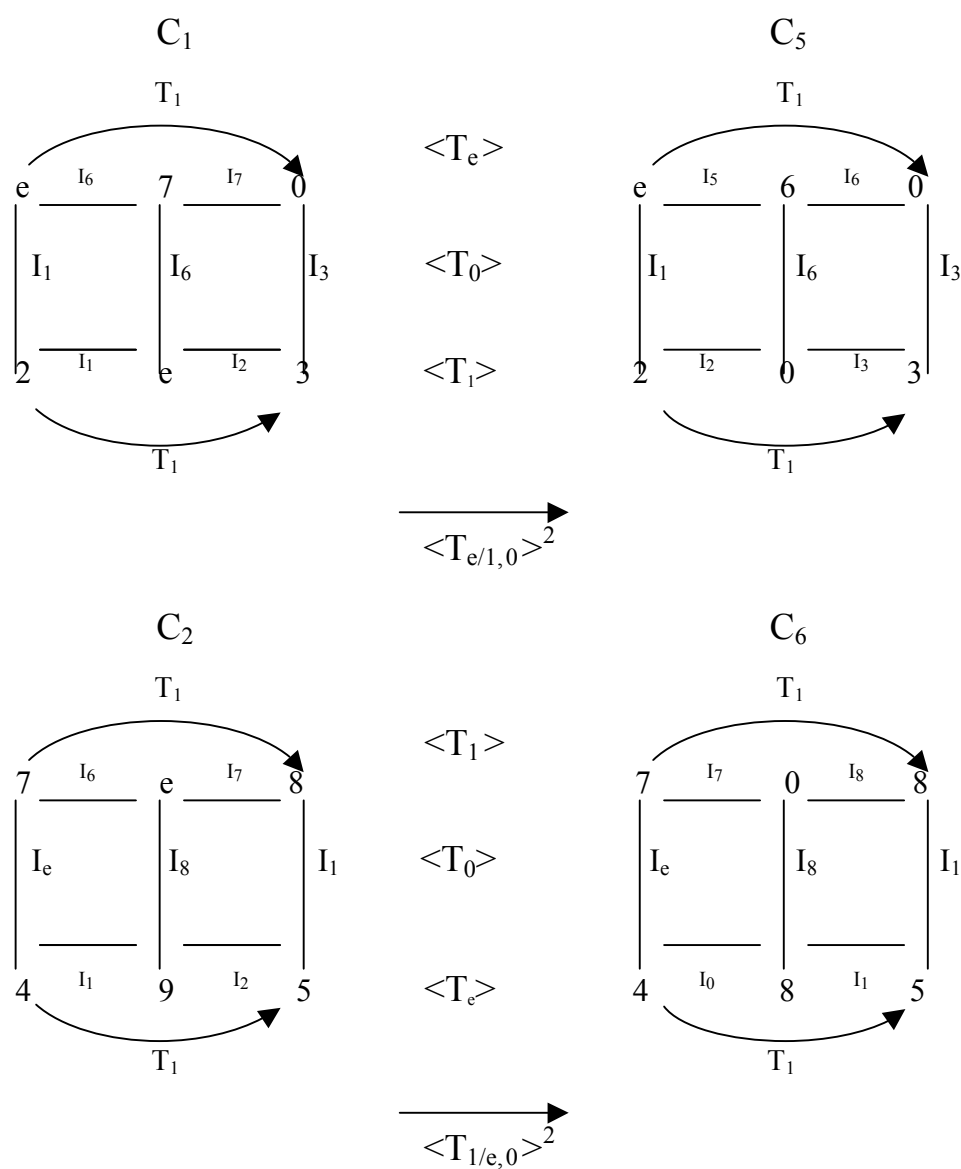
axis-dyad chords of m.1 retain the neighbor notes of the corresponding chords in m.12 while transposing the axis notes by T_1 or T_e .

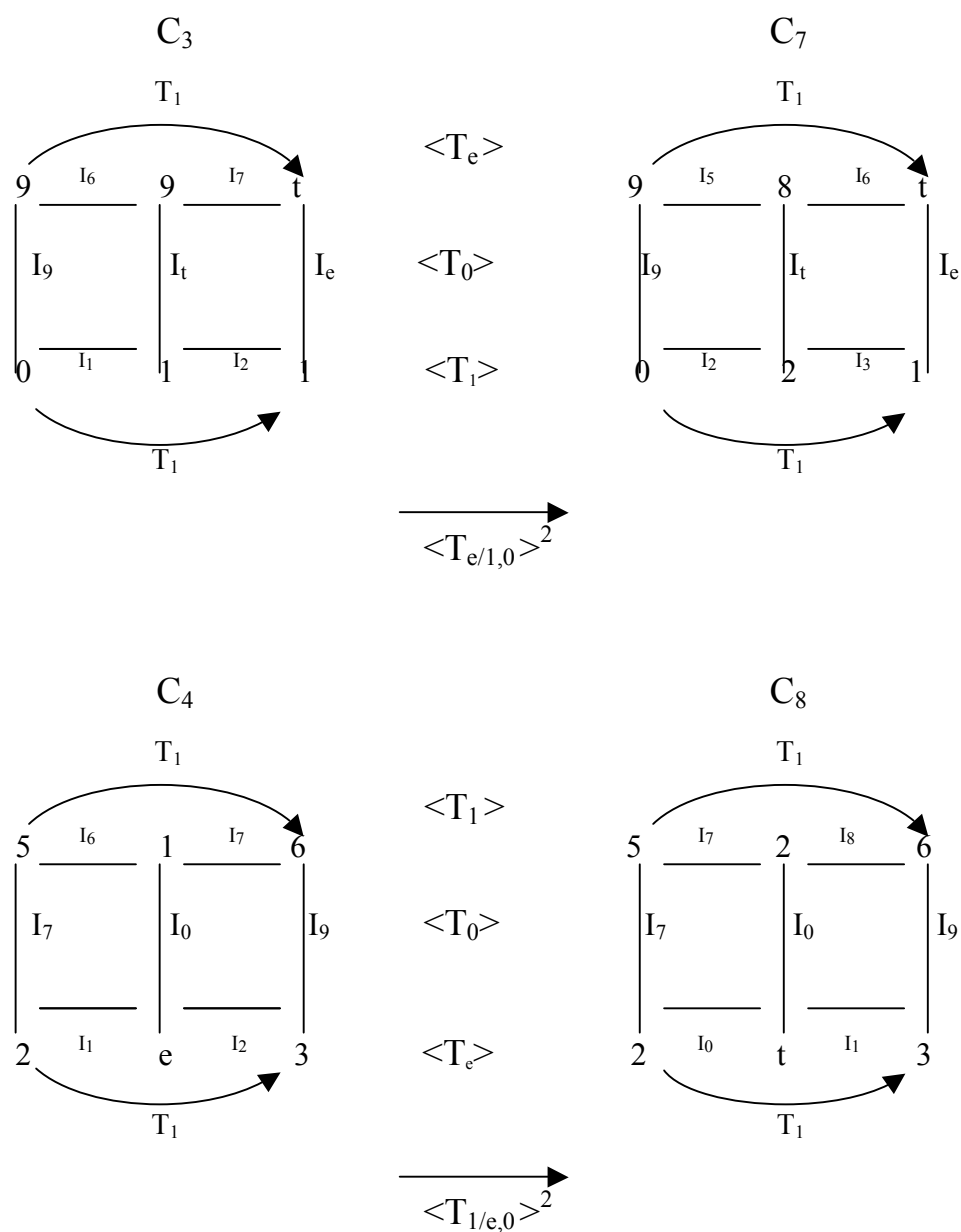
Example 4.21(b) Cyclic arrays and axis-dyad chords for Perle's Six New Etudes for Piano, No.6, m.1 and m.12



That is, the alternations of axis dyad are realized in the bass part, bringing new arrays, 5,6/2,3 and 7,8/0,1, in m.12. Example 4.21(c) interprets the eight axis-dyad chords as K-nets, presented in m.1 and m.12: they are labeled C_1 - C_8 .

Example 4.21(c) Isographic relationships between axis-dyad chords from m.1 and m.12



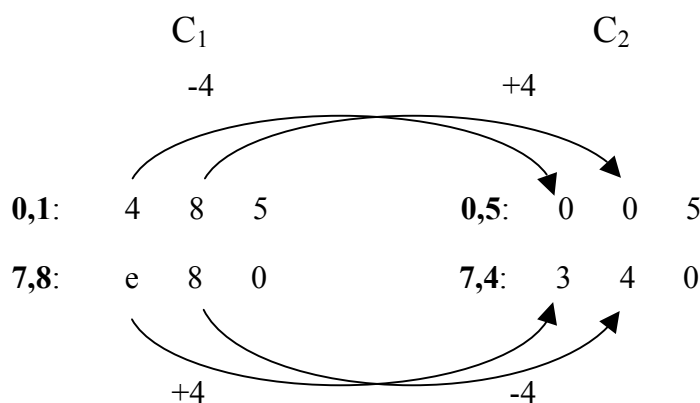
(Example 4.21c continued)

The K-nets of C_1 to C_4 share the same internal I_n arrows with each of the corresponding K-nets of C_5 to C_8 , while their outer I_n arrows present complementary $\langle T_e \rangle$ and $\langle T_1 \rangle$ relations. For this type of composite isography, we can derive the labels $\langle T_{e/1,0} \rangle^2$ or

$\langle T_{1/e,0} \rangle^2$ as the pertinent mappings of the K-nets of C_1 - C_4 to the corresponding K-nets of C_5 - C_8 . In other words, the $\langle T_{n/-n,0} \rangle^2$ relations among the axis-dyad chords of this example display traditional positive isography in the outer segment relationships while their internal I_n arrows are identical. In Perle's twelve-tone tonality, this type of composite isography also describes the shift between arrays in the same key, while positively composite isography allows for the shift between arrays in the same mode.¹³³

For the second instance of this type of composite isography, a modulation may occur between arrays which share only a single tonic sum in each of the upper and the lower trichordal segments. Example 4.22 illustrates that the shift from 0,1/7,8 to 0,5/7,4 is achieved by moving one of a pair of neighbor dyads and the axis dyad in opposite directions while holding the remaining neighbor dyad as it is.

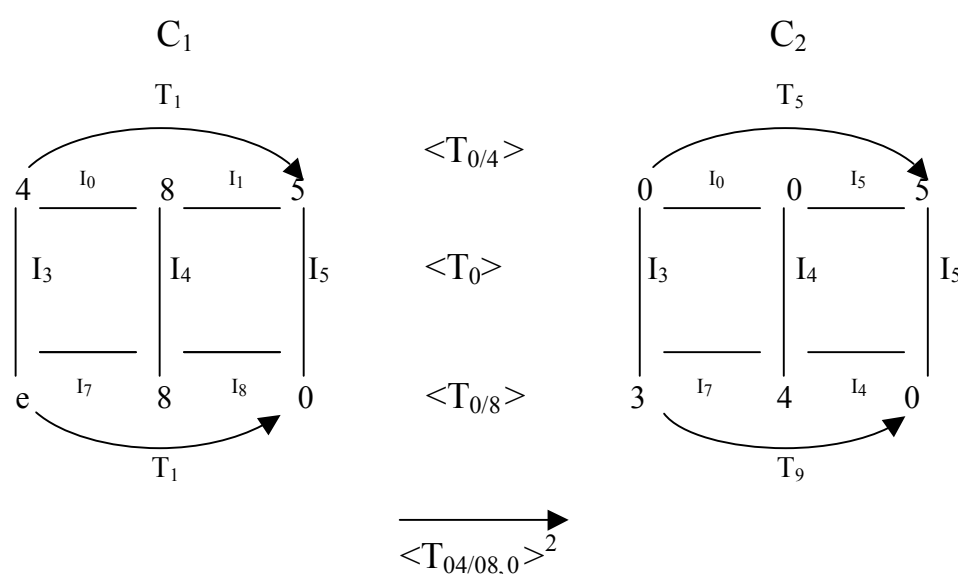
Example 4.22 Axis-dyad chords in an asymmetrical transpositional relationship drawn from arrays 0,1/7,8 and 0,5/7,4 of differing cyclic intervals



¹³³ The key and mode relationships can be derived from the sums and differences between tonic sums. The given arrays, 6,7/1,2 and 5,6/2,3, have the shared key of 8,8: $6+2,7+1=8,8$ and $5+3,6+2=8,8$. See Perle, *Twelve-Tone Tonality*, 2ed., 47.

By doing this the arrays in different interval systems are connected. Example 4.23 presents network interpretations of C_1 and C_2 , demonstrating axial isography between the corresponding upper and lower segments while the analogous internal I_n arrows are identical.

Example 4.23 Composite isographic relationship between axis-dyad chords from arrays 0,1/7,8 and 0,5/7,4, comprising the same internal I_n arrows



In this example, we can derive a $\langle T_{0/4} \rangle$ relation from the axial isography between the upper trichords and a $\langle T_{0/8} \rangle$ relation between the lower trichords.¹³⁴ The K-nets, however, share the same internal inversive relations, 3/4/5. Throughout the course of this chapter, any $\langle T_n \rangle$ relations between the corresponding upper and lower segments are always represented by the numerical subscript prior to the comma ; $\langle T_{04/08,0} \rangle^2$ in Example 4.23 is the result of combining $\langle T_{0/4} \rangle$ between the corresponding upper

¹³⁴ In chapter 2 I have changed Stoecker's labeling for axial isography, $\langle T_{n/-n} \rangle$, into $\langle T_{0/n} \rangle$ for indicating the spatial transformation in the same inversive zone.

segments and $\langle T_{0/8} \rangle$ between the corresponding lower segments. The middle $\langle T_0 \rangle$ relation between corresponding internal I_n arrows is indicated by the numerical subscript 0 after the comma.

Example 4.24(a) provides one of multiple instances of this type of composite isography in Perle's music. The example excerpts the last passage (mm.83-89) from the fourth etude of *Six Etudes for Piano*.¹³⁵

Example 4.24(a) Perle, *Etude No.4*, mm.83-89.

0,4/11,9 C₁ C₂ C₃ C₄

C₅ C₆ C₇ C₈

0,3/11,10

C₉ C₁₀ C₁₁ C₁₂ C₁₃

0,2/11,11 0,1/11,0

¹³⁵ This is Perle's Example 64 in *Twelve-Tone Tonality*, 2ed., 143.

The subject of this piece is restated in mm.83-84 and in mm.85-86 with only small changes in pitch. In the following measures, however, a truncated repeat of the subject omits the initial chord at m.87 and the first three chords at m.89. In Example 4.24(a), I have labeled all thirteen axis-dyad chords as C_1 to C_{13} except for some tetrachordal segments. The cyclic array for this passage is provided in Example 4.24(b).

Example 4.24(b) Axis-dyad chords for Perle's Etude No.4, mm.83-89

m.83						m.84					
C ₁			C ₂			C ₃			C ₄		
t	2	2	2	t	6	t	2	6	2	t	6
4	7	2	9	2	7	4	7	2	5	6	3
						0,4/11,9					
						m.86					
C ₅			C ₆			C ₇			C ₈		
t	2	2	2	t	6	3	9	7	1	e	4
4	7	2	9	2	7	4	7	2	6	5	5
						0,3/11,10					
						m.88					
						C ₉			C ₁₀		
						2	t	5	3	9	6
						9	2	8	4	7	3
						0,2/11,11					
						m.89					
						C ₁₂			C ₁₃		
						3	9	4	0	0	1
						4	7	5	e	0	0
						0,1/11,0					

In order to examine how the modulations among the four different arrays occur in this passage, Example 4.24(c) provides only the four axis-dyad chords which are enclosed by rectangles in Example 4.24(b), each presented in the corresponding place within its array in which the array modulates to another array.

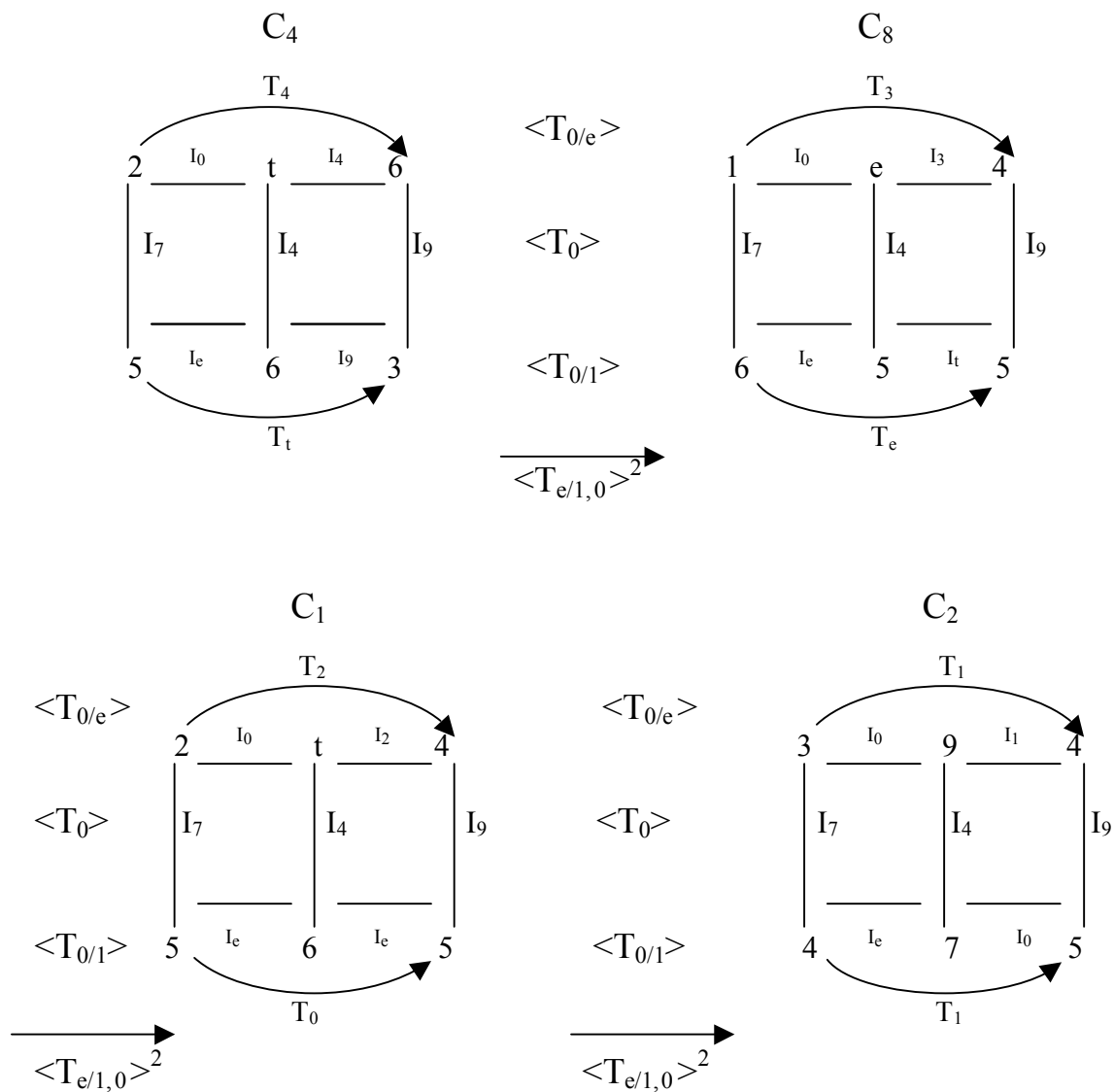
Example 4.24(c) Systematic changes of cyclic arrays in mm.83-89

C ₄				C ₈				C ₁₁				C ₁₂			
0,4:	2	t	6	0,3:	1	e	4	0,2:	2	t	4	0,1:	3	9	4
9,11:	5	6	3	10,11:	6	5	5	11,11:	5	6	5	0,11:	4	7	5

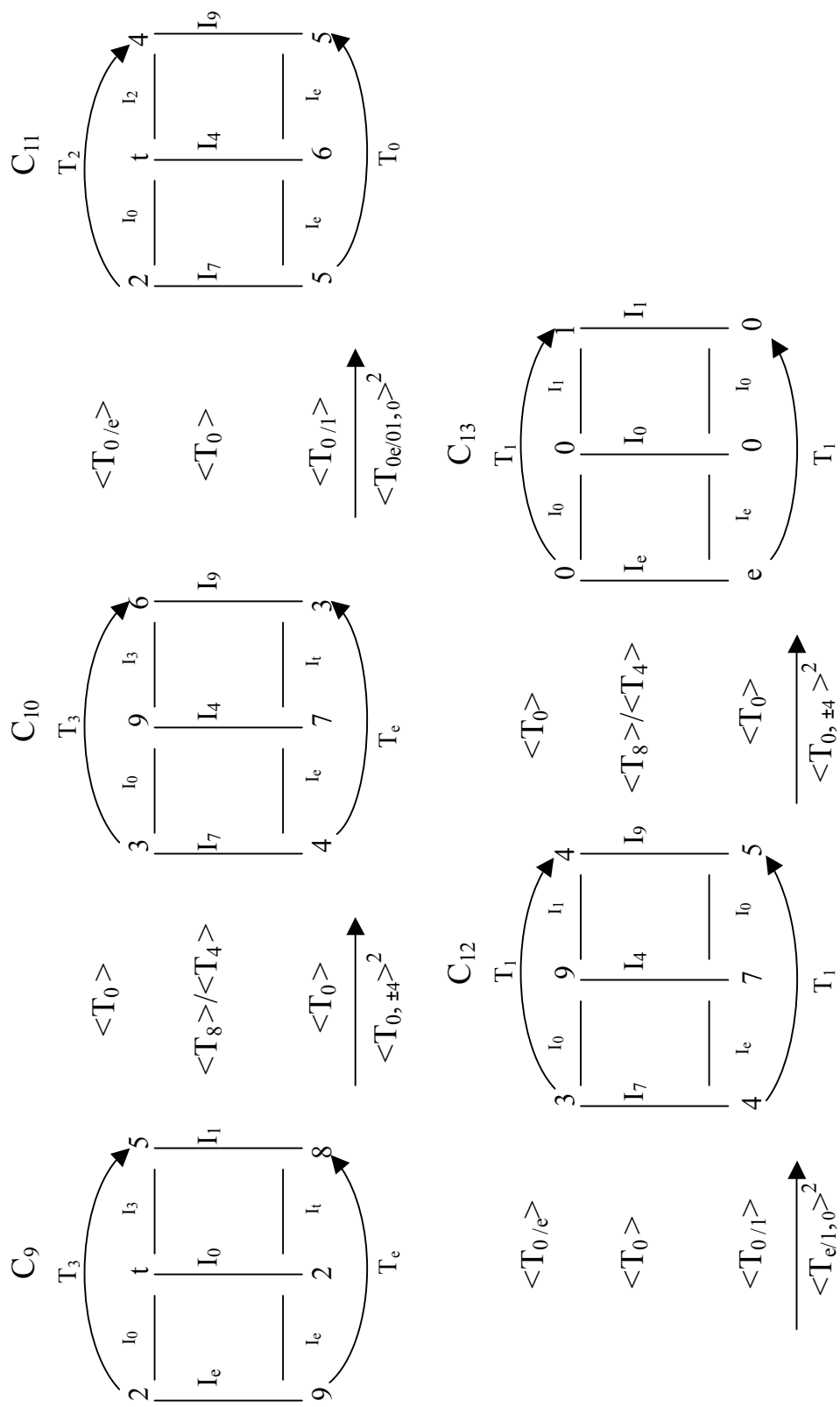
For the modulations, Perle decreases the second tonic sum in each of the upper cyclic sets and increases the first tonic sum in each of the lower cyclic sets.¹³⁶ Although the four cyclic arrays in this passage each retain a different interval cycle, the corresponding vertical dyads of each array yield identical secondary sums, 7/4/9. Accordingly, while the network interpretations of these chords, shown in Example 4.24(d), display axially isographic relationships between both the upper trichordal segments via $\langle T_{0/e} \rangle$ and the lower trichordal segments via $\langle T_{0/l} \rangle$, they all have the same internal I_n arrows.

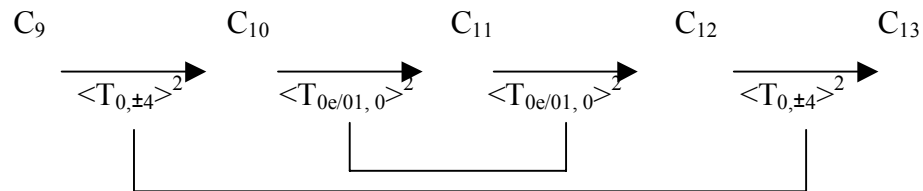
¹³⁶ Gretchen Foley, "Mutation as Compositional Process in George Perle's 'Etude No.4' from *Six Etudes for Piano*," *University of Western Ontario* 19-20 (2000-2001), 267. In this article, she also provides the four arrays in this passage to explain how Perle uses mutation by modifying an earlier subject.

Example 4.24(d) Isographic relation between successive axis-dyad chords C_4 , C_8 , C_{11} and C_{12}



This reveals that there are the same $\langle T_{0e/01,0} \rangle^2$ relationships between successive chords in this example. As we have seen in Example 4.24(a), the second phrase simply restates the subject of the first phrase, although a new chord appears at the end of the second phrase producing a new array. The following two phrases, however, provide a truncated repeat of the subject to form a symmetrical layout between corresponding transformational relationships as shown in Example 4.24(e).

Example 4.24(e) Composite isographic relationships between successive chords C_9 - C_{13} 

(Example 4.24(e) continued)

That is, there are two pairs with strongly composite isography, C_9 - C_{10} and C_{12} - C_{13} , each represented by $\langle T_{0,\pm 4} \rangle^2$. In addition, the middle three successive K-nets of C_{10} - C_{12} feature the same $\langle T_{0e/01, 0} \rangle^2$ relations.

Let us view another of Perle's works which exemplifies such modulations, those yielding the composite isographies examined above. Example 4.25 excerpts the first section (mm.1-13) from Perle's *Toccata* (1969).

Example 4.25 Perle, *Toccata*, mm.1-12

C_1 C_2
 C_3 C_4
 C_5

$5,4/4,3$ $5,4/3,2$
 $5,4/2,1$

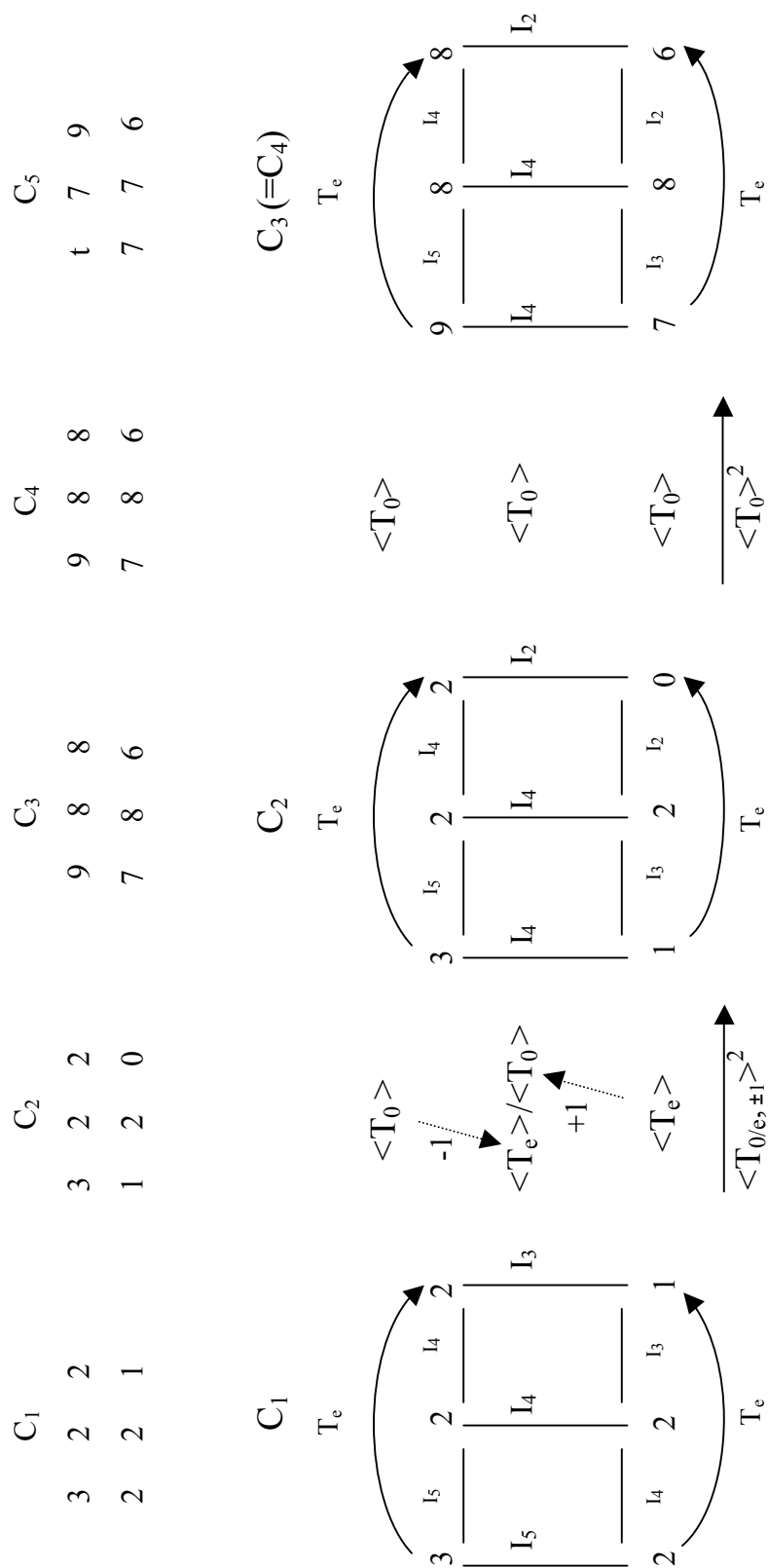
C_1	C_2	C_3	$\text{C}_{3.1}$	C_4	C_5
5,4: 3 2 2	5,4: 3 2 2	9 8 8	8 9 7	9 8 8	5,4: t 7 9
4,3: 2 2 1	3,2: 1 2 0	7 8 6	7 8 6	7 8 6	2,1: 7 7 6

The section consists of three brief motives (1-3, 4-6, 7-12) and also three different cyclic arrays as notated below the music in Example 4.25.¹³⁷ As we see in this example the changes of arrays do not coincide with the formal divisions. Example 4.25 provides five axis-dyad chords, labeled C_1 - C_5 , the first and the last chords drawn from each motive. Perle, however, suggests two different readings of C_3 .¹³⁸ In C_3 and $C_{3,1}$, although the pitch-class content is preserved, the new ordering of the pitches yields different secondary sums or differences. We can see in this example that the arrays share the tonic sums of the upper cyclic sets while the tonic sums in each of the lower cyclic sets show a consistent difference of 11. The network interpretations of these chords are provided in Examples 4.26 and 4.27.

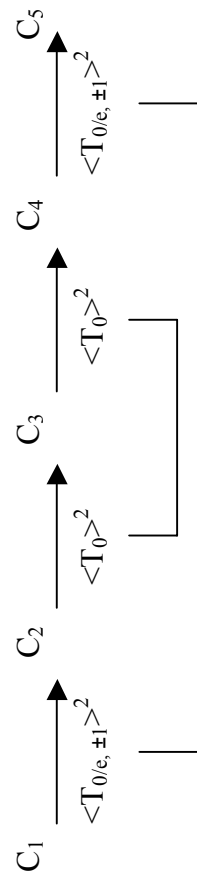
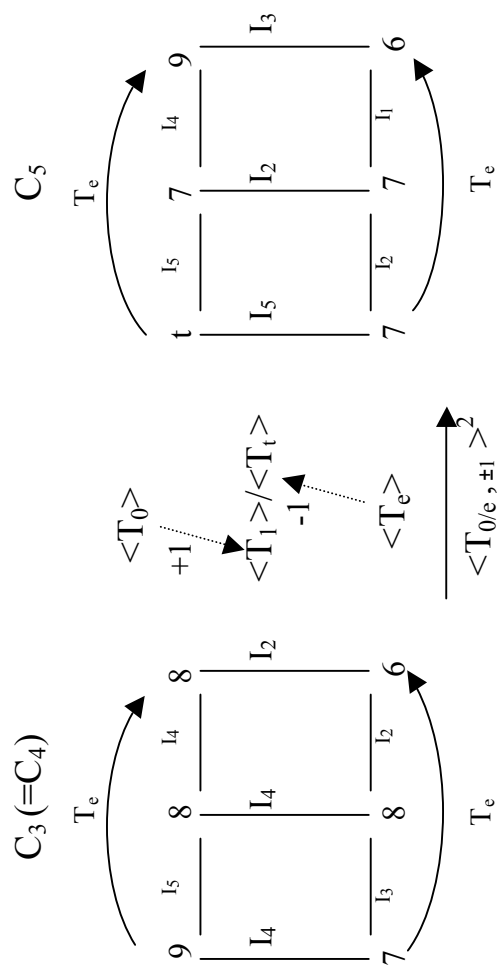
¹³⁷ Perle offers the cyclic arrays and lays out the axis-dyad chords for this section in *Twelve-Tone Tonality*, 2ed., 63-65.

¹³⁸ Perle, *Twelve-Tone Tonality*, 2ed., 65.

Example 4.26 First possible network interpretation and composite isographic relationships between successive chords C_1 - C_5



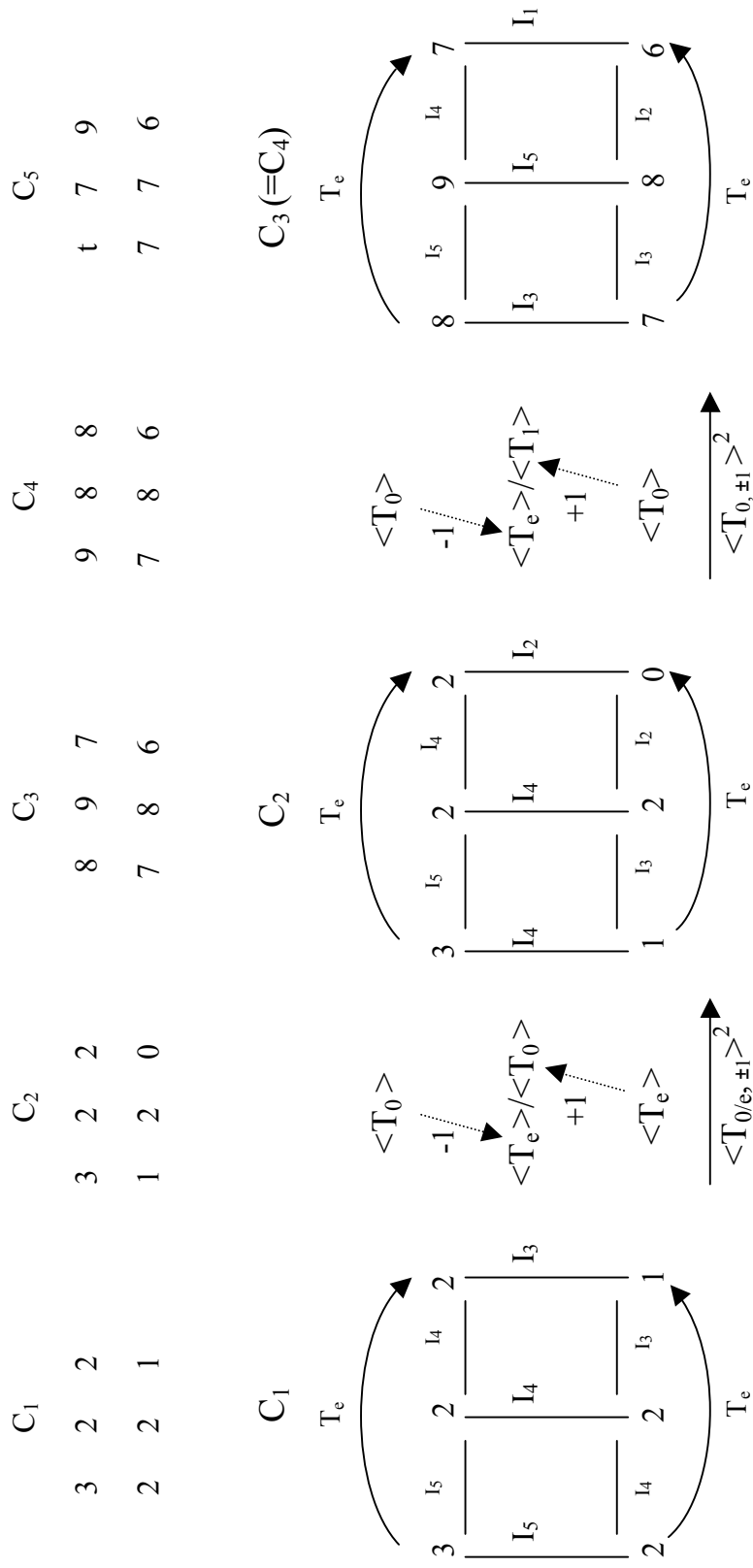
(Example 4.26 continued)



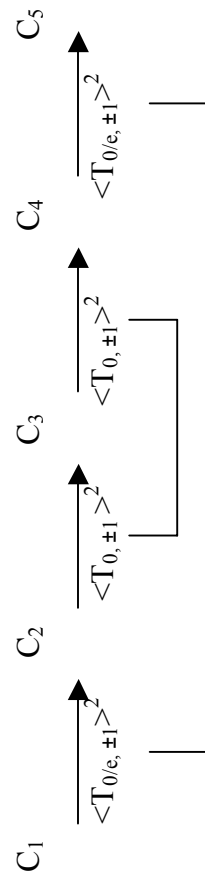
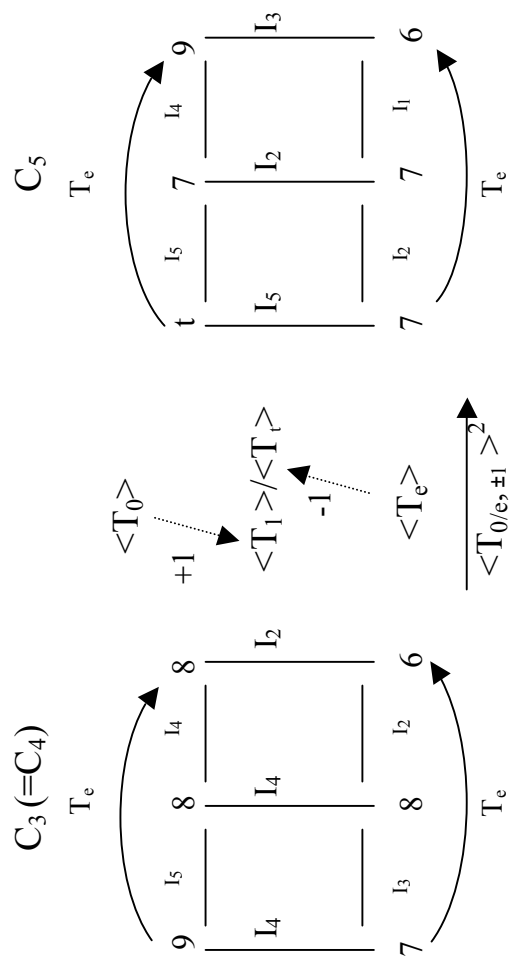
In Example 4.26, the K-nets of C_1 and C_2 share the same T_n and I_n values between the upper segments while the lower I_n values of C_1 increase by 11 over the lower I_n values of C_2 . In addition, their internal I_n values vary by 11 and 0. This type of composite isography involving strong and positive isography between the outer segments, at $\langle T_0 \rangle$ and $\langle T_e \rangle$, is represented by $\langle T_{0/e, \pm 1} \rangle^2$; the numerical subscript after the comma, ± 1 , indicates the differences among the outer $\langle T_0 \rangle / \langle T_e \rangle$ values and the internal $\langle T_e \rangle / \langle T_0 \rangle$ values. The K-net of C_4 also has the same type of composite isography with C_5 , represented by $\langle T_{0/e, \pm 1} \rangle^2$. In addition, this type of composite isography allows for modulations between arrays within the same master array of the synoptic key and the synoptic mode, but also in different keys, modes, and tonalities.¹³⁹ In Example 4.26, the K-nets of the middle three chords, C_2 , C_3 , and C_4 , are all strongly composite isographic at $\langle T_0 \rangle^2$. On the other hand, Example 4.27 provides another interpretation for the middle three chords, C_2 , $C_{3.1}$, and C_4 , labeled as the same $\langle T_{0, \pm 1} \rangle^2$.

¹³⁹ In Perle's theory, master mode and master keys are the difference and the sum of cyclic intervals. The arrays, 4,5/3,4 and 4,5/2,3, which retains cyclic intervals 1,1, are both members of master array(0) of the synoptic mode (1-1, 1-1=0) and master array (2) of the synoptic key (1+1, 1+1=2). See Perle, *Twelve-Tone Tonality*, 2ed., 92.

Example 4.27 Second possible network interpretation and composite isographic relationships between successive chords C_1 - C_5



(Example 4.27 continued)



Although the labels between the K-nets of the chords are not exactly the same in both of the network interpretations displayed in Example 4.26 and 4.27, both reveal a symmetrical display of the same transformational relationships in the succession of chords C_1 - C_5 . As we have seen in this passage, the analytic possibilities for interpreting the same chords in a new order or occasionally the same passage in different arrays are also analogous to the possibilities of different K-net readings.¹⁴⁰ Because of the analogies, the K-nets can also be used to interpret different readings of the same axis-dyad chords or the same passage, even in different arrays, featuring another aspect of the new chord or passage.

As we examined above, the K-nets involving internal I_n arrows, which I proposed here, can broadly interpret both the symmetrical motions between axis-dyad chords within an array and the various relationships between cyclic arrays. With these composite isographies, K-nets bring into the open the transformational relationships that remain latent between Perle's axis-dyad chords within an array or between arrays and also model the abstract relations among arrays. In other words, it is worthwhile to connect axis-dyad chords within an array or between arrays with the composite isographies since there is no other way of describing the indistinct relations among arrays other than their having the same mode, key or tonality or synoptic array.¹⁴¹ We have also seen that all the composite isographies examined above, except for positively composite isography, may belong to a large family of traditional axial isographies,

¹⁴⁰ Perle also provides a different reading of this passage in new arrays in his *Twelve-Tone Tonality*, 2ed., 65.

¹⁴¹ Foley addresses the value of composite isography among axis-dyad chords in her "Arrays and K-nets," 94.

which share some of their I_n arrows between the K-nets of axis-dyad chords while the remaining I_n arrows change. As we examined above, the K-nets involving internal I_n arrows can broadly interpret both the symmetrical motions between axis-dyad chords within an array and the various relationships between cyclic arrays.

Figure 1 (see below) shows all the types of composite isographies defined here, those which for the most part exhibit complementary values or the same value between the corresponding internal I_n arrows and also provide traditional strong, positive, and axial isography in the outer segment relationships. In addition, Figure 1 illustrates how the composite isographies represent motions between arrays, whether of the same mode or the same key or synoptic arrays.

Figure 1.

type	notation	outer relationship	internal	modulations
Strongly composite	$\langle T_{0, \pm n} \rangle^2$	strong, strong	$\langle T_0 \rangle$ or $\langle T_{\pm n} \rangle$	
Positively composite	$\langle T_{n, \pm m} \rangle^2$	positive, positive	$\langle T_n \rangle$ or $\langle T_{\pm n} \rangle$	same mode
Other composite	I $\langle T_{n/-n, 0} \rangle^2$	positive, positive	$\langle T_0 \rangle$	same key
	II $\langle T_{0n/0-n, 0} \rangle^2$	axial, axial	$\langle T_0 \rangle$	same key
	III $\langle T_{0/n, \pm m} \rangle^2$	strong, positive	$\langle T_n \rangle / \langle T_m \rangle$	synoptic mode, synoptic key

The analysis that follows will examine how these K-nets of axis-dyad chords interpret the diverse modulations between arrays and how inversive symmetry can be revealed by network interpretations in Perle's "Etude No.5" from *Six Etudes for*

Piano.¹⁴² The six etudes date between 1973 and 1976. The fifth etude consists of three sections (A, B, and A') and a coda. While each of the first and second sections comprises a pair of subsections, the last section consists of a single subsection, as shown in Figure 2.

Figure 2. Formal structure of Etude No.5

A (mm.1-16)		B (mm.17-32)		A' (mm.33-40)	Coda (mm.41-43)
Aa	Aa'	Bb	Bb'	A'a'	

The right-hand and the left-hand parts consist of different arrays which are treated as two contrapuntal voices. In the opening two measures, as given in Example 4.28(a), the axis-dyad chords in each of the right- and left-hand parts are based on the 3,7/1,2 and 2,6/0,1 arrays.

¹⁴² Foley, Gretchen C, "Pitch and Interval Structures in George Perle's Theory of Twelve Tone Tonality," Ph.D. diss., University of Western Ontario, 1999.

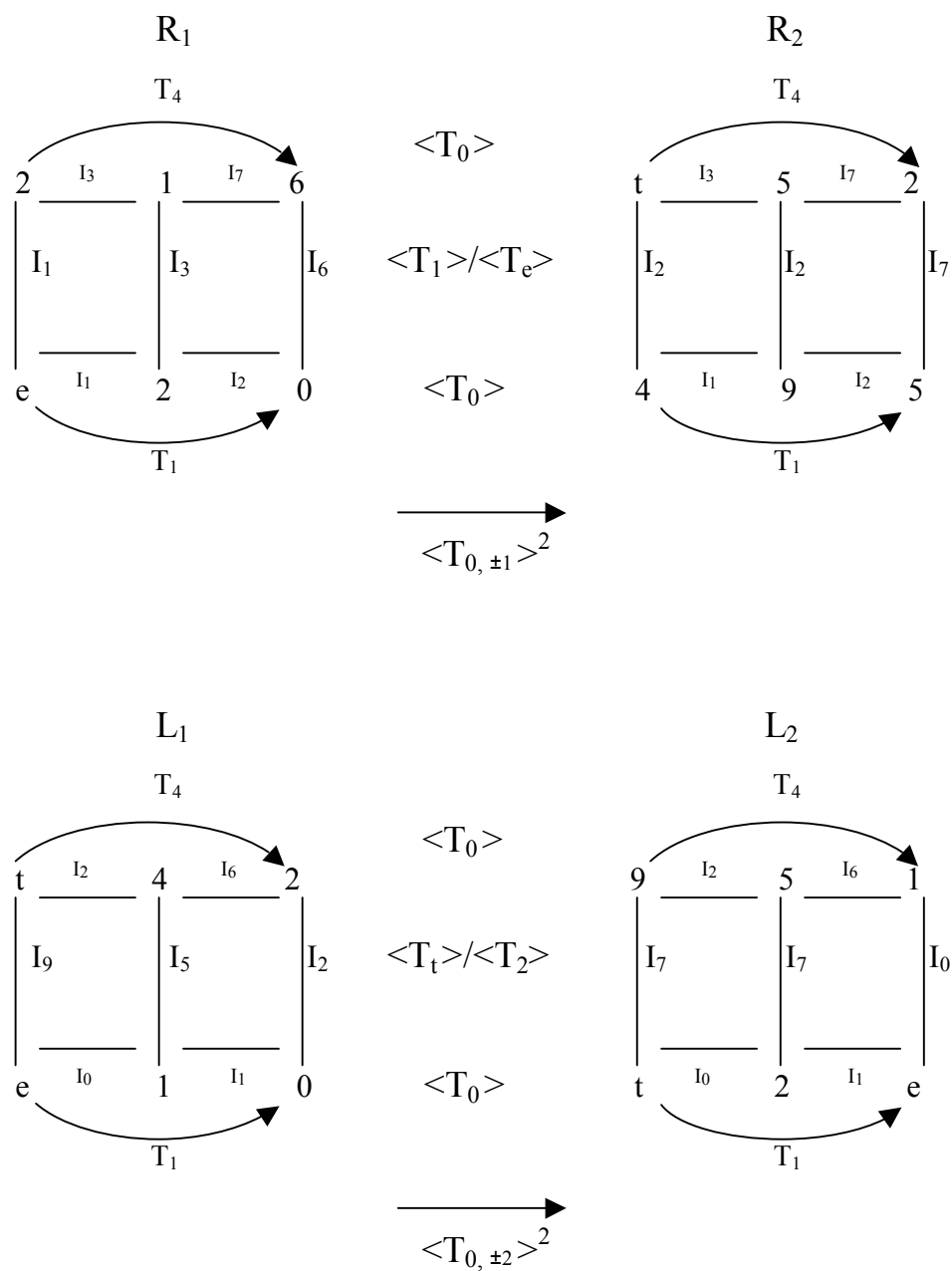
Example 4.28(a) Perle, *Six Etudes for Piano*, No.5, mm.1-2

$\text{♩} = 84$
 pp
 mf
 R_1 R_2
 L_1 L_2
 I_1 I_1 I_2 I_0 I_7 I_7

	R_1				R_2		
3,7:	2	1	6	t	5	2	
1,2:	e	2	0	4	9	5	
	L_1				L_2		
2,6:	t	4	2	9	5	1	
0,1:	e	1	0	t	2	e	

Both arrays have the same cyclic intervals, 4,1. Example 4.28(b) interprets a pair of axis-dyad chords in the right hand as R_1 and R_2 , and in the left hand as L_1 and L_2 .

Example 4.28(b) Strongly composite isography between axis-dyad chords R_1 - R_2 and L_1 - L_2



We can also see from the music given in Example 4.28(a) that the left hand projects one of the primary sums of L_1 , I_2 and I_1 , and the secondary sums of L_2 , I_7 and I_0 , on the

musical surface.¹⁴³ Note that each of the hands shows strongly composite isography, providing complementary differences between the internal I_n arrows. In this piece, since the significant modulations occur in the beginning of each of the subsections, for ease of comparison Example 4.29(a) provides the initial chords of the first two subsections, Aa and Aa'.

¹⁴³ In this piece, the inversional relations represented as the outer and the inner relations are rarely reflected in the musical surface.

Example 4.29(a) Perle, *Six Etude for Piano*, No.5, m.1, 9-10

m.1 R_1

$\text{♩} = 84$ *pp*

L_1

mm.9-10 R_3 R_4

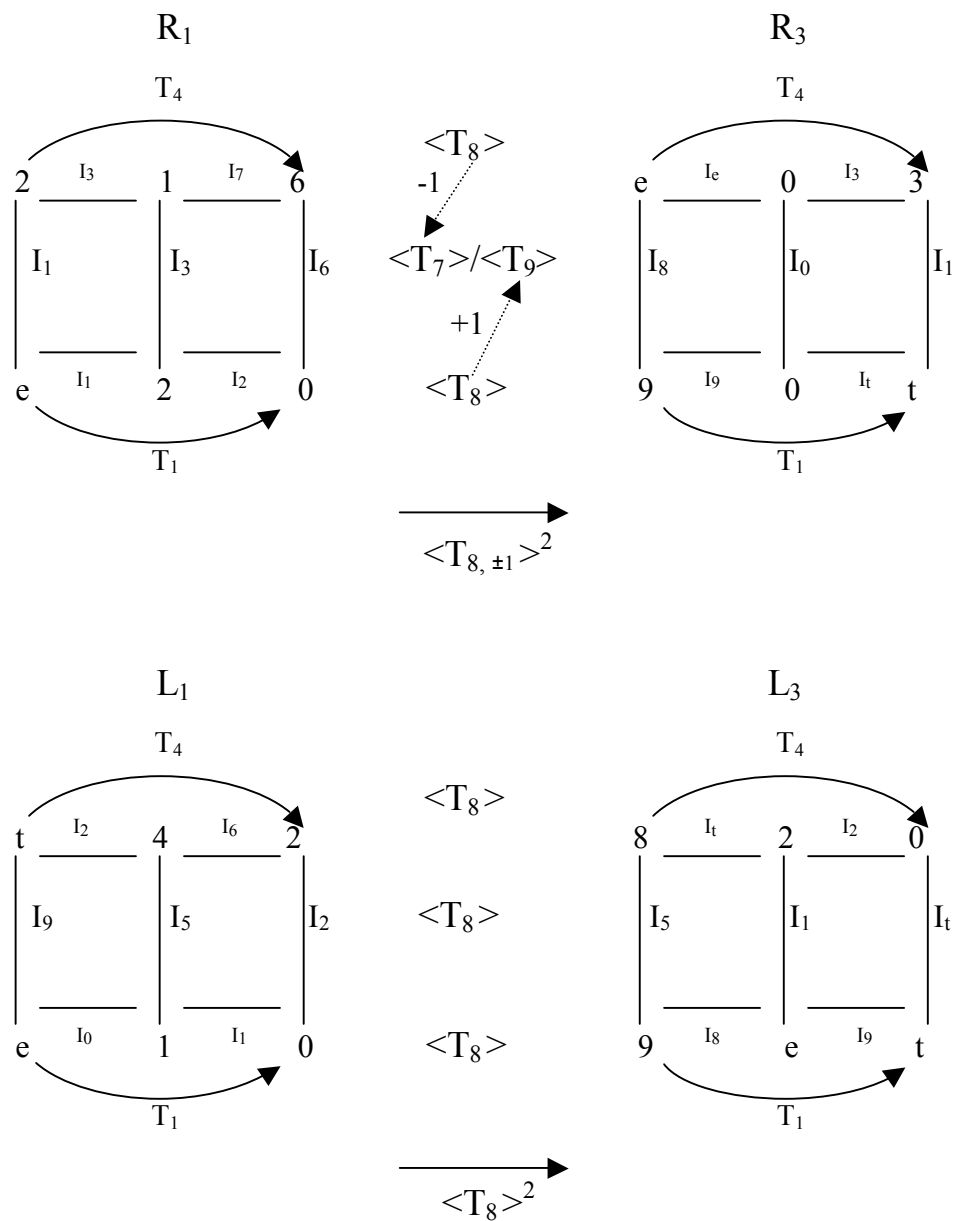
L_3 L_4

	R_1				R_3				R_4		
3,7:	2	1	6	11,3:	e	0	3		e	0	3
1,2:	e	2	0	9,10:	9	0	t		7	2	8

	L_1				L_3				L_4		
2,6:	t	4	2	10,2:	8	2	0		7	3	e
0,1:	e	1	0	8,9:	9	e	t		8	0	9

In this the Aa' subsection changes arrays to 11,3/9,10 in the right hand and to 10,2/8,9 in the left hand. Example 4.29(b) interprets the modulations between the Aa and the Aa' sections as positively composite isographic, mapping R_1 and L_1 onto the K-nets that model R_3 and L_3 at $\langle T_{8,\pm 1} \rangle^2$ and $\langle T_8 \rangle^2$ (or $\langle T_{8,\pm 0} \rangle^2$).

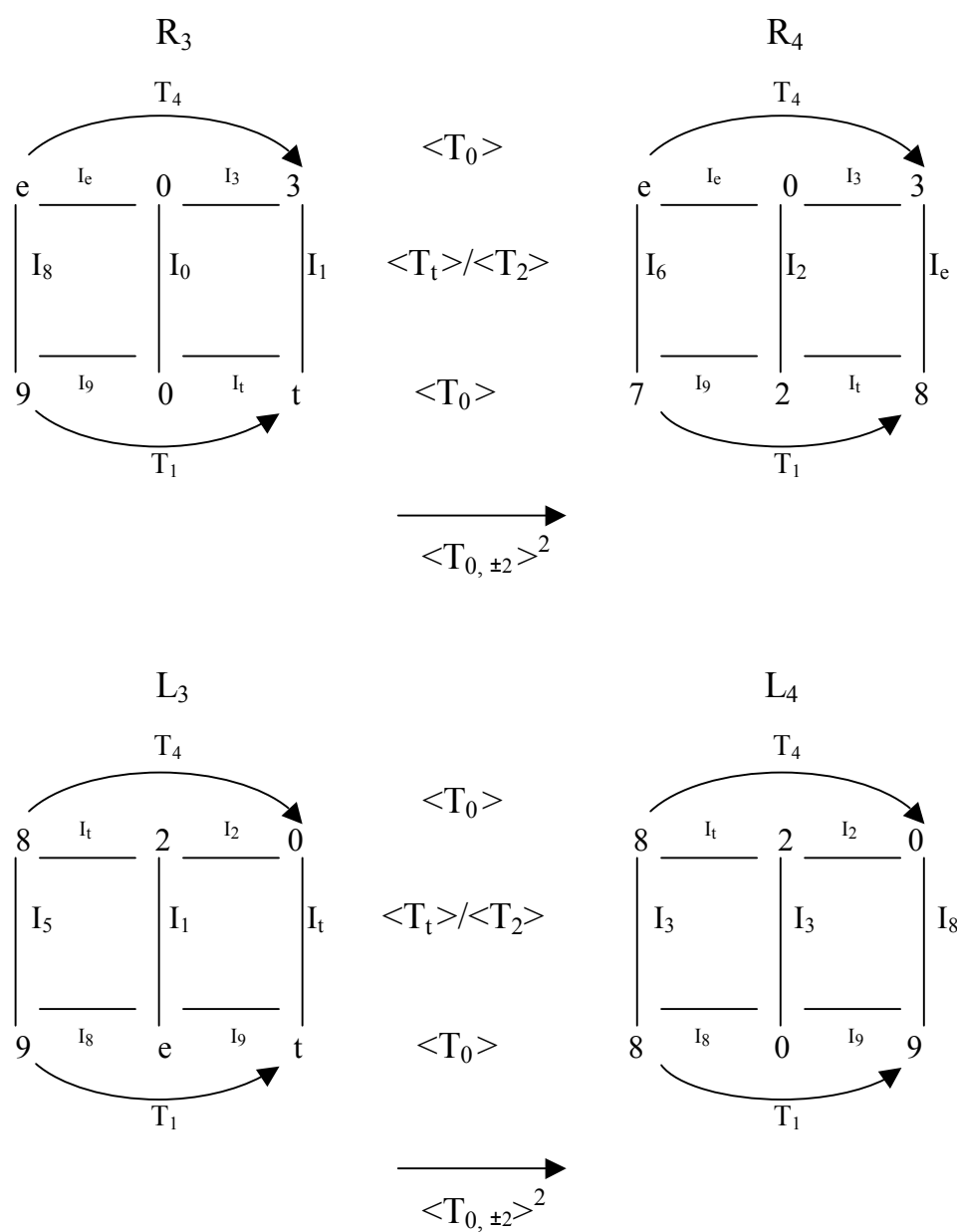
Example 4.29(b) Positively composite isography between axis-dyad chords R_1 - R_3 and L_1 - L_3



Here, $\langle T_8 \rangle^2$ represents the identical positively isographic relationships between the outer trichordal segments and ± 1 (or ± 0) within the angle brackets represents the degree by which the upper trichordal segments of both chords move in symmetrical motion

against their lower trichordal segments. Example 4.29(c) shows that the Aa' section begins with the axis-dyad chords labeled R₃-R₄ for the right hand and L₃-L₄ for the left hand, yielding the same strongly composite isography as denoted by $\langle T_{0,\pm 2} \rangle^2$.

Example 4.29(c) Strongly composite isography between axis-dyad chords R₃-R₄ and L₃-L₄



In the middle sections, the right hand of both the Bb and Bb' sections also begins with strongly composite isography in each of the 7,11/5,6 and 3,7/1,2 arrays as shown in Example 4.30(a) and 4.30(b).

Example 4.30(a) Perle, *Six Etude for Piano*, No.5, mm.17-18, 25-26

Bb section (mm.17-18)

R₅ R₆

L₅ L₆

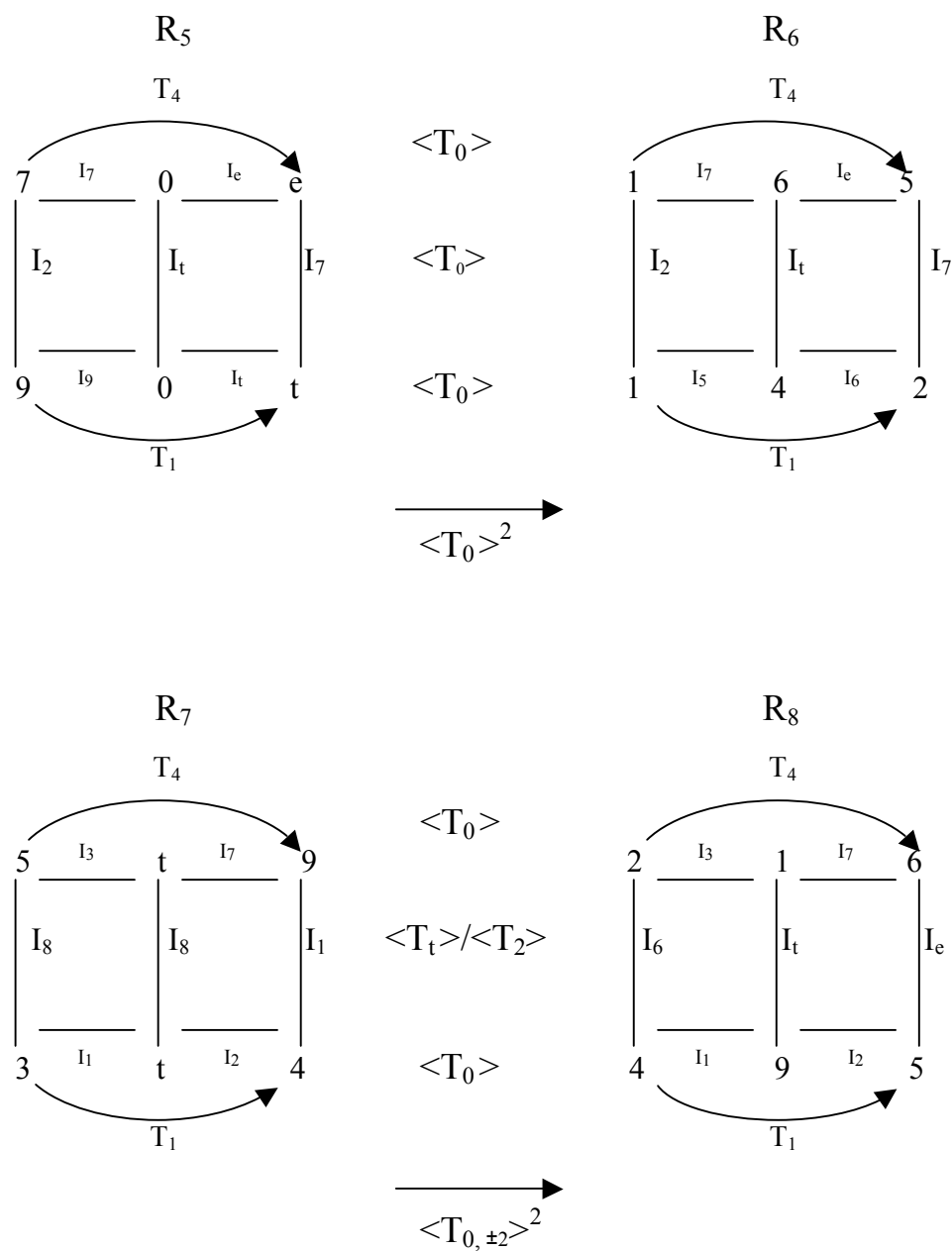
Bb' section (mm.26-27)

R₇ R₈

L₇ L₈

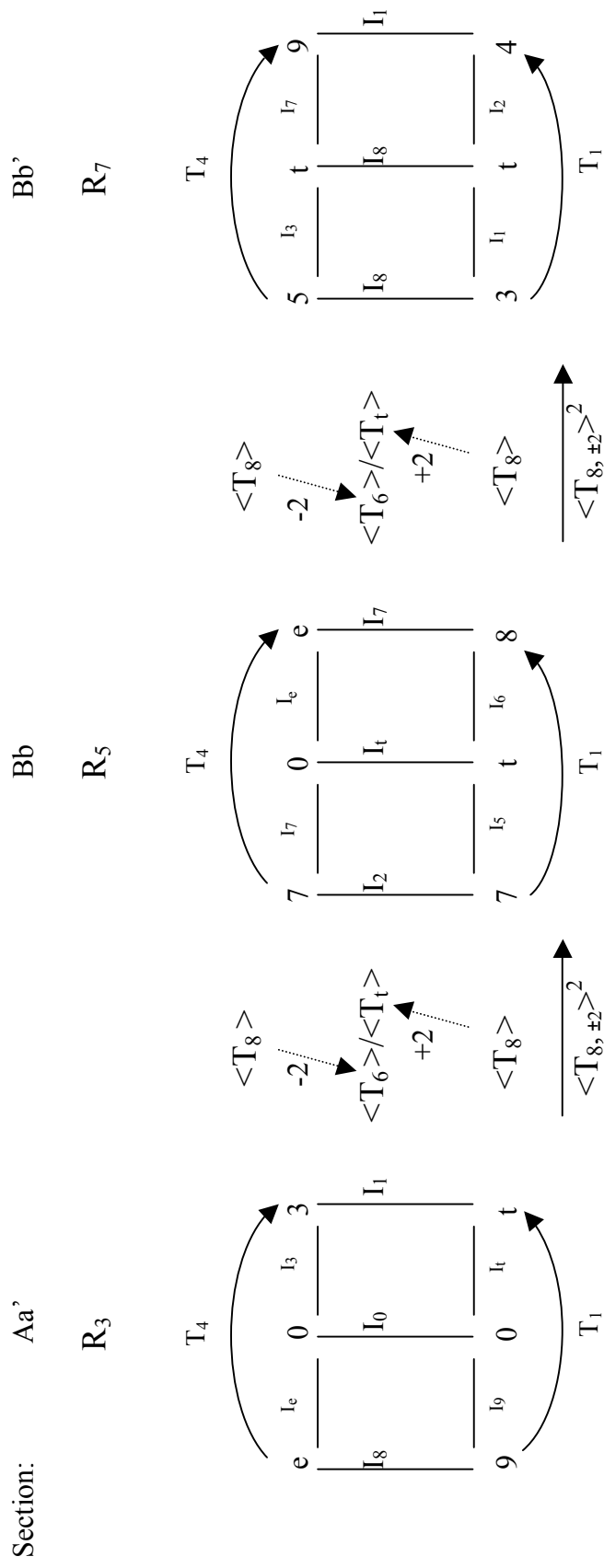
	R ₅				R ₆		
7,11:	7	0	e	7,11:	1	6	5
5,6:	7	t	8	5,6:	1	4	2
	R ₇				R ₈		
3,7:	5	t	9	3,7:	2	1	6
1,2:	3	t	4	1,2:	4	9	5

Example 4.30(b) Strongly composite isography between axis-dyad chords R_5 - R_6 and R_7 - R_8



In addition, Example 4.30(c) illustrates how the K-nets of the chords in the right hand can interpret the modulations between arrays from the Aa' to the Bb' section.

Example 4.30(c) Positively composite isography between axis-dyad chords drawn from arrays in Aa' - Bb' sections

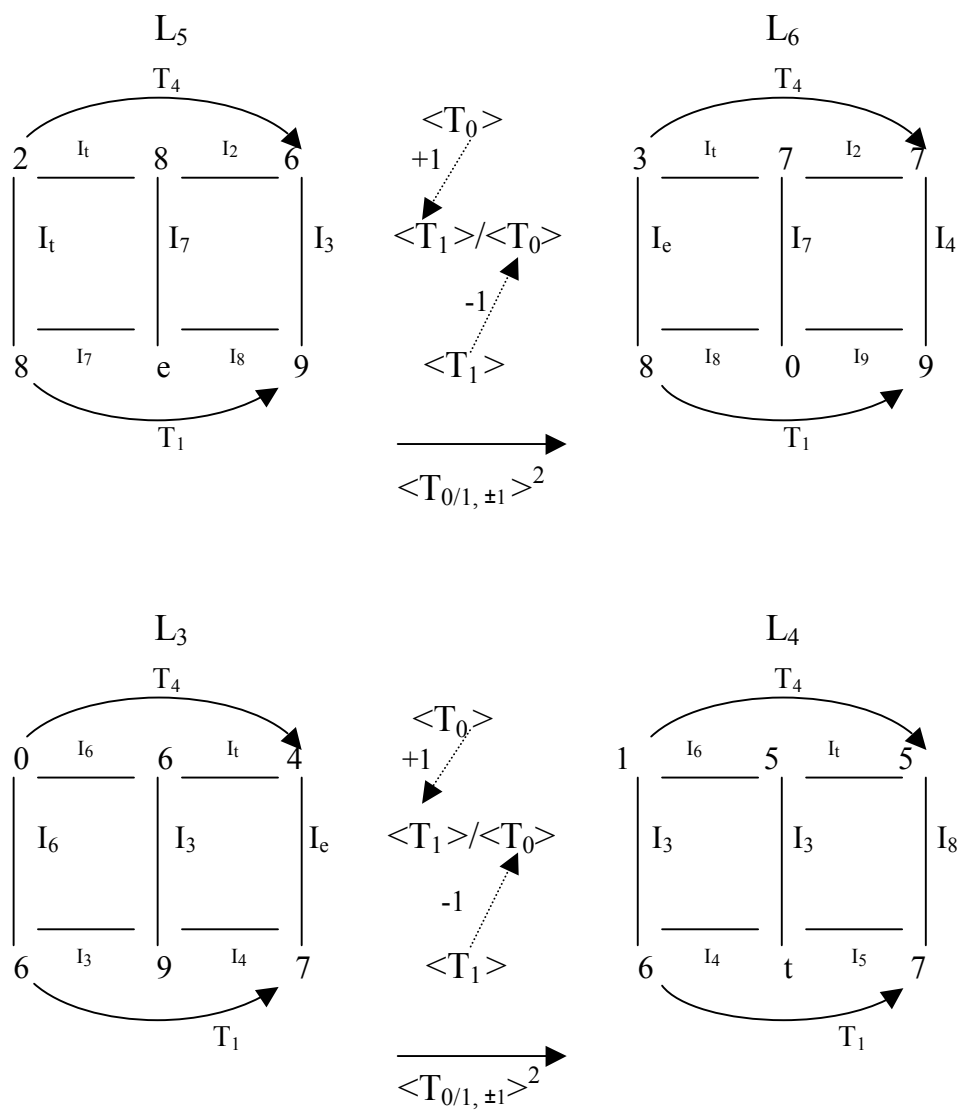


Three successive K-nets in this example display the same positively composite isography at $\langle T_{8,\pm 2} \rangle^2$. On the other hand, the left hand in each of the Bb and the Bb' subsections share a single cyclic set, 10,2 or 6,10, but changes another cyclic set from 7,8 to 8,9 or from 3,4 to 4,5 as shown in Example 4.30(d). The left hand in the middle section thus involves four different arrays as given in Example 4.30(d).

Example 4.30(d) Composite isographic relationship between chords L_5 - L_6 and L_7 - L_8

	L_5				L_6		
10,2:	2	8	6	10,2:	3	7	7
7,8:	8	e	9	8,9:	8	0	9

	L_7				L_8		
6,10:	0	6	4	6,10:	1	5	5
3,4:	6	9	7	4,5:	6	t	7



The K-nets in this example display the modulations between arrays occurring in the left-hand part of this section. Note that there are two pairs with composite isography, represented by the same $\langle T_{0/1, \pm 1} \rangle^2$ labels.

In the final A'a' section, which occupies only eight measures, for the left hand part there is a return of the material from section Aa' at T_8 , 10,2/8,9 becoming 6,10/4,5 and finally the coda brings back the opening array 2,6/0,1 as shown in Example 4.31(a).¹⁴⁴

¹⁴⁴ For the left hand part in the last two measures, a tetrachordal segment is excepted in this example.

Example 4.31(a) Perle, *Six Etude for Piano*, No.5, mm.33-34, 41-43

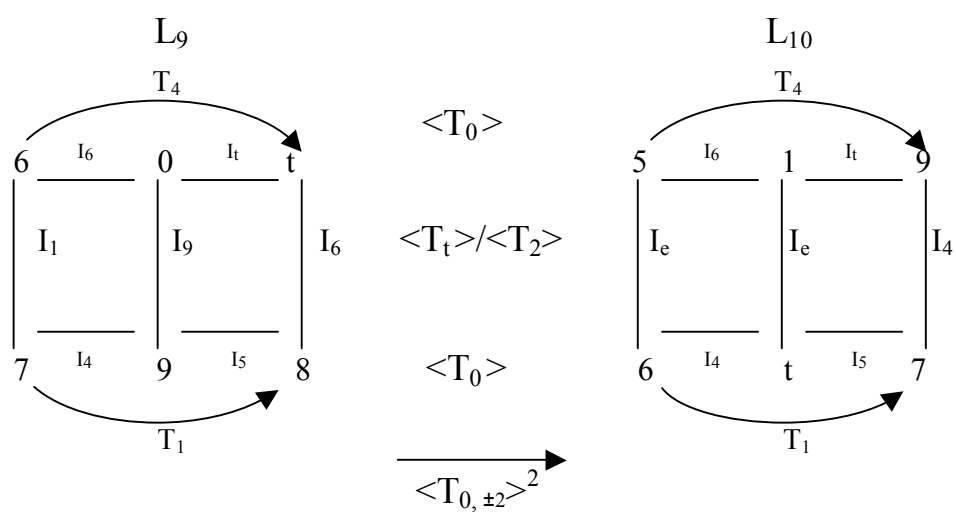
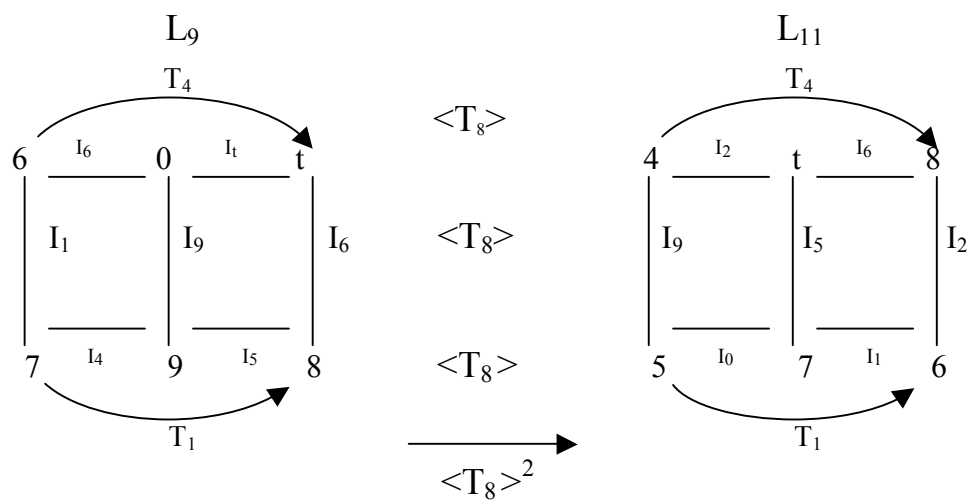
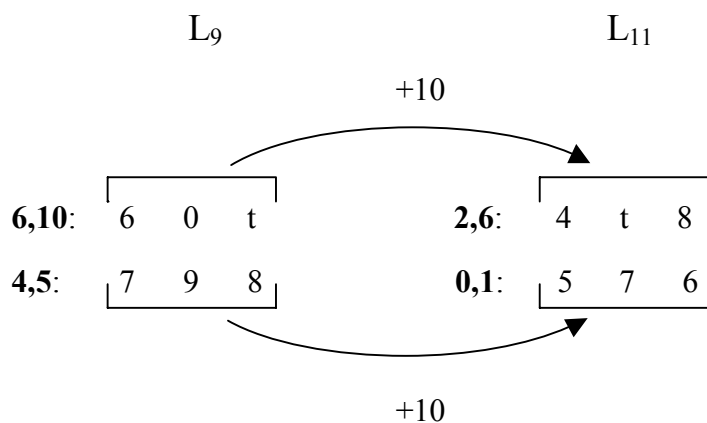
A'a' section (mm.33-34)

Coda (mm.41-43)

	b'			A'a'				Coda		
	L ₇			L ₉			L ₁₀	L ₁₁		
t,2:	8	2	0	6,10:	6	0	t	5	1	9
8,9:	9	e	t	4,5:	7	9	8	6	t	7
								0,1:	5	7
									6	

Example 4.31(b) interprets the first chords in both the last section and the coda as K-nets.

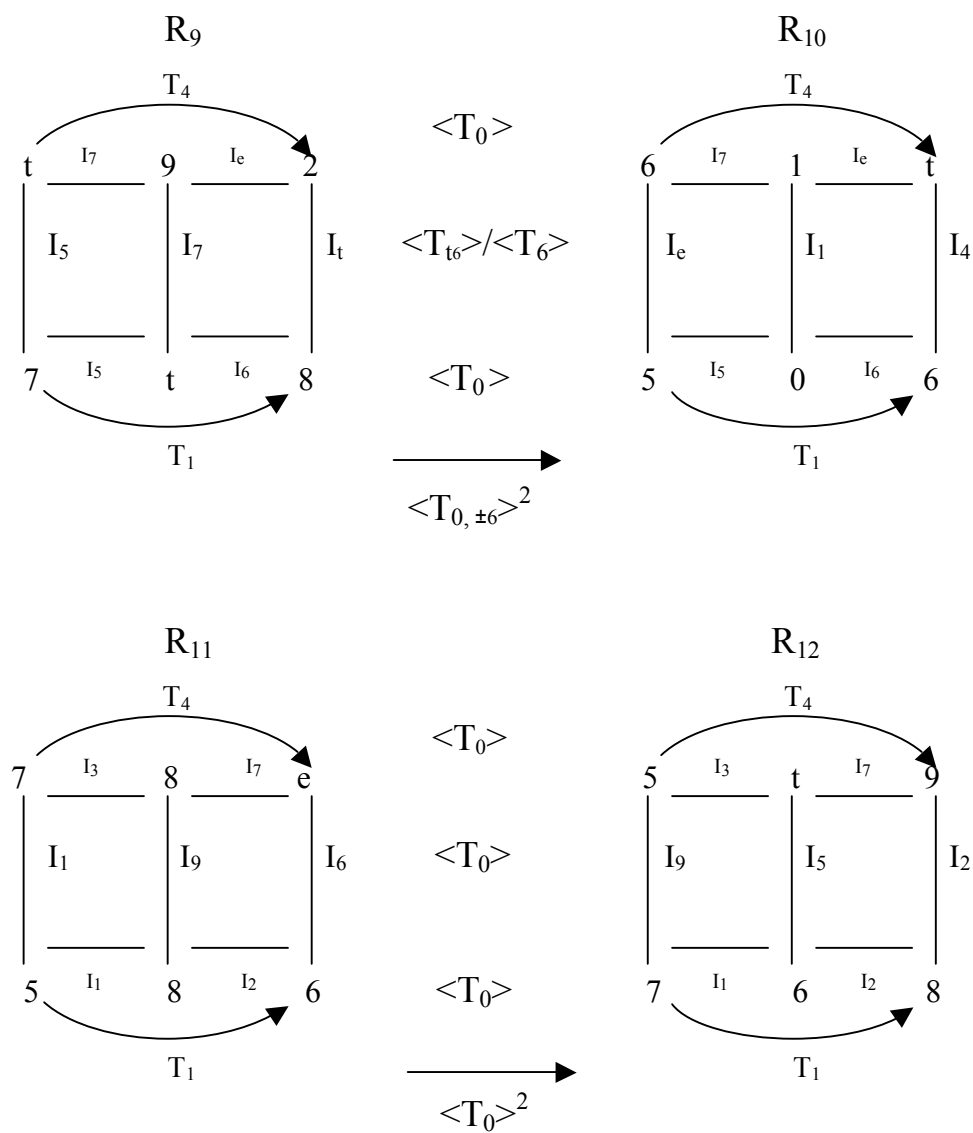
Example 4.31(b) Positively and strongly composite isography between chords L_9 - L_{11} and L_9 - L_{10}



In this, the modulation is effected by transposing all notes by 10 and thus the K-nets of these axis-dyad chords are positively composite isographic at $\langle T_8 \rangle^2$. Example 4.31(b) also interprets the initial chords of the last A'a' section and shows strongly composite isography as denoted by $\langle T_{0,\pm 2} \rangle^2$. On the other hand, for the right hand part, the array 3,7/1,2 in the Bb' section modulates to the 7,11/5,6 in the A'a' section and finally the coda brings back the opening array 3,7/1,2 as shown in Example 4.31(c).

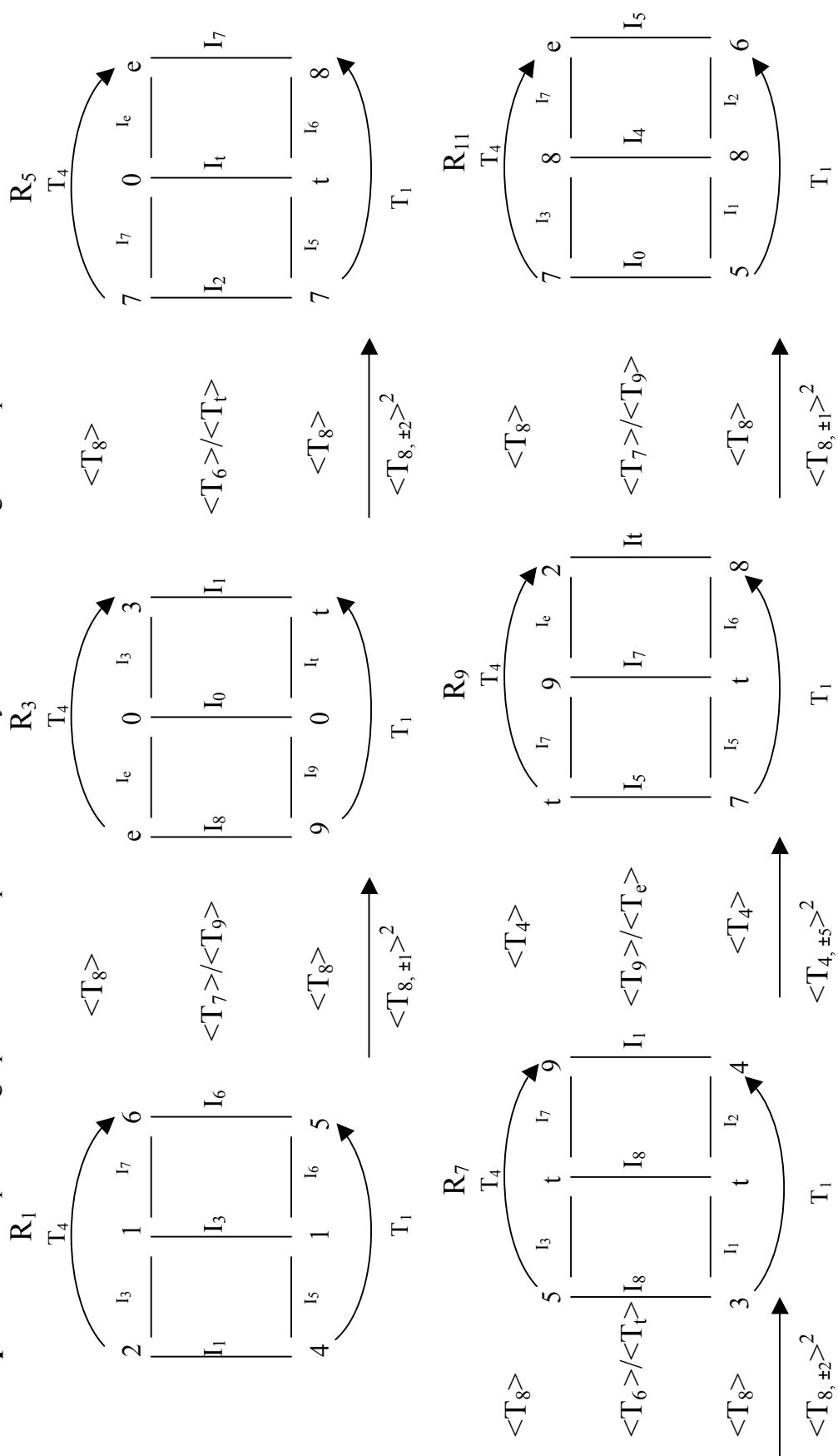
Example 4.31(c) Strongly composite isography between chords R_9 - R_{10}
and R_{11} - R_{12}

	R ₉			R ₁₀			R ₁₁			R ₁₂			
7,11:	t	9	2	6	1	t	3,7:	7	8	e	5	t	9
5,6:	7	t	8	5	0	6	1,2:	5	8	6	7	6	8



In addition, Example 4.31(c) illustrates that the right hand in both the last section and the coda begins with strongly composite isography in both 7,11/5,6 and 3,7/1,2. Finally, the transformational relationship between formal sections in the right hand emerges in Example 4.32, in which the first axis-dyad chords of each subsection are interpreted as K-nets. In this example the right-hand part provide the series of transformations

$$\langle \langle T_{8,\pm 1} \rangle^2, \langle T_{8,\pm 2} \rangle^2, \langle T_{8,\pm 2} \rangle^2, \langle T_{4,\pm 5} \rangle^2, \langle T_{8,\pm 1} \rangle^2 \rangle.$$

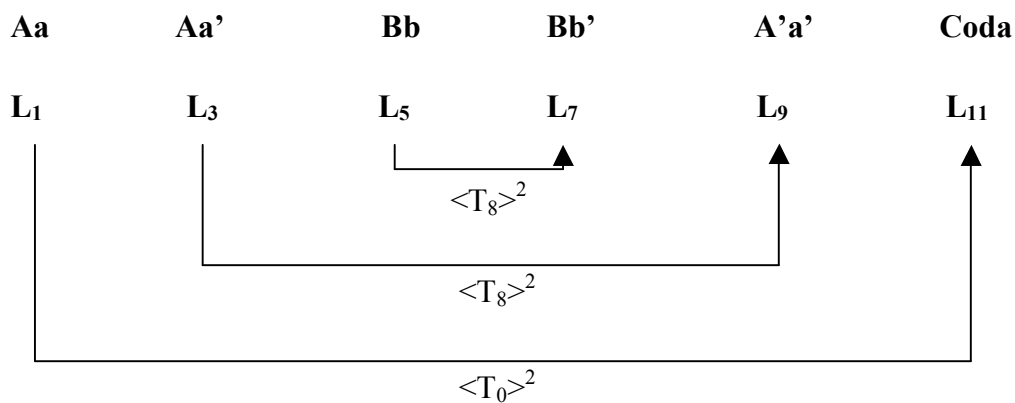
Example 4.32 Composite isographic relationship between axis-dyad chords of the right-hand part drawn from each section

That is, the right-hand part shows that changes in cyclic arrays generating positively composite isography at $\langle T_8 \rangle^2$ or $\langle T_4 \rangle^2$ coincide exactly with the formal divisions as displayed in Example 4.32. On the other hand, the left-hand part's transformational relationships between formal sections emerge in Example 4.33, providing the series of transformations $\langle \langle T_{8, \pm 2} \rangle^2, \langle T_{0/e, \pm 5} \rangle^2, \langle T_8 \rangle^2, \langle T_{0/1, \pm 5} \rangle^2$, and $\langle T_8 \rangle^2 \rangle$.

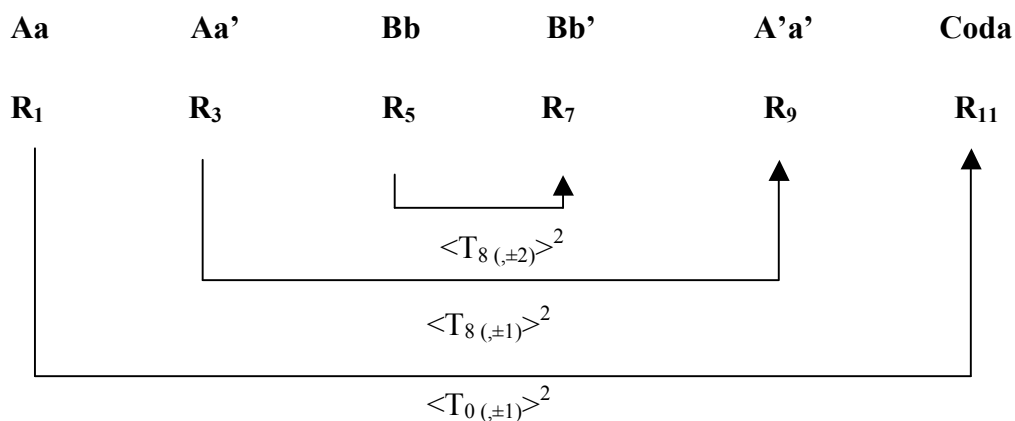
We can see in this example that the changes clearly made through positively composite isography do not coincide with the formal divisions and that the composite isographies at $\langle T_{0/e, \pm 5} \rangle^2$ and $\langle T_{0/1, \pm 5} \rangle^2$ sharing a cyclic set seem to obscure the formal boundaries. Meanwhile, the composite isographies at $\langle T_{0/e, \pm 5} \rangle^2$ and $\langle T_{0/1, \pm 5} \rangle^2$ instead delineates the middle subsections by yielding a different tonality: by the sum of all the tonic sums of each array (10,2/7,8, $10+2+7+8=3$ and the sums of 6,10/3,4 is 11), their aggregate sums 3 and 11 are tonality 3 (mod 4) but other sections are unified by the same tonality 1. Since each hand provides different types of composite isographies between the K-nets drawn from each subsection in individual transformational paths, they do not seem to be related to each other. Example 4.34, however, shows that the left-hand part in this piece begins and ends the same array yielding strongly composite isography and goes through other arrays providing positively composite isographic relationships $\langle T_0 \rangle^2 / \langle T_8 \rangle^2$ / $\langle T_8 \rangle^2$ (or $\langle T_{0, \pm n} \rangle^2 / \langle T_{8, \pm n} \rangle^2 / \langle T_{8, \pm n} \rangle^2$) as the right-hand part.

Example 4.34 Symmetrical layout in succession of chords drawn from each section

Left-hand Part

Section

Right-hand Part

Section

That is, this example displays that the left- and right-hand parts are unified by identical transformational relationships each provided by the same type of composite isography.¹⁴⁵

¹⁴⁵ To emphasize that both parts display the same type of composite isographic relationship, the additional numerical subscript after the comma within the angle brackets is parenthesized in this example.

In this chapter, I have examined how the K-nets of axis-dyad chords feature the symmetrical inversions that not only inherently reside in a cyclic set and in a combination of cyclic sets, an array, but also work between trichords and between axis-dyad chords, each drawn from different cyclic sets or arrays. In fact, since the network interpretations of axis-dyad chords are dual K-nets themselves and often barely occur at all in pitch space, one may question the practicality of these composite isographies. It is, nevertheless, helpful to grasp such complex relationships among the axis-dyad chords and the underlying arrays in music of twelve-tone tonality, which can not be done easily using only Perle's abstract concepts of keys, tonalities, and synoptic arrays. The network interpretations of axis-dyad chords retaining internal I_n arrows also bring to light the diverse potentials of inversional properties among arrays that Perle has explored in twelve-tone tonality. Thus, in this chapter I have attempted to show the ways in which the K-nets of axis-dyad chords can interpret a variety of arrays including cyclic intervals which are the same as well as those which are different, and also how K-nets can be used to describe the symmetrical properties inherent both within a single one of Perle's cyclic arrays as well as between arrays. I hope the device I have provided will be useful for the analysis of Perle's compositions and will be one which can also be applied with considerable flexibility for describing the inversional symmetry with which Perle's music is saturated.

Conclusion

In this study, I have provided a way of understanding the full role and value of inversive relations in post-tonal music. It is not easy to hear inversive relations, especially when they are not realized in pitch space, whereas transposition is easily perceptible in all cases. In addition, even when the wedge motion is apparent in the musical surface, the wedge motion can be perceived most easily by the pattern of increasing and decreasing intervals of transposition. Thus, in some writings, inversive relations are treated only as abstract properties which are far removed from the musical surface and regarded as unnecessary for drawing isography. Indeed, in a number of pieces in which transposition is a prominent compositional feature, the inversive relations have been considered to be much less important or have been ruled out entirely. There are, however, also passages or pieces which do show the inversive context clearly in the musical surface. In these passages or pieces, inversive relations are necessary both to interpret the musical context and also to draw isography. More importantly, in many pieces, I find that transposition and inversion can compete and interact rather than there being only one property for the entire passage or piece. These pieces require listeners to alternate focusing on transpositional and inversive relations. Note that such focusing on or listening to a single property does not mean that only that property exists. Even when it is not possible to focus on or to listen to the inversive relations, those relations are still there and play their role in cooperation with the transpositional relations.

Since I believe that K-nets offer a way to bring out a potentially strong role for inversive properties, in chapters 2 and 3 I examined all isographies which can be

organized and interpreted using twelve inversional zones. In particular when an axial isography highlights a consistent wedge motion this isography plays an important role in expressing the musical context. In addition, to express diverse aspects of the musical context, it is helpful to use all types of traditional, axial and reductive isographies; each of the various isographies can most economically feature a pervasive transpositional or inversional feature regardless of the number of nodes.

Chapters 2 and 3 have also examined the use of hyper-transpositional and -inversional relations. It cannot be denied that $\langle \text{In} \rangle$ relations are farther removed from the musical surface and thus can be treated as an abstraction. In contrast, $\langle \text{Tn} \rangle$ relations may be considered in a positive way since they are audible and reflect the transpositional context. However, in the spatial transformations among inversional zones, the $\langle \text{Tn} \rangle$ relations represent the distance between inversional zones at a higher level and also identify transformations of transformations just as hyper-inversional relations do. Thus, $\langle \text{Tn} \rangle$ relations can also be considered as abstract even though they are relatively more perceptible.

In addition, $\langle \text{In} \rangle$ relations do not simply indicate the dual pc inversion which can not be perceived. Instead hyper-inversional and inversional relations can represent a structural similarity at a different level of the inversional structure. That is, hierarchical levels may be seen as the result of considering the ways in which the structural chords are arranged or organized in a structural uniformity among inversional zones whether the hyper-relations are audible or not. A hyper-transformation thus involves diverse aspects of the inversional and also the transpositional contexts at a different level and as a whole.

Chapter 4 describes Perle's twelve-tone tonality using a transformational approach in order to examine the inversive relations which inherently reside in the relations among axis-dyad chords. There are also some reasons to prefer the use of K-nets to Perle's compositional method. First of all, the K-nets, including T and I relations together, can efficiently represent a segment of a cyclic set or a cyclic array, noting that Perle's system is based on the conjunction of interval cycles and inversive symmetry. In particular, although the interval couples are measured for axis-note dyads and for neighbor-note dyads, Perle's arrays are founded only on the inversionally related set forms for the axis-dyad chords. Secondly, isography for axis-dyad chords can offer an overall transformational path which often shows symmetrical structure. They show that the transformational relationships in Perle's passages or pieces also embody a compositional design both among the chords and among larger sections. In addition, as Foley has addressed the complex relations among the axis-dyad chords and the arrays cannot easily be grasped by using only Perle's abstract concepts of keys, modes, tonalities, and synoptic arrays; instead they can be much more readily comprehended by a transformational approach. Chapter 4 thus demonstrates that composite isography with internal I_n arrows can more properly interpret axis-dyad chords and also the diverse relationships between axis-dyad chords within an array or between arrays.

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