GEOMETRIC UNIFICATION OF SCHRÖDINGER AND YANG-MILLS EQUATIONS AND RIEMANNIAN SPECTRA OF VECTOR BUNDLES

by

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Abstract

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Artur Sowa

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We consider a Riemannian submersion metric on a total space of a principal fiber bundle and its geometry. It turns out that a natural second order PDE, constructed from intrinsic geometric quantities, leads to a weakly coupled system of Yang-Mills and stationary Schrödinger equations.

The system leads to numerical invariants depending on both the topology of a bundle and the Riemannian geometry of the base manifold. In the "limiting" case of a trivial bundle they become the eigenvalues of the Laplacian of the base-manifold.

We display solutions of the nonlinear eigenvalue problem associated with this system in case of linear bundles over compact two-manifolds and prove their uniqueness.

We further show, that existence of solutions of the nonlinear eigenvalue problem on manifolds of more than four dimensions is equivalent to the existence of pure Yang-Mills fields on those manifolds with conformally deformed metrics.

We consider also the elliptic theory of this system on the Euclidean plane with $U(1)$ as the gauge group, where it reduces to a single scalar equation: $-\triangle f = f^{-3}$. We prove in particular that there are no finite energy solutions. We further show that there are radially symmetric solutions and we give a complete description of their properties. The results generalize in part to $-\triangle f = f^{-p}$, where $p > 0$.

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To my Mother, my wife Jola and my son Izaak.

Contents

1 Introduction

In the 1920's Kaluza and Klein unified General Relativity and Electromagnetism by "lifting" the geometric Einstein's picture of a manifold-universe to a total space of a principal fiber-bundle. Their bundle had an abelian structure group $SU(1)$. It was noticed later (see [8]) that the same procedure may be successfully applied to unify the nonabelian $SU(2)$-Yang-Mills theory with General Relativity.

On the other hand Klein and Gordon introduced their equation

$$\triangle f + m^2 f = 0,$$

where $\triangle$ is a hyperbolic (Lorentzian) Laplacian, to describe quantum-mechanical properties of a relativistic particle. The motivation for the equation's form came from the requirement of its geometric invariance. It should be noted that in the case of elliptic theory (static states) the Klein-Gordon equation is equivallent to the Schrödinger equation (with m^2 replaced by a potential), and there is no essential difference between them as long as the space-time splits off the time axis.

In this paper we explore the simplest possible second order (nonlinear) partial differential equation emerging from the Kaluza-Klein geometry.

Namely, a connection on a principal bundle P over a (pseudo-) Riemannian manifold induces a Riemannian submersion metric on the total space of P. A metric constructed in this way gives rise to the Laplace-Beltrami operator (depending on the connection) on 1-forms on P. Roughly speaking, we ask whether a form defining the connection (defined up to a pointwise proportionality) can be in the null-space of this operator. This leads to the system (cf. (14) below)

$$\begin{cases} D\Omega = 0 \\ fD^{\star}\Omega = -2\Omega(\nabla f, .) \\ -\triangle f + |\Omega|^2 f = \nu f, \end{cases}$$

which contains weakly coupled Yang-Mills and Klein-Gordon (or stationary Schrödinger) equations.

In Section 5 we show that there is a *unique* solution of this eigenvalue problem for linear bundles over compact manifolds of two dimensions. The uniqueness is in accord with the fact, that having only one numerical invariant (the eigenvalue in this case) suffices to classify linear bundles up to "torsion".

In Section 6 we show that existence of solutions of the nonlinear eigenvalue problem on manifolds of more than four dimensions is equivalent to the existence of pure Yang-Mills fields on those manifolds with conformally deformed metrics. Using the developed techniques, we also obtain an interesting result about harmonic symplectic forms (Proposition (6.3).

Since the elliptic version of the system of equations with $U(1)$ as the gauge group assumes a specially simple form on two-manifolds, i.e. $-\triangle f = f^{-3}$, we are able to give an a priori estimate on growth of a solution in terms of the first Dirichlet eigenvalues of geodesic balls of the manifold (Theorem 7.1). This enables us to show that there are no finite energy solutions on the Euclidean plane. However, we prove existence of smooth radially symmetric solutions on R^2 (Theorem 8.6) and give a complete description of their properties. Some of our results generalize to $-\triangle f = f^{-p}$, where $p > 0$ and to higher dimensional spaces. The only reference to this problem of which we know is [2]. Our method is more general and gives stronger results, e.g. pointwise estimates of the radially symmetric solution. It can be generalized to similar equations on radially symmetric (non-euclidean) domains. Even though the asymptotics of the solutions are given already in [2], we have been able to simplify the proofs adapting a simple argument from [6] instead of using nonlinear integro-differential equations.

Our motivation is twofold.

1. *Geometric.* The structure of a principal (or equivalently vector) fiber bundle P over a Riemannian manifold M is fundamental in geometry. The problem we consider is stated invariantly with respect to this structure. Our ultimate goal is to look for numerical invariants of (M, P).

   The present approach is different from the well known topological one, which deals with topological rather than Riemannian manifolds and provides us with cohomological invariants of bundles—the characteristic classes. Our invariants intertwine topological properties of bundles with geometric (isometry) invariants of the base manifolds. Although both approaches make use of connections, there is a significant "philosophical" disparity: In the process of constructing, say, Chern classes one eventually demonstrates that the resulting object does not depend on the choice of a connection used. We do distinguish certain connections, the eigenforms, but ultimately hope to reduce their number drastically (up to a gauge transformation). However only the simplest case of the first eigenform, and for linear bundles over two-manifolds only, is worked out in this paper.

   We also want to mention one clear analogy with characteristic classes.

During the construction we use some auxiliary tools, such as the Riemannian submersion metrics, which finally disappear just as connections do in case of the characteristic classes.

2. *Physical.* Bundles over Riemannian manifolds appear to be of great interest to physicists. Moreover, our equations seem to reflect in a unified form the two great aspects of physics: the Schrödinger and the Yang-Mills equations. The geometric nature of the equations considered in this paper seems to invite new interpretations of the Schrödinger equation. Their adequacy and usefulness in the context of physics remain however unclear to the author.

2 The geometry

Let (M, g) be a Lorentzian manifold and $\pi : P \to M$ a principal fiber-bundle with a compact semisimple or compact abelian structure group G. We equip the bundle with a connection and let ω denote the connection 1-form. Then P carries a natural pseudo-Riemannian structure μ defined in the following way.

The tangent space of P at each point decomposes into the vertical, i.e.

tangent to the fiber, and the horizontal part defined by the connection. We consider them orthogonal. The scalar product $\mu = \mu(\omega)$ is defined by

$$\mu(X,Y) = h(\omega(X),\omega(Y)) + g(\pi_\star X, \pi_\star Y),$$

where h is a bi-invariant scalar product on the Lie algebra G of G. This metric defines the Laplace-Beltrami operator on forms

$$\Delta_\omega = d\delta + \delta d,$$

where d is the exterior derivative and its formal adjoint δ acting on forms of degree q is defined by $\delta = (-1)^{dim P(q+1)+1} \star d \star$ (where $\star$ denotes the μ-Hodge star and therefore depends on ω).

Our aim is to explicitely evaluate this Laplacian on the form ω. Let us notice that since the metric μ depends on ω itself, we should expect a nonlinear expression. Note also, that ω is a G-valued 1-form, but we fix an orthonormal basis in G and consider the componentwise action of Δ.

Let us first collect a few facts concerning geometry of the group G, and geometry of a Riemannian submersion metric on P that we are going to need later. The general reference is [1], where the theory is developed in a Riemannian setting but, as long as it comes to *algebraic* properties, everything remains valid in the pseudo-Riemannian case. A semisimple compact group

G carries a bi-invariant Riemannian structure h, determined up to a constant by its Killing form. Let V_i, $i = 1, 2, \ldots, dimG$, denote left-invariant orthonormal vector fields on G. It is well known, cf. [10], that for the Levi-Civita connection ∇^G of h

$$\nabla^G_{V_i} V_j = \frac{1}{2}\left[V_i, V_j\right], \tag{1}$$

where $[.,.]$ denotes the Lie bracket in G. In particular

$$\nabla^G_{V_i} V_i = 0. \tag{2}$$

Consequently, it is easy to see, that the Ricci tensor Ric_G of this metric satisfies

$$Ric_G V = -\frac{1}{4}\left[V_i, \left[V_i, V\right]\right] \tag{3}$$

with summation over repeated indices. These formulae are trivially satisfied for an abelian group.

This implies that in fact (G, h) has a constant positive Ricci curvature ρ, unless it is an abelian group. Finally

$$Ric_G V = \rho V, \tag{4}$$

where ρ is either positive or 0. (The most interesting example $SU(2)$ has constant positive curvature. Tori have constant curvature 0.)

Recall that a connection ω on a principal fiber bundle P defines the covariant derivative D on the bundle's geometric objects and its formal adjoint acting on vector valued forms of degree q: $D^{\star} = (-1)^{dim M(q+1)+1} \star D \star$ (where $\star$ denotes the g-Hodge star). It also defines its curvature $\Omega = D\omega$, i.e. an anti-symmetric tensorial 2-form with values in G. ω and Ω are related by the structure equation

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega. \tag{5}$$

By the G-action on P, V_i's generate orthonormal vertical vectors denoted for simplicity by V_i as well. Similarly, we will identify V_i with $\omega(V_i)$ and $\Omega(X, Y)$ with the corresponding vertical vector fields.

For the pseudo-Riemannian submersion metric μ on P let V_i be as above, and let X_j be horizontal vector fields, so that $\omega(X_j) = 0$. It turns out that the Levi-Civita connection ∇ of the metric μ satisfies the following identities

$$\begin{aligned} \nabla_{X_k} X_j &= \nabla^M_{X_k} X_j - \Omega(X_k, X_j) \\ \nabla_{V_i} X_j &= \nabla_{X_j} V_i = \hat{\Omega}(X_j, V_i) \\ \nabla_{V_i} V_l &= \nabla^G_{V_i} V_l, \end{aligned} \tag{6}$$

where ∇^M is the Levi-Civita connection of the pseudo-Riemannian structure on M and $\hat{\Omega}$ is a $(2,1)$ tensor of one vertical and one horizontal argument and horizontal values, defined uniquely by

$$\mu(\hat{\Omega}(X_j, V_i), X_k) = \mu(\Omega(X_j, X_k), V_i),$$

for all X_k. In particular for mutually orthogonal X_i of lengths 1 or -1

$$|\Omega|^2 = \mu(\Omega(X_j, X_k), \Omega(X_j, X_k)) = \mu(\hat{\Omega}(X_j, V_i), \hat{\Omega}(X_j, V_i)). \tag{7}$$

We are now prepared for the main computation. For simplicity, we carry it out in the Riemannian setting, where one has $\delta\alpha(\ldots) = -(\nabla_{Y_j}\alpha)(Y_j, \ldots)$ for an orthonormal basis Y_j. Since all the formulae are local and tensorial, one may consider a complexification of any simple region in M, where the Lorentzian and Riemannian geometries coincide, or just formally substitute $\sqrt{-1}Y_i$ for Y_i of length -1 in the formula above. One can also use the Weitzenböck formula to obtain another proof of Theorem 2.2 below. We remark that this theorem can not be obtained directly from variational principles, although it is a limiting case of a certain Euler-Lagrange equation (cf. below). We find it a very important source of motivation.

Lemma 2.1 *If a function f is constant along fibers, then $\delta(f\omega) = 0$, i.e. $f\omega$ is co-closed with respect to the metric μ.*

Proof. Fix an orthonormal basis V_i, X_j and its extension to local vector fields. We have

$$\begin{aligned}\delta(f\omega) &= -(\nabla_{V_i}(f\omega))(V_i) - (\nabla_{X_j}(f\omega))(X_j) \\ &= f\left\{-(\nabla_{V_i}\omega)(V_i) - (\nabla_{X_j}\omega)(X_j)\right\} - V_i f\omega(V_i) - X_j f\omega(X_j) \\ &= f\left\{-V_i\omega(V_i) + \omega(\nabla_{V_i}V_i) - X_j\omega(X_j) + \omega(\nabla_{X_j}X_j)\right\} - X_j f\omega(X_j),\end{aligned}$$

since $V_i f = 0$. It follows from (2) and (6) that

$$\nabla_{V_i}V_i = 0,$$

and at a point

$$\nabla_{X_j}X_j = 0,$$

if we take for X_j horizontal vector fields projecting to such fields on M that $\nabla^M_{X_j}X_j = 0$, which is possible and easily seen using geodesic coordinates. On the other hand $\omega(X_j) = 0$, since X_j are horizontal and $\omega(V_i)$ is a constant vector in the Lie algebra. Thus, the right-hand side of the formula above vanishes, which proves the lemma.

Theorem 2.2 *For the metric μ on a total space of a principal fiber bundle with the structure group G as above, the following formula holds*

$$\Delta_\omega\omega = -D^*\Omega + (\rho + |\Omega|^2)\omega, \tag{8}$$

where $\rho = 0$ if and only if G is abelian.

Proof. By the Lemma and (5), we obtain

$$\triangle_\omega \omega = \delta d\omega = -\frac{1}{2}\delta[\omega,\omega] + \delta\Omega.$$

For a horizontal $X = X_k$ for some k

$$\begin{aligned}
\delta[\omega,\omega](X) &= -(\nabla_{X_j}[\omega,\omega])(X_j, X) - (\nabla_{V_i}[\omega,\omega])(V_i, X) \\
&= -X_j([\omega,\omega](X_j, X)) + [\omega,\omega](\nabla_{X_j}X_j, X) + [\omega,\omega](X_j, \nabla_{X_j}X) \\
&\quad -V_i([\omega,\omega](V_i, X)) + [\omega,\omega](\nabla_{V_i}V_i, X) + [\omega,\omega](V_i, \nabla_{V_i}X),
\end{aligned}$$

and since $\nabla_{V_i}X$ is horizontal by (6), each term contains at least one horizontal vector as an argument of $[\omega,\omega]$. But ω annihilates horizontal vectors, so the right-hand side is zero.

Now, we evaluate the form on a vertical vector $V = V_j$ for some j.

$$\begin{aligned}
\delta[\omega,\omega](V) &= -(\nabla_{X_j}[\omega,\omega])(X_j, V) - (\nabla_{V_i}[\omega,\omega])(V_i, V) \\
&= -X_j([\omega,\omega](X_j, V)) + [\omega,\omega](\nabla_{X_j}X_j, V) + [\omega,\omega](X_j, \nabla_{X_j}V) \\
&\quad -V_i([\omega,\omega](V_i, V)) + [\omega,\omega](\nabla_{V_i}V_i, V) + [\omega,\omega](V_i, \nabla_{V_i}V).
\end{aligned}$$

If we eliminate terms containing at least one horizontal vector field as above, and notice that for vectors chosen as in the lemma $\nabla_{X_j}X_j = \nabla_{V_i}V_i = 0$ and $[\omega,\omega](V_i, V)$ is constant, we are left with

$$\delta[\omega,\omega](V) = [\omega,\omega](V_i, \nabla_{V_i}V).$$

In view of (1) this is equal to

$$\left[\omega(V_i), \frac{1}{2}[V_i, V]\right].$$

By (3) the last expression is nothing but $-2Ric_G V$, and it follows from (4) that

$$-\frac{1}{2}\delta[\omega,\omega] = \rho\omega. \tag{9}$$

Next we compute $\delta\Omega$. Keeping in mind that $\Omega(A,B)=0$ whenever at least one of the arguments is a vertical vector, we obtain

$$\begin{aligned}
\delta\Omega(V) &= -(\nabla_{X_j}\Omega)(X_j, V) - (\nabla_{V_i}\Omega)(V_i, V) \\
&= -X_j(\Omega(X_j, V)) + \Omega(\nabla_{X_j}X_j, V) + \Omega(X_j, \nabla_{X_j}V) \\
&\quad -V_i(\Omega(V_i, V)) + \Omega(\nabla_{V_i}V_i, V) + \Omega(V_i, \nabla_{V_i}V) \\
&= \Omega(X_j, \nabla_{X_j}V) \\
&= \Omega(X_j, \hat{\Omega}(X_j, V)),
\end{aligned}$$

and by (7)

$$\delta\Omega(V) = |\Omega|^2\omega(V). \tag{10}$$

The last step is an evaluation of $\delta\Omega(X)$ on a horizontal X as above. It is straightforward to see (cf. [7]), that

$$\delta\Omega(X) = -X_i\Omega(X_i, X) = -D^*\Omega(X). \tag{11}$$

Equations (9), (10) and (11) imply the theorem.

Remark. It follows from (8) that the connection 1-form ω is never harmonic with respect to the pseudo-Riemannian metric μ, unless

$$\rho + |\Omega|^2 = 0$$

and in particular the norm of the curvature is a negative constant. The reason for this is that it satisfies the norming condition

$$\omega(V) = V.$$

This is also the condition forcing connection forms to lie in an affine hypersurface rather than a linear subspace of the space of all the 1-forms on P. Observe also that one obtains the Yang-Mills equations by simply neglecting the vertical part of (8), and requiring that the horizontal part be zero.

Our next step is to admit a pointwise renormalization of the above condition and consider $f\omega$ instead of ω, where f is a scalar function constant on fibers. Now, forms $f\omega$ lie in a linear subspace of the space of all 1-forms. As a consequence of the above theorem, we obtain

Corollary 2.3 *For a function $f : P \to R$ constant on fibers, we have*

$$\triangle_\omega(f\omega) = -fD^\star\Omega - 2\Omega(\nabla f, .) + [-\triangle f + (\rho + |\Omega|^2)f]\omega. \qquad (12)$$

Caution! The sign convention for the Laplacian $\triangle$ on functions (in the elliptic case) is such that $\triangle = \frac{d^2}{dx^2}$ on the real line. This is opposite to the sign of $\triangle_\omega \omega$.

Proof. Since f is constant on geodesically complete fibers in P, it follows easily from (6) that Laplacian of f on P does not in fact depend on the metric μ and is itself constant on fibers. Moreover, it is equal to the pull-back of $\triangle f$, where $\triangle$ is the scalar Laplacian on M and with a slight abuse of notation we denote it by $\triangle f$ as well.

Let us now use the well known identity

$$\triangle_\omega(f\alpha) = -\triangle f\alpha - 2\nabla_{\nabla f}\alpha + f\triangle_\omega \alpha,$$

which holds for any 1-form α and a function f. If we now substitute $\alpha = \omega$, then, since f is constant on fibers and hence ∇f is a horizontal vector, it follows from (6) that

$$\nabla_{\nabla f}\omega = \Omega(\nabla f, .).$$

The above formulae together with (8) prove the Corollary.

Later on we are going to need an explicit formula for the Laplacian on 1-forms, which are pull-backs of 1-froms on the base manifold. Such a formula was proven in [5], but we need to specify an invariant version of it for $U(1)$-bundles. For simplicity, we denote by a both a 1-form on M and its pull-back—a tensorial 1-form on P. We let d^M, δ^M, $\triangle^M$ denote the familiar operators on M. Since the proof is very similar to those above, we only sketch it here.

Lemma 2.4 *Let $P \to M$ be a $U(1)$ bundle and let a be a tensorial 1-form on P. Then*

$$\triangle_\omega a = \triangle^M a + \langle \Omega, da \rangle \omega.$$

Proof. It is a direct calculation that $da = d^M a$, $\delta a = \delta^M a$ and $d\delta a = d^M \delta^M a$. The calculation of δda is a little more subtle. Namely $= \delta da(X_i) = \delta^M da(X_i)$, but

$$\delta da(V) = da(X_i, \hat{\Omega}(X_i, V)) = \langle \Omega, da \rangle \omega(V),$$

where the first equality follows from a computation similar to that in Lemma 2.1, and the second one is a consequence of the definition of $\hat{\Omega}$ and does not hold if the structure group has higher dimension.

3 The equation

Consider the condition of harmonicity of the "relaxed" connection 1-form $f\omega$

$$\triangle_\omega(f\omega) = 0,$$

and, more generally, the eigenvalue problem

$$\triangle_\omega(f\omega) = \lambda f\omega, \tag{13}$$

for a real constant ν.

Since the expression (12) decomposes naturally into the horizontal (or tensorial) and the vertical parts, we obtain the following equivalent system.

$$\begin{cases} D\Omega = 0 \\ fD^\star\Omega = -2\Omega(\nabla f, .) \\ -\triangle f + |\Omega|^2 f = \nu f, \end{cases} \tag{14}$$

where the first equation is the Bianchi identity, which is always satisfied by the *curvature* tensor Ω and $\nu = \lambda - \rho$. The first two equations form the well known Yang-Mills system with a source (or self-source) term $\frac{-2}{f}\Omega(\nabla f,.)$. The third equation is the Klein-Gordon equation on M with energy density instead of the usual mass constant, or, in the elliptic case, the stationary Schrödinger equation.

Remarks.

1. Let us consider two solutions of (14), say (f_1, ω_1) on $P_1 \to M_1$ and (f_2, ω_2) on $P_2 \to M_2$. It is straightforward that the product $f_1 f_2$ and the obvious Cartesian "product" connection is then a solution on $P_1 \times P_2 \to M_1 \times M_2$. If in addition, the bundles are circle bundles or at least one of them is trivial, we obtain a solution on $P_1 \otimes P_2 \to M_1 \times M_2$. Here $P_1 \otimes P_2$ denotes the principal bundle associated via the adjoint representation to the tensor product of the linear bundles associated via the adjoint representation to P_1 and P_2 respectively.

2. If M is a compact manifold, then multiplying the last equation in (14) by f and integrating by parts we see that there are no smooth solutions unless ν is nonnegative and strictly positive if the connection is not flat. In particular the system (14) does not admit smooth solutions on compact manifolds.

3. We are interested in solutions of (14), i.e. pairs (ω, f) such that the connection ω is at least continuous and thus preserves its geometric interpretation.

4. The system (14) is gauge-invariant and therefore every solution ω is determined only up to a gauge transformation.

5. It follows from the construction of the system (14) that real numbers ν corresponding to solutions are numerical invariants of the structure (M, P), which depend (on complex bundles of higher dimensions) also on a real parameter ρ, i.e. the choice of one of the mutually proportional Killing forms on the semisimple group G. (Addition of ρ shifts the spectral set on the real axis by ρ.)

4 Variational principles

Since, as shown in Lemma 2.1, $\delta(f\omega) = 0$, the equation $\Delta_\omega(f\omega) = 0$ is the Euler-Lagrange equation of the functional

$$A(f, \omega) = \frac{1}{2}\int_P \langle d(f\omega), d(f\omega)\rangle_\omega dV(\omega).$$

It is however an immediate linear-algebraic remark that the volume element $dV(\omega) = dV$ does not in fact depend on the connection form. Similarily, df is orthogonal to ω, and the norms $|df|_\omega = |df|$, $|\Omega|_\omega = |\Omega|$, $\frac{1}{4}|[\omega,\omega]|_\omega^2 = \rho$ (cf.(3), (4)) do not depend on ω. Therefore using (5) we obtain

$$2A(f,\omega) = \int_P \langle df \wedge \omega + fd\omega, df \wedge \omega + fd\omega\rangle_\omega dV$$

$$= C\left\{\int_M |df|^2 + \int_M f^2|\Omega|^2 + \tfrac{1}{4}\int_M f^2\langle[\omega,\omega],[\omega,\omega]\rangle\right\}$$

$$= C\left\{\int_M |df|^2 + \int_M f^2|\Omega|^2 + \rho\int_M f^2\right\},$$

where $C = (dimG)(volumeG)$.

The system (14) may be obtained *formally* by a calculation of the Euler-Lagrange equation of the functional of two independent variables

$$\hat{A}(f,\omega) = \int_M |df|^2 + \int_M f^2|\Omega|^2 + \rho\int_M f^2. \tag{15}$$

The functional $\hat{A}$ suggests that the scalar potential f plays role of a density function modifying the total energy of the field ω. It also gives us a hint that the system (14) may be rewritten in the following form

$$\begin{cases} D\Omega = 0 \\ D^\star(f^2\Omega) = 0 \\ -\triangle f + |\Omega|^2 f = \nu f. \end{cases} \tag{16}$$

This can also be verified easily by a direct calculation.

Surprisingly, it is rather easy to guess a nontrivial smooth solution (Ω, f) (and not (ω, f)!) to the system (16) on a $U(1)$-bundle (where $D = d$ and $\rho = 0$) over a noncompact complete Riemannian three-manifold M^3. For this purpose we use a nonconstant harmonic function on M. It is well known that such a function always exists.

Example 4.1 *Let $u : M^3 \to R$ be a harmonic function. Then $f = a \exp u + b \exp(-u)$ for any constants a, b and $\Omega = \star du$ satisfy (16) with $\nu = 0$.*

Proof. We have $d\Omega = d \star du = - \star \triangle u = 0$ and $d^\star(f^2 \Omega) = \star d \star (a \exp u + b \exp(-u)) \star du = 0$. Moreover $- \triangle f + |\Omega|^2 f = - \triangle (a \exp u + b \exp(-u)) + |du|^2 (a \exp u + b \exp(-u)) = 0$. It is interesting that we can always choose the constants a, b so that f will be changing its sign.

5 Eigenvalue problem on $U(1)$-bundles over compact manifolds

Throughout this section we assume that M is a compact manifold and $\nu > 0$. If the gauge group is $U(1)$, it is well known that the curvature 2-form is simply a scalar real (or purely imaginary) valued form, the covariant derivative becomes the ordinary exterior derivative, i.e. $D = d$, and $\rho = 0$. Thus the system (16) becomes

$$\begin{cases} d\Omega = 0 \\ d^*(f^2\Omega) = 0 \\ -\triangle f + |\Omega|^2 f = \nu f. \end{cases} \tag{17}$$

We prove the following

Proposition 5.1 *If (f,ω) satisfies (17) with $d\omega = \Omega$, then $d^*\Omega = 0$, $\Omega(\nabla f, .) = 0$. If in addition $dim M = 2$, then $f = const$.*

Proof. Every $U(1)$-invariant 1-form on P can be written in the form $f\omega + a$, where a is tensorial. Integrating scalar products of $U(1)$-invariant geometric objects over P or M is equivalent up to a constant and we will not specify the domain of integration in what follows.

Consider Δ_ω as an operator acting on 1-forms on P and let $f_i\omega + a_i$ denote the family of all those 1-eigenforms corresponding respectively to eigenvalues λ_i (in a nondecreasing sequence), which are invariant with respect to the $U(1)$-action on P. These forms satisfy $\Delta_\omega(f_i\omega + a_i) = \lambda_i(f_i\omega + a_i)$, and it follows from Corollary 2.3 and Lemma 2.4 that

$$\begin{cases} \frac{-1}{f_i} d^\star(f_i^2\Omega) + \Delta^M a_i = \lambda_i a_i \\ -\Delta f_i + |\Omega|^2 f_i + \langle \Omega, da_i \rangle = \lambda_i f_i. \end{cases} \tag{18}$$

It is well known from the general theory (cf. [4]) that every L_2 1-form can be written as an infinite linear combination of eigenforms, and the decomposition is unique up to a choice of basis in every (finite dimensional) eigenspace. It follows from the uniqueness that every $U(1)$-invariant 1-form can be written as (possibly infinite) linear combination of the forms $f_i\omega + a_i$. Also $f_i\omega + a_i$ is orthogonal to $f_k\omega + a_k$ if $\lambda_i \neq \lambda_k$. In particular, if a $U(1)$-invariant 1-form is orthogonal to all eigenforms corresponding to $\lambda_i \neq \lambda_k$ for a fixed k, it is either an eigenform with eigenvalue λ_k or it is trivial.

Since we assume that $f\omega$ is an eigenform (recall that (17) is equivalent to (13)), we have $f = f_k$, $\nu = \lambda_k$ and $a_k = 0$ for a certain k. Multiplying the second equation of (18) by f and integrating by parts, we obtain

$$\int \langle \nabla f_i, \nabla f \rangle + \int |\Omega|^2 f_i f + \int \langle \Omega, da_i \rangle f = \lambda_i \int f_i f = 0,$$

where the last equality follows from the fact that $f\omega$ and $f_i\omega + a_i$ are orthogonal. We obtain a similar (with $a_k = 0$) equation multiplying the third equation of (17) by f_i and so on.

It follows that

$$\int \langle \Omega, da_i \rangle f = 0.$$

Since by (14)

$$d^\star(f\Omega) = \star(df \wedge \star\Omega) + fd^\star(\Omega) = -\Omega(\nabla f, .)$$

integrating the last integral by parts we obtain

$$0 = \int \langle d^\star(f\Omega), a_i \rangle = -\int \langle \Omega(\nabla f, .), a_i \rangle.$$

Therefore, $\Omega(\nabla f, .)$ is an eigenform with eigenvalue $\nu = \lambda_k$ as well. Using Lemma 2.4 with $a = \Omega(\nabla f, .)$ we obtain in particular that $\langle d(\Omega(\nabla f, .)), \Omega \rangle = 0$, and

$$\int \langle d(\Omega(\nabla f, .)), f\Omega \rangle = 0.$$

Integrating by parts again we obtain

$$0=\int\langle\Omega(\nabla f,.),d^{\star}(f\Omega)\rangle=-2\int|\Omega(\nabla f,.)|^{2},$$

Thus $\Omega(\nabla f,.)=0$ and $d^{\star}\Omega=0$.

If the underlying manifold M has two dimensions, then $\Omega=\star\varphi$ for a scalar function φ, and by (17)

$$0=\star d^{\star}(f^{2}\Omega)=d\star(f^{2}\star\varphi)=d(f^{2}\varphi),$$

which implies that $\varphi=\frac{A}{f^{2}}$ for a certain constant A, if the function f never vanishes. Therefore Ω is pointwise-proportional to the volume form dV, but $dV(\nabla f,.)=0$ forces $\nabla f=0$. This completes the proof.

Proposition 5.1 implies the following

Corollary 5.2 *Let $c_1[P]$ denote the first Chern class of a bundle P over an even-dimensional compact Riemannian manifold (M,g). If the harmonic 2-form Ω in the cohomology class $c_1[P]$ is nonsingular and the system (17) admits solutions on the bundle P, then $|\Omega|=const$.*

Proof. Since Ω is nonsingular and $\Omega(\nabla f,.)=0$, we have $\nabla f=0$ and f is a constant function. Since in addition f satisfies the third equation of (17), it follows that $|\Omega|=const$ as claimed.

Using the last part of the proof of Proposition 5.1, we see that on a two-manifold $|\Omega|^2 = \varphi^2 = \frac{A^2}{f^4}$ and the last equation of the system (17) becomes

$$-\triangle f + \frac{A^2}{f^3} = \nu f.$$

Rescaling f by the factor of $\sqrt{A}$ we can further reduce the equation to

$$-\triangle f + \frac{1}{f^3} = \nu f, \qquad (19)$$

unless $A = 0$ (and therefore $\Omega = 0$) when the equation reduces to

$$-\triangle f = \nu f, \qquad (20)$$

Now, if we can show existence of solutions for (19), then we can recover Ω from $\star\Omega = \frac{A}{f^2}$ and solve it for ω. Indeed, it is well known (cf. [1], p. 255) that a 2-form in the cohomology class $c_1[P]$ of a $U(1)$-bundle P is a curvature of a certain connection on this bundle. On the other hand, any nontrivial 2-form on a two-manifold can be put into $c_1[P]$ by multiplication by a constant.

We obtain the following

Proposition 5.3 *If M is a compact two-manifold, then the system (17) admits a (unique) smooth solution with $f > 0$. In addition*

$$\nu = \left(2\pi \frac{c_1[P]([M])}{VolM}\right)^2, \qquad (21)$$

where $VolM$ is the volume of M.

Proof. Assume first that P is nontrivial. Then (cf. [1])

$$c_1[P]([M]) = \frac{1}{2\pi}\int_M \Omega = \frac{1}{2\pi}\int_M f^{-2}.$$

We know from the previous proposition that $f = const$ and it follows from (19) that $f = \nu^{-\frac{1}{4}}$. Substituting this value of f to the integral identity above we obtain the formula (21).

If the bundle is trivial, then the first Chern class is 0. Thus $|\Omega|^2 = \frac{A^2}{f^4} = 0$ and $\Omega = 0$. The system (17) reduces to (20), whose only positive solution is constant with an eigenvalue $\nu = 0$ and (21) is satisfied. This completes the proof.

Remark. On a trivial bundle we have infinitely many solutions with f changing sign. Indeed, every eigenfunction of the Laplacian and the trivial connection give us a solution. Since on nontrivial bundles $\Omega = \star\frac{1}{f^2}$, we must have $f > 0$, if we insist on smooth connections, and the solutions above are unique in general.

6 Solutions for nonabelian bundles over manifolds of more than four dimensions

Suppose that we have a solution (ω, f) of (16) on a Riemannian manifold (M, g) of $n \neq 4$ dimensions. If we deform the metric g conformally and introduce a new metric $g_2 = f^{\frac{4}{n-4}} g$, then the connection ω is a solution of the system of Yang-Mills equations on (M, g_2). Indeed, if we let $\star_2$ denote the Hodge star operator with respect to g_2, then for a 2-form Ω we have $\star_2 \Omega = f^2 \star \Omega$ and therefore $D^\star(f^2\Omega) = 0$ implies $D^{\star_2}\Omega = 0$ as claimed.

In this section we will show that the inverse theorem is true if $n > 4$. Indeed, let us assume that for a principal bundle over a Riemannian manifold (M, g) as above, we have a connection ω satisfying the Yang-Mills equations $D^\star\Omega = 0$. We define a metric $g_1 = f^{\frac{-4}{n-4}} g$ and obtain $D^{\star_1}(f^2\Omega) = 0$. If in addition the function f satisfied

$$- \triangle_1 f + |\Omega|_1^2 f = \nu f, \tag{22}$$

the system (16) would be satisfied on (M, g_1). Here the operator $\triangle_1$ and the norm $|\Omega|_1$ depend on f. If we make a substitution $h := f^{\frac{-n}{n-4}}$ and perform an elemantary calculation, we see that (22) is equivalent to the following semilinear elliptic equation

$$\frac{n-4}{n} \triangle h + |\Omega|^2 h^{\frac{n-4}{n}} = \nu h^{\frac{n+4}{n}}. \tag{23}$$

Therefore, if we can find a positive solution of the equation (23), we obtain a solution (ω, f) of the system (16) on (M, g_1).

We now prove the following

Lemma 6.1 *Let M be a compact manifold of $n > 4$ dimensions and $V : M \to R$ a nonnegative smooth function. Then there is a number $\nu > 0$ such that the semilinear eigenvalue problem*

$$\triangle h + V h^{\frac{n-4}{n}} = \nu h^{\frac{n+4}{n}} \tag{24}$$

admits a smooth positive solution.

Proof. To show existence of solutions of (24) we find a maximizer of the functional

$$-\frac{1}{2}\int |\nabla h|^2 + \frac{n}{2n-4}\int V h^{\frac{2n-4}{n}},$$

subject to the constraint $\frac{n}{2n+4}\int h^{\frac{2n+4}{n}} = 1$ in the class of $H^{1,2}$ functions. By virtue of the Hölder inequality, the second integral of the functional is bounded on the constraint submanifold and therefore the whole functional is bounded above. We denote its supremum by ν. Since the exponent $\frac{2n+4}{n}$ is subcritical, i.e. the Sobolev imbedding $H^{1,2} \hookrightarrow L^{\frac{2n+4}{n}}$ is compact, we can find a maximizing function, say h, by the standard method of choosing a maximizing sequence.

The function h is a weak solution of the equation (24). Using the critical imbedding $H^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$ and starting the standard bootstrap argument, we will easily obtain that $h \in C^\infty$ and that it is a classical solution of (24).

We now show that we can find a *positive* solution. Indeed, let $\varphi = |h| \geq 0$. Then $|\nabla \varphi| = |\nabla h|$ almost everywhere, and φ is a maximizer as well. Therefore $\varphi \in C^\infty$ as before and φ is a solution of (24). Since φ is bounded above, we can choose a sufficiently large constant $c > 0$, so that

$$-\triangle \varphi + c\varphi = c\varphi + V\varphi^{\frac{n-4}{n}} - \nu\varphi^{\frac{n+4}{n}} \geq 0.$$

The strong maximum principle implies that either $\varphi \equiv 0$, which is impossible because of the constraint, or $\varphi > 0$ everywhere. Thus φ is a positive solution of (24). This concludes the proof.

Retaining the above notation, we obtain as a corollary

Theorem 6.2 *Assume that a Riemannian manifold* (M, g), $dimM = n > 4$

admits a Yang-Mills connection ω with curvature Ω. Then there is a constant $\nu > 0$ and a function $f > 0$ such that (ω, f) is a solution of (16) on (M, g_1).

Proof. Since $n > 4$ and $|\Omega| \geq 0$, the equation (22) is of the form (24) and the theorem follows from Lemma 6.1.

REMARK. For $n = 3$ the equation (24) admits weak solutions, but we do not know if at least one of the solutions is smoth and positive.

The following fact about harmonic symplectic forms follows from Theorem 6.2 and Proposition 5.1.

Theorem 6.3 *Let Ω be a real-valued nonsingular 2-form on a manifold M of $n = 2k$ dimensions such that $d\Omega = 0 = d^*\Omega$, and there is a constant c such that the cohomology class of $c\Omega$ is in the image of the natural homomorphism $H^2(M, Z) \to H^2(M, R)$. Then $|\Omega| = const.$*

Proof. Let us assume first that $k > 2$. Multiplying the form Ω by a constant if necessary, we can assume without loss of generality that $c = \frac{1}{2\pi}$. Then by the hypothesis, the 2-form Ω is the curvature form of a connection on the $U(1)$-bundle with the first Chern class equal to the cohomology class $[\frac{1}{2\pi}\Omega]$. Assume that $|\Omega|^2 \neq const.$ The equation (23) admits a positive smooth solution h. Then, defining $f := h^{\frac{4-n}{n}}$, we obtain a solution of (17) on (M, g_1). But since Ω is nondegenerate, by virtue of Proposition 5.1 we have $\nabla^1 f = 0$ and therefore $\nabla^1 h = 0$, i.e. $h = const.$ Therefore h cannot be a solution of (23) and the contradiction proves the Theorem for $k > 2$.

If $k = 2$, we consider a new manifold $N := M \times S^1$ with the product metric. Using the natural projection of N onto M, we pull-back the bundle and the connection to N. With abuse of notation we denote the pull-back connection by ω and its curvature by Ω again. Now we repeat the above reasoning. In particular, there is a function f such that (ω, f) is a solution of (17) with respect to a new metric on N. Since Ω is symplectic on M, it follows from Proposition 5.1 that ∇f depends only on the S^1-variable. Thus f depends only on the S^1-variable and the third equation of (17) implies that $|\Omega| = const.$

Finally, the case of $k = 1$ is trivial. This completes the proof.

7 Static states for $U(1)$-fields on noncompact surfaces

If $\nu \neq 0$, then $f = \nu^{-\frac{1}{4}}$ is a solution of (19) on any surface. If $\nu = 0$ the equation (19) can be solved explicitely on R^2. Namely

$$f(x, y) = f(r) = \sqrt{2r},$$

where $r = \sqrt{x^2 + y^2}$, is a solution. Unfortunately this solution is not smooth. In this and the next section we prove existence of smooth solutions with $\nu = 0$ and describe their properties.

First, we give an a priori estimate for the growth at infinity of solutions to (19) in terms of the first Dirichlet eigenvalue of the Laplacian on geodesic balls. Note that a solution is subharmonic, and therefore assumes its suprema over given domains on their boundaries. Let $p \in M$ be a fixed point in our manifold, let $B(r)$ be the geodesic ball of radius r about p, $clB(r)$ its closure, and let $\lambda_r > 0$ denote the first Dirichlet eigenvalue of $B(r)$ for those r for which the boundary $\delta B(r)$ is smooth.

Theorem 7.1 *If f is a positive solution of*

$$\Delta f \geq \frac{1}{f^3} \tag{25}$$

on $clB(r)$ and r is such that $\delta B(r)$ is smooth then

$$\left(\sup_{\delta B(r)} f\right)^4 \geq \frac{256}{27}\frac{1}{\lambda_r}. \tag{26}$$

Proof. Let ϕ denote the first eigenfunction, i.e. $\triangle\phi = -\lambda_r\phi$, $\phi|_{\delta B(r)} \equiv 0$. It is known that one may assume $\phi > 0$ in $B(r)$ and the exterior normal derivative on the boundary $\langle\nabla\phi, n\rangle \leq 0$ (in fact < 0 but we do not need this stronger fact). Multiplying both sides of (19) by ϕ and integrating over $B(r)$, we obtain

$$\int_{B(r)} \phi \triangle f \geq \int_{B(r)} \phi \frac{1}{f^3}.$$

Using the Green's formula

$$\begin{aligned} \textstyle\int_{B(r)} \phi \triangle f &= \textstyle\int_{B(r)} f \triangle \phi - \int_{\delta B(r)} f\langle\nabla\phi, n\rangle \\ &= -\lambda_r \textstyle\int_{B(r)} f\phi - \int_{\delta B(r)} f\langle\nabla\phi, n\rangle. \end{aligned}$$

On the other hand, since $f > 0$ we have

$$\begin{aligned} \textstyle\int_{\delta B(r)} f\langle\nabla\phi, n\rangle \geq &\ \textstyle\sup_{\delta B(r)} f \int_{\delta B(r)}\langle\nabla\phi, n\rangle = \sup_{\delta B(r)} f \int_{B(r)} \triangle\phi \\ &= -\lambda_r \textstyle\sup_{\delta B(r)} f \int_{B(r)} \phi, \end{aligned}$$

where we apply the Green's formula again. It follows that

$$\int_{B(r)} \phi \left\{-\lambda_r f + \lambda_r \sup_{\delta B(r)} f - \frac{1}{f^3}\right\} \geq 0. \tag{27}$$

Thus, since ϕ is positive, there is a point $q \in B(r)$ such that

$$-\lambda_r f(q)^4 + \lambda_r \sup_{\delta B(r)} f \cdot f(q)^3 \geq 1. \tag{28}$$

Calculating extrema we notice that as a polynomial of $f(q)$ for $0 \leq f(q) \leq \sup_{\delta B(r)} f$, the expression

$$-\lambda_r f(q)^4 + \lambda_r \sup_{\delta B(r)} f \cdot f(q)^3$$

attains its supremum at $\frac{3}{4} \sup_{\delta B(r)} f$. Evaluating the left–hand side of (28) on this supremum, we obtain the inequality (26).

Let f be a function on R^n and let S_r denote the sphere with center at the origin and radius r. We introduce the following notation

$$\bar{f}(r) := \frac{1}{|S_r|} \int_{S_r} f,$$

where $|S_r|$ denotes volume of the sphere.

We obtain the following

Corollary 7.2 *If f is a positive solution to (25) on R^n, then*

$$\bar{f}(r)^4 \geq \frac{256}{27} \frac{1}{\lambda_r}. \tag{29}$$

Proof. $\bar{f}$ is a radially symmetric function. A direct calculation shows that

$$\triangle \bar{f}(r) = \bar{f}''(r) + \frac{n-1}{r}\bar{f}'(r) = \frac{1}{|S_r|}\int_{S_r} \triangle f.$$

On the other hand, it follows from the Jensen's inequality that

$$\frac{1}{|S_r|}\int_{S_r} \triangle f \geq \frac{1}{|S_r|}\int_{S_r} \frac{1}{f^3} \geq \frac{1}{\bar{f}^3}.$$

Therefore

$$\triangle \bar{f} \geq \frac{1}{\bar{f}^3}$$

and the claim follows from the theorem above.

Next, we show that there are no finite energy solutions to (16) or equivalently (19) on the Euclidean plane. In fact, the following theorem holds.

Theorem 7.3 *If f is a positive solution to (19) on the Euclidean plane, then*

$$\int_{R^2} |df|^2 = \infty.$$

Proof. A direct calculation shows that

$$\bar{f}'(r) = \frac{1}{2\pi r}\int_{B_r} \triangle f,$$

where B_r is the disc with center at the origin and radius r. Integrating this inequality we obtain

$$\bar{f}(R) = \frac{1}{2\pi}\int_0^R \frac{1}{r}\int_{B_r} \triangle f + \bar{f}(0). \tag{30}$$

On the other hand, by the Fubini's Theorem

$$\int_0^R \frac{1}{r}\int_{B_r} \triangle f =$$

$$\int_0^R \int_0^r \frac{\rho}{r}\int_0^{2\pi} \triangle f(\rho,\theta) d\theta d\rho dr =$$

$$\int_0^R \int_\rho^R \frac{\rho}{r}\int_0^{2\pi} \triangle f(\rho,\theta) d\theta dr d\rho = \tag{31}$$

$$\ln R \int_0^R \int_0^{2\pi} \rho \triangle f(\rho,\theta) d\theta d\rho - \int_0^R \int_0^{2\pi} \rho \ln\rho \triangle f(\rho,\theta) d\theta d\rho \leq$$

$$\ln R \int_{B_R} \triangle f + c_1,$$

where $0 < c_1 := -\int_0^1 \int_0^{2\pi} \rho \ln\rho \triangle f(\rho,\theta) d\theta d\rho$. Since it is known that λ_r is proportional to r^{-2}, this inequality together with (30) and Corollary 7.2 imply that there is a positive constant c_2 such that

$$\int_{B_R} \triangle f \geq c_2 \frac{\sqrt{R}}{\ln R} \qquad \text{for } R \text{ sufficiently large.} \tag{32}$$

However,

$$\int_{B_R} \triangle f = \int_{S_R} \langle \nabla f, n \rangle \leq \left(\int_{S_R} |df|^2 \right)^{\frac{1}{2}} \left(\int_{S_R} 1 \right)^{\frac{1}{2}},$$

and therefore by (32) there is a constant $c_3 > 0$ such that

$$\int_{S_R} |df|^2 \geq \frac{c_3}{(\ln R)^2} \geq c_3 (\ln \ln)'(R), \tag{33}$$

and the integral $\int \int_{S_R} |df|^2 dR$ diverges. This completes the proof.

Remarks.

1. The Theorem 7.1 is true on any noncompact complete manifold of arbitrary dimension. The proof generalizes to the case $\triangle f \geq f^{-p}$ with any $p > 0$ and gives

$$\left(\sup_{\delta B(r)} f \right)^{p+1} \geq \frac{(p+1)^{p+1}}{p^p} \frac{1}{\lambda_r}.$$

   It also gives an upper estimate for minima over geodesic spheres of solutions to $\triangle f \leq -f^q$ with any $q > 1$. We shall see later that it gives the best possible estimate in this generality (cf. Proposition 8.3).

2. The proof of the above theorem remains almost unchanged for

$$\triangle f = f^{-p} \tag{34}$$

on R^n for $n \geq 2$ and any $p > 0$. The only difference is that the estimate (33) must be replaced by

$$\int_{S_R} |df|^2 \geq c\frac{R^\alpha}{(\ln R)^2},$$

where c is a positive constant and $\alpha = \frac{4}{p+1} - (n-1)$. The integral of the last expression diverges if and only if $\alpha > -1$. Thus, if $0 < p < \frac{4}{n-2} - 1$ (in particular p is any positive number for $n = 2$), then the equation (34) does not admit finite energy solutions on R^n.

8 Radially symmetric solutions on R^2

Our next step is to further restrict attention to the Euclidean plane $M = R^2$ and prove existence of radially symmetric solutions f to (19) on it.

Let (r, θ) denote the polar coordinates on R^2. If we assume that $f(r, \theta) = u(r)$ is a solution of (19), then u satisfies

$$u''(r) + \frac{u'(r)}{r} - \frac{1}{u^3(r)} = 0. \tag{35}$$

In addition, f is a smooth solution of (19) with $f = 1$ at the origin if and only if u is a solution to the following initial value problem

$$\begin{cases} u''(r) + \frac{u'(r)}{r} - \frac{1}{u^3(r)} = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases} \tag{36}$$

The following two lemmas show that the solutions (possibly singular!) do not blow up to infinity in finite r and a singularity can occur only at $r = 0$.

Lemma 8.1 *If u is a positive solution to (35) and $\limsup_{r \to r_0^-} u(r) = \infty$ then $r_0 = \infty$, the function is increasing for sufficiently large r, and*

$$\limsup_{r \to \infty} u'(r) < \infty.$$

Proof. It follows from (35) that if $u' = 0$ then $u'' > 0$, so u can have at most one extremum and if one exists, it must be a minimum. We may therefore assume that the function grows for r sufficiently large.

Let us consider the following "energy" of u

$$E(r) = \frac{1}{2}u'(r)^2 + \frac{1}{2u(r)^2}.$$

An expression of this type was introduced in [9].

Since by (35)

$$E'(r) = -\frac{u'(r)^2}{r} \leq 0,$$

$E(r)$ is a nonincreasing function. Let us fix $r_1 < r_0$. Then for all $r \in (r_1, r_0)$ we have $2E(r) \leq 2E(r_1)$, i.e.

$$u'(r)^2 \leq u'(r_1)^2 + \frac{1}{u(r_1)^2} - \frac{1}{u(r)^2}.$$

Assume now $\lim_{r \to r_0^-} u(r) = \infty$. Passing to the limit we obtain

$$\limsup_{r \to r_0^-} u'(r)^2 \leq u'(r_1)^2 + \frac{1}{u(r_1)^2}$$

This means $\limsup_{r \to r_0^-} u'(r) < \infty$ which implies that $r_0 = \infty$. This completes the proof.

Remark. We will see later that for smooth solutions in fact

$$\limsup_{r \to \infty} u'(r) = 0. \tag{37}$$

We show next that $r = 0$ is the only point at which the solutions to (35) can have singularities.

Lemma 8.2 *If u is a positive solution to (35) and* $\lim_{r \to r_0^+} u(r) = \infty$ *then* $r_0 = 0$.

Proof. The hypothesis can be satisfied only if r_0 is to the left of the unique infimum. Then $u' < 0$ which implies

$$u'' = \frac{1}{u^3} - \frac{u'}{r} > 0.$$

Thus u' must in fact be an increasing function and we have $\lim_{r \to r_0^+} u'(r) = -\infty$.

If we fix $\varepsilon > 0$, there is $r_1 > r_0$ such that for all $r \in (r_0, r_1)$

$$\frac{1}{u(r)^3} < \varepsilon < -\frac{u'(r)}{r}.$$

For such r

$$u''(r) < \varepsilon - \frac{u'(r)}{r} < -2\frac{u'(r)}{r}.$$

We consider this as a differential inequality satisfied by u'. Let w be a solution of

$$w'(r) = -2\frac{w}{r},$$

i.e. $w(r) = -\frac{k^2}{r^2}$. If we choose k to be a constant such that $w(r_2) = u'(r_2)$ for a certain $r_2 \in (r_0, r_1)$, then

$$u'(r) > -\frac{c^2}{r^2}$$

for all $r \in (r_0, r_2)$. This contradicts $\lim_{r \to r_0^+} u'(r) = -\infty$ unless $r_0 = 0$, which completes the proof.

Finally, we have

Proposition 8.3 *The initial value problem (36) admits a unique smooth solution u_1. Moreover, the function u_1 is strictly increasing and there is a constant $c > 0$ such that* [1]

$$c\sqrt{r} \leq u_1(r) \leq \frac{4}{c^3}\sqrt{r} + 1 \tag{38}$$

for all r.

The function $u_a(r) = \sqrt{a}u(\frac{r}{a})$ is a solution with the initial value $u_a(0) = \sqrt{a}$.

In addition, the equation (35) for $r > 0$ admits solutions satisfying $lim_{r \to 0} u(r) = \infty$ and $lim_{r \to 0} u(r) = 0$. Moreover, the singular solution assumes a unique minimum at a certain $r_0 > 0$. This exhausts all possibilities.

Proof. The lower estimate follows immediately from (26), because for a radially symmetric solution $\sup_{\delta B(r)} f = u(r)$ and it is known that λ_r is proportional to r^{-2}.

[1]The upper estimate was proven by L. Karp.

To obtain the upper estimate, let us rewrite the equation (35) in the form

$$(ru'(r))' = \frac{r}{u(r)^3}.$$

Assume $r > \varepsilon > 0$. Integrating the identity above and using the lower bound for u we have

$$\begin{aligned} ru'(r) - \varepsilon u'(\varepsilon) &= \int_\varepsilon^r (\rho u'(\rho))' d\rho = \int_\varepsilon^r \frac{\rho}{u(\rho)^3} d\rho \\ &\leq \frac{1}{c^3} \int_\varepsilon^r \frac{\rho}{\sqrt{\rho^3}} d\rho = \frac{2}{c^3} \left(\sqrt{r} - \sqrt{\varepsilon}\right). \end{aligned}$$

Thus, if we let $\varepsilon \to 0$, we obtain

$$u'(r) \leq \frac{2}{c^3} \frac{1}{\sqrt{r}}, \tag{39}$$

which proves (37).

Integrating from ε to r again, we obtain

$$u(r) \leq \frac{4}{c^3} \left(\sqrt{r} - \sqrt{\varepsilon}\right) + u(\varepsilon).$$

Since $u(0) = u_1(0) = 1$, passing with ε to the limit completes the proof of the upper estimate.

Local existence of (36) may be verified directly by writing a uniquely defined power series in r satisfying all the conditions and convergent for r small enough (see also [2]). Uniqueness for $r > 0$ follows from the standard theory of ODE. We leave details to the reader.

The upper estimate or Lemma 8.1 show that the solution does not blow up to infinity in finite r. The lower estimate shows that the solution does not decrease to 0. This proves that the solution extends to all $r > 0$.

It follows from the equation (36) that $u'' > 0$ for small r and therefore u' grows above 0 for such r. Since in addition the function admits at most one extremum (cf. the proof of Lemma 8.1) we have $u'(r) > 0$ for all r and u is strictly increasing.

A direct calculation shows that u_a is as claimed.

If we attempt to solve (35) with the initial condition $u'(1) < 0$ then $\lim_{r\to 0} u(r) = \infty$. Indeed,

$$u'' > -\frac{u'}{r},$$

and if $u' < 0$, this differential inequality allows us to compare u' to the solution $w(r) = -\frac{k^2}{r}$ of the corresponding equation so that

$$u'(r) < -\frac{k^2}{r},$$

for $r < 1$ with the constant k chosen so that $w(1) = u'(1)$. Therefore $\lim_{r\to 0} u'(r) = -\infty$ and thus $\lim_{r\to 0} u(r) = \infty$. This singular solution is defined for all $r > 0$ by Lemma 8.1. As noticed above, if an extremum exists it must be a unique minimum. To see that u assumes the minimum, it suffices to show that $u(r) \to \infty$ as $r \to \infty$. However, it follows from (26) that $u(r-a) > c\sqrt{r-a}$ for all $a > 0$ and $0 < r < 2a$, where c is as above. In particular $u(2a) > c\sqrt{a}$ which proves the claim.

It is easy to guess an example of the continuous solution, namely $u(r) = \sqrt{2r}$. We do not know if there are other continuous solutions vanishing at the origin. This completes the proof.

In fact, we are able to do better describing the behavior of (smooth) solutions at infinity, showing that they asymptotically converge to $\sqrt{2r}$. The structure of the proof is rather standard and we borrow it from [6]. However, since the equation is essentially different from those considered in [6] some of the arguments must be new and they follow from the Proposition above.

Proposition 8.4 *Smoth solutions to (36) satisfy*

$$\lim_{r\to\infty} \frac{u(r)}{\sqrt{2r}} = 1.$$

Proof. First, we show that if the limit exists, then it equals 1 as above. Let $\lim_{r\to\infty} \frac{u(r)}{\sqrt{r}} = L$. Then, since by (38) and (39) we have $\lim_{r\to\infty} u(r) = \infty$

and $\lim_{r\to\infty} u'(r) = 0$, we can apply the l'Hospital Rule twice as follows

$$L = \lim_{r\to\infty} \frac{u(r)}{\sqrt{r}} = 2 \lim_{r\to\infty} \frac{u'(r)}{\frac{1}{\sqrt{r}}} = -4 \lim_{r\to\infty} u''(r) r \sqrt{r}.$$

On the other hand by (36)

$$u''(r) r \sqrt{r} + u'(r)\sqrt{r} = (\frac{\sqrt{r}}{u(r)})^3,$$

and therefore

$$-\frac{1}{4}L + \frac{1}{2}L = \frac{1}{L^3},$$

which means $L = \sqrt{2}$ as required.

We now prove that the limit exists. By substitutions $r = e^t$ and $x(t) = r^{-\frac{1}{2}}u(r)$, we obtain an equation equivalent to (36) in the form

$$x''(t) + x'(t) + \frac{1}{4}x(t) = \frac{1}{x(t)^3}, \tag{40}$$

If we further substitute $y(t) = x'(t)$, this becomes equivalent to the system

$$\begin{cases} x' &= y \\ y' &= -y - \frac{1}{4}x + x^{-3}. \end{cases} \tag{41}$$

This system of first order equations has only one stationary point $(\sqrt{2}, 0)$. Linearizing it about the point, we obtain

$$\begin{cases} x' &= y \\ y' &= -y - (x - \sqrt{2}) + o[(x-\sqrt{2})^2]. \end{cases} \tag{42}$$

Therefore the stationary point is asymptotically stable. It suffices to show that a solution converges to the stationary point.

Let us associate to (41) the following "energy"

$$E(t) = \frac{1}{2}x'^2 + \frac{1}{8}x^2 + \frac{1}{2x^2}. \tag{43}$$

We have $E'(t) = -x'^2 \le 0$ and therefore $E(t)$ is a nonincreasing function. Thus $\lim_{t\to\infty} E(t) = E_0$ exists and E_0 is a finite number. Since

$$E(s_0) - E(s_1) = \int_{s_0}^{s_1} x'(t)^2 dt,$$

we see that

$$\int_{s_0}^{\infty} x'(t)^2 dt < \infty. \tag{44}$$

In addition, since

$$x'(t) = u'(r)\sqrt{r} - \frac{u(r)}{2\sqrt{r}}$$

we observe that (38) and (39) imply boundedness of $x'(t)$ for t bounded away from $-\infty$.

We are now able to repeat the argument in [6] to obtain $\lim_{t\to\infty} x'(t) = 0$. Namely, suppose to the contrary that either $\limsup_{t\to\infty} x'(t) > 2\delta > 0$ or $\liminf_{t\to\infty} x'(t) < -2\delta < 0$. Let us consider the first case only, since the second one is analogous. By (44) there is a sequence $\zeta_1 < \zeta_2 < \ldots$ of numbers growing to ∞ and such that $\lim_{k\to\infty} x'(\zeta_k) = 0$. Therefore, there are two sequences $\{t_k\}$ and $\{s_k\}$ such that $t_1 < s_1 < t_2 < s_2 \ldots \uparrow \infty$, $x'(t_k) = \delta$, $x'(s_k) = 2\delta$ and $\delta < x'(s) < 2\delta$ for $s \in (t_k, s_k)$. By the Mean Value Theorem we obtain a sequence $\{\eta_k\}$ satisfying

$$x''(\eta_k) = \frac{x'(s_k) - x'(t_k)}{s_k - t_k} = \frac{\delta}{s_k - t_k},$$

where $t_k < \eta_k < s_k$. But since by (40) $x''(t)$ is bounded for t bounded away from $-\infty$, there exists a constant $\varepsilon > 0$ such that $s_k - t_k > \varepsilon$. Thus the integral in (44) can be estimated from below by

$$\sum_{k=1}^{\infty} \int_{t_k}^{s_k} x'(t)^2 dt \geq \sum_{k=1}^{\infty} \delta^2 \varepsilon = \infty,$$

which contradicts (44). Therefore $\lim_{t\to\infty} x'(t) = 0$.

By (43) we conclude that

$$lim_{t\to\infty}(\frac{x^2}{8} + \frac{1}{2x^2}) = E_0,$$

and therfore $\lim_{t\to\infty} x(t) = \lim_{r\to\infty} \frac{u(r)}{\sqrt{r}}$ exists by convexity of $\frac{x^2}{8} + \frac{1}{2x^2}$. This completes the proof.

Corollary 8.5 *The linear part of the system (42) describes a pendulum with small friction. In particular, there are infinitely many intersection points* *of a smooth solution $u(r)$ with the singular (stationary) solution $\sqrt{2r}$. This sequence of numbers is determined up to a multiplicative constant, since any two smooth solutions are related as described in Proposition 8.3.*

Finally, we obtain the following

Theorem 8.6 *For the Euclidean plane, the system (14) admits a smooth solution (f, ω) which is flat at infinity, i.e. $|\Omega| \to 0$ as $|x| \to \infty$.*

Proof. By Proposition 8.3, we have a smooth solution $f(r, \theta) = u(r)$ to (19). Moreover, as stated above the curvature 2-form must then satisfy

$$\star\Omega = \frac{1}{u(r)^2},$$

and in particular $|\Omega| \to 0$ as $|x| \to \infty$. Let us fix a global gauge over R^2 and make an additional Ansatz, that in this gauge the connection 1-form ω is also radially symmetric, i.e. $\omega = \alpha(r)d\theta$. Then

$$\Omega = d\omega = \alpha'(r)dr \wedge d\theta$$

and thus $\star\Omega = \frac{1}{r}\alpha'(r)$. Therefore, if we define $\alpha(r)$ by

$$\alpha(r) = \int_1^r \frac{\rho}{u(\rho)^2} d\rho$$

(f, ω) will satisfy (14). This completes the proof.

Remarks.

1. Apparently, the argument above remains valid in the case of singular or continuous u and shows that the system (14) admits also continuous and singular solutions.

2. There is also a simple explicit solution descending from f depending on one variable only, e.g. $f = \sqrt{x^2+1}$. We find it less interesting however, since it is not flat at infinity.

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