75-13,643

VENTURA, Joseph, 1938-
DIELECTRIC PROPERTIES OF THE MAGNETIC SURFACE OF NEUTRON STARS.

The City University of New York, Ph.D., 1975
Physics, general

Xerox University Microfilms, Ann Arbor, Michigan 48106

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DIELECTRIC PROPERTIES OF THE MAGNETIC SURFACE OF NEUTRON STARS

by

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Submitted to the Graduate Faculty in Physics
in partial fulfillment of the requirements for
the Degree of Doctor of Philosophy,
The City University of New York.

1974

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

DIELECTRIC PROPERTIES OF THE MAGNETIC SURFACE OF NEUTRON STARS

by

Joseph Ventura

Adviser: Professor Vittorio Canuto

In this work we examine the effect of the pulsar magnetic field estimated at $\sim 10^{12} - 10^{13}$ G on the dielectric properties of the nonrelativistic surface plasma of the neutron star. Making use of quantum theory we obtain explicit expressions for the dielectric tensor $\epsilon_{\alpha\beta}$ which governs the propagation of electromagnetic waves. Since the magnetoionic expressions for $\epsilon_{\alpha\beta}$ have frequently been used in the pulsar literature, emphasis is given to recovering these expressions and establishing the domain of their validity. Deviations from the magnetoionic theory have been carefully explored in the long wave-length approximation. A numerical study is also given for a "one-dimensional" electron gas, where only the ground state Landau level is occupied. It was possible to establish the magnetoionic modes for a surprisingly broad range of physical parameters.

The last part of our study is devoted to the effect of collisions in the magnetic surface. We find that the geometry of a scattering event is drastically affected by the presence of the magnetic field. In order to incorporate such effects, we were led to a one-dimensional multichannel formulation of scattering theory. Explicit and general expression were obtained for the scattering cross-section and collision frequency in the magnetic field. A reduction of the effective collision frequency with increasing magnetic field strength is found.

ACKNOWLEDGEMENTS

It is a pleasure to thank Professor Vittorio Canuto for his devoted guidance. I would also like to thank Professors Joel Gersten, Robert Goldman, Donald Kelly, Frank Martino, Marvin Mittleman, Malvin Ruderman, and Narkis Tzoar for very helpful and stimulating discussions.

I am further indebted to Drs. Tzvi Gal-Chen, John Lodenquai, and Jack Teller, and to several other colleagues at NASA for their continued interest during the course of this work and for innumerable comments and ideas, from which my work has benefited.

All numerical computations were performed on the IBM 360/95 computer of the Goddard Institute for Space Studies. I am indebted to Professor Robert Jastrow for his kind hospitality at GISS.

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PART I

GENERAL INTRODUCTION

1. Anisotropic Emission from Pulsars

The puzzling properties of pulsar radiation has posed, since the discovery in 1968, a choice between two general classes of possible emission sources. The observed pulses could be produced either by the pulsations of a star, giving rise to successive increase and decrease of its brightness, or by a rotating "lighthouse" type of source. Furthermore, the emitting region could be no larger than one hundred kilometers in diameter as evidenced by the sharp detail of the observed pulses.[1] (Interference would give rise to further smoothening of the signal in a more extended source.)

As no possible stellar candidate pulsating at the characteristic period of less than a second could be sustained, the discussion soon settled on rotating cosmic sources of small dimension, namely dense collapsed stars. The existence of such stars as the end products of stellar evolution was postulated on theoretical grounds in a brief paper by Landau in 1932.[2] The argument used is that a celestial body, which has depleted its thermonuclear fuels, is bound to decrease its size, and radiate its thermal energy until Fermi degeneracy takes effect, and builds up enough pressure to stop any further collapse.

Several years later white dwarfs were identified as members of nearby binary stars. White dwarfs are made of normal atomic matter, compressed by their own gravity to characteristic densities of $10^{5-8}/\text{g cm}^3$. Their weight is sustained by the pressure of the atomic electrons, which form a degenerate, almost relativistic, electron gas. The structure and properties of these stars were investigated by Chandrasekhar in 1939.[3] Electron degeneracy can hold the weight of stars with masses up to a critical mass $M_c = 1.2\ M_\odot$. The end of more massive stars may lie in considerable mass loss during the Cefeid variable period, or in an explosive supernova event, with the remaining core becoming a neutron star, or a black hole.[4,5]

In the neutron star state it is neutron degeneracy (occurring at nuclear densities) which is responsible for holding the considerably increased weight of the star. Just as in the case of electron degeneracy, neutron degeneracy can sustain stable stars up to a critical mass, which is estimated at $1.5 - 3\ M_\odot$. Beyond this limit the star is unstable. The uncertainty in determining the critical mass is due to our incomplete knowledge of the equation of state for dense nuclear matter.

The third possible state of the black hole is a still denser state reached when the star collapses below its

Schwarzschild radius at which the star's gravitational energy equals its rest mass. This represents the state of total collapse. At this density, the behavior of the object is described by the laws of general relativity.[6]

Pulsars are now identified as rotating magnetic neutron stars, and as such, offer the first observational evidence for the existence of such objects in space. Quite remarkably, this identification is based only on negative evidence, namely, that it has been possible to exclude all other proposed candidates on theoretical grounds.[1,7] The microscopic mechanism responsible for their characteristically nonthermal emission still eludes a satisfactory explanation. Any proposed emission mechanism has, first of all, to account for the anisotropy in the radiation, needed in order to produce a collimated beam. It has been argued that the possible existence of enormous magnetic fields, of the order of 10^{12} Gauss, threading these stars could give rise to anisotropic emission.[8]

Even though the radiation mechanism is still unknown, considerable progress has been made in understanding the structure and properties of neutron stars. All the knowledge thus gained confirms, in what might seem like a series of coincidences, the original hypothesis: that the pulsar is a rotating magnetic neutron star.[7,9] In the next section we will review the general properties of neutron stars.

2. Structure of Neutron Stars[5,7,9,10]

The properties of degenerate matter at high densities have been studied for about 40 years, and are summarized in Landau & Lifschitz.[11] When normal atomic matter is placed under high pressure, its electronic structure is disrupted by pressure ionization. A plasma results, composed of atomic nuclei, which have been stripped of all their electrons, and a degenerate electron gas. At densities $\rho \sim 2 \times 10^6$ g/cm^3 the electrons become relativistic. Normal nuclei immersed in a sea of relativistic electrons are unstable to the inverse beta decay and this results in the formation of neutron-rich nuclei. At densities of 4.3×10^{11} g/cm^3 free neutrons begin to appear outside the nuclei, and at $\rho \sim 10^{12}$ g/cm^3 the matter density becomes dominated by the degenerate neutron sea. At densities of 3×10^{14} g/cm^3, the swollen neutron-rich nuclei begin to merge and lose their identity. Below that level one has neutron matter. Zero temperature matter thus consists of a degenerate neutron sea with Fermi energy 30 MeV, and a sea of degenerate protons and electrons, each of which has about 4% of the neutron abundance. At a density $\rho \sim 4 \times 10^{14}$ g/cm^3 the electron Fermi energy reaches 120 MeV. For densities $\rho \sim 10^{15}$ g/cm^3 the contribution of hyperons becomes important, cf. Fig. 1.

The whole spectrum of densities described above is encountered in the typical neutron star, so that the determination of its detailed structure in hydrostatic equilibrium becomes a matter of considerable complexity. The very general features of a degenerate neutron sphere in equilibrium under the influence of its gravity, were first studied by Openheimer and Volkov[12] in the context of the general theory of relativity, as is appropriate for dense and heavy objects such as neutron stars. Treating the neutrons as noninteracting Fermi particles at zero temperature, they obtained a critical mass $M_c = 0.72\ M_{\odot}$[*], above which the star would be unstable. Even though hydrostatically stable, the above neutron sphere is not realistic in its composition. Chemical equilibrium predicts a considerably richer structure as described above, giving rise to deviations in the equation of state from that of a $T = 0$ neutron gas. Including these effects, more realistic models of neutron stars[13] have revised the critical mass M_c upwards to values between 1.6 - 2 $M_{\odot}$.

Finally, two body forces can have a profound effect on the structure of the neutron star. Neutrons with opposite spins (in the S state) experience a small attractive force

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[*] $M_{\odot}$ is the solar mass.

in the densities considered. In the degenerate neutron gas this small attraction is expected to give rise to the formation of pairs characteristic of the superfluid state.[7,9] This is expected to happen at densities $\rho_n \sim 10^{13} - 10^{14}$ g/cm^3, and temperatures below the critical temperature T_c estimated 10^{10} °K. The energy gap Δ, characteristic of the superfluid state, decreases, however, at higher densities, and finally disappears at $\rho_n \sim 1.5 - 2 \times 10^{14}$ g/cm^3.[14] However, the superfluidity need not vanish at this point because at $\rho_n > 1.5 \times 10^{14}$ g/cm^3 the P state (in the triplet state with spin 1) becomes attractive. The density $\rho_n = 1.5 \times 10^{14}$ g/cm^3 corresponds to the neutron density in atomic nuclei, the total density in the nuclei being about double that value.

The proton liquid is superconducting, and behaves approximately like the neutron liquid, except that its density is smaller by one or two orders of magnitude. As a result of the Coulomb repulsion between protons, the energy gap is probably smaller by one order of magnitude than for the neutron liquid. This results in a critical temperature for proton superconductivity $T_c \sim 10^9$ °K. The electrons, however, do not become superconducting. This happens because of their highly relativistic nature. For relativistic electrons the ratio of the Coulomb energy to the kinetic energy is of the order of the fine structure constant α, and therefore, as pointed out by Ginzburg,[15] is too small for pairing to exist at any finite temperature.

The temperature in neutron stars is $T < T_c$, and therefore superfluid neutrons and superconducting protons as discussed above are expected in the interior of the neutron star. The point is that due to neutrino and electromagnetic radiation the star cools rapidly from its original temperature of formation (estimated at 10^{11} °K for a supernova core[16]). Furthermore, due to its exceedingly high thermal and electrical conductivity, the entire star, soon after its formation, obtains a temperature lower than 1-5 x 10^8 °K.

Another many-body feature, characteristic of this low temperature, arises near the surface of the star, where a solid crust is expected to form.[17] This region is populated by fully ionized nuclei and relativistic electrons. The melting temperature T_m is determined from the condition $\Gamma\, k\, T_m = e^2\, Z^2/r_i$, where $r_i = n_i^{-1/3}$ is the average distance between nuclei with charge eZ. The numerical factor is $\Gamma \sim$ 100-200, i.e. melting sets in when the kinetic energy of the ions is about two orders of magnitude smaller than their Coulomb interaction. The melting temperature is therefore

$$T_m \sim 10^3 \rho^{1/3} Z^{5/3} \; {}^\circ K$$

If $Z \gtrsim 10$ and $\rho \gtrsim 10^{10}$ g/cm^3 the temperature is $T_m \gtrsim 10^8$ °K. Thus besides a thin plasma (liquid or gas) outer layer, an appreciable plasma layer of the star should be solid.

In summary, then, the typical neutron star of mass $M = 0.5\ M_\odot$ has a size of approximately 8 km and the following structure, cf. Fig. 1:

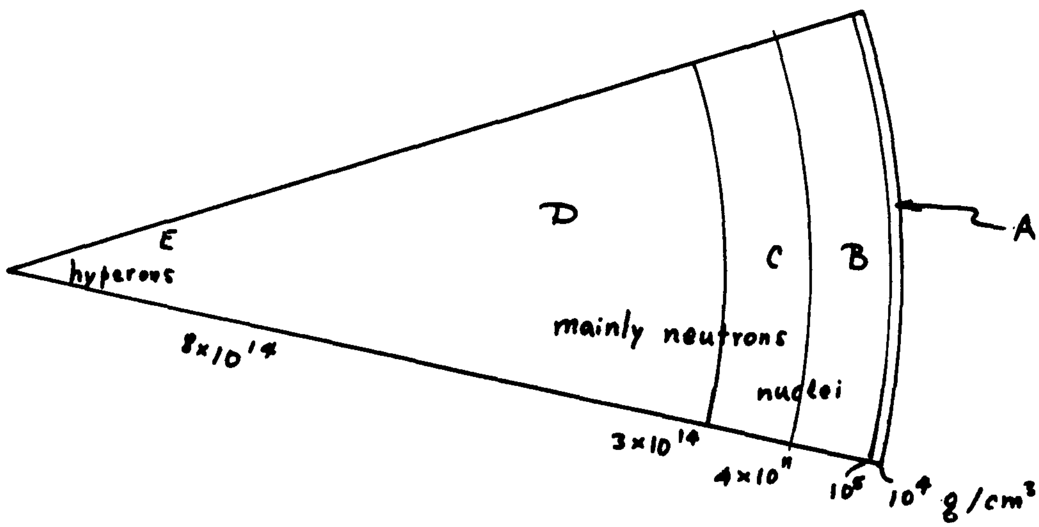

Fig. 1. Cross-section of the typical neutron star.

A. The outer crust. - The nuclei arrange themselves into a body-centered cubic lattice whose melting temperture ranges from 10^8 °K to near 10^{10} °K. The lattice shear modulus μ of the crust is enormous, about 10^{17} times stiffer than steel[18], and its conductivity is temperature dependent and of the order 10^6 times greater than that of copper.

B. The inner crust. - Is composed of neutron-rich nuclei still forming a lattice. The neutron sea between the nuclei is probably a superfluid.

C. The superfluid core. - The core consists mainly of neutrons. According to some authors[19] the existence of superfluidity in this region has observable manifestations seen in the healing of the "glitches" (the sudden speed-ups in rotation) of the Crab and Vela pulsars[18]. Their argument is based on the observation that the loose crust-core coupling due to superfluidity can allow different rates of rotation of the two bodies for microscopic periods of time (ranging from a week to a year).

D. Neutron crystalline inner core. - A recent calculation by Canuto and Chitre[19] has confirmed a previous suggestion[20] that there will be crystallization of neutron matter at high density. Their calculation based on realistic nuclear potentials gives a solid neutron phase for densities above 1.5×10^{15} g/cm^3.

E. The hyperon core. - At densities $\rho > 10^{15}$ g/cm^3 hyperons become important constituents. The interactions between all the relevant particles are not well known as yet, and this results in considerable uncertainty in our knowledge concerning the structure of the neutron star. A recent assessment of this problem is found in the current literature.[21]

3. Huge Magnetic Fields

The results given in the previous section on the structure of neutron stars have not included the effect of the magnetic field $\vec{H}$ of the order of 10^{12} Gauss which, as mentioned earlier, seems to be a necessary attribute of the pulsar surface. For the bulk of the neutron star, the anisotropy introduced on the electron plasma due to the presence of the magnetic field is insignificant, being of the order $\hbar \omega_H / E_F$, where ω_H and E_F are the Larmor frequency and Fermi energy of the electron, and $\hbar$ is Plank's constant. Numerically, $\hbar \omega_H \cong 10 \times H_{12} \times$ KeV (H_{12} is the magnetic field in units of 10^{12} Gauss.) Since the electrons in the main body of the star are relativistic ($E_F \gg \hbar \omega_H$), it is clear that the thermodynamic properties are not affected by the presence of the magnetic field. It also turns out that the properties of nuclei remain unaffected by $\vec{H}$ for similar reasons.

The presence of the magnetic field, however, is expected to affect very dramatically the properties of the plasma at the outer surface of the neutron star, where $E_F \leqslant 10 \times H_{12}$ KeV. As this outer layer of a few centimeters of thickness may prove significant in understanding the features of pulsar radiation, it has naturally attracted considerable attention in the last few years.[22,23,24]

The characteristic feature of this medium is its anisotropy. An electron imbedded in a magnetic field $\vec{H}$ can move freely in the direction of $\vec{H}$, while its transverse motion is confined in the lowest quantum orbital of radius $\sim 2.6 \times 10^{-10} \times H_{12}^{-1/2}$ cm. Thus, for $H = 10^{12}$ Gauss the electron obtains a strongly guided motion along the magnetic lines of force. What are the macroscopic features which will doubtless arise due to this anisotropic mobility of the individual electrons?

Before we enter the more technical details of this question, and discuss the recent progress achieved in understanding the properties of the surface, it will be necessary to review in a quantitative way the basic properties of an electron plasma immersed in a strong magnetic field.

Classical theory. - A nonrelativistic electron in the presence of a uniform magnetic field $\vec{H}$ moves along a helix. Its motion is uniform parallel to $\vec{H}$, i.e., the longitudinal component of its velocity $v_{\parallel}$ = constant, while the component $v_{\perp}$ is that of a uniform rotation with a characteristic frequency of gyration

$$\omega_H = \frac{eH}{mc} = 1.76 \times 10^{19} H_{12} \ \mathrm{sec}^{-1}.$$

$-e$ and m are the charge and mass of the electron, c is the velocity of light. Since the radius of gyration, $R_H = v_\perp / \omega_H$, decreases in inverse proportion to H, there is a limiting field beyond which the classical picture is no longer valid. This happens when R_H becomes comparable to the deBroglie wave length $\hbar / m v_\perp$. The classical description is thus justified, only when the following condition holds:

$$\frac{m v_\perp^2}{\hbar \omega_H} \cong \frac{E_F}{\hbar \omega_H} \gg 1 .$$

It should be mentioned here, however, that even in cases when this condition is satisfied (classical limit), the classical theory cannot explain all of the observed phenomena. The diamagnetism in metals[25] is one such phenomenon, for example; the de Haas-Van Alphen oscillations[25] observed in the magnetic susceptibility at low temperatures (for a magnetic field $H \sim 5 \times 10^5$ Gauss, $E_F / \hbar \omega_H \sim 10^4$) is another such example of phenomena not explained by the classical theory. These are properties which derive from the quantum mechanical nature of electron degeneracy, and can be fully understood only in the context of quantum theory.

Quantum theory. - In the opposite limit $m v^2 / \hbar \omega_H \lesssim 1$, which will primarily interest us here, the classical theory is altogether meaningless, and again one has to employ quantum mechanics. The Schrondinger equation for an electron in a constant and homogeneous field $\vec{H}$ can be solved exactly. A number of wave functions have been given in the literature, each class depending on the gauge chosen for $\vec{A}_0$, the vector potential which represents the external field. Details of the wave functions will be given in Appendix A and Secs. II2, and III2 as they are needed for our computation. For the present it will suffice to discuss the general features of the electron's motion.

The Hamiltonian (nonrelativistic) is

$$\mathcal{H} = (2m)^{-1}\, \pi^2 = (2m)^{-1} \left(\vec{p} + \frac{e}{c} \vec{A}_0 \right) . \tag{1}$$

Here we distinguish between two kinds of momenta appearing in the theory, $\vec{p} \equiv i\hbar \vec{\nabla}$ is the canonical momentum, while $\vec{\pi} \equiv \vec{p} + \frac{e}{c} \vec{A}_0$ is the physical momentum (so the velocity is $\vec{v} = \dot{\vec{r}} = \pi/m$). The Hamiltonian is separable into two commuting parts according to $\mathcal{H} = (2m)^{-1} (\pi_\perp^2 + \pi_\parallel^2)$ where $\pi_\perp$ and $\pi_\parallel$ stand for the transverse (relative to $\vec{H}$) and the longitudinal components of the momentum. The operator $\pi_\parallel^2 = p_z^2 = (i\hbar \partial_z)^2$ has a continuous spectrum of eigenvalues corresponding to a free motion parallel to H. $\pi_\perp^2$ on the other hand contributes a discrete harmonic oscillator type spectrum of eigenvalues:

$$\pi_{\perp}^{2} = 2m\hbar\omega_{H}(n + \tfrac{1}{2}), \text{ where } n = 0,1,2,3... \tag{2}$$

The energy spectrum is thus given by

$$E_{np_z} = \hbar\omega_H(n + \tfrac{1}{2}) + p_z^2/2m.$$

This equation does not include the effect due to the electron's spin. The spin's magnetic dipole couples to H giving an additional term in the energy equation equal to $\pm\tfrac{1}{2}\hbar\omega_H$ with the $\pm$ corresponding to a parallel or antiparallel spin alignment respectively. With the addition of this term one obtains

$$E_{np_z} = \hbar\omega_H n + p_z^2/2m , \tag{3}$$

where all levels except the lowest are doubly degenerate.

The generalization of the above concepts into a relativistic Dirac theory is simple. The corresponding equation for the energy spectrum becomes

$$E_{np_z}^2 = m^2c^4 + 2mc^2\hbar\omega_H n + c^2 p_z^2 .$$

The electrons become relativistic at densities $\rho \gtrsim 10^8\ g/cm^3$, i.e. at the neutron star crust, cf. Fig. 1. The properties of a relativistic electron plasma have been treated systematically in the literature[26], and will not concern us here.

Discrete phase-space. - The discrete levels of equation (2), known as "Landau levels" after Landau's pioneering work on the diamagnetism of metals[27], are equally spaced levels, separated by the characteristic energy $\hbar\omega_H \sim 11.58 \times H_{12}$ KeV. This energy exceeds the typical thermal energy at the surface of neutron stars kT $\sim$ 0.86 KeV for $T = 10^6$ °K, and the situation thus arises (at low densities) in which all of the electrons populate the lowest n=0 level. After a brief non-degenerate region at the surface (less than 1 cm thick for the typical equilibrium atmosphere), electron degeneracy takes over ($E_F > 0.86$ KeV), and at $E_F = \hbar\,\omega_H$ the first (doubly degenerate) excited state begins to be populated. Higher Landau levels will likewise become populated at still higher densities. The phase space here is continuous in the z-direction but discrete in the transverse direction, so that the energy eigenstates are arranged on the surface of concentric cylinders in phase space, cf. Figure 2.

For isotropic electrons ($\vec{H} = 0$), the relation between the number density and the Fermi momentum $p_F = (2\ m\ E_F)^{1/2}$ (degenerate case) is easily found by counting the number of states inside the Fermi sphere of radius p_F. The well known result is

$$N = \frac{4\pi}{3}\, p_F^3 \,/\, (2\pi\hbar)^3 \ .$$

For the anisotropic electrons, however, occupying the lowest part of the Landau spectrum one has to count the number of states on the cylinders, so that

$$N = \sum_{n=0}^{n_F} g_n \int_{-p_F^n}^{p_F^n} dp_z \quad , \tag{4}$$

where p_F^n is the Fermi momentum in the n-th Landau level, n_F is the highest level below the Fermi surface ($n_F \cong E_F/\hbar\omega_H$), and g_n stands for the degree of degeneracy of the Landau level and can be determined either from the quantum theory,[26] cf. Appendix A, or through the principle of correspondance in the classical limit $n >> 1$.[28]

In this limit the discrete levels merge into a continuum, cf. Fig. 2. We can then write the continuum expression for the number density

$$N = \frac{2\pi}{(2\pi\hbar)^3} \int_0^\infty p_\perp \, dp_\perp \int dp_z \quad ,$$

where $p_\perp \equiv mv_\perp$ is the <u>physical</u> momentum here, and the integrals are restricted by the constraint $p_\perp^2 + p_z^2 \leq 2\, mE_F$. Noting now that $p_\perp^2 = 2\, m\, \hbar\, \omega_H n$, and writing the integral as a limiting sum we find

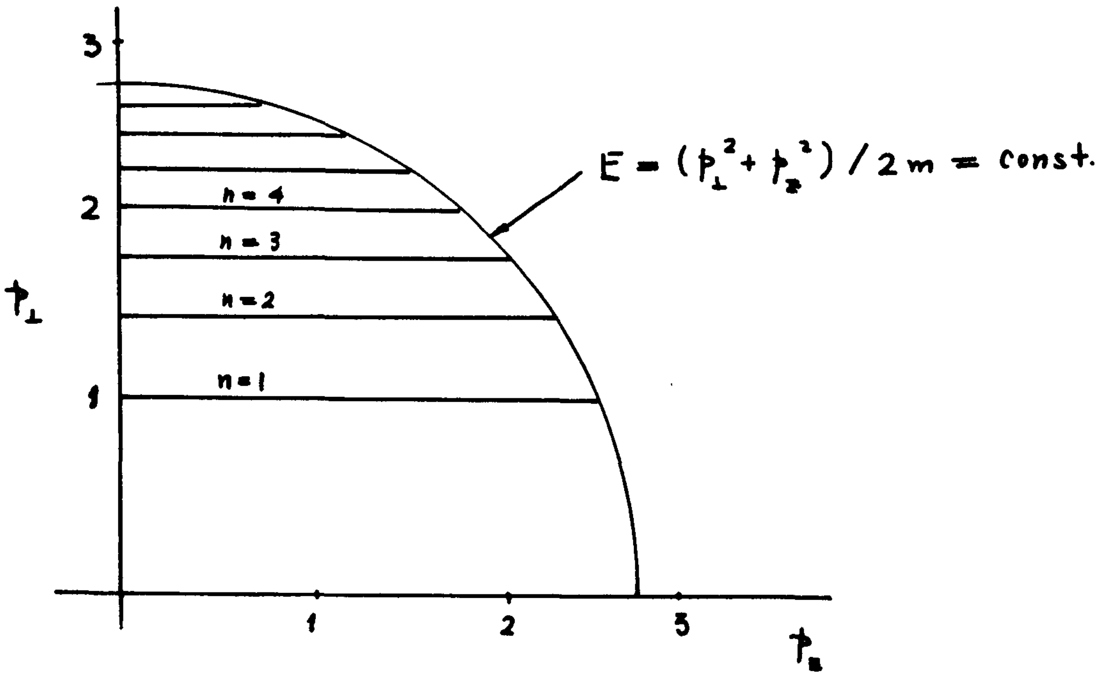

Fig. 2. The discrete nature of the phase-space is shown including the seven lowest Landau levels. The transverse momentum is quantized according to the law $p_\perp \propto \sqrt{n}$. In the limit $n >> 1$ the difference $\sqrt{n+1} - \sqrt{n} \cong \frac{1}{2} n^{-\frac{1}{2}}$, and the discrete phase-space merges into a continuum.

$$N = \frac{2m\omega_H}{(2\pi\hbar)^2} \sum_{n=0}^{n_F} a_n p_F^{\,n} \qquad (5)$$

where $a_o = 1$, $a_n = 2$, $(n \geqq 1)$, and the Fermi momentum in the n-th level is given by

$$p_F^{\,n} = \sqrt{2m(E - \hbar\omega_H n)}$$

It is seen from this equation that the value of the Fermi momentum is different at each energy level.

4. Properties of the Magnetic Surface.[9,10,22]

The importance of the probable existence of magnetic fields of the order of 10^{12}G was recognized early in the history of pulsars, and extensive studies were undertaken by a number of authors, to determine the structural changes in the medium[23,26] as well as the modifications in many physical processes characterizing the plasma.[22] In this section we shall summarize some of the knowledge thus gained.

Equation of State. - The thermodynamic properties of a magnetized electron gas at arbitrary density, temperature, and field intensity H were studied in a series of papers by Canuto and Chiu.[26] Their results indicate a qualitative modification in the equation of state not predicted classically. The pressure obtained is anisotropic even in thermodynamic equilibrium. The pressure is defined quantum mechanically as the expectation value of the stress energy tensor giving the result (in the nonrelativistic limit)

$$
\begin{aligned}
P_\perp &= \langle p_\perp^2/2m \rangle = \hbar\omega_H \langle n \rangle \\
P_\parallel &= \langle p_\parallel^2/2m \rangle
\end{aligned}
\qquad (6)
$$

The anisotropy is most obvious in the intense field limit (low quantum numbers) and tends to disappear in the high quantum number regime. Thus classical plasmas, in which high n-levels are occupied, are still expected to have $P_{\perp} = P_{\parallel}$ in thermodynamic equilibrium.

Cylindrical atoms. - Recent research has indicated that atomic physics and chemistry at the surface of the neutron star have no terrestial analogue.

In the absence of an external $\vec{H}$ field, the spherical symmetry of the Coulomb force determines many properties of atomic levels including their degeneracy. This degeneracy is lifted in the presence of a magnetic field, and the Zeeman splitting of the atomic levels increases proportionally with H. At a field of 10^7 G the level spacing is about 1 e V, and the chemical properties of matter are dramatically affected.

At 10^{12} G electrons in the n=0 Landau level combine with protons to form "one-dimensional" atomic states. The magnetic constriction of the electron's transverse motion makes these atoms more tightly bound than the terrestial hydrogen atom by a factor of about ten. Rough estimates give the ground state at $E \sim - 133$ eV for the hydrogen atom in a field of 10^{12} G. The corresponding electron distribution is confined in the transverse direction within the characteristic radius of gyration $\rho_0 \sim 2.6 \times 10^{-10}$ cm, and extends parallel to the field to a distance of $\sim 3.1 \times 10^{-9}$ cm from the nucleus.[23] The large binding energy has obvious implications on the nature and degree of ionization of the surface plasma on the neutron star.

These values of the binding energy and length strictly correspond to the ground state, which can be adequately approximated by a separable wave function in cylindrical coordinates

$$\psi_0 = e^{-\gamma\varrho^2 - \beta|z|} , \tag{7}$$

where $\gamma = \frac{1}{2}\, eH/\hbar c$. The atom's transverse dimension is thus determined exclusively by the magnetic field, a reasonable assumption since the Coulomb energy $e^2/r \ll \hbar\omega_H$. β is a parameter to be adjusted by the variational principle (the atom's length is β^{-1}). The resulting length is of the order of the Bohr radius and a rather insensitive function of the field strength H.

Atomic Spectrum. - Another important qualitative modification appears in the spectrum of the hydrogen excited states. The cylindrical atom has approximately independent longitudinal and transverse modes of excitation.

To a fair approximation the excited states can also be written in a separable form, cf. Appendix B.

$$\psi_{os} = \phi_{os}(\varrho,\theta)\; f_{sm}(z) ; \tag{8}$$

where s and m are positive integers of zero. Here again the transverse dependence is that of free electron in the magnetic field, cf. Appendix A:

$$\phi_{0s}(\rho,\theta) = e^{-is\theta} \rho^{s} e^{-\gamma\rho^2} . \qquad (8')$$

The ground state given in Eq. (7) is recovered from Eqs. (8), (8') by assigning s=0, m=0, for the transverse and longitudinal quantum numbers, and by substituting

$$f_{00}(z) = e^{-\beta|z|} .$$

It has been shown that excitations of the orbital quantum number s result in a set of tightly-bound states whose z-dependence,

$$f_{s0}(z) = e^{-\beta_s|z|} ,$$

is again determined from the variational principle. These states along with the ground state are characterized by large binding energies in the order of 100 eV for hydrogen.

Unlike these tightly-bound states the excitation of the longitudinal mode (m > 0) results in a loosely-bound hydrogen-like spectrum. An in-depth study of the whole hydrogen spectrum in an intense magnetic field based on the one-dimensional Schrodinger equation (cf. Appendix B) has been given by Canuto and Kelly.[29] In Sections III and IV, we shall consider the important problem of Coulomb scattering in the magnetic field, which can be treated by extending the above problem of the hydrogen bound states to the continuum.

Multielectron atoms were considered by Cohen, Lodenquai and Ruderman.[23] They calculate the ionization energy of the last electron E_I, which shows the surprising trend to increase with Z (the total number of electrons). They have explained that this tendency is due to the imperfect screening of the central charge, when electron correlations are taken into account. This effect can have important consequences on the surface plasma in neutron stars. At the typical temperature $T \sim 10^6$ °K only the very lightest elements (i.e. hydrogen and helium) will be ionized. As most formation models predict a surface consistency mostly of iron peak elements, this means that most of the surface matter will remain unionized with the exception of the relatively rare light elements.

It should be pointed out, however, that such conclusions are only tentative, since the properties of the magnetic surface have not been studied exhaustively as yet. It has been suggested[30] that in addition to the magnetic fields already considered, huge electric fields, induced by the fast rotation of the magnetic neutron star, may also be present at the surface. These fields may provide an alternate mechanism for ionization.

Radiation processes. - Both classical and quantum electrodynamics predict a number of processes in a plasma involving the emission of photons. Classically the acceleration of a charged particle alone results in the emission of E.M. radiation, while according to quantum theory radiative transitions between states of different energy give rise to the emission of photons. While most of these processes are severely affected by the presence of an external magnetic field, some are entirely due to the presence of such a field (synchrotron radiation).

The study of radiation processes in the magnetic field has traditionally evolved along the lines of the classical theory, which is strictly not applicable in the conditions presently associated with the surface of neutron stars. This has prompted a number of research groups to examine the modifications in the theory brought about when the full machinery of quantum electrodynamics is properly utilized. An assessment of the present status of this effort may be found in the recent review articles.[22] Here we shall merely summarize the processes involved.

(a) Synchrotron radiation. - An electron in the magnetic field can emit a photon through the process

$$e^- \rightarrow e^- + \gamma \ .$$

This process is strictly forbidden in the absence of an external field, as momentum and energy cannot be conserved simultaneously. In the presence of $\vec{H}$, however, it becomes the main limitation in the design of synchrotron accelerators due to the resulting energy losses associated with relativistic electrons in $\vec{H}$. Because of its importance in this practical application, this process has been studied exhaustively in both classical and quantum approaches.[31,32] Sokolov and Ternov[31] in their monograph have demonstrated the merging of the quantum and classical predictions in the limit of large quantum numbers, i.e. when the electrons occupy high energy Landau levels ($n \gg 1$). Recovering the classical formulas from quantum theory in this case has proven to be a particularly tedious task.

The low quantum number limit of this process has been recently examined[33] in view of the possible application to pulsar theory. It turns out from the kinematics that most of the radiation will be near the Larmor frequency and its multiples (in the X-ray region), so that this process is unimportant in explaining the radio-emission from pulsars. It may, however, be an important X-ray emission mechanism, and as such is being studied in relation to the newly discovered X-ray pulsars.[34]

(b) Electron Bremsstrahlung. - An electron in the magnetic field collides with an ion making a radiative transition from a state n to a state n' of lower energy

$$e_n^- + (Z, A) \rightarrow e_{n'}^- + (Z, A) + \gamma .$$

It has recently been suggested that a maser mechanism involving Bremsstrahlung transitions between low-lying Landau levels may be at the heart of the pulsar emission problem.[35] Stimulated emission produced by such transitions (operating near the magnetic pole of the neutron star) has the attractive feature that it can produce a continuous spectrum of radiation, unlike the atomic masers, where only one well-defined frequency is emitted. A number of fundamental features of the observed radio-pulsars were possible to explain in the context of this model such as the surprisingly high radio-intensity, the double peaked shape of pulses, and linear polarization.

It is well-known that the Bremsstrahlung process becomes increasingly important in a dense medium. The effect of the medium on the process cannot be neglected in such cases, and instead of free photons one has to consider the propagation of plasma modes (plasmons). This remark, of course, applies to other processes as well. All processes involving the emission or absorption of photons in a dense plasma are similarly affected by the dispersive properties of the medium. This question will constitute one of our main concerns later on, and will be examined in detail in Section I6, and much of parts II and III.

Photon absorption. - Processes involving the absorption of photons determine the opacity of the stellar plasma, and therefore are essential in calculations of the rate of stellar evolution. In the case of neutron stars where all nuclear burning has been deployed, these processes, together with neutrino processes, determine the cooling rate of the star from the initial post-implosion temperature of $\sim 10^{11}$ °K.

At the surface the opacity becomes strongly anisotropic because of the huge magnetic field. The processes contributing to the opacity, and the modification in the ionization and the cooling induced by the presence of the magnetic field have recently been explored.[16] For several of these processes, it is found, that the absorption cross-section is considerably reduced in the direction of the magnetic field, resulting in a substantial reduction of the star's cooling rate. Again plasma effects are important in evaluating these processes, and are introduced in terms of the medium dielectric constant.

The dispersive properties of a magnetized classical plasma are reviewed in the next two sections.

5. Propagation of Electromagnetic Waves in a Plasma.[34,35]

We here give some general considerations involving the dielectric behavior of a fully ionized plasma. An electromagnetic wave travelling through a medium of free charges is affected by the motion of the charges the wave induces. The resulting charge and current densities will either sustain in a self-consistent way, or interupt the propagation of the wave.

Wave equation. - The exact nature of the induced charges and currents will constitute the subject of the next section. Here we shall merely accept their presence, and continue to describe the macroscopic response of the medium. This is done by studying Maxwell's equations (for a fully ionized plasma):[34,35]

$$\begin{aligned} &(\mathrm{I})\quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho\,, &&(\mathrm{III})\quad \vec{\nabla}\cdot\vec{B} = 0 \\ &(\mathrm{II})\quad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} &&(\mathrm{IV})\quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \end{aligned} \tag{9}$$

Here E and B stand for the electromagnetic wave in question; ρ and j are the appropriate charge and current densities. It follows from these equations, that the $\vec{E}$ and $\vec{B}$ fields can propagate through the medium in the form of waves with a predictable speed and dispersion. As this is a very well known result we shall only summarize some of the relations involved for convenient reference.

The wave equation for the $\vec{E}$ field is obtained by combining equations II and IV. One finds:

$$-\nabla \times (\nabla \times E) = \frac{4\pi}{c^2} \frac{\partial j}{\partial t} + \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \ .$$

For the Fourier component we further find

$$\begin{aligned} \nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) &= -\frac{\omega^2}{c^2}\left(\vec{E} + i\,\frac{4\pi}{\omega}\,\vec{j}\right) \\ &\equiv -\frac{\omega^2}{c^2}\,\boldsymbol{\epsilon} \cdot \vec{E} \end{aligned} \qquad (10)$$

Here the symbol $\boldsymbol{\epsilon}$ stands for the dielectric tensor $\varepsilon_{\alpha\beta}$ (α, β = 1,2,3), and the dot product $\boldsymbol{\epsilon} \cdot \vec{E}$ represents the vector $\varepsilon_{\alpha\beta} E_{\beta}$. This effective dielectric tensor is defined macroscopically in terms of the current density, as shown in Equation (10).

For a plane wave $e^{i\vec{k}\cdot\vec{r}}$ we finally obtain using the operational substitution $\nabla \rightarrow i\vec{k}$

$$\left[\frac{c^2}{\omega^2}\left(k^2 \delta_{\alpha\beta} - k_\alpha k_\beta\right) - \varepsilon_{\alpha\beta}\right] E_\beta = 0 \qquad (11)$$

This equation has a solution only for those values of the refractive index $n = ck/\omega$, which make the determinant equal to zero. Quite generally for a given direction of propagation, one finds at least one solution or mode, characterized by its index of refraction n. This is because the equation of the determinant (dispersion equation) is at least first order in n^2 depending on the exact form of $\epsilon_{\alpha\beta}(k,\omega)$. Propagating modes correspond only to positive definite values of n^2. Examples of these notions will be given in Section 6.

Polarization densities. - The densities , and j appearing in Equation (9) refer in general to the total charge and current densities, i.e. the sum of external and induced polarization densities. One may thus introduce the electric displacement $\vec{D}$ as

$$\vec{D} \equiv \vec{E} + 4\pi \vec{P}$$

where P is the polarization field, which is related to the induced densities as follows:

$$\begin{aligned} \rho_{pol} &= -\vec{\nabla}\cdot\vec{P} \\ j_{pol} &= \frac{1}{c}\frac{\partial P}{\partial t} \end{aligned} \qquad (12)$$

In the absence of external charges and currents one has $\rho = \rho_{pol}$. We thus find from (9I):

$$\rho_{pol} = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{E} = \frac{i}{4\pi}\vec{k}\cdot\vec{E}$$

It is concluded from this equation that, unless a particular propagation mode is transversely polarized ($E\perp k$), there will be an induced charge density, such as to off-balance the longitudinal part of $\vec{E}$,[35] i.e.

$$i\,\vec{k}\cdot\vec{D} = 4\pi\rho_{ext} = 0 \quad .$$

This charge density fluctuation will be of the same frequency ω , and will propagate along with the field E at the same speed ω/k.

6. Dielectric Constant for a Cold Plasma.[35,36]

In the previous section we studied the macroscopic aspects of the interaction of an electromagnetic wave and a medium containing free charges. We shall now take up the problem of determining the dielectric response of the medium from the microscopic theory. We shall first examine a one component plasma made up of electrons only. Since the effect of the ion component is simply additive, this separation is convenient.

Isotropic case, $\vec{H} = 0$. - In the absence of an external magnetic field an electron will respond to the wave $\vec{E}(t) = \vec{E}\, e^{-i\omega t}$ according to the equation (classical)

$$m\dot{\vec{v}} = -e\vec{E}\, e^{-i\omega t} ,$$

where e, m, and v stand for the charge, mass and velocity of the electron. Fourier transforming this equation we find

$$\vec{v} = \frac{e}{im\omega}\vec{E}(t) .$$

The corresponding induced current is

$$\vec{j} = \frac{Ne^2}{im\omega}\vec{E}(t) , \tag{14}$$

where N is the number density of electrons. According to Equation (10), the dielectric constant $\varepsilon_{\alpha\beta}$ is now defined in terms of $\vec{j}$ from the relation

$$\vec{j}_{\alpha} = i \frac{\omega}{4\pi} (\delta_{\alpha\beta} - \varepsilon_{\alpha\beta}) E_{\beta} \quad . \tag{15}$$

We thus find from Equations (14) and (15)

$$\begin{aligned} \varepsilon_{\alpha\beta} &= (1 - \omega_p^2 / \omega^2) \delta_{\alpha\beta} \, , \\ \omega_p^2 &= 4\pi N e^2 / m \quad . \end{aligned} \tag{16}$$

ω_p is known as the plasma frequency, and takes the value $\omega_p = 5.6 \times 10^4 \times N \sec^{-1}$. It is seen from Equation (16) that $\varepsilon_{\alpha\beta}$ is proportional to the unit tensor, thus exhibiting the isotropic character of the medium. Each direction of linear polarization propagates independently in this case, and the midium will not be optically active.

It is therefore convenient to examine separately the propagation of the transverse and longitudinal polarizations.

(a) Transverse waves. - We find from Eq. (11)

$$\left(\frac{c^2 k^2}{\omega^2} - \varepsilon_0 \right) E_{\perp} = 0 \, ,$$

where $\varepsilon_0 = 1 - \omega_p^2/\omega^2$. The condition on the determinant gives the dispersion relation

$$n^2 = \varepsilon = 1 - \omega_p^2/\omega^2 \quad . \tag{17}$$

The case of propagation in the vacuum corresponds to $\varepsilon_0 = 1$, so that the dispersion relation becomes $n^2 = 1$ giving the expected value for the refractive index. Equation (17) implies that the medium is transparent only to waves with frequency $\omega >> \omega_p$, so that there is a low frequency cut-off for the transverse waves. This behavior is characteristic of the presence of free charges.

(b) Longitudinal waves. - Equation (12) gives in this case $\varepsilon E_{\parallel} = 0$, which is realized when $\varepsilon = 0$, or by Eq. (17), when $\omega = \omega_p$. This implies the possibility of longitudinal plasma oscillations at the plasma frequency. Such waves can not exist in the vacuum according to the classical theory. It should be emphasized that these waves are density fluctuations within the plasma unlike the transverse waves, which do not involve charge densities.

Let us now state the limitations which were implicit in the preceding discussion. Apart from the assumption that the electron's motion can be described classically, mentioned in the beginning of this section, two additional assumptions have been implied: (a) the thermal motion of the electrons has been neglected, and (b) the effect of collisions has also been neglected. The first of these assumptions is justified when the electron's kinetic displacement during one period $2\pi/\omega$ is small compared to the wavelength, or when the electron's velocity $v << \omega/k$. This condition is usually satusfied by rapidly oscillating fields. The second assumption on collisions is justified when $\omega >> \nu$, where ν is the collision frequency. Electron-ion collisions are responsible for the DC resistivity of the plasma, and generally tend to damp the motion of the electrons. The evaluation of the collision frequency in the presence of an intense magnetic field will constitute the subject of Part IV of this study.

Magnetoionic Theory. - We shall now turn our attention to the properties of anisotropic plasmas, and will examine what new effects are introduced by the presence of an external and homogeneous magnetic field under the same simplifying assumptions of a cold, collisionless plasma introduced earlier.

The electron's equation of motion is now

$$m\dot{\vec{v}} = e\vec{E}e^{-i\omega t} - \frac{e}{c}\vec{v}\times\vec{H} \quad . \tag{18}$$

This equation is again solved for $\vec{v}$, which depends linearly on $\vec{E}$. The relation, however, is now of a tensor form. The current is readily defined as $\vec{j} = - e N \vec{v}$, and the dielectric tensor $\varepsilon_{\alpha\beta}$ is again obtained from the defining equation, e.g. Eq. (15). The details of this derivation can be found in the textbooks on plasma theory, and will not be repeated. The resulting expression for $\varepsilon_{\alpha\beta}$ is

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} S & -iD & \cdot \\ iD & S & \cdot \\ \cdot & \cdot & \varepsilon \end{pmatrix} ,$$

$$S = 1 - \frac{\omega_p^2}{\omega^2 - \omega_H^2} \; ; \quad D = \frac{-\omega_p^2}{\omega^2 - \omega_H^2} \cdot \frac{\omega_H}{\omega} , \tag{19}$$

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2} .$$

Here the z-axis was taken along the direction of the magnetic field $\vec{H}$. ω_p and ω_H are the plasmas frequency, and Larmor frequency defined earlier. The tensor $\varepsilon_{\alpha\beta}$ is hermitian, and can be put in a diagonal form

$$\begin{pmatrix} \varepsilon_+ & & \\ & \varepsilon_- & \\ & & \varepsilon_0 \end{pmatrix} ,$$

with the eigenvalues defined as

$$\varepsilon_{\pm} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_H)} , \quad \varepsilon_0 = 1 - \omega_p^2 / \omega^2 . \tag{20}$$

The constants S and D. The significance of these eigenvalues will become apparent in the following. $(\varepsilon_{\pm} = S' \mp D)$.

Faraday rotation. - One of the typical properties of a magnetized plasma is that it changes the direction of linear polarization of an incident electromagnetic wave. The presence of the magnetic field makes the medium optically active in other words, and, for this reason, the magnetized plasma is often called "magnetoactive."

The simplest manifestation of this property occurs for parallel propagation, i.e. when $\vec{k} \parallel \vec{H}$. Equation (11) now takes the form

$$\begin{pmatrix} n^2 - S & iD & \cdot \\ -iD & n^2 - S & \cdot \\ \cdot & \cdot & -\varepsilon_0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 . \tag{21}$$

It is easy to see that the transverse $(\vec{E} \perp \vec{k})$, and longitudinal $(\vec{E} \parallel \vec{k})$ directions of polarization decouple in this case, making this direction of propagation particularly easy to describe analytically. Eq. (21) is then separable into

$$\begin{pmatrix} n^2 - S & iD \\ -iD & n^2 - S \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0 \,, \qquad \text{(transverse)}$$

$$\varepsilon_0 E_z = 0 \,. \qquad \text{(longitudinal)}$$

One readily obtains the requirement for longitudinal wave as $\varepsilon_0 = 0$, implying $\omega = \omega_p$. Just as in the field free case, we thus have a longitudinal mode at the plasma frequency. At any other frequency one can only have purely transverse plasmons of circular polarization. The index of refraction is found at the zeros of the determinant

$$\det \begin{Vmatrix} n^2 - S & iD \\ -iD & n^2 - S \end{Vmatrix} = (n^2 - S)^2 - D^2 = 0$$

or

$$n_\pm^2 = S \mp D = \varepsilon_\pm \,.$$

It is easy to check by direct substitution that the ± signs correspond to the polarizations $E_x \mp i E_y$.

Perpendicular propagation $\vec{k} \perp \vec{H}$. - Without loss of generality we can assume $\vec{k} = (0,k,0)$. Eq. (11) becomes in this case

$$\begin{pmatrix} n^2 - S & iD & \cdot \\ -iD & n - S & \cdot \\ \cdot & \cdot & n^2 - \varepsilon_0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \,.$$

The z-polarization is again separable giving a (transverse) mode with dispersion relation

$$n^2 = \varepsilon_0 = 1 - \omega_p^{\,2} / \omega^2 \,. \tag{23}$$

For the x and y polarizations the vanishing of the corresponding determinant gives

$$n^2 = (S^2 - D^2) / S = \varepsilon_+ \varepsilon_- / S \tag{24}$$

This mode has transverse and longitudinal components. One finds for this mode

$$\frac{E_y}{E_x} = -i \frac{D}{S} \,. \tag{25}$$

Let us remark here that the transverse mode given by Eq. (23) has exactly the same dispersion relation we found for the field-free situation, cf. Eq. (17). This is not surprising since the magnetic field does not affect the motion of electrons in the z-direction.

Propagation at an arbitrary angle. - Let us now consider the more general case of propagation at an angle with respect to $\vec{H}$, i.e. $\vec{k} = (0, \sin\theta, \cos\theta)$. Equations (11) now take the form

$$\begin{pmatrix} n^2 - S & iD & 0 \\ -iD & n^2\cos^2\theta - S & -n^2\cos\theta\sin\theta \\ 0 & -n^2\cos\theta\sin\theta & n^2\sin^2\theta - \varepsilon \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 .$$

The vanishing of the determinant defines two independent modes characterized by the dispersion relation

$$n^4(\cos^2\theta\ \varepsilon + \sin^2\theta\ S) - n^2\left[\varepsilon S(\cos^2\theta + 1) + (S^2 - D^2)\sin^2\theta\right] + \varepsilon(S^2 - D^2) = 0. \qquad (26)$$

The two modes are customarily labeled ordinary and extraordinary corresponding to the positive or negative sign in front of the square root in the solution of Eq. (26).

Special cases of the above expression, corresponding to values of the parameters of astrophysical interest, will be considered in more detail in Part III.

The magnetoionic theory outlined above has found considerable application in problems related to pulsar theories. The reason for its popularity is, besides its simplicity, that it incorporates the basic feature of anisotropy introduced by the magnetic field. Refinements of this theory are necessary in order to understand certain plasma effects, as the Landau damping and a number of other absorption mechanisms. The basic features of the propagating modes, however, are already present in the magnetoionic treatment, with only minor quantitative modifications introduced by the more refined treatments.

In Part II we will investigate the quantum mechanical features of the dielectric tensor, which have been ignored in the above treatment. It will thus become necessary to treat the magnetized electron in quantum theory, and then use the methods of quantum statistics to obtain an expression for the current density $\vec{j}$ induced by the external electromagnetic field. The thermal motion of the electron will thus also be incorporated. Making use of perturbation theory we will then arrive at a quantum mechanical expression for the tensor $\varepsilon_{\alpha\beta}$ having a rather rich and complex structure. In interpreting these complexities it will often be useful to refer to the magneto-ionic formulae, Eq. (19), as a guideline.

Another very important question to be settled will, of course, be to determine the domain of validity of Eqs. (19), and the nature of possible deviations from the classical dispersion formulae discussed in this section.

Appendix A. Wave Function for an Electron in a Constant Magnetic Field.

In this appendix we give a summary of the quantum mechanical description of an electron's motion in the presence of an external magnetic field.

The mathematics of solving Schrodinger's equation has been given more than forty years ago and is fairly straightforward.[37] Aside from these early papers, however, there have been more recent contributions usually aimed at illuminating the more formal aspects of the theory, and at interpreting the various sets of solutions. Most notable in this respect is the well known paper of Johnson and Lippman.[38]

Schrodinger Hamiltonian. - We begin our discussion with the nonrelativistic Hamiltonian already given in Eq. (1)

$$\mathcal{H} = (2m)^{-1}\pi^2$$

$$\vec{\pi} = \vec{p} + \frac{e}{c}\vec{A}_0 = \begin{pmatrix} p_x - \frac{1}{2}\frac{eH}{c}y \\ p_y + \frac{1}{2}\frac{eH}{c}x \\ p_z \end{pmatrix} \qquad \text{(A-1)}$$

The vector potential has been represented here in the so-called symmetric gauge $A_0 = -\frac{1}{2}\, r \times H$.

By inspection of the Hamiltonian one readily determines two constants of the motion defined by the operators

$\pi_\perp^2 = \pi_x^2 + \pi_y^2$, and $\pi_z = -i\hbar\partial_z$. Both these operators commute with the Hamiltonian, $\mathcal{H}$, which is thus separable into independent transverse and longitudinal parts. The longitudinal part, being proportional to π_z^2, contributes a plane-wave factor e^{ikz} to the wave function as follows:

$$\pi_z e^{ikz} = \hbar k e^{ikz}$$

$$\pi_z^2 e^{ikz} = \hbar^2 k^2 e^{ikz}$$

Even though $\pi_\perp^2$ is a constant, it is never possible to define completely the transverse velocity, as can be seen from the commutation relation

$$[\pi_x, \pi_y] = -i\hbar \frac{eH}{c} = -i\hbar m\omega_H .$$

This indeterminacy in the velocity represents one of the most significant deviations from the classical theory.

Harmonic oscillator spectrum. - The algebra defined by the transverse momentum operators π_x and π_y is recognized as that of the one-dimensional harmonic oscillator. It is thus convenient to introduce raising and lowering operators

$$\pi_\pm = \pi_x \pm i\pi_y ,$$

having the following commutation properties:

$$[\pi_- , \pi_+] = 2m\hbar\omega_H ,$$
$$[\pi_\perp^2 , \pi_\pm] = \pm (2m\hbar\omega_H)\pi_\pm \qquad \text{(A-2)}$$

The eigenvalues of $\pi_\perp^2$ are thus readily found to be

$$\pi_\perp^2 = 2m\hbar\omega_H (n + 1/2) , \qquad (n = 0, 1, 2 \ldots)$$

One further notes that the corresponding eigenfunctions can be built from the ground state by the successive application of the raising operator π_+. This procedure is exactly similar to the one used to generate the wave functions of the harmonic oscillator.

The ground state is defined by the equation

$$\pi_- \phi_0 = 0 . \qquad \text{(A-3)}$$

This equation is most easily solved with the help of the coordinate transformation $\rho_\pm = x \pm i y$. One easily finds

$$\pi_\pm \equiv -i\hbar\left(\frac{\partial}{\partial\rho_\mp} + \gamma\rho_\pm\right) ,$$

where $\gamma = \frac{1}{2} e H / h c$. One readily obtains the solution of Equation (A-3) as[38]

$$\phi_0 = f(\rho_-)\, e^{-\gamma\rho^2} , \qquad \text{(A-4)}$$

where $\rho^2 = \rho_+ \rho_- = x^2 + y^2$, and f is an arbitrary function. The presence of this function corresponds to an infinite degeneracy in the harmonic oscillator energy spectrum defined earlier.

Degeneracy of the Landau levels. - In order to resolve the degeneracy one can ask that ϕ_0 be the eigenfunction of an additional operator, which commutes with both $\pi_\perp^2$, and π_z. Such operators can be defined as follows:[38]

$$\begin{aligned} x_0 &= x - \pi_y / m\omega_H , \\ y_0 &= y + \pi_x / m\omega_H . \end{aligned} \tag{A-5}$$

These operators commute with the Hamiltonian, but do not commute with one another,

$$[x_0 , y_0] = i\, 2\gamma^{-1} . \tag{A-6}$$

Johnson and Lippman show by classical correspondence that x_o and y_o can be identified as the position operators for the center of the circular orbit (guiding center). It is thus seen that in the quantum mechanical treatment, just as in the classical case, a degeneracy is associated with the indefiniteness in the position of the transverse orbit. This happens because two helical orbits of the same size but with different guiding centers will have the same energy.

Geometrical interpretation of the wave functions. - The size of the transverse orbit can now be identified as

$$
\begin{aligned}
R^2 &= (x-x_0)^2 + (y-y_0)^2 \\
&= (m\omega_H)^{-2}(\pi_x^2 + \pi_y^2) = \pi_\perp^2 / (m\omega_H)^2
\end{aligned} \tag{A-7}
$$

and is, therefore, given exactly in the energy representation in terms of the quantum number n

$$
R_n = \sqrt{(n + 1/2)\gamma^{-1}}
$$

It should further be emphasized that according to Eq. (A-6) x_o and y_o can not be defined simultaneously. If we thus chose to define x_o in order to resolve the level degeneracy, y_o will remain completely undefined thus giving rise to quantum mechanical solutions corresponding to families of classical orbits as indicated in Figure A-1.

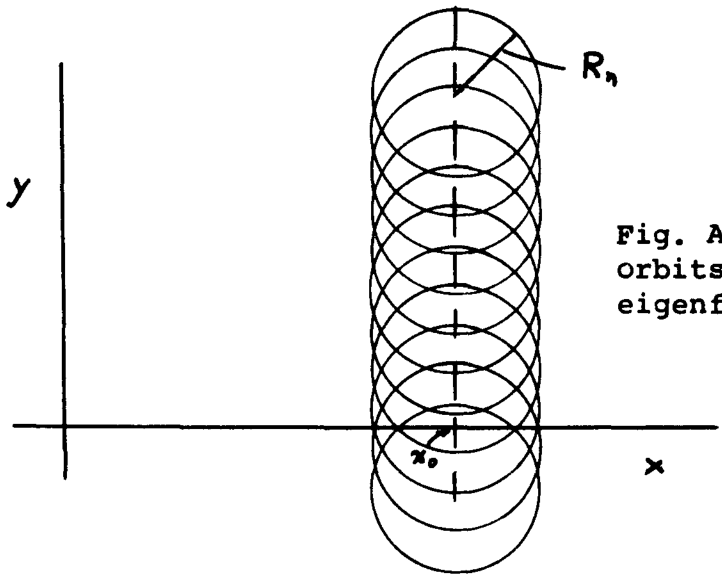

Fig. A-1. Family of classical orbits corresponding to the eigenfunctions of x_o.

Solutions of this type will be given explicitly in Section II2.

Another way to resolve the level degeneracy is obtained if we seek eigenfunctions of the operator

$$\rho_0^{\,2} = x_0^{\,2} + y_0^{\,2} \quad . \tag{A-8}$$

The eigenvalue problem contained in Eqs. (A-6) and (A-8) is formally identical to that of the one-dimensional harmonic oscillator. We may thus immediately conclude that the eigenvalues of ρ_0^2 are $(s + \frac{1}{2})^{-1}$, where s is any positive integer or zero.

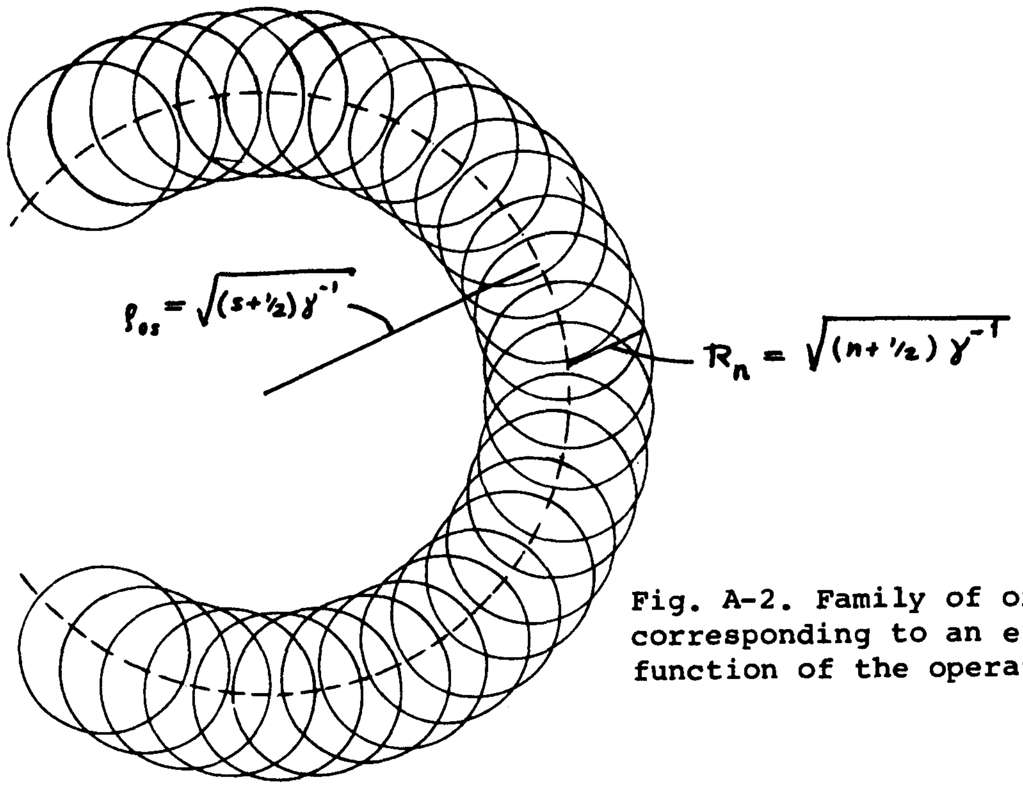

Fig. A-2. Family of orbits corresponding to an eigenfunction of the operator ρ_0^2.

Angular momentum. - It turns out that the cylindrically symmetrical solutions thus obtained are also eigenfunctions of the angular momentum operator $L_z = (r \times p)_z$ as well as of ρ_0^2. L_z can be expressed in a more suggestive form with the help of Eqs. (A-5) and (A-7):

$$L_z = x\pi_y - y\pi_x - m\omega_w(x^2+y^2)$$

After some algebra one finds

$$L_z = \tfrac{1}{2}\, m\omega_H (R^2 - \rho_0^2) \qquad \text{(A-9)}$$
$$= \hbar\,(n - s)\ .$$

The last form is of course meaningful only in the particular (ns) representation defined earlier.

The solution of the eigenvalue problem contained in Eqs. (A-6) and (A-8) defines a set of auxiliary functions f, cf. Eq. (A-4)

$$f_s = \rho_-^{\,s} = \rho^s\, e^{-is\theta}\ .$$

where ρ, θ are the cylindrical coordinates. The ground state degeneracy is thus resolved by defining an infinite set of independent solutions

$$\phi_{0s} = C_{0s}\, \rho^s\, e^{-is\theta}\, e^{-\gamma\rho^2}\ , \qquad \text{(A-10)}$$

where C_{0s} are normalization constants. From these n = 0 solutions one can generate the eigenfunctions corresponding to higher energy states by the successive application of the raising operator π_+:[38]

$$\phi_{n,n+s} = (-\pi)^n\, \phi_{0s} \qquad \text{(A-11)}$$
$$= (\gamma/\pi)^{1/2}\, e^{i(n-s)\theta}\, I_{ns}(\gamma\rho^2)\ .$$

The functions I_{ns} are an orthonormal set related to the Laguerre functions as follows:[39]

$$I_{ns}(t) = (-)^{n-s}(n!s!)^{-1/2} e^{-t/2} t^{(s-n)/2} Q_n^{s-n}(t)$$

$$Q_n^{s-n}(t) = e^t t^{n-s} \frac{\partial^n}{\partial t^n}(t^s e^{-t}) \quad .$$

The choice of phases adopted here corresponds to the wave-functions of Sokolov[39] rather than those of Johnson and Lippman.

The above set of cylindrical wavefunctions will be found useful in discussing the hydrogen atom as well as a number of scattering problems in the magnetic field calling for the use of cylindrically symmetrical solutions.

Appendix B. One-dimensional Atoms.

The Hamiltonian for the hydrogen atom in the presence of a magnetic field (nonrelativistic) as:

$$\mathcal{H} = \mathcal{H}_\perp + \frac{p_z^2}{2m} - \frac{e^2}{r}$$
$$\mathcal{H}_\perp = \frac{1}{2m}\left[\left(p_x - \frac{eH}{2c}y\right)^2 + \left(p_y + \frac{eH}{2c}x\right)^2\right] \tag{B-1}$$

It was stressed in the main text that at the typical neutron star magnetic fields, the electron's motion is dominated by the magnetic field, so that the Coulomb interaction can be considered as a perturbation.

It is customary to adopt the so-called "adiabatic approximation",[42] which consists in writing the wavefunction in separable form:

$$\psi_{os} = \phi_{os}(\rho, \theta)\ f(z). \tag{B-2}$$

The transverse dependence is that of the free electron in the magnetic field (as detailed in Appendix A) corresponding to the lowest Landau orbital n=0, and occupying a cylindrical shell of mean radius $\rho_s = \gamma^{-1/2}(s+\frac{1}{2})^{1/2}$. As a result of the Coulomb force, however, the z-dependence is no longer that of a plane-wave. One should note that there is no magnetic force acting along this direction, and therefore the

Coulomb force can no longer be ignored. Its effect can be determined by inserting the wavefunction (B-2) in the appropriate Schrodinger Equation

$$\left(H_{\perp} + \frac{p_z^2}{2m} - \frac{e^2}{r} \right) \phi_{os}(\rho,\theta)\, f(z) = E\, \phi_{os}(\rho,\theta)\, f(z) \qquad \text{(B-3)}$$

where $H_{\perp}$ is given by Eq. (B-1). We also know from the discussion in Appendix A that

$$H_{\perp}\, \phi_{os}(\rho,\theta) = 0 \quad .$$

for the n=0 level. Multiplying (B-3) by $\phi_{os}^{*}(\rho,\theta)$ and integrating over the variables ρ,θ the following equation for f(z) is obtained:

$$\begin{aligned} &\left(\frac{d^2}{dz^2} + U_s(z) \right) f_s(z) = E\, f_s(z) \\ &U_s(z) \equiv \frac{2me^2}{\hbar^2} \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\theta\, |\phi_{os}|^2 (\rho^2 + z^2)^{-1/2} \end{aligned} \qquad \text{(B-4)}$$

By solving this equation one obtains a discrete spectrum for $E < 0$ corresponding to bound one-dimensional atomic states. For $E > 0$ a continuous spectrum is found corresponding to scattering states. In what follows we will discuss only the bound states. The problem of Coulomb Scattering will be taken up in Part III.

The validity of this procedure can be established as follows. The wavefunction, corresponding to an eigenstate of the Hamiltonian (B-1) can in all generality be expanded as a sum in the Landau spectrum, i.e.

$$\psi = \sum_{n's'} \phi_{n's'}(\rho,\theta) \; f_{n's'}(z)$$

Substitution of this form in the Schrodinger Equation (B-3) results - after projecting into a given Landau level (n,s) - in the following system of equations($s' = n' - n + s$) :

$$\left(\frac{d^2}{dz^2} + U^{ss}_{nn}(z) \right) f_{ns}(z) =$$

$$= E\, f_{ns}(z) - \sum_{n' \neq n} U^{s's}_{n'n}(z) \; f_{n's'}(z) \qquad \text{(B-5)}$$

$$U^{s's}_{n'n}(z) \equiv \frac{2me^2}{\hbar^2} \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\theta \; \phi^*_{n's'}(\rho,\theta) \, (\rho^2 + z^2)^{-1/2} \, \phi_{ns}(\rho,\theta)$$

Equation (B-5) should be regarded as a system of coupled equations in the functions $f_{ns}(z)$, which reduces to Eq. (B-4) if the off diagonal terms $V_{n'n}(z)$ with $n' \neq n$ vanish. This means that in general there would be additive terms on the right hand side of Eq. (B-4) which couple the n=0 level to higher Landau levels. The presence of such terms indicates that as a result of the Coulomb interaction, the electron cannot exist in a "pure" n=0 state, and that there will be admixtures from higher levels.

The presence of such admixtures, however, is expected to be in the ratio of $V_{Coul} / \hbar\omega_H \sim 10^2/10^4 = 10^{-2}$ (for a magnetic field of 10^{12}G). In other words the admixture from higher levels is expected to be minimal, since the energy separating these levels from the n=0 level is large as compared with the Coulomb interaction. In addition it is also possible to exclude the admixture of different angular momentum states (different s), since the interaction commutes with the angular momentum operator, i.e. $[V_{Coul}, L_z] = 0$, as was mentioned in Appendix A. It is on these grounds that such admixtures have been neglected in the preceding argument.

The solution of Equation (B-4) is far from trivial. The potential $U_s(z)$ is expressible in terms of the error-function, and a rich literature has been devoted to this problem since it was formulated by Schiff and Snyder. For a recent assessment, relating to the application to the neutron star surface, References 23 and 29 may be consulted.

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PART II

QUANTUM THEORY OF THE DIELECTRIC TENSOR FOR A MAGNETIZED PLASMA

1. Introduction

In Sections I5 and I6 we obtained the magneto-ionic (MI) expressions for the dielectric constant of a cold classical plasma. These expressions have been frequently used in the recent pulsar literature to account for the plasma effect in the transport of radiation. The common assumption has generally been that MI theory represents a fair approximation at least for the transport of long wavelengths (as compared to the dimension of the Landau orbit and the de Broglie wavelength).

One is reminded however, that the quantum mechanical aspects were found to play a profound role in determining the bulk behavior of the magnetic surface. Since the quantum theory of transport processes is a subtle subject not totally understood at the moment, we felt that there was the possibility of having missed some quantum mechanical properties by just adapting an equation that is essentially classical. Besides, we wanted to investigate under what conditions the magneto-ionic theory is reproduced by the quantum mechanical treatment.

After obtaining the quantum mechanical formulae for the dielectric constant, we will show that quite independently of the specific form of the distribution function, the long wavelength approximation reproduces the results of the magneto-ionic theory. This puts an end to the question, indeed asked many times, as to whether the degeneracy of electrons often encountered in problems related to pulsars is properly taken into account by the magneto-ionic formulae. We also recover the classical polarization tensor $\Pi_{ij}(k,\omega)$ in the limit of large quantum numbers. We want to emphasize that our treatment does not rely on the Boltzmann equation or Wigner distribution function. It follows from the very quantum mechanical definition of $\varepsilon_{\alpha\beta}$.

The results thus obtained will be applied in Part III to determine the type and extent of possible deviations from the magneto-ionic theory that we may expect to find at the neutron star surface.

2. Wavefunction and Eigenvalues of an Electron in a Magnetic Field

For the sake of completeness, we shall briefly review here the quantum mechanical properties of an electron in a constant and uniform magnetic field H. [For a detailed study see(5,10)].

In the presence of such a field, electrons are trapped in quantum orbits which emerge from the Schrodinger Hamiltonian:

$$\mathcal{H} = \frac{\vec{\pi}^{\,2}}{2M}\ , \qquad \vec{\pi} \equiv \vec{p} + \frac{e}{c}\vec{A}_0$$

Here and in the following M stands for the mass of the electron. Taking the z-direction to coincide with that of $\vec{H}$, one has (Landau gauge) $\vec{A}_0 = (-\,yH,\ 0,\ 0)$. The Hamiltonian and its eigen-states then take the form:

$$\left.\begin{aligned} \mathcal{H} &= \frac{1}{2M}\left[(p_x - M\omega_H y)^2 + p_y^{\,2} + p_z^{\,2}\right] \\ \phi_{np_xp_z} &= w_n(y - a^2 p_x/\hbar)\, e^{i(p_x x + p_z z)/\hbar} \end{aligned}\right\} \qquad (1)$$

where $a \equiv (\hbar/M\omega_H)^{1/2}$; $\omega_H \equiv eH/Mc$ is the cyclotron frequency; $w_n(u)$ are the normalized harmonic oscillator wavefunctions, i.e.,

$$w_n(u) = (a\sqrt{\pi}\, 2^n n!)^{-1/2}\, e^{-u^2/2a^2}\, H_n(u/a) \qquad (2)$$

where H_n stand for the Hermite polynomials.[11]

The energy spectrum is correspondingly given by:

$$E_n(p_z) = \hbar\omega_H(n + 1/2) + \frac{p_z^2}{2M} \quad . \qquad (3)$$

The momenta π_x and π_y are not constants of the motion, as one can easily check that

$$[\pi_x, \pi_y] = -i M\hbar\omega_H , \qquad [\pi_\perp^2, \pi_\pm] = \pm 2 (M\hbar\omega_H)\pi_\pm \qquad (4)$$

Here, $\pi_\pm \equiv \pi_x \pm i\pi_y$ act as raising and lowering operators:

$$\pi_\pm |n\, p_x p_z\rangle = C_n^{(\pm)} |n \pm 1\, p_x p_z\rangle \quad , \qquad (5)$$

with $\quad C_n^{(\pm)} \equiv -\sqrt{2M\hbar\omega_H}\, (n + \frac{1}{2} \pm \frac{1}{2})^{1/2}$.

3. The Dielectric Tensor

An electromagnetic wave of frequency ω, applied on a plasma, gives rise to induced charge and current densities which can either sustain, in a self-consistant way, or damp the propagation of the wave. Thus, the plasma will be transparent to the frequency ω, if the set of Maxwell's equations (or equivalently, the wave equation) accepts a solution at this frequency.

For a particular Fourier component (k,ω), the wave equation is[12]

$$\left(k^2 \delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2} \varepsilon_{\alpha\beta} \right) E_\beta = 0 \quad . \qquad (6)$$

The dielectric properties of the medium are contained in the tensor $\varepsilon_{\alpha\beta}$ known as the dielectric tensor. The induced current density is given in terms of $\varepsilon_{\alpha\beta}$ as[12]

$$j_\alpha = i \frac{\omega}{4\pi} \left(\delta_{\alpha\beta} - \varepsilon_{\alpha\beta} \right) E_\beta \qquad (6a)$$

as may be easily be proved from Maxwell's equation

$$(\vec{\nabla} \times \vec{B})_\alpha - \frac{4\pi}{c} j_\alpha - i \frac{\omega}{c} E_\alpha = - i \frac{\omega}{c} \varepsilon_{\alpha\beta} E_\beta$$

while the induced charge density is determined from the continuity equation, $\rho = \vec{k} \cdot \vec{j} / \omega$.

Eq. (6) has the form of an eigenvalue problem having solutions only for those values of $k^2 c^2 / \omega^2 = n^2$ which make the determinant vanish, i.e.,

$$\det \left[n^2 (\delta_{\alpha\beta} - n_\alpha n_\beta) - \epsilon_{\alpha\beta} \right] = 0 \;, \tag{7}$$

where $n = k_\alpha / |k|$. To each positive definite solution of the dispersion eq. (7), there is a corresponding wave which propagates with a refractive index $n = kc/\omega$.

The dispersive properties of the medium thus depend on the precise form of $\epsilon_{\alpha\beta}$. In what follows, seeking the dielectric response of a magnetized quantum plasma to an electromagnetic perturbation (A, ϕ), we will choose a guage such that the scalar potential $\phi = 0$, and therefore $\vec{E} = i(\omega/c)\vec{A}$. The vector potential will therefore obey eq. (6) just like $\vec{E}$.

To derive a quantum mechanical expression for $\epsilon_{\alpha\beta}$, we shall now evaluate, in perturbation theory, the $(\vec{k}, \omega)$ Fourier component of the current induced by the external field

$$A(t) = A\; e^{i(\vec{k}\cdot\vec{r} - \omega t)} + c.c.$$

The current operator in second quantized form is

$$\vec{j}(\vec{r},t) = -\frac{e}{M}\psi^{\dagger}(\vec{r})\left(\vec{p} + \frac{e}{c}\vec{A}_0 + \frac{e}{c}\vec{A}(t)\right)\psi(\vec{r}) + \\ + cc \tag{8}$$

The unperturbed Hamiltonian is

$$\mathcal{H}_0 = \frac{1}{2M}\int d^3r\, \psi^{\dagger}(\vec{r})\left(\vec{p} + \frac{e}{c}\vec{A}_0\right)^2\psi(\vec{r}) = \sum_n E_n\, a_n^{\dagger} a_n$$

where a_n is the destruction operator for an electron of the corresponding state in the Landau spectrum, eq. (1). The operators a_n obey the usual anticommutation rules, and the expansion $\psi(\vec{r}) = \sum_n \phi_n(\vec{r})\; a_n$ was implied in the above equation.

Treating A(t) as a small perturbation, we write the interaction as

$$\mathcal{H}_I = \frac{e}{Mc}\int d^3r\, \psi^{\dagger}(\vec{r})\left(\vec{\pi}\cdot\vec{A}(t)\right)\psi(\vec{r}) \\ \cong \frac{e}{Mc}\sum_{nm}\left[\langle n|\vec{\pi}^{\,k}\cdot\vec{A}|m\rangle e^{-i\omega t} + \langle n|\vec{\pi}^{\,-k}\cdot\vec{A}|m\rangle e^{i\omega t}\right] a_n^{\dagger} a_m \tag{9}$$

with

$$\vec{\pi}^{k} = \tfrac{1}{2}\left(e^{i\vec{k}\cdot\vec{r}}\,\vec{\pi} + \vec{\pi}\, e^{i\vec{k}\cdot\vec{r}} \right) \tag{10}$$

Let the state vector ϕ_{No} represent a quantum state of N electrons in the absence of external field, i.e., $H_0\, \phi_{No} = E_{No}\, \phi_{No}$. In the presence of the perturbation, one has to first order in $\vec{A}$:

$$\Phi_{No} = \phi_{No}^{(+)} e^{-i\omega t} + \phi_{No}^{(-)} e^{i\omega t} + \phi_{No} \tag{11}$$

$$\begin{aligned} \phi_{No}^{(+)} &= \frac{e}{Mc} \sum_{nm} \frac{\langle n | \vec{\pi}^{k}\cdot A \,|\, m\rangle}{E_{No} - H_0 + \hbar\omega}\, a_n^{\dagger} a_m\, \phi_{No} \\ &= \frac{e}{Mc\hbar} \sum_{nm} a_n^{\dagger} a_m\, \phi_{No}\, \frac{\langle n | \vec{\pi}^{k}\cdot\vec{A} \,|\, m\rangle}{\omega + \omega_{mn}} \end{aligned} \tag{11a}$$

$$\phi_{No}^{(-)} = -\frac{e}{Mc\hbar} \sum_{nm} a_m^{\dagger} a_n\, \phi_{No}\, \frac{\langle m | \vec{\pi}^{-k}\cdot\vec{A} \,|\, n\rangle}{\omega - \omega_{mn}} \tag{11b}$$

here $\omega_{mn} = \hbar^{-1}(E_m - E_n)$, and $\langle m| \quad |n\rangle$ is the matrix element between one-electron states.

It is easy to see that the resulting expression for the current is:

$$j(k,\omega) = j^{(1)}(k,\omega) + j^{(2)}(k,\omega) + j^{(3)}(k,\omega) \tag{12}$$

$$\vec{j}^{(1)}(\vec{k},\omega) = -\frac{e^2}{Mc}\vec{A}\left(\phi_{No}\, , \int d^3r\, \psi^\dagger\psi\ \phi_{No}\right)$$

$$= -\frac{e^2 N}{Mc}\vec{A}$$

$$\vec{j}^{(2)}(\vec{k},\omega) = -\frac{e^2}{M^2c}\left(\phi_{No}\, , \int d^3r\, \psi^\dagger \pi_\alpha^{-k}\psi\ \phi_{No}^{(+)}\right)$$

$$= -\frac{e^2}{M^2c\hbar}\sum_{nmpq}\langle a_n^\dagger a_m\, a_p^\dagger a_q\rangle \frac{\langle n|\pi_\alpha^{-k}|m\rangle\langle p|\pi^k\cdot A|q\rangle}{\omega-\omega_{pq}}$$

$$\vec{j}^{(3)}(\vec{k},\omega) = -\frac{e^2}{M^2c}\left(\phi_{No}^{(-)}\, , \int d^3r\, \psi^\dagger \pi_\alpha^{-k}\psi\ \phi_{No}\right)$$

$$= \frac{e^2}{M^2c\hbar}\sum_{nmpq}\langle a_n^\dagger a_m\, a_p^\dagger a_q\rangle \frac{\langle n|\vec{\pi}^k\cdot\vec{A}|m\rangle\langle p|\pi_\alpha^{-k}|q\rangle}{\omega+\omega_{mn}}$$

Comparing this expression with eq. (6a) we obtain the quantum mechanical expression for $\varepsilon_{\alpha\beta}$ as

$$\varepsilon_{\alpha\beta} = \left(1-\frac{\omega_p^2}{\omega^2}\right)\delta_{\alpha\beta} + \frac{\omega_p^2}{\omega^2}\,\tau_{\alpha\beta}$$

$$\tau_{\alpha\beta} = \frac{-1}{NM\hbar}\sum_{nm}\left[\frac{\langle n|\pi_\alpha^{-k}|m\rangle\langle m|\pi_\beta^{k}|n\rangle}{\omega-\omega_{mn}} - \frac{\langle n|\pi_\beta^{k}|m\rangle\langle m|\pi_\alpha^{-k}|n\rangle}{\omega+\omega_{mn}}\right] f_n(1-f_m) \tag{13}$$

where we have put $f_n \equiv \langle a_n^\dagger a_n\rangle$, $\omega_p^2 \equiv 4\pi\, Ne^2/M$.

By interchanging the dummy indices in the second term, eq. (13) can be recasted in the following more instructive form

$$\tau_{\alpha\beta} = \frac{-1}{NM\hbar} \sum_{nm} \frac{\langle n| \pi_\alpha^{-k} |m\rangle\langle m| \pi_\beta^{k} |n\rangle}{\omega - \omega_{mn}} \left(f_n - f_m \right) \tag{13a}$$

$$= \frac{-1}{NM\hbar} \sum_{nm} \left[\frac{\langle n| \pi_\alpha^{-k} |m\rangle\langle m| \pi_\beta^{k} |n\rangle}{\omega - \omega_{mn}} - \frac{\langle n| \pi_\beta^{k} |m\rangle\langle m| \pi_\alpha^{-k} |n\rangle}{\omega + \omega_{mn}} \right] f_n \tag{13b}$$

Either of the expressions (13a) and (13b) will be used on convenience in the following. It is worthwhile to notice that the statistical factor $(1-f_m)$ in eq. (13), which is a manifestation of the Pauli exclusion principle, has disappeared in (13b).

The matrix elements appearing in the expression for $\epsilon_{\alpha\beta}$ may be evaluated by making use of the wavefunctions, e q. (1). It is easy to see that ($k_x = 0$):

$$\langle n' p_x' p_z' | \pi_\alpha^{\vec{k}} | n p_x p_z \rangle = \delta_{p_x}^{p_x'} \delta_{p_z + \hbar k_z}^{p_z'} \langle n' | \pi_\alpha^{k_y} | n \rangle \tag{14}$$

where $\pi_\alpha^{k_y} \equiv \exp\left[i k_y y \right] \left(\pi_\alpha + \tfrac{1}{2} \hbar k_\alpha \right)$. By making use of eq. (5) one can cast the resulting integrals in the form[13]

$$\langle m|\, e^{ik_y Y}\,|n\rangle = \frac{1}{a}\int_{-\infty}^{\infty} du\ w_m^*(u)\ w_n(u)\ e^{ik_y y}$$
$$= C_{mn}\ I_{mn}(\tfrac{1}{2}a^2k_y^2) \tag{15}$$

Here

$$u \equiv y - \frac{a^2 p_x}{\hbar}\ , \qquad C_{mn} \equiv i^{m-n}\, e^{i a^2 k \hbar^{-1} p_x}$$
$$I_{mn}(\rho) \equiv (m!\,n!)^{-1/2}\, e^{-\rho/2}\, \rho^{(m-n)/2}\, Q_n^{m-n}(\rho) \tag{16}$$

and $Q_m^{\ell}(\rho)$ are the Laguerre polynomials. The functions $I_{mn}(\rho)$ of eq. (16) form an orthonormal set and have been studied extensively in the literature[14].

One finds

$$\langle m|\, \pi_x^{k_\perp}|n\rangle = -i(\tfrac{1}{2}M\hbar\omega_H)^{1/2}\, C_{mn}\, I_{mn}^{(-)}$$
$$\langle m|\, \pi_y^{k_\perp}|n\rangle = (\tfrac{1}{2}M\hbar\omega_H)^{1/2}\, C_{mn}\Big(I_{mn}^{(+)} + \frac{a k_\perp}{\sqrt{2}}\, I_{mn}\Big) \tag{17}$$
$$\langle m|\, \pi_z^{k_\perp}|n\rangle = (p + \tfrac{1}{2}\hbar k_z)\ C_{mn}\, I_{mn}$$

where $I_{mn} \equiv I_{mn}(\tfrac{1}{2}a^2k^2)$, $I_{mn}^{(\pm)} = \sqrt{n}\, I_{m,n-1} \pm \sqrt{n+1}\, I_{m,n+1}$, and $k_\perp \equiv k_y$.

4. Statistical Averaging

It is now time to examine more closely the nature of the summations $\sum_{mn}$ and of the statistical quantities f_n appearing in eq. (13). The quantum states may be defined by the quantum numbers n, p_z, p_x of eq. (1) and the spin quantum number, $s = \pm 1$. The inclusion of the spin is straightforward in our calculation. It modifies the spectrum eq. (3) by an additive quantity, $\pm \frac{1}{2} \hbar\omega_H$, making all but the lowest Landau levels doubly degenerate. The modified spectrum is

$$E_{np} = \frac{p^2}{2M} + \hbar\omega_H n$$

where $p \equiv p_z$. The level degeneracy due to spin is

$$a_n = 1 \quad \text{for} \quad n = 0,$$
$$a_n = 2 \quad \text{for} \quad n > 0.$$

while the additional degeneracy in the quantum number p_x has been discussed in the literature.[15]

The density of electrons is expressed by

$$N = \frac{1}{V} \sum_{n p_x p s} f_{n p_x p s} = \frac{M\omega_H}{(2\pi\hbar)^2} \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} dp \, f(E_{np}) \qquad (18)$$

where V is a macroscopic normalization volume.

Turning again to eqs. (13), we note that the summation over p , p_x, s of the intermediate state $|m\rangle \equiv |m, p', p_x', s'\rangle$ becomes trivial because of the δ-functions in the matrix element eq. (14). The corresponding conservation laws are $p' = p \pm \hbar k_z$, $p_x' = p_x$, $s' = s$, the upper and lower signs refer to the first and second terms in eq. (13b). We may therefore write eq. (13a) and (13b) as:

$$\tau_{\alpha\beta} \equiv \frac{-1}{N} \sum_{mnpp_xs} \frac{(t_{\alpha\beta})_{mn}}{\omega - \omega_{mn}(p, p+\hbar k_z)} \left[f(E_{np}) - f(E_{mp'}) \right] \tag{19}$$

$$= \frac{-1}{N} \sum_{mnpp_xs} \left[\frac{(t_{\alpha\beta})_{mn}}{\omega - \omega_{mn}(p, p+\hbar k_z)} - \frac{(t^{(1)}_{\alpha\beta})_{mn}}{\omega + \omega_{mn}(p, p-\hbar k_z)} \right] f(E_{np}) \tag{20}$$

with

$$(t_{\alpha\beta})_{mn} = \frac{1}{M\hbar} \langle n| \pi_\alpha^{-k_\perp} |m\rangle\langle m| \pi_\beta^{k_\perp} |n\rangle$$

$$(t^{(1)}_{\alpha\beta})_{mn} = \frac{1}{M\hbar} \langle n| \pi_\beta^{k_\perp} |m\rangle\langle m| \pi_\alpha^{-k_\perp} |n\rangle$$

$$\omega_{mn}(p, p+\hbar k_z) = (m-n)\omega_H \pm \frac{p}{M} k_z + \frac{\hbar k_z^2}{2M}$$

Substituting eqs. (17) for the matrix elements, we find for the tensor $t_{\alpha\beta}$:

$$t_{xx} = \frac{\omega_H}{2} \left| I_{mn}^{(-)} \right|^2 , \qquad t_{yy} = \frac{\omega_H}{2} \left(I_{mn}^{(+)} + \frac{a k_\perp}{\sqrt{2}} I_{mn} \right)^2$$

$$t_{xy} = i \, \frac{\omega_H}{2} \left(I_{mn}^{(+)} + \frac{a k_\perp}{\sqrt{2}} I_{mn} \right) I_{mn}^{(-)} = - t_{yx}$$

$$t_{zz} = \frac{1}{M\hbar} \left(p + \tfrac{1}{2}\hbar k_z \right)^2 I_{mn}^2 \tag{21}$$

$$t_{zx} = - t_{xz} = - i \left(\frac{\omega_H}{2M\hbar} \right)^{1/2} \left(p + \tfrac{1}{2}\hbar k_z \right) I_{mn} \, I_{mn}^{(-)}$$

$$t_{zy} = t_{yz} = \left(\frac{\omega_H}{2M\hbar} \right)^{1/2} \left(p + \tfrac{1}{2}\hbar k_z \right) I_{mn} \left(I_{mn}^{(+)} + \frac{a k_\perp}{\sqrt{2}} I_{mn} \right)$$

The argument in the functions I_{mn} is $\tfrac{1}{2} a^2 k_\perp^2$ as before. Similar expressions may be easily written for the tensor $t_{\alpha\beta}^{(1)}$.

The integral over p in eqs. (19) and (20) is performed under the analyticity convention[16]

$$\lim_{\eta \to 0} \frac{1}{\omega \pm \omega_{mn} + i\eta} = P \frac{1}{\omega \pm \omega_{mn}} - i\pi\delta(\omega \pm \omega_{mn}) \tag{22}$$

Thus the principal value of the integral in question will give rise to the hermitian (refractive) part of the tensor $\tau_{\alpha\beta}$, while the integral over the δ-function gives the antihermitian (absorptive) part.

The case of propagation along the magnetic field ($k \parallel H$) is of particular interest because it brings out simple properties of the medium. In the following paragraph we shall give a detailed study of absorption and refraction in this direction.

5. Longitudinal Propagation k ∥ H

In this case ($k_\perp = 0$), one has

$$I_{mn}(0) = \delta_{mn} \quad , \quad I^{(\pm)}_{mn}(0) = \sqrt{n}\,\delta^m_{n-1} \pm \sqrt{n+1}\,\delta^m_{n+1} \tag{23}$$

Eqs. (21) then give $t_{xx} = t_{yy}$, $t_{xy} = - t_{yx}$ so that we can express all four components by introducing the linearly independent quantities

$$t_\pm = t_{xx} \mp i\, t_{xy} = \begin{cases} n\omega_H\, \delta^m_{n-1} \\ (n+1)\,\omega_H\, \delta^m_{n+1} \end{cases}$$

$$t_{zz} = \frac{1}{M\hbar}\left(p + \tfrac{1}{2}\hbar k_z\right)^2 \delta^m_n \tag{23a}$$

$$t_{zx} = t_{xz} = 0 \quad , \quad t_{zy} = t_{yz} = 0$$

Similarly, we find

$$t^{(1)}_\pm = t^{(1)}_{xx} \mp i\, t^{(1)}_{xy} = \begin{cases} (n+1)\,\omega_H\, \delta^m_{n+1} \\ n\,\omega_H\, \delta^m_{n-1} \end{cases}$$

$$t^{(1)}_{zz} = t_{zz}$$

Accordingly, we find from eq. (20)

$$\tau_{\pm} = \mp \frac{\omega_H}{P_0} \sum_n a_n \int_{-\infty}^{\infty} dp \left[\frac{n}{\omega_{\pm} - \frac{pk}{M} \mp \frac{\hbar k^2}{2M}} - \frac{n+1}{\omega_{\pm} - \frac{pk}{M} \pm \frac{\hbar k^2}{2M}} \right] f(E_{np}) \tag{24a}$$

$$\tau_{zz} = - \frac{1}{M\hbar P_0} \sum_{n=0} a_n \int_{-\infty}^{\infty} dp \left[\frac{(p+\frac{1}{2}\hbar k)^2}{\omega - \frac{pk}{M} - \frac{\hbar k^2}{2M}} - \frac{(p+\frac{1}{2}\hbar k)^2}{\omega - \frac{pk}{M} + \frac{\hbar k^2}{2M}} \right] f(E_{np}) \tag{24b}$$

where $\omega_{\pm} = \omega \pm \omega_H$. We have also introduced (cf. eq. (18)) the quantity

$$P_0 = \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} dp \, f(E_{np}) = N \frac{(2\pi\hbar)^2}{M\omega_H} .$$

The principal value of the above integrals gives the refractive part of $\tau_{\alpha\beta}$ in accordance with eq. (22). For the absorptive part we note that the integral over the δ-function in (19) gives quite generally

$$\tau_{\alpha\beta}^{A} = \frac{i\pi}{P_0} \frac{M}{k_z} \sum_{mn} a_n (t_{\alpha\beta})_{mn} \left[f(E_{np_l}) - f(E_{np_l} + \hbar\omega) \right] \tag{25}$$

with

$$p_\ell = \frac{M}{k_z}\left[\omega - \ell\omega_H\right] - \tfrac{1}{2}\hbar k_z \quad , \qquad \ell = m-n$$

Substituting (23a) we find, accordingly

$$\tau_{\pm}^{A} = i\pi \frac{M\omega_H}{k P_0} \sum_n a_n \left\{ \begin{matrix} n \\ n+1 \end{matrix} \right\} \left[f(E_{np_\pm}) - f(E_{np_\pm} + \hbar\omega) \right]$$

(25a)

$$\tau_{zz}^{A} = i\pi \frac{M^2\omega^2}{\hbar k^3 P_0} \sum_n a_n \left[f(E_{np_0}) - f(E_{np_0} + \hbar\omega) \right]$$

where $p_\pm = k^{-1} M \cdot \omega_\pm \pm \frac{1}{2}\hbar k$, $\quad p_0 = k^{-1} M\omega - \frac{1}{2}\hbar k$

6. Special Cases

a) Classical Limit. In the limit of large quantum numbers, $\langle n \rangle \gg 1$, the discrete spectrum approaches a continuum. The quantum mechanical formulae then lead to the corresponding classical expressions by the following procedure:[15]

$$\sum_{n p p_x s} = \frac{M\omega_H}{(2\pi\hbar)^2} \sum_n a_n \int_{-\infty}^{\infty} dp \rightarrow \frac{2M^2\omega_H}{(2\pi\hbar)^2} \int_0^{\infty} dn \int_{-\infty}^{\infty} dv_z$$

Now $2M^{-1}\hbar\,\omega_H\, n \cong v^2 = v_x^2 + v_y^2$ so that $\int dv^2 = \frac{1}{\pi}\iint dv_x dv_y$, and therefore:

$$\frac{M\omega_H}{(2\pi\hbar)^2} \sum_n a_n \int_{-\infty}^{\infty} dp \rightarrow \frac{2M^3}{(2\pi\hbar)^3} \int d^3v \tag{26}$$

Eq. (18) correspondingly becomes $N = 2M^3(2\pi\hbar)^{-3} \int d^3 v\, f(v)$. In rewriting eq. (19) we have

$$E_{mp'} - E_{np} = \Delta E = \hbar\left[l\omega_H + k_z v_z \left(1 + \frac{\hbar k_z}{2Mv_z}\right)\right]$$

$$\cong \hbar\left[l\omega_H + k_z v_z \right].$$

where $\ell \equiv m-n$, $\vec{v} \equiv \vec{\pi}/M$, $p' = p + \hbar k_z$.

Assuming that $\ell \ll n$, $\hbar k_z \ll p$, we may write

$$f(E_{mp'}) - f(E_{np}) \cong \Delta f =$$

$$= \frac{\hbar}{M} \left(\frac{\ell\omega_H}{v_\perp} \frac{\partial f}{\partial v_\perp} + k_z \frac{\partial f}{\partial v_z} \right) ;$$

consequently, eq. (19) becomes:

$$\tau_{\alpha\beta} = \frac{2M^3}{N(2\pi\hbar)^3} \sum_{\ell=-\infty}^{\infty} \int d^3v \left(\frac{\ell\omega_H}{v_\perp} \frac{\partial f}{\partial v_\perp} + k_z \frac{\partial f}{\partial v_z} \right) \frac{\frac{\hbar}{M} t_{\alpha\beta}}{\omega - \ell\omega_H - k_z v_z} \quad (27)$$

We can now make use of the asymptotic expression ($n \gg 1$)[17]:

$$I_{mn}\left(\frac{a^2 k_\perp^2}{2}\right) \cong J_\ell(a k_\perp \sqrt{2n}) = J_\ell\left(\frac{k_\perp v_\perp}{\omega_H}\right) ,$$

to cast eqs. (21) in the following form:

$$\frac{\hbar}{M} t_{\alpha\beta} = \begin{pmatrix} v_\perp J_\ell'^2 & -v_\perp^2\left(\frac{\ell J_\ell}{\rho}\right) J_\ell' & -i v_\perp v_z J_\ell J_\ell' \\ i v_\perp^2 \left(\frac{\ell J_\ell}{\rho}\right) J_\ell' & v_\perp^2 \left(\frac{\partial J_\ell}{\rho}\right)^2 & v_\perp v_z \left(\frac{\ell J_\ell}{\rho}\right) J_\ell \\ i v_\perp v_z J_\ell J_\ell' & v_\perp v_z \left(\frac{\partial J_\ell}{\rho}\right) J_\ell & v_z^2 J_\ell^2 \end{pmatrix} \tag{27a}$$

where $\rho \equiv k_\perp v_\perp / \omega_H$, $J_\ell \equiv J_\ell(\rho)$, $J_\ell' \equiv dJ_\ell / d\rho$. Eqs. (27) and (27a) are identical with the well-known expressions obtained in the classical kinetic theory.[18] In deriving eq. (27a) we have also made use of the identities $J_{\ell-1} - J_{\ell+1} = 2J_\ell'$ and $J_{\ell-1} + J_{\ell+1} = 2(\ell J_\ell / \rho)$ to write $(n >> 1)$:

$$\begin{aligned} \left(\frac{\hbar\omega_H}{2M}\right)^{1/2} I_{mn}^{(-)} &\cong -v_\perp J_\ell' \\ \left(\frac{\hbar\omega_H}{2M}\right)^{1/2} \left(I_{mn}^{(+)} + \frac{a k_\perp}{\sqrt{2}} I_{mn}\right) &\cong v_\perp \left(\frac{\ell J_\ell}{\rho}\right)\left[1 + \frac{a^2 k_\perp^2}{2\ell}\right] \end{aligned} \tag{28}$$

The second term in the bracket of eq. (28) has been omitted. For strong magnetic fields, this is an excellent approximation except for very short wavelengths. Numerically $a^2/2 = \hbar/2M\omega_H \sim 3.3 \times 10^{-8}\ H^{-1}\ cm^2$, where H is the magnetic field in Gauss.

For long wavelengths when $\rho = k_\perp v_\perp / \omega_H << 1$ (numerically, $v_\perp / \omega_H \cong 1.7 \times 10^3\ (v_\perp / c)\ H^{-1}$ cm), one has $J_\ell(\rho) \cong J_\ell(0) = \delta_{\ell,0}$. In this case, the tensor $t_{\alpha\beta}$ becomes independent of the direction of propagation.

For the absorptive part our equations give in this limit

$$\tau^A_{\alpha\beta} = -i\pi \frac{2M^3}{N(2\pi\hbar)^3} \sum_{\ell=-\infty}^{\infty} \int d^3v \left(\frac{\ell\omega_H}{v_\perp} \frac{\partial f}{\partial v_\perp} + k_z \frac{\partial f}{\partial v_z} \right) \times$$

$$\times \frac{\hbar}{M} t_{\alpha\beta}\, \delta(\omega - \ell\omega_H - k_z v_z)$$

which coincides with the classical prediction for the Landau damping.

b) Degenerate Limit. This limit is of particular theoretical interest since the description of a many-body system of interacting electrons in terms of a single-particle Fermi spectrum (quasi-particles) is known to be very good.[19] (Throughout the derivation of eq. (13) we have implicitly assumed that only single-particle intermediate states are important).

In this limit, the distribution function is $f(E) = \vartheta(E_F - E)$. A Fermi momentum is defined for each Landau level as

$$p_F^{(n)} = \sqrt{2ME_F}\left(1 - \frac{n\hbar\omega_H}{E_F}\right)^{1/2} \tag{29}$$

and eq. (18) becomes

$$N = \frac{2M\omega_H}{(2\pi\hbar)^2}\sum_n a_n\, p_F^{(n)} \equiv \frac{M\omega_H}{(2\pi\hbar)^2}\,P_0 \tag{30}$$

The integrals in eq. (24) can be evaluated exactly, and one finds

$$\begin{aligned} \tau_\pm &= \sum_n a_n\, \tau_\pm^{(n)} \\ \tau_\pm^{(n)} &= \mp\frac{M\omega_H}{kP_0}\left[\log\left|\frac{M\omega_\pm - k(p_F^{(n)} \mp \frac{1}{2}\hbar k)}{M\omega_\pm + k(p_F^{(n)} \pm \frac{1}{2}\hbar k)}\right| + n\log\left|\frac{M^2\omega_\pm^2 - k^2(p_F^{(n)} \mp \frac{1}{2}\hbar k)^2}{M^2\omega_\pm^2 - k^2(p_F^{(n)} \pm \frac{1}{2}\hbar k)^2}\right|\right] \\ \tau_{zz} &= 1 + \frac{M^2\omega^2}{\hbar k^3 P_0}\sum_n a_n \log\left|\frac{\omega^2M^2 - k^2(p_F^{(n)} + \frac{1}{2}\hbar k)^2}{\omega^2M^2 - k^2(p_F^{(n)} - \frac{1}{2}\hbar k)^2}\right| \end{aligned} \tag{31}$$

Here P_0 is defined through eq. (30).

In the limit $\hbar k/2p_F^{(n)} << 1$, it is straightforward to see that[20]

$$\tau_\pm^{(n)} \cong \frac{M}{k} \log \left| \frac{M\omega_\pm - k p_F^{(n)}}{M\omega_\pm + k p_F^{(n)}} \right| \pm (2n+1) \frac{\hbar M k^2 p_F^{(n)}}{M^2\omega_\pm - k^2 p_F^{(n)\,2}} \tag{31a}$$

A general and more systematic study of the long wavelength limit is reserved for the next paragraph.

For the absorptive part we find from (25a)

$$\tau_\pm^A = i\pi \frac{M\omega_H}{P_0 k} \sum_{n=n_F-n_\omega} a_n \left\{ \begin{matrix} n \\ n+1 \end{matrix} \right\} \tag{32}$$

where $n_F \equiv E_F/\hbar\omega_H$, and $n_\omega \cong \omega/\omega_H$ gives the number of occupied levels which can be excited above the Fermi surface by the radiation $\hbar\omega$.

7. Long Wavelength Limit and Magneto-Ionic Theory

We first remark that for $a^2k^2/2 \ll n^{-1}$, $I_{mn}(a^2k^2/2) \cong I_{mn}(0) = \delta_{mn}$ and therefore eqs. (21) approach the values of eqs. (23) in this limit. The tensor $\tau_{\alpha\beta}$ then approaches the values given by eqs. (24) and (25a) if we substitute k_z for k.

We now look for the limit of these equations when $k_z \to 0$. The bracket in eq. (24a) may be written as

$$\frac{\pm(2n+1)\frac{\hbar k^2}{2M} - \left(\omega_\pm - \frac{pk}{M}\right)}{\left(\omega_\pm - \frac{pk}{M}\right)^2 - \frac{\hbar^2 k^4}{4M}}$$

Dropping the k^4 term in the denominator we obtain

$$\tau_\pm = \mp \frac{\omega_\mu}{P_0} \sum_n a_n \int_{-\infty}^{\infty} dp\, f(E_{np}) \left(p + \tfrac{1}{2}\hbar k\right)^3 \tag{33}$$

and similarly

$$\tau_{zz} = -\frac{2k^2}{\omega^2 M^2 P_0} \sum_n a_n \int_{-\infty}^{\infty} dp\, f(E_{np}) \left(p + \tfrac{1}{2}\hbar k\right)^3 \tag{33a}$$

These expressions approach their k = 0 values providing that

(a) $\langle |p| \rangle \frac{k}{M} \ll \omega, \omega_{\pm}$

(b) $\frac{\hbar k^2}{2M} \langle 2n+1 \rangle = \frac{\langle \pi_\perp^2 \rangle k^2}{2M^2 \omega_H} \ll \omega_{\pm}$ (34)

(c) $k \ll p / \hbar$

Correspondingly, one finds in this limit

$$\tau_{\pm} \cong \pm \frac{\omega_H}{\omega_{\pm}} \quad , \qquad \tau_{zz} \cong - \frac{3k^2}{\omega^2} \frac{\langle p^2 \rangle}{M^2} \ll 1 , \tag{35}$$

or, by eq. (13)

$$\begin{aligned} \varepsilon_{\pm} &\cong 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_H)} , \\ \varepsilon_{zz} &\cong 1 - \frac{\omega_p^2}{\omega^2} . \end{aligned} \tag{35a}$$

These results are identical with the predictions of the classical magneto-ionic theory.[8] It is noteworthy that eqs. (35a) were derived independently of the form of the distribution function $f(E_{np})$, and therefore are expected to hold equally well for equilibrium or nonequilibrium situations.

Eqs. (34) may be used to check whether the classical results, eq. (35), are applicable in cases of physical interest. Should eqs. (35) give indices of refraction kc/ω inconsistent with eqs. (34), one would have to use a refined theory based on eqs. (24) and (25).

8. The One Dimensional Gas

(a) General Considerations. At the superstrong magnetic fields which are probably associated with neutron stars, an interesting situation arises, when the characteristic energy of the electrons is lower than the excitation energy of the Landau levels ($p^2/2M \ll \hbar\omega_H$). Only the lowest $n = 0$ level is then populated, and the mobility of the electrons is therefore entirely determined by the value of p_z, thus giving rise to a one-dimensional electron gas. Low density, as well as an intense magnetic field, is necessary for this situation to be realized, since at densities $N \gtrsim 10^{28} \times H_{12}^{3/2}\ \mathrm{cm}^{-3}$, where $H_{12} = H/10^{12}$ Gauss, the Fermi energy of the electrons becomes as high as $\hbar\omega_H$. Electron densities and fields satisfying this condition may arise in the plasma, which forms the atmosphere of a neutron star.[1]

Retaining only the $n = 0$ term in eqs. (24), we find after a partial integration

$$\tau_{\pm} = \pm \frac{\omega_H}{k P_0} \int_{-\infty}^{\infty} dp \, \frac{\partial f}{\partial p} \log \left| p \mp \tfrac{1}{2}\hbar k - \omega_{\pm} M/k \right|$$

$$\tau_{zz} = 1 - \frac{\omega^2 M^2}{\hbar k P_0} \int_{-\infty}^{\infty} dp \, \frac{\partial f}{\partial p} \log \left| (p + \tfrac{1}{2}\hbar k)^2 - \omega^2 M^2/k^2 \right| \qquad (37)$$

for the absorptive part of $\tau_{\alpha\beta}$ eq. (25a) give

$$\tau_{-}^{A} = i\pi \frac{M\omega_{\mu}}{kP_0}\left[f(E_{p_-}) - f(E_{p_-} + \hbar\omega)\right] \quad ;$$

$$\tau_{zz}^{A} = i\pi \frac{M^2\omega^2}{\hbar k^3 P_0}\left[f(E_{p_0}) - f(E_{p_0} + \hbar\omega)\right] \quad ,$$

while the component τ_{+}^{A} vanishes in this case. In the case of a degenerate electron plasma these expressions reduce to the n = 0 term of eqs. (31). We will consider the degenerate case in greater detail in Part III. We will establish, for this case, the domain of validity of the magneto-ionic theory and will also discuss the properties of $\varepsilon_{\alpha\beta}$ outside this domain.

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PART III

SEARCH FOR NEW MODES OF PROPAGATION

1. Introduction.

In Part II we made use of the machinery of quantum statistics to obtain general expressions for the dielectric tensor. It was then possible, in Section II7, to obtain the magnetoionic expressions as the long wavelength limit of $\varepsilon_{\alpha\beta}$. By studying a number of other limits it was further demonstrated that $\varepsilon_{\alpha\beta}$ has a much richer structure than is contained in the magnetoionic formulae.

In Part III we will turn our attention to the problem of a degenerate one-dimensional electron gas, both because of its analytic simplicity and because of its relevance to the pulsar problem. After defining, in parameter space, the borders within which the M.I. theory is valid, cf. Fig. 1, we shall embark in a systematic search for new modes of propagation not predicted classically.

A simply modified version of the M.I. theory involving corrections in the longitudinal component ε_{zz} of the dielectric tensor, is thus found to apply within an extended long-wavelength regime (Regions I and II of Fig. 1.). Details of the Landau orbits are of no importance in this regime, but are expected to become important for the propagation of short wavelengths, comparable to the dimension of the Larmor radius, i.e. $\lambda \lesssim (\hbar/m\omega_\mu)^{1/2}$.

Thus, while the description of the propagation of short wavelengths becomes relatively complicated, long wavelengths can be studied within a very simple framework. We will be primarily interested to determine deviations from the magnetoionic theory within this framework.

The modification of ϵ_{zz} from the respective M.I. expression is found to alter considerably the dispersion relations, giving rise to new plasma modes. These modes, however, are found to be strongly absorbed due to Landau damping, so that we can say in conclusion that the M.I. modes are the only propagating modes in the long wavelength region, $\lambda >> (\hbar/m\omega_\mu)^{1/2}$.

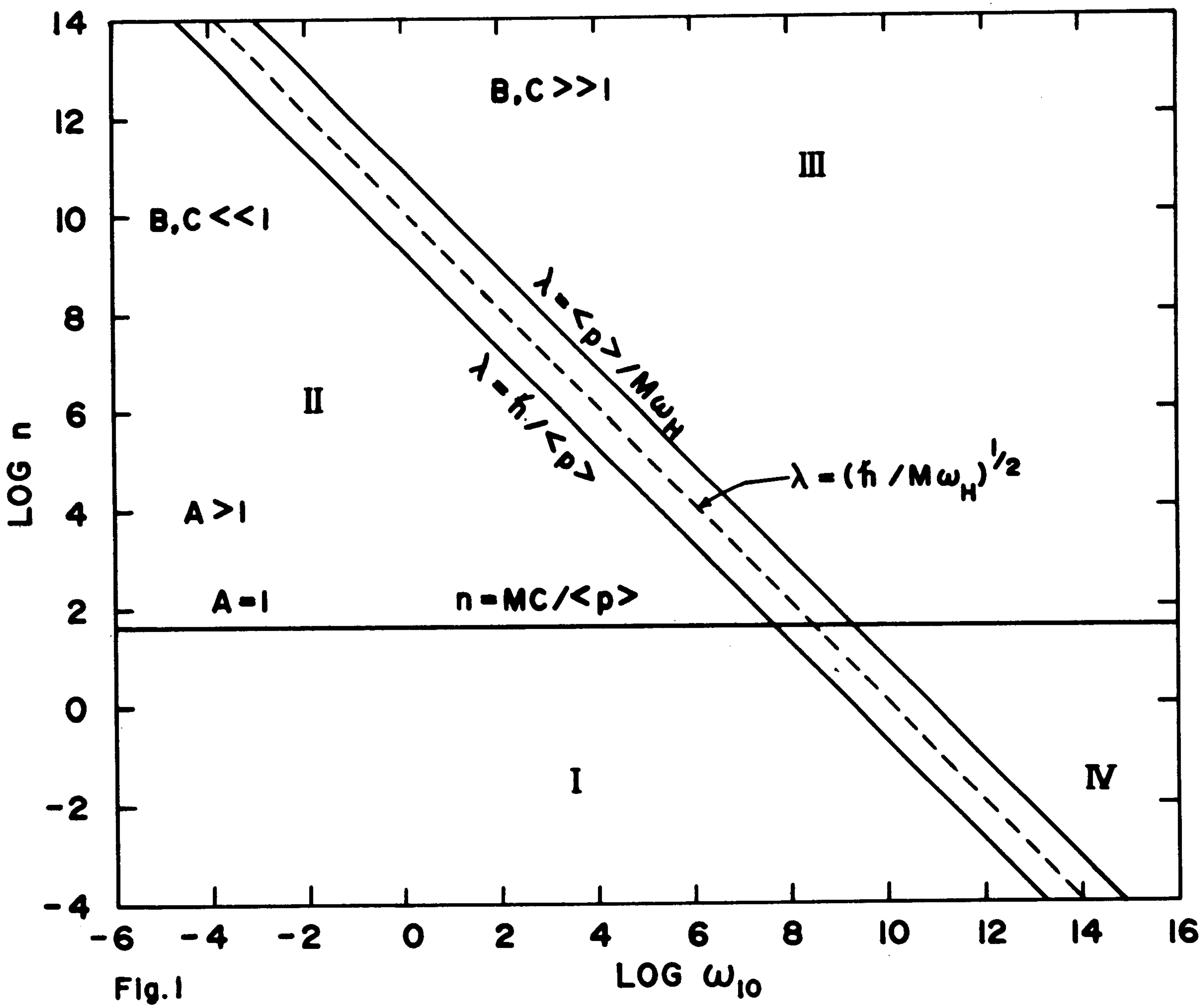

B,C >>1
III
B,C <<1
$\lambda = \langle p \rangle / M\omega_H$
$\lambda = \hbar / \langle p \rangle$
II
$\lambda = (\hbar / M\omega_H)^{1/2}$
A >1
A = 1
n = MC/<p>
I
IV
LOG n
LOG ω_{10}
14
12
10
8
6
4
2
0
-2
-4
-6
-4
-2
0
2
4
6
8
10
12
14
16

Fig. 1

2. Domains of Approximation

We now wish to establish numerically the regime of validity of the magnetoionic formulae, eq. (II35), in applications pertinent to the atmosphere of neutron stars. We shall also seek new collective modes of propagation in the plasma within the framework of the one-dimensional gas model of the previous paragraph.

We thus visualize[(1)] a totally ionized atmosphere at a temperature of $\simeq 10^6$ °K giving rise to a degenerate electron gas of density $N \simeq 10^{27}\text{cm}^{-3}$ embedded in a magnetic field $H = 10^{12}$ Gauss. Under these conditions the electrons (non-relativistic) are confined in the lowest Landau level n = o. The Fermi momentum for this level, cf. eq. (II30), is $p_F = 2\pi^2\hbar^2 N/M\omega_H$ giving a Fermi energy $E_F = p_F{}^2/2M \cong .64 \times N_{27}^2$ keV (where $N_{27} \equiv N/10^{27}\text{cm}^{-3}$), while the energy of the first Landau excitation, n = 1, is $\hbar\omega_H \cong 11.58$ keV.

It will be convenient for our discussion to introduce the following dimensionless parameters $A \equiv \langle |p| \rangle \cdot k/M\omega$, $B = \langle |p| \rangle \cdot k/M\omega_H$, $B_\pm = \langle |p| \rangle \cdot k/M\omega_\pm$, $C = k/\langle |p| \rangle$. As before, p is the momentum of the "one dimensional" electron in the z direction.

According to eq. (II34) these parameters must be small compared to unity for the magnetoionic (M.I.) theory to be applicable. This corresponds to the region of small refractive index and long wavelength, i.e. region I in the n - ω diagram of Fig. 1. Turning back to our one-dimensional gas, we may now ask how the dielectric properties of the plasma are modified outside this region. Eqs. (II36), which represent an exact expression of the dielectric tensor, may be written in terms of the above parameters as

$$\tau_{\pm} = \frac{1}{4} B^{-1} \log \left| \frac{\frac{1}{2} C \pm B_{\pm}^{-1} + 2}{\frac{1}{2} C \pm B_{\pm}^{-1} - 2} \right|$$

$$\tau_{zz} = 1 + \frac{1}{4} A^{-2} C^{-1} \log \left| \frac{A^{-2} - (\frac{1}{2} C + 2)^2}{A^{-2} - (\frac{1}{2} C - 2)^2} \right| \tag{1}$$

Numerically we find $\langle |p| \rangle = \frac{1}{2} p_F \cong 2.51 \times 10^{-2} \times N_{27} \times Mc$, $\omega_H = eH/Mc \cong 1.76 \times 10^{19}\ \sec^{-1}$

$$A = \frac{1}{2} \frac{p_F}{Mc} \frac{kc}{\omega} \cong 2.51 \times 10^{-2}\, n\, N_{27}\, H_{12}^{-1}$$

$$B = A \frac{\omega}{\omega_H} \cong 1.43 \times 10^{-11}\, n\, \omega_{10}\, N_{27}\, H_{12}^{-2} \tag{2}$$

$$C = \frac{100}{2.51} \frac{\hbar}{Mc} k \cong 5.13 \times 10^{-10}\, n\, \omega_{10}\, N_{27}^{-1}\, H_{12}$$

Where $n = kc/\omega$ stands for the index of refraction, and $\omega_{10} = \omega/10^{10}\sec^{-1}$.

The domains for the various approximations are summarized in Fig. 1. Two such domains are defined for the transverse components $\tau_{\pm}$. In the long wavelength regime B, C << 1 the magnetoionic expressions are valid, and the corresponding transverse plasmons are given by the classical dispersion relation $n_{\pm}^2 = 1 - \omega_p^2/\omega(\omega \pm \omega_H)$. In the short wavelength region B, C >> 1, however, one finds $n_{\pm}^2 \cong 1 - \omega_p^2/\omega^2$, a result, which is independent of the magnetic field.[2] Correspondingly four separate domains of approximation are defined for τ_{zz}. Notably in region II (Fig. 1) while are still given by the M.I. theory the longitudinal component τ_{zz} is not. In fact τ_{zz} becomes quite singular near the separation line of regions I and II (see Fig. 3). Approximate expressions for $\tau_{\alpha\beta}$ valid in each one of these domains are given in Appendices A and B.

These approximate expressions as well as the exact expressions eq. (1) are valid only for propagation along the magnetic lines of force ($\theta = 0$). In any other direction of propagation $\tau_{\alpha\beta}$ depends on the angle θ as discussed at the end of the previous paragraph. The tensor $t_{\alpha\beta}$, however, being a manifestation of the details of the Landau orbits of

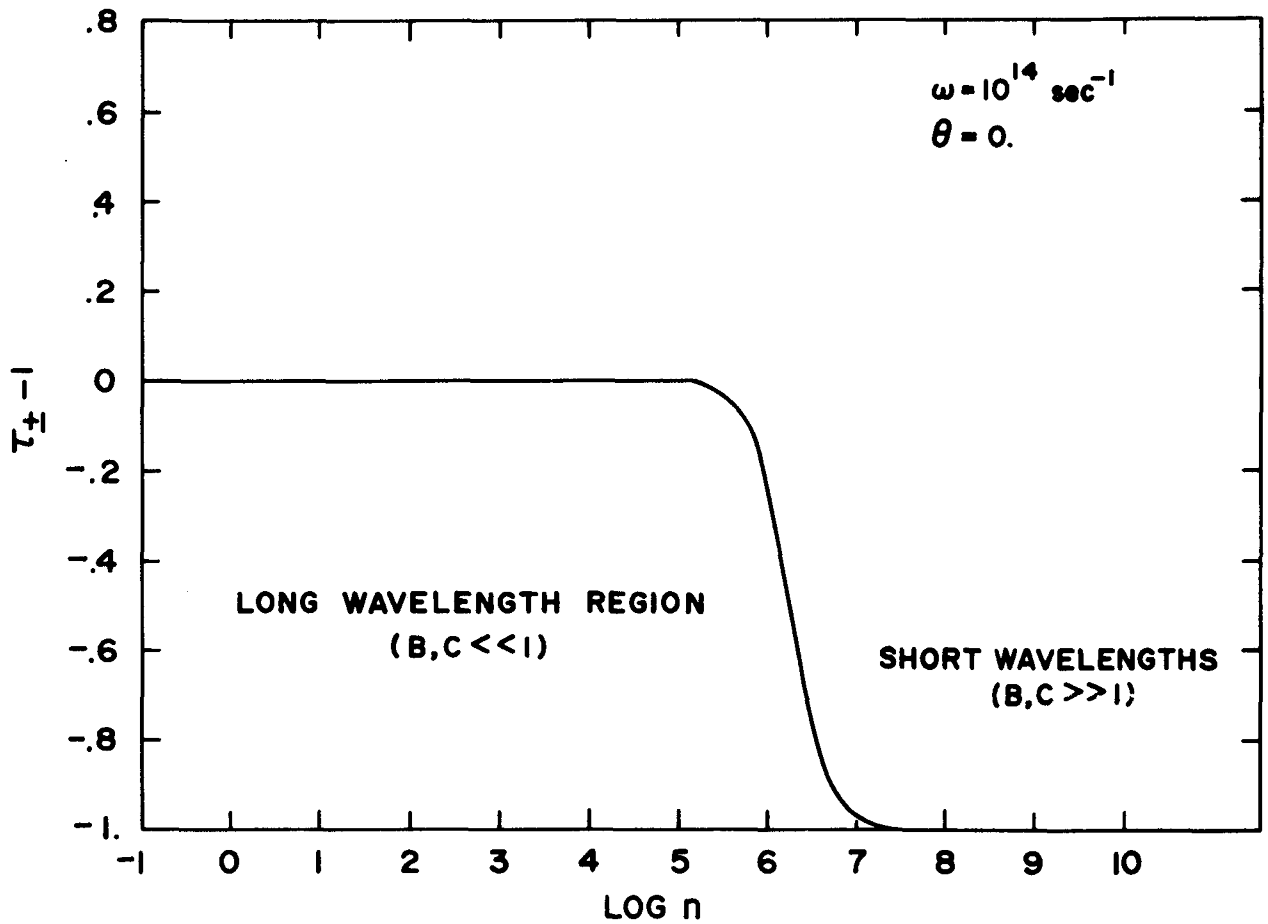


Fig. 2. The transverse components of the tensor $\tau_{\alpha\beta}$ as a function of n at $\theta = 0$ displaying the characteristic features of the low frequency regime ($\omega << \omega_H$). For $n < \lesssim 10^5$ the functions approach assymptotically the magnetoionic values.

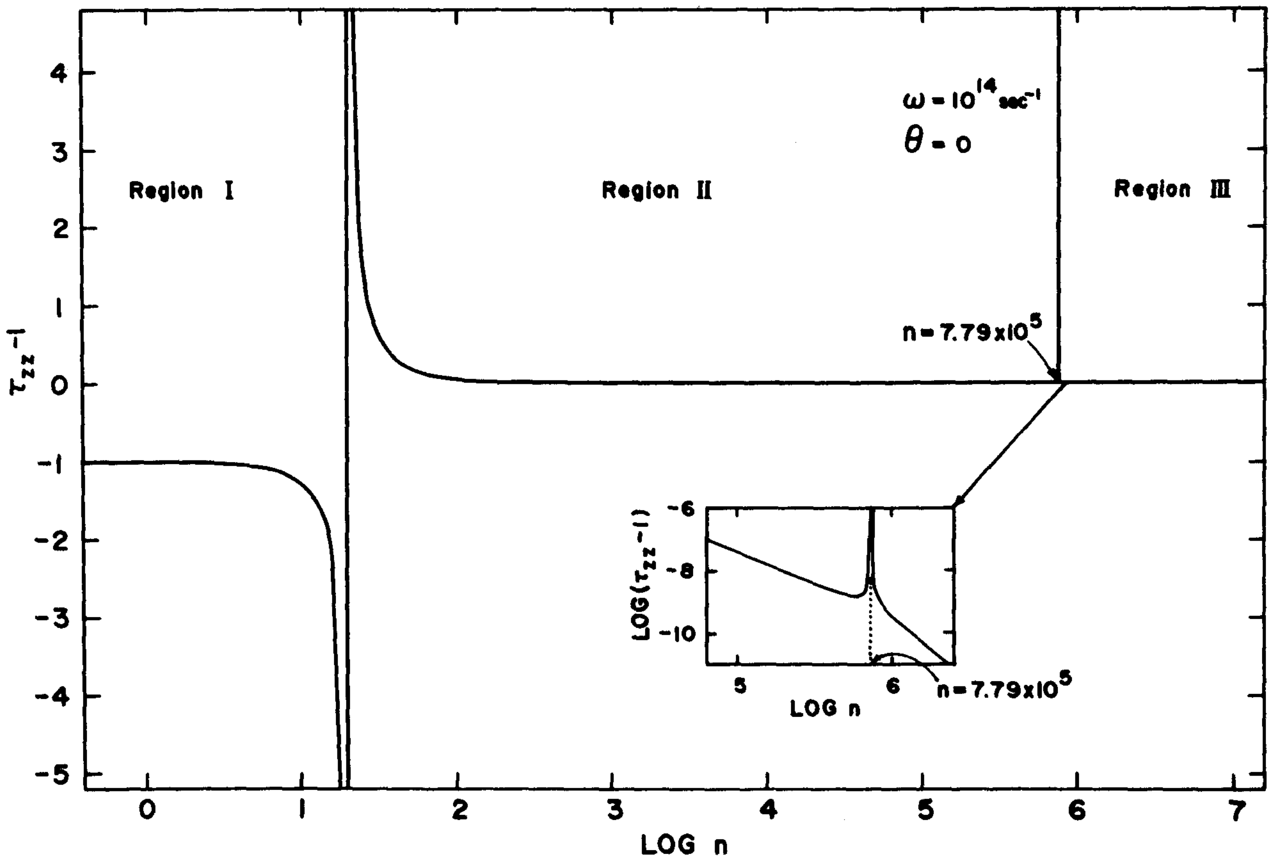


Fig. 3. The longitudinal component of the tensor $\tau_{\alpha\beta}$ as a function of n at $\theta = 0$, displaying the singularities and assymptotic features characteristic of the low frequency regime ($\omega \ll \omega_b$).

the electron, becomes essentially independent of the angle, when the wavelength is long compared to the Larmor radius i.e. when $ak \ll 1$. It is thus concluded, that the region $C < 1$ (which includes both regions I and II in Fig. 1) is free of this dependence. The only remaining dependence on the angle is found by replacing $k_z = k \cos\theta$ instead of k in eqs. (2). The dependence on the structure of the Landau orbits is expected to become important in the short wavelength regions III and IV.

In the following, we shall take a detailed look at the dispersion relations defining plasmons at various angles of propagation, with our interest aimed at the modifications to the M.I. predictions brought about by the use of the exact $\mathcal{I}_{\alpha\beta}$. Plasmons propagating in the direction of the magnetic field ($\theta = 0$) are thus taken up in the following paragraph. Long wavelength plasmons at oblique directions of propagation are discussed §4, while in §5 we investigate the rather singular properties of propagation at $\theta = 90^\circ$, and the angular dependence of the short wavelength modes.

3. Transverse and Longitudinal Plasmons at $\vartheta = 0$.

It is characteristic of this direction of propagation that four components of $\varepsilon_{\alpha\beta}$ vanish identically. These are the components ε_{xz}, ε_{zx}, ε_{zy}, ε_{yz}. The transverse and longitudinal polarizations are thus decoupled in this direction for precisely the same reasons as in the M.I. theory. Two transverse modes of circular polarization are thus defined, as well as a longitudinal mode (plasmon), through the dispersion relations

$$n^2 = \varepsilon_{\pm} = 1 + \frac{\omega_p^2}{\omega^2}\left(\tau_{\pm} - 1\right) \tag{3a}$$

$$\varepsilon_{zz} = 1 + \frac{\omega_p^2}{\omega^2}\left(\tau_{zz} - 1\right) = 0 \tag{3b}$$

Since the parameters A, B, C of eqs. (1) are functions of n and ω (see eq. 2), each one of the eq. (3) defines a line in the (n, ω) plane, which represents the corresponding dispersion relation. For a given value of ω the structure of the functions $\tau_i(n,\omega)$ is shown in Fig. 2 and 3. Their assymptotic behaviour and analytic structure is discussed in the appendices.

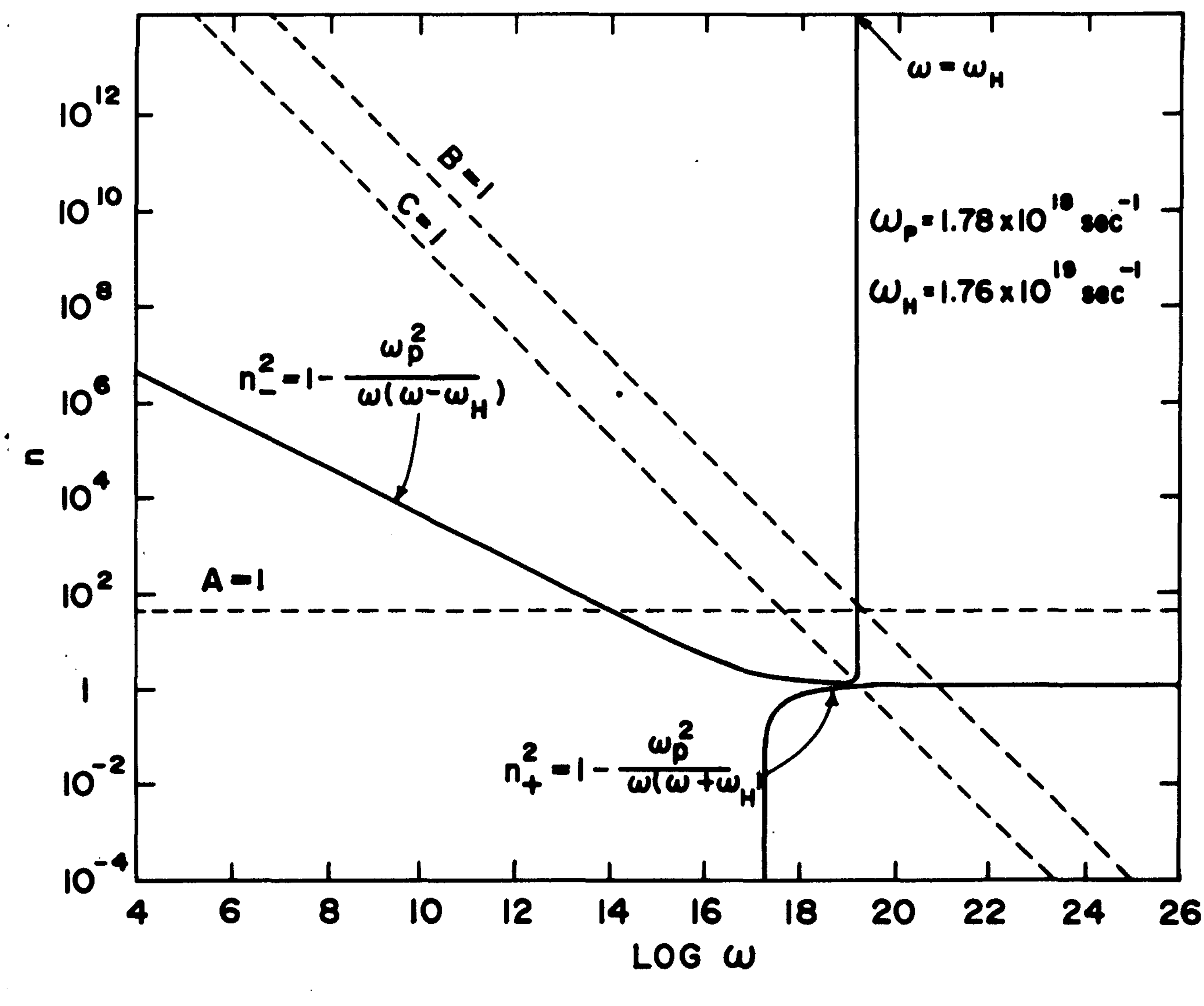


Fig. 4. Dispersion relations for the transverse plasmons represented as curves in the (n, ω) plane.

For transverse plasmons we find from (3a), modes, which are essentially identical with the M.I. modes, see Fig. 4. This result of our numerical computation may be understood in terms of Fig. 5 which represents a graphical solution of eq. (3a) for a given value of the frequency ω .

For the longitudinal plasmons, on the other hand, we predict considerable deviations from the magnetoionic theory. The line representing eq. (3b) in the (n, ω) plane is shown in Fig. 6. In the low frequency region $\omega \ll \omega_p$, we predict a mode of collective excitation for every value of the frequency ω, while the M.I. theory predicts no longitudinal plasmons in these frequencies. It is noted that the branch of our solution which enters region I of the (n, ω) diagram coincides very closely with the straight line $\omega = \omega_p$ of the classical theory, just as expected.

Characteristically the upper branch of the solution of Fig. 6 almost coincides with a branch of the singularities of the function $\tau_{zz}(n, \omega)$ cf. Fig. 7. This most important connection of the plasmon dispersion relation, and the analytic structure of the dielectric tensor, may again be demonstrated by representing the solution of (3b) as the intersection of two curves, cf. Fig. 8, for a given value of the frequency ω. The nature of the singularities of $\tau_{zz}(n, \omega)$ near n = 20 is

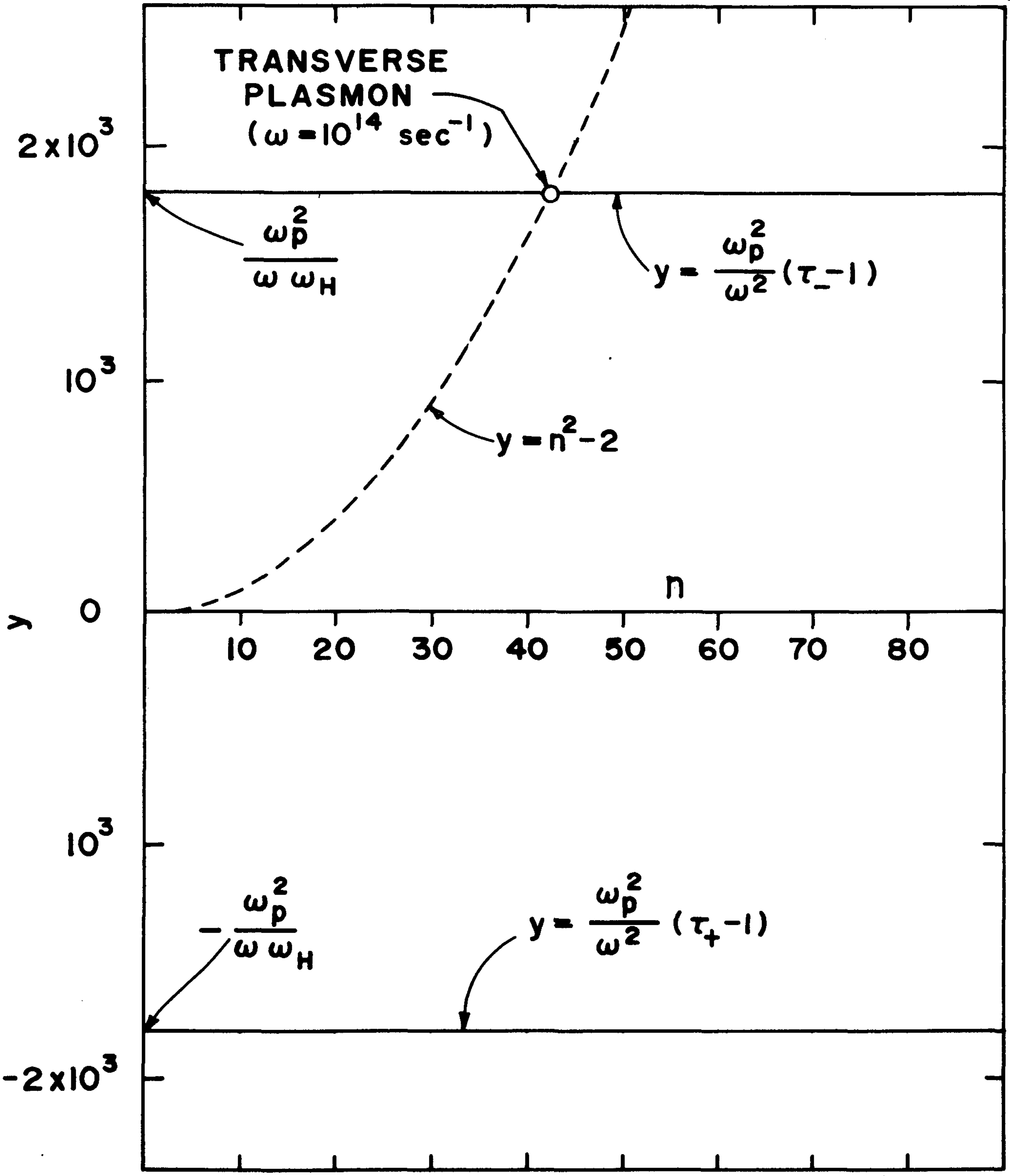


Fig. 5. Graphical solution of Eq. (39a) at $\theta = 0$ defining the transverse plasmon n_-. It is seen that this mode is well defined within the range of the magnetoionic theory. n_+ is not defined at this particular frequency.

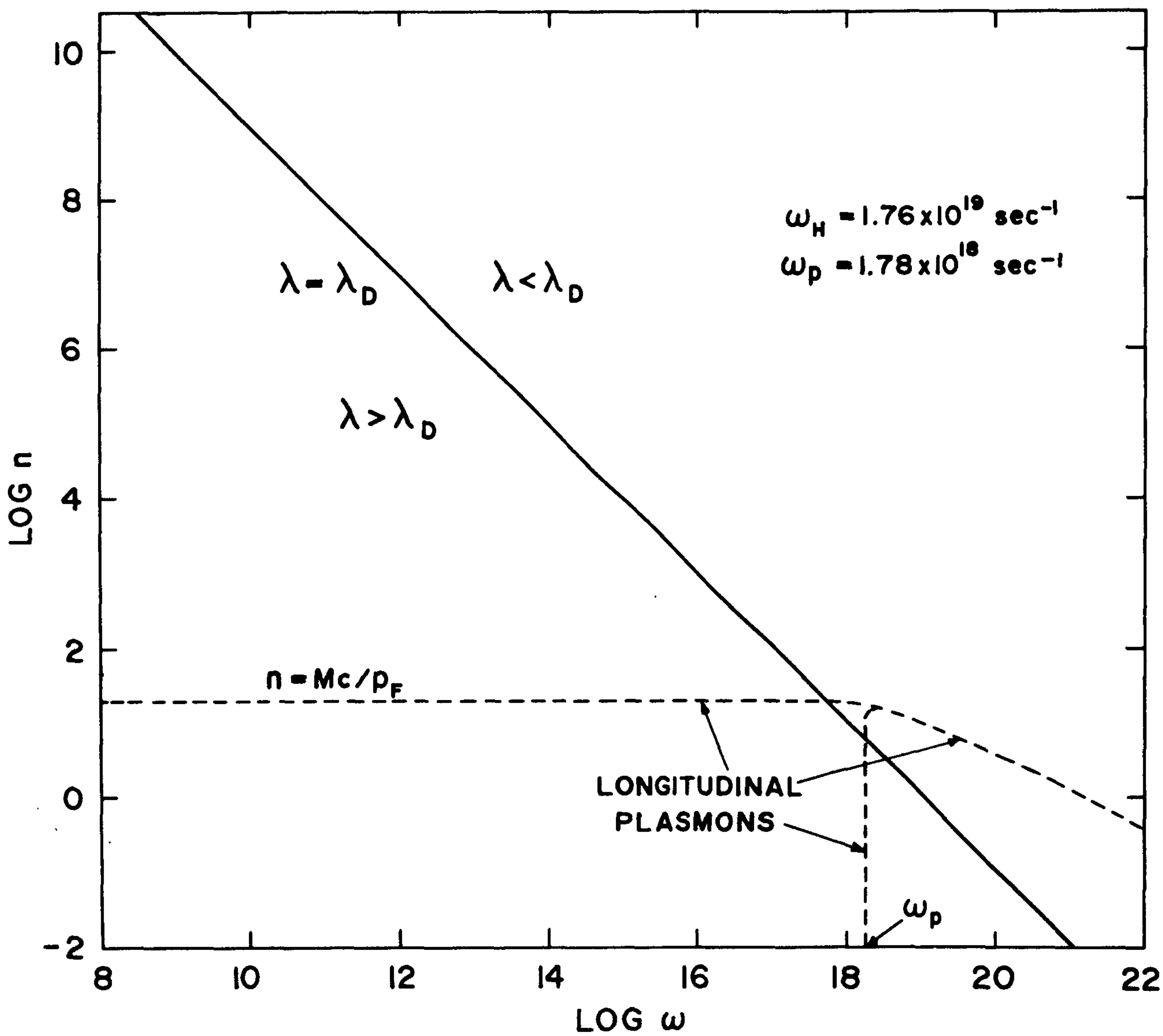


Fig. 6. The longitudinal plasmon at $\theta = 0$ according to the exact theory. Of the two branched curve representing the dispersion relation, the upper branch is new. The magneto-ionic mode $\omega = \omega_p$ corresponds here to the long-wavelength part of the upper branch. λ_D is the electron's deBroglie wavelength $\lambda_D = \hbar / p_F \cong 7.7 \times 10^{-10}$ cm.

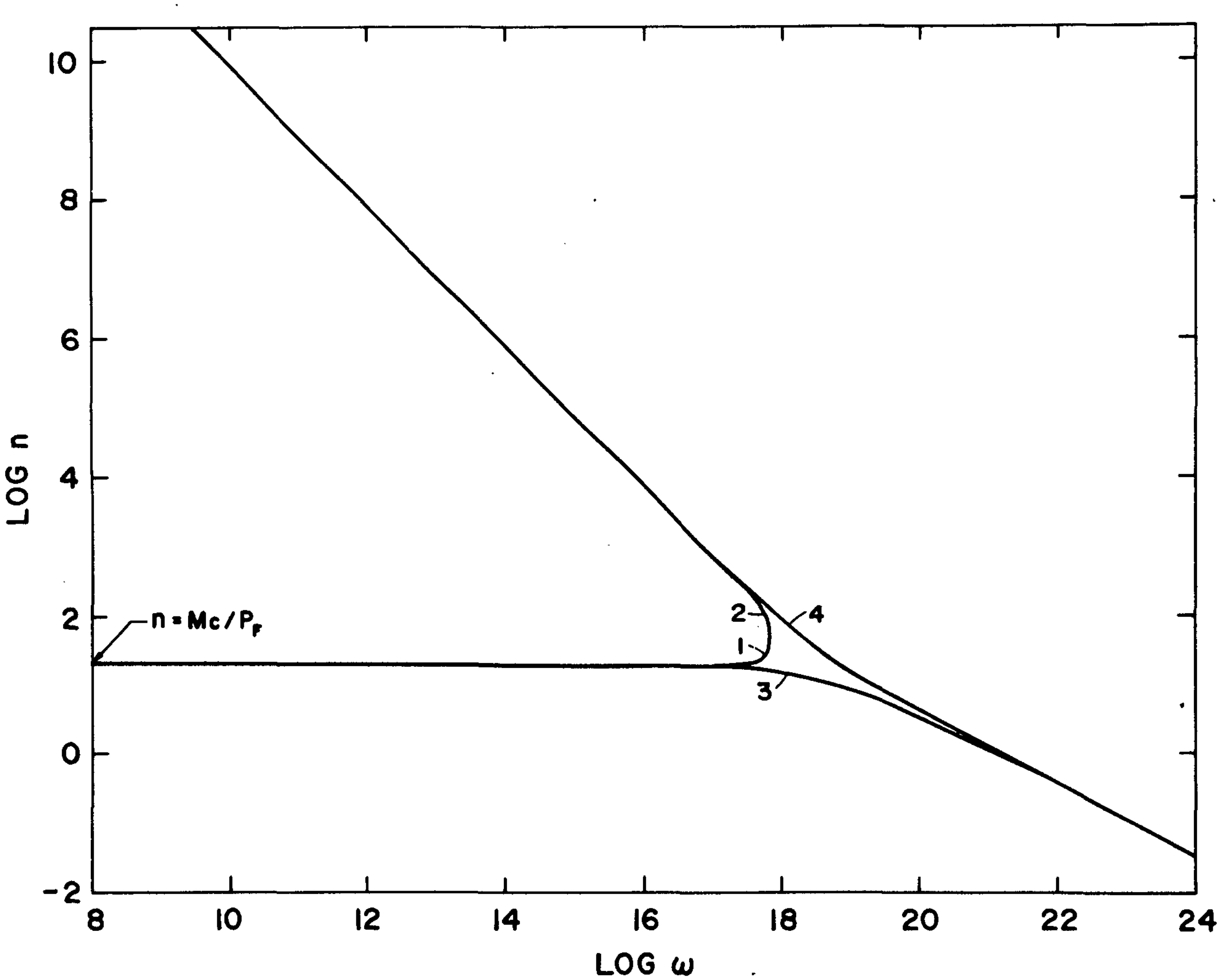


Fig. 7. Singularities of τ_{23}. The numbers on the curves correspond to the labeling of the singularities according to Appendix B.

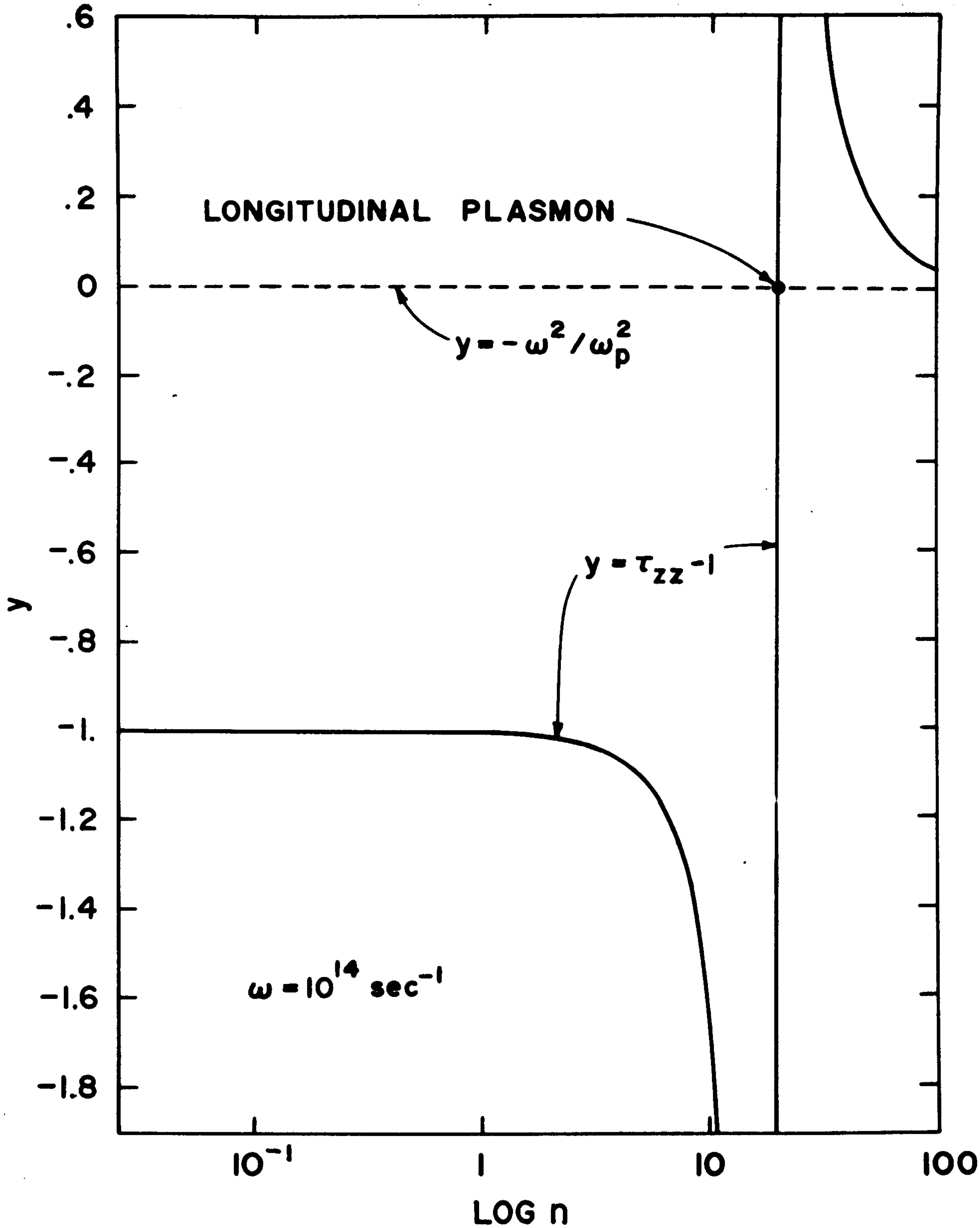


Fig. 8. Graphical representation of the solution to Eq. (39b) defining a longitudinal plasmon distinct from the magnetoionic mode.

such that this function acquires all the values between $-\infty$ and $+\infty$ within a brief interval on the n axis. It is therefore bound to intersect the line $y = -\omega^2/\omega_p^2$ within this neighborhood.

It is worth noting that singularities of the components $\tau_{\pm}$ might give rise to a new set of transverse modes, similar to the longitudinal modes predicted here. The existence of such a singularity, however, requires (cf. Appendix B) that the Fermi energy $E_F \geq \hbar\omega_H$, and is thus incompatible with our assumption of a one-dimensional gas.[2]

4. Long Wavelength Plasmons at Oblique Directions ($\theta \neq 0$)

In the previous paragraph it was found that at $\theta = 0$ the transverse plasmon n_- remains virtually classical at low frequencies, since it is found within region II in the n, ω diagram. It is expected however that this may not continue to be so, as we deviate to other directions of propagation $\theta \neq 0$, for the reason that $n(\theta)$ is no longer independent of the value of τ_{zz}, which does not retain its classical value in region II, as we found previously.

We shall investigate this problem by keeping the M.I. expressions for $\tau_{\pm}$ while introducing the exact expression of eq. (1) for τ_{zz}. This approximation should adequately describe the plasma modes in the long wavelength regime (B, C $\ll$ 1). The angular dependence of τ_{zz} is found by substituting $n \rightarrow n_z = n \cos\theta$ in the last of eqs. (1). We thus end up with a simple extension of the M.I. theory, the consequences of which, we shall now examine in some detail.

The dispersion equation can be written in the form:

$$\epsilon_{zz} = \frac{n^2 \sin^2\theta \, (S^2 - D^2 - n^2 S^2)}{n^4 \cos^2\theta - n^2 S (1 + \cos^2\theta) + S^2 - D^2} \tag{4}$$

with

$$\varepsilon_{zz} = 1 + \frac{\omega_p^2}{\omega^2}\left(T_{zz} - 1 \right)$$

$$S = \tfrac{1}{2}\left(\varepsilon_- + \varepsilon_+ \right) = 1 + \frac{\omega_p^2}{\omega_H^2 - \omega^2}$$

$$D = \tfrac{1}{2}\left(\varepsilon_- - \varepsilon_+ \right) = -\frac{\omega_H}{\omega}\,\frac{\omega_p^2}{\omega_H^2 - \omega^2}$$

Eq. (4) is easily obtained from (I26) by replacing $\varepsilon_o \rightarrow \varepsilon_{zz}$.

To solve eq. (4) one has in general to make use of numerical computation. For low frequencies however $\omega << \omega_p$, it is possible to study this equation analytically. Since the L. H. S. of eq. (4) is a large number of the order of ω_p^2/ω^2, it turns out that (4) is satisfied very close to the poles of the R. H. S. (i.e. the zeros of the denominator).

The only positive root of the denominator occurs approximately at

$$n^2(\theta) \cong \frac{|D|}{\cos\theta} + \tfrac{1}{2}\, S\left(1 + \frac{1}{\cos^2\theta} \right) \tag{5}$$

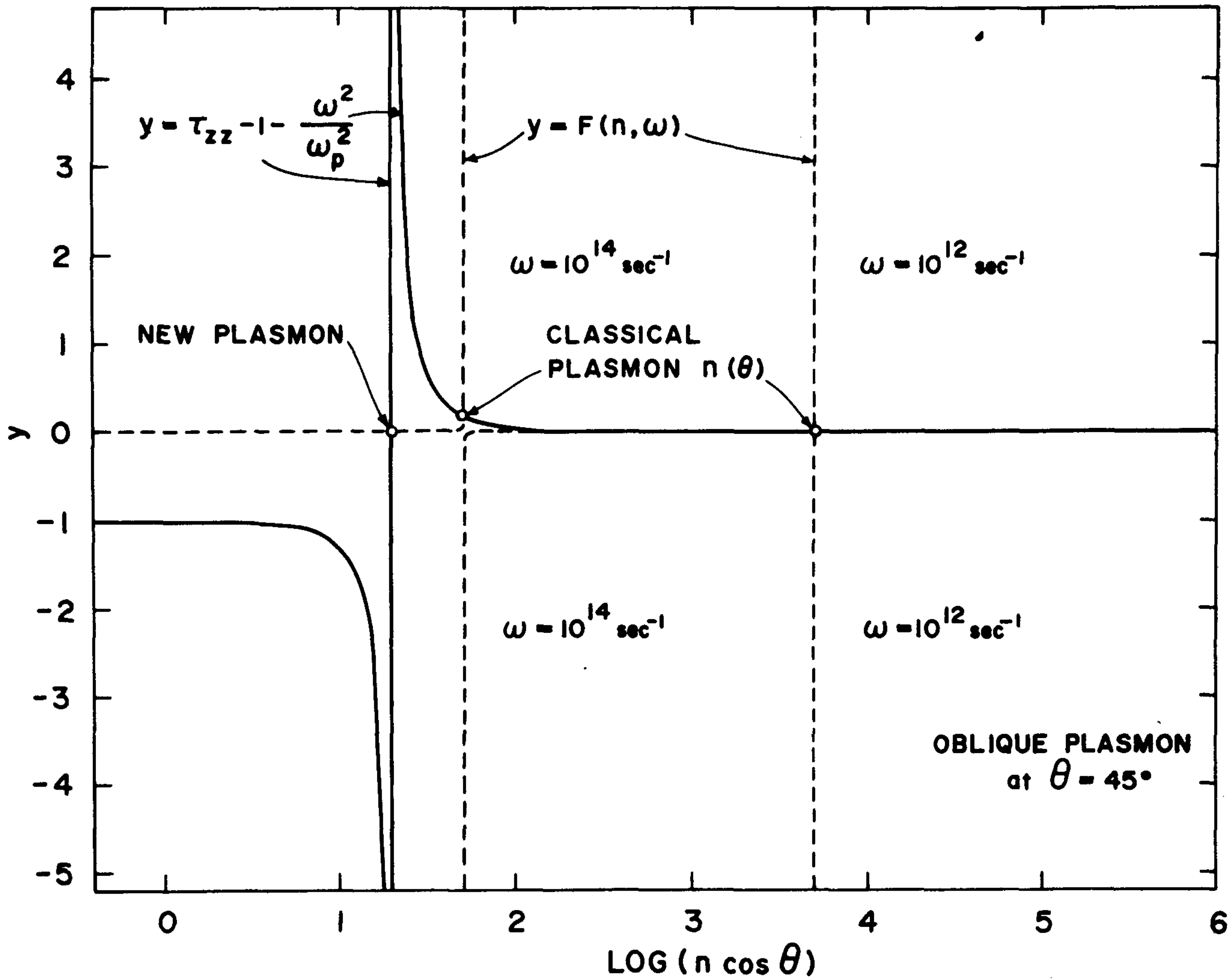
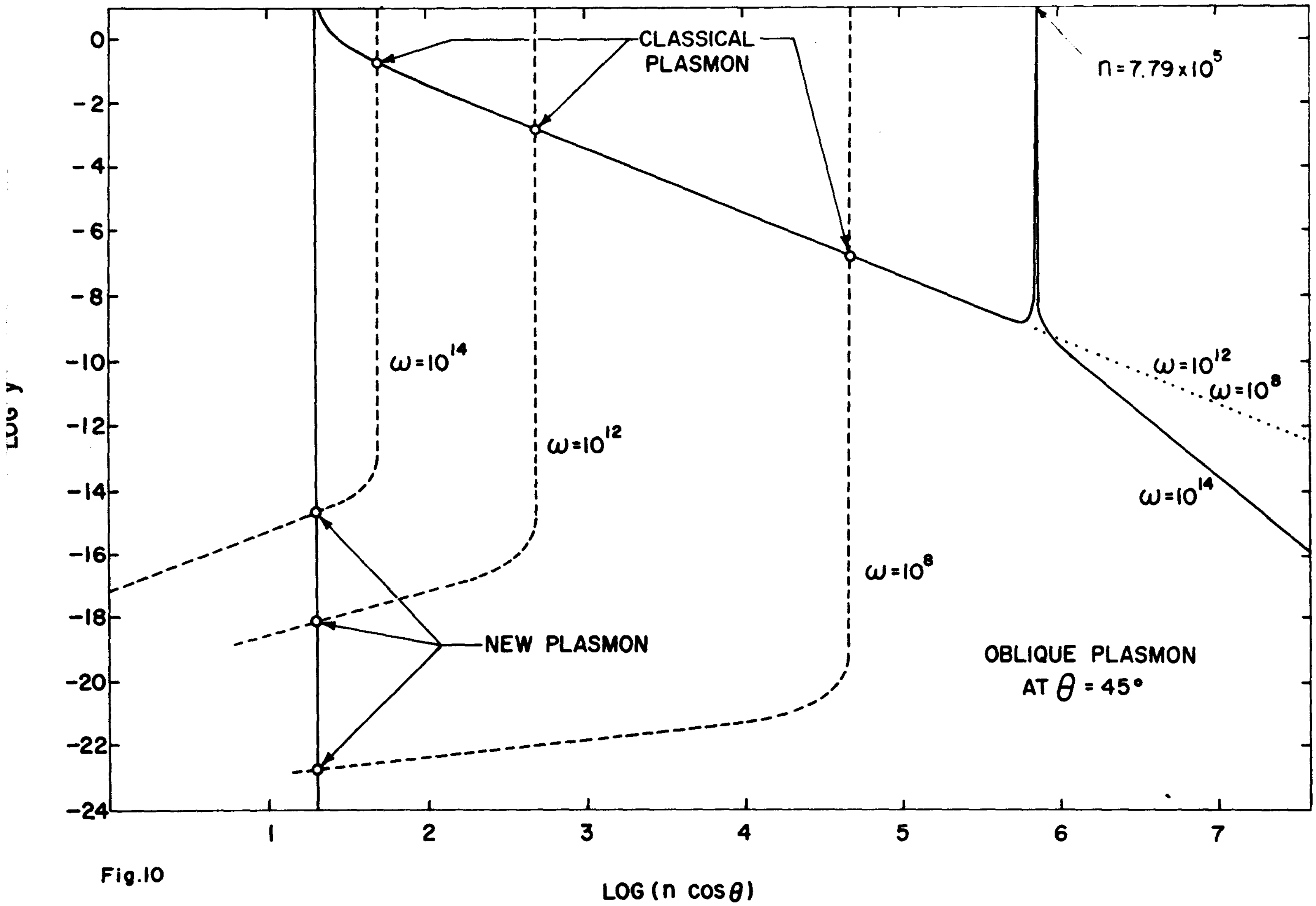


Fig. 9. Graphical solution of Eq. (4) at $\theta = 45^\circ$. $F(n,\omega)$ represents the R. H. S. of Eq. (4) multiplied by the factor ω^2/ω_p^2.



CLASSICAL PLASMON
n = 7.79 x 10^5
ω = 10^14
ω = 10^12
ω = 10^8
ω = 10^12
ω = 10^8
ω = 10^14
NEW PLASMON
OBLIQUE PLASMON AT θ = 45°
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
-22
-24
1
2
3
4
5
6
7
LOG y
LOG (n cos θ)

Fig.10

110

the next order correction being given by the additive term

$$\frac{\omega_p^2}{\omega_H^2}\left(1+\frac{\omega\,\omega_H}{\omega_p^2\cos\theta}\right)\sin^2\theta$$

At $\theta = 0$ eq. (5) reproduces the expected value $n^2 = 1 + \omega_p^2/(\omega_H - \omega)$, and since $S << D$ in eq. (5) we establish the approximate dependence on the angle $n(\theta) \cong n_-(\cos\theta)^{-\frac{1}{2}}$.

The above analysis turns out to be extremely insensitive to the exact value of τ_{zz} insofar as it is a smoothly varying function of n. The extent over which this statement holds, may be witnessed with the help of Figs. 9 and 10, and depends critically on our assumption of low frequencies. It is seen from these diagrams, that the classical M.I. dispersion relation remains virtually unchanged even though the quantity $\tau_{zz} - 1$ is modified very drastically from its classical value of -1 to the small positive values, which this quantity takes in region II of the n, ω diagram.

The foregoing arguments, however, do not hold in the neighborhood of $n \cos \eta = 20$, where τ_{zz} was found to be singular. A new plasmon, not predicted by the classical theory, is defined at precisely this value of n, at the intersection of the two curves[2] in Figs. 9 and 10 which represent the graphical solution of eq. (4). The dependence of the refractive index on the angle is now of the form $n_o(\vartheta) \propto (\cos\vartheta)^{-1}$. As we approach $\vartheta = 0$ this mode will fall on the longitudinal plasmon at $n \cong 20$ of the previous paragraph, and correspondingly eq. (4) becomes $\epsilon_{zz} = 0$.

5. Propagation at $\theta = 90^\circ$ Short Wavelength Modes

No new modes can be expected in the long wavelength regime B, C << 1 at $\theta = 90^\circ$ since the substitution $k \to k_z = 0$ in the expressions eqs. (1), gives precisely the magnetoionic limit for $\tau_{\alpha\beta}$. The only new features are thus the short wavelength modes associated with the Landau features are thus the short wavelength modes associated with the Landau orbits.

The general expressions for the tensor $\tau_{\alpha\beta}$ are found from eqs. (II20) and (II21) which give for n = 0, $\theta = \pi/2$:

$$\tau_{\alpha\beta} = \sum_{m=1}^{\infty} \frac{\chi_{\alpha\beta}^{(m)}}{m^2 - \omega^2/\omega_H^2} \tag{6}$$

with

$$\chi_{xx} = I_{m1}^2 , \qquad \chi_{yy} = (I_{m1} + 2^{-1/2} ak\, I_{m0})^2$$

$$\chi_{xy} = -\chi_{yx} = -i \frac{\omega}{\omega_H} I_{m1} (I_{m1} + 2^{-1/2} ak\, I_{m0})$$

$$\chi_{zz} = \frac{2 \langle p^2 \rangle}{M \hbar \omega_H} I_{m0}^2$$

The remaining elements of $\chi_{\alpha\beta}$ vanish identically.

For $ak \ll 1$ the sum in (6) is dominated by the $m = 1$ term and we recover the M.I. limit. The resonances at $\omega = m\,\omega_H$ are a characteristic feature of eqs. (6). The resulting "oscillatory properties in the refractive index are a well known effect for weak magnetic fields. For the strength of the magnetic field visualized here however, these resonances constitute excitations at very high frequencies, and their contribution to the physical properties of the plasma, is expected to be unimportant from thermodynamic considerations alone.[3]

6. Absorption

With the aid of the formalism developed in Part I, the magnitude of the absorptive part of $\epsilon_{\alpha\beta}$ is easy to calculate for any given plasmon. Let us first examine the longitudinal plasmons of §3.

From the second of eq. (II25a) we find that whether or not there is absorption depends on the magnitude of the quantity.

$$(2M\hbar)^{-1}(p_F^{\,2}-p_0^{\,2}) = \tfrac{1}{2}\, A C^{-1}\left[4-(A^{-1}-\tfrac{1}{2}C)^2\right]$$

There is no absorption unless the value of this quantity lies between zero, and one. This condition defines a region of width $\hbar\omega$ below the Fermi surface; it is only in this region that the bracket in eq. (II25a) gives a nonvanishing contribution. In the long wavelength regime this condition is satisfied on the horizontal branch of Fig. 6 where we have $2 - \frac{1}{2}C < A^{-1} < 2 + \frac{1}{2}C$. We thus conclude that absorption occurs for the corresponding plasmon. The magnitude of the absorptive part of the dielectric constant is in this case.

$$\epsilon_{zz}^{A} = i\pi\,\frac{M^2\omega_p^{\,2}}{2\hbar k^3 p_F} = \frac{i\pi\omega_p^{\,2}}{4\omega^2}\,A^{-2}C^{-1} \gg \frac{\omega_p^{\,2}}{\omega^2}$$

The branch in Fig. 6, which falls into the classical plasmon $\omega = \omega_p$ in region I, can be shown, on these grounds, to have no collisionless absorption associated with it.

Similarly we find for propagation at oblique directions, that only the new plasmon associated with the singularity in τ_{zz} at $n_z \sim 20$ (in the long wavelength regime) suffers collisionless absorption, while none of the classical M.I. modes do.

We therefore conclude that the magneto-ionic modes are the only propagating modes at low frequencies $\omega << \omega_p$. The new mode found in § 4 near the pole of the polarization tensor τ_{zz} with $n \cong 20/\cos\theta$ does not propagate due to strong Landau absorption.

7. Conclusions.

The longitudinal component of the dielectric tensor ϵ_{zz} was found in § 2 to deviate grossly from its classical value, obtaining large positive values in Region II of the parameter space, cf. (Fig. 1), as opposed to the large negative values predicted classically. Since ϵ_{zz} appears explicitly in the dispersion relation defining plasmons, cf. Eq. (4) one might expect to find profound deviations from the classical magnetoionic theory. Contrary to such expectations, however, Eq. (4) was found to be extremely insensitive on the value of ϵ_{zz} for low frequencies $\omega << \omega_p$, so that the magnetoionic dispersion relations remain valid to an excellent approximation. The new mode which is related to a singularity in ϵ_{zz} will not propagate due to strong Landau absorption.

The magnetoionic expressions are thus reliable for the propagation of radio waves in the pulsar magnetosphere. For optical and X-ray frequencies, however, one should make use of the more detailed theory presented in Part II.

Appendix A. Analytic Structure of $\tau_{\pm}(n, \omega)$

Introducing the variable $\chi = 10^{10} \quad \omega_{10}^{-1}$ one has from eq. (36a)

$$\tau_{\pm} = \frac{\chi}{4B_o} \log \left| \frac{D_{\pm}(\chi,\omega) + 2}{D_{\pm}(\chi,\omega) - 2} \right| \tag{A-1}$$

$$D_{\pm}(\chi,\omega) = B_{\pm}^{-1} + \tfrac{1}{2} C = B_o^{-1} \left(1 \pm \frac{\omega}{\omega_H}\right)\chi + C_o \chi^{-1} \tag{A-2}$$

with $B_o = .143 \times N_{27} \times H_{12}^{-2}$, $C_o = 2.565 \times N_{27}^{-1} \times H_{12}$. The dependence of the above forms on ω is, clearly, very weak at low frequencies ($\omega << \omega_H$), so that χ becomes a convenient parameter for studying the properties of this function.

$\tau_{\pm}$ has singularities when $D_{\pm}(\chi, \omega) = 2$. This equation has two roots, which are real if the condition $B_o(1 \pm \omega/\omega_H)^{-1} > C_o$ is satisfied. For $\omega << \omega_H$ this condition becomes essentially

$$\frac{B_o}{C_o} = \frac{B}{\tfrac{1}{2}C} = \frac{p_F^2/2M}{\hbar\omega_H} > 1 \tag{A-3}$$

and therefore contradicts our initial assumption of a one-dimensional gas. We thus conclude, that $\tau_{\pm}$ cannot be singular at low frequencies, so long as only the lowest Landau level is populated.

The following limiting cases are of some interest (a) Long Wavelength Limit $\chi >> 1$. One finds $D_{\pm}(\chi, \omega) \cong B_o^{-1}(1 \pm \omega/\omega_H)$ so that expanding the logarithm we find $\tau_{\pm} \cong \pm\, \omega_H/(\omega \pm \omega_H)$. (b) Short Wavelength Limit $\chi << 1$. Here one finds $D_{\pm}(\chi, \omega) \cong C_o\, \chi^{-1}$ (unless ω becomes very large $\omega \sim \omega_H\, \chi^{-2}$). Consequently

$$\tau_{\pm} \cong B_o^{-1} C_o^{-1} \chi^2 << 1 \quad . \tag{A-4}$$

The corresponding dispersion relation takes the form $n^2 = 1 - \omega_p^2/\omega^2$ in this limit.

Appendix B. Analytic Structure of $\tau_{zz}(n,\omega)$

From the second of eq. (36a) it is evident that τ_{zz} is singular whenever any one of the four conditions $A^{-1} \pm \frac{1}{4}C = (\pm)\ 2$ is satisfied. For a given value of ω these conditions yield four values of the refractive index, at which τ_{zz} is singular, as follows

$$
\begin{aligned}
n &= n_{1,2}(\omega) = a\left(1 \pm \sqrt{1-b\omega}\right)^{-1} \\
n &= n_{3,4}(\omega) = a\left(\sqrt{1+b\omega} \pm 1\right)^{-1}
\end{aligned}
\qquad \text{(B-1)}
$$

with $a \cong 19.93 \times N_{27}^{-1} \times H_{12}$, $b \cong 1.02 \times 10^{-18} \times N_{27}^{-2} \times H_{12}^{2}$. At $n = n_3(\omega)$, τ_{zz} obtains large negative values, while the remaining three singular points correspond to positive infinities of τ_{zz}. Eqs. (B-1) define a line in the (n,ω) plane which, as shown in Fig. 8, is composed of three separate branches. At low frequencies these curves fall assymptotically on the lines $n = a$ and $n = 2a/b$. At high frequencies the first of eqs. (B-1) gives complex values of n, while the second one gives two real values, which converge assymptotically on the line $n = a/(b\,\omega)^{\frac{1}{2}}$.

Approximate expressions for τ_{zz} can be easily obtained for each one of the assymptotic regions in the (n, ω) plane, cf. Fig. 1. Here we merely give the leading terms in the corresponding expansions

$$\tau_{zz}^{I} \cong -4A^{2}(1 + C^{2}/16) \quad \text{in Reg. I } (A, C \ll 1)$$

$$\tau_{zz}^{II} \cong 1 + \tfrac{1}{4}A^{-2} \quad \text{in Reg. II } (A \gg 1, C \ll 1)$$

$$\tau_{zz}^{III} \quad 1 + 2A^{-2}C^{-2} \quad \text{in Reg. III } (A C \gg 1)$$

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1. Chiu, H. Y., 1970, Publications Astron. Soc. of the Pacific, 82, p. 486. Canuto, V. and Chiu, H. Y., 1971, Space Science Reviews, 12, (1971) 3-74.

2. When the n $\geq$ 1 Landau level becomes populated such singularities do in fact contribute new transverse plasmons. As $E_F/\hbar\omega_H$ increases, a new set of such modes will appear each time that this quantity crosses an integer, cf. Eq. (II31) and (II31a).

3. For $\theta \neq 90^\circ$ the condition for the resonances becomes $\omega/\omega_H = m + B_0 n_z \omega_{10} (C_0 \cdot n_z \omega_{10} \pm 2)$ with the constants B_0 and C_0 as defined in Eq. (A-2) of the Appendix. Since m is a positive integer it is seen again that this condition is never satisfied at low frequencies.

4. G. Rickayzen in "The Many Body Problem" (Bergen School of Physics, 1961) edited by C. Fronsdal, Benjamin, New York, 1962. References to earlier work can be found in this article.

PART IV

COLLISION FREQUENCY AND COULOMB SCATTERING IN AN INTENSE MAGNETIC FIELD

1. Introduction

At the neutron-star surface, electrons imbedded in the huge magnetic fields can move freely in the direction of the magnetic field lines, while their motion perpendicular to the field is confined in the lowest Landau quantum orbitals of radius $2.6 \times 10^{-10} \times H_{12}^{-\frac{1}{2}}$ cm, where H_{12} is the magnetic field in units of 10^{12} Gauss. The strongly guided character of the electron's motion may be expected to dominate the dynamics of any collision event in these gigantic fields, and deviations in the cross-section may result as compared to the field free case. The effect of the magnetic field on several processes which characterize the properties of the surface plasma was summarized in Part I. In Part II we gave a quantum mechanical description of the dielectric properties of a magnetized plasma.[3] The effect of collisions however was only superficially considered due to uncertainties in the cross-section of electron-ion scattering in an intense magnetic field, and in the formulation of the collision frequency for electrons confined in low lying Landau orbitals.

In Part IV we shall try to remedy these omissions. We shall concern ourselves both with some formal aspects of the scattering problem in the magnetic field, which have thus far not been emphasized in the literature, and with the specific application of Coulomb scattering. In Sections 2 and 3 we thus examine the assymptotic properties of the scattering state, and give an exact expression for the scattering cross-section in the magnetic field. A one-dimensional Boltzman theory is employed in Section 4, to obtain an expression for the collision frequency, which takes proper account of the discrete nature of the phase space. In Section 5 we obtain general and analytically simple expressions for the Coulomb matrix elements in the IMF. Previously known expressions are recovered in the classical limit (large quantum numbers). In the application to the problem of electron-ion scattering discussed in Sections 6 and 7, the collision frequency is found to exhibit giant oscillations as a function of the electron energy in resonance with the discrete Landau levels. The reliability of the Born approximation is finally discussed in Section 8, a brief description of which follows.

According to the formal considerations developed in Section 2, no infinite matrix elements are allowed in the theory. The known divergence of the forward matrix elements (corresponding to the transmission process nk nk in a given Landau level) is accordingly found to be a feature of the Born approximation not reflected by the exact T matrix. This is contrary to a well-known property of the field free Coulomb scattering, in which case the Born approximation coincides with the prediction of the exact theory, and both matrix elements become infinite in the limit of zero momentum transfer.

2. Description of the Collision Event

Let the heavy scatterer, placed at the origin of our coordinates, be represented by a potential $V(\vec{r})$. The electron, approaching along the magnetic field lines, is described assymptotically by the unperturbed Hamiltonian and the corresponding wavefunction[4]

$$H_0 = (2m)^{-1}\left(\vec{p} - \frac{e}{2c}\,\vec{r}\times\vec{H}\right)^2$$

$$\psi_{nsk} = e^{ikz}\,\phi_{ns}(\rho,\theta) \tag{1}$$

$$\phi_{ns}(\rho,\theta) = \sqrt{\gamma/\pi}\;e^{i(n-s)\theta}\,I_{ns}(\gamma\rho^2)$$

Here $\vec{H}$ is the magnetic field, $-e$ and m the charge and mass of the electron, and $\gamma \equiv \frac{1}{2}\,eH/\hbar c$. n and s are non-negative integers, $I_{ns}(t)$ is an orthonormal set of functions constructed from the Laguerre polynomials, and given explicitly in Appendix A. The magnetic field has been assumed to act along the z-direction, and $\hbar k$ is the momentum of the electron along z.

The cylindrically symmetrical solutions of Sokolov were chosen here, to match the symmetry of the scattering problem at hand. The corresponding energy spectrum is

$$E_{nsk} = \frac{\hbar^2 k^2}{2m} + \hbar\omega_H (n + 1/2) \qquad (2)$$

where ω_H = eH/mc is the usual Larmor frequency.

This equation is actually modified by the addition of the spin coupling term $\hbar\omega_H$ which gives rise to a spectrum of doubly degenerate Landau levels, except for the lowest level, corresponding to n = o and spin anti-parallel to the magnetic field, which is not degenerate.

Eq. (1) describes a particle which can move freely in the direction of the magnetic field, while its transverse motion is bounded in the n-th Landau level within a characteristic radius of gyration $R_n = \gamma^{-1/2}(n+1/2)^{1/2}$ and located at a mean distance from the origin $\rho_s = \gamma^{-1/2}(s+1/2)^{1/2}$.

We shall now formulate the collision problem, keeping in mind that the geometry of the outgoing waves changes from the usual spherical waves e^{ikr}/r to the cylindrical waves $e^{ikz}\ \phi_{ns}(\rho, \theta)$ defined in equation (1). The scattering state is given by an equation of the Lippman-Schwinger type

$$\psi^{(+)}_{nsk} = e^{ikz}\phi_{ns} + (E - H_0 + i\eta)^{-1} V \psi^{(+)}_{nsk} \qquad (3)$$

As usual, the assymptotic behavior of this wavefunction will give an explicit expression for the scattering amplitude in the new cylindrical geometry. Substituting the coordinate representation of the propagator

$$\langle r'|\,(E-H_0+i\eta)^{-1}\,|\,r\rangle = \frac{-im}{\hbar^2}\sum_{ns}\phi_{ns}(\rho',\theta')\,\phi^*_{ns}(\rho\theta)\,\frac{e^{ik_n|z-z'|}}{k_n}\;,$$

$$k_n = \left\{(2m/\hbar^2)\left[E-\hbar\omega_H(n+1/2)\right]\right\}^{1/2}\;, \tag{4}$$

cf. Appendix B, we find the following assymptotic expression

$$\psi^{(+)}_{nsk}\underset{|z|\to\infty}{\sim}\phi_{ns}\,e^{ikz}+\sum_{n's'}\phi_{n's'}\left[A_{n's'}(k')\,\theta(z)\,e^{ik'z}+A_{n's'}(-k')\,\theta(-z)\,e^{-ik'z}\right]\;,$$

$$A_{n's'}(k') = \frac{-im}{\hbar^2|k'|}\left(\psi_{n's'k'}\,,\,V\,\psi^{(+)}_{nsk}\right)\;, \tag{5}$$

$$\hbar^2k'^2 = \hbar^2k^2 + 2m\hbar\omega_H(n-n')\;.$$

where $\theta(z)$ is the step function.

The last of these equations is an expression of the conservation of energy in the transition nsk $\to$ n's'k'. As expressed by equations (5) the problem has the form of a multi-channel, one-dimensional, scattering event with an incoming cylindrical wave in the state nsk, and outgoing waves n',s',k' and n', s',-k'. According to the results of one-dimensional scattering, the relative probabilities for transmission and reflection in the state n's'k' are given by the expression $(|k'|/k)\left|\delta_{n'n}\,\delta_{s's}\,\delta_{k'k} + A_{n's'}(k')\right|^2$ with the positive values of k' corresponding to transmission and negative k' corresponding to reflection (backward scattering).

The transition matrix is defined by the equation

$$T_{\alpha'\alpha} \equiv (\psi_{\alpha'}, V\,\psi_{\alpha}^{(+)}) \tag{6}$$

where we have introduced the shortened notation $\alpha \equiv (nsk)$.

The statement that all probabilities add up to one, or equivalently that flux is conserved during the scattering, can take the form

$$\begin{aligned} -2\,\mathrm{Im}\,T_{\alpha\alpha} &= \sum_{\alpha'} m(\hbar k)^{-1}\,|T_{\alpha'\alpha}|^2 \\ &= 2\pi \sum_{\alpha'} \rho(E_{\alpha'})\,|T_{\alpha'\alpha}|^2 \end{aligned} \tag{7}$$

where $\rho(E_{\alpha'})$ stands for the density of states in the interval $E_{\alpha'}$, $E_{\alpha'} + dE_{\alpha'}$ Equation (7) in the last form is recognized as the well known optical theorem. Since the partial probabilities individually cannot exceed one, we also have the following weaker constraint on the matrix element

$$\left| \delta_{\alpha'\alpha} - i m (\hbar^2 k')^{-1} T_{\alpha'\alpha} \right| < k / k' \tag{8}$$

In view of this equation no infinite matrix elements are allowed. These constraints will be found useful later in exploring the limitations of the Born approximation.

It should also be noted that the probabilities mentioned above are probabilities per single incident particle. The transition probability per unit time is obtained by multiplying by the incident flux $\hbar k m^{-1}$.

3. The Scattering Cross Section

The notion of the scattering cross section has often been avoided in the literature, possibly due to an inherent difficulty in formally describing an incident beam of uniform density in the magnetic field. In this respect it should be noted, that the wavefunction $_{nsk}$ of Equation (1) represents a cylindrically symmetrical beam of current density

$$(j_z)_{nsk} = \frac{\gamma}{\pi}\,\frac{\hbar k}{m}\,\left|I_{ns}(\gamma\rho^2)\right|^2 \tag{9}$$

This distribution is spacially confined in the transverse direction, and corresponds to an impact parameter $\rho_s = \left[\gamma^{-1}(s+1/2)\right]^{1/2}$. For the description of a scattering experiment, however, one has to visualize an incident beam of uniform density representing all possible impact parameters.

A uniform density beam can only be represented as a statistical mixture of electron states of different values of s with the same weight. Summing the partial currents in Equation (9) over s, and making use of the identity[5]

$$\sum_s \left|I_{ns}(t)\right|^2 = 1$$

we obtain a flux $(\gamma/\pi)\hbar km^{-1}$ for this beam. The cross section can now be obtained as the total current going into the state n'k' according to Equation (5), divided by the incident flux

$$\sigma_{nk \to n'k'} = \frac{\pi}{\gamma} \left|\frac{k'}{k}\right| \sum_{s's} \left|A_{n's'}(k')\right|^2 \tag{10}$$

One can, of course, avoid the notion of the cross section altogether, by calculating directly the total transition probability per unit time from the partial probabilities. It should be noted however, that while the value of the transition probability depends on the normalization of the beam, the cross section is independent of the normalization.[6]

We further note in connection with Equation (10), that throughout this paper we shall be dealing with spin independent interactions V, so that the spin of the electron will always be conserved. Moreover, for cylindrically symmetrical potentials $V(\rho, z)$, s' is determined from the conservation of the angular momentum $L_z = -i\hbar(x\partial_y - y\partial_x)$, whose eigenvalue is $\hbar(n-s)$. In this case one has $s' = n' - n+s$.

4. The Collision Frequency

An external electric field $\vec{\mathcal{E}}$, parallel to the magnetic field H, tends to accelerate electrons along the magnetic flux lines, while collisions with ions give rise to a drag-force. The role of collisions in this limited problem can be understood by referring to the one-dimensional Boltzman equation governing the evolution of $f_n(z,k,t)$, the distribution function in the n Landau level. For a spatially homogeneous situation we have

$$\frac{\partial f_n(k,t)}{\partial t} - \frac{e\mathcal{E}}{\hbar}\frac{\partial f_n(k,t)}{\partial k} = \left(\frac{\partial f_n}{\partial t}\right)_{coll} \tag{11}$$

k is the momentum of the electron along H as before. Such a one-dimensional kinetic equation has been previously used in the literature,[7] and found adequate for the description of disturbances in the motion of the electron acting in the direction of $\vec{H}$. The collision term appearing at the RHS of this equation is our next objective.

Collisions give rise to transitions between Landau levels. The rate of transitions nk $\rightarrow$ n'k' leading out of the element n,dk of the phase space is

$$dN_- = \frac{\gamma}{\pi} \frac{dk}{2\pi} N_i f_n(k) [1 - f_{n'}(k')] (\hbar k/m) \sigma_{nk \to n'k'}$$

where N_i is the number density of ions. Transitions in the reverse direction occur at a rate

$$dN_+ = \frac{\gamma}{\pi} \frac{dk'}{2\pi} N_i f_{n'}(k') [1 - f_n(k)] (\hbar k'/m) \sigma_{n'k' \to nk}$$

$$= \frac{\gamma}{\pi} \frac{dk}{2\pi} N_i f_{n'}(k') [1 - f_n(k)] (\hbar k/m) \sigma_{nk \to n'k'}$$

Here we have put $dk' = (k/k')dk$ as dictated by energy conservation, and made use of the microreversibility property $\sigma_{n'k' \to nk} = \sigma_{nk \to n'k'}$. Combining the above results we finally obtain for the total rate of transitions per single incident particle

$$\left(\frac{\partial f_n}{\partial t}\right)_{coll} = S_n(k) = N_i \sum_{n'k'} [f_{n'}(k') - f_n(k)] \frac{\hbar k}{m} \sigma_{nk \to n'k'} \qquad (12)$$

where k' is determined up to a sign from the energy conservation condition. We have thus obtained an expression which, while similar in form to the usual collision term one obtains for an isotropic electron gas, also takes proper account of the discrete nature of phase space in the IMF. It is easy to check that substitution of the Fermi Dirac distribution function in equation (12) gives zero identically, just as expected for the case of thermal equilibrium.

Combining equations (11) and (12) we obtain a system of coupled differential equations in k and t, which takes the place of the usual integrodifferential equation for f(p,t) in the isotropic case. A collision frequency can now be defined for each Landau level in the relaxation time approximation[8] from the equation $(\partial f_n/\partial t)_{coll} = -\nu_n(f_n - f^0)$, where $f^0 = f^0(E_{nk})$ is the equilibrium distribution function, and is a function of the electron energy only. Clearly ν_n will depend on the nature of the deviation from equilibrium at hand. Here we are interested in investigating the steady state, which arises as a result of the external field . Proceeding along the standard Boltzman approach[8] we write $f_n = f^0 + f^1_n$ and evaluate f^1_n to lowest order in $\mathcal{E}$. Accordingly, the LHS of Equation (11) becomes

$$
\begin{aligned}
LHS &= -\frac{e\mathcal{E}}{\hbar}\frac{\partial f_n}{\partial k} \cong -\frac{\hbar k}{m}\, e\mathcal{E}\,\frac{\partial f_0}{\partial E} \\
RHS &= -S_n(k) = \frac{\hbar|k|}{m}\, N_i \sum_{n'k'} \left(f^1_{n'} - f^1_n\right) \sigma_{nk\to n'k'}
\end{aligned}
\tag{13}
$$

where $E = E_{nk}$. By equating these expressions we obtain a linear system of algebraic equations in $f^1_n(k)$

$$
\begin{aligned}
&\sum_{n'} A_{nn'}\, f^1_{n'} = \frac{e\mathcal{E}}{N_i}\frac{\partial f_0}{\partial E} \\
&A_{nn} = 2\,\sigma^{(-)}_{nn} + \sum_{n'\neq n}\left(\sigma^{(+)}_{n'n} + \sigma^{(-)}_{n'n}\right) \\
&A_{n'n} = -\left(\sigma^{(+)}_{n'n} - \sigma^{(-)}_{n'n}\right), \qquad n' \neq n
\end{aligned}
\tag{13a}
$$

The superscripts ($\pm$) indicate forward and backward scattering respectively. The indices n,n' run over all levels which are compatible with a given value of the electron energy E. In obtaining (13a) use was made of the property $f^1_n(-k) = -f^1_n(k)$, which is a consequence of the reflection properties of equation (13).

Closer inspection of equation (13) reveals the remarkable property

$$\frac{S_n(k)}{k} = \text{constant} \tag{14}$$

The above ratio is independent of n, but does depend on the energy E. Accordingly the collision frequencies at different Landau levels are also related to one another. It is therefore sufficient to have the expression for ν in a particular level (n=o, say) and determine ν_n as follows

$$\nu_0(E) = \frac{\hbar k_0}{m} N_i \sum_{nk} (1 - \eta_{nk})\, \sigma_{0k \to nk} , \tag{15}$$

$$\nu_n(E) = \nu_0(E)\, k / (k_0 \eta_{nk}) ,$$

$$\frac{2mE}{\hbar^2} = k_0^2 = k^2 + 4n\gamma \quad , \quad (\eta_{nk} \equiv f_n' / f_0')$$

The quantities η_{nk} are determined by solving the system of equations (13a). From the properties of $f_n(k)$ mentioned earlier however it is easy to deduce that $-\eta_{n,-k} = \eta_{nk} \equiv \eta_n$, and $\eta_0 = 1$. The solution of the system of linear equations itself is of course straightforward, even though the number of the equations involved may become prohibitive in the limit $E >> \hbar\omega_H$. In this paper we are mainly interested in the mathematically simple case of electrons energetically confined in the low lying Landau levels. The computation of the collision frequency in this limit is given in Sections 6 and 7. Before this can be done, however, it is necessary to evaluate the matrix elements for the transitions of interest. This will be the subject of the following section.

The collision frequency $\nu_{\parallel}$, as defined above, is clearly associated with the D.C. conductivity in the direction of H. As our one-dimensional formulation does not account for the balancing of forces in the transverse direction, the quantity $\nu_{\perp}$, associated with the transverse conductivity cannot be given along these lines. This problem will be treated in a separate communication.

5. The Matrix Element for Coulomb Scattering

Coulomb scattering in the magnetic field has been previously dealt with in the literature. Tannenwald,[9] in an early paper, used a W.K.B. approximation of the wavefunctions in eq. (1) to obtain manageable expressions for the matrix elements of the unscreened Coulomb potential in the limit of large quantum numbers. A more general derivation of Tannenwald's expression, not involving the W.K.B. approximation, was given by Goldman.[10] As none of these results, however, is applicable in the region of low quantum numbers, which is our primary concern in this paper, it will be necessary to take a fresh look at the problem.

In a Born type approximation, one finds for a screened Coulomb potential $V = e^{-\beta r}/r$

$$
\begin{aligned}
V_{n'n}^{s's}(\xi) &= (\psi_{n's'k'}, V\psi_{nsk}) \\
&= 2\int_0^\infty dt\, I_{n's'}(t)\, I_{ns}(t)\, K_0(2\sqrt{t\xi})
\end{aligned}
\tag{16}
$$

where s' = n - n + s.

K_o is the modified Bessel function and is the result of the z-integration in equation (16). The parameter ξ is defined as $\xi = (4\gamma)^{-1}(\beta^2 + K^2)$ where K is the momentum transfer k-k', and β is the screening parameter.[11]

The matrix elements of equation (16) are not independent of one another. Those which correspond to negative angular momentum n-s > 0 are related to the ones with n-s < 0 simply by interchanging n with s, and n' with s'. This symmetry is easily traced to equation (A2) of the Appendix. Furthermore, the recurrence formula (A4) derived in Appendix A links the matrix elements of adjoining indices. Making use of these properties it is possible to evaluate exactly the integral in equation (16) obtaining the result

$$
\begin{aligned}
n \leq s, \quad & V^{s's}_{n'n}(\xi) = \left[\frac{s!\,s'!}{n!\,n'!}\right]^{1/2} \Psi(s+1, s-s'+1, \xi)\, Q^{n'-n}_{n}(-\xi) \\
n > s, \quad & V^{s's}_{n'n}(\xi) = \left[\frac{n!\,n'!}{s!\,s'!}\right]^{1/2} \Psi(n+1, n-n'+1, \xi)\, Q^{s'-s}_{s}(\xi)
\end{aligned}
\qquad (17)
$$

where Q^{ℓ}_{n} is the associated Laguerre polynomial, and Ψ is the confluent hypergeometric function.[12] Details of the calculation are given in Appendix A.

In the limit of large quantum numbers one can write[13]

$$s'!\,\Psi(s+1, s-s'+1, \xi) \cong 2(\xi s)^{\frac{1}{2}(s'-s)} K_{s-s'}(2\sqrt{\xi s})$$

$$Q_n^{n'-n}(-\xi) \equiv n'!\, e^{-\xi}(\xi n')^{\frac{1}{2}(n'-n)} I_{n'-n}(2\sqrt{\xi n'})$$

Here I_m and K_m are the usual Bessel functions. Using Stirling's approximation for the factorials we find:

$$n \leq s\,, \qquad V_{n'n}^{s's}(\xi) = 2\left(\frac{s}{s'}\right)^{s'/2}\left(\frac{n'}{n}\right)^{n/2} e^{-\xi} K_{n-n'}(2\sqrt{\xi s})\, I_{s-s'}(2\sqrt{\xi n'})$$

If we also assume that the quantum numbers n,n',s are of the same order of magnitude in addition to being large, and that $\xi << 1$, we recover Goldman's result.[10] The condition on ξ can be understood in terms of the expression $4\xi = (E_\perp/E_\parallel)(n'-n)^2/n$, where $E_\perp$ and $E_\parallel$ are the transverse and parallel to the field components of the electron's energy, and where the screening parameter β is assumed to vanish.

6. Collisions in the Lowest Landau Level

Of particular interest for the properties of plasma near the surface of a magnetic neutron star, is the case of electrons occupying only the lowest orbital in the Landau spectrum, corresponding to n = 0, and the electron's spin aligned antiparallel to the magnetic field. The energy of the electron is in this case $\hbar^2 k^2/2m < \hbar\omega_H$.

The only transitions, which are energetically allowed, are of the type $0s \rightarrow 0s$, with the possibilities of reflection and transmission ($k' = \pm k$) mentioned in section 2. The corresponding matrix element according to equation (17) is

$$V_{00}^{ss}(\xi) = s!\,\Psi(s+1, 1, \xi) \tag{18}$$

where depends on the magnetic field, the screening radius and the momentum transfer in accordance with $\xi = (4\gamma)^{-1}$ $(\beta^2 + 4k^2)$. Referring to equation (15), we see that only backward scattering events (reflections) contribute to the collision frequency. The corresponding cross section is from equations (5) and (10)

$$\sigma_{0,k\rightarrow 0,-k} = \frac{\pi}{\gamma}\left(\frac{Z_i e^2 m}{\hbar^2 k}\right)\sum_{s=0}^{\infty}\left|f_{s+1}(\xi)\right|^2 \tag{19}$$

The quantity $f_{s+1}(\xi) = s!\Psi(s+1,1,\xi)$ is a monotonically decreasing function s. For s = 0 we thus get the largest contribution

$$f_1(\xi) = \Psi(1,1,\xi) = e^{\xi}\int_{\xi}^{\infty}\frac{dt}{t}e^{-t}$$

expressed here in terms of the exponential integral function. The remaining terms can be evaluated with the help of the recurrence equation[12]

$$f_2(x) = (1-x)\ f_1(x) - 1$$

$$f_{n+1}(x) = \left(2+\frac{x-1}{n}\right)f_n(x) - \left(1-\frac{1}{n}\right)f_{n-1}(x) \ ; \qquad n \geq 2$$

For values of $\xi \gtrsim 1$, the sum in equation (19) is dominated by the first few terms. For $\xi << 1$ use of the assymptotic expressions for the confluent hypergeometric function lead to the result

$$W(\xi) \equiv \sum_{s=0}^{\infty} |f_{s+1}|^2 \cong 4\xi^{-1}\int_0^{\infty} dx\ K_0^2(2\sqrt{x}) = \xi^{-1} \ ; \qquad \xi << 1 \qquad (20)$$

For $\xi >> 1$ one finds $W(\xi) \cong \xi^{-2}$. Fig. 1 shows the result of the numerical evaluation of $W(\xi)$. Our results can be fitted very accurately by the empirical expression $W(x) = x^{-2}(1+x^{-\alpha})^{-1/\alpha}$, $\alpha = 0.756$, which has the same assymptotic behaviour as described above. The value of the parameter is chosen to fit our numerical value at x=1.

From equations (15) and (17) we now find the collision frequency

$$\nu(k) = \frac{2\pi}{\gamma} \frac{Z_i^2 e^4 m N_i}{\hbar^3 k} W(\xi) \tag{21}$$

with $\xi = (4\gamma)^{-1}(\beta^2 + 4k^2)$. Comparing this result to the value ν_{iso} one gets for the isotropic case H=0, we find

$$\frac{\nu}{\nu_{iso}} = k^{-1} \gamma^{1/2} \left(1 + \frac{k^2}{2\gamma}\right) \frac{W(\xi)}{\log(1 + \cot^2 \theta_0/2)}$$

where the minimum angle of scattering θ_o is the result of the screening.[14]

It is noted that ν_{iso} diverges logarithmically in the limit of no screening ($\beta = 0$), in sharp contrast to the intense magnetic field case, where $\nu(k)$ does not diverge for $\beta = 0$. Aside from the logarithmic factor, this ratio is of order one for $k^2/\gamma \cong 1$, and diverges for small values of k^2/γ.

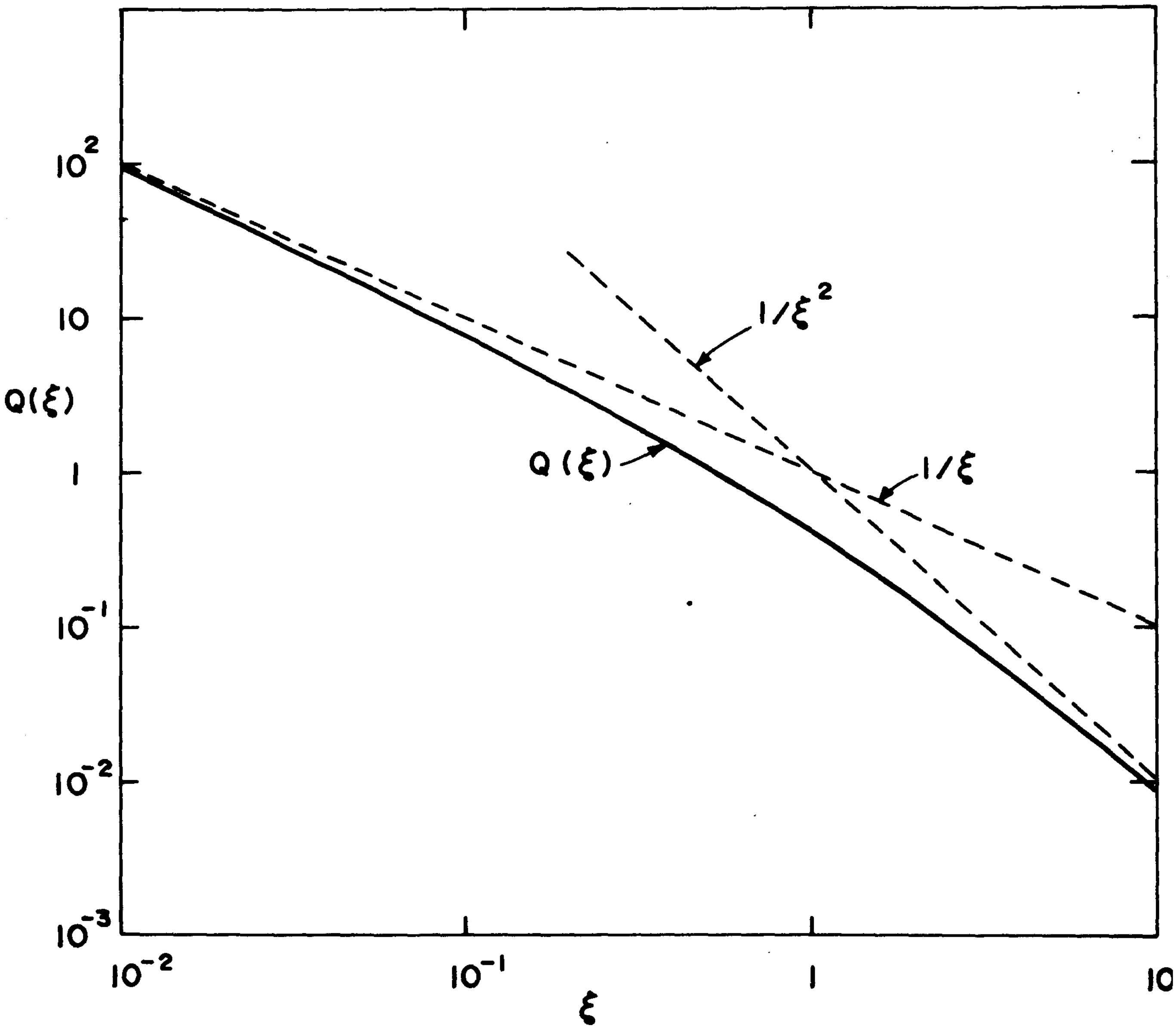

Figure 1. Plot of the numerical evaluation of the strength function $W(\xi)$.

7. Giant Oscillations in the Collision Frequency

When the kinetic energy of the electron exceeds the separation energy between Landau levels $\hbar\omega_H \cong 1\ 2 \times H_{12}$ keV (H_{12} is the magnetic field in units of 10^{12} Gauss) the collision frequency as given by Equation (15) exhibits discontinuities as a function of E in resonance with the discrete levels. In this section we explore the details of this oscillatory behavior in the low lying Landau levels. The relevant matrix elements were computed numerically along lines similar to those detailed in the previous section, making use of the appropriate recursion formulae for the hypergeometric function. The resulting collision frequencies are shown in Figure 2.

The cross sections of interest are given below in terms of the dimensionless quantities $W_{n'n}(\xi)$ representing the strength of a particular transition, and defined by

$$W_{n'n}(\xi) = \sum_{s=0}^{\infty} \left| V_{n'n}^{s's}(\xi) \right|^2 ; \qquad (s' = s + n' - n) \tag{22}$$

Introducing the quantities $\chi_n = k_n^2/4\gamma$ which are a measure of the electron's longitudinal kinetic energy in units of the separation energy between Landau levels, we may rewrite our results of equations (10) and (15) in a form more suitable for numerical evaluation as follows:

z	W_{oo}	W_{11}	W_{1o}	W_{22}	W_{21}	W_{2o}	W_{33}	W_{32}	W_{31}	W_{3o}
0.010	95.921	90.611	3.119	86.617	4.873	0.464	83.304	6.231	0.741	0.164
0.020	46.578	42.477	2.491	39.585	3.700	0.441	37.291	4.570	0.686	0.161
0.030	30.284	26.847	2.141	24.534	3.067	0.422	22.757	3.694	0.643	0.158
0.040	22.209	19.217	1.902	17.279	2.646	0.406	15.829	3.124	0.607	0.155
0.050	17.406	14.741	1.724	13.071	2.339	0.392	11.851	2.715	0.577	0.153
0.060	14.229	11.821	1.583	10.355	2.101	0.379	9.304	2.403	0.550	0.151
0.070	11.979	9.778	1.468	8.473	1.910	0.368	7.554	2.156	0.526	0.148
0.080	10.304	8.276	1.371	7.102	1.752	0.357	6.288	1.954	0.504	0.146
0.090	9.012	7.130	1.288	6.064	1.619	0.348	5.337	1.785	0.485	0.144
0.100	7.985	6.230	1.216	5.256	1.505	0.338	4.001	1.643	0.467	0.142
0.200	3.507	2.466	0.792	1.970	0.874	0.271	1.668	0.887	0.343	0.125
0.300	2.111	1.389	0.589	1.078	0.602	0.228	0.900	0.584	0.271	0.112
0.400	1.452	0.912	0.467	0.697	0.451	0.197	0.578	0.423	0.223	0.102
0.500	1.077	0.655	0.384	0.495	0.355	0.173	0.410	0.325	0.188	0.093
0.600	0.839	0.498	0.325	0.374	0.289	0.154	0.309	0.259	0.162	0.086
0.700	0.676	0.394	0.280	0.296	0.242	0.139	0.244	0.214	0.141	0.079
0.800	0.559	0.322	0.244	0.241	0.206	0.126	0.199	0.180	0.125	0.074
0.900	0.471	0.269	0.216	0.201	0.178	0.115	0.166	0.154	0.112	0.069
1.000	0.404	0.229	0.193	0.171	0.156	0.105	0.142	0.134	0.101	0.064
2.000	0.139	0.080	0.084	0.060	0.063	0.055	0.050	0.052	0.047	0.038
3.000	0.071	0.043	0.048	0.033	0.035	0.034	0.028	0.029	0.028	0.025
4.000	0.044	0.028	0.032	0.022	0.023	0.024	0.018	0.019	0.019	0.018
5.000	0.030	0.020	0.023	0.015	0.017	0.018	0.013	0.014	0.014	0.014
6.000	0.021	0.015	0.017	0.012	0.013	0.014	0.010	0.011	0.011	0.011
7.000	0.016	0.012	0.013	0.009	0.010	0.011	0.008	0.008	0.009	0.009
8.000	0.013	0.009	0.011	0.008	0.008	0.009	0.006	0.007	0.007	0.007
9.000	0.010	0.008	0.009	0.006	0.007	0.007	0.005	0.006	0.006	0.006
10.000	0.008	0.006	0.007	0.005	0.006	0.006	0.005	0.005	0.005	0.005

Table 1. Values of the functions $W_{n'n}(z)$ for the lowest four Landau levels.

$$\sigma_{n'n}^{(\pm)} = \sigma_H \left(\chi_{n'} \chi_n \right)^{-1/2} W_{n'n}(\xi^{\pm})$$

$$\nu_0(\chi_0) = \nu_H \sum_n \sum_{\pm} (1 \mp \eta_n) \chi_n^{-1/2} W_{n0}(\xi^{\pm}) \tag{23}$$

$$\sigma_H = \frac{\pi}{4} \left(\frac{Z}{a_0 \gamma} \right)^2 , \qquad \nu_H = \frac{\pi}{2} \frac{Z^2 e^2}{a_0 \hbar} N_i \gamma^{-3/2}$$

where $a_o = \hbar^2/me^2$ is the Bohr radius, the superscripts refer to forward and backward scattering as previously, and the dimensionless quantity $\xi = \xi^{\pm}$ is defined in terms of the longitudinal momentum transfer as described in Section 5. The collision frequencies $\nu_n(\chi_o)$ are related to $\nu_o(\chi_o)$ by Equation (15).

The transition strength quantities $W_{n'n}(\xi)$ have been tabulated for convenient reference, cf. Table 1. In Table 2 we give the value of the ratios $\eta_n = f_n^1/f_o^1$ as computed from Equation (14). The resulting collision frequencies are shown in Figure 2 as a function of the electron energy χ_o. The characteristic collision frequency ν_H is numerically $\nu_H \cong 3.1 \times 10^8 \times N_{20} \times H_{12}^{-3/2} \sec^{-1}$, where $N_{20} = N/10^{20} \text{cm}^{-3}$,

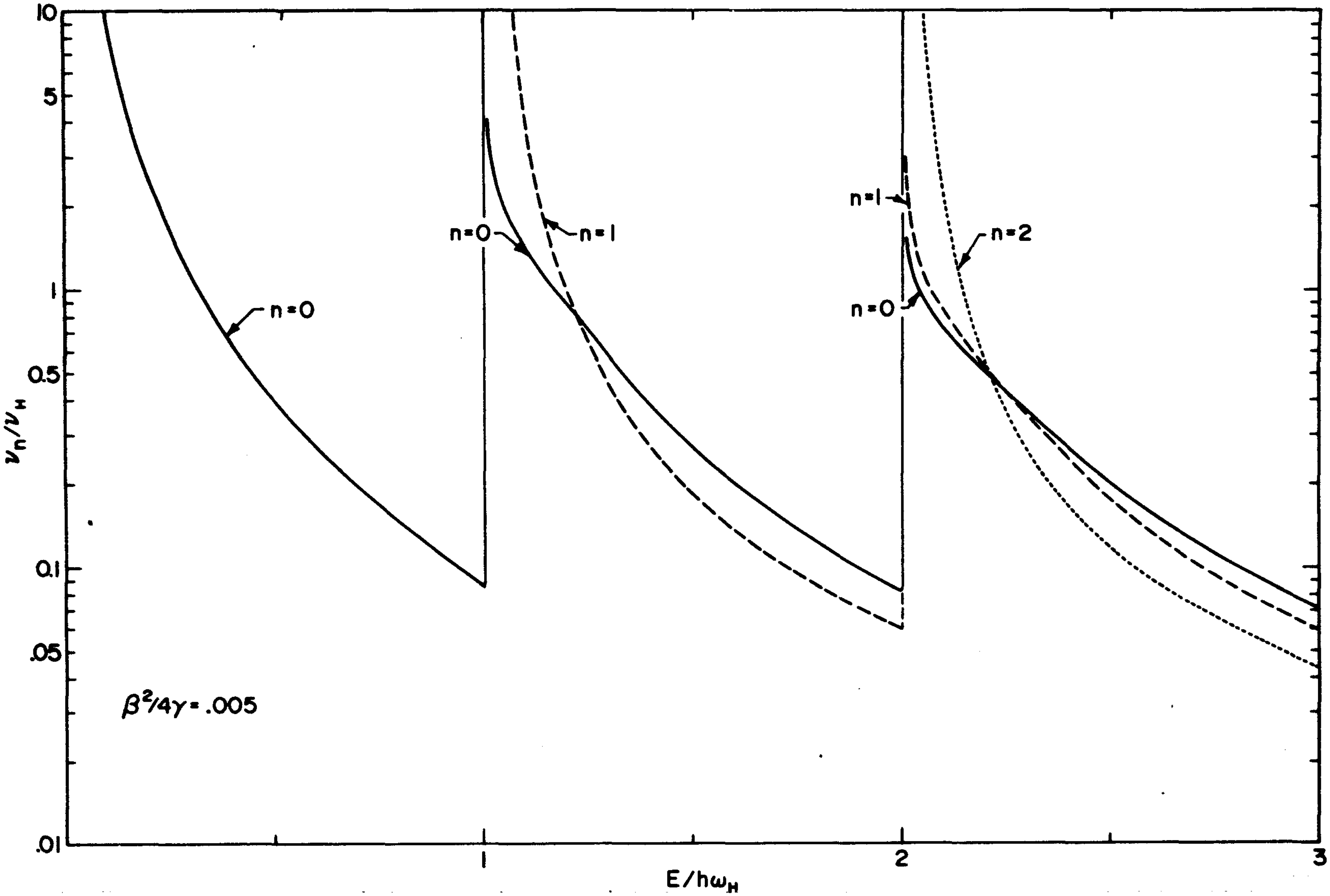

Figure 2. The collision frequency ν_n as a function of the electron energy. The dependence on the screening parameter $\beta^2/4\gamma$ is weak providing that $\beta^2/4\gamma \ll 1$.

$$\sigma_{n'n}^{(\pm)} = \sigma_H \, (\chi_{n'} \chi_n)^{-1/2} \, W_{n'n}(\xi^{\pm})$$

$$\nu_0(\chi_0) = \nu_H \sum_n \sum_{\pm} (1 \mp \eta_n) \, \chi_n^{-1/2} \, W_{n0}(\xi^{\pm}) \qquad (23)$$

$$\sigma_H = \frac{\pi}{4} \left(\frac{Z}{a_0 \gamma} \right)^2 , \qquad \nu_H = \frac{\pi}{2} \, \frac{Z^2 e^2}{a_0 \hbar} N_i \gamma^{-3/2}$$

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$H_{12} = H/10^{12}$ Gauss. The dependence of η_n on the screening distance is weak so long as the parameter $\beta^2/4\gamma << 1$. This parameter is equal to the ratio of the Larmor radius over the screening distance, and is indeed expected to be small in the limit of H considered. No mention of the electron's spin was made until this point. This has been possible due to the spin independence of the Coulomb potential. Figure 2 gives the collision frequency for electrons with spin antiparallel to H. ν_n for electrons with the opposite spin orientation may be found from the same graph, if we change the abcissa to read $(E - \hbar\omega_H)/\hbar\omega_H$.

A comparison of the quantities η_n to the ratio k_n/k_o is also indicated in Table 2, and has the following significance. The case $\eta_n \to k_n/k_o$, according to equation (15), represents a situation where the collision frequency is the same in any Landau level for a given value of E, i.e. $\nu_n(E) = \nu_o(E)$. This symmetry property resembles the situation, known to occur in isotropic plasmas. In the IMF this situation arises when $\sigma^{(+)}_{n'n} = \sigma^{(-)}_{n'n}$, i.e., the cross section displays a forward to backward symmetry. The matrix $A_{n'n}$ defined in equation (13a) becomes diagonal in this case, and the equations for f^1_n become decoupled. In the isotropic case a similar decoupling (of the different directions in

$E/\hbar\omega_H$	η_1	k_1/k_0	$E/\hbar\omega_H$	η_1	η_2	k_1/k_0	k_2/k_0
1.05	.028	.218	2.05	.565	.015	.716	.156
1.1	.111	.302	2.1	.621	.072	.724	.218
1.2	.373	.408	2.2	.714	.283	.739	.302
1.3	.604	.480	2.3	.797	.500	.752	.361
1.4	.750	.535	2.4	.856	.657	.764	.408
1.5	.837	.577	2.5	.896	.760	.775	.447
1.6	.889	.612	2.6	.923	.826	.784	.480
1.7	.921	.642	2.7	.941	.870	.793	.509
1.8	.942	.667	2.8	.954	.901	.802	.535
1.9	.956	.688	2.9	.963	.922	.809	.557
2.0	.966	.707	3.0	.970	.938	.816	.577

Table 2. Values of the ratios $\eta_n = f_n^1 / f_o^1$ as a function of the electron energy.

the momentum p) occurs, and can be traced to the symmetry of the cross section $\sigma_{\vec{p} \rightarrow \vec{p}'}$ under the rotation of the vector $\vec{p}'$ by 180° about the $\vec{p}$ direction (azimuthal rotation). The resulting collision frequency $\nu(p)$ depends on the magnitude of p but not on its direction.[8]

8. Limitations of the Born Approximation

The Born type approximation used to calculate the matrix elements in the last two sections has consisted of substituting the free particle form exp(ikz) to replace the z- dependence of the scattering state. From general considerations one expects to obtain accurate results in this approximation, when the kinetic energy of the electron is large compared to the effective one-dimensional potential "seen" by the particle as it moves along the magnetic lines. For electrons in the low lying Landau levels this effective potential is of the order $Ze^2 \gamma^{1/2}(s+1/2)^{-1/2}$. In view of these qualitative considerations it has often been stated[9,10] that the Born approximation is expected to improve at high values of n,s, and k.

Here we wish to add some remarks motivated from the one-dimensional picture of scattering developed in Section 2, and in particular from the constraint given by Equation (8). A brief examination of the matrix element is enough to convince us that this inequality is violated for sufficiently small values of the quantities k',k, and ξ. The divergence at $k' = 0$ can be understood as a typical resonance behavior, since the kinetic energy obtains the value required to excite one of the higher Landau levels, and the propagator becomes infinite at this value of k', cf. Eq. (4). k = 0 corresponds to bound states in the incident channel. These atomic states in the IMF have recently received attention in connection to the opacity and cooling in neutron stars.[15] Finally the quantity ξ is minimized for zero momentum transfer ($K = k'-k = 0$) which corresponds to the transmission process $nsk \rightarrow nsk$. The Born matrix element diverges in this case logarithmically (for weak screening)

$$V_{nn}^{ss}(\xi) \underset{s\xi \to 0}{\cong} -\left(\frac{n!}{s!}\right)^{1/2}\left[\log \xi + \psi(s+1)\right] .$$

Here $\psi(s) \equiv \Gamma'(s)/\Gamma(s)$ is the logarithmic derivative of the gamma function. This difficulty was recognized long ago, and ad hoc assumptions have often been made to cut off the infinite matrix elements in the computation of total transition probabilities.[9,10] It is also worth noting that the problem persists even at large values of n,s and k, when the Born approximation is expected to be accurate.

An improved representation of the scattering state must be used in order to understand the source of this difficulty.

Writing the wavefunction in a separable form[16] $\psi_{nsk}^{(+)} = \phi_{ns}(\rho,\theta) f_k(z)$ we find for the matrix element in question,

$$\langle nsk|T|nsk\rangle = \int_{-\infty}^{\infty} dz\, e^{-ikz}\, V_{ns}(z)\, f_k(z) \ ,$$
$$V_{ns}(z) = \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\theta\, |\phi_{ns}(\rho,\theta)|^2\, V(\vec{r}) \ . \qquad (24)$$

Here $V_{ns}(z)$ is the effective potential in the z-direction. The Born matrix element is just the integrated potential which can become arbitrarily large as was pointed out above. However, even a slight deviation of $f_k(z)$ from its Born value can drastically alter this situation. This point can be illustrated by making use of the eikonal approximation.[17]

Following the standard procedure[18] we write $f_k(z) = \exp\left[i(kz + \phi(z))\right]$, where $\phi(z)$ is to be determined by substitution into the one dimensional Schrödinger equation

$$\left[\frac{d^2}{dz^2} + k^2 + U(z)\right] f_k(z) = 0 \ , \qquad (25)$$

where $U(z) = -2m\hbar^{-2} V_{ns}(z)$. The resulting equation for $\phi(z)$ is simplified by making the standard assumption of smooth behavior in z, whereby the second derivative of ϕ, as well as, the second power of the first derivative can be neglected. The result is

$$\phi(z) = (2k)^{-1} \int_{-\infty}^{z} dz' \; U(z') .$$

the corresponding expression for the transmission matrix element is easily obtained by examining the limit of $f_k(z)$ at $z \to \infty$, or performing the integral in equation (21). In either case one finds

$$\langle nsk | T | nsk \rangle = -i\hbar k m^{-1} (1 - e^{i\phi(\infty)}) \tag{26}$$

where $\phi(\infty) = (2k)^{-1} \int_{-\infty}^{\infty} dz \; U(z)$. The transmission probability is accordingly equal to one. What becomes infinite here is not the matrix element but the phase $\phi(\infty)$. By expanding the exponential in the above equation and retaining terms up to first order in $\phi(\infty)$ we recover the Born matrix element. This expansion is of course not legitimate in our case since $\phi(\infty)$ diverges. The source of our previous difficulties with the Born approximation thus becomes clear. We further note that the eikonal approximation predicts no reflection. Clearly, it is not a sensitive tool for calculating backward scattering, which alone contributes to the electrical resistivity in the lowest Landau level, as discussed in Section 6.

For a more complete investigation one can alternately ask for the solution of the one-dimensional Schrödinger equation (25). This problem has been treated in the literature, mostly in relation to the bound state solutions mentioned earlier.[15] Since no exact solutions can be found for the effective potentials $V_{ns}(z)$, a variety of numerical and analytical approaches have been developed to handle the problem approximately. These methods can easily be extended to study the continuum problem of scattering. Calculations on this line, already in progress, indicate that in the limit $k^2 \ll \gamma$ the transmission probability goes to zero and the wave is almost totally reflected. This comes in sharp contrast to the qualitative features predicted by either the Born or the Eikonal approach. Further work on this line is needed in order to determine the behaviour of the matrix element in the intermediate region of k^2.

9. Discussion

It is of some interest to review our formulation of the collision frequency in the light of previous work. A serious attempt to understand long wavelength features of magnetized quantum plasmas in the framework of the Boltzmann-Vlasov equation has been presented several years ago by Kelly.[19] The inclusion of binary collisions in this very general approach, however, has proved to be no trivial task.[20] The discussion of the collision term, presented in Section 4 may thus be viewed as a step in resolving this difficulty. Admittedly, we have limited our discussion to problems involving only longitudinal external forces, for which it was possible to account for the effect of electron-ion collisions in a simple, as well as rigorous way. The natural extension of this work will be to include the effect of transverse external forces, not discussed here. The results already obtained should affect, at least quantitatively, some of the transport properties of plasmas in strong magnetic fields.

APPENDIX A. EVALUATION OF THE COULOMB MATRIX ELEMENT

We summarize here the mathematics of evaluating the integral in equation (16). The function $I_{ns}(t)$ is defined as[4]

$$I_{ns}(t) = (n!\,s!)^{-1/2}\, e^{-t/2}\, t^{(n-s)/2}\, Q_s^{n-s}(t)$$

$$Q_s^{n-s}(t) = e^{t}\, t^{s-n} \frac{\partial^s}{\partial t^s}\left(t^n e^{-t}\right) \qquad \text{(A-1)}$$

Q_s is the Laguerre polynomial and the normalization is such that

$$\int_0^\infty dt\, \left|I_{ns}(t)\right|^2 = 1$$

The following symmetry property will be repeatedly needed below

$$I_{ns}(t) = (-)^{n-s}\, I_{sn}(t) \qquad \text{(A-2)}$$

Consider now the matrix element in equation (16) for the case $n < s$. Using (A1) and (A2) we find

$$\mathcal{V}_{nn'}^{ss'} \equiv (n!\,s!\,n'!\,s'!)^{1/2}\, V_{nn'}^{ss'} = 2\int_0^\infty dt\; Q_n^{\ell}(t)\, K_0(2\sqrt{\xi t})\, \partial_t^{n'}\left(t^{s'} e^{-t}\right)$$

with $\ell = s-n = s'-n'$. After one partial integration, and making use of the identity $\partial_t Q_n^{\ell} = -n\, Q_{n-1}^{\ell+1}$ we find

$$U_{nn'}^{ss'} = n\, U_{n-1,n'-1}^{s\,s'} - 2\int_0^{\infty} dt\, Q_n^{\ell}(t)\, \partial_t K_0\, \partial_t^{n'-1}(t^{s'} e^{-t})$$

Using the definition of the Laguerre polynomial and remembering that $\ell = s-n = s'-n'$ we may write $Q_n^{\ell}\ \partial_t^{n'-1}(t^{s'} e^{-t}) = t\, Q_{n'-1}^{\ell+1}\ \partial_t^{n}(t^{s}\ e^{-t})$ and the above equation takes the form

$$U_{nn'}^{ss'} = n\, U_{n-1,n'-1}^{s\,s'} - 2\int_0^{\infty} dt\, Q_{n'-1}^{\ell+1}\ t\,\partial_t K_0 \partial_t^{n}(t^{s} e^{-t}) \qquad \text{(A-3)}$$

After one more partial integration, and making use of the identity $\partial_t(t\ \partial_t K_0) = \xi\ K_0$, we establish the recurrence formula

$$U_{nn'}^{ss'} = (n+n'-1+\xi)\, U_{n-1,n'-1}^{s\,s'} - (n-1)(n'-1)\, U_{n-2,n'-2}^{s\,s'} \qquad \text{(A-4)}$$

for $n' = 1$ the second term in this equation vanishes so long as $s' > 0$.

In addition to this formula, which links matrix elements of different orders, it is possible to write an explicit analytic expression for the matrix element $V_{n0}^{ss'}$, as follows:

$$(n!\,s!\,s'!)^{1/2}\, V_{n0}^{ss'} = 2\int_0^\infty dt\; K_0(2\sqrt{t\xi})\, \partial_t^n (t^s e^{-t})$$

$$= 2\,\xi^{n/2} \int_0^\infty dt\; t^{s-n/2}\, e^{-t}\, K_n(2\sqrt{\xi t}) \qquad \text{(A-5)}$$

here s' = s-n and the last form was obtained after n partial integrations, and with the help of the identity

$$\partial_t^n K_0(2\sqrt{t\xi}) = (-)^n \xi^{n/2} t^{-n/2} K_n(2\sqrt{t\xi})$$

The last integral in equation (A5) is recognized as one of the known integral representations of the confluent hypergeometric function of the second kind[12], giving the result

$$U_{n0}^{ss'}(\xi) = s!s'!\,\xi^n\, \Psi(s+1, s-s'+1, \xi)$$

Using equation (A4) we can now generate the remaining matrix elements. The result is:

$$U_{nn'}^{ss'}(\xi) = s!s'!\, Q_n^{n'-n}(-\xi)\, \Psi(s+1, s-s'+1, \xi) \qquad \text{(A-6)}$$

where $s > n$, $s-n = s'-n'$. The validity of equation (A6) becomes apparent, as soon as one notes that equation (A4) is independent of the indices s', s, and actually coincides with the recursion formula for the associated Laguerre polynomial $Q_n^{n'-n}(-\xi)$.

APPENDIX B: Electron Propagator.

To prove equation (4) one expands the propagator in a complete set of eigenstates obtaining

$$\sum_{ns} \phi_{ns}(\rho', \theta') \phi^{*}(\rho, \theta) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(z'-z)}}{E - E_{nk} + i\eta}$$

with $E_{nk} = \hbar\,\omega_H(n+1/2) + \hbar^2 k^2/2m$. The k integral can now be performed by complex integration around the poles at $k^2 = k_n^2 + i\,\eta$, where we have introduced the quantity $k_n^2 = (2m/\hbar^2)\left[E - \hbar\,\omega_H(n + 1/2)\right]$. The result of the integration is $(m/i\hbar^2 k_n) \exp(ik_n |z-z'|)$, and equation (4) is proved.

APPENDIX C. SCATTERING CROSS SECTION

We visualize an electron beam of uniform density in the incoming channel nk, as discussed in Section 3. The flux of the incident beam was found to be $(\gamma/\pi)\,\hbar\,km^{-1}$. Electrons in this beam will be scattered into other channels n'k' due to the presence of the potential V. The probabilities for such transitions were obtained in Section 2.

The total current being scattered into the quantum state n's'k' is computed as follows:

$$J_{n's'k'} = \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\theta\, \left(\psi^{scat}_{n's'k'}\right)^* \frac{\hbar}{im} \frac{\partial}{\partial z} \psi^{scat}_{n's'k'}$$

$$= \frac{\hbar k'}{m} \left| A_{n's'}(k') \right|^2 \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\theta \left| \phi_{n's}(\rho\theta) \right|^2$$

$$= \frac{\hbar k'}{m} \left| A_{n's}(k') \right|^2 .$$

$\psi^{scat}_{n's'k'}$ represents the scattered wave :

$$\psi^{scat}_{n's'k'} \equiv A_{n's'}(k')\ \phi_{n's'}(\rho\theta)\ e^{ik'z}$$

Use was also made of the normalization condition for the function ϕ_{ns}

$$\int_0^\infty \rho\, d\rho \int_0^{2\pi} d\theta \; | \phi_{ns} |^2 = 1 \quad .$$

We thus conclude that the total current flowing into the channel n'k' is

$$J_{n'k'} = \sum_{s's} J_{n's'k'} = \frac{\hbar k'}{m} \sum_{s's} | A_{n's'}(k') |^2$$

The corresponding cross section is therefore found to be

$$\sigma_{nk \to n'k'} = \frac{J_{n'k'}}{(\gamma/\pi)\hbar k m^{-1}} = \frac{\pi}{\gamma} \left| \frac{k'}{k} \right| \sum_{s's} | A_{n's'}(k') |^2 \quad .$$

APPENDIX D. FUNDAMENTAL JUSTIFICATION OF THE KINETIC EQUATION

We here wish to present an addendum in justification of the kinetic equation used in Section 4.

The application of the methods of quantum statistics (the grand partition function for example) establishes for the magnetized electron gas the following state of thermodynamic equilibrium:

$$f_n(k) = \left(e^{-E_{nk}/kT} + 1 \right)^{-1} \tag{D-1}$$

It is only natural then, to attempt to describe small deviations from the thermodynamic equilibrium situation by means of a statistical distribution function $f_n(z,k,t)$, which will describe the evolution of the electron gas after a perturbation has been switched on. The function $f_n(z,k,t)$ will still be interpreted as the number density of electrons in the state n, k at time t after the perturbation was switched on. The presence of only the coordinate z suggests that only longitudinal perturbations are being considered.

It is then possible to write a Boltzman-type equation for $f_n(z,k,t)$ to study the evolution of the electron gas under the influence of an external longitudinal force, as was done in Section 4.

A long debate, started by Wigner in 1932, has existed in the theory of quantum plasmas to determine the circumstances under which a quantum gas can adequately be described by a kinetic equation. Wigner's own method starts with the description of the system in quantum statistics by means of the density matrix, and introducing the so-called Wigner function as

$$f(\vec{x},\vec{p}) = (\pi h)^{-3} \int d^3y \; e^{2i(\vec{p}\cdot\vec{y})/\hbar} \rho(\vec{x}-\vec{y}, \vec{x}+\vec{y})$$

where $\rho(\vec{x}', \vec{x}'')$ is the density matrix. The equation for $f(x,p,t)$ is then determined by Fourier transforming the equation of motion of the ensemble density matrix, $\rho(x', x'',t)$. In the classical limit $\hbar \rightarrow 0$ this equation becomes identical with the Boltzman equation. An assessment of this effort has been given by Mori, Oppenheim and Ross.[22]

Most of the above studies have dealt with isotropic plasmas (H=0). A discussion of the Wigner approach in the presence of a strong magnetic field has been given by Kelly.[19] Here we will consider a simpler approach based on the one-dimensional Schrodinger equation discussed previously (Section IV8).

The equation for f_n can now be found by taking the Fourier transform of Eq. (3). The following integrals result:

$$(A) = \int_{-\infty}^{\infty} \frac{d\lambda}{\pi\hbar} \frac{\partial \rho}{\partial t} e^{2ik\lambda} = \frac{\partial f_n}{\partial t}$$

$$B = \frac{1}{i\hbar} \int_{-\infty}^{\infty} \frac{d\lambda}{\pi\hbar} \frac{\hbar^2}{2m} e^{2ik\lambda} \left(\partial_{z''}^2 - \partial_{z'}^2 \right) \rho_n (z-\lambda, z+\lambda, t)$$

where $z' = z-\lambda$; $z'' = z+\lambda$. It is easy to see that

$$\partial_{z''}^2 - \partial_{z'}^2 = (\partial_{z''} - \partial_{z'})(\partial_{z''} + \partial_{z'}) = \partial_\lambda \partial_z .$$

We thus find

$$B = \frac{1}{im} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{2ik\lambda} \partial_\lambda \partial_z \rho_n (z-\lambda; z+\lambda, t)$$

$$= -\frac{\hbar k}{m} \partial_z f_n (z k t)$$

where we have made use of a partial integration and have taken into account that the contribution of the boundaries is zero.

In the presence of an external longitudinal force given by the potential function W(z,t) the one-dimensional Schrodinger equation takes the form

$$i\hbar \frac{\partial F_n}{\partial t} = \mathcal{H}_n(z)\, F_n(z,t) =$$

$$= \left[-\frac{\hbar^2}{2m} \partial_z^{\,2} + W(z,t) \right] F_n(z,t)$$

where F_n is the appropriate wavefunction.

The corresponding density matrix, defined as

$$\rho_n(z',z'',t) \equiv F_n^{*}(z',t)\, F_n(z,t) \tag{D-2}$$

satisfies the equation

$$\frac{\partial \rho_n(z',z'',t)}{\partial t} = \frac{1}{i\hbar} \left[\mathcal{H}_n(z') - \mathcal{H}_n(z'') \right] \rho_n(z',z'',t) \tag{D-3}$$

We now define the Wigner distribution function as

$$f_n(z,k,t) = 2\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi\hbar}\, e^{i2k\lambda} \rho_n(z-\lambda, z+\lambda, t) \tag{D-4}$$

The equation for f_n can now be found by taking the Fourier transform of Eq. (3). The following integrals result:

$$(A) = \int_{-\infty}^{\infty} \frac{d\lambda}{\pi\hbar} \frac{\partial \rho_n}{\partial t} e^{2ik\lambda} = \frac{\partial f_n}{\partial t}$$

$$(B) = \frac{1}{i\hbar} \int_{-\infty}^{\infty} \frac{d\lambda}{\pi\hbar} \frac{\hbar^2}{2m} e^{2ik\lambda} \left(\partial_{z''}^2 - \partial_{z'}^2 \right) \rho_n(z-\lambda, z+\lambda, t)$$

where z' = z−λ ; z" = z+λ . It is easy to see that

$$\partial_{z''}^2 - \partial_{z'}^2 = (\partial_{z''} - \partial_{z'})(\partial_{z''} + \partial_{z'}) = \partial_\lambda \partial_z$$

We thus find

$$(B) = \frac{1}{im} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{2ik\lambda} \partial_\lambda \partial_z \rho_n(z-\lambda, z+\lambda, t)$$

$$= - \frac{\hbar k}{m} \partial_z f_n(z, k, t) \quad .$$

where we have made use of a partial integration and have taken into account that the contribution of the boundaries is zero.

$$(c) = \frac{1}{i\hbar}\int_{-\infty}^{\infty}\frac{d\lambda}{\pi\hbar}\, e^{2ik\lambda}\Big(W(z-\lambda) - W(z+\lambda)\Big)\,\rho(z-\lambda, z+\lambda, t)$$

$$= \frac{1}{\hbar}\,\frac{\partial W}{\partial z}\int_{-\infty}^{\infty}\frac{d\lambda}{\pi\hbar}\,\frac{\partial e^{2ik\lambda}}{\partial k}\,\rho(z-\lambda, z+\lambda, t)$$

$$= \frac{\partial W}{\partial z}\,\frac{\partial f_n(z\,k\,t)}{\partial \hbar k} = -F\cdot\frac{\partial f}{\partial \hbar k}$$

where $F = -\dfrac{\partial W}{\partial z}$ is the force.

Collecting the three terms together we finally obtain the equation

$$\frac{\partial f_n}{\partial t} + \frac{\hbar k}{m}\,\frac{\partial f_n}{\partial z} + \frac{1}{\hbar}\,F\cdot\frac{\partial f_n}{\partial k} = 0\,, \qquad \text{(D-5)}$$

which precisely coincides with the kinetic equation.

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