An unstable variant of the Morava Change of Rings theorem for $K(n)$ theory

by

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Abstract

An unstable variant of the Morava Change of Rings theorem for $K(n)$ theory

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Advisor: Robert Thompson

We formulate a very general criteria for a base change comonads for Ext computations. We then use this criteria to prove a generalized version of the Morava change of rings theorem from stable homotopy theory.

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Contents

Chapter 1

Introduction

The aim of the present work is twofold: to give a foundational account of comonadic techniques in unstable homotopy theory; and to apply this foundation to prove an unstable variant of the celebrated Morava change of rings theorem from stable homotopy theory.

Comonads have traditionally been used to define resolving objects, and hence derived functors, in both abelian and non-abelian settings. In recent work of Bousfield (see [4]), it is shown that a comonad G defines a model structure on the category of cosimplicial objects over the category of coalgebras over G. In this model structure, the traditional "cobar" resolutions that have been used to compute derived functors are suitable for computing Quillen derived functors over this model structure.

One can then inquire as to the effect of morphisms of comonads on these model structures. Under suitable hypothesis, a morphism of comonads will induce a functor on the categories of cosimplicial coalgebras over the respective comonads that maps resolving objects to resolving objects. This is the content of our "generalized Shapiro's lemma".

The Morava change of rings theorem states that an Ext computation over the category of comodules over Brown and Peterson theory can be reduced to a computation in a simpler, but purely algebraic, theory called "$\Sigma(n)$". We generalize this theorem from the category of comodules over BP theory to the category of unstable comodules over BP theory. We then define an unstable version of $\Sigma(n)$ and use the generalized Shapiro's lemma to prove that the change of rings isomorphism exists in a certain range of dimensions.

Chapter 2

The theory of comonads

2.1 Comonads and coalgebras

We give an overview of the theory of comonads. Of particular interest are maps of comonads and the functors that they induce. We use the terminology of [8], and note that much of what is written here can be found in [11], [7], and [8].

Definition 2.1.1. A comonad is a quadruple $(\mathcal{C}, G, \Delta, \epsilon)$ where $\mathcal{C}$ is a category, G is an endofunctor on $\mathcal{C}$, $\epsilon : G \to 1_{\mathcal{C}}$ and $\Delta : G \to G^2$ are natural transformations, and the triple (G, Δ, ϵ) is a comonoid object in the category of endofunctors over $\mathcal{C}$. In other words, the following diagrams commute:

$$\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[d, "\Delta"] & G^2 \arrow[d, "\Delta_G"] & G \arrow[r, "\Delta"] \arrow[dr, "1"] & G^2 \arrow[d, "\epsilon_G"] & G \arrow[r, "\Delta"] \arrow[dr, "1"] & G^2 \arrow[d, "G\epsilon"] \\
G^2 \arrow[r, "G(\Delta)"] & G^3 & & G & & G
\end{tikzcd}$$

Example 2.1.1. Let C be a counital coalgebra over a base ring R. Then there is a comonad on the category of R-modules given by $M \mapsto C \otimes_R M$. The comultiplication and counit is induced by the coalgebra structure of C.

Definition 2.1.2. A counital coalgebra (X, ψ_X) over a comonad G consists of an object X in $\mathcal{C}$ and a morphism $\psi_X : X \to G(X)$ such that the following diagrams commute:

$$\begin{tikzcd}
X \arrow[r, "\psi_X"] \arrow[d, "\psi_X"] & G(X) \arrow[d, "G(\psi_X)"] \\
G(X) \arrow[r, "\Delta_X"] & G(G(X))
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\psi_X"] \arrow[dr, "1_X"] & G(X) \arrow[d, "\epsilon_X"] \\
 & X
\end{tikzcd}$$

A morphism $X \to Y$ between coalgebras over a comonad G is a map $f : X \to Y$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\psi_X"] & Y \arrow[d, "\psi_Y"] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd}$$

Example 2.1.2. A counital coalgebra over the comonad defined by a coalgebra C (as above) is a left C-comodule.

Definition 2.1.3. The category of Eilenberg-Moore coalgebras over a comonad G has as objects counital coalgebras over G and morphisms as defined above. It is denoted by $\mathcal{C}^G$.

If G is a comonad on $\mathcal{C}$, then there is a functor $\mathcal{C} \to \mathcal{C}^G$ defined on objects by $X \mapsto (GX, \Delta_X)$. This is right adjoint to the forgetful functor $\mathcal{C}^G \to \mathcal{C}$.

There is another natural algebraic category that arises from a comonad. It models the "cofree objects" over G.

Definition 2.1.4. The Kleisli category over G, denoted $\mathcal{C}_G$, has the same objects as $\mathcal{C}$, and $\mathcal{C}_G(X,Y) = \mathcal{C}(GX,Y)$. The composite of two morphisms $f : GX \to Y$ and $g : GY \to Z$ is defined to be

$$\begin{tikzcd} GX \arrow[r, "\Delta_X"] & G^2X \arrow[r, "Gf"] & GY \arrow[r, "g"] & Z \end{tikzcd}$$

The identity of an object X in $\mathcal{C}_G$ is given by the counit $\epsilon_X : GX \to Y$.

The associativity and identity axioms for a category are easily verified, and are standard exercises in, e.g., [7]. There is also a functor $\mathcal{C} \to \mathcal{C}_G$ given by $X \mapsto X$ and $(X \xrightarrow{f} Y) \mapsto (GX \xrightarrow{Gf} GY \xrightarrow{\epsilon_Y} Y)$. This is right adjoint to the functor $\mathcal{C}_G \to \mathcal{C}$ defined by $X \mapsto GX$ and $(GX \xrightarrow{f} Y) \mapsto (GX \xrightarrow{\Delta_X} G^2X \xrightarrow{Gf} GY)$.

That the Kleisli category models the "cofree objects" over G is made explicit by the following standard proposition.

Proposition 2.1.1. *There are adjoint functors $i : \mathcal{C}_G \to \mathcal{C}^G$ and $j : \mathcal{C}^G \to \mathcal{C}_G$, with j left adjoint to i, defined by*

- $i(X) = (GX, \Delta_X)$
- $i(GX \xrightarrow{f} Y) = GX \xrightarrow{\Delta_X} G^2X \xrightarrow{Gf} GY$
- $j(X, \psi_X) = X$
- $j(X \xrightarrow{f} Y) =$ $$\begin{tikzcd} GX \arrow[r, "Gf"] \arrow[d, "\epsilon_X"] & GY \arrow[d, "\epsilon_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Proof. That $i(f)$ is a morphism of coalgebras $(X, \Delta_X) \to (Y, \Delta_Y)$ is clear from the definition. That i respects composites is given by the following diagram:

$$\begin{tikzcd} GX \arrow[r, "\Delta_X"] \arrow[d, "\Delta_X"] & G^2X \arrow[r, "Gf"] \arrow[d, "\Delta_{GX}"] & GY \arrow[d, "\Delta_Y"] & \\ G^2X \arrow[r, "G\Delta_X"] & G^3X \arrow[r, "G^2f"] & G^2Y \arrow[r, "Gg"] & GZ \end{tikzcd}$$

The upper path is $i(g) \circ i(f)$ while the lower path is $i(g \circ f)$.

The following diagram demonstrates that j respects compositions:

$$\begin{tikzcd}
GX \arrow[r, "\Delta_X"] & G^2X \arrow[r, "G^2f"] \arrow[d, "G\epsilon_X"] & G^2Y \arrow[d, "G\epsilon_Y"] & & \\
& GX \arrow[r, "Gf"] & GY \arrow[r, "Gg"] & GZ \arrow[r, "\epsilon_Z"] & Z
\end{tikzcd}$$

The top path is equal to $j(g) \circ j(f)$ while the bottom is equal to $j(g \circ f)$.

The adjunction isomorphism $\alpha_{X,Y} : \mathcal{C}_G(j(X), Y) \simeq \mathcal{C}^G(X, i(Y))$ is given by

$$\alpha_{X,Y}(\, GX \xrightarrow{f} Y \,) = \; X \xrightarrow{\psi_X} GX \xrightarrow{\Delta_X} G^2X \xrightarrow{Gf} GY$$

In other words, the unit $1_{\mathcal{C}^G} \to i \circ j$ is given by ψ_-, the coaction of the objects in $\mathcal{C}^G$. The counit $j \circ i \to 1_{\mathcal{C}_G}$ is ϵ, the counit of G.

□

2.2 Morphisms of comonads and induced functors

There are two kinds of morphisms between comonads that are immediately interesting.

Definition 2.2.1. A functor of comonads $(F, \alpha) : (\mathcal{C}, G) \to (\mathcal{D}, H)$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\alpha : F \circ G \to H \circ F$ such that

1. $\Delta^H \circ \alpha = (H * \alpha) \circ (\alpha * G) \circ (F * \Delta^G)$
2. $(\epsilon^H * F) \circ \alpha = F * \epsilon^G$.

This definition is motivated by the following proposition.

Proposition 2.2.1. *A functor of comonads* $(F, \alpha) : (\mathcal{C}, G) \to (\mathcal{D}, H)$ *induces a functor* $\alpha_* : \mathcal{C}^G \to \mathcal{D}^H$*. Moreover, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{C}^G \arrow[r, "\alpha_*"] \arrow[d] & \mathcal{D}^H \arrow[d] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

Proof. Given a G coalgbra (X, ψ_X) define its image under α_* to be $(FX, \alpha_X \circ F(\psi_X))$. This is a H coalgebra if the following outer diagram commutes:

$$\begin{tikzcd}
FX \arrow[r, "F(\psi_X)"] \arrow[d, "F(\psi_X)"] & FG(X) \arrow[r, "\alpha_X"] \arrow[d, "FG(\psi_X)"] & HF(X) \arrow[d, "HF(\psi_X)"] \\
FG(X) \arrow[r, "F(\Delta)"] \arrow[d, "\alpha_X"] & FG^2(X) \arrow[r, "\alpha_{G(X)}"] & HFG(X) \arrow[d, "H(\alpha_X)"] \\
HF(X) \arrow[rr, "\Delta"] & & H^2F(X)
\end{tikzcd}$$

The upper left diagram commutes since (X, ψ_X) is a G-coalgebra. The upper right diagram commutes since α is a natural transformation. Finally, the commutativity of the lower box is precisely the first condition on α stated in definition 2.2.1. It remains to check that $(FX, \alpha_X \circ F(\psi_X))$ is counital. Consider the following diagram:

$$\begin{tikzcd}
FX \arrow[r, "\psi_X"] \arrow[dr, "1_FX"] & FGX \arrow[r, "\alpha_X"] \arrow[d, "F(\epsilon^G_X)"] & HFX \arrow[dd, "\epsilon^H_{FX}"] \\
& FX \arrow[dr, "1_{FX}"] & \\
& & FX
\end{tikzcd}$$

The left triangle commutes since (X, ψ_X) is counital, and the right trapezoid commutes by the second condition on α as stated in 2.2.1.

If $f : (X, \psi_X) \to (X', \psi_{X'})$ is a morphism of G-coalgebras, then $F(f) : (FX, \alpha_X \circ F\psi_X) \to (FX', \alpha'_X \circ F\psi_{X'})$ is a morphism of H-coalgebras:

$$\begin{tikzcd}
FX \arrow[r, "f"] \arrow[d, "F\psi_X"] & FX' \arrow[d, "F\psi_{X'}"] \\
FGX \arrow[r, "FGf"] \arrow[d, "\alpha_X"] & FGX' \arrow[d, "\alpha_{X'}"] \\
HFX \arrow[r, "HFf"] & HFX'
\end{tikzcd}$$

The top square commutes since f is a morphism of G-coalgebras. The bottom square commutes since α is natural.

$\square$

There is a dual notion with respect to the Kleisli categories.

Definition 2.2.2. An opfunctor of comonads $(U,\beta) : (\mathcal{D},H) \to (\mathcal{C},G)$ consists of a functor $U : \mathcal{D} \to \mathcal{C}$ together with a natural transformation $\beta : GU \to UH$ such that

1. $(U * \Delta^H) \circ \beta = (\beta * H) \circ (G * \beta) \circ (\Delta^G * U)$
2. $(U * \epsilon^H) \circ \beta = (\epsilon^G * U)$.

Proposition 2.2.2. *A comonad opfunctor* (U,β) *induces a functor* $\beta_* : \mathcal{D}_H \to \mathcal{C}_G$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{C}_G & \mathcal{D}_H \arrow[l, "\beta_*"'] \\
\mathcal{C} \arrow[u] & \mathcal{D} \arrow[l, "U"] \arrow[u]
\end{tikzcd}$$

Proof. Define $\beta_*(Y) = (GUY, \Delta^G_{UY})$. If $f : HY \to Y'$ is a map in $\mathcal{D}_H$, then define $\beta_*(f) = GUY \xrightarrow{\beta_Y} UHY \xrightarrow{Uf} UY' \in \mathcal{C}_G$. We must verify that β respects composites and identities. The latter is given by:

$$\begin{tikzcd}
GUY \arrow[r, "\beta_Y"] \arrow[dr, "\epsilon^G_{UY}"] & UHY \arrow[d, "U\epsilon^H_Y"] \\
& UY
\end{tikzcd}$$

This diagram commutes by definition of comonad opfunctor.

Now suppose $f : HY \to Y'$ and $g : HY' \to Y''$ are composable arrows in $\mathcal{D}_H$.

$$\begin{tikzcd}
GUY \arrow[r, "\Delta^G_{UY}"] \arrow[d, "\beta_Y"] & G^2UY \arrow[r, "G\beta_Y"] & GUHY \arrow[r, "GUf"] \arrow[d, "\beta_{HY}"] & GUY' \arrow[d, "\beta_{Y'}"] & \\
UHY \arrow[rr, "U\Delta^H_Y"] & & UH^2Y \arrow[r, "UHf"] & UHY' \arrow[r, "Ug"] & UY''
\end{tikzcd}$$

The commutativity of the leftmost rectangle is exactly condition (1) of the definition of opfunctor of comonads. The upper path is $\beta_* g \circ \beta_* f$, while the lower path is $\beta_*(g \circ f)$.

$\square$

We have proven that β_* exists between Kleisli categories. However, we are particularly interested in β_* as a functor of the subcategories of cofree objects in $\mathcal{D}^H$, the full Eilenberg-Moore category. We examine the composite

$$\mathcal{D}^H \xrightarrow{j} \mathcal{D}_H \xrightarrow{\beta_*} \mathcal{C}_H \xrightarrow{i} \mathcal{C}^H$$

By definition,

- $(i \circ \beta_* \circ j)(Y, \psi_Y) = (GUY, \Delta^G_{UY})$
- $(i \circ \beta_* \circ j)(\, Y \xrightarrow{f} Y' \,) = GUf \circ GU\epsilon^H_Y \circ G\beta_Y \circ \Delta^G_{UY} = GUf$

As to be expected, the coaction on any Y is ignored by this functor.

Suppose that one is interested in extending β_* to the entire category of coalgebras over H. Let (Y, ψ_Y) by an arbitrary H-coalgebra. Consider the following split equalizer (the first two stages of the cobar resolution):

$$\begin{tikzcd} Y \arrow[r, "\psi_Y"] & HY \arrow[r, shift left, "H(\psi_Y)"] \arrow[r, shift right, "\Delta_Y"'] & H^2Y \end{tikzcd}$$

We can "drag" part of this diagram over via $G \circ U$ and β:

$$\begin{tikzcd} GUY \arrow[r, shift left, "GU(\psi_Y)"] \arrow[r, shift right, "G\beta_Y \circ \Delta^G_{UY}"'] & GUHY \end{tikzcd}$$

If the equalizer of the above diagram exists in $\mathcal{C}$, then we have a decent candidate for $\beta_*(Y, \psi_Y)$.

Proposition 2.2.3. *Let (U, β) be an opfunctor of comonads $(\mathcal{D}, H) \to (\mathcal{C}, G)$ such that for all H-coalgebras (Y, ψ_Y), the following pair has an equalizer in $\mathcal{C}$:*

$$\begin{tikzcd} GUY \arrow[r, shift left, "GU(\psi_Y)"] \arrow[r, shift right, "G\beta_Y \circ \Delta^G_{UY}"'] & GUHY \end{tikzcd}$$

Then $\beta_* : \mathcal{D}_H \to \mathcal{C}_G$ *can be extended to a functor* $\beta_* : \mathcal{D}^H \to \mathcal{D}^G$.

Proof. We first need to demonstrate that the equalizer of the above pair has a G-coaction. We note that there is a morphism of diagrams:

$$\begin{tikzcd}
\beta_* Y \arrow[r] & GUY \arrow[r, shift left, "GU(\psi_Y)"] \arrow[r, shift right, "\beta_Y \circ \Delta^G_{UY}"'] \arrow[d, "\Delta^G_{UY}"] & GUHY \arrow[d, "\Delta^G_{UHY}"] \\
G(\beta_* Y) \arrow[r] & G^2UY \arrow[r, shift left, "G^2U(\psi_Y)"] \arrow[r, shift right, "G\beta_Y \circ G\Delta^G_{UY}"'] & G^2UHY
\end{tikzcd}$$

This induces a map $\psi_{\beta_* Y} : \beta_* Y \to G(\beta_* Y)$. Coassociativity and counitalness follow from the the respective properties on the morphisms of diagrams.

□

2.3 Induced adjunctions

Suppose that $F : \mathcal{C} \longleftrightarrow \mathcal{D} : U$ is an adjoint pair of functors. Furthermore, suppose that G and H are comonads over $\mathcal{C}$ and $\mathcal{D}$, respectively. We are interested in when this adjunction “lifts” to an adjunction between $\mathcal{C}^G$ and $\mathcal{D}^H$. The following is essentially a paraphrase of some results in [8] and [11].

Proposition 2.3.1. *Suppose* $(\mathcal{C}, G)$ *and* $(\mathcal{D}, H)$ *are comonads,* $(F, \alpha) : (\mathcal{C}, G) \to (\mathcal{D}, H)$ *is a comonad functor, and* F *has a right adjoint* $U : \mathcal{D} \to \mathcal{C}$*. Then there exists* $\beta : GU \to UH$ *such that* $(U, \beta) : (\mathcal{D}, H) \to (\mathcal{C}, G)$ *is a comonad opfunctor.*

Proof. We define β as follows:

$$\beta_Y = GUY \xrightarrow{\sigma_{GUY}} UFGUY \xrightarrow{U(\alpha_{UY})} UHFUY \xrightarrow{UH(\tau_Y)} UHY$$

where σ and τ are the unit and counit of the adjunction between F and U (we are avoiding the usual notation since ϵ is already taken). We now verify that (U, β) is a comonad opfunctor.

We first derive a fundamental relation governing $\tau, \alpha, \beta, F, U, G$, and H.

Proposition 2.3.2. *As defined above,*

$$(\tau * H) \circ (F * \beta) = (H * \tau) \circ (\alpha * U)$$

Proof.

$$\begin{tikzcd}
FGUY \arrow[r, "F\sigma_{GUY}"] \arrow[dr, "1"] & FUFGUY \arrow[r, "FU(\alpha_{UY})"] \arrow[d, "\tau_{FGUY}"] & FUHFUY \arrow[r, "FUH(\tau_Y)"] \arrow[d, "\tau_{HFUY}"] & FUHY \arrow[d, "\tau_{HY}"] \\
& FGUY \arrow[r, "\alpha_{UY}"] & HFUY \arrow[r, "H\tau_Y"] & HY
\end{tikzcd}$$

The top path is the left side of the equation, and the bottom path is the right hand side. □

Armed with this relation, we will now demonstrate that β satisfies the relations defining a comonad opfunctor.

First, we need to verify that

$$(\beta * H) \circ (G * \beta) \circ (\Delta^G * U) = (U * \Delta^H) \circ \beta$$

The strategy is to adjoint both sides over and verify that they are equal in $\mathcal{D}$. We begin with the adjoint of the left hand side:

$$\begin{array}{rl} & (\tau * H^2) \circ (F * \beta * H) \circ (FG * \beta) \circ (F * \Delta^G * U) \\ = & (H * \tau * H) \circ (\alpha * UH) \circ (FG * \beta) \circ (F * \Delta^G * U) \\ = & (H * \tau * H) \circ (HF * \beta) \circ (\alpha * GU) \circ (F * \Delta^G * U) \\ = & (H^2 * \tau) \circ (H * \alpha * U) \circ (\alpha * GU) \circ (F\Delta^G * U) \\ = & (H^2 * \tau) \circ (\Delta^H * FU) \circ (\alpha * U) \\ = & \Delta^H \circ (H * \tau) \circ (\alpha * U) \\ = & \Delta^H \circ (\tau * H) \circ (F * \beta) \\ = & (\tau * H^2) \circ (FU * \Delta^H) \circ (F * \beta) \end{array}$$

The bottom expression is the adjoint of $(U * \Delta^H) \circ \beta$.

We now verify the condition

$$(U * \epsilon^H) \circ \beta = (\epsilon^G * U)$$

This is given by the following diagram:

$$\begin{tikzcd}
GU \arrow[r, "\sigma_{GU}"] \arrow[d, "\epsilon^G_U"] & UFGU \arrow[r, "U\alpha_U"] \arrow[dr, "UF\epsilon^G_U"] & UHFU \arrow[r, "UH\tau"] \arrow[d, "U\epsilon^H_{FU}"] & UH \arrow[d, "U\epsilon^H"] \\
U \arrow[rr, "\sigma_U"] \arrow[rrr, bend right, "1"'] & & UFU \arrow[r, "U\tau"] & U
\end{tikzcd}$$

□

Proposition 2.3.3. *Suppose that F and U are an adjoint pair of functors between $\mathcal{C}$ and $\mathcal{D}$, G is a comonad on $\mathcal{C}$, and H is a comonad on $\mathcal{D}$. Furthermore, suppose there is an α such that (F, α) is a functor of comonads, and that the induced β extends to a functor $\mathcal{D}^H \to \mathcal{C}^G$.*

Under these hypotheses, α_* is left adjoint to β_*.

Proof. Let $\gamma : \mathcal{C}(-, U-) \to \mathcal{D}(F-, -)$ denote the adjunction isomorphism between F and U with unit σ and counit τ. We will proceed by defining lifts $\tilde{\sigma}$ and $\tilde{\tau}$ that define an adjunction $\tilde{\gamma}$.

We begin with the unit. Let (X, ψ_X) be a G-coalgebra. We need a map $\tilde{\gamma} : X \to \alpha^*(FX)$. Consider the diagram:

$$\begin{tikzcd}
\alpha^*(FX) \arrow[r] & GUFX \arrow[r, shift left, "GU(\alpha_X \circ F(\psi_X))"] \arrow[r, shift right, "G\beta_{FX} \circ \Delta^G_{UFX}"'] & GUHFX \\
X \arrow[r, "\psi_X"] & G(X) \arrow[u, "G(\sigma_X)"] &
\end{tikzcd}$$

It will be shown that $G(\sigma_X) \circ \psi_X$ equalizes the above pair, and thus factors through a unique map $\tilde{\sigma}_X : X \to \alpha^* FX$. Thus, we need to show that:

$$GU\alpha_X \circ GUF\psi_X \circ G(\sigma_X) \circ \psi_X = G(\beta_{FX}) \circ \Delta^G_{UFX} \circ G(\sigma_X) \circ \psi_X$$

We first note that $\Delta^{\mathcal{C}}$ is a natural transformation and that (X, ψ_X) is a G-coalgebra.

$$\begin{tikzcd}
X \arrow[r, "\psi_X"] \arrow[d, "\psi_X"] & GX \arrow[r, "G(\sigma_X)"] \arrow[d, "\Delta^G_X"] & GUFX \arrow[d, "\Delta^G_{UFX}"] & \\
GX \arrow[r, "G\psi_X"] & G^2X \arrow[r, "G^2(\sigma_X)"] & G^2UFX \arrow[r, "G\beta_{FX}"] & GUHFX
\end{tikzcd}$$

We now examine the bottom line of the above diagram, suppressing the outer application of G:

$$\begin{tikzcd}
X \arrow[r, "\psi_X"] & GX \arrow[r, "G(\sigma_X)"] & GUFX \arrow[r, "\beta_{FX}"] & UHFX
\end{tikzcd}$$

We now use the naturality of β and η; and the relation $(\beta * F) \circ (G * \sigma) = (U * \alpha) \circ (\sigma * G)$ as proven in proposition 2.3.2.

$$\begin{tikzcd}
X \arrow[r, "\psi_X"] \arrow[d, "\sigma_X"] & GX \arrow[r, "G(\sigma_X)"] \arrow[d, "\sigma_{GX}"] & GUFX \arrow[d, "\beta_{FX}"] \\
UFX \arrow[r, "UF\psi_X"] & UFGX \arrow[r, "U\alpha_X"] & UHFX
\end{tikzcd}$$

Therefore,

$$\begin{aligned}
& G(\beta_{FX} \circ G(\sigma_X) \circ \psi_X) \circ \psi_X \\
= \; & G(U\alpha_X \circ UF\psi_X \circ \sigma_X) \\
= \; & GU\alpha_X \circ GUF\psi_X \circ G(\sigma_X) \circ \psi_X
\end{aligned}$$

as required.

To see that $\tilde{\sigma}_X$ is a map of coalgebras, consider the diagram:

$$\begin{tikzcd}
 & G\alpha^*FX \arrow[rr] & & G^2UFX \\
\alpha^*(FX) \arrow[ur, "\psi_{\alpha^*FX}"] \arrow[rr] & & GUFX \arrow[ur, "\Delta_{UFX}"] & \\
 & G(X) \arrow[uu, "G(\tilde{\sigma}_X)"] \arrow[rr, "\Delta_X"] & & G^2(X) \arrow[uu, "G^2(\sigma_X)"'] \\
X \arrow[uu, "\tilde{\sigma}_X"] \arrow[ur, "\psi_X"] \arrow[rr, "\psi_X"] & & G(X) \arrow[uu, "G\sigma_X"] \arrow[ur, "\Delta_X"] &
\end{tikzcd}$$

The right hand side of the cube is a commutative square of morphisms of parallel pairs. Therefore, the induced maps on the left hand side commute.

We will now lift the counit τ. Let Y be am H-coalgebra. Then $\tilde{\tau}_Y$ is induced by:

$$\begin{tikzcd}
\alpha_*\alpha^*Y \arrow[r] & FGUY \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\alpha_{UY}"] & FGUHY \arrow[d, "\alpha_{UHY}"] \\
 & HFUY \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "H\tau_Y"] & HFUHY \arrow[d, "H\tau_{HY}"] \\
Y \arrow[r, "\psi_Y"] & HY \arrow[r, shift left, "H\psi_Y"] \arrow[r, shift right, "\Delta^H_Y"'] & H^2Y
\end{tikzcd}$$

We will now verify that $(\alpha_* * \tilde{\sigma}) \circ (\tilde{\tau} * \alpha_*) = 1$. To this end, let X be a G-coalgebra. Consider the following diagram:

$$\begin{tikzcd}
FX \arrow[r, "F\psi_X"] \arrow[d, "F\tilde{\sigma}_X"] & FGX \arrow[d, "FG\sigma_X"] \\
F\alpha^*\alpha_*X \arrow[r] & FGUFX \arrow[d, "\alpha_{UFX}"] \\
 & HFUFX \arrow[d, "H\tau_{FY}"] \\
FX \arrow[r, "\alpha_X \circ F\psi_X"] & HFX
\end{tikzcd}$$

We would like to identify the vertical composite $H\tau_{FY} \circ \alpha_{UFX} \circ FG(\sigma_X)$ with α_X, thereby showing that the induced map $FX \to FX$ is the identity. To this end, consider the following diagram:

$$\begin{tikzcd}
FGX \arrow[r, "FG\sigma_X"] \arrow[d, "\alpha_X"] & FGUFX \arrow[d, "\alpha_U FX"] & \\
HFX \arrow[r, "HF\sigma_X"] & HFUFX \arrow[r, "H\tau_{FY}"] & HFX
\end{tikzcd}$$

This diagram commutes by naturality of α, and the composite of the bottom is equal to 1_{HFX} sicne σ and τ are the unit and counit, respectively, of an adjunction.

$\square$

2.4 Derived Functors

The construction of derived functors with respect to a monad or comonad originally appeared in [1]. Here, we will give an overview of the construction for the category of G-coalgebras for a given comonad G. Our perspective will be in light of recent work by Bousfield.

2.4.1 The induced monad on the category of coalgebras

Given a comonad G over a category $\mathcal{C}$, consider the category $\mathcal{C}^G$ of Eilenberg-Moore coalgebras over G. The comonad G factors into a pair of adjoint functors (see [7]):

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "G"] & G\text{-coalg} \arrow[l, shift left, "U"]
\end{tikzcd}$$

Here, we confuse the comonad G with the right adjoint $\mathcal{C} \to G$-coalg. In fact, the image of X under the right adjoint is simply $(G(X), \Delta_X)$, the cofree coalgebra over X.

It is worth noting the unit and counit of the adjunction. The unit, $\sigma : 1_{G-\text{coalg}} \to GU$ is simply $\sigma_X = \psi_X$, as the following diagram commutes by definition:

$$\begin{tikzcd}
X \arrow[r, "\psi_X"] \arrow[d, "\psi_X"] & G(X) \arrow[d, "\Delta_X"] \\
G(X) \arrow[r, "G(\psi_X)"] & G^2(X)
\end{tikzcd}$$

Thus, σ_X is a morphism of G-coalgebras.

The counit, $\tau : GU \to 1_{\mathcal{C}}$, is simply the counit of G considered as a comonad. That is, $\tau_Y = \epsilon_Y$ for all objects Y in $\mathcal{C}$.

By forming the composite $T = G \circ U : G\text{-coalg} \to G\text{-coalg}$, we obtain a monad on the category of coalgebras over G. We refer to [7] for a proof of this. It will be advantageous, however, to note the structure maps for T. The multiplication $\mu : T^2 \to T$ is given by

$$\mu = G * \tau * U$$

Of course, in this case, τ is merely ϵ, the counit of the comonad G. Hence, at a G-coalgebra (X, ψ_X), the multiplication on T is simply:

$$G^2X \xrightarrow{G\epsilon_X} G(X)$$

We quickly verify that this is a map of G-coalgebras:

$$\begin{tikzcd}
G^2X \arrow[r, "G\epsilon_X"] \arrow[d, "\Delta_{GX}"] & G(X) \arrow[d, "\Delta_X"] \\
G^3X \arrow[r, "G^2(\epsilon_X)"] & G^2X
\end{tikzcd}$$

This commutes since Δ is a natural transformation.

Of course, a monad T also comes with a unit map $\nu : 1 \to T$. In this case, it is given by $\nu_X = \psi_X : X \to G(X)$, the unit of the induced adjunction of the comonad G.

2.4.2 Cosimplicial resolutions

We begin with a familiar notion.

Definition 2.4.1. A cosimplicial object in a category $\mathcal{C}$ is a functor $X^\bullet : \underline{\Delta} \to \mathcal{C}$, where $\underline{\Delta}$ is the category of finite ordinals and monotonic maps.

The category of cosimplicial objects over a category $\mathcal{C}$ shall be denoted $c\mathcal{C}$.

We will be primarily interested in cosimplicial objects over the category of G-coalgebras for the purpose of computing derived functors.

Definition 2.4.2. Let T be the monad induced on the category of Eilenberg-Moore coalgebras over a comonad G. Given a coalgebra (X, ψ_X), we define a cosimplicial object $T^\bullet(X)$ as follows:

$$\begin{aligned} T^\bullet(X)^n &= T^{n+1}(X) \\ d^i : T^\bullet(X)^n \to T^\bullet(X)^{n+1} &= T^i \nu_{T^{n-i+1}(X)} \\ s^j : T^\bullet(X)^n \to T^\bullet(X)^{n-1} &= T^j \mu_{T^{n-j+1}(X)} \end{aligned}$$

where $\nu : 1 \to T$ and $\mu : T^2 \to T$ are the structure maps for T. The proof that $T^\bullet(X)$ is actually a cosimplicial object can be found in [7], and only depends on the relations between μ and ν as monad structure maps.

Of course, its a bit silly to use extra notation when all of this can be stated in terms of the structure of the comonad G. We will immediately specialize:

$$\begin{aligned} G^\bullet(X)^n &= G^{n+1}(X) \\ d^i : G^\bullet(X)^n \to G^\bullet(X)^{n+1} &= G^i \psi_{G^{n-i+1}(X)} \\ s^j : G^\bullet(X)^n \to G^\bullet(X)^{n-1} &= G^j \epsilon_{G^{n-j+1}(X)} \end{aligned}$$

Henceforth, we shall not bother with the extra notation (T, μ, ν) for the induced monad on $\mathcal{C}^G$.

It is standard practice to then define derived functors via these cosimplicial resolutions. Before doing this, however, we will give an account of work by Bousfield that shows that these dervied functors give “the right answer” with respect to a model structure on cosimplicial coalgebras.

2.4.3 The model structure on cosimplicial coalgebras over a comonad

In [4], Bousfield shows that, given a left-proper simplicial model category $\mathcal{M}$ and a class $\mathcal{G}$ of group objects in $Ho(\mathcal{M})$ such that any object in $\mathcal{M}$ can be "resolved" by an object in $\mathcal{G}$, there is a model structure on the category of cosimplicial objects over $\mathcal{M}$ in which the weak equivalences are "$\mathcal{G}$-equivalences". We shall give an account of a special case of this construction for the case of a discrete model category with a monad, which is covered in chapter 7 of [4]. For basic material on model categories, see [6].

Let G be a comonad on $\mathcal{C}$. We also use G to denote the induced monad on $\mathcal{C}^G$.

Proposition 2.4.1 (The discrete model structure). *If $\mathcal{C}^G$ has all finite limits and colimits, then $\mathcal{C}^G$ is endowed with the discrete model structure:*

- *Weak equivelances are isomorphisms*
- *Every map is a fibration and a cofibration*

Proof. The first axiom of the model category definition is that the category be complete and cocomplete. The rest of the axioms follow trivially. □

Since every map is a cofibration, $\mathcal{C}^G$ is also left proper (the pushout of a cofibration along a weak equivalence is a cofibration). We now with to use the objects $G(X)$ as "resolving objects". If $G(X)$ is a group object for all G-coalgebras X, then we can apply Bousfield's machinery.

Theorem 2.4.1. *Under the hypotheses that*

- $\mathcal{C}^G$ *has finite limits and colimits*
- $G(X)$ *is a group object for all* X *in* $\mathcal{C}^G$

There exists a $\mathcal{G}$*-resolution model structure on the category of cosimplicial objects over* $\mathcal{C}^G$ *(in the sense of [4]) where a map* $f : X^\bullet \to Y^\bullet$ *is*

- *a weak equivalence if the induced*

  $$Hom_G(Y^\bullet, G(Z)) \to Hom_G(X^\bullet, G(Z))$$

  is a weak equivalence of simplicial groups for all $G(Z)$*,*
- *a cofibration if* f *is a Reedy cofibration and the induced*

  $$Hom_G(Y^\bullet, G(Z)) \to Hom_G(X^\bullet, G(Z))$$

  is a fibration of simplicial groups,
- *a fibration when* $f : X^n \to Y^n \times_{M^n Y^\bullet} M^n X^\bullet$ *is a* $\mathcal{G}$*-injective fibration.*

Proof. This is a restatement of results in sections 3 and 7 of [4]. □

In general, it can be rather difficult to identify the fibrant objects in a model category. However, in section 6 of [4], Bousfield defines the notion of a weak resolution of an object, which in the case of the model stucture induced by a monad, coincides with the traditional way of making resolutions, e.g. in [3]. This is summarized by the following:

Proposition 2.4.2. *For a general model category $\mathcal{M}$ with a class of $\mathcal{G}$-injectives, a weak $\mathcal{G}$-resolution of an object $A \in \mathcal{M}$ consists of a $\mathcal{G}$-equivalence $cst(A) \to Y^\bullet$ in $c\mathcal{M}$. such that Y^n is $\mathcal{G}$-injective for $n \geq 0$. Such a $Y^\bullet$ is called termwise $\mathcal{G}$-injective .*

The condition of being termwise $\mathcal{G}$-injective is evidently weaker than being $\mathcal{G}$-fibrant. The upshot of this is the following:

Proposition 2.4.3. *If $A \to Y^\bullet \in \mathcal{M}$ is a weak $\mathcal{G}$-resolution for an object $A \in \mathcal{M}$, and $F : \mathcal{M} \to \mathcal{A}$ is a functor to an abelian category that carries weak equivalences to isomorphisms, then there is a natural isomorphisms*

$$\mathcal{R}^s_{\mathcal{G}} F(A) \simeq \pi^s F Y^\bullet$$

for $s \geq 0$.

Proof. Again, this is from section 6 of [4]

$\square$

Applying these propositions with $\mathcal{M} = G$-coalg and the induced monad $G : G$-coalg $\to G$-coalg, we have the following:

Proposition 2.4.4. *Let $F : G$-coalg $\to \mathcal{A}$ be a functor to an abelian category. Then the right derived functors of F, evaluated at a coalgebra X, are given by*

$$\mathcal{R}^s_{\mathcal{G}} F(X) \simeq \pi^s F \mathcal{G}^\bullet X$$

Proof. We will show that $X \to \mathbf{G}^\bullet X$ is a weak resolution in the $\mathcal{G}$-resolution model structure defined by G. We note that $\mathbf{G}^\bullet$ is termwise $\mathcal{G}$-injective by construction. We will now show that $cst(X) \to \mathbf{G}^\bullet X$ is a $\mathcal{G}$-equivalence.

An arbitrary $\mathcal{G}$-injective object is a split subobject of $G(I)$ for some I. We will show that for all $G(I)$,

$$d^0 : Hom_G(\mathbf{G}^\bullet X, G(I)) \to Hom_G(cst(X), G(I))$$

is a weak equivalence of simplicial sets. Since the target is constant, we think of this map as an augmentation of $Hom_G(\mathbf{G}^\bullet X, G(I))$ (hence the naming).

Let $K = Hom_G(\mathbf{G}^\bullet X, G(I))$ and $K_{-1} = Hom_G(cst(X), G(I))$. The map $d^0 : K_0 \to K_{-1}$ is an augmentation operator, and we define left contraction maps $s_{-1} : K_n \to K_{n+1}$ by $s_{-1}(f) = G(\epsilon_I) \circ G(f)$. We must verify the identities for a left contraction, namely that $d_0 s_{-1} = 1$, $d_{i+1} s_{-1} = s_{-1} d_i$ for $i \geq 0$ and $s_{j+1} s_{-1} = s_{-1} s_j$.

□

2.5 Criteria for Change of Ext

2.5.1 The Classical Shapiro's Lemma

Here, we recount the statement and proof of Shapiro's lemma as in [9].

Proposition 2.5.1 (Shapiro). *Let $\pi : (A, \Gamma) \to (B, \Sigma)$ be a morphism of Hopf algebroids. Let Γ be A-flat and assume that $\Gamma \otimes_A B$ is injective as a right Σ-comodule. Then*

$$Ext^*_\Gamma(A, \pi^* N) \simeq Ext^*_\Sigma(B, N)$$

naturally in the Σ-comodule N, in such a way that for a Γ-comodules M the following diagram commutes

$$\begin{tikzcd}
 & Ext^*_\Gamma(A, M) \arrow[dl] \arrow[dr] & \\
Ext^*_\Gamma(A, \pi^*\pi_* M) \arrow[rr] & & Ext^*_\Sigma(B, \pi_* M)
\end{tikzcd}$$

Proof. We will argue directly using the cobar complex for N. To this end, let $N \to \Sigma^\bullet \otimes N$ denote the cobar complex over Σ. We apply the functor $\pi^* = (\Gamma \otimes_A B)\Box_\Sigma -$, observing that for any right Σ-comodule M, $M\Box_\Sigma \Sigma \otimes N = M \otimes_B N$.

$$\begin{tikzcd}
\pi^* N \arrow[r] & \Gamma \otimes_A N \arrow[r] & \Gamma \otimes_A \Sigma \otimes_B N \arrow[r] & \cdots
\end{tikzcd}$$

We need to show that homming with any $\Gamma \otimes_A I$ gives an acyclic complex. Let $K = Hom_\Gamma(\Gamma \otimes_A \Sigma^{\bullet -1} \otimes_B, \Gamma \otimes I)$, which, by adjointness, is $Hom_A(\Gamma \otimes_A \Sigma^{\bullet -1} \otimes_B, I)$. Let $s^k_{-1} = 1_\Gamma \otimes 1^{k+1}_\Sigma \otimes \epsilon$ where $\epsilon : \Sigma \to 1$ is the counit of Σ. These are maps of A-modules, and hence induce a contraction s_{-1*} on K.

□

2.5.2 Generalized Shapiro's Lemma

To attempt a generalized version of Shapiro's Lemma, we must examine the structure of resolutions over coalgebra categories defined by comonads. As above, a comonad G defining the category of G-coalgebras gives rise to a monad G over that same category. Given a G-coalgebra M, a resolution of M looks like $M \to G^\bullet M$ where $G^\bullet M$ is the cosimplicial G-coalgebra gotten by iterating the monad G on M.

Let $(\mathcal{C}, G)$ and $(\mathcal{D}, H)$ as above. Consider the Bousfield $\mathcal{G}$-resolution model structures on cosimplicial G-coalgebras and cosimplicial H-coalgebras.

Proposition 2.5.2 (Generalized Shapiro's Lemma)**.** *Let* $(F, \alpha) : G \to H$ *be a morphism of comonads. Suppose that* α^* *has a right adjoint, and suppose that* α^**, considered as taking values in* $\mathcal{C}$*, is a retract of* $U \circ X$ *for some functor* $X : \mathcal{D} \to \mathcal{D}$*.*

Then

$$Ext(M, \alpha^* N) \simeq Ext(\alpha_* M, N)$$

For M *a* G*-coalgebra and* N *a* H*-coalgebra.*

Proof. Let $N \to H^\bullet N$ be the canonical resolution of N. Then,

$$Ext^i_H(\alpha_* M, N) = \mathcal{R}^i \mathcal{D}^H(\alpha_* M, N) \simeq \pi^i \mathcal{D}^H(\alpha_* M, H^\bullet N)$$

By the adjunction of α_* and α^*,

$$\pi^i \mathcal{D}^H(\alpha_* M, H^\bullet N) \simeq \pi^i \mathcal{C}^G(M, \alpha^* H^\bullet N)$$

We now wish to observe that $\alpha^* N \to \alpha^* H^\bullet N$ is a weak G-resolution of $\alpha^* Y$. By the hypothesis that, for all $X \in \mathcal{D}$, $\alpha^*(HX) = GJ$ for some $J \in \mathcal{C}$, we have that $\alpha^* H^\bullet$ is termwise G-injective. It remains to show that $\alpha^* N \to \alpha^* H^\bullet N$ is a G-equivalence.

Recall that $\alpha^* H^\bullet N$ is a split subobject of $UX(H^\bullet N)$. We then have $\mathcal{C}^G(\alpha^* H^\bullet N, GJ) = \mathcal{C}(\alpha^* H^\bullet N, J)$ is a retract of $\mathcal{C}(UX(H^\bullet N), J)$. However, $UX(\epsilon^{\mathcal{D}})^*$ is a contraction for this simplicial set, since ϵ^D is a contraction of $H^\bullet N$ considered as a cosimplicial object in $\mathcal{D}$. It follows that $\mathcal{C}^G(\alpha^* H^\bullet N, GJ)$ is contractible.

Hence, $\alpha^* N \to \alpha^* H^\bullet N$ is a weak G-resolution, which means that

$$\pi^i \mathcal{C}^G(M, \alpha^* H^\bullet N) \simeq R^i \mathcal{C}^G(M, \alpha^* N) = Ext^i_G(M, \alpha^* N)$$

and the thereom follows.

$\square$

Chapter 3

An unstable change of rings

3.1 The Stable Case

We now apply the collection of abstractions of the previous sections to derive the classical stable change of rings theorem.

3.1.1 Background of the stable change of rings theorem

There are several tools in stable homotopy for computing the stable homotopy groups of the spheres. Among these, there is a cohomology theory, called Brown-Peterson theory, or BP theory, and its associated mod-p Adams-Novikov spectral sequence:

$$Ext_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi_*^S(S)$$

There are various computational advantages of this spectral sequence, and a full account of it can be found in [10]. In particular, one can filter the E_2 term of the mod p Adams-Novikov spectral sequence by v_n periodicity, where the operators v_n are coefficients in BP theory. The resulting spectral sequence is called the *chromatic sprectral sequence*.

A key step in setting up the chromatic spectral sequence is the *Morava change of rings theorem*, which reduces a certain Ext computation over BP to an Ext computation over $K(n)$, which is an algebraically simpler theory.

3.1.2 The Hopf algebroid of stable cooperations in BP

This section is a summary of relevent facts that can be found in, e.g., [10] and [5].

Given any cohomology theory E with E_*E flat over E_*, there is an associated algebraic object (E_*, E_*E) called a Hopf algebroid.

Definition 3.1.1. A Hopf algebroid (A, Γ) is a cogroupoid object in the category of rings. That is, for any ring R, $Hom(A, R)$ and $Hom(A, \Gamma)$ form the set of objects and morphisms of a groupoid, respectively.

This implies, via the Yoneda lemma, that there exist maps:

$$\begin{aligned}
&\eta_L, \eta_R : A \to \Gamma \\
&\epsilon : \Gamma \to A \\
&\Delta : A \to \Gamma_{\eta_R} \otimes_{A\ \eta_L} \Gamma \\
&c : \Gamma \to \Gamma
\end{aligned}$$

such that the following diagrams commute:

In particular, the pair (BP_*, BP_*BP) of BP coefficients and BP cooperations forms a Hopf algebroid. Given a ring R, the pair

$$(Hom(BP_*, R), Hom(BP_*BP, R))$$

is the groupoid with objects the p-typical formal groups over R and morphisms strict isomorphisms of formal groups. As graded rings,

$$\begin{array}{l} BP_* \simeq \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \\ BP_*BP \simeq BP_*[h_1, h_2, \ldots] \end{array}$$

where $|v_i| = |h_i| = 2(p^i - 1)$.

There are various choices of generators that one can pick for these objects, and here we use the generators from [5] to facilitate the unstable computations later.

The unit map $\eta_L : BP_* \to BP_*BP$ is, by this choice of generators, the inclusion of algebras $BP_* \hookrightarrow BP_*BP$. We then define η_R in terms of η_L and the dual of the universal formal group law F over BP_*:

$$\sum^{F^*} v_j^{p^i} h_i = \sum^{F^*} h_j^{p^i} \cdot v_i$$

We adopt the convention that the left action of an element v in BP_* on an element of h in BP_*BP is denoted by vh, whereas the right action of v on h is denoted $v \cdot h$ or $h\eta_R(v)$.

In practice, one examines this infinite sum at specific degrees to obtain expressions for the right action of the v_i.

Let $K(n) = \mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ and $\Sigma(n) = K(n) \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)$. Then $(K(n), \Sigma(n))$ is a Hopf algebroid inheriting structure from (BP_*, BP_*BP).

3.1.3 The stable cotensor

Suppose (A, Γ) is a Hopf algebroid. The category of left Γ-comodules can be gotten as the category of Eilenberg-Moore coalgebras over the comonad $M \mapsto \Gamma \otimes_A M$, where M is an A-module. It is unfortunate that such "comodules" arise as "coalgebras" over a comonad, but this is simply one of those incongruences in notation that arise when two branches of mathematics collide.

Suppose $\pi : (A, \Gamma) \to (B, \Sigma)$ is a morphism of Hopf algebroids. Note that this induces an adjoint pair of functors

$$F : A\text{-mod} \leftrightarrow B\text{-mod} : U$$

where $F(M) = B \otimes_A M$ and U restricts scalars to A along π.

Define $\alpha_M : B \otimes_A (\Gamma \otimes_A M) \to \Sigma \otimes_B (B \otimes M)$ as the B-linear extension of $\pi \otimes_A 1_M : \Gamma \otimes_A M \to \Sigma \otimes_A M$.

Proposition 3.1.1. *As defined above, the pair (F, α) is a functor of comonads* $(\Gamma \otimes_A -) \longrightarrow (\Sigma \otimes_B -)$.

Proof. We must verify that the following diagram commutes for all M:

$$\begin{tikzcd}
B \otimes_A (\Gamma \otimes M) \arrow[r, "1 \otimes_A \Delta^\Gamma \otimes 1"] \arrow[dd, "\alpha_M"] & B \otimes_A (\Gamma \otimes \Gamma \otimes M) \arrow[dr, "\alpha \otimes 1 \otimes 1"] & \\
 & & \Sigma \otimes_B (B \otimes_A \Gamma \otimes_A M) \arrow[dl, "1 \otimes \alpha"] \\
\Sigma \otimes_B (B \otimes_A \Gamma) \arrow[r, "\Delta^\Sigma \otimes 1 \otimes 1"] & \Sigma \otimes \Sigma \otimes_B (B \otimes \Gamma) &
\end{tikzcd}$$

$\square$

This diagram amounts to the relation $(\pi \otimes \pi) \circ \Delta^\Gamma = \Delta^\Sigma \circ \pi$, which is holds because π is a morphism of Hopf algebroids. Similarly, α is compatible with the counits.

By proposition 2.3.1, we can recover a $\beta_N : \Gamma \otimes_A N \to \Sigma \otimes_B (B \otimes_A N)$ such that the pair (U, β) is an opfunctor of comonads. According to the formula, this is:

$$\Gamma \otimes_A N \simeq \Gamma \otimes_A A \otimes_A B \xrightarrow{\pi \otimes \pi \otimes 1} \Sigma \otimes_B B \otimes_A N$$

Since the category of A-modules has all equalizers, by proposition 2.3.3, there is right adjoint from Σ-comodules to Γ-comodules given by the following equalizer:

$$\begin{tikzcd}
\pi^*N \arrow[r] & (\Gamma \otimes B) \otimes N \arrow[r, shift left] \arrow[r, shift right] & \Gamma \otimes B \otimes \Sigma \otimes N
\end{tikzcd}$$

This is the cotensor product $(\Gamma \otimes_A B)\square_\Sigma N$ as in, e.g., [9].

3.1.4 Applying Shapiro's Lemma

In this section, we let

$$\begin{aligned}
&(A, \Gamma) = (BP_*, BP_*BP)\\
&(B, \Sigma) = (K(n)_*, \Sigma(n))\\
&B(n)_* = v_n^{-1}BP_*/I_n\\
&K(n)_*BP = K(n)_* \otimes_{BP_*} BP_*BP\\
&BP_*K(n) = BP_*BP \otimes_{BP_*} K(n)_*
\end{aligned}$$

There is an evident projection morphism of Hopf algebroids $\pi : (A, \Gamma) \to (B, \Sigma)$ which induces a functor $\pi_* : \Gamma\text{-comod} \to \Sigma\text{-comod}$. The right adjoint to this functor is $\pi^* = ((\Gamma \otimes_A B)\square_\Sigma -)$.

We must verify that π^* is a subobject of $U \circ X$ for some endofunctor X of B-modules. The following algebraic lemma establishes this.

Proposition 3.1.2. *There is a map*

$$K(n)_*BP \to \Sigma(n) \otimes_{K(n)_*} B(n)_*$$

which is an isomorphism of left Σ*-comodules and of* $B(n)_*$ *modules and carries* 1 *to* 1.

This is proven in [9]. If we apply the conjugation c to the above, we get an isomorphism of right Σ-comodules:

$$BP_*K(n) \simeq B(n)_* \otimes_{K(n)_*} \Sigma(n)$$

However, the right adjoint π^* is given by $\alpha^*N = BP_*K(n)\Box_\Sigma N$. Thus,

$$\begin{aligned}\pi^*(N) &= BP_*K(n)\Box_\Sigma N\\ &= B(n)_* \otimes_{K(n)_*} \Sigma\Box_\Sigma N\\ &= B(n)_* \otimes_{K(n)_*} N\end{aligned}$$

Thus, Shapiro's lemma applies, and we can conclude that

$$Ext_{BP_*BP}(M, \alpha^*N) \simeq Ext_{\Sigma(n)}(\pi^*M, N)$$

3.2 The Unstable Case

3.2.1 The category of unstable BP_*-comodules

Let M be a free module over BP_*. We define $U(M) \subseteq \Gamma \otimes_{BP_*} M$ to be generated by

$$\{h^I \otimes m | 2len(I) < deg(m)\}$$

where $\Gamma = BP_*BP$. There are maps $U \to U^2$ and $U \to 1$ induced by the Hopf algebroid structure of (BP_*, Γ) which form the structure of a cotriple over the category of free BP_*-modules.

Suppose M is an arbitrary (-1)-connected BP_*-module. We can extend U to M by considering a free resolution of M

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r, "p_0"] & F_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

and applying the functor U to the first two terms:

$$\begin{tikzcd} U(F_1) \arrow[r] & U(F_0) \end{tikzcd}$$

We define $U(M)$ to be the cokernel of $U(p_0)$. Suppose $g : M \to N$ is a morphism of BP_*-modules. Consider the following diagram:

$$\begin{tikzcd} U(F_1) \arrow[r] \arrow[d, "U(g_1)"] & U(F_0) \arrow[r] \arrow[d, "U(g_0)"] & U(M) \arrow[r] & 0 \\ U(F_1') \arrow[r] & U(F_0') \arrow[r] & U(N) \arrow[r] & 0 \end{tikzcd}$$

The vertical maps arise from lifting g to a map of resolutions of M and N. The desired $U(g)$ now arises from an application of the short 5 lemma.

Remark 3.2.0.1. An element free definition of the category of unstable BP_*-comodules can be found in [3]. The excess condition appears in [2].

3.2.2 The category of unstable $\Sigma(n)$-comodules

Consider the category of $K(n)_*$-modules. We will define a comonad over this category whose coalgebras are the unstable $\Sigma(n)$-comodules.

Definition 3.2.1. Let N be a $K(n)$-modules, then

$$U_\Sigma(N) = K(n)_* \otimes_{BP_*} U_\Gamma(N)$$

where N is regarded as a BP_*-module for the purpose of applying U_Γ. We define a comultiplication as follows.

$$\begin{tikzcd}
U_\Sigma(N) = K(n)_* \otimes_{BP_*} U_\Gamma(N) \arrow[r, "1\otimes\Delta_N^\Gamma"] \arrow[dr, "\Delta_N^\Sigma"] & K(n)_* \otimes_{BP_*} U_\Gamma^2(N) \arrow[d, "1\otimes U_\Gamma(\pi\otimes 1_{U_\Gamma(N)})"] \\
 & K(n)_* \otimes_{BP_*} U_\Gamma(K(n)_* \otimes_{BP_*} U_\Gamma(N))
\end{tikzcd}$$

The counit is given by

$$U_\Sigma(N) = K(n)_* \otimes_{BP_*} U_\Gamma(N) \xrightarrow{1\otimes\epsilon_N^\Gamma} K(n)_* \otimes_{BP_*} N = N$$

Theorem 3.2.1. *U_Σ is a comonad over the category of $K(n)_*$-modules.*

Proof. We must demonstrate that the coproduct as defined is coassociative. We examine the diagram:

$$\begin{tikzcd}
U_\Sigma(N) = K(n)_* \otimes U_\Gamma(N) \arrow[r, "1\otimes\Delta^\Gamma_N"] \arrow[d, "1\otimes\Delta^\Gamma_N"] & K(n)_* \otimes U^2_\Gamma(N) \arrow[r, "1\otimes U_\Gamma(\pi\otimes 1_{U_\Gamma(N)})"] & K(n)_* \otimes U_\Gamma(K(n)_* \otimes U_\Gamma(N)) = U^2_\Sigma(N) \arrow[d, "1\otimes\Delta^\Gamma_{K(n)_*\otimes U_\Gamma(N)}"] \\
K(n)_* \otimes U^2_\Gamma(N) \arrow[d, "1\otimes U_\Gamma(\pi\otimes 1_{U_\Gamma(N)})"] & & K(n)_* \otimes U^2_\Gamma(K(n)\otimes U_\Gamma(N)) \arrow[d, "1\otimes U_\Gamma(\pi\otimes 1)"] \\
K(n)_* \otimes U_\Gamma(K(n)_* \otimes U_\Gamma(N)) \arrow[dr, "1\otimes U_\Gamma(1\otimes\Delta^\Gamma_N)"] & & K(n)_* \otimes U_\Gamma(K(n)_* \otimes U_\Gamma(K(n)_* \otimes U_\Gamma(N))) = U^3_\Sigma(N) \\
& K(n)_* \otimes U_\Gamma(K(n)_* \otimes U^2_\Gamma(N)) \arrow[ur, "1\otimes U_\Gamma(1\otimes U_\Gamma(\pi\otimes 1_{U_\Gamma(N)}))"] &
\end{tikzcd}$$

That this diagram commutes is a matter of observing that all of the arrows consist of the comultiplication in BP_*BP and changing coefficients to $K(n)_*$. The order of these operations does not matter.

□

3.2.3 The morphism of comonads $U_\Gamma \to U_\Sigma$

We now define a morphism of comonads $U_\Gamma \to U_\Sigma$. This is given by the pair $(K(n) \otimes_{BP_*} -, \alpha)$ where

Definition 3.2.2. We define $\alpha : K(n)_* \otimes_{BP_*} U_\Gamma(-) \to U_\Sigma(K(n)_* \otimes_{BP_*} -)$ as

$$\alpha_M = K(n)_* \otimes U_\Gamma(M) \to K(n)_* \otimes U_\Gamma(K(n)_* \otimes_{BP_*} M)$$

for $M \in BP_*$-mod.

That is, $K(n)_* \otimes U_\Gamma(-)$ applied to the map $M \to K(n)_* \otimes_{BP_*} M$. We must verify that this satisfies the axioms for a morphism of comonads.

Theorem 3.2.2. *The pair* $(K(n)_* \otimes_{BP_*} -, \alpha)$ *is a morphism of comonads* $U_\Gamma \to U_\Sigma$.

Proof. We must check that $\Delta^\Sigma \circ \alpha = (U_\Sigma * \alpha) \circ (\alpha * U_\Gamma) \circ (F * \Delta^\Gamma)$. That is, for every BP_*-module M, the following diagram must commute:

$$\begin{tikzcd}
K(n)_* \otimes U_\Gamma(M) \arrow[r, "1\otimes\Delta^\Gamma_M"] \arrow[d, "\alpha_M"] & K(n)_* \otimes_{BP_*} U^2_\Gamma(M) \arrow[r, "\alpha_{U_\Gamma(M)}"] & U_\Sigma(K(n)_* \otimes U_\Gamma(M)) \arrow[d, "U_\Sigma(\alpha_M)"] \\
U_\Sigma(K(n)_* \otimes M) \arrow[rr, "\Delta^\Sigma_M"] & & U^2_\Sigma(K(n)_* \otimes M)
\end{tikzcd}$$

After applying all of the definitions, this diagram reduces to:

$$\begin{tikzcd}
U_\Sigma(M) \arrow[r, "1\otimes\Delta^\Gamma_M"] \arrow[d, "\alpha_M"] & K(n)_* \otimes_{BP_*} U^2_\Gamma(M) \arrow[r, "U_\Sigma(\pi\otimes 1)"] & U^2_\Sigma(M) \arrow[d, "U_\Sigma(\alpha_M))"] \\
U_\Sigma(K(n)_* \otimes M) \arrow[r, "1\otimes\Delta^\Gamma_{K(n)_*\otimes M}"] & K(n)_* \otimes U^2_\Gamma(K(n)_* \otimes M) \arrow[r, "1\otimes U_\Gamma(\pi\otimes 1)"] & U^2_\Sigma(K(n)_* \otimes M)
\end{tikzcd}$$

which evidently commutes.

We must now check that $\epsilon^\Sigma \circ \alpha = \pi_* * \epsilon^\Gamma$. This is expressed by the following diagram:

$$\begin{tikzcd}
K(n)_* \otimes U_\Gamma(M) \arrow[r, "\alpha_M"] \arrow[d, "\pi_*(\epsilon^\Gamma_M)"] & U_\Sigma(K(n)_* \otimes M) \arrow[d, "\epsilon^\Sigma_{\pi_*(M)}"] \\
K(n)_* \otimes M \arrow[r, "1"] & K(n)_* \otimes M
\end{tikzcd}$$

Applying the definitions, this reduces to:

$$\begin{tikzcd}
K(n)_* \otimes U_\Gamma(M) \arrow[r, "1\otimes U_\Gamma(\pi\otimes 1)"] \arrow[d, "1\otimes\epsilon^\Gamma_M"] & U_\Sigma(K(n)_* \otimes M) \arrow[d, "1\otimes\epsilon^\Gamma_{\pi_*(M)}"] \\
K(n)_* \otimes M \arrow[r, "1"] & K(n)_* \otimes M
\end{tikzcd}$$

which commutes, since the same coefficients are being killed in M in both paths. ☐

There is thus a functor $\alpha_* : U_\Gamma$-comod $\to U_\Sigma$-comod.

3.2.4 The unstable cotensor

We will now apply the criterion for the existence of an adjoint to α_*. Since, in the stable case, this is given by a certain cotensor product, it is not a stretch to think of this as an unstable variant of the cotensor product.

Theorem 3.2.3. *$\alpha_* : U_\Gamma$-comod $\to U_\Sigma$-comod has a right adjoint $\alpha^* : U_\Sigma$-comod $\to U_\Gamma$-comod.*

Proof. We will check the criterion for the existence of an adjoint to the functor induced by a morphism of comonads.

1. There is a right adjoint to $\pi_* = K(n)_* \otimes_{BP_*} -$. It is the forgetful functor $K(n)_*$-mod $\to BP_*$-mod. We do not bother to denote this, in general.

2. We now define a natural transformation $\beta : U_\Gamma \to U_\Sigma$.

$$\beta_N = \pi \otimes 1 : U_\Gamma(N) \to K(n)_* \otimes U_\Sigma(N)$$

3. We now check the compatibility condition. The unit of the adjunction between $K(n)_* \otimes -$ and the forgetful functor is given by $\pi \otimes 1 : M \to K(n)_* \otimes M$. The compatibility diagram is thus:

$$\begin{tikzcd}
U_\Gamma(M) \arrow[r, "U_\Gamma(\pi\otimes 1)"] \arrow[d, "\pi\otimes 1"] & U_\Gamma(K(n)_* \otimes M) \arrow[d, "\pi\otimes 1"] \\
K(n)_* \otimes U_\Gamma(M) \arrow[r, "1\otimes U_\Gamma(\pi\otimes 1)"] & U_\Sigma(K(n)_* \otimes M)
\end{tikzcd}$$

This diagram is easily seen to commute.

4. The equalizer of

$$\begin{tikzcd}
U_\Gamma(N) \arrow[r, shift left, "U_\Gamma(\psi_N)"] \arrow[r, shift right, "U_\Gamma(\beta_N)\circ\Delta_N^\Gamma"'] & U_\Gamma U_\Sigma(N)
\end{tikzcd}$$

exists since the category of BP_*-modules has all equalizers.

$\square$

Therefore, there is a right adjoint α^* that is given by the equalizer of the above diagram. It now remains to find the subcategory of unstable BP_* modules such that the pair of adjoints induces an equivalence of categories.

3.2.5 The counit of the adjunction

In the stable case, we have that for any $\Sigma(n)$-comodule N,

$$N \simeq K(n)_* \otimes_{BP_*} (\Gamma \otimes_{BP_*} K(n)_*)\Box_{\Sigma(n)} N$$

via the counit of the adjunction between π_* and π^*. One would expect this to happen in the unstable case as well.

Theorem 3.2.4. *For an arbitrary U_Σ-comodule (N, ψ_N),*

$$N \simeq K(n)_* \otimes \alpha^* N$$

as U_Σ-comodules

Proof. We first compute $\alpha^* N$ for an arbitrary U_Σ-comodule N. It is the equalizer of the following pair:

$$\begin{tikzcd} U_\Gamma(N) \arrow[r, shift left, "U_\Gamma(\psi_N)"] \arrow[r, shift right, "U_\Gamma(\beta_N)\circ\Delta_N^\Gamma"'] & U_\Gamma U_\Sigma(N) \end{tikzcd}$$

Note that $U_\Gamma(\beta_N) = U_\Gamma(\pi \otimes 1_{U_\Gamma(N)})$. Since tensoring by $K(n)_*$ is flat, equalizers are preserved, and $K(n)_* \otimes \alpha^* N$ is the equalizer of the above diagram tensored with $K(n)_*$:

$$\begin{tikzcd}
K(n)_* \otimes U_\Gamma(N) \arrow[r, shift left, "1\otimes U_\Gamma(\psi_N)"] \arrow[r, shift right, "1\otimes U_\Gamma(\pi\otimes 1)\circ 1\otimes\Delta^\Gamma_N"'] & K(n)_* \otimes U_\Gamma U_\Sigma(N)
\end{tikzcd}$$

If we recall the definition of U_Σ and Δ^Σ, the above is actually:

$$\begin{tikzcd}
U_\Sigma(N) \arrow[r, shift left, "U_\Sigma(\psi_N)"] \arrow[r, shift right, "\Delta^\Sigma_N"'] & U_\Sigma U_\Sigma(N)
\end{tikzcd}$$

The equalizer of the above pair is (N, ψ_N).

□

3.2.6 The unit of the adjunction

We now must determine the right subcategory of U_Γ-comodules for which the unit of the adjunction between α_* and α^* is an isomorphism.

We will proceed by "destabilizing" the algebraic argument found in [9]. Some notation will be required.

Definition 3.2.3. Let

- $B(n)\{\iota_k\} = v_n^{-1}BP_*/I_n\{\iota_k\}$
- $K(n)_*BP\{\iota_k\} = K(n)_* \otimes U_\Gamma(BP_*\{\iota_k\})$

Proposition 3.2.1. *Let $k \geq 2p^n + 1$. Then there is a map*

$$K(n)_*BP\{\iota_k\} \to U_\Sigma(K(n)_* \otimes_{BP_*} BP_*\{\iota_k\}) \otimes B(n)_*$$

which is an isomorphism of U_Σ-comodules and of $B(n)$-modules, and which carries 1 to 1.

Proof. The stable version of this theorem follows from a counting argument over the connective case. We thus introduce a "connective" version of U_Σ.

The formal group law in BP_*BP gives the following relation between the left and right actions:

$$\eta_R(v_{n+k}) \equiv h_k^{p^n} \cdot v_n - v_n^{p^k} h_k \bmod I_n \cup (\eta_R(v_{n+1}), \dots, \eta_R(v_{n+k-1}))$$

The idea is to divide out by v_n so that any element of $k(n)_*BP\{\iota_k\}$ can be expressed in terms of monomials in h^I with no power greater than or equal to p^n appearing. This would make $k(n)_*BP\{\iota_k\}$ free and of finite type. This is easily done stably, since $\eta_R(v_n) \equiv v_n \bmod I_n$.

Unstably, we have

$$\eta_R(v_{n+k} \otimes \iota_k) \equiv (h_k^{p^n} \cdot v_n - v_n^{p^k} h_k) \otimes \iota_k \bmod I_n \cup (\eta_R(v_{n+1}), \dots, \eta_R(v_{n+k-1}))$$

We examine the excess of $h_k^{p^n} \cdot v_n \otimes \iota_k = h_k^{p^n} \otimes v_n \iota_k$. The left hand side of the tensor is of length p^n and the right hand side of the tensor is of degree $k + (2p^n - 2)$. This means that $k - 2 > 0$, ie, that $k > 3$. Thus, this relation is valid on the 3-sphere and above.

Let

- $k(0)_* = \mathbb{Z}_{(p)} \subset K(0)_*$
- $k(n)_* = \mathbb{F}_p[v_n] \subset K(n)_*$
- $k(n)_*BP\{\iota_k\} = k(n)_* \otimes_{BP_*} BP_*BP \otimes BP_*/I_n\{\iota_k\}$
- $b(n)\{\iota_k\} = k(n)_*[u_1, u_2, \ldots]\{\iota_k\} \subset B(n)_*\{\iota_k\}$

where $u_l = \frac{v_{n+l}}{v_n}$. We want to make $k(n)_*BP\{\iota_k\}$ a $b(n)\{\iota_k\}$-module via η_R. This would entail defining the map as:

$$u_l \otimes \iota_k \mapsto (h_l^{p^n} - v_n^{p^l-1}h_l) \otimes \iota_k$$

However, we need $k > 2(p^n)$ for this relation to exist. We therefore define the above map for $k > 2(p^n)$.

Hence, $k(n)_*BP\{\iota_k\}$ is a right $b(n)\{\iota_k\}$-module for $k > 2(p^n)$. It is free and of finite type, generated by monomials of the form h^I where all $i_l < p^n$.

Mimicking the proof in the stable case, we now define

$$\sigma(n)_*\{\iota_k\} = k(n)_*BP\{\iota_k\} \otimes_{b(n)_*\{\iota_k\}} k(n)_*$$

The pair $(k(n)_*, \sigma(n)_*\{\iota_k\})$ is a Hopf algebroid with structure induced by BP_*BP.

We claim that $k(n)_*BP\{\iota_k\}$ is isomorphic to $\sigma(n)\{\iota_k\}\otimes_{k(n)_*\{\iota_k\}} b(n)_*\{\iota_k\}$. Define a $b(n)_*\{\iota_k\}$-linear map $f : k(n)_*BP\{\iota_k\} \to b(n)_*\{\iota_k\}$ by

$$f(h^I) = \begin{array}{l} 1 \text{ if } I = (0,0,0,\ldots) \\ 0 \text{ otherwise} \end{array}$$

By definition, $f \circ \eta_R = 1_{b(n)_*\{\iota_k\}}$ and $f \otimes 1_{k(n)_*} = \epsilon : \sigma(n)_*\{\iota_k\} \to k(n)_*$.

We lift f to a map of $\sigma(n)\{\iota_k\}$-modules:

$$\begin{tikzcd}
k(n)_*BP\{\iota_k\} \arrow[r] \arrow[dr, "\overline{f}"'] & \sigma(n)_*\{\iota_k\} \otimes k(n)_*BP\{\iota_k\} \arrow[d, "1\otimes f"] \\
& \sigma(n)_*\{\iota_k\} \otimes b(n)_*\{\iota_k\}
\end{tikzcd}$$

We claim that $\overline{f}$ is an isomorphism. Since both sides are free and of finite type over $b(n)_*\{\iota_k\}$, it suffices to prove that $\overline{f} \otimes 1_{k(n)_*}$ is an isomorphism. Tesoring the above diagram on the right by $k(n)_*$ (over $b(n)_*\{\iota_k\}$) yields:

$$\begin{tikzcd}
\sigma(n)_*\{\iota_k\} \arrow[r, "\Delta"] \arrow[dr, "\overline{f}\otimes 1_{k(n)_*}"'] & \sigma(n)_*\{\iota_k\} \otimes \sigma(n)_*\{\iota_k\} \arrow[d, "U_{\sigma(n)}(\epsilon)"] \\
& \sigma(n)_*\{\iota_k\}
\end{tikzcd}$$

Counitarity of $U_{\sigma(n)}$ demonstrates that the required map is an isomorphism.

$K(n)_* \otimes \overline{f}$ now witnesses the isomorphism claimed in the theorem.

□

Definition 3.2.4. Let U_Γ-comod$_k$ denote the category of U_Γ-comodules which can be exhibited as a colimit of $BP_*\{\iota_l\}$ for $l \geq k$. We call such a comodule k-connected.

Proposition 3.2.2. *Let M be an unstable U_Γ comodule of connectivity k. Then*

$$U_\Gamma(K(n)_* \otimes M) \simeq B(n)_* \otimes_{K(n)_*} U_\Sigma(K(n)_* \otimes M)$$

if $k \geq 2p^n + 1$.

Proof. We recall the isomorphism of proposition 3.2.1:

$$K(n) \otimes U_\Gamma(BP_*\{\iota_k\}) \simeq U_\Sigma(K(n)_* \otimes BP_*BP_*\{\iota_k\}) \otimes_{K(n)} B(n)_*$$

We wish to dualize this in the manner of proof of the stable change of rings theorem. To this end, consider the following computation:

$$\begin{aligned} v_n(1 \cdot v_n^{-1} - v_n^{-1}) \cdot v_n &= (v_n \cdot v_n^{-1} - 1) \cdot v_n \\ &= v_n - 1 \cdot v_n \end{aligned}$$

That is, the left and right action of v_n^{-1} are the same precisely when the left and right action of v_n are the same. As we observed that the formula relating the left and right action exists in dimension $2p^n + 1$ and above (see proposition 3.2.1), it follows that these actions are the same for $k \geq 2p^n + 1$.

Therefore, we have the following diagram:

$$\begin{tikzcd}
K(n) \otimes U_\Gamma(BP_*\{\iota_k\}) \arrow[r] \arrow[d, "c \circ T"] & U_\Sigma(K(n) \otimes BP_*\{\iota_k\}) \otimes B(n)_* \arrow[d, "c \circ T"] \\
U_\Gamma(K(n) \otimes BP_*\{\iota_k\}) \arrow[r] & B(n)_* \otimes U_\Sigma(K(n)_* \otimes BP_*\{\iota_k\})
\end{tikzcd}$$

Here, the vertical maps are the conjugation following the twist map. They have no kernel since the left and right action of v_n^{-1} are the same, and therefore a non-zero element will conjugate to a non-zero element. They are onto since the map in the stable case is onto, and therefore they are isomorphisms.

We thus have an isomorphism:

$$U_\Gamma(K(n)_* \otimes_{BP_*} BP_*\{\iota_k\}) \simeq B(n)_* \otimes_{K(n)_*} U_\Sigma(K(n)_* \otimes_{BP_*} BP_*\{\iota_k\})$$

for $k \geq 2p^n + 1$.

Since colim is an exact functor and U_Γ is defined via a resolution, U_Γ commutes with colimits. The same argument holds for U_Σ. Hence, if
$M = \operatorname{colim}_{\mathcal{D}} BP_*\{\iota_{k_i}\}$,

$$\begin{aligned} U_\Gamma(K(n)_* \otimes M) &\simeq U_\Gamma(K(n)_* \otimes \operatorname{colim}_{\mathcal{D}} BP_*\{\iota_{k_i}\}) \\ &\simeq \operatorname{colim}_{\mathcal{D}} U_\Gamma(K(n)_* \otimes BP_*\{\iota_{k_i})\} \\ &\simeq \operatorname{colim}_{\mathcal{D}} B(n)_* \otimes_{K(n)_*} U_\Sigma(K(n)_* \otimes M) \\ &\simeq B(n)_* \otimes U_\Sigma(K(n)_* \otimes M) \end{aligned}$$

□

Proposition 3.2.3. *Let M be an unstable U_Γ comodule of connectivity k. Then*

$$\alpha^*\alpha_* M \simeq B(n)_* \otimes_{BP_*} M$$

if $k \geq 2p^n + 1$.

Proof. We examine the equalizer diagram that defines $\alpha^*\alpha_* M$:

$$\begin{tikzcd} \alpha^*\alpha_* M \arrow[r] & U_\Gamma(K(n)_* \otimes M) \arrow[r, shift left] \arrow[r, shift right] & U_\Gamma U_\Sigma(K(n)_* \otimes_{BP_*} M) \end{tikzcd}$$

We now apply the isomorphism of proposition 3.2.2:

$$\begin{tikzcd} \alpha^*\alpha_* M \arrow[r] & U_\Gamma(K(n)_* \otimes M) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & U_\Gamma U_\Sigma(K(n)_* \otimes_{BP_*} M) \arrow[d] \\ & B(n)_* \otimes_{K(n)} U_\Sigma(K(n)_* M) \arrow[r, shift left] \arrow[r, shift right] & B(n)_* \otimes U_\Sigma^2(K(n)_* M) \end{tikzcd}$$

The coeqeualizer of the bottom parallel pair is

$$B(n)_* \otimes_{K(n)_*} K(n)_* \otimes_{BP_*} M = B(n)_* \otimes_{BP_*} M$$

□

Definition 3.2.5. Let M be a U_Γ comodule of connectivity k. Then M is of height n iff $B(n)_* \otimes_{BP_*} M = M$

Proposition 3.2.4. *Let $U_\Sigma(N)$ be a cofree U_Σ-comodule. Then*

$$\alpha^* U_\Sigma(N) = B(n)_* \otimes_{BP_*} N$$

Proof. This follows from proposition 3.2.3. □

Proposition 3.2.5. *Let M be a U_Γ-comodule of height n and of connectivity $2p^n + 1$. The unit of the adjunction at M, $M \to \alpha^* \alpha_* M$, is an isomorphism of U_Γ comodules.*

Proof. By proposition 3.2.3, $\alpha^* \alpha_* M \simeq B(n)_* \otimes_{BP_*} M$. This is an isomorphism precisely when M is of height n. □

Theorem 3.2.5 (Unstable Change of Rings)**.** *If M is a U_Γ-comodule of height n and connectivity greater than or equal to $2p^n + 1$, then*

$$Ext_{U_\Gamma}(BP_*, M) \simeq Ext_{U_\Sigma}(K(n)_*, K(n)_* \otimes_{BP_*} M)$$

Proof. Since $\alpha_* \circ U_\Sigma = B(n)_* \otimes -$, the hypothesis of the Generalized Shapiro's Lemma are satisfied, and for all comodules M,

$$Ext_{U_\Gamma}(BP_*, \alpha^*\alpha_* M) \simeq Ext_{U_\Sigma}(K(n)_*, K(n)_* \otimes_{BP_*} M)$$

If M is of height n and of dimension greater than or equal to $2p^n + 1$, then $M \simeq \alpha^*\alpha_* M$.

□

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