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ON CORES AND BARGAINING SETS FOR n-PERSON COOPERATIVE GAMES WITHOUT SIDE PAYMENTS

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FOREWORD

The theory of n-person games with side payments allowed was first developed by Von Neumann and Morgenstern [18]. In the years since their classic work, the theory has grown and various solution concepts have been developed. The theory of n-person games without side payments, a generalization of the classical theory, is relatively new, and as yet not as much is known about such games.

In this dissertation, two well known solution concepts for side payment games, the core and the bargaining set, are studied in the context of games without side payments. Roughly, the core is the set of outcomes of a game for which no coalition can do better by playing alone. The bargaining set is a set of outcomes which is stable under given standards of negotiation.

In the first chapter, the problem of determining when a game without side payments has a core is studied. Known sufficient conditions are generalized and are shown in some cases to be necessary. Finally necessary and sufficient conditions are derived. These generalize known results for side payment games.

The second chapter treats the problem of defining a bargaining set for all non-side payment games which exists and reduces to the classical notion of the bargaining set for side payment games. A general existence theorem is proved, and various possible bargaining sets are discussed. Finally, a particular bargaining set is defined, shown to exist for all games, and shown to reduce to the known bargaining set for side payment games.

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CHAPTER I. SOME THEOREMS ON THE CORE OF AN n-PERSON GAME WITHOUT SIDE PAYMENTS

1. INTRODUCTION

In [13], Scarf proved a theorem on linear inequalities which he used to derive a sufficient condition that a non-side payment game has a core. In this chapter, this theorem will be used to derive more general sufficient conditions for the existence of a core. These conditions will be shown to be necessary for certain classes of games. Finally, a necessary and sufficient condition for a non-empty core will be derived for all games whose payoff sets are assumed to be convex. This condition reduces to Shapley's condition for the core of a side-payment game.

2. DEFINITIONS

Let the set of players be $N = \{1,\ldots,n\}$. For each $S \subset N$, let E^S be the Euclidean space of dimension $|S|$ whose coordinates are indexed by the players in S . If $u \in E^N$, then u^S will denote its projection onto E^S . By $x \geqq 0$, $x \in E^N$, we mean each coordinate is non-negative. By $x \geq 0$ we mean $x \geqq 0$ and $x \neq 0$.

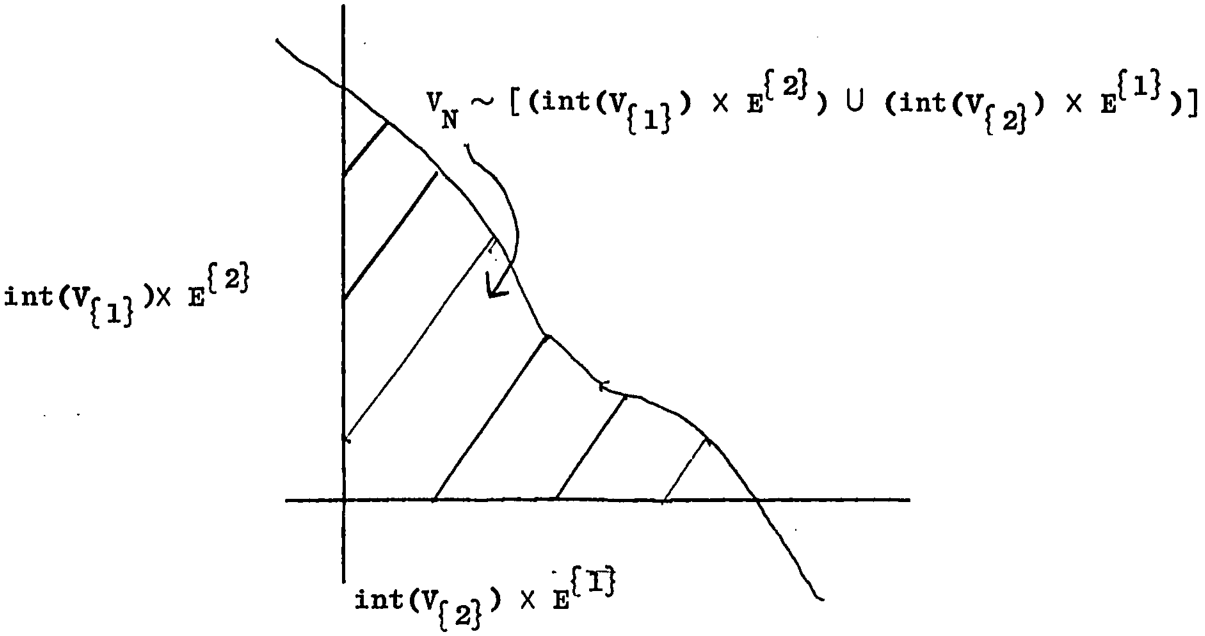
Definition: An n-person game without side-payments $\Gamma = \{V_S\}_{S \subset N}$ is a collection of sets satisfying the following conditions:

1) For each $S \subset N$, V_S is a closed, non-empty subset of E^S .

2) If $x \in V_S$ and $y \in E^S$ is such that $y \leqq x$, then $y \in V_S$.

3) $V_N \sim \bigcup_{i \in N} [\text{interior } (V_{\{i\}}) \times E^{N \sim \{i\}}]$ is non-empty and bounded.

Note that since $V_{\{i\}}$ is considered to be a subset of $E^{\{i\}}$, interior $(V_{\{i\}})$ is an open interval in $E^{\{i\}}$, and interior $(V_{\{i\}}) \times E^{N \sim \{i\}}$ is an open half-space in E^N. For example, for $n = 2$, the set described in 3 would look like the following.



Each V_S can be interpreted as the utility levels that the players in S can achieve by acting cooperatively without outside help. The game defined above is often called a game in "characteristic function" form. For a discussion of games without side payments and an extensive bibliography, see Aumann [1].

<u>Definition</u>: <u>A point</u> $u \in V_N$ <u>is said to be in the core of</u> Γ <u>if for all</u> $S \subset N$, $u^S \notin \text{interior } (V_S)$.

This is equivalent to saying u is in the core if and only if for all $S \subset N$, there exists no $y^S \in V_S$ such that $y^S > u^S$. Since $S = N$ is included here, a point in the core is necessarily Pareto optimal. Intuitively, no coalition can form and give all its members <u>strictly more</u> than they get from a point in the core.

3. GENERAL SUFFICIENT CONDITIONS

In this section we will state Scarf's theorem and use it to derive some sufficient conditions for the existence of a non-empty core. We need the following definition.

Definition: Let A *be an* $n \times m$ *matrix and* b *an n-vector. A collection* $j_1, \ldots, j_k$ *of columns of* A *will be called a feasible sub-basis for the system of linear inequalities*

$$Ax \leqq b \quad \underline{\text{if}}:$$

1) *the columns* $j_1, \ldots, j_k$ *are linearly independent n-vectors, and*

2) *there exists an* $x \in R^m$ *such that* $Ax \leqq b$ *and* $x_i = 0$ *for* $i \neq j_1, \ldots, j_k$.

We can now state Scarf's theorem.

Theorem (Scarf [13]): Let A *be an* $n \times m$ *matrix and* b *a non-negative n-vector such that the convex set*

$$\{x \in R^m \mid x \geqq 0, Ax \leqq b\}$$

is bounded. Let C *be an arbitrary* $n \times m$ *matrix. Then there exists a feasible sub-basis for the system* $Ax \leqq b$ *so that if we define*

$$u_i = \min\{c_{ij} \mid \text{for all } j \text{ in this sub-basis}\},$$

then for each column k *of* C, *there exists a row* i *corresponding to a zero slack (i.e., for row* i, $\sum_{j=1}^{m} a_{ij}x_j = b_i$) *such that*

$$u_i \geq c_{ik} .$$

Scarf proves this theorem by means of a constructive algorithm which yields the required sub-basis.

We will now generalize the notion of a balanced collection and a balanced game (see Shapley [15], Scarf [13]). Let there be given a set of vectors $\{b^{(S)}\}_{S \subset N}$ such that

1) $b^{(S)} \in E^S$ for all $S \subset N$

2) $b^{(N)} > 0$

3) for all $S \neq N$, $b^{(S)} \geq 0$.

We will sometimes consider each $b^{(S)}$ to be a vector in E^N having a zero in each coordinate corresponding to a player not in S .

(Note: We use the notation $b^{(S)}$ rather than b^S to point out that $b^{(S)}$ is not the projection of a fixed vector $b \in E^N$ for all $S \subset N$.)

Definition: Let Λ be a collection of proper, non-empty subsets of N . Λ is said to be $\{b^{(S)}\}_{S \subset N}$ balanced if there exist $\delta_S > 0$ for all $S \in \Lambda$ such that

$$\sum_{S \in \Lambda} \delta_S \, b^{(S)} = b^{(N)} .$$

The numbers δ_S, $S \in \Lambda$, are called $\{b^{(S)}\}$ balancing coefficients for Λ .

Definition: An n-person game $\Gamma = \{V_S\}_{S \subset N}$ is said to be $\{b^{(S)}\}_{S \subset N}$ balanced if whenever Λ is a $\{b^{(S)}\}_{S \subset N}$ balanced collection,

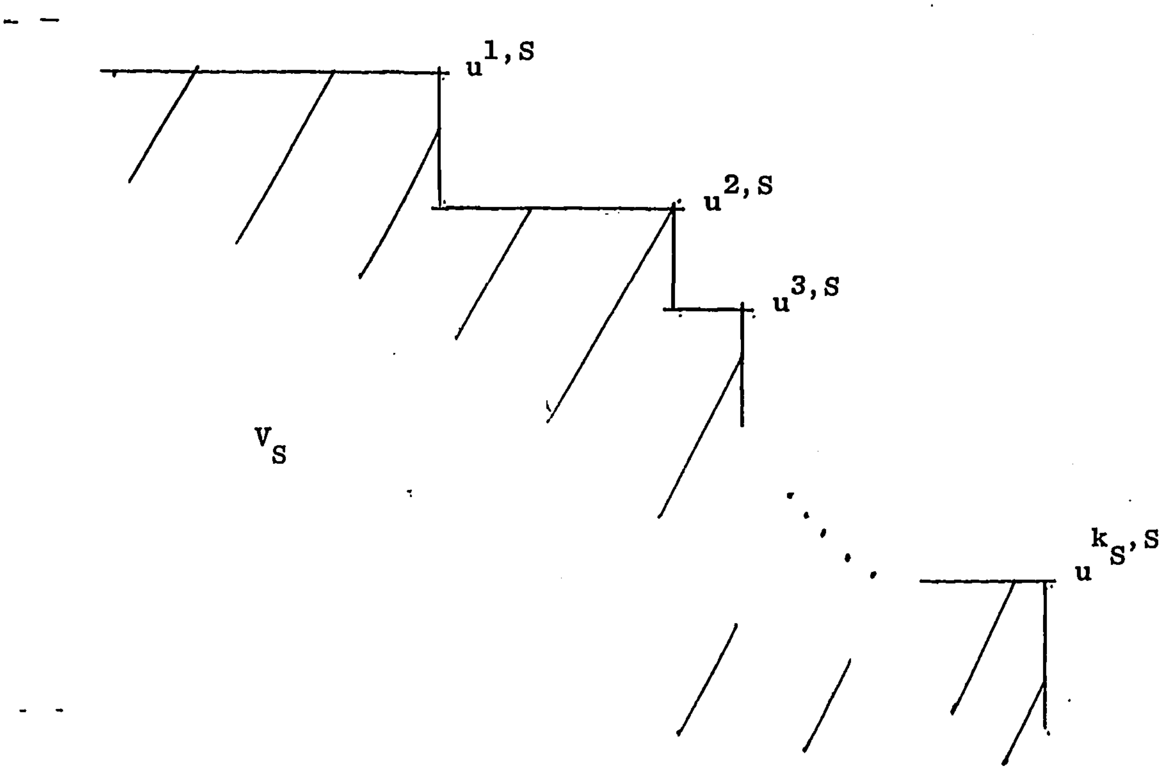
(1) $(u \in E^N$ and $u^S \in V_S$ for all $S \in \Lambda)$ implies $u \in V_N$.

Note that in the case where for each S $b^{(S)}$ is the characteristic vector of the coalition S, $\{b^{(S)}\}$ balanced means balanced in the sense of Shapley and Scarf.

Theorem 1: A $\{b^{(S)}\}_{S \subset N}$ balanced n-person game always has a core.

Proof: The proof follows Scarf's proof for balanced games. It will be in two parts. First the theorem will be proved for games such that for all $S \subsetneqq N$, V_S is generated by a finite number of vectors $u^{1,S}, \ldots, u^{k_S,S}$ in E^S in the sense that

$$(2) \qquad V_S = \bigcup_{\alpha=1}^{k_S} \{v^S \in E^S | v^S \leqq u^{\alpha,S}\} \ .$$



The proof will then be extended to arbitrary games.

Suppose each V_S, $S \neq N$, is of the form (2) above. Let C be the matrix whose entries are:

$$c_{i,(j,S)} = \begin{cases} u_i^{j,S} & \text{if} \quad i \in S \\ M & \text{if} \quad i \notin S \end{cases}$$

where M is chosen so that $M > u_i^{j,S}$ for all $i \in S$, $j = 1,\ldots,k_S$, and for all $S \neq N$. Let A be the matrix whose entries are:

$$a_{i,(j,S)} = \begin{cases} b_i^{(S)} & \text{if} \quad i \in S \\ 0 & \text{if} \quad i \notin S \ . \end{cases}$$

For both A and C, $i = 1,\ldots,n$; $j = 1,\ldots,k_S$; and S varies over all proper non-empty subsets of N. A column $b^{(S)}$ of A is repeated once for each generator of V_S.

$$C = \left[\begin{array}{ccccccc} & u_1^{1,S} & u_1^{2,S} & \cdots & u_1^{k_S,S} & \\ & M & M & \cdots & M & \\ & u_3^{1,S} & u_3^{2,S} & \cdots & u_3^{k_S,S} & \\ \cdots\cdots & \vdots & \vdots & & \vdots & \cdots\cdots \\ & M & M & \cdots & M & \\ & \vdots & \vdots & & \vdots & \\ & u_n^{1,S} & u_n^{2,S} & \cdots & u_n^{k_S,S} & \end{array}\right]$$

$$A = \left[\begin{array}{ccccccc} & b_1^{(S)} & b_1^{(S)} & \cdots & b_1^{(S)} & \\ & 0 & 0 & \cdots & 0 & \\ \cdots\cdots & b_3^{(S)} & b_3^{(S)} & \cdots & b_3^{(S)} & \cdots\cdots \\ & \vdots & \vdots & & \vdots & \\ & 0 & 0 & \cdots & 0 & \\ & \vdots & \vdots & & \vdots & \\ & b_n^{(S)} & b_n^{(S)} & \cdots & b_n^{(S)} & \end{array}\right]$$

Let $m = \sum_{S \neq N} k_S$. Since each column of A contains at least one positive entry, the convex set

$$\{x \in R^m \,|\, x \geqq 0, \; Ax \leqq b^{(N)}\}$$

is bounded. By Scarf's theorem, there exists a feasible sub-basis for the system $Ax \leqq b^{(N)}$ so that if we define

$$u_i = \min\{c_{i,(j,S)} \,|\, \text{for all } (j,S) \text{ in the sub-basis}\}$$

then for each column (k,S) of C, there exists a row i, corresponding to a zero slack, such that $u_i \geqq c_{i,(k,S)}$. Therefore the vector $u = (u_1,\ldots,u_n)$ does not lie in the interior of any V_S, $S \neq N$. If we can show that $u \in V_N$ then any Pareto optimal point $v \geqq u$ will be in the core.

Let $S_1, \ldots, S_p$ be those distinct coalitions which are represented by the vectors of the sub-basis. The vectors $u^{S_i} \in V_{S_i}$ for $i = 1, \ldots, p$ since u^{S_i} is less than or equal to some generator of V_{S_i} which appears in the sub-basis.

By the definition of a sub-basis, there exists $x \in R^m$ such that $Ax \leqq b^{(N)}$ and $x_{(k,S)} = 0$ for all columns (k,S) not in the sub-basis. If we could show that x in fact satisfies $Ax = b^{(N)}$ then we would have that the collection $\{S_1, \ldots, S_p\}$ contains a $\{b^{(S)}\}_{S \subset N}$ balanced collection. By hypothesis, the game is $\{b^{(S)}\}_{S \subset N}$ balanced, and since $u^{S_i} \in V_{S_i}$, for $i = 1, \ldots, p$, we could conclude that $u \in V_N$.

Suppose i^* is a row with

$$(3) \qquad \sum_{(k,S)} a_{i^*,(k,S)} x_{(k,S)} = \sum_{S \ni i^*} b^{(S)}_{i^*} \left(\sum_{j=1}^{k_S} x_{(j,S)} \right) < b^{(N)}_{i^*} .$$

Consider the column of C corresponding to $S = \{i^*\}$. It is of the form

$$\begin{bmatrix} M \\ \vdots \\ M \\ c_{i^*,(1,\{i^*\})} \\ M \\ \vdots \\ M \end{bmatrix} .$$

There exists a row i corresponding to a zero slack (by (3), $i \neq i^*$) such that $u_i \geqq M$. Row i corresponding to a zero slack means

$$\sum_{S \ni i} b_i^{(S)} \left(\sum_{j=1}^{k_S} x_{(j,S)} \right) = b_i^{(N)} > 0$$

which implies $x_{(j_0,S_0)} > 0$ for some $S_0 \ni i$ and $i \leq j_0 \leq k_{S_0}$. Hence, column (j_0,S_0) is in the sub-basis. Therefore

$$\begin{aligned} u_i &= \min\{c_{i,(j,S)} \mid \text{for all } (j,S) \text{ in the sub-basis}\} \\ &\leq c_{i,(j_0,S_0)} < M . \end{aligned}$$

The contradiction implies that $x \geq 0$ in fact satisfies $Ax = b^{(N)}$. This completes the proof for "finitely generated" games.

Let $\Gamma = \{V_S\}_{S \subset N}$ be an arbitrary game, and suppose it is $\{b^{(S)}\}_{S \subset N}$ balanced. Let $(r_1, r_2, \ldots\ldots)$ be a denumeration of the rational numbers with r_1 chosen so that for each $S \neq N$ the $|S|$-vector $(r_1, r_1, \ldots, r_1) \in V_S$. For each S, $1 < |S| < n$ and each $\alpha = 1,2,\ldots.$ define V_S^α to be the set generated by those vectors in V_S all of whose coordinates are in the set $\{r_1, r_2, \ldots, r_\alpha\}$. For $|S| = 1,n$ define $V_S^\alpha = V_S$ for all α. Let $\Gamma_\alpha = \{V_S^\alpha\}_{S \subset N}$ for $\alpha = 1,2,\ldots.$.

For each α, Γ_α is "finitely generated" and since $V_S^\alpha \subset V_S$ for $S \neq N$, Γ_α is $\{b^{(S)}\}_{S \subset N}$ balanced. Therefore for each α, there exists a point x^α in the core of Γ_α. For all α,

$$x^\alpha \in V_N \sim \bigcup_{i \in N} [\text{interior } (V_{\{i\}}) \times E^{N \sim \{i\}}] .$$

That this set is compact follows from property 3) in the definition of a game. Let x be a limit point of the x^α. We shall show that x is in the core of Γ.

Suppose for some $S \subset N$, there exists a $u \in V_S$ such that $u > x^S$.

Case 1: Suppose u has all rational coordinates. Then choose α so large that $u \in V_S^\alpha$ and

$$|x_\alpha - x| < \eta \quad \text{where} \quad \eta = \min_{i \in S} (u_i - x_i) .$$

This gives $u > x_\alpha^S$, a contradiction to the fact that x_α is in the core of Γ_α .

Case 2: Arbitrary u . Let

$$R = \{y \in E^S | x_i < y_i \leq u_i \quad \text{for} \quad i \in S\} .$$

There exists $\tilde{u} \in R$ such that $\tilde{u}$ has all rational coordinates; i.e., $\tilde{u} > x^S$, contradicting Case 1.

This completes the proof of Theorem 1.

We say a collection Λ is a minimal $\{b^{(S)}\}_{S \subset N}$ balanced collection if it is $\{b^{(S)}\}_{S \subset N}$ balanced and no proper sub-collection of Λ is $\{b^{(S)}\}_{S \subset N}$ balanced. The following lemmas are easy to prove.

Lemma 1: A $\{b^{(S)}\}_{S \subset N}$ balanced collection Λ is minimal $\{b^{(S)}\}_{S \subset N}$ balanced if and only if the vectors $b^{(S)}$, $S \in \Lambda$ (considered as elements of E^N) are linearly independent. Hence a minimal $\{b^{(S)}\}_{S \subset N}$ balanced collection has unique $\{b^S\}$-balancing coefficients, and conversely.

Lemma 2: A game $\Gamma = \{V_S\}_{S \subset N}$ is $\{b^{(S)}\}_{S \subset N}$ balanced if and only if whenever Λ is a minimal $\{b^{(S)}\}_{S \subset N}$ balanced collection,

$$(u \in E^N \quad \text{and} \quad u^S \in V_S \quad \text{for all} \quad S \in \Lambda) \text{ implies } \quad u \in V_N .$$

We now give an example of a game which is not balanced in the sense of Scarf, but is $\{b^{(S)}\}_{S \subset N}$ balanced for proper choice of vectors $b^{(S)}$.

Let $n = 3$, and let

$$V_{\{i\}} = \{x^i \in E^{\{i\}} | x^i \leqq 0\}, \; i = 1,2,3 ,$$
$$V_{12} = \{x^{12} \in E^{\{12\}} | x^1 \leqq 4, \; x^2 \leqq 3\} ,$$
$$V_{23} = \{x^{23} \in E^{\{23\}} | x^2 \leqq 4, \; x^3 \leqq 3\} ,$$
$$V_{13} = \{x^{13} \in E^{\{13\}} | x^1 \leqq 2, \; x^3 \leqq 5\} , \quad \text{and}$$

$$V_{123} = \{x^{123} \in E^{\{123\}} | x^1 \leqq 4, x^2 \leqq 3, x^3 \leqq 0\} \cup \{x^{123} \in E^{\{123\}} | x^1 \leqq 2, x^2 \leqq 4, x^3 \leqq 3\} ,$$

This game is not balanced in the sense of Scarf since $(2,0,5) \notin V_N$. However the game is $\{b^{(S)}\}_{S \subset N}$ balanced for $b^{(N)} = (1,2,3)$, $b^{(12)} = (1,1)$, $b^{(13)} = (1,1)$, $b^{(23)} = (1,1)$, and $b^{(i)} = 1$ for $i = 1,2,3$. In fact, $(2,4,3)$ is a point in the core, as required by Theorem 1.

4. GAMES FOR WHICH THE CONDITION IS ALSO NECESSARY

From here on we will consider only games $\Gamma = \{V_S\}_{S \subset N}$ for which V_S is a convex set. It is easy to justify the convexity assumption on the basis that if two points x and y are attainable by a coalition, then an appropriate lottery could be arranged having any point on the line segment joining x and y as its expected value.

We will first study a very special class of games which will be called hyperplane or weighted side-payment games. These are games for which V_S is a half-space in E^S, i.e.,

$$V_S = H^{b^{(S)},v(S)} = \{x \in E^S | b^{(S)} \cdot x \leqq v(S)\} \text{ where } b^{(S)} \geq 0 .$$

This class obviously includes all side-payment games.

Weighted side payment games can be realized in situations similar to side payment games. Coalition S has "value" $v(S)$. If S forms and agrees on a payoff vector $x \in V_S$, the result would be a) player $i \in S$ receives x_i, b) player i's "agent" receives $(b_i^{(S)}-1)x_i$. Intuitively, there is a cost involved in forming the coalition, and its members must pay according to the benefit they received from its formation. The number $b_i^{(S)}- 1$ is i's "tax rate" in coalition S. It may depend on such factors as traveling distance between i and the other players in S, desirability of i as a partner, etc. Side payment games may be considered "tax free".

Lemma 3: *A hyperplane game* $\Gamma = \{H^{b^{(S)}, v(S)}\}_{S \subset N}$ *is* $\{b^{(S)}\}_{S \subset N}$ *balanced if and only if whenever* Λ *is a minimal* $\{b^{(S)}\}_{S \subset N}$ *balanced collection and* δ_S, $S \in \Lambda$, *are the* $\{b^{(S)}\}$ *balancing coefficients for* Λ,

$$v(N) \geqq \sum_{S \in \Lambda} \delta_S \, v(S) \quad . \tag{4}$$

Proof: Suppose Λ is a minimal $\{b^{(S)}\}$ balanced collection and (4) holds. Suppose $x \in E^N$ and $x^S \in H^{b^{(S)}, v(S)}$ for all $S \in \Lambda$. Then for all $S \in \Lambda$,

$$b^{(S)} \cdot x \leqq v(S) \; .$$

Combining with (4), we get

$$v(N) \geqq \sum_{S \in \Lambda} \delta_S \, v(S) \geqq \sum_{S \in \Lambda} \delta_S \, b^{(S)} \cdot x = b^{(N)} \cdot x \quad .$$

Hence $x \in H^{b^{(N)}, v(N)}$.

Suppose now that the game is $\{b^{(S)}\}$ balanced. Let Λ be a minimal $\{b^{(S)}\}$ balanced collection. Since the $b^{(S)}$, $S \in \Lambda$, are independent, the system of linear equations

$$b^{(S)} \cdot x = v(S), \; S \in \Lambda$$

has a solution $x \in E^N$. Since $x^S \in H^{b^{(S)}, v(S)}$ for all $S \in \Lambda$, we can assert $x \in H^{b^{(N)}, v(N)}$. Thus $v(N) \geqq b^{(N)} \cdot x = \sum_{S \in \Lambda} \delta_S \, b^{(S)} \cdot x = \sum_{S \in \Lambda} \delta_S \, v(S)$.

Lemma 4: *A hyperplane game* $\Gamma = \{H^{b^{(S)}, v(S)}\}_{S \subset N}$ *has a core if and* only if whenever Λ is a minimal $\{b^{(S)}\}_{S \subset N}$ balanced collection and δ_S, $S \in \Lambda$, are the $\{b^{(S)}\}_{S \subset N}$ balancing coefficients for Λ, (4) holds.

Proof: Let A be the $2^n - 1 \times n$ matrix whose first $2^n - 2$ rows are $b^{(S)}$, $S \neq N, \emptyset$, in any order, and whose last row is $-b^{(N)}$. Let c be the column vector with coordinates $v(S)$, for $S \neq N, \emptyset$, and $-v(N)$, ordered as in A. Thus Γ has a core if and only if the system of inequalities $Ay \geqq c$ has a solution $y \in R^n$. By a transposition theorem for inequalities, this system has a solution if and only if the system of linear equations

(5) $$xA = 0, \quad xc = 1$$

has no non-negative solution $x \in R^{2^n-1}$. (See Gale [7, p. 46]). We rewrite (5) as:

(5′) $$\sum_{S \neq N} x_S b^{(S)} - x_N b^{(N)} = 0$$

$$\sum_{S \neq N} x_S v(S) - x_N v(N) = 1$$

Since each $b^{(S)} \geq 0$, any non-negative solution to (5′) must have $x_N > 0$. Otherwise $x_S = 0$ for all $S \subset N$, which is impossible. If (5′) has a non-negative solution x then it would have a non-negative basic solution y (Gale [7, p. 50]), i.e.,

$$V = \{(b^{(S)}, v(S)) \mid y_S > 0\}$$

would be a linearly independent set of vectors in R^{n+1}. Let $\{S_1, \ldots, S_r, N\} = \{S \subset N \mid y_S > 0\}$. The vectors $b^{(S_1)}, \ldots, b^{(S_r)}, b^{(N)}$ span an r-subspace of R^n. The vectors $b^{(S_1)}, \ldots, b^{(S_r)}$ must be linearly independent. This can be seen as follows. Choose a basis for the r-subspace from these vectors, and suppose $b^{(S_1)}$ is not in this basis. By (5′) we have

$$\sum_{S \neq N} y_S b^{(S)} - y_N b^{(N)} = 0 \ .$$

y_{S_1} must be greater than zero since $b^{(S_2)}, \ldots, b^{(S_r)}, b^{(N)}$ are linearly independent. Since we know already that $y_N > 0$, the vectors $\sum_{S \neq N} y_S b^{(S)}, b^{(S_2)}, \ldots, b^{(S_r)}$ span the r-subspace. Hence $b^{(S_1)}, b^{(S_2)}, \ldots, b^{(S_r)}$ are linearly independent. We have shown that if (5′) has a non-negative solution x , then it has a non-negative solution y where $\{b^{(S)} | y_S > 0\}$ are linearly independent. Therefore (5′) has no non-negative solution if and only if whenever $b^{(S_1)}, \ldots, b^{(S_m)}$ are linearly independent and there exists $\delta_{S_i} > 0$, $i = 1, \ldots, m$ such that $\sum_{j=1}^{m} \delta_{S_i} b^{(S_i)} = b^{(N)}$, we must have

$$\sum_{j=1}^{m} \delta_{S_i} v(S_i) \leq v(N) \ .$$

By Lemma 1, $\Lambda = \{S_1, \ldots, S_m\}$ is a minimal $\{b^{(S)}\}_{S \subset N}$ balanced collection and Lemma 4 is proved.

Theorem 2: A hyperplane game $\Gamma = \{H^{b^{(S)}, v(S)}\}_{S \subset N}$ has a core if and only if it is $\{b^{(S)}\}_{S \subset N}$ balanced.

Proof: Lemma 3 and Lemma 4.

Lemma 5: Let $\Gamma = \{V_N; \overline{V_S}, S \neq N\}$ be a game where for each $S \neq N$, V_S is a convex set. Then Γ has a core if and only if for each $S \neq N$ there exists a vector $b^{(S)} \geq 0$, $b^{(S)} \in E^S$ and a real number $v(S)$ such that $H^{b^{(S)}, v(S)} \supset V_S$ and $\Gamma' = \{V_N; H^{b^{(S)}, v(S)}, S \neq N\}$ has a core.

Proof: Suppose we have such a Γ', and let x be in the core of Γ' .

By definition, $x \in V_N$ and $x^S \notin$ interior $H^{b^{(S)},v(S)}$ for all $S \subset N$. Then for all $S \neq N$, $x^S \notin$ interior (V_S). Hence x is in the core of Γ.

Suppose now that x is in the core of Γ. Then $x \in V_N$ and $x^S \notin$ interior V_S for all $S \subset N$. Therefore by the theorem of the separating hyperplane, for each $S \neq N$ there exists a non-zero vector $b^{(S)} \in E^S$ and a number $v(S)$ such that

$$x^S \cdot b^{(S)} \geqq v(S) \quad \text{and}$$

$$u \cdot b^{(S)} \leqq v(S) \quad \text{for all} \quad u \in V_S \, .$$

By property 2 in the definition of a game, we must have each $b^{(S)} \geq 0$. If we define

$$\Gamma' = \{V_N;\ H^{b^{(S)},v(S)},\ S \neq N\}$$

then x is in the core of Γ'.

<u>Theorem 3</u>: <u>Let</u> $\Gamma = \{V_S\}_{S \subset N}$ <u>be a game where</u> 1) $V_N = H^{b^{(N)},v(N)}$ <u>for some</u> $v(N)$ <u>and some</u> $b^{(N)} \in E^N$, $b^{(N)} > 0$. 2) V_S <u>is convex for all</u> $S \neq N$. <u>Then</u> Γ <u>has a core if and only if there exist vectors</u> $b^{(S)} \in E^S$, $b^{(S)} \geq 0$ <u>for all</u> $S \neq N$ <u>such that</u> Γ <u>is</u> $\{b^{(S)}\}_{S \subset N}$ <u>balanced</u>. (<u>Note here that</u> $b^{(N)}$ <u>is the normal vector to</u> V_N).

<u>Proof:</u> Sufficiency follows from Theorem 1. We will prove necessity.

By Lemma 5, if Γ has a core then there exist $b^{(S)} \geq 0$ and $v(S)$ for $S \neq N$ such that

$$\Gamma' = \{H^{b^{(N)},v(N)};\ H^{b^{(S)},v(S)},\ S \neq N\}$$

has a core, where $H^{b^{(S)},v(S)} \supset V_S$ for $S \neq N$. By Theorem 2, Γ' must be $\{b^{(S)}\}_{S \subset N}$ balanced. Because $H^{b^{(S)},v(S)} \supset V_S$ for all $S \neq N$, this implies that Γ is in fact $\{b^{(S)}\}_{S \subset N}$ balanced.

The following is an example to show that if V_N is not a halfspace, then a game may have a core without being balanced for any system of vectors $\{b^{(S)}\}_{S \subset N}$.

Let $N = \{1,2,3\}$ and let $\Gamma = \{V_S\}_{S \subset N}$ be

$$V_{123} = \{(x_1,x_2,x_3) \mid x_1 \leqq \tfrac{1}{2},\ x_2 \leqq \tfrac{1}{2},\ x_3 \leqq 0\} ,$$

$$V_{12} = \{(x_1,x_2) \mid x_1 + x_2 \leqq 1\} , \text{ and}$$

$$V_S = \{x^S \in E^S \mid x_i \leqq 0 \text{ for all } i \in S\} \text{ for all other } S \subset N .$$

Γ has a unique core point $(\frac{1}{2},\frac{1}{2},0)$. Suppose Γ were $\{b^{(S)}\}$ balanced for some $\{b^{(S)}\}$. Then the collection $\{\{12\},\{3\}\}$ must be $\{b^{(S)}\}$ balanced. But this implies that

$$\{(x_1,x_2,x_3) \mid x_1 + x_2 \leq 1,\ x_3 \leq 0\} \subset V_{123}$$

which is not the case. Hence Γ is not balanced for any choice of vectors $b^{(S)}$.

This leads us to look for somewhat different conditions which may be necessary and sufficient for all games for which the V_S are arbitrary convex sets. This is done in the next section.

5. NECESSARY AND SUFFICIENT CONDITIONS FOR GAMES WITH CONVEX V_S

We will first discuss the concept of the support function of a convex subset of Euclidean n-space.

<u>Definition</u>: <u>Let</u> C <u>be a convex subset of</u> E^n. <u>The extended support function of</u> C <u>is defined for each</u> $\alpha \in E^n$ <u>to be</u>

$$h_C(\alpha) = \sup_{x \in C} x \cdot \alpha .$$

If h_C is defined only where it is finite, it is sometimes called the support function of C. For example if $C = H^{\beta,a} = \{x \in E^n | x \cdot \beta \leqq a\}$ the support function $\tilde{h}_C$ is defined only for vectors of the form $c\beta$ where c is a non-negative real number, and $\tilde{h}_C(c\beta) = ca$. On the other hand, the extended support function h_C is defined on all of E^n, and

$$h_C(\alpha) = \begin{cases} ca & \text{if } \alpha = c\beta,\ c \geqq 0 \\ \infty & \text{otherwise.} \end{cases}$$

We will however mean extended support function whenever we say support function.

If C is a non-empty closed convex set in E^n, and if h_C is its (extended) support function, then

$$C = \{x \in E^n | x \cdot \alpha \leq h_C(\alpha) \quad \text{for all} \quad \alpha \in E^n\}$$

(Valentine [17, p. 59]).

The following lemma gives the support function of a closed convex polytope (not necessarily compact) in terms of its finitely many defining halfspaces.

Lemma 6: *Suppose* a^i, $i = 1, \ldots, m$, *are non-zero vectors in* E^n *and* c^i, $i = 1, \ldots, m$, *are real numbers*. *Let* $P = \bigcap_{i=1}^{m} H^{a^i, c^i} \equiv \bigcap_{i=1}^{m} \{x \in E^n | a^i \cdot x \leq c^i\} \neq \emptyset$. *Then the support function of* P *is given by*

$$h_P(\alpha) = \min \{ \sum_{i=1}^{m} \lambda_i c^i | \sum_{i=1}^{m} \lambda_i a^i = \alpha,\ \lambda_i \geq 0 \ \underline{\text{and}}$$

$$\{a^i | \lambda_i > 0\} \ \underline{\text{is linearly independent}}\}$$

for all $\alpha \in E^n$.

(We define the minimum over the empty set to be $+\infty$.)

Proof: Given $\alpha \in E^n$, we want to maximize $\alpha \cdot x$ subject to: $a^i \cdot x \leq c^i$ for $i = 1, \ldots, m$.

This is a linear programming problem. The dual problem is

$$\text{minimize } \lambda_1 c^1 + \dots + \lambda_m c^m$$

$$\text{subject to: } \lambda_i \geq 0,\ i = 1,\dots,m, \text{ and } \sum_{i=1}^{m} \lambda_i a^i = \alpha .$$

Since $P \neq \emptyset$, the first program always has a feasible vector. Hence by the duality theorem of linear programming, if the dual problem is feasible we have

$$\max \alpha \cdot x = \min \sum_{i=1}^{m} \lambda_i c^i .$$

If the dual problem is infeasible, we have by our convention $\min \sum_{i=1}^{m} \lambda_i c^i = \infty$. But infeasibility of the dual problem implies the first program has no optimal vector, hence

$$\max \alpha \cdot x = \infty .$$

The dual program is a canonical linear program (see Gale [7,p.84]), and if it has an optimal vector then it has a basic optimal vector. Hence we can choose the λ_i so that

$$\{a^i \mid \lambda_i > 0\}$$

is linearly independent, and we have proven

$$h_P(\alpha) = \sup_{x \in P} \alpha \cdot x \quad \text{for all } \alpha \in E^n .$$

Definition: We say two convex subsets of E^n C and D have the strict separation property (s.s.p) if $C \cap D = \emptyset$ implies there exists $\beta \in E^n$ such that $\sup_{x \in C} x \cdot \beta < \inf_{y \in D} y \cdot \beta$.

For example, if C is compact and D is closed, then C and D have s.s.p. (See Berge and Ghouila-Houri [3, p. 55].)

Lemma 7: Let C and D be two closed convex subsets of E^n with s.s.p., and let h_C and h_D be their support functions. Then $C \cap D \neq \emptyset$ if and only if for all $\alpha \in E^n$,

$$h_C(\alpha) + h_D(-\alpha) \geqq 0 \ .$$

Proof: Suppose $x \in C \cap D$. Then for all $\alpha \in E^n$,

$$\alpha \cdot x \leqq \min (h_C(\alpha), h_D(\alpha))$$

$$-\alpha \cdot x \leqq \min (h_C(-\alpha), h_D(-\alpha)) \ .$$

Hence,

$$0 \leqq \min (h_C(\alpha), h_D(\alpha)) + \min (h_C(-\alpha), h_D(-\alpha)) \leqq h_C(\alpha) + h_D(-\alpha) \ .$$

Suppose $C \cap D = \emptyset$. By s.s.p., there exists a vector $\beta \in E^n$ such that

$$h_C(\beta) = \sup_{x \in C} x \cdot \beta < \inf_{y \in D} y \cdot \beta = -\sup_{y \in D} y \cdot (-\beta) = -h_D(-\beta) \ .$$

Thus

$$h_C(\beta) + h_D(-\beta) < 0 \ .$$

Lemma 8: Let V be a closed non-empty convex set in E^n such that if $x \in V$, $y \in E^n$, $y \leqq x$, then $y \in V$. Let P be a closed non-empty convex polytope such that $x \in P$ implies $x \geqq 0$. Then V and P have the strict separation property.

Proof: Decompose P as the vector sum

$$P = C + K = \{y + z \mid y \in C, \ z \in K\}$$

where C is a closed polyhedral convex cone and K is a compact polyhedron (Goldman [8]). Note that if $y \in C$ then $y \geqq 0$, for if C had a point with a negative coordinate, then it would have points for which this coordinate was arbitrarily large and negative. This would imply that P would have points with this coordinate negative, contrary to assumption.

Suppose $V \cap P = \emptyset$. Suppose $x \in V \cap K$. Then $x = x + 0 \in V \cap P$.

Hence $V \cap K = \emptyset$. Since K is compact, V and K have s.s.p., i.e., there exists $\beta \in E^n$ such that

$$\sup_{x \in V} x \cdot \beta < \inf_{z \in K} z \cdot \beta \ .$$

Note that $\beta \geq 0$ since V has points with arbitrarily large negative coordinates.

Now

$$\begin{aligned} \inf_{u \in P} u \cdot \beta &= \inf_{\substack{y \in C \\ z \in K}} (y + z) \cdot \beta \\ &\geq \inf_{y \in C} y \cdot \beta + \inf_{z \in K} z \cdot \beta \\ &\geq \inf_{z \in K} z \cdot \beta > \sup_{x \in V} x \cdot \beta \end{aligned}$$

since $y \cdot \beta \geqq 0$ for $y \in C$. Hence V and P have s.s.p..

We are now prepared to give a necessary and sufficient condition for games with convex V_S to have a core.

<u>Theorem 4:</u> <u>Let</u> $\Gamma = \{V_N; V_S, S \neq N\}$ <u>be a game in which</u> V_S <u>is convex for all</u> $S \subset N$, <u>and let</u> h_S <u>be the support function of</u> V_S. <u>Then</u> Γ <u>has a core if and only if for each</u> $S \neq N$ <u>there exists a vector</u> $b^{(S)} \in E^S$, <u>with</u> $b^{(S)} \geq 0$ <u>and</u> $h_S(b^{(S)}) < \infty$, <u>such that for each</u> $\alpha \in E^N$;

$$h_N(\alpha) \geqq \max \{\Sigma \lambda_S h_S(b^{(S)}) | \lambda_S \geqq 0, \Sigma \lambda_S b^{(S)} = \alpha \ , \tag{6}$$

$$\underline{\text{and}} \ \{b^{(S)} | \lambda_S > 0\} \ \underline{\text{is linearly independent}}\} \ .$$

<u>Proof:</u> By Lemma 5, Γ has a core if and only if there exist vectors $b^{(S)} \in E^S$, $b^{(S)} \geq 0$, and real numbers $v(S)$ for each $S \neq N$ such that $H^{b^{(S)}, v(S)} \supset V_S$ and

$$\Gamma' = \{V_N; H^{b^{(S)}, v(S)}, S \neq N\}$$

has a core. By definition of support function we have $h_S(b^{(S)}) \leq v(S)$. It is easy to see that we need only consider $v(S) = h_S(b^{(S)})$ since by reducing the value of $v(S)$ we make it "easier" to have a core.

Now Γ' (with $v(S) = h_S(b^{(S)})$) has a core if and only if $V_N \cap P \neq \emptyset$ where P is the polytope of "undominated points", i.e.,

$$P = \bigcap_{S \neq N} \{x \in E^N | x \cdot b^{(S)} \geq h_S(b^{(S)})\}$$

$$= \bigcap_{S \neq N} \{x \in E^N | x \cdot (- b^{(S)}) \leq - h_S(b^{(S)}\} .$$

We may assume without loss of generality that for all $i \in N$, $V_{\{i\}} = \{x \in R^{\{i\}} | x \leqq 0\}$. This implies that P is contained in the nonnegative orthant. By Lemma 7, $V_N \cap P \neq \emptyset$ if and only if for all $\alpha \in E^N$,

$$h_N(\alpha) + h_P(- \alpha) \geq 0 . \tag{7}$$

By Lemma 6,

$$h_P(- \alpha) = \min \{ \sum_{S \neq N} \lambda_S(- h_S(b^{(S)})) | \sum_{S \neq N} \lambda_S(- b^{(S)}) = - \alpha , \lambda_S \geqq 0 , \text{ and } \{b^{(S)} | \lambda_S > 0\} \text{ is linearly independent}\}. \tag{8}$$

$$= - \max \{ \sum_{S \neq N} \lambda_S h_S(b^{(S)}) | \sum_{S \neq N} \lambda_S b^{(S)} = \alpha , \lambda_S \geqq 0 , \text{and } \{b^{(S)} | \lambda_S > 0\} \text{ is linearly independent}\} .$$

Combining (7) and (8) we get that Γ has a core if and only if for all $\alpha \in E^N$, (6) is true. This completes the proof.

Notice that Theorem 4 is a generalization of Lemma 4 which in turn is a generalization of Shapley's theorem (Shapley [15]). Our first proof of Lemma 4 is similar to that of Bondareva [4], who independently obtained

Shapley's results for side-payment games.

It would be interesting to see if Theorem 4 can yield a proof of Theorem 1 for games with convex V_S which does not use Scarf's theorem on inequalities. Thus far the author has not been able to attain this result.

CHAPTER II. EXISTENCE OF BARGAINING SETS FOR COOPERATIVE GAMES WITHOUT SIDE PAYMENTS

1. INTRODUCTION

In [2], Aumann and Maschler introduced the concept of the bargaining set for cooperative games with side payments. In [5] and [10] the bargaining set $M_1^{(i)}$ is defined, and it is proved that for every coalition structure B, there exists a payoff vector x such that the individually rational payoff configuration $(x\,;B) \in M_1^{(i)}$. In [11], Peleg defines a bargaining set $\tilde{M}_1^{(i)}$ for non-side payment games which generalizes $M_1^{(i)}$. He shows, however, that the above existence theorem is not true for $\tilde{M}_1^{(i)}$.

In this chapter we prove a general existence theorem for a class of bargaining sets for games with no side payments. Relatively weak restrictions are put on these games; for example, the payoff sets V_S are not required to be convex. In [11], Peleg assumes convexity. At that time, he proved that the bargaining set $\tilde{M}_1^{(i)}$ is always non-empty for games of pairs, that is, games for which only two person coalitions have power. We shall prove this result without the convexity assumption as a consequence of the general existence theorem.

We shall also define and discuss several different notions of bargaining set for games without side payments. In particular, we define the bargaining set M which generalizes $M_1^{(i)}$ and is non-empty for all non-side payment games.

2. DEFINITIONS AND LEMMAS

Let the set of players be $N = \{1,\dots,n\}$. For each $S \subset N$, let E^S be the Euclidean space of dimension $|S|$ whose coordinates are indexed by the players in S. If $u \in E^N$ the u^S will denote its projection onto E^S. If x and y are vectors we say $x \geq y$ if $x \geqq y$ and $x \neq y$.

For the purposes of this chapter we will use the following definition of an n-person game with no side payments.

Definition: An n-person game without side payments $\Gamma = \{V_S\}_{S \subset N}$ is a collection of sets satisfying the following conditions:

1) For each $S \subset N$, V_S is a closed subset of E^S.
2) If $x \in V_S$ and $y \in E^S$ is such that $y \leqq x$, then $y \in V_S$.
3) For each $i \in N$, $V_{\{i\}} = \{x \in E^{\{i\}} | x \leqq 0\}$.

For a discussion of these and other possible assumptions, and for an extensive bibliography, see Aumann [1]. Property 2 is called comprehensiveness.

We use Ω_S and Ω_S^+ to denote respectively the non-negative and the strictly positive orthant in E^S, i.e.,

$$\Omega_S = \{x \in E^S | x \geqq 0\}$$

$$\Omega_S^+ = \{x \in E^S | x > 0\} .$$

Definition: For $S \subset N$, $\bar{V}_S^+$ is defined as follows:

1) If $V_S \cap \Omega_S^+ \neq \emptyset$, then $\bar{V}_S^+ = \{x \in E^S | x \in \text{closure } (V_S \cap \Omega_S^+)$ and there is no $y \in \text{closure } (V_S \cap \Omega_S^+)$ such that $y > x\}$.

2) If $V_S \cap \Omega_S^+ = \emptyset$ and $V_S \cap \Omega_S \neq \emptyset$, then $\bar{V}_S^+ = \{0^S\}$, where 0^S is the origin in E^S.

3) *If* $V_S \cap \Omega_S = \emptyset$, *then* $\bar{V}_S^+ = \emptyset$.

Definition: *An individually rational payoff configuration (i.r.p.c.) is a pair* (x;B) *where* B *is a partition of* N *into non-empty mutually disjoint subsets, called a coalition structure, and* $x \in E^N$ *is such that for all* $S \in B$, $x^S \in \bar{V}_S^+$.

We call a coalition structure B *acceptable if* $S \in B$ *implies* $V_S \cap \Omega_S \neq \emptyset$ *and* $V_S \cap \Omega_S^+$ *is bounded.*

In [11], Peleg defines an i.r.p.c. differently. In place of $\bar{V}_S^+$ he uses $\bar{V}_S = \{x \in V_S | x \geqq 0$ and there is no $y \in V_S$ such that $y > x\}$. In Lemma 1 we show $\bar{V}_S^+ \subset \bar{V}_S$ for all S . Hence any i.r.p.c. that we consider is also a valid outcome in Peleg's sense.

Lemma 1: $\bar{V}_S^+ \subset \bar{V}_S$ *for all* $S \subset N$.

Proof: We need only consider those S for which $V_S \cap \Omega_S \neq \emptyset$.

Suppose $V_S \cap \Omega_S^+ = \emptyset$. Since $V_S \cap \Omega_S \neq \emptyset$, $0^S \in V_S$ by comprehensiveness. $V_S \cap \Omega_S^+ = \emptyset$ implies there is no $y \in V_S$ such that $y > 0^S$.

Suppose $V_S \cap \Omega_S^+ \neq \emptyset$, and suppose $x \in \bar{V}_S^+ \sim \bar{V}_S$. Then there exists $y \in V_S \sim$ closure $(V_S \cap \Omega_S^+)$ such that

$$y > x \geq 0 \quad .$$

But this implies that

$$y \in V_S \cap \Omega_S^+ \subset \text{closure } (V_S \cap \Omega_S^+)$$

which is impossible. Thus $\bar{V}_S^+ \subset \bar{V}_S$.

Lemma 2: *If* $V_S \cap \Omega_S^+ \neq \emptyset$, *then* $\bar{V}_S^+ = \{x \in E^S | x \in$ closure $(V_S \cap \Omega_S^+)$ *and there is no* $y \in$ closure $(V_S \cap \Omega_S^+)$ *such that* $y \geq x$ *and if* $y^i = x^i$ *then* $y^i = x^i = 0\}$.

Proof: $\bar{V}_S^+$ obviously contains the set on the right. We show the other inclusion. Let $x \in \bar{V}_S^+$. Suppose $y \in$ closure $(V_S \cap \Omega_S^+)$ is such that

$$y^i > x^i \quad \text{for } i \in P \subset S,\ P \neq \emptyset ,$$
$$y^i = x^i = 0 \quad \text{for } i \in S \sim P .$$

Let $\epsilon = \min_{i \in P} (y^i - x^i) > 0$. Let 0_y^ϵ be the open sphere with center y and radius ϵ . Since $y \in$ closure $(V_S \cap \Omega_S^+)$, there exists $z \in 0_y^\epsilon \cap V_S \cap \Omega_S^+$.

For $i \in P$, we have

$$z^i - x^i = z^i - y^i + y^i - x^i \geq z^i - y^i + \epsilon > 0 \quad .$$

For $i \in S \sim P$, $z_i > 0 = x^i$. Hence $z > x$. Since $z \in$ closure $(V_S \cap \Omega_S^+)$ we contradict $x \in \bar{V}_S^+$. Hence $\bar{V}_S^+$ is also contained in the right hand set.

Definition: Suppose S is such that $V_S \cap \Omega_S \neq \emptyset$. We define H_S as follows. If $V_S \cap \Omega_S^+ \neq \emptyset$, then

$$H_S = \{x \in \Omega_S | \sum_{i \in S} x^i = 1\} .$$

Otherwise

$$H_S = \{x \in \Omega_S | \sum_{i \in S} x^i = 0\} = \{0^S\} .$$

Lemma 3: Suppose S is a coalition such that $V_S \cap \Omega_S \neq \emptyset$ and $V_S \cap \Omega_S^+$ is bounded. Then there exists a continuous positive real valued function $d_S(h)$ defined for all $h \in H_S$ such the function

$$\varphi_S\colon H_S \to \bar{V}_S^+ ,$$

defined for $h \in H_S$ by

$$\varphi_S(h) = d_S(h)\, h ,$$

is a homeomorphism.

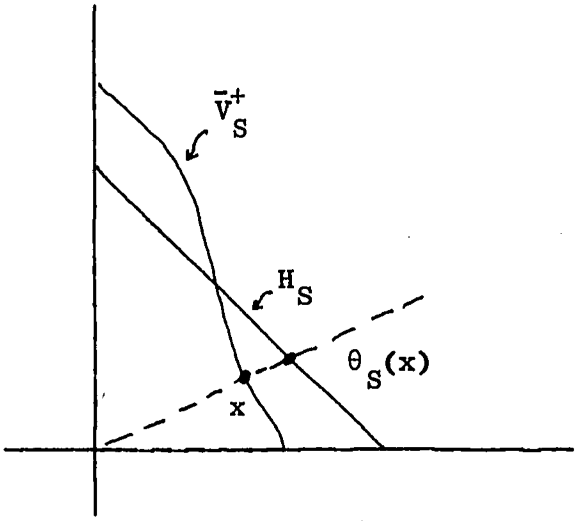
Proof: If $V_S \cap \Omega_S^+ = \emptyset$, then $\bar{V}_S^+ = H_S = \{0^S\}$ and $d_S \equiv 1$ is the required function. Suppose now that $V_S \cap \Omega_S^+ \neq \emptyset$. Define

$$\theta_S : \bar{V}_S^+ \to H_S$$

for $x \in \bar{V}_S^+$ by

$$\theta_S(x) = \frac{x}{\sum_{i \in S} x^i} \quad .$$

Since $V_S^- \cap \bar{\Omega}_S^+ \neq \emptyset$, it is clear that $0^S \notin \bar{V}_S^+$. Thus θ_S is continuous. It is also clear that for $x \in \bar{V}_S^+$, $\theta_S(x) \in H_S$. $\theta_S(x)$ is merely the projection of x onto H_S along the ray from the origin through x .



We first show θ_S is one to one. Suppose that $x, y \in \bar{V}_S^+$, $x \neq y$, and

$$\frac{x}{\sum_{i \in S} x^i} = \frac{y}{\sum_{i \in S} y^i} \quad .$$

Then

$$x = \frac{\sum_{i \in S} x^i}{\sum_{i \in S} y^i} \, y \quad ,$$

which implies that x and y lie on the same ray through the origin.

Since $0^S \notin \bar{V}_S^+$,

$$\frac{\sum_{i\in S} x^i}{\sum_{i\in S} y^i} \neq 0 .$$

Thus we have either $x \geq y$ or $y \geq x$, and if $x^i = y^i$ then $x^i = y^i = 0$. In either case by Lemma 1 we have that one of the points is not in $\bar{V}_S^+$, a contradiction. Thus θ_S is one to one.

We now show θ_S is onto. Let $h \in H_S$. We need to show that there exists $x \in \bar{V}_S^+ \cap r(h)$, where $r(h)$ is the ray from the origin through h. Let $K = \text{closure } (V_S \cap \Omega_S^+)$ and consider $K \cap r(h)$. This is compact since $V_S \cap \Omega_S^+$ is bounded. Hence there exists $x \in K \cap r(h)$ such that

$$\|x\| = \max_{y\in K\cap r(h)} \|y\| .$$

Since $V_S \cap \Omega_S^+ \neq \emptyset$, there is a strictly positive point in V_S. Therefore, by comprehensiveness, $K \cap r(h) \neq \{0\}$, and we must have $\|x\| > 0$.

We show $x \in \bar{V}_S^+ \cap r(h)$. Suppose $x \notin \bar{V}_S^+$. Then there exists $z \in K$ such that $z > x$. By comprehensiveness again,

$$\{u \in \Omega_S | u \leqq z\} \subset K .$$

Choose a real number t such that

$$1 < t \leq \min \left\{ \frac{z^i}{x^i} \,\middle|\, \text{all } i \text{ where } x^i > 0 \right\} .$$

Then for all i, $z^i \geq t\, x^i \geq 0$. Thus $0 \leqq t\, x \leqq z$, and hence $t\, x \in K$. Since $x \in r(h)$, we have also $t\, x \in r(h)$, hence $t\, x \in K \cap r(h)$ and

$$\|tx\| = t\, \|x\| > \|x\| .$$

This contradiction proves $x \in \bar{V}_S^+ \cap r(h)$. Thus θ_S is onto.

We have θ_S is continuous, one to one and onto. It is easy to see that $\bar{V}_S^+$ is closed, hence compact. Thus θ_S is a homeomorphism.

Let $\varphi_S = \theta_S^{-1} : H_S \to \bar{V}_S^+$. Since φ_S is also radial projection, it must be of the form

$$\varphi_S(h) = d_S(h)\, h$$

for each $h \in H_S$, where d_S is a real valued function defined on H_S. Since $0^S \notin \bar{V}_S^+$, we must have for all $h \in H_S$,

$$d_S(h) > 0 .$$

To complete the proof, we must show that d_S is continuous. Let p^i be the projection map from E^S onto the i^{th} coordinate. For $h \in H_S$, $\max\limits_{i \in S} h^i > 0$ and so

$$d_S(h) = d_S(h) \frac{\max\limits_{i \in S} h^i}{\max\limits_{i \in S} h^i} = \frac{\max\limits_{i \in S} d_S(h)\, h^i}{\max\limits_{i \in S} h^i}$$

$$= \frac{\max\limits_{i \in S} p^i\,(\varphi_S(h))}{\max\limits_{i \in S} p^i\,(h)}$$

which is continuous. This completes the proof of Lemma 3.

Let B be an acceptable coalition structure. Define $X^+(B) = \times_{S \in B} \bar{V}_S^+$. Equivalently,

$$X^+(B) = \{x \in E^N \mid (x;B) \text{ is an i.r.p.c.}\} .$$

Let $H(B) = \times_{S \in B} H_S$. The following lemma was proved by Peleg [4].

Lemma (Peleg [10]): Let $\tilde{c}^1(h),\ldots,\tilde{c}^n(h)$ be n non-negative continuous real valued functions defined on $H(B)$. If for each $h \in H(B)$ and each $S \in B$, there is an $i \in S$ such that $\tilde{c}^i(h) \geqq h^i$, then there is an $h_0 \in H(B)$ such that $\tilde{c}^j(h_0) \geqq h_0^j$ for all $j \in N$.

The next lemma is an analogue of Peleg's lemma for $X^+(B)$.

Lemma 4: *Let* B *be an acceptable coalition structure, and let* $c^1(x),\dots,c^n(x)$ *be* n *non-negative continuous real valued functions defined on* $X^+(B)$. *If for each* $x \in X^+(B)$ *and each* $S \in B$ *there is an* $i \in S$ *such that* $c^i(x) \geq x^i$, *then there is an* $x_0 \in X^+(B)$ *such that* $c^j(x_0) \geq x_0^j$ *for all* $j \in N$.

Proof: Let φ_S and d_S be the maps from Lemma 3 for each $S \in B$. Define

$$\varphi_B = \mathop{\times}_{S \in B} \varphi_S : H(B) \to X^+(B)$$

to be the product homeomorphism. Let $\varphi_B^i(h)$ be the i^{th} coordinate of $\varphi_B(h)$ and notice that if $i \in S$, then

$$\varphi_B^i(h) = d_S(h)\, h^i \quad .$$

For each $i \in N$, let

$$\tilde{c}^i : H(B) \to R$$

be defined as follows. If $i \in S$, then for $h \in H(B)$,

$$\tilde{c}^i(h) = \frac{c^i(\varphi_B(h))}{d_S(h)}$$

Since $d_S > 0$ and c^i is non-negative on $X^+(B)$, and all the functions are continuous, we have $\tilde{c}^i$ is a continuous non-negative real valued function on $H(B)$ for all $i \in N$.

For each $h \in H(B)$ and each $S \in B$, there is an $i \in S$ such that

$$c^i(\varphi_B(h)) \geq \varphi_B^i(h) = d_S(h)\, h^i \quad .$$

Thus

$$\tilde{c}^i(h) = \frac{c^i(\varphi_B(h))}{d_S(h)} \geq h^i \quad .$$

Hence the functions $\tilde{c}^i$ satisfy the hypotheses of Peleg's lemma, and there must be an $h_0 \in H(B)$ such that

$$\tilde{c}^j(h_0) \geqq h_0^j \quad \text{for all} \quad j \in N \ .$$

Suppose $j \in S$. Then

$$c^j(\varphi_B(h_0)) = d_S(h_0) \quad \tilde{c}^j(h_0) \geqq d_S(h_0) \ h_0^j \quad .$$

Let $x_0 = \varphi_B(h_0) \in X^+(B)$. We have

$$c^j(x_0) \geqq d_S(h_0) \ h_0^j = \varphi_B^j(h_0) = x_0^j \quad .$$

We have shown that for all $j \in N$

$$c^j(x_0) \geqq x_0^j \quad .$$

This completes the proof.

The next result is also an extension of a result of Peleg. The proof is similar to Peleg's.

Lemma 5: Let B be an acceptable coalition structure, and let $A_1, \ldots, A_n$ be closed subsets of $X^+(B)$. If for each $i \in N$, $A_i \supset \{x \in X^+(B) | x^i = 0\}$, and for each $S \in B$, $\bigcup_{i \in S} A_i = X^+(B)$, then $\bigcap_{i \in N} A_i \neq \emptyset$.

Proof: For $x, y \in X^+(B)$, define

$$\rho(x,y) = \max_{i \in N} |x^i - y^i| \quad .$$

If $X^+(B)$ has only one point, it is easy to see that it must be the origin in E^N . By hypothesis, the origin must be in each A_i, $i = 1, \ldots, n$.

If $X^+(B)$ has more than one point, define $m = \max_{x,y \in X^+(B)} \rho(x,y) > 0.$ Since $X^+(B)$ is compact, $m < \infty$. For each $i \in N$, define for $x \in X^+(B)$

$$c^i(x) = x^i - \frac{x^i}{m} \rho(x,A_i)$$

where $\rho(x,A_i) = \min_{y \in A_i} \rho(x,y) \leq m$. Each $c^i(x)$ is non-negative and continuous. If $x \in X^+(B)$ and $S \in B$, there is a $j \in S$ such that $x \in A_j$, hence $\rho(x,A_j) = 0$ and $c^j(x) = x^j$.

By Lemma 4, there exists an $x_0 \in X^+(B)$ such that $c^j(x_0) \geq x_0^j$ for all $j \in N$. This implies that for all $j \in N$ either $x_0^j = 0$ or $\rho(x_0^j, A_i) = 0$. In the first case, $x_0 \in A_j$ by hypothesis. In the second case, the compactness of A_j implies $x_0 \in A_j$. Hence $x_0 \in \bigcap_{j \in N} A_j$.

3. GENERAL BARGAINING SETS

Let $B = \{S_1, \ldots, S_m\}$ be an acceptable coalition structure for an n-person game Γ. For each $x \in X^+(B)$, let $\tilde{R}^1(x), \ldots, \tilde{R}^m(x)$ be m binary relations defined on $S_1, \ldots, S_m$ respectively. Let $i \in N$, and suppose that $i \in S_k \in B$. Then define

$$E_i = \{x \in X^+(B) \mid i \ \tilde{R}^k(x) \ j \text{ for all } j \in S_k \sim \{i\}\} .$$

We assume the relations $\tilde{R}^1(x), \ldots, \tilde{R}^m(x)$ satisfy the following properties:

1) For each $i \in N$, E_i is closed.

2) For each $i \in N$, $E_i \supset \{x \in X^+(B) \mid x^i = 0\}$.

3) For each $x \in X^+(B)$, and each $S_j \in B$, there is an $i \in S_j$ such that $x \in E_i$.

Intuitively, we can interpret $i \ \tilde{R}^k(x) \ j$ to mean that j has no justified complaint against i in the payoff x, i.e., j is not "stronger" than i at the point x. Then property 2 says that if player i gets zero at x, then no player can complain to i. Property 3 says that for each payoff x, there is a player in each $S \in B$ who is immune from complaints.

This treatment is similar to that of Peleg [12], but his relations $R^k(x)$ are to be interpreted oppositely, i.e., $i\ R^k(x)\ j$ means i is "stronger" than j at x.

Finally we define the *bargaining set for the coalition structure* B *and relations* $\tilde{R}^1,\ldots,\tilde{R}^m$ to be

$$M(B;\ \tilde{R}^1,\ldots,\tilde{R}^m) = \{(x;B) \mid x \in X^+(B) \text{ and for each } S_k \in B, \text{ if } i,j \in S_k \text{ then } i\ \tilde{R}^k(x)\ j\}$$

$$= \{(x;B) \mid x \in \bigcap_{i \in N} E_i\} .$$

Theorem 1: Let $B = \{S_1,\ldots,S_m\}$ *be an acceptable coalition structure and for each* $x \in X^+(B)$ *let* $\tilde{R}^1(x),\ldots,\tilde{R}^m(x)$ *be binary relations defined on* $S_1,\ldots,S_m$ *respectively so that properties 1, 2, and 3 above are satisfied. Then* $M(B;\ \tilde{R}^1,\ldots,\tilde{R}^m) \neq \emptyset$.

Proof: By properties 1, 2, and 3, the E_i satisfy the conditions of Lemma 5. Hence $\bigcap_{i \in N} E_i \neq \emptyset$.

Intuitively, Theorem 1 says that if an acceptable coalition structure B forms, and each coalition S_k in B adopts a "reasonable" standard of stability $\tilde{R}^k(x)$ (i.e., $\tilde{R}^k(x)$ satisfies 1, 2, and 3), then there is a point x in $X^+(B)$ such that the i.r.p.c. $(x;B)$ is simultaneously stable for all coalitions in B.

In the remaining sections we will define various standards of stability and investigate the resulting bargaining sets.

4. THE BARGAINING SET M' AND RELATED BARGAINING SETS

In this section we shall define a standard of stability of the type treated in the last section. This will lead to a bargaining set M' which exists for all non-side payment games. It is shown that in the side-payment case, M' contains the classical bargaining set $M_1^{(i)}$. An example is given where this inclusion is strict. We will then consider two closely related standards of stability and their bargaining sets.

Definition: Let $(x;B)$ *be an* i.r.p.c. *Let* $i,j \in S \in B$. *We say* $i \succ j$ *in* $(x;B)$ *if and only if*

1) *there exists a* $C \in T_{ij} = \{S \subset N \mid i \in S,\ j \notin S\}$ *and a* $y^C \in V_C$ *such that* $y^C > x^C$, *and*

2) *there exists no* $D \in T_{ji}$ *and* $z^D \in V_D$ *such that* $z^D \geqq x^D$ and $z^{D \cap C} \geqq y^{D \cap C}$.

Definition: Let $(x;B)$ *be an* i.r.p.c. *Let* $i,j \in S \in B$. *We say* $i \gg j$ *in* $(x;B)$ *if and only if* $x \in \text{closure } \{x \in X^+(B) \mid i \succ j$ *in* $(x;B)\}$.

In other words, we say $i \succ j$ in $(x;B)$ if i has a justified objection y^C against j in $(x;B)$ in the classical sense (see [10] and [5]). We say $i \gg j$ in $(x;B)$ if arbitrarily close to x there are points $x^{(k)}$ such that $i \succ j$ in the i.r.p.c. $(x^{(k)};B)$. Note that $i \succ j$ in $(x;B)$ implies $i \gg j$ in $(x;B)$.

Definition: Let $(x;B)$ *be an* i.r.p.c. *Let* $i,j \in S \in B$. *We say* $i \, R' \, j$ *in* $(x;B)$ *if and only if*

1) $i \succ j$ *in* $(x;B)$, *and*

2) *there exists no sequence of distinct players* $i_1, i_2, \ldots, i_\alpha \in S \sim \{i,j\}$ *such that* $j \gg i_1 \gg i_2 \gg \ldots \gg i_\alpha \gg i$ *in* $(x;B)$.

We say $i \, \tilde{R}' \, j$ *in* $(x;B)$ *if and only if* $j \, \not{R}' \, i$ *in* $(x;B)$.

Lemma 6: R' *is an acyclic relation for each* i.r.p.c. (x;B), *that is, if* $i_1,\dots,i_\alpha \in S \in B$ *and* $i_1\, R' i_2 R' \dots R' i_\alpha$ *in* (x;B) *then* $i_\alpha \not R' i_1$.

Proof: Suppose $i_1 R' \dots R' i_\alpha$ in (x;B) . Then in particular $i_1 \succ \dots \succ i_\alpha$ in (x;B), which implies $i_\alpha \not R' i_1$.

We define the bargaining set M' to be

$$M' = \{(x;B) \mid (x;B) \text{ is an i.r.p.c., and for each } S \in B, \text{ if } i,j \in S \text{ then } i\tilde{R}'j \text{ in } (x;B)\} .$$

Theorem 2: *Let* B *be an acceptable coalition structure. Then there exists an* $x \in X^+(B)$ *such that* $(x;B) \in M'$.

Proof: Suppose $B = \{S_1,\dots,S_m\}$ is the acceptable coalition structure. For each $x \in X^+(B)$, each $k = 1,\dots,m$ and each $i,j \in S_k$ define $i\ \tilde{R}^k(x)\ j$ if and only if $i\ \tilde{R}'\ j$ in (x;B). We wish to show $M(B;\ \tilde{R}^1,\dots,\tilde{R}^m) \neq \emptyset$. By Theorem 1 we need only show that the relations $\tilde{R}^k$ satisfy the tree properties of the last section.

Let $i \in N$ and suppose that $i \in S \in B$. Then define

$$E_i = \{x \in X^+(B) \mid i\ \tilde{R}'\ j \text{ in } (x;B) \text{ for all } j \in S \sim \{i\}\} .$$

1) E_i is closed:

Let $x \in X^+(B) \sim E_i$. Then there is a $j \in S \sim \{i\}$ such that $j\ R'\ i$ in (x;B) . That is, $j \succ i$ in (x;B), and for every sequence of distinct players in S of the form $i_0 = i,\ i_1,\dots,i_\alpha,\ i_{\alpha+1} = j$, there is a ν, $0 \le \nu \le \alpha$, such that $i_\nu >/> i_{\nu+1}$ in (x;B) .

To show that $X^+(B) \sim E_i$ is open in $X^+(B)$ we need only show that for any $i,j \in S \in B$,

$$O_1 = \{x \in X^+(B) \mid i \succ j \text{ in } (x;B)\} \text{ and}$$

$$O_2 = \{x \in X^+(B) \mid i >/> j \text{ in } (x;B)\}$$

are open in $X^{+}(B)$. It is clear from the definition that O_2 is open in $X^{+}(B)$.

We now consider O_1 . Suppose $i,j \in S \in B$, and $i \succ j$ in $(x;B)$. Then there exists a $C \in T_{ij}$ and a $y^C \in V_C$ such that $y^C > x^C$, and if $D \in T_{ji}$, then $z^D \in V_D$ implies $z^D \not\geqq (x^{D\sim C}, y^{D\cap C})$. Let

$$\delta = \min_{D\in T_{ji}} \inf_{z^D\in V_D} \max_{i\in D} \left[(x^{D\sim C}, y^{D\cap C})^i - z^i\right] .$$

By the above, $\delta \geq 0$. Suppose $\delta = 0$. Then for some $D \in T_{ji}$ there exists a sequence $z^D_{(k)} \in V_D$, $k = 1,2,\ldots,$ such that

$$0 < \max_i \left[(x^{D\sim C}, y^{D\cap C})^i - z^i_{(k)}\right] < \frac{1}{k} .$$

Define another sequence $\bar{z}^D_{(k)}$, $k = 1,2,\ldots,$ as follows. For $i \in D$,

$$\bar{z}^i_{(k)} = \begin{cases} (x^{D\sim C}, y^{D\cap C})^i & \text{if } z^i_{(k)} \geq (x^{D\sim C}, y^{D\cap C})^i \\ z^i_{(k)} & \text{otherwise.} \end{cases}$$

By comprehensiveness, $\bar{z}^D_{(k)} \in V_D$ for $k = 1,2,\ldots$. But $\bar{z}^D_{(k)} \to (x^{D\sim C}, y^{D\cap C})$ as $k \to \infty$, and since V_D is closed, we must have $(x^{D\sim C}, y^{D\cap C}) \in V_D$, a contradiction. Hence we have $\delta > 0$.

Let ϵ be a real number such that

$$0 < \epsilon < \min\,(\delta, \min_{i\in C} |y^i - x^i|) .$$

We will show

$$O' = \{\bar{x} \in X^{+}(B) \mid \|\bar{x} - x\| < \epsilon\} \subset O_1$$

and hence O'_1 is open in $X^{+}(B)$. Let $\bar{x} \in O$. We must show $i \succ j$ in $(\bar{x};B)$.

First note that $y^C > \bar{x}^C$ by choice of ϵ . Suppose for some $D \in T_{ji}$, there is a $z^D \in V_D$ such that $z^D \geqq (\bar{x}^{D\sim C}, y^{D\cap C})$. Then we have

$$0 \geqq \max_{i \in D} \left[(\bar{x}^{D \sim C}, y^{D \cap C})^i - z^i \right]$$

$$= \max_{i \in D} \Big[(\bar{x}^{D \sim C}, y^{D \cap C})^i - (x^{D \sim C}, y^{D \cap C})^i + (x^{D \sim C}, y^{D \cap C})^i - z^i \Big] .$$

Let $i_0 \in D$ be such that

$$(x^{D \sim C}, y^{D \cap C})^{i_0} - z^{i_0} = \max_{i \in D} \left[(x^{D \sim C}, y^{D \cap C})^i - z^i \right] .$$

Then we have

$$0 \geqq \max_{i \in D} \left[(x^{D \sim C}, y^{D \cap C})^i - z^i \right] + (\bar{x}^{D \sim C}, y^{D \cap C})^{i_0} - (x^{D \sim C}, y^{D \cap C})^{i_0}$$

$$\geqq \delta - \epsilon > 0 .$$

This contradiction shows that $\bar{x} \in O_1$, and we have proven that O_1 is open in $X^+(B)$. This concludes the proof that E_i is closed for each $i \in N$.

2) $E_i \supset \{x \in X^+(B) \mid x^i = 0\}$:

Since $0 \in V_{\{i\}}$, if $x^i = 0$ then there is no j such that $j \succ i$ in $(x;B)$. Therefore there is no j such that $j \; R' \; i$ in $(x;B)$.

3) For each $x \in X^+(B)$, and each $S \in B$, there is an $i \in S$ such that $x \in E_i$:

Suppose not. Then for some $x \in X^+(B)$ and some $S \in B$, for each $i \in S$ there is a $j \in S$ such that $j \; R' \; i$ in $(x;B)$. Thus there exist players $i_1, \ldots, i_\alpha \in S$ such that $i_1 \; R' \; i_2 \; R' \ldots R' \; i_\alpha \; R' \; i_1$ in $(x;B)$, which is a contradiction to acyclicity.

This completes the proof of Theorem 2.

It would have been reasonable to define a game so that for each $S \subset N$, $V_S \cap \Omega_S \neq \emptyset$ and compact. Then every coalition structure would be acceptable, and thus would satisfy Theorem 2. Note that for side payment games, if the usual requirement is made that for all $S \subset N$, $v(S) \geq 0$, then every coalition structure is acceptable. Also, in this case

$$X^{+}(B) = X(B) = \underset{S \in B}{X} \left\{ x \in \Omega_S \middle| \sum_{i \in S} x^i = v(S) \right\},$$

which is the payoff space usually considered for the coalition structure (see Peleg [10]).

Recall that the bargaining set $M_1^{(i)}$ for side-payment games is the set of all i.r.p.c.'s $(x;B)$ where for all $i,j \in S \in B$, $i \not\succ j$ in $(x;B)$. Thus for side-payment games, $M' \supset M_1^{(i)}$. The following example shows this inclusion may be strict.

Example 1: Let $n = 7$. Suppose $v(1234) = 100$, $v(15) = 26$, $v(1256) = 51$, $v(367) = 26$, and $v(475) = 26$. Let $v(S) = 0$ for all other $S \subset N$. Let $B = \{1234,5,6,7\}$, and let

$$(x;B) = (25,25,25,25,0,0,0;\ 1234,5,6,7) .$$

To see that $(x;B) \notin M_1^{(i)}$, note that $1 \succ 2$ via the justified objection $y^{15} = (25\frac{1}{2}, \frac{1}{2})$. (See the discussion following the definition of the relation $\succ$). To see that $(x;B) \in M'$, one can first verify by inspection that there is no other justified objection other than $1 \succ 2$, hence for $0 \leq i, j \leq 4$, $(i,j) \neq (2,1)$, $i\ \tilde{R}'\ j$. Finally, $2\ \tilde{R}'\ 1$ is true since $2 \gg 3$, $3 \gg 4$, $4 \gg 1$ in $(x;B)$. To check these three, notice that for $0 < \epsilon \leq 25$:

a) $2 \succ 3$ in $(25-\epsilon, 25-\epsilon, 25+\epsilon, 25+\epsilon, 0,0,0;B)$ via $y^{1256}=(25, 25-\frac{\epsilon}{2}, \frac{\epsilon}{2}, 1)$

b) $3 \succ 4$ in $(25,25,25-\epsilon,25+\epsilon,0,0,0;B)$ via $y^{367} = (25-\frac{\epsilon}{2}, \frac{\epsilon}{2}, 1)$

c) $4 \succ 1$ in $(25+\epsilon,25,25,25-\epsilon,0,0,0;B)$ via $y^{475} = (25-\frac{\epsilon}{2}, \frac{\epsilon}{2}, 1)$.

At this point, two other bargaining sets will be discussed. Both have the desired property that for side-payment games, they reduce to $M_1^{(i)}$.

The first one, however, does not exist for all nonside-payment games. The existence of the second is still an open problem.

The first set is defined as M' was defined, except that R' is replaced by R'' where $i\ R''\ j$ means $i \succ j$ and $j \not\gg i$. To see that this reduces to $M_1^{(i)}$ for side-payment games, we show that in such a game $i\ R''\ j$ in $(x;B)$ if and only if $i \succ j$ in $(x;B)$. Suppose $i \succ j$ and also $j \gg i$ in $(x;B)$. Then there is an open neighborhood of x in $X(B)$ such that $i \succ j$ in $(\bar{x};B)$ for all $\bar{x}$ in this neighborhood. By definition, there is an $\bar{x}_0$ in the neighborhood such that $j \succ i$ in $(\bar{x}_0;B)$. Hence $i \succ j \succ i$ in $(\bar{x}_0;B)$, which contradicts the acyclicity of $\succ$ for side-payment games (see Davis and Maschler [5]).

Example 2: Let $n = 6$. Define

$$V_{123} = \{x^{123} | x^1 + x^2 + x^3 \leqq 1\}$$
$$V_{145} = \{x^{145} | x^1 \leqq 2,\ x^4 \leqq 4,\ x^5 \leqq 3\}$$
$$V_{256} = \{x^{256} | x^2 \leqq 2,\ x^5 \leqq 4,\ x^6 \leqq 3\}$$
$$V_{346} = \{x^{346} | x^3 \leqq 2,\ x^4 \leqq 3,\ x^6 \leqq 4\}$$
$$V_S = \{x^S \in E^S | x^S \leqq 0^S\} \text{ for all other } S \subset N .$$

Consider the coalition structure $B = \{123,4,5,6\}$. It is easy to check that for any i.r.p.c. $(x;B)$, $1\ R''\ 3$ unless $x^3 = 0$ and
$3\ R''\ 2$ unless $x^2 = 0$ and
$2\ R''\ 1$ unless $x^1 = 0$.

Since in any i.r.p.c. $(x;B)$, not all of x^1, x^2 and x^3 are zero, there is no stable point with respect to R'' .

The second set is defined similarly except R' is replaced by R''' where for $i,j \in S \in B$, $i\ R'''\ j$ in $(x;B)$ means $i \succ j$ in $(x;B)$

and there is no sequence of distinct players $i_1, i_2, \ldots, i_\alpha \in S \sim \{i,j\}$ such that $j \succ i_1 \succ i_2 \succ \ldots \succ i_\alpha \succ i$ in $(x;B)$. By the acylicity of $\succ$ in side-payment games, it is clear that in a side-payment game $i\ R''' j$ if and only if $i \succ j$. Hence this bargaining set also reduces to $M_1^{(i)}$. At this time it is not known whether Theorem 2 is true for this set for all non-side-payment games. The difficulty lies in the fact that the sets E_i (defined for R''') are not necessarily closed.

5. GAMES OF PAIRS

In this section we extend Peleg's result on games of pairs.

Definition: An n-person game $\Gamma = \{V_S\}_{S \subset N}$ is called a game of pairs if whenever $S \subset N$ and $|S| \neq 2$, then

$$V_S = \{x \in E^S | x \leqq 0^S\} .$$

The bargaining set $\tilde{M}_1^{(i)}$ for non-side payment games is defined exactly as $M_1^{(i)}$ for side-payment games, i.e., the set of all i.r.p.c.'s $(x;B)$ where for all $i,j \in S \in B$, $i \not\succ j$ in $(x;B)$.

Theorem 3: Let Γ be a game of pairs and let B be an acceptable coalition structure for Γ. Then there exists an $x \in X^+(B)$ such that $(x;B) \in \tilde{M}_1^{(i)}$.

Proof: Let $B = \{S_1, \ldots, S_m\}$ be the acceptable coalition structure. For each $x \in X^+(B)$, each $k = 1, \ldots, m$ and each $i,j \in S_k$, define $i\ \tilde{R}^k(x) j$ if and only if $j \not\succ i$ in $(x;B)$. Again we wish to show $M(B; \tilde{R}^1, \ldots, \tilde{R}^k) \neq \emptyset$.

Define

$$E_i = \{x \in X^+(B) | \text{for all } j \in S \sim \{i\},\ j \not\succ i \text{ in } (x;B)\ ,$$

where $i \in N$ is such that $i \in S \in B$. In the proof of Theorem 2, we showed for all $i,j \in S \in B$,

$$\{x \in X^+(B) \mid i \succ j \text{ in } (x;B)\}$$

is open in $X^+(B)$. Hence each E_i is closed. We also showed in the proof of Theorem 2 that if $x^i = 0$, then there is no j such that $j \succ i$ in $(x;B)$. Hence

$$E_i \supset \{x \in X^+(B) \mid x^i = 0\} \text{ for all } i \in N .$$

We need only show that for each $x \in X^+(B)$, and each $S \in B$, there is an $i \in S$ such that $x \in E_i$. But this is true since in a game of pairs it is easy to see that the relation $\succ$ is acyclic.

6. RELATIVE BARGAINING POSITION - THE BARGAINING SET M

In this section we shall discuss the notion of a player's relative bargaining position at a given payoff. This will lead us to define another standard of stability which gives rise to a bargaining set M.

M will be shown to exist for all non-side-payment games and to reduce to $M_1^{(i)}$ in the case of a side-payment game.

Let B be an acceptable coalition structure. Let $i \in N$, and suppose $i \in S \in B$. If $\bar{V}_S^+ \neq \{0^S\}$, let $c^i = \max\limits_{z \in \bar{V}_S^+} z^i > 0$. Otherwise let $c^i = 1$. Also define

$$F_i = \{x \in X^+(B) \mid \text{for all } j \in S \sim \{i\},\ j \not\succ i \text{ in } (x;B)\} .$$

For $x,y \in X^+(B)$, define

$$d_B(x,y) = \sum_{i \in N} \frac{|x^i - y^i|}{c^i} .$$

d_B is a metric on $X^+(B)$ (in fact on E^N) equivalent to the Euclidean distance metric.

For $x \in X^+(B)$, let

$$d_B(x,F_i) = \min_{y \in F_i} d_B(x,y) .$$

Since F_i is compact, we have $d_B(x,F_i)$ is a continuous function of x, and $d_B(x,F_i) = 0$ if and only if $x \in F_i$.

Notice that the quantity $\frac{|x^i - y^i|}{c^i}$ measures the difference in x^i and y^i relative to the maximum that player i can get as a member of S . It is easy to see that if each player's utility units are multiplied by positive scaling factors, then the relative distance between payoffs in the original game is the same as the relative distance between the rescaled payoffs in the rescaled game. Since it is also true that the set F_i for the rescaled game is just a rescaling of F_i for the original game, we have that $d_B(x,F_i)$ is invariant under changes in the player's utility units.

We can interpret $d_B(x,F_i)$ as follows: it is the least total relative change that the payoff x must undergo in order for player i to achieve stability, that is, to have no justified objections against him. We will consider player i to have a stronger bargaining position than player j relative to a payoff x if $d(x,F_i) < d(x,F_j)$, i.e., less total change is necessary for player i to achieve stability than for player j . In some sense, it is "easier" for i to arrange a change to become stable than for j .

Definition: *Let* B *be an acceptable coalition structure, and let* $(x;B)$ *be an* i.r.p.c. . *Let* $i,j \in S \in B$. *We say* $i \, R \, j$ *in* $(x;B)$ *if and only if*

1) $i \succ j$ *in* $(x;B)$, *and*

2) $d_B(x,F_i) < d_B(x,F_j)$.

We say $i \, \tilde{R} \, j$ *in* $(x;B)$ *if and only if* $j \not{R} \, i$ *in* $(x;B)$.

Thus, $i \, R \, j$ means i has a justified objection against j at

$(x;B)$ and, in addition, i can enforce this objection since he has a better bargaining position relative to the payoff x. Intuitively, j would not take i's justified objection seriously if i was "deep in debt" while j was fairly "solvent".

Lemma 7: R *is an acyclic relation for each i.r.p.c.* $(x;B)$.

Proof: This follows immediately from Property 2 in the definition of R.

We now define the bargaining set for the relation $\tilde{R}$ to be

$$M = \{(x;B) | B \text{ is an acceptable c.s.}, \ x \in X^{+}(B), \text{ for each } S \in B,$$
$$\text{if } i,j \in S \text{ then } i \ \tilde{R} \ j \text{ in } (x;B)\} .$$

Theorem 4: *For any acceptable c.s.* B, *there exists an* $x \in X^{+}(B)$ *such that* $(x;B) \in M$.

Proof: Let B be an acceptable coalition structure. Define

$$E_i = \{x \in X^{+}(B) | i \ \tilde{R} \ j \text{ in } (x;B) \text{ for all } j \in S \sim \{i\}\} .$$

From earlier considerations with the relation $\succ$, it follows that

$$E_i \supset \{x \in X^{+}(B) | x^{i} = 0\} .$$

In addition, the continuity of $d_B(x,F_k)$ for all $k \in N$ implies that E_i is closed. Also as before, the acyclicity of R implies that for each $x \in X^{+}(B)$, and each $S \in B$, there is an $i \in S$ such that $x \in E_i$. Hence by Theorem 1 there is an $x \in X^{+}(B)$ such that $(x;B) \in \bar{M}$.

Theorem 5: *For side-payment games*, $M = M_1^{(i)}$.

Proof: Since $i \not\succ j$ implies $i \not{\tilde{R}} \ j$, we have $M_1^{(i)} \subsetneq M$. Suppose $(x;B) \in M \sim M_1^{(i)}$. Then there exist players $i,j \in S \in B$ such that $i \succ j$ at $(x;B)$ and $d_B(x,F_i) \geq d_B(x;F_j)$. But $i \succ j$ at $(x;B)$ implies $d_B(x;F_j) > 0$, hence $d_B(x,F_i) > 0$. This implies, however, that there exists a player $k \in S$ such that $k \succ j$ at $(x;B)$ and

$d_B(x,F_k) \geq d_B(x,F_i) \succ 0$. This process continues indefinitely (i.e. there is an $\ell \succ k$, etc.), and since there are only a finite number of players, we obtain a contradiction to the acyclicity of $\succ$ for side-payment games. Thus $M \subset M_1^{(i)}$.

7. CONCLUDING REMARKS

The bargaining set M defined in section 6 offers a solution to the problem of generalizing $M_1^{(i)}$ to games without side-payments. However, new problems now arise. For side-payment games, the bargaining set $M_1^{(i)}(= M)$ led to the definition of two other solution concepts, the kernel K and the nucleolus ν, which satisfy $M_1^{(i)} \supset K \supset \nu \neq \emptyset$. (For the kernel, see Davis and Maschler [6], and Maschler and Peleg [9]. For the nucleolus, see Schmeidler [14]). One virtue of the kernel is that it is easier to compute than $M_1^{(i)}$. The nucleolus has the property that for the coalition structure N, it is a unique point which varies continuously with changes in the game.

It would be interesting to know if it is possible to generalize these and get a kernel or nucleolus for the bargaining set M for all non-side-payment games. At this time, this question has not been answered.

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AUTOBIOGRAPHICAL STATEMENT

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