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CONCERNING FIXED-POINT FREE ORIENTATION PRESERVING HOMEOMORPHISMS OF THE EUCLIDEAN PLANE

by

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INTRODUCTION

The investigations presented here are motivated by the following general problems:

A. Given two homeomorphisms t, u of a topological space X onto itself, when is t conjugate to u? That is, when does there exist a homeomorphism k of X onto itself such that $kt k^{-1} = u$?

B. Given a homeomorphism t from a topological space X onto itself, when is t embeddable in a flow?

There are certain obvious necessary conditions for an affirmative answer to problem A. For example, the induced maps t_*, u_* on the homotopy groups $\pi(X)$ must be conjugate and the orbits of t must correspond homeomorphically to the orbits of u. Similarly for an affirmative solution to problem B, it is clear that t_* must be the identity and that it must be possible to decompose X into a disjoint union of curves in such a way that each t orbit is contained in a single curve.

Invariants of these kinds are reasonably well understood. This paper is an attempt to shed some light on the nature of some less understood types of invariants by studying special cases in detail. In particular, we will be concerned with a fixed-point free orientation preserving homeomorphism t of the Euclidean plane R^2 onto itself. Mappings t with these properties are called free mappings. In the case of problem A the map u will always be the unit translation in the positive horizontal direction.

The invariants described above are all useless for determining if t and u are conjugate because $\pi_i(R^2) = 0$ for $i \geq 1$ and the orbits of all points under t and u are all homeomorphic by a Theorem of Brouwer (cf. Sperner [7, p. 14]). Also $t_{::}$ is the identity, so the first type of invariant described for problem B is trivial; however, we shall make some use of the second.

It is well known that there are free mappings which are not conjugate to translations [4]. In addition it is shown by an example in Andrea [1, p. 68] that there do exist free mappings which are not embeddable in flows. A different example with some interesting properties will be given in chapter II.

A free mapping of the plane can also be viewed as a homeomorphism of $S^2 = R^2 \cup \{\infty\}$ with a unique fixed point at ∞. An examination of the example in chapter II suggests that all of the obstructions to embedding a free mapping t in a flow or to making t conjugate to a translation may have to do with the properties of t (extended to S^2) near ∞. Our main purpose here is to investigate this possibility. To this end, we make the definitions below which give a more precise meaning to the question.

We shall say that a free mapping t is <u>boundedly conjugate to a translation</u> if for each bounded subset B of the plane there exists a homeomorphism h of the plane onto itself such that the restriction of hth^{-1} to h (B) is the translation $(x,y) \mapsto (x+1,y)$.

We shall say that a free mapping t is <u>boundedly embeddable in a flow</u> if for each bound subset B of R^2, there exists a family $\{T^a : a \in R^1\}$ of homeomorphisms of the plane onto itself with the properties:

1. T^{o} = identity;
2. $T^{a+b} = T^{a} \circ T^{b}$;
3. $T^{1}(x,y) = t(x,y)$ for each (x,y) in B;
4. T^{a} has no fixed points for $a \neq 0$;
5. $T^{a}(x,y)$ is continuous in (a,x,y).

We shall call such a family of homeomorphisms a free flow.

Our main concern from now on will be the problems analogous to A and B for free mappings and translations of the Euclidean plane:

A'. When is a free mapping boundedly conjugate to a translation?

B'. When is a free mapping boundedly embeddable in a flow?

Kamke [5] has essentially solved the problem which is analogous to problem A' for the case of a flow. We will employ his result in the first chapter to obtain a relation between the problems A' and B'. By means of this relation we will see that the study of the problems A' and B' reduces to the consideration of the single apparently simpler problem B'.

That B' is itself a difficult problem will be shown in chapter II where an example will be given which illustrates some of the difficulties encountered in trying to embed a free mapping in a flow. The example points out how much the global behavior of a free map can differ from one which is embeddable in a flow. Nevertheless, we will develop a technique which will boundedly embed the free mapping under consideration in a flow. Hence that free mapping will be shown to be boundedly conjugate to a translation.

Finally, in chapters III and IV an attempt will be made at boundedly embedding a C^1 free mapping in a flow. Although the attempt is not successful, we feel that the methods developed are of some interest and perhaps some variation of them could be used to solve the general bounded problems.

In this paper p_i, $i=1,2$ will denote the projection onto the i^{th} coordinate axis of the Euclidean plane R^2. Z will denote the integers, d and $|\ |$ will denote Euclidean distance. $\bar{A}$, $\mathring{A}$, A' will denote respectively the closure, interior, boundary of a subset A of the plane.

A mapping f of a topological space χ^1 into a topological space χ^2 is said to be proper if the inverse image by f of each compact set in χ^2 is compact in χ^1. We will be concerned with the case $\chi^1=R^1$ and $\chi^2=R^2$. The image of R^1 by a proper mapping is said to be properly embedded in the plane.

Chapter I

Bounded Conjugacy and Bounded Embeddability

In this and the following chapters all homeomorphisms involved in the definitions of "boundedly embeddable in a flow" and "boundedly conjugate to a translation" as well as their inverses will be assumed to be Lipschitz continuous.

Theorem I. 1. A free mapping of the Euclidean plane is boundedly conjugate to a translation if and only if it is boundedly embeddable in a flow.

Proof. The proof will be sketched since the details omitted can be found in Kamke [5].

Let t be a free mapping of the plane which is boundedly conjugate to a translation. Then for each bounded subset B of the plane, there exists a Lipschitz continuous homeomorphism h of the plane onto itself such that $hth^{-1}(x,y) = (x+1,y)$ for each point (x,y) that belongs to $h(B)$. Let $\{t^a: a \in R^1\}$ be the flow given by $t^a(x,y) = (x+a,y)$ for each a in R^1 and (x,y) in R^2. Then the flow $\{ht^ah^{-1}: a \in R^1\}$ is easily seen to have the properties required to show that t is boundedly embeddable in a flow.

Conversely suppose that T is a Lipschitz continuous free mapping of R^2 which is boundedly embeddable in a flow. Since each bounded subset B of the plane is contained in a closed disk, we can assume that B is a closed disk.

Let $\{F^a: a \in R^1\}$ be a Lipschitz continuous free flow in the plane such that F^1 agrees with T on B. By part of the proof of

Satz I (Kamke [5, p.292]) there exists a Lipschitz continuous free flow $\{T^s : s \in R^1\}$ such that T^s agrees with F^s on B and $T^s(x,y)=(x+s,y)$ for $0 \leq s \leq 1$ on the exterior of some open disk D containing B. Thus there exists x_o in R^1 such that $x_o \leq x$ implies that $T^s(x,y) = (x+s,y)$ for $0 \leq s \leq 1$. Since $\{T^s : s \in R^1\}$ is a Lipschitz continuous free flow we have by the Poincaré-Bendixon Theorem (cf. Kamke [6]) that for each (x,y) in R^2 the flow line containing (x,y) (i.e. $\bigcup \{T^s(x,y) : s \in R^1\}$) is properly embedded in the plane. Clearly it also follows from the Poincaré-Bendixon Theorem that to each point (x_o,t) in the plane (x_o as above) and real number s, there exists a point (x,y) in the plane such that $T^s(x_o,t) = (x,y)$. Therefore the mapping $h: R^2 \to R^2$ given by $h(x,y) = (s,t)$ is a Lipschitz continuous homeomorphism. Thus for each point (x,y) in B, $hT(x,y) = (p_1h(x,y)+1,\ p_2h(x,y))$. Hence T is boundedly conjugate to a translation.

Remark I.1. One could also prove that the free mapping T above is "boundedly conjugate to a translation" by observing that the free mapping T^1 above is such that any disk has a non-empty intersection with at most finitely many images of itself by T^n, n in Z, or by noting that any arc joining two points is such that at most finitely many images of itself by powers of T^1 have a non-empty intersection with a given bounded set. Then by applying Brouwer's Plane Translation Theorem (cf. Sperner [7], Andrea [1,2], or Barrar [3, p. 343] for additional references) we have that T^1 is conjugate to a translation of the plane. Using this approach we prove only that T^1 is conjugate to a translation by a homeomorphism, not by a Lipschitz continuous homeomorphism as was obtained using the results of Kamke.

Remark I. 2. If a free mapping T is embeddable in a flow, then clearly T is boundedly embeddable in a flow. Hence T is boundedly conjugate to a translation.

Chapter II

Examples

We begin by describing a well known free mapping which is not conjugate to a translation.

Example II.1. (cf. Andrea [1, p. 67]).

Let $\{F^a: a \in R^1\}$ be the flow that is defined by the solutions of the differential equation

$$\frac{dx}{dt} = \begin{cases} 1 - x^2 & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases};$$

$$\frac{dy}{dt} = \begin{cases} x & \text{for } |x| < 1 \\ 1 & \text{for } |x| \geq 1 \end{cases}.$$

Clearly F^a is a Lipschitz continuous mapping of R^2 for each $a \neq 0$. However, no F^a, $a \neq 0$ is conjugate to a translation since each disk containing $[-1,1] \times \{0\}$ has a non-empty intersection with each of its images by F^a, $a \neq 0$. Nevertheless, F^a, $a \neq 0$ is boundedly conjugate to a translation by the remark at the end of the previous chapter.

In the next example a free mapping is described which has global properties very different from those of a flow. In order to discuss these properties we make the following definition. A bounded subset B of the plane diverges with respect to a free mapping t of the plane if at most finitely many images of B by powers of t have a non-empty intersection with any bounded subset of the plane.

The above definition is motivated by Sperner's version of Brouwer's Plane Translation Theorem (c.f. Sperner [7, p.47] or Andrea [1, p. 71]): A fixed-point free orientation preserving homeomorphism t of the Euclidean plane onto itself is conjugate to a translation if and only if every bounded subset of the plane diverges with respect to t.

The free mapping in the next example has the surprising property that there is a point in the plane such that no connected set properly containing the point diverges. Such points are not contained in any properly embedded invariant curve. Since the Poincaré-Bendixon Theorem implies that every flowline of a fixed-point free mapping is properly embedded in the plane, the free mapping in the second example can not be embedded in a flow. Nevertheless, that free mapping will be shown to be boundedly embeddable in a flow and hence boundedly conjugate to a translation.

We now proceed to define the free mapping described above.

Example II. 2.

Let Φ be a C^1 diffeomorphism of the real line satisfying the following conditions:

1. $\Phi(0) = 0$;
2. $\Phi(x) < x$ for $x \neq 0$;
3. $\Phi(x) = x - 3$ for $|x+1| \geq 2$.

It is clear that such a function Φ can be defined and will leave no point fixed except 0.

Let $S(x,y) = (x-3) \sin^2 \pi y + \Phi(x) \cos^2 \pi y$.

Define $t(x,y) = (S(x,y), y+1)$. Clearly t is a C^1 free mapping of R^2. Incidentally, t is also conjugate to a translation, since it is easily seen that every bounded subset of the plane diverges with respect to t.

We now map the (x,y) plane into the (u,v) plane by defining $k(x,y)=(u,v)$ for each (x,y) in R^2 where $u=\arctan x$ and $v=y+x$. Clearly k is a C^1 diffeomorphism from the (x,y) plane onto the region $G=\{(u,v) : |u|<\frac{\pi}{2}\}$.

Define $f(u,v)= kt\,(x,y)$ so that $f^n(u,v)=kt^n k^{-1}(u,v)$ for each integer n. We then have

1. $\lim_{u\to\pm\frac{\pi}{2}}(p_1f(u,v)) - p_1(u,v)) = \lim_{x\to\pm\infty}(\arctan(x-3)-\arctan x)=0$;
2. $\lim_{u\to\pm\frac{\pi}{2}}(p_2(f(u,v)) - p_2(u,v))= \lim_{x\to\pm\infty}(y+x-2-(y+x))= -2$.

We are now in a position to define a Lipschitz continuous free mapping on the (u,v) plane with the interesting properties previously mentioned. We proceed by extending the mapping f to a Lipschitz continuous free mapping T defined as follows:

$$T(u,v) = (u,v-2) \text{ for } |u| \geq \frac{\pi}{2};$$

$$T(u,v) = f(u,v) \text{ for } |u|<\frac{\pi}{2}.$$

The properties 1. and 2. above imply that T is Lipschitz continuous.

To see that any bounded connected set properly containing (0,0) does not diverge we look at T^n for large n. In particular we will show that infinitely many images by T^n ($n\in Z$) of such a connected set have a non-empty intersection with $L=\{(u,v): |u|\leq\frac{\pi}{2},\ v=0\}$.

Suppose that C is a connected set properly containing (0,0). Let (u_0, v_0) be a point in the component of $C\cap G$ containing (0,0) and different from (0,n) (n in Z) and let $(x_0, y_0) = k^{-1}(u_0,v_0)$. We then have $u_0 = \arctan x_0$ and $v_0 = y_0 + x_0$.

We now examine $\lim_{m \to \pm\infty} T^m \ (u_0, v_0)$.

Case 1: $y_0 \notin Z$.

For sufficiently large integers n we have $p_1 \ t^n \ (x_0, y_0) < -1$. Then $\lim_{m \to +\infty} T^m \ (u_0, v_0) = (-\frac{\pi}{2}, -\infty)$.

Case 2: $y_0 \in Z$, $x_0 > 0$.

Then $\lim_{m \to -\infty} T^m \ (u_0, v_0) = (\frac{\pi}{2}, +\infty)$.

Case 3: $y_0 \in Z$, $x_0 < 0$.

Then $\lim_{m \to +\infty} T^m \ (u_0, v_0) = (-\frac{\pi}{2}, -\infty)$.

We now observe that for each integer m, $p_2 \ (T^m \ (0,0)) = m$.

In case 1, there exists an integer $M > 0$ such that for each integer $m > M$ $p_2 \ (T^m \ (u_0, v_0)) < 0$. But also $p_2 \ (T^m \ (0,0)) > 0$.

In case 2, there is an integer $M > 0$ such that $m < -M$ implies that $p_2 \ (T^m \ (u_0, v_0)) > 0$. Here we also have $p_2 \ (T^m \ (0,0)) < 0$.

In case 3, there exists an integer $M > 0$ such that $m > M$ implies that $p_2 \ (T^m \ (u_0, v_0)) < 0$. In this case $p_2 \ (T^m \ (0,0)) > 0$.

The above three cases imply that for all but finitely many integers $T^m (C)$ contains points both above and below the line segment L. Since $T^m \ (C)$ is connected and contained in G, the previous statement implies that infinitely many $T^m \ (C)$ have a non-empty intersection with L. Therefore C can not diverge with respect to T.

Thus we see in particular that there is no curve through $(0,0)$ which is invariant under T and is properly embedded in the plane. Therefore as mentioned before, it follows from the Poincaré-Bendixon Theorem that T can not be embedded in a flow. Because of this fact, it is not readily seen that T is boundedly embeddable in a flow.

Nevertheless, we now show that T is, indeed, boundedly embeddable in a flow and hence is boundedly conjugate to a translation.

Our purpose now is to define for each integer N a free mapping of the (u,v) plane which is embeddable in a Lipschitz continuous flow and agrees with T except on the set $H=\{(u,v): |u|<\frac{\pi}{2}, |v|>N\}$. To do this we define a free mapping t^* on the (x,y) plane which agrees with t on the set $k^{-1}(E)$ where $E=G\setminus H=\{(u,v): u<\frac{\pi}{2}, |v|\leq N\}$, so that $k^{-1}(E)=\{(x,y): |x+y|\leq N\}$.

Clearly there exists a non-decreasing C^1 function g on R^1 satisfying:

1. $g(y) = y$ for $-N-4\leq y\leq N+1$;
2. $g(y) = N+3/2$ for $y\geq N+3/2$;
3. $g(y) = -N-9/2$ for $y\leq -N-9/2$.

Put $S^*(x,y) = (x-3)\sin^2(\pi g(y)) + \phi(x)\cos^2(\pi g(y))$ where ϕ is as before. Then we note that

$S^*(x,y) = S(x,y)$ for $-N-4\leq y\leq N+1$, $x\leq -1$, or $x\geq 4$;

$S^*(x,y) = x-3$ for $y\leq -N-9/2$ or $y\geq N+3/2$.

Define $t^*(x,y) = (S^*(x,y), y+1)$. Clearly t^* is a free mapping of the (x,y) plane which agrees with t on $k^{-1}(E)$.

Let $B(b)$ be the intersection of the line $y=-(1/3)x+b$ with the strip $-N-13/2\leq y\leq -N-11/2$. Then $t^{*-1}(B(b))\cup B(b)\cup t^*(B(b))$ is a single straight line segment and $B^*(b)=\bigcup_{i=-\infty}^{\infty} t^{*i}(B(b))$ is a C^1 curve for each real number b. We further observe that if $b\leq -N-29/6$ or $b\geq N+17/6$, then $B^*(b)$ is just the straight line $y=-(1/3)x+b$.

We now embed t^* in a flow.

Let a be a real number. Whenever both $y\leq -9/2$ and

$y + a \leq -9/2$, define $F^{*a}(x,y) = (x-3a, y+a)$. Then the line segment joining (x,y) and $F^{*a}(x,y)$ has slope $-1/3$. To define $F^a(x,y)$ for any point (x,y) in the plane let M be a non negative integer such that both $y-M \leq -9/2$ and $y-M+a \leq -9/2$ and put $F^a(x,y) = t^{*M} F^{*a} t^{*-M}(x,y)$. F^a is easily seen to be well defined.

Since $F^1 = t^*$, t^* is embedded in the flow $\{F^a\colon a \in R^1\}$.

Let $T^*(u,v) = kt^* k^{-1}(u,v)$ for $u < \frac{\pi}{2}$ and $T^*(u,v) = T(u,v)$ for $u \geq \frac{\pi}{2}$. Then T^* is embedded in the flow $\{T^a\colon a \in R^1\}$ given by $T^a(u,v) = k F^a k^{-1}(u,v)$ for $u < \frac{\pi}{2}$ and $T^a(u,v) = (u, v-2a)$ for $u \geq \frac{\pi}{2}$.

To check that T^a is Lipschitz continuous recall that for $|b|$ sufficiently large, $B(b)$ is just a straight line with slope $-1/3$. We then have as in the computation for $\lim_{u \to \pm\frac{\pi}{2}} ktk^{-1}(u,v)$ that

$$\lim_{u \to \pm\frac{\pi}{2}} kt^{*a}k^{-1}(u,v) = (\pm\tfrac{\pi}{2}, v-2a).$$

A simple computation shows that the vector field defined by T^a is Lipschitz continuous. But this just means that the flow $\{T^a\colon a \in R^1\}$ is Lipschitz continuous.

Since T^1 agrees with T on $R^2 \setminus H$ we have that the free mapping

$$T^* = \begin{cases} T & \text{on } R^2 \setminus H \\ kt^* k^{-1} & \text{on } H \end{cases}$$

is embeddable in a Lipschitz continuous flow; hence, T is boundedly conjugate to a translation.

Chapter III

Minimal Relations

In this and the next chapter, an attempt will be made to boundedly embed a C^1 free mapping T in a flow. That is, given a bounded subset B (which can be assumed to be a closed disk) of the plane and a free mapping T of R^2, we will attempt to find a flow $\{T^a: a \in R^1\}$ such that T^1 agrees with T on B. To explore this possibility, we will make use of certain equivalence relations on the plane. The equivalence classes will be either closed strips or lines. We would hope that each equivalence class belonging to a minimal relation would turn out to be an orbit of the desired flow.

Let B be a closed disk and let D be an open disk containing B. The equivalence relations employed here will depend on T, D, B, real numbers $K > 0$, and a C^1 function φ satisfying:

1. $\varphi(p) = 0$ if and only if p is in B;

2. $\varphi(p) \to \infty$ as $p \to D'$, p in D.

Now we let $G(K, \varphi)$ denote the class of all pairs (Q,X) consisting of an equivalence relation Q on R^2 and a unit vector field X defined on R^2 which have the following properties:

1. X is a non-vanishing Lipschitz continuous vector field with Lipschitz constant K.

2. X is a positively directed horizontal vector field on the exterior of D. x_0 and x_1 will denote real numbers such that $x \le x_0$ and $x \ge x_1$ imply that (x,y) is not in D for each y in R^1. It can be assumed that $x_0 = 0$. Let H denote the half plane

$\{(x,y)\in R^2 : x\leq 0\}$ and H^+ denote the half plane $\{(x,y)\in R^2:\ x\geq x_1\}$.

3. Each class of Q contains every orbit of X that has a nonempty intersection with it.

4. Each class in Q is connected and closed.

5. For each p in R^2 $d^*(E(p), E(T(p)))\leq\Phi(p)$, where d^* and $E(p)$, p in R^2 are defined as follows:

i. if p is a point in the plane, then $E(p)$ is the class of Q which contains p;

ii. if E_1 and E_2 are classes of Q, then $d^*(E_1, E_2)$ $= d(E_1\cap H,\ E_2\cap H)$.

We define a partial ordering on $G(K,\Phi)$ by saying for (Q_i, X_i), (Q_j, X_j) in $G(K,\Phi)$ that <u>(Q_j, X_j) refines (Q_i, X_i)</u> or <u>$(Q_i, X_i)\succ(Q_j, X_j)$</u> if $Q_i\neq Q_j$ and each equivalence class of Q_j is contained in an equivalence class of Q_i. In this case we say that $i<j$.

We will show that each chain in $G(K,\Phi)$ partially ordered by the relation "refines" has a lower bound. To prove this property we use the following results.

<u>Lemma III.1.</u> <u>Let $\{(Q_i, X_i):\ i\in I\}$ be a chain in $G(K,\Phi)$ according to the relation "refines" defined above. For each p in R^2 let $E(p) = \bigcap\{E_i(p): i\in I\}$. Then $Q=\{E(p):\ p\in R^2\}$ is a partition of R^2 such that each element of Q is connected and separates R^2 into at most two unbounded residual domains.</u>

<u>Proof.</u> It is clear that for two points p, q in R^2, either $E(p)$ $=E(q)$ or $E(p)$ is disjoint from $E(q)$.

Let $H_i(p) = E_i(p)\cap H$;

$$H^+_i(p) = E_i(p) \cap H^+;$$
$$C_i(p) = E_i(p) \cap (R^2 \setminus (H \cup H^+)^\circ).$$

Then for each p in R^2 and i,j in I we have that $i < j$ implies that $C_j(p) \subset C_i(p)$. For every p in R^2 and i in I, $C_i(p)$ is obviously compact and easily seen to be connected; hence, $C(p) = \bigcap\{C_i(p) : i \in I\}$ is compact and connected. Furthermore, it is clear from the definition of $H_i(p)$ and $H^+_i(p)$ that for every p in R^2, $H(p) = \bigcap\{H_i(p): i \in I\}$ and $H^+(p) = \bigcap\{H^+_i(p): i \in I\}$ are closed and connected. Since $H(p) \cap C(p)$ and $H^+(p) \cap C(p)$ are closed intervals (possibly degenerating to a point), we have that $E(p) = C(p) \cup H(p) \cup H^+(p)$ is connected and closed.

From the facts that $H(p)$ separates H into at most two components, $H^+(p)$ separates H^+ into at most two components, $E(p)$ is connected, and $C(p)$ is compact it follows that there are at most two unbounded components in $R^2 \setminus E(p)$.

In the following we shall assume the hypothesis and notation of lemma III.1.

Lemma III.2. If E, F are distinct classes of Q, then every neighborhood of the boundary of the residual domain W of $R^2 \setminus (E \cup F)$ that separates E and F contains points w in W with the property that every neighborhood of w has a non-empty intersection with uncountably many distinct classes of Q.

Proof: Since W is open and all classes of Q are closed, every neighborhood of each component of W' contains uncountably many distinct classes of Q contained in W as one can verify by standard arguments. Applying this fact to two of these classes and the residual domain in W determined by them, we have the desired result.

<u>Lemma III. 3. Given a chain $\{(Q_i, X_i): i \in I\}$ in $G(K,Q)$, there exists a countable subset J of I such that for all p in R^2, $\bigcap\{E_j(p): j \in J\} = \bigcap\{E_i(p): i \in I\}$.</u>

<u>Proof:</u> Let $\{O_n: n=1,2,\ldots\}$ be a countable collection of open sets which form a basis for the topology of the plane.

Let i_o be any element of I. Suppose that $i_o, i_1, \ldots, i_{n-1}$ have been selected so that $i_o < i_1 < \ldots < i_{n-1}$. We define i_n as follows. If $O_n \subset E(q)$ for some q in R^2, then let i_n be any element of I such that $i_n > i_{n-1}$. If $O_n \not\subset E(p)$ for any p in R^2, then there are points q, r in O_n such that $E(q) \cap E(r) = \emptyset$, and we can find an i in I such that $E_i(r) \cap E_i(q) = \emptyset$ and $i > i_{n-1}$. In this case let $i_n = i$.

The above procedure defines $i_o, \ldots, i_n, \ldots$ inductively. Let $J = \{i_n: n = 0, 1, \ldots\}$.

To see that J has the desired properties suppose that for some u in R^2 $F(u) = \bigcap_{j \in J} E_j(u) \neq E(u)$. Then there exists a v in $F(u)$ such that v is not in $E(u)$. Let W be the residual domain separating $E(u)$ from $E(v)$. By lemma III. 2. there is a point w in W

such that each basis element containing w has non-empty intersection with infinitely many classes of Q. In particular, there is a basis element O_m containing w which is contained in W. By construction of i_m there exist points q, r in O_m such that $E_{i_m}(q)$ is disjoint from $E_{i_m}(r)$. But the fact that q and r belong to W implies that q and r belong to F(u). Hence $F(q) = F(u) = F(r)$. Thus we have a contradiction and $F(u) = E(u)$.

Lemma III. 4. Given a partition $Q = \{E(p): p \in R^2\}$ of the plane as defined in the hypothesis of lemma III.1, there exists a non-vanishing unit vector field X on the plane with the following properties:

1. X is a Lipschitz continuous vector field with Lipschitz constant K;

2. X is a positively directed horizontal vector field on the exterior of D;

3. For each p in R^2 the orbit of X that contains p, O(p), is contained in E(p).

Proof: Let J be as in lemma III. 3. Let (X_j) be the sequence of vector fields in the chain $\{(Q_j, X_j): j \in J\}$.

By Ascoli's theorem, there is a subsequence (X_{j_n}) of (X_j), $j_n > j$ $(n=1,2,\ldots)$, which converges uniformly on compact sets to a vector field X. Let $M = \{j_n \in J: n=1,2,\ldots\}$. Clearly X is a non-vanishing unit vector field and satisfies conditions 1. and 2. in the statement of this lemma since all the X_j have these properties.

Now we want to show that O(p) is contained in E(p) for each p in R^2. For each m in M let $O_m(p)$ denote the orbit of

X_m that contains p. Then $\{O_m(p)\}$ is the sequence of orbits containing p which is such that $\{X_m\}$ converges to X uniformly on $\bar{D}$. Since all X_m are equal on the exterior of D, we have that $\{X_m\}$ converges uniformly to X on R^2. Clearly $\bigcap\{E_m(p): m \in M\} = \bigcap\{E_j(p): j \in J\}$; hence by lemma III.3 $O(p) \subset E(p)$.

<u>Lemma III.5 For every p, q in R^2; Lim $d^*(E_m(p), E_m(q))$ as $m \to \infty$ through elements of M, is equal to $d^*(E(p), E(q))$.</u>

The proof is straight-foreward, so we omit it.

<u>Corollary.</u> <u>$d^*(E(p), E(T(p))) \leq \phi(p)$.</u>

This follows directly from lemma III.5 and the definition of ϕ.

We now have proved the following.

<u>Theorem III.1.</u> <u>Each chain in $G(K,\phi)$ has a lower bound.</u>

<u>Corollary.</u> <u>$G(K,\phi)$ has a minimal element according to the previously defined relation "refines."</u>

<u>Proof.</u> Apply Zorn's lemma and theorem III.1.

Chapter IV

Extensions of Minimal Elements

Here we examine the structure of the minimal classes of the last chapter. We first look at the following result.

Theorem IV. 1. Let T be a Lipschitz continuous free mapping of R^2. Suppose that for each closed disk B in R^2 and some open disk D containing B there exist K, Φ depending on B, D, and T such that an element (Q,X) in $G(K,\Phi)$ (as defined in chapter III) has the property that each class of Q is a line. Then T is boundedly conjugate to a translation.

Proof. Let $\{f^a\colon\ a \in R^1\}$ be the flow induced on R^2 by X. Then for each p in B, $T(p) = f^a(p)$ for some $a \neq 0$. Since the mapping $A\colon B \to R^1$ given by $A(p) = a$ if $T(p) = f^a(p)$ is continuous, and B is compact and connected, we have that $A(B)$ is compact, connected, and bounded away from zero. Thus there exist real numbers b, c, $0 < b \leq c$ such that either $A(B) \subset [b,c]$ or $A(B) \subset [-c,-b]$.

We can now extend A to a continuous function on R^2 with the following properties:

1. $A(p) = 1$ (resp., $A(p) = -1$) for p in $R^2 \sim D$;

2. $A(p) > 0$ (resp., $A(p) < 0$) for p in D if $A(q) > 0$ (resp., $A(q) < 0$) for some q in B.

Define $T^*(p) = f^{A(p)}(p)$ for each p in R^2. Clearly T^* is a free map of R^2 which agrees with T on B and is translation by $(1,0)$ (resp., $(-1,0)$) on the exterior of D.

The proof can be completed in two different ways.

Method 1. Reparametrize the flow $\{f^a: a \in R^1\}$ such that $A(B)=1$. Then T^* is embedded in a free flow. Hence by the theorem I. 1. T is boundedly conjugate to a translation.

Method 2. Observe that since $\{f^a: a \in R^1\}$ is a free flow, every bounded subset of R^2 diverges with respect to T^*. Hence by Brouwer's Translation Theorem, T^* is conjugate to a translation of the plane. Therefore T is boundedly conjugate to a translation.

One would like to prove that any free mapping is boundedly conjugate to a translation by applying theorem IV.1. Any element (Q,X) satisfying the hypothesis of that theorem is clearly minimal. Therefore one is led to ask if every minimal element of $G(K,\Phi)$ satisfies the hypothesis if K and Φ are suitably chosen. We have not been able to answer this question, but the following theorem gives some information in this direction.

Theorem IV.2. Let T be a C^1 free mapping of R^2. Suppose (Q,X) is an element of $G(K,\Phi)$ and that Q contains a class E which has a non-empty interior and is such that X is C^1 on a neighborhood of one component E^* of E'. Then for every $\lambda > 1$ there exists a $K(\lambda) \geq K$ such that (Q,X) is not minimal in $G(K(\lambda), \lambda\phi)$.

To prove this theorem, we first define a homeomorphism from the (x,y) plane to the (s,t) plane so that the flow induced by the vector field X on the (x,y) plane transforms to a positively directed horizontal flow on the (s,t) plane. To define such a homeomorphism we let $\{f^s : s \in R^1\}$ be the flow induced by X. Define $h(s,t) = f^s(0,t)$ for every (s,t) in R^2. By the method in Kamke [5, p. 292] we see that each (x,y) in R^2 can be written $(x,y) = f^s(0,t)$ for a unique (s,t) in R^2. Thus we have that h^{-1} is a well defined homeomorphism which is the identity on the half plane $H = \{(x,y): x \leq 0,\ y \in R^1\}$.

We now define the metric d^* on the (x,y) plane to be the metric induced from the Euclidean metric on the (s,t) plane by the map h. The notation d^* is justified by the fact that if p, q are points in $E'(p)$, $E'(q)$ resp. such that $p_1 h^{-1}(p) = p_2 h^{-1}(q)$ and the components of $E'(p)$, $E'(q)$ containing p, q bound a strip which separates $E(p)$ from $E(q)$ then $d^*(p,q) = d^*(E(p),E(q))$ (where $d^*(E(p), E(q))$ is meant in the sense defined in the previous chapter).

Now we are in a position to prove the following.

Proposition IV. 1. Let (Q,X) be in $G(K,\phi)$. Let E be a class of Q. If p belongs to the intersection with B of one component of E', then $T(p)$ belongs to the same component of E'.

Proof. If $\mathring{E} \neq \emptyset$, let H^* be the component of the complement of E in R^2 such that p is in $\overline{H^*}$.

Since ϕ is a C^1 function on D which is zero on B, ϕ' is also zero on B. Hence for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that p in B and $d^*(p,q) < \delta(\varepsilon)$ imply that $\phi(q) < \varepsilon\, d^*(p,q)$. For each $\varepsilon > 0$ let $V(\varepsilon,p) = \{q \in H^*:\ p_1 h^{-1}(q) = p_1 h^{-1}(p)$ and $d^*(E(p), E(q)) = d^*(p,q) < \delta(\varepsilon)\}$. By lemma III. 2. we have that for each $\varepsilon > 0$ and p in $B \cap E^*$ $V(\varepsilon,p) \neq \emptyset$. Since $\phi(p)=0$ and $\phi(q) \geq 0$ we have by the above that $d^*(E(q), E(T(q))) \leq \phi(q)$ $< \varepsilon d^*(p,q) = \varepsilon\, d^*(E(p), E(q))$ for each q in $V(\varepsilon,p)$. In particular for q in $V(p) = V(1,p)$ we have that $d^*(E(q), E(T(q)))$ $< d^*(E(p), E(q))$. Since $E(q) \subset H^*$ for each q in $V(p)$, the preceding inequality means that $E(T(q)) \subset \overline{H^*}$ for each q in $V(p)$. Therefore, $T(p)$ is in $\overline{H^*}$ and $T(p)$ is in $\overline{H^*} \cap E$.

We now assume that the lower boundary E^* of E' is such that $p_2 h^{-1}(E^*) = 0$. Assume also that X is C^1 in a neighborhood of E^*. Let $L = p_1 h^{-1}(D \cap E^*)$. For each (s,t) in R^2 let $u(s,t)$ $= p_1 h^{-1} Th(s,t)$ and $v(s,t) = p_2 h^{-1} Th(s,t)$. We now prove the following results, which we will use in the proof of theorem IV. 2.

<u>Lemma IV. 1.</u> $\frac{\partial}{\partial t}(v(s,t) - t)\Big|_{t=0} = 0$ if $\phi(h(s,0)) = 0$.

<u>Proof.</u> For s in L and $t \leq 0$ we have that $d^*(E(h(s,t)), E(T(h(s,t))))$ $\leq \phi(h(s,t))$. Since ϕ is a C^1 function which is equal to zero only on the closed disk B and h is C^1 in a neighborhood of the curve $t=0$ we have that $\phi(h(s,0))=0$ implies that $(\phi h)_t(s,t))\Big|_{t=0}$ $=0$.

Let $S = \{s \in L:\ \phi(h(s,0))=0\}$. For each s in S consider a sequence of points $h(s,t_n)$ where $t_n < 0$ and $h(s,t_n)$ is in $V(\frac{1}{n}, h(s,0))$. Since $h(s,t_n)$ is in H^* we have that $d^*(T(h(s,t_n)),$ $h(s,t_n)) = |v(s,t_n) - t_n| \leq \phi(h(s,t_n))$. Thus

$\left| \frac{v(s,t_n)}{t_n} - 1 \right| \leq \frac{\Phi(h(s,t_n))}{t_n}$. Since $\Phi(h(s,0)) = 0$ implies $v(s,0) = 0$ we have the desired result.

Lemma IV.2. Let $S_\varepsilon = \{ s \in L: \ v_t(s,0) + (\Phi h)_t(s,0) > \varepsilon \}$ and $W_\varepsilon = \{ s \in R^1: \ \Phi(h(s,0)) > \varepsilon \ \text{ or } \ s \notin L \}$. Then for a sufficiently small $\varepsilon > 0$.

1. $S_\varepsilon \cup W_\varepsilon = R^1$;

2. $S_\varepsilon' \cap W_\varepsilon' = \emptyset$.

Proof. Clearly W_ε is non-empty. By lemma IV.1. S_ε is non-empty.

Let $S = \{ s \in R^1: \ h(s,0) \in B \}$. By the continuity of $(\Phi h)_t$ at $t=0$ and the fact that S is compact we have that there exists an $a > 0$ such that $d(s,S) < a$ implies that $h(s,0)$ is in D and $|(\Phi h)_t(s,0) + v_t(s,0)| > \frac{1}{2}$. It is clear that for sufficiently small ε, $0 < \varepsilon < \frac{1}{2}$, there exists a $b > 0$, $b < a$ such that if $d(s,S) > b$, s in L, then $\Phi(h(s,0)) > \varepsilon$. It is also easily seen that $d(s,S) \leq a$ implies that $|v_t(s,0) + (\Phi h)_t(s,0)| > \varepsilon$.

From the above facts follow the desired results.

Lemma IV.3. There exists a C^1 function $g: R^1 \to [0,1]$ such that $g = 1$ on $S_\varepsilon \setminus (\overline{S}_\varepsilon \cap \overline{W}_\varepsilon)^\circ$, $g = 0$ on $W_\varepsilon \setminus (\overline{S}_\varepsilon \cap \overline{W}_\varepsilon)^\circ$, and $0 \leq g(s) \leq 1$ on each component of $\overline{S}_\varepsilon \cap \overline{W}_\varepsilon$.

Proof. The proof essentially follows from Urysohn's Lemma.

Lemma IV. 4. Given $\lambda > 1$, there exists a $\delta^*(\lambda) > 0$ such that for $0 \leq t \leq \delta^*$ we can define $\Delta^1(s,t)$ for s in S_ε and $\Delta^2(s,t)$ for s in W_ε such that $\Delta_t^1(s,t) > -1$ on S_ε and $\Delta_t^2(s,t) > -1$ on W_ε. In addition, for $h(s,t)$ in D, $0 \leq t \leq \delta^*$ implies that $v(s,t) - t - \lambda\Phi(s,t) \leq \Delta^i(s,t)$

$\leq v(s,t) - t + \lambda\phi(s,t)$, $i = 1,2$.

Proof. On S_ε define $\Delta^1(s,t) = v(s,t) - v(s,0) - t + \phi(h(s,t)) - \phi(h(s,0))$. For $t=0$ we have $\Delta^1(s,t) = 0$. Hence for sufficiently small $\delta_0 > 0$ and $0 \leq t \leq \delta_0$ we have that h is C^1 at (s,t) and $v(s,t) - t - \lambda\phi(h(s,t)) \leq \Delta^1(s,t) \leq v(s,t) - t + \lambda\phi(h(s,t))$. Since also $v_t(s,0) - 1 + \phi(h(s,0)) > -1 + \varepsilon$ for s in S_ε we have by continuity of v_t and $(\phi h)_t$ at $t=0$ that there exists a $\delta_1 > 0$, $\delta_1 \leq \delta_0$ such that $v_t(s,t) - 1 + (\phi h)_t(s,t) > -1 + \varepsilon > -1$. Because $\Delta^1_t(s,t) = v_t(s,t) - 1 - (\phi h)_t(s,t)$ we have $\Delta^1_t(s,t) > -1$ for $0 \leq t \leq \delta_1$, s in S_ε.

On $W_\varepsilon \cap h^{-1}(D)$ we define $\Delta^2(s,t) = 0$ for $0 \leq t \leq \delta_1$. Since we have $v(s,0) - \phi(h(s,0)) \leq \Delta^2(s,0) \leq v(s,0) + \phi(h(s,0))$ we have for any $\lambda > 1$ that $v(s,0) - \lambda\phi(h(s,0)) < \Delta^2(s,0) < v(s,0) + \lambda\phi(h(s,0))$. Hence by continuity of v, t, h and ϕ we have that there exists $\delta^* > 0$, $\delta^* \leq \delta_1$ such that for $0 \leq t \leq \delta^*$, $v(s,t) - t - \lambda\phi(h(s,t)) \leq \Delta^2(s,t) \leq v(s,t) - t + \lambda\phi(h(s,t))$ and $\Delta^2_t(s,t) > -1$ on $W_\varepsilon \times [0, \delta^*]$.

Lemma IV. 5. There exists a $\delta'' > 0$ and a C^1 function Δ defined on $(S_\varepsilon \cup W_\varepsilon) \times [0, \delta'']$ such that

1. for $0 \leq t \leq \delta''$ and $h(s,t)$ in D, $|v(s,t) - t - \Delta(s,t)| \leq \lambda\phi(h(s,t))$;
2. $h(s+1, t + \Delta(s,t))$ is in the interior of E unless $t=0$;
3. the map $F: h(s,t) \mapsto h(s+1, t + \Delta(s,t))$ is a C^1 diffeomorphism from $h(R^1 \times [0, \delta''])$ into a subset of R^2.

Proof. Since v and ϕ are C^1 on $L \times \{0\}$, both $\Delta^1(s,0)$ and $\Delta^2(s,0)$ are zero on $h(L \times \{0\})$. If we take δ'' sufficiently small, $0 < \delta'' < \delta^*$, then $t + |\Delta(s,t)| \leq 1/3\, d^*(E^*, E' \setminus E^*)$ and we have 2 for $0 \leq t \leq \delta''$.

Define $\Delta(s,t) = g(s)\Delta^1(s,t) + (1-g(s))\Delta^2(s,t)$ for $0 \leq t \leq \delta''$, s in R^1. Since Δ is a convex combination of Δ^1 and Δ^2 which is independent of t we then clearly have 1. By the form of F we see that F is a homeomorphism if $\Delta_t(s,t) > -1$. This clearly follows from the construction of Δ.

We are now in a position to complete the proof of theorem IV.2. First we take $0 < \delta' \leq \delta''$ such that $0 \leq t \leq \delta'$ implies for n in Z that $P_2(h^{-1}(F^n(h(s,t)))) \leq \delta''$.

We embed in a flow the homeomorphism F in lemma IV. 5. For $s \leq 0$ and t in R^1 we have (s,t) is in the exterior of $h^{-1}(D)$. For $0 \leq t \leq \delta'$ define $F^a(h(s,t)) = F^M F^a F^{-M}(h(s,t))$ where M is an integer such that $p_1(F^{-M}(h(s,t))) + a < 0$. It is easily seen that $\{F^a: a \in R^1\}$ is well defined and $F^1 = F$.

Let $Y = (z_1, z_2)$ be the vector field induced by the flow $\{F^a: a \in R^1\}$. Let $X = (w_1, w_2)$. Since X is non-vanishing at $t = 0$ and Y agrees with X at $t = 0$ we can find a $\delta^o > 0$, $\delta^o \leq \delta'$ such that Y does not vanish for $0 \leq t \leq \delta^o$. Thus for a given s in R^1 we have for either $i=1$ or $i=2$ that $w_i(h(s,0)) \neq 0$ and $z_i(h(s,0)) \neq 0$. Since $w_i(h(s,0)) = z_i(h(s,0))$ we have for one i that $w_i(h(s,0)) \neq 2z_i(h(s,0))$. Hence by the continuity of w_i and z_i there is a $\delta > 0$, such that for $0 \leq t \leq \delta$

$$\frac{2t-\delta}{\delta} w_i(h(s,t)) \neq \frac{2t-2\delta}{\delta} z_i(h(s,t)) \text{ for this } i.$$

Therefore the vector field

$$X^* = \begin{cases} X & \text{for } t \leq 0 \text{ and } t \geq \delta \\ Y & \text{for } 0 \leq t \leq \tfrac{1}{2}\delta \\ \dfrac{2t-\delta}{\delta} X + \dfrac{2\delta - 2t}{\delta} Y & \text{for } \tfrac{1}{2}\delta \leq t \leq \delta \end{cases}$$

is a non-vanishing vector field on R^2.

Let $\{T^a: a \in R^1\}$ be the flow induced by X^*. Let Q^* consist of the orbits of X^* $(O^*(h(s,t)))$ for $0 < t < \frac{\delta}{2}$, the classes of Q other than E, and $E \setminus \bigcup \{O^*(h(s,t)): 0 \leq t \leq \frac{\delta}{2}\}$. Then clearly (Q^*, X^*) belongs to $G(K(\lambda), \lambda\Phi)$. Therefore (Q,X) is not minimal in $G(K(\lambda), \lambda\Phi)$. This proves Theorem IV. 2.

Perhaps in a further study it would be possible to use the method developed in Theorem IV. 2 to prove that for suitable K and Φ all equivalence classes belonging to a minimal element of $G(K,\Phi)$ are lines.

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AUTOBIOGRAPHICAL STATEMENT

Elinor Ruth Evenchick was born in New York City on October 25, 1942. In the spring of 1945 her family moved to Peekskill, New York where they resided until the summer of 1949 when they moved to California. After living in Los Angeles and Santa Monica for five years, her family returned east to Baltimore, Maryland.

Elinor's primary and secondary school education was obtained in the various public school systems where she lived. She received her B.S. degree (with High Honors and with Honors in Mathematics) from the University of Maryland, College Park in 1963. Two years later she received her M.A. degree in Mathematics from Brandeis University. Deciding to pursue the Ph.D. degree, Elinor came to the City University of New York where she studied under the supervision of Professor Richard Sacksteder.

It was during her years as a graduate student that both of Elinor's parents passed away, her mother in 1965 and her father in 1966.

In May of 1969 Elinor was married to her husband Mark Berger, a graduate student in history at the City University of New York.