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Rigorous approaches to critical phenomena at phase transitions

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RIGOROUS APPROACHES TO CRITICAL PHENOMENA AT PHASE TRANSITIONS

by

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ABSTRACT

RIGOROUS APPROACHES TO CRITICAL PHENOMENA AT PHASE TRANSITIONS

by

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The theoretical study of critical phenomena at phase transitions is one of the most challenging problem in condensed matter physics. As has been shown by renormalization group (RG) theory, critical fluctuations can cause drastic changes in the nature of the phase transition which have been observed experimentally. However, there are many discussions about the reasons for these changes. Since physically different phenomena, such as additional interactions, or the dependance of system parameters upon temperature or critical fluctuations, can result in similar effects, it is very difficult to conclusively ascribe critical fluctuations as the real reason for the some striking effects in phase transitions. Unfortunately, RG theory is not a big help in settling this dispute. This theory can only determine the critical asymptotics. For the case of qualitative effects this

theory does not work very well and can only make intuitive predictions which very often totally contradict the picture obtained from mean field theory. Consequently, hoping to find the true description of critical phenomena at phase transitions, we study various important systems through an alternative approach. We use an exactly solvable model that takes into account fluctuation interactions partially. It is, therefore, conceptually somewhere between mean field and RG theories. The advantage of the model is that it is exactly solvable, and can therefore provide us with the description of phase transitions within the whole range of variation of thermodynamical quantities. In systems with coupled order parameters the model finds the first order phase transition induced by fluctuations. This type of transition is replaced by a second order when fluctuations are suppressed. Systems with two interacting order parameters which additionally are coupled to two random fields exhibit second order transitions. Finally the model is applied to the case of a random field coupled to an order parameter in d dimension and proves similar critical behavior with the pure case of (d - 2) dimension. At the end we use the fundamental formalism of RG in systems described by the most arbitrary symmetry Hamiltonian in order to construct an exact RG equation which contains no redundant operators.

DEDICATION

This dissertation is dedicated to my brother who so sincerely and solidly stands by me

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

A situation for which interparticle interactions cause correlations between many particles in a system manifests itself most strikingly in phase transitions. In general the theoretical study of systems undergoing phase transitions is so complicated that it inevitably requires some kind of approximation. The three main approaches to critical phenomena in phase transitions are the mean field (MF) theory [1], the renormalization group (RG) theory [2-7], and the use of exactly solvable models [8]. MF is a relatively simpler mathematically and conceptually theory but its results do not hold when they are considered close to the critical point of the phase transition. The most sophisticated mathematically as well as conceptually more precise of the approaches is the RG theory which has, however, a substantial limitation, in that its results are based on various approximations of general equations which cannot be solved exactly. Therefore, very often one cannot be absolutely sure about the correctness of some of the theory's predictions. In this connection some exactly solvable models were invented which conceptually lay between MF and RG theories, that allow for exact solutions and therefore have their results accessible for comparison with those of RG analysis, to hopefully create a clearer, more established and well organized picture for phase transitions. This idea is the common thread between the various research topics that we would like to consider in this thesis. First we will use an exactly solvable model for the study of the following realistic systems: A system containing two competing order parameters, another system having a random field coupled to an order parameter, then a system with a random temperature, a system with two coupled order parameters which are in addition coupled to two random fields, and finally we will derive an exact RG equation which contains no redundant operators for a system of the most general type Hamiltonian. But before we consider these topics in detail, we would like to present a general discussion of the three approaches to the field of phase transitions in order to point out similarities, differences and difficulties.

1.2 FUNDAMENTAL CONCEPTS

The first approach in the physics of phase transitions is Landau's MF theory [1] that provided the basis of almost all many-body theories developed prior to 1970. The basic idea of MF theory is to focus attention on a particular particle in the system and to assume that all other neighboring particles create a mean field which acts on the one designated particle. Hence, this approach neglects the effects of fluctuations that may extend beyond the length scale associated with the designated particle. Since only those fluctuations that occur within the associated length scale of the particle are included, this approach has therefore succeeded in reducing the many-body statistical mechanics problem into a one-body problem. Such procedures can never be exact, but they often can be accurate and very useful. MF theory is a least accurate for systems near the critical (phase transition) points where fluctuations extend over large scales. The theoretical results for transition temperatures can be significantly improved if, for example, the designated "particle" is really a pair of particles which is influenced by the mean field of all other particles in the system. Hence the problem is now reduced to a two-body problem and it accounts for fluctuations on a larger length scale, that defined by the pair. Still such an improvement is not enough for the prediction of the correct critical exponents. For example MF always finds $\beta = 1/2$ independent of the space dimensionality d, where β is defined through the temperature dependance of the order parameter which can be expressed as $(T_c - T)^{\beta}$. T_c is a temperature of a phase transition. Actually for $d > 4$, MF theory can become correct. This is not so obvious at first but it is somehow expected since the neglect of fluctuations predicts order-disorder transition temperatures which are higher than the true ones but their accuracy increases with increasing dimensionality.

A significantly more sophisticated method that considers all length scale fluctuations is the RG theory developed in 1971 by Wilson [2]. For reviews of RG theory see Refs. [3-7]. Wilson's method is very general and has a wide range of applications even outside of the field of phase transitions. Within the area of phase transitions, one might say that Wilson's theory is an extension of ideas on the subject by Kadanoff in the 1960s. The first main idea of RG is to remove from the partition function some degrees of freedom by averaging over them. The second important step of RG is to try to rewrite the partially summed partition function into the same form as the original one with obviously fewer degrees of freedom and perhaps different coupling constants which are related to the old ones through RG equations. These equations are generated from the RG procedure and they can be used to extract physical results. RG theory provides a powerful tool of looking at many-body problems. While the theory was developed to study second order phase transitions, it is being used for the investigation of other kinds of transitions such as first order phase transitions, and in general other areas of physics as well. Very good results have been obtained from RG analysis [3,4]. Critical asymptotics were calculated with an accuracy higher than that of experiment in some cases [9-12]. Despite this however the theory is still not well-grounded since these results are based on perturbation theory that uses parameters which are not generally small. This creates the danger that a theoretically calculated result may be an artifact of an approximation. Among the most known expansions used are the ϵ-expansion, where $\epsilon = d - 4$, [13], the $1/n$ -expansion where n is the number of components of a vector order parameter, [14,15], and the expansion in small coupling constants in three dimensional space [16,17]. Furthermore, there is another direction in the field of critical phenomena which uses RG ideas but it is based on the exact functional RG equation for a system of a general type, first derived by Wilson in 1974 [18]. An important advantage of using this approach is that various other approximation schemes of the exact equation are constructed to find very good values of critical exponents [19-25]. On the other hand, these schemes are not well grounded as well.

In spite of the relative success of RG theory and although it is described mathematically, it must be said in all fairness that our understanding of the theory is not yet thorough. This is so because all results of RG theory have been obtained using some form of truncation, such as the small-ϵ truncation, where higher order terms are neglected, or the $n \to \infty$ truncation, where $1/n$ terms are neglected, or in the case of Wilson's recursion formula the source of the truncation is the plausibility arguments used to obtain it. No proof exists that guarantees that the results obtained by a truncated RG will be the same as those from an exact RG theory. For example, a serious problem with a RG transformation is that it does not insure the existence of fixed points. Fixed points are special points of the parameter space defined by the coupling constants of the Hamiltonian which are invariant under a RG transformation, and critical phenomena are related to the properties of the RG transformation near them. More specifically, critical asymtotics are related to the eigenvalues of the linearized RG transformation near fixed points. The problem arises from the fact that a simple approximation to the RG transformation can falsely provide us with a fixed point even when the full transformation cannot. Therefore, this is a serious reason for lessening the confidence one has in the correctness and reliability of RG results. Furthermore, questions concerning the term limit, the convergence or divergence, etc. of expansions are unresolved. In this connection, we hope that other approaches, such as exactly solvable models, can provide a more lucid picture of phase transitions.

In addition to MF and RG theories one has exactly solvable models for the studying of critical phenomena [8]. Since this approach considers fluctuations in a limited fashion, conceptually lays somewhere between MF and RG and can be used to compare its results with both other approaches. For example, having in mind the unreliability of many of the results of RG analysis due to the various approximations used, and the impreciseness of MF results due to a conceptually inaccurate theory, exactly solvable models provide an alternative route to a problem with the advantage that they are exactly solvable. Because of that such models can provide us with an exact expression of the partition function which can then be used to obtain thorough pictures of phase transitions around or at the transition temperature and within the whole range of variation of thermodynamical quantities. This is one of the benefits of exactly solvable models over RG theory since within the context of RG theory one gets the asymptotic behavior of a system only. Very often the interpretation of an asymptotic behavior does not provide us with a clean cut answer on the true description of the phenomenon. Therefore, if the appropriate exactly solvable model is developed which after using a different point of view among other things can recover RG results, then this may be strongly suggesting the correctness of RG predictions. In particular, when the system of competing order parameters or the random field system are considered within the RG theory that takes into account all fluctuation interactions, demonstrate drastic changes in the entire picture of phase transitions in comparison with MF. However, some of the RG results which are intuitively obtained using the perturbation theories mentioned above are not guaranteed to be correct. Hence, the results of these two systems obtained within the context of an exactly solvable model may be used to determine whether indeed the RG results are not artifacts of an approximation. Furthermore, the RG formalism and consequently the field of phase transitions can be better understood if the overall picture obtained from exactly solvable models in comparison to RG analysis is demonstrated to be qualitatively the same.

Let us now review the structure of this dissertation concerning the various research topics. The exactly solvable model is used in chapters 2 and 3 where we consider "pure" systems, and in chapters 4, 5 and 6 where we deal with random systems. In chapter 7 we have a different direction in mind to the field of critical phenomena at phase transitions. We use the fundamental formalism of RG theory to derive an exact RG equation of the most arbitrary symmetry Hamiltonian. More explicitly, in the second chapter we demonstrate the important features of the exactly solvable model that considers fluctuation interactions partially, and we first apply it in the third chapter to the problem of two coupled scalar order parameters. The critical behavior of such a system is interesting even in MF theory. However, the study of the same problem by

RG analysis predicts some significant differences with the MF results, such as the existence of a fluctuation-induced first order phase transition. Nevertheless, one is not absolutely sure about the correctness of these predictions, since there is a danger that a theoretically predicted result may be a consequence of the approximation used by RG. Having this in mind, we attack the problem within the context of the exactly solvable model and we show qualitative similarities between the model's results and RG predictions, [50,58]. Therefore we can present this as a strong argument that qualitative RG results may indeed be a fact.

The exactly solvable model can also be used to study phase transitions in disordered systems. These problems are considered in chapters 4, 5 and 6. The study of such systems is of great importance because real systems always contain some impurities. Perhaps one of the biggest examples of how the physics of a system drastically changes with the introduction of impurities is semiconductors. The band structure of these devices is modified when the impurities create another energy level between the valence and conduction band which consequently results in the increase of the electrical conductivity of semiconductors.

Different kinds of impurities have different effects. For example, a system may be randomly distorted which may create a preferred direction of spins which otherwise are randomly distributed. Also, some sites of a ferromagnetic system may not be magnetic, and depending on the number of these sites, the phase transition might be completely destroyed or simply the transition temperature might be lowered, [3]. In the latter case, the critical exponents may change. It has been proven [26-33] that to all orders in perturbation expansion, the critical exponents of a phase transition in a d dimensional $(4<d<6)$ system with short-range interactions and random quenched field coupled to the order parameter are the same as those of a (d-2)-dimensional pure system. More specifically, when the ϵ expansion is performed for random field systems having space dimension $d = 6 - \epsilon$, one finds the same critical exponents as when it is performed for pure systems having space dimension $d = 4 - \epsilon$. In addition, the critical exponents are identical to all orders when the $1/n$ expansion is performed with $4<d<6$ for random systems, and $2<d<4$ for pure systems. The effective lowering of the space dimension was intuitively explained by saying that it is the random field and not the thermal fluctuations that becomes the dominant cause for disorder near the phase transition. Also, the lower limit of space dimensionality for which a phase transition is not possible changes from 2 to 4, and the upper limit for which MF results hold changes from 4 to 6. A lot of the known results [13,26-36], are obtained using RG technics which are fairly complicated mathematically and sometimes results are intuitively interpreted, as for example in ref. [29,34]. Our purpose is to study the effect of random fields on phase transitions within the framework of the exactly solvable model. Since the model is exactly solvable, it makes it possible for us to explicitly recover and support results obtained by RG by means of perturbation methods. Specificly, in chapter 4, we prove the destruction of a phase transition for $d \leq 4$ when a random field is interacting with the order parameter in a Ginzburg-Landau-Wilson (GLW) type Hamiltonian, [55,57]. We also show how a phase transition resumes for $d>4$ as predicted by [29], how we obtain critical behavior for $4 < d \leq 6$, and how the MF theory exponents of the pure φ^4 model are again reproduced by the random system when $d > 6$. In addition we prove that the critical exponents of the random system of arbitrary dimension d agrees with those of the pure φ^4 model of dimension d - 2. The dimension $d = 6$ is proven to be a marginal one. Crossover effects are also discussed. In chapter 5 we consider the random temperature problem where the model finds no new changes in the critical behavior in comparison to the pure case chapter 2, [57]. Finally the model is used in chapter 6. There we prove that when random fields are added in the free energy functional of the two coupled order parameters of chapter 3, the fluctuation induce jump-like phase transition found in the pure case may be replaced by a continuous one, [59]. The way we attack these random problems is by using the replica method [34,35]. One of the important advantages of this approach is that it starts with a quenched Hamiltonian which is not translationally invariant, and derives an effective Hamiltonian which is translationally invariant, a fact that the ϵ expansion in its standard form [13] assumes.

In chapter 7 we have a relatively different direction in mind as far as critical phenomena. Unlike the previous chapters where we solve equations for particular classes of systems and perform relatively practical calculations, in the last chapter we would like to use the fundamental formalism of RG theory in order to derive an exact RG transformation equation that contains no redundant operators for a free energy functional with an arbitrary symmetry. All known results of RG theory can be obtained using this equation. In general, exact RG equations were derived before [18,37-39,56], which even though are impossible to solve they are still considered very important because using them various approximation schemes can be developed [19-22,25]. However, there is a very important difference between the RG equation of the most general isotropic Hamiltonian ref. [39], as well as an anisotropic one derived in chapter 7,[56], and the rest of equations, ref.[18,37,38]. That is, they do not contain any redundant operators, which bare no physical meaning and must be transformed away. The task of their exclusion is very cumbersome, so it is desirable that these operators do not exist in exact RG transformation equations in the first place. Finally chapter 8 summarizes all work.

CHAPTER 2

FEATURES OF THE EXACTLY SOLVABLE MODEL

2.1 INTRODUCING THE MODEL

In this chapter we would like to demonstrate the features of the exactly solvable model by reviewing the results of the simple φ^4 model as derived by MF, RG and by the exactly solvable model. Therefore we start from the isotropic Ginzburg-Landau free energy functional with a scalar order parameter $\varphi(\boldsymbol{x})$

$$H = \frac{1}{2}\int d^dx\left[\tau\varphi^2(\boldsymbol{x}) + c(\nabla\varphi(\boldsymbol{x}))^2 + \frac{1}{4}g\varphi^4(\boldsymbol{x}) - h\varphi(\boldsymbol{x})\right]$$

$$H = \frac{1}{2}\int d^dx\left[\tau\varphi^2(\boldsymbol{x}) + c(\nabla\varphi(\boldsymbol{x}))^2 + \frac{1}{4}g\varphi^4(\boldsymbol{x}) - h\varphi(\boldsymbol{x})\right] \tag{2.1}$$

where $\tau \propto T - T_c$ with T_c being a trial critical temperature, h a constant conjugate field, and c, g are constants of interaction. The physical meaning of c is that it is the scale of microscopic interactions in the initial system. Functional (2.1) may describe, for example, a paramagnetic to ferromagnetic phase transition, or a liquid to vapor phase transition. In these two examples τ, c, and g have different values. Our model uses the approximation

$$\int d^dx\,\varphi^4(\boldsymbol{x}) \rightarrow \frac{a[\varphi]}{V}\int d^dx\,\varphi^2(\boldsymbol{x}) \quad , \quad a[\varphi] \equiv \int d^dx\,\varphi^2(\boldsymbol{x}) \tag{2.2}$$

where V is the volume of the system. This was originally proposed by Schneider [40] for the isotropic φ^4 model and was developed and generalized in Refs [41-50]. In momentum space such a reduction corresponds to the splitting of the factor of the φ^4 term, $\delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 + \boldsymbol{p}_4)$, which provides momentum conservation, into a product of two δ-functions, $\delta(\boldsymbol{p}_1 + \boldsymbol{p}_2)\,\delta(\boldsymbol{p}_3 + \boldsymbol{p}_4)$. This split has a clear physical meaning. While the model is preserving the symmetry of the system, it is also considering fluctuation interactions having equal and opposite momenta only. Using the above reduction the exponent in the partition function becomes quadratic in the functional $a[\varphi]$, and after the use of a transformation analogous to that of Hubbard-Stratonovich,

$$\exp\left[-\frac{V}{2}K\left(\frac{a[\varphi]}{V}\right)\right] = \frac{1}{2\pi}\int dxdy\, \exp\left[-\frac{V}{2}K\left(\frac{x}{V}\right) + i(xy - ya[\varphi])\right] \tag{2.3}$$

which is true for an arbitrary function K, we obtain

$$Z = \int D\varphi_q \int_{-\infty}^{\infty} dxdy$$

$$\times \exp\left\{-\frac{V}{2}\left[\tau x + \frac{1}{4}gx^2 - xy + \frac{1}{V}\sum_q (cq^2 + y)\varphi_q^2 - \frac{h}{2V}\varphi_{q=0}\right]\right\} \tag{2.4}$$

We now perform all Gaussian integrals with respect to the order parameter, except $\varphi_{q=0} \equiv \varphi_0$ which may condense at the phase transition to obtain

$$Z = \int_{-\infty}^{\infty} d\varphi_0\, dx\, dy$$

$$\times \exp\left\{-\frac{V}{2}\left[\tau x + \frac{1}{4}gx^2 - xy + y\varphi_0^2 - h\varphi_0 + \frac{1}{V}\sum_{q\neq 0} \ln(cq^2 + y)\right]\right\} \tag{2.5}$$

where $\varphi_0 \to \varphi_0/\sqrt{V}$ and $h \to h/2\sqrt{V}$. First we note that the summation of the kind $\sum_q \ln(cq^2 + y)$ is divergent if we allow the upper limit of the cutoff momentum to tend to infinity. Therefore, we must keep the cutoff momentum finite. However, critical asympotics should not depend upon momentum cutoff. When $2 < d < 4$ this can be handled by renormalizing the summation and then setting the momentum cutoff to be equal to infinity [41]. For $d \geq 4$ the sum becomes nonrenormalizable and we must maintain the momentum cutoff explicitly. However, as we demonstrate below the dependance upon the momentum cutoff is absorbed into a renormalization of the trial value of the critical temperature and into an insignificant constant addition to the free energy. Let us suppose that the cutoff momentum is equal to A. Then

$$\sum_q \ln(cq^2 + y) = \left[y\theta(A) + f_d(y;c)\right]V \tag{2.6}$$

and with S_d the surface area of a d-dimensional unit sphere we have

$$\theta(A) \equiv \theta(A;c) = \frac{S_d}{(2\pi)^d}\begin{cases} \dfrac{A^{d-2}}{c(d-2)} & d \neq 2 \\ \dfrac{1 + \ln(cA^2)}{2c} & d = 2 \end{cases}$$

$$f_d(y;c) = \frac{S_d}{(2\pi)^d}\begin{cases} \dfrac{\pi y^{d/2}}{d\, c^{d/2} \sin\left[\dfrac{\pi d}{2}\right]} \equiv \kappa(c) y^{d/2} & d \neq \text{even} \\ \dfrac{1}{d}\left(-\dfrac{y}{c}\right)^{d/2} \ln y \equiv \mu(c) y^{d/2} \ln y & d = \text{even} \end{cases} \tag{2.7}$$

Note the requirement $y \geq 0$. $\theta(A)$ is used to renormalize x, $x \rightarrow x + \theta(A)$, which consequently results in the renormalization of the trial critical temperature τ, $t = \tau + 1/2\, g\theta(A;c)$. As a result the partition function becomes

$$Z = \int dx\, dy\, d\varphi_0 \exp\left[-\frac{V}{2} F(x, y, \varphi_0)\right], \tag{2.8}$$

with

$$F(x,y,\varphi_0) = x(t-y) + \frac{gx^2}{4} + y\,\varphi_0^2 + f_d(y;c) - h\,\varphi_0 \tag{2.9}$$

We now note that since V is a multiplicative constant then in the thermodynamic limit $V \to \infty$ one can exactly calculate the partition function using the method of the steepest descend. Hence, all thermodynamic quantities can be calculated. The critical free energy is found by considering the equilibrium equations $\partial F/\partial x = 0$, $\partial F/\partial y = 0$, and the equation of state $\partial F/\partial \varphi_0 = 0$. After substituting x away we obtain the system

$$t - y + \frac{g}{2}\left[\varphi_0^2 + f_d'(y;c)\right] = 0 \tag{2.10.a}$$

$$y\,\varphi_0 = h \tag{2.10.b}$$

with the free energy

$$F(\varphi_0) = t\left[\varphi_0^2 + f_d'(y;c)\right] + \frac{g}{4}\left[\varphi_0^2 + f_d'(y;c)\right]^2 - h\,\varphi_0 \tag{2.11}$$

Now we are ready to derive and discuss various results.

2.2 RESULTS

If $h = 0$ then, for any $d > 2$ in order to have a nontrivial $\varphi_0 \neq 0$, from Eq. (2.10.b) one must assume $y = 0$. The solution of Eq. (2.10.a) gives

$$\varphi_0 = \sqrt{-\frac{2t}{g}} \tag{2.12}$$

When $d \leq 2$ the solution $y = 0$ is incompatible with Eq.(2.10.a). Therefore nontrivial solution $\varphi_0 \neq 0$ does not exist in this case. This demonstrates that $d = 2$ is a marginal dimension which agrees with RG analysis. The transition is of the second order with the critical exponent $\beta = 1/2$. MF finds the second order phase transition with the same critical exponent β but with no restrictions on d. Furthermore, the model gives more interesting results when calculating the critical exponent δ, which is defined through the dependance of the order parameter φ_0 on the constant field h at $t = 0$, that is $\varphi_0 \propto h^{1/\delta}$. More specifically, when the constant conjugate field h is small but not zero then at the critical temperature $t = 0$ the system of Eqs. (2.10) gives

$$-\frac{h}{\varphi_0} + \frac{g}{2}\varphi_0^2 + \frac{g\kappa(c)d}{4} h^{\frac{d-2}{2}} \varphi_0^{\frac{2-d}{2}} = 0 \qquad d \neq \text{even} \tag{2.13.a}$$

$$-\frac{h}{\varphi_0} + \frac{g}{2}\varphi_0^2 + \frac{g}{2}\left(-\frac{1}{c}\right)^{d/2}\left(\frac{h}{\varphi_0}\right)^{\frac{(d-2)}{2}}\left[\frac{1}{d} + \frac{1}{2}\ln\left(\frac{h}{\varphi_0}\right)\right] = 0 \qquad d = \text{even} \tag{2.13.b}$$

One the one hand, for $2 < d < 4$ and small h the term $-h/\varphi_0$ of Eq. (2.13.a) is unimportant, and on the other hand, for $d > 4$ and small h the last terms of Eqs. (2.13.a) and (2.13.b) are unimportant, therefore if for these two cases we solve for the order parameter we obtain,

$$\varphi_0 = \begin{cases} \left(\dfrac{-\kappa(c)\, d}{2}\right)^{\frac{2}{(d+2)}} h^{\frac{1}{\delta}} & 2 < d < 4 \\ \left(\dfrac{2}{g}\right)^{\frac{1}{3}} h^{\frac{1}{\delta}} & d > 4 \end{cases} \tag{2.14}$$

with $\delta = (d + 2) / (d - 2)$ for $2 < d < 4$, and $\delta = 3$ for $d > 4$. The result of MF theory is $\delta = 3$ regardless of dimensionality, and the result of RG theory is $\delta = 3$ for $d > 4$ and $\delta = (d + 2 - \eta) / (d - 2 + \eta)$ for $2 < d < 4$, where η is the critical exponent for the correlation function. Since η is numerically small, $\eta \sim 0.05$, we have a pretty good agreement between the model and RG theory. Furthermore, at space dimensionality $d = 4$, the model finds from Eq. (2.13.b) logarithmic corrections for δ,

$$\varphi_0 = \left(\frac{h\,|\ln h\,|}{3c^2}\right)^{1/3} \tag{2.15}$$

which also is in agreement with RG analysis. We see how the model, as it is the case with RG theory, shows critical behavior for $2 < d < 4$ and MF results for $d > 4$. MF theory derives $\delta = 3$ independently of d. In addition, the model finds that the space dimensionality $d = 4$ is a marginal one since above it MF results hold.

Furthermore, a crossover effect for $2 < d < 4$ where critical behavior is occurring can be observed. In other words, even though for $2 < d < 4$ and h small $\delta = (d + 2)/(d - 2)$ a crossover to MF behavior can be obtained where $\delta = 3$, if h, g and c are properly chosen together in such a way so that

$$h > g^{\frac{d+2}{2(4-d)}} \, |\kappa(c)|^{\frac{3}{4-d}} \, 2^{\frac{d+8}{2(d-4)}} \tag{2.16}$$

The above inequality is derived from the system of Eqs. (2.10) by requiring that $|-y| \geq |g/2 f_d'(y;c)|$. We notice how upon suppression of fluctuations, ($\kappa(c) \rightarrow 0$), Eq. (2.16) is always true and the behavior is therefore MF one as expected. Furthermore, we see from Eqs. (2.10) and (2.13) that in this limit the model finds the expected from MF theory results. Hence we notice that what really causes the critical behavior found by the model is the effect of fluctuation interactions.

2.3 CONCLUSION

The above analysis is really a strong indication that the model is a good choice for use in other systems. It has been already successfully used in a series of papers [41-50]. It was applied to finite systems [41] and for various crossover phenomena [43-44], it was generalized for the Ginzburg-Landau-Wilson functionals, containing all even powers of an order parameter [44]. Within the framework of the model, the possibility of a first order phase transition induced by fluctuations in cubic systems [43,45] was demonstrated as it is the prediction from RG theory [51]. After generalization the model gives critical exponents numerically close to the experimental ones, that is when $d = 3$ $\beta = 1/3$, $\gamma = 1$, $\delta = 5$, $\eta = 0$, [44]. These exponents are defined as follows: susceptibility $\propto (T_c - T)^{-\gamma}$, specific heat $\propto (T_c - T)^{-\alpha}$, correlation length $\propto |T_c - T|^{-\nu}$, which measures the distance over which fluctuations are correlated, and correlation function $<\varphi(q)\varphi(-q)> \propto q^{-2+\eta+d}$ at T_c for small q. The model was also used to investigate physical effects in real systems. Consequences of phase transitions in orthorombic high T_c superconducting systems with d-pairing were studied [47] and it was found that the results are close enough to the ones of the RG theory [51]. The model also gives reasonable results in studying oxygen ordering near a structural phase transition in *Y-Ba-Cu-O* ceramic superconductors [49] Finally RG methods were applied to the model and it was shown that direct calculation of the partition function and solution of the RG equation for the model leads to identical results [48]. Even though RG is a very good theory for the study of critical phenomena near the critical region, which is defined by Ginzburg constant, it is however fairly mathematically complicated theory and can provide us with critical exponents only. RG theory takes into account all fluctuation interactions. On the contrary, MF which ignores all fluctuation effects, is a relatively easier theory and as far as the critical exponents, this theory is correct only away from the critical region defined by the correlation length, or for $d > 4$. It turns out that the effect of fluctuations which is such a real effect on the one hand, one the other hand is such a difficult thing to be considered mathematically something that RG attempts to do. Therefore it is always desired that we have exactly solvable models that at least consider fluctuation effects partially and can hopefully support intuitive predictions of the fluctuation theory. Chapters 3, 4, 5, and 6 are devoted to the application of the exactly solvable model in different systems.

CHAPTER 3

COUPLED ORDER PARAMETERS

3.1 INTRODUCTION

It would have been very fortunate if one could find the critical behavior of any functional when fluctuation interactions corresponding to all length scales from microscopic sizes up to the correlation length are been considered. It can be shown that for $d > 4$ fluctuations need not be considered and the Landau picture of phase transitions is correct [13]. Below dimension 4, however, the fluctuation effects must be considered, but as mentioned before, consideration of all fluctuation interactions from the RG theory stand points may be a fairly complex task for certain systems, and some of the theory's predictions may be a result of the approximation. For example, RG theory predicts a series of qualitative effects, such as the transformation of a continuous phase transition in some anisotropic systems into jump-like phase transitions due to the fluctuation interaction [43]. Unfortunately, these transformations cannot be proven rigorously within the framework of RG. It is just a general belief that a flow of RG trajectories from the region of stability of a Ginzburg- Landau functional corresponds to a first order phase transition driven by fluctuations. There is some chance that either such a treatment is incorrect or that such a flow away is a result of approximations, e.g., the ϵ-expansion. Therefore, the treatment of these systems with the help of exactly solvable models, which at least partially take into account fluctuation effects, may clarify the situation. If an exactly solvable model yields results analogous to the ones of RG theory, it would be a serious argument in support of the reality of these qualitative effects. Such a system studied by both RG and MF theories is described by the Ginzburg-Landau functional

$$F[\varphi_1, \varphi_2] = \frac{1}{2}\int d^d x \left\{ \sum_{i=1}^{2}\left[\tau_i\, \varphi_i^2(x) + c_i\,(\nabla\varphi_i(x))^2 + \frac{1}{4} g_i\, \varphi_i^4(x)\right] + \frac{1}{2} w\, \varphi_1^2(x)\, \varphi_2^2(x) \right\} \quad (3.1)$$

where $\tau_i = (T - T_{ci})/T_{ci}$, and T_{ci} is a trial critical temperature for the order parameter $\varphi_i(x)$. This functional can describe transitions in anisotropic antiferromagnetic systems [52,53]. The MF phase diagram of above functional demonstrates bicritical or tetracritical points, respectively for negative or positive $\Delta \equiv g_1 g_2 - w^2$, see figures (1-3). Explicitly, lines $\tau_2 = 0$ and $\tau_1 = 0$ on the (τ_1,τ_2) plane are lines of the second order phase transitions into phases $\varphi_2 \neq 0$, $\varphi_1 = 0$ and $\varphi_1 \neq 0$, $\varphi_2 = 0$, respectively [52]. On the contrary, RG analysis shows that the phase diagram of this system contains lines of the first order transition induced by fluctuations [53]. That is, RG theory predicts that when τ_1 and τ_2 are close to each other the disorder-order phase transition may become of the first order [53]. Our goal is to compare results of the RG theory with results of the exactly solvable model. In fact, using the exactly solvable model we prove the existence of first kind order-disorder phase transition. The possibility of first order phase transitions in systems with cubic anisotropy was analyzed using this model in papers [43,45]. Results similar to the RG predictions have been obtained in other cases as well [47,49,50,58].

3.2 GENERAL RELATIONS

In the functional (3.1) we reduce the terms $\varphi_i^4(x)$ and $\varphi_1^2(x)\varphi_2^2(x)$ as follows,

$$\begin{aligned} \int d^d x\, \varphi_i^4(x) &\rightarrow \frac{a_i^2[\varphi_i]}{V}; \\ \int d^d x\, \varphi_1^2(x)\, \varphi_2^2(x) &\rightarrow \frac{a_1[\varphi_1]\, a_2[\varphi_2]}{V}. \end{aligned} \tag{3.2}$$

where the functional $a_i[\varphi_i]$ is defined by Eq. (2.2). As well as in the case of the isotropic Ginzburg-Landau functional this reduction does not change the symmetry of the system but reduces the number of interacting modes to those with equal and antiparallel momenta.

After the reduction (3.2) the exponent in the partition function becomes quadratic in form with respect to functionals $a_i[\varphi_i]$,

$$Z \propto \int D\varphi_1(x)\, D\varphi_2(x)\, \exp$$
$$\left\{-\frac{V}{2}\left[\sum_{i=1}^{2}\left(\frac{\tau_i a_i[\varphi_i]}{V} + \frac{1}{V}\int d^3x\, c_i(\nabla\varphi_i(x))^2 + \frac{1}{4V^2} g_i a_i^2[\varphi_i]\right) + \frac{1}{2V^2} w a_1[\varphi_1] a_2[\varphi_2]\right]\right\} \tag{3.3}$$

Using the transformation Eq. (2.3) we obtain

$$Z = \int D\varphi_{1q} D\varphi_{2q} \int_{-\infty}^{\infty} dx_1\, dx_2\, dy_1\, dy_2$$
$$\times \exp\left\{-\frac{V}{2}\sum_{i=1}^{2}\left[\tau_i x_i + \frac{1}{4} g_i x_i^2 - x_i y_i + \frac{1}{V}\sum_q \left(c_i q^2 + y_i\right)|\varphi_{iq}|^2\right] + \frac{1}{2} w x_1 x_2\right\}. \tag{3.4}$$

We may now perform integrations over all modes except $\varphi_{i,\,q=0}$ which condenses at the phase transition to obtain,

$$Z = \int_{-\infty}^{\infty} d\varphi_1\, d\varphi_2\, dx_1 dx_2\, dy_1 dy_2$$

$$\times \exp\left\{-\frac{V}{2}\sum_{i=1}^{2}\left[\tau_i x_i + \frac{1}{4}g_i x_i^2 - x_i y_i + y_i|\varphi_i|^2 + \frac{1}{4}w x_1 x_2 + \frac{1}{V}\sum_{q\neq 0}\ln(c_i q^2 + y_i)\right]\right\} \tag{3.5}$$

where we define $\varphi_i = \varphi_{i,\,q=0}/\sqrt{V}$. The summation of the kind $\sum_q \ln(cq^2 + y)$ is treated as in chapter 2. $\theta(A)$ is used to renormalize x_i, $x_i \rightarrow x_i + \theta(A;c_i)$, and the trial critical temperature τ_i, $t_i = \tau_i + 1/2\,[\,g_i\theta(A;c_i) + w\theta(A;c_{i'\neq i})] \equiv (T - T_i)/T_i$. As a result we have the partition function in the form,

$$Z \propto \int\left(\prod_i dx_i\, dy_i d\varphi_i\right)\exp\left[-\frac{V}{2}F(x_i,y_i,\varphi_i)\right], \tag{3.6}$$

with function F defined to be,

$$F(x_i,y_i,\varphi_i) = \sum_{i=1}^{2}\left[x_i(t_i - y_i) + \frac{g_i x_i^2}{4} + \frac{w x_1 x_2}{4} + y_i\,\varphi_i^2 + f_d(y_i;c_i)\right]. \tag{3.7}$$

Due to the fact that V is a large multiplicative constant, one can use the steepest descent method to calculate the partition function exactly. The saddle points with respect to x_i, y_i and φ_i of the integral (3.6) are defined by equations, $\partial F/\partial x_i = 0$, $\partial F/\partial y_i = 0$, and the equation of state $\partial F/\partial\varphi_i = 0$. After eliminating x_i we obtain the system

$$t_i - y_i + \frac{g_i}{2}\left[\varphi_i^2 + f_d'(y_i;c_i)\right] + \frac{w}{2}\left[\varphi_{j\neq i}^2 + f_d'(y_{j\neq i};c_j)\right] = 0$$
$$y_i\,\varphi_i = 0 \tag{3.8}$$

with the nonequilibrium free energy

$$F(\varphi_i) = \sum_{i=1}^{2}\left\{t_i\left[\varphi_i^2 + f_d'(y_i;c_i)\right] + \frac{g_i}{4}\left[\varphi_i^2 + f_d'(y_i;c_i)\right]^2 + \right.$$
$$\left. + \frac{w}{4}\left[\varphi_1^2 + f_d'(y_1;c_1)\right]\left[\varphi_2^2 + f_d'(y_2;c_2)\right] + f_d(y_i;c_i) - y_i f_d'(y_i;c_i)\right\} \tag{3.9}$$

Solving the system of Eqs. (3.8) we can find the temperature dependance of the order parameters. The physical solution of these equations must correspond to the minimal value of the free energy Eq. (3.9).

3.3 SOLUTIONS FOR VARIOUS PHASES

There are four different solutions for the system of Eqs. (3.8). The first one corresponds to the disordered phase $\varphi_1 = \varphi_2 = 0$. The second one is a mixed phase with both φ_1 and φ_2 not equal to zero. The third one (and analogously the fourth one) is when one of the order parameters is not zero and the other one is, say $\varphi_1 \neq 0$ and $\varphi_2 = 0$. Let us agree to call such an ordered phase, phase 1. Then phase 2 will be the ordered phase having $\varphi_1 = 0$ and $\varphi_2 \neq 0$. At first we will demonstrate that regardless of the existence of fluctuation interactions, systems having space dimension $d > 4$ always exhibit second order phase transition. Without any loss of generality we do so by studying a disorder-order phase transition into phase 1. From the system of Eqs. (3.8) we derive

$$\frac{\Delta d \kappa(c_2) y_2^{\frac{(d-2)}{2}}}{4g_1} - y_2 + t_2 - \frac{wt_1}{g_1} = 0 \qquad d \neq \text{even} \tag{3.10.a}$$

$$\frac{\Delta \mu(c_2) y_2^{\frac{(d-2)}{2}}}{2g_1}\left(1 + \frac{d}{2}\ln y_2\right) - y_2 + t_2 - \frac{wt_1}{g_1} = 0 \qquad d = \text{even} \tag{3.10.b}$$

where $\Delta = g_1 g_2 - w^2$. Now suppose that $|t_1| << 1$ and $|t_2| << 1$, then the first terms of Eqs. (3.10.a) and (3.10.b) may be omitted, and using the system of Eqs. (3.8) separately with (3.10.a) and (3.10.b) we find $\varphi_1 = \surd(-t_1/g_1)$ which therefore indicates always a second order phase transition.

The picture of phase transition for 3-dimensional systems is much more interesting. Let us elaborate on that. For the mixed phase Eqs. (3.8) give $y_i = 0$ and one arrives at the solution given by mean field theory [52],

$$\varphi_1^2 = \frac{2(wt_2 - g_2 t_1)}{\Delta}, \quad \varphi_2^2 = \frac{2(wt_1 - g_1 t_2)}{\Delta},$$
$$F(\varphi_1, \varphi_2) = \frac{2wt_1 t_2 - g_1 t_2^2 - g_2 t_1^2}{\Delta}. \tag{3.11}$$

This solution exists only when,

$$\Delta > 0; \quad wt_2 - g_2 t_1 > 0; \quad wt_1 - g_1 t_2 > 0. \tag{3.12}$$

The mixed phase free energy does not have a minimum when $\Delta < 0$.

Using Eqs. (3.8) we may solve for phase 1 to obtain two physically possible solutions,

$$\varphi_{1\pm}^2 = \frac{2w|\kappa_2| y_{2\pm}^{\frac{1}{2}}}{g_1} - \frac{2t_1}{g_1}$$
$$\kappa_i = -\frac{c_i^{-3/2}}{8\pi} \tag{3.13}$$

with the two values for y_2 given by,

$$y_{2\pm}^{1/2} = -\frac{|\kappa_2|\Delta}{2g_1} \pm \sqrt{\left(\frac{\kappa_2 \Delta}{2g_1}\right)^2 + \left(t_2 - \frac{wt_1}{g_1}\right)} \tag{3.14}$$

and the requirement that $y_{2\pm}^{1/2}$ is greater than or equal to zero. The free energy of phase 1 is given by,

$$F_{\pm} \equiv F(\varphi_{1\pm} \neq 0, \varphi_2 = 0) =$$
$$-\frac{t_1^2}{g_1} + 2|\kappa_2|(\frac{wt_1}{g_1} - t_2)y_{2\pm}{}^{1/2} + \frac{2}{3}|\kappa_2|y_{2\pm}{}^{3/2} + \frac{\Delta\kappa_2^2}{g_1}y_{2\pm} \tag{3.15}$$

The replacement of subscript 1 with 2 (and 2 with 1) in Eqs (3.13) through (3.15) gives results concerning phase 2.

An interesting observation about Eq. (3.15) is that it is the free energy of phase 1 but does not have κ_1 involved. κ_1 controls fluctuations of order parameter φ_1 which by assumption is the non-zero order parameter of phase 1. In general, the radii of interactions between the order parameters in the original system which critical behavior is described by the Ginzburg-Landau functional (3.1), are proportional to $c_i^{1/2}$ or to $\kappa_i^{-1/3}$. Hence, by setting $c_i \to \infty$ or ($\kappa_i \to 0$) we suppress all fluctuations. In this infinite range of interaction the expressions of the free energy Eq. (3.15), and of the order parameter Eq. (3.13), become those obtained by MF theory as expected. For a finite range of interaction, it can be shown that F_+ is always a lower than or equal to free energy F_-. We will also show that Eqs. (3.13-3.15) can correspond to a first order phase transition caused by fluctuation interactions, in accordance with the predictions of RG theory. The different systems one can have are the following three: first a system having $w > 0$ and $\Delta < 0$, second that of $w > 0$ and $\Delta > 0$, and third that of $w < 0$ and $\Delta > 0$. A system having $w < 0$ and $\Delta < 0$ cannot exist since that would create an infinite value for the partition function.

3.3.1 SYSTEM OF $w>0, \Delta<0$

Let us first deal with the system having $w>0$ and $\Delta<0$. The plane (t_1, t_2) is separated into three regions, (see figure 4). The region for which the disorder phase occurs, and those that phases 1 and 2 occur. In addition the coexistence line between the two ordered phases is shown. Such a phase diagram is constructed having in mind that for say phase 1 φ_{1+} must be greater than or equal to zero. Then, phase 1 realizes in the region enclosed by the parabola and straight line,

$$t_2 \geq \frac{t_1^2}{w^2\kappa_2^2} + \frac{g_2}{w}t_1 \equiv z(t_1) \tag{3.16.a}$$

$$t_1 \geq \frac{-w\kappa_2^2\Delta}{2g_1} \equiv b \tag{3.16.b}$$

simultaneously, as well as in the common region of inequalities

$$t_1 \leq b \quad , \quad t_2 \geq \frac{w}{g_1}t_1 - \left(\frac{\kappa_2\Delta}{2g_1}\right)^2 \equiv z_1(t_1) \tag{3.17}$$

Similarly, phase 2 is physical in an analogous region given by above lines after the replacement of subscript 1 with 2 (and 2 with 1). The disorder phase occupies only part of the first quadrant in the (t_1,t_2) plane. The coexistence curve between phases 1 and 2 is given by $F(\varphi_{1+}\neq 0, \varphi_2=0) = F(\varphi_1=0, \varphi_{2+}\neq 0)$, and an order-order phase transition between the two low symmetry phases is of the first kind.

We will now discuss a phase transition into phase 1 from disorder, or from phase

1 into disorder. (An analogues discussion can be extended for phase 2). The straight line $t_2 = z_1(t_1)$ is tangent on the parabola at $(t_1, t_2) = (\ b,\ b[w/(2g_1) + g_2/(2w)]\ )$. This is an important point because it separates systems that exhibit first kind disorder-order transitions with those that exhibit second kind. Let us elaborate on that. Having in mind figure 4, entering phase 1 from disorder by crossing the parabola, dashed line, which means that the condition $t_1 \geq b$ is satisfied, the transition is of the second kind. The transition temperature lays on the parabola $t_2 = z(t_1)$. The temperature dependance of the order parameter is

$$\varphi_1^2 = -\frac{2\bar{T}}{g_1 T_1} + \frac{w^2 \kappa_2^2 (g_1 T_1 - w T_2)\,\bar{T}}{g_1^2 T_1 T_2 \left[\dfrac{T_{crl}}{T_1} - 1 - \dfrac{w\kappa_2^2 \Delta}{2g_1}\right]} \tag{3.18.a}$$

$$\frac{T_{crl}}{T_1} > 1 + \frac{w\kappa_2^2\Delta}{2g_1}, \tag{3.18.b}$$

where $\bar{T} \equiv T - T_{crl}$ with

$$T_{crl} = T_1 + \frac{w^2\kappa_2^2 T_1^2}{2T_2} - \frac{g_2 w \kappa_2^2 T_1}{2} + \frac{|\kappa_2| T_1}{2}\sqrt{\kappa_2^2 w^2 \left(\frac{g_2 T_2 - w T_1}{T_2}\right)^2 + \frac{4w^2 (T_1 - T_2)}{T_2}} \tag{3.19}$$

On the other hand, if phase 1 is entered from disorder by crossing the straight line $t_2 = z_1(t_1)$, (solid line), which implies that $t_1 < b$ is true, then the transition is of the first kind. The transition temperature lays on $t_2 = z_1(t_1)$ and the temperature dependance of the order parameter is,

$$\varphi_1^2 = \frac{w\kappa_2^2|\Delta|}{g_1^2} + \frac{2}{g_1} - \frac{2T_{cr2}}{g_1 T_1} - \frac{2T'}{g_1 T_1} + \frac{2w\kappa_2}{g_1}\sqrt{\frac{(g_1 T_1 - wT_2)T'}{g_1 T_1 T_2}}\,;$$

$$\frac{T_{cr2}}{T_1} < 1 + \frac{w\kappa_2^2|\Delta|}{2g_1} \tag{3.20}$$

where $T' \equiv T - T_{cr2}$ and

$$T_{cr2} = \frac{T_1 T_2\left(g_1 - w - \dfrac{\kappa_2^2\Delta^2}{4g_1}\right)}{(g_1 T_1 - wT_2)}\,. \tag{3.21}$$

The above critical temperature corresponds to the first saddle point of the curve $F(\varphi_1 \neq 0, \varphi_2 = 0)$ vs φ_1, (see figure 8). The dependance of the free energy on φ_1 for a temperature higher than T_{cr2} is shown in figure 7 and corresponds to the disorder phase. T_{cr2} is the temperature at which the system jumps from phase 1 into disorder, (assuming it is originally in phase 1 and the temperature is being raised), figure 8. It is now that $F(\varphi_{1+} \neq 0, \varphi_2 = 0)$ = $F(\varphi_{1-} \neq 0, \varphi_2 = 0)$ and $\varphi_{1+} = \varphi_{1-}$, with the disorder free energy described by a local minimum which is still lower than the free energy of phase 1. Hence, if for example is now assumed that the system is originally in the disorder phase, it will still remain so for this first-saddle-point transition temperature T_{cr2} Eq. (3.21). As the temperature is lowered the first saddle point slowly changes into a local maximum and a local minimum corresponding to free energies F_- and F_+ respectively (see figure 9). At the critical temperature for which F_- equals the disorder free energy $F(\varphi_1 = 0, \varphi_2 = 0)$, (or equivalently when $\varphi_{1-} = 0$), the second saddle point appears (figure 10). This critical temperature is given by $t_2 = z(t_1)$ or explicitly by T_{cr1} Eq. (3.19). Approaching this temperature from above makes the unstable local maximum corresponding to F_- and the local minimum of the disorder phase to shift towards one another, and at T_{cr1} together they create just one local unstable maximum. Therefore, the system makes a first order disorder-order phase transition into phase 1 described by F_+. The temperature dependance of the order parameter evaluated close to this critical point is,

$$\varphi_1^2 = -\frac{2\bar{T}}{2g_1} + \frac{w^2\kappa_2^2(g_1T_1 - wT_2)\bar{T}}{g_1^2T_1T_2\left[-\frac{T_{crl}}{T_1} + 1 + \frac{w\kappa_2^2|\Delta|}{2g_1}\right]} + \frac{2w\kappa_2^2|\Delta|}{2g_1} + \frac{4}{g_1} - \frac{4T_{crl}}{g_1T_1}; \qquad (3.22)$$
$$\frac{T_{crl}}{T_1} < 1 + \frac{w\kappa_2^2|\Delta|}{2g_1}$$

The first order phase transition is in agreement with the predictions of RG theory. In the limit of $\kappa_2 \to 0$, or equivalently $c \to \infty$, (where fluctuations are suppressed), the free energy F_+, Eq. (3.15), as well as T_{cr1} Eq. (3.19) and T_{cr2} Eq. (3.21) reduce to MF expressions. To realize that in this limit Eq. (3.21) indeed reduces to T_1 which is the MF result, one must note that the point $(t_1, t_2) = (\ b,\ b[w/(2g_1) + g_2/(2w)]\ )$ at which the straight line $t_2 = z_1(t_1)$ Eq. (3.17) is tangent on the parabola $t_2 = z(t_1)$ Eq. (3.16.a) becomes $(t_1, t_2) \to (0,0)$ which implies that at this particular point $T_1 = T_2$. But this point is the only one which after the limit $\kappa_2 \to 0$ still separates disorder and phase 1 and in addition lays on the line $t_2 = z_1(t_1)$ where points corresponding to first order transition may be found. Consequently, the expressions of the order parameter Eqs.

(3.13,3.20,3.22) reduce to expressions of strictly second order transitions with transition temperature $t_1 = 0$. We see that in this limit the jump of the first order phase transition implied by Eqs. (3.20) and (3.22) vanishes. Moreover, in this limit the cumbersome expression of the coexistence curve becomes the straight line, $t_2\sqrt{g_1} = t_1\sqrt{g_2}$ and the boundaries of disorder and phase 1 and disorder and phase 2 become $t_1 = 0$ and $t_2 = 0$ respectively. These are the expected from MF theory results. Hence, we realize that what really causes first order disorder-order transitions are fluctuation interactions which in the above limit are suppressed.

An interesting point about the phase diagram t_2 vs t_1, figure 4, is the following. The MF theory analysis finds that the first quadrant of the plane corresponds to complete disorder, figure 1. The model's first quadrant, however, contains phases 1 and 2 in addition with the disorder phase. This feature is also in agreement with RG and is caused by the fluctuations. When these are suppressed the model's phase diagram reduced to the MF one figure 1. At a first look of this result one would conclude that the fluctuations raise the transition temperatures. This is not necessarily so however because the model's phase plane is the (t_1,t_2) as opposed to the MF one which is the (τ_1,τ_2), and the renormalized temperatures t_i, are shifted down with respect to τ_i.

Let us consider the special case of $t_1 = t_2 \equiv \tau$, and the vertex w greater than both vertices g_1 and g_2. In MF theory this corresponds to the phase diagram with the bicritical point at $\tau = 0$, figure (1). The model shows that this bicritical point is replaced by a first order transition [50], and the phase diagram given by the model, (figure 4), is quite different from the MF theory one, and qualitatively agrees with the result of RG analysis.

Indeed, if inequalities

$$\kappa_2^2 g_2 (w - g_2) > \kappa_1^2 g_1 (w - g_1); \quad w(w - g_1) > g_1 (w - g_2) \tag{3.23}$$

are fulfilled then one can show that the lowest free energy corresponds to the phase transition with the temperature dependance of the order parameter defined by the relationship

$$\varphi_1^2 = \left| \frac{\kappa_2^2 \Delta (w^2 - 2g_1 w + g_1 g_2)}{2g_1 (w - g_1)} \right| + \frac{2|\kappa_2| w}{g_1} \sqrt{\frac{(g_1 - w)\tau'}{g_1}} - \frac{\tau'}{g_1}$$
$$\tau' \equiv \tau - \tau_c^{(1)}; \qquad \tau_c^{(1)} = \frac{\kappa_2^2 \Delta^2}{4g_1 (w - g_1)} \tag{3.24}$$

At the temperature $\tau_c^{(1)}$ the order parameter φ_1 has a jump, therefore the phase transition is of the first order. Since first order phase transitions in systems described by functional (3.1) in accordance to MF theory do not occur, this means that such transitions are induced by fluctuation interactions. The inequalities (3.23) together with $\Delta < 0$ and $w > g_i$ bound the domain in the space of vertices g_1, g_2 and w where disorder-order transition is of the first order.

On the other hand, for the set of inequalities

$$w(w - g_1) < g_1 (w - g_2); \quad \kappa_1^2 \Delta^2 < 4\kappa_2^2 w g_2 (w - g_2)^2 \tag{3.25}$$

we have the second order phase transition into the phase $\varphi_1 \neq 0$, $\varphi_2 = 0$ occurring at the temperature $\tau_c^{(2)} = \kappa_2^2\, w\, (w - g_2)$ with the order parameter

$$\varphi_1^2 = -\frac{2\tau'}{g_1}\left[1 + \frac{\kappa_2^2 w^2 (w - g_1)}{g_1^2}\right], \quad \tau' = \tau - \tau_c^{(2)}. \tag{3.26}$$

If we interchange indices 1 and 2 in Eqs. (3.23-3.26) we then have conditions for first and second order phase transitions into the phase $\varphi_1 = 0$, $\varphi_2 \neq 0$.

In conclusion, we rigorously proved the existence of a fluctuation-induced first order phase transition. In addition, even though within the context of RG theory such a transition was a mere prediction and not an explicit proof, after the results of the model one may indeed claim that this is the true behavior of systems with competing order parameters and that RG results after all are not artifacts of an approximation or of a wrong interpretation.

3.3.2 SYSTEM OF $w>0$, $\Delta>0$

Systems having $w>0$ and $\Delta>0$ exhibit second kind order-disorder transitions. The plane (t_1,t_2) is split into 4 different regions, (see figure 5): Phase 1 occurs in the joined region enclosed by that part of inequality (3.16.a) that obeys $t_1 \geq 0$, and that part of $g_1\, t_2 \geq w\, t_1$ that obeys $t_1 < 0$. The analogous result for phase 2 is obtained after subscripts 1 and 2 are replaced by 2 and 1 respectively. The mixed phase occurs in the common region defined by inequalities (3.12).

The boundaries of disorder and phase 1 are created by that part of the $t_2 = z(t_1)$ parabola Eq. (3.16.a), that lays in the first quadrant of the (t_1,t_2) plane, that is when $t_1 \geq 0$. This part of the parabola gives the critical temperature for the second kind disorder-order transition into phase 1 Eq. (3.19), and the temperature dependance of the order parameter is given by Eq. (3.18.a). A transition from phase 1 into the mixed phase occurs through the line $g_1\, t_2 = w\, t_1$,hence

$$T_{cr3} = \frac{T_1 T_2 (w - g_1)}{(w T_2 - g_2 T_1)} \tag{3.27}$$

and is of second order. More explicitly, if $\mathcal{T} \equiv T - T_{cr3}$ then near the critical temperature the order parameters of phase 1 are given by

$$\varphi_1^2 = \frac{2}{\Delta T_1 T_2}(w T_1 - g_2 T_2)\, \mathcal{T} + \frac{2}{g_1} - \frac{2 T_{cr3}}{g_1 T_1} \tag{3.28.a}$$

$$\varphi_2 = 0 \tag{3.28.b}$$

and the order parameters of the mixed phase are given by Eq. (3.28.a) and

$$\varphi_2^2 = \frac{2}{\Delta T_1 T_2} (w T_2 - g_1 T_1)\, \mathcal{T} \tag{3.29}$$

Equations (3.28-3.29) show that at criticality ($\mathcal{T}=0$), the expressions of the order parameters for phase 1 and the mixed phase coincide indicating a second order phase transition between the two ordered phases. This result agrees with MF theory. The kind of such an order-order phase transition cannot be rigorously concluded by RG theory. Similar results concerning phase 2 are easily obtained. The second kind order-disorder transition is in agreement with MF theory. RG theory predicts that in addition to a second order phase transition, a first order may be possible as well.

3.3.3 SYSTEM OF $w<0,\ \Delta>0$

The last type of a system exhibits a second kind order-disorder phase transition. The phase diagram of (t_1,t_2) is qualitatively the same as that of $w>0,\ \Delta>0$ (see figure 6). Phase 1 occurs in the region that inequalities $g_1\, t_2 \geq w\, t_1$, $t_2 \leq z(t_1)$ and $t_1 \leq 0$ are simultaneous. Similarly, the analogous result for phase 2 is obtained from the inequalities $g_2\, t_1 \geq wt_2$, $t_2 \leq 0$ and $t_1 \leq z(t_2)$, where the last inequality is obtained from the parabola (3.16.a) after subscripts 1 and 2 are replaced by 2 and 1 respectively. The mixed phase occurs in the common region defined by inequalities (3.12). The critical temperature for a disorder-order transition into phase 1 is given by Eq. (19), and the temperature dependance of the order parameter is given by Eq. (18.a). The second kind order-order phase transition from phase 1 into the mixed phase occurs through the boundary $g_1\, t_2 = w\, t_1$. A similar discussion can be extended for phase 2.

The second kind order-disorder transition proven within the model, is in agreement with the MF analysis. In addition to the second order phase transition, RG theory predicts a first order as well. Not having an absolute agreement between the model's results and RG predictions is somehow expected since the model takes into account partially fluctuation interactions.

3.4 PHASE DIAGRAM

By defining $y \equiv w/g_2$ and $x \equiv g_1/g_2$, and assuming $T_1 > T_2$ one can construct the phase diagram of a disorder-order phase transition into say phase 1 (see figure 11). This diagram shows qualitatively how the parameter space is divided into three main regions, those of the first and second kind order-disorder transitions and the forbidden regions. When y is negative we have a region of second order transition for relations of the free energy satisfying $\Delta \geq 0$, which is separated from the forbidden region through line $\Delta = 0$. When y is positive the regions corresponding to first and second order transitions are separated through line,

$$x = \frac{y^3 - y^2\sqrt{(y-1)^2 + p}}{(2y - 1 - p)} \qquad p \equiv \frac{(T_1 - T_2)4g_2^2}{T_1\kappa_2^2} \tag{3.30}$$

which is obtained from Eqs. (3.17). The requirement $w^2 - 2wg_1 + g_1g_2 > 0$ follows as well. This line, Eq. (3.30), runs through the region defined by $\Delta < 0$, as expected since we showed that when $\Delta < 0$ systems that exhibit first or second order transitions are possible. Upon suppression of fluctuations, Eq. (3.30) becomes $x = 0$ and having in mind figure (11) this implies that no relationship between the coupling parameters can create a first order phase transition as expected.

3.5. CONCLUSION

The fact that our model's results are in a qualitative agreement with RG predictions is an encouraging sign for the truth of RG results, which after all do not seem to be artifacts of approximations. Indeed, on the phase diagram of the RG equations of functional (3.1) the point $g_1 = g_2 = w$ is the stable fixed point corresponding to the second order phase transition, as it is also inferred by the model from (3.24). Any phase trajectories started outside of the domain of attraction to this point are interpreted as first order phase transitions. Within RG the analytical expression for the boundary of this domain does not exist, so we cannot make quantative comparison of our results and the results of RG theory. Moreover, we would not expect that these results coincide quantitatively because our model treats fluctuations in a limited manner. However, similarly to the RG results the model demonstrates that in the space of parameters g_1, g_2 and w there is a domain where order-disorder phase transitions are of the first order.

Since the phase transitions obtained here are due to the fluctuation interaction, in the case when fluctuations are suppressed we must obtain the regular MF transitions. To test if this is true we must increase the radii of interactions between the order parameters in the Ginzburg-Landau functional (3.1). Therefore, in the limit $\kappa_i \to 0$, (or $c_i \to \infty$) all fluctuations are suppressed. Having in mind Eq. (3.13) it is easy to see that when radii of interactions increase, order parameter jumps decrease and in the limit of the infinite range of interaction the first order phase transition disappears, therefore the model's solutions corresponding to first order phase transitions reduce to the results of

MF theory as expected. In conclusion, the model's result of the existence of first order transition induced by fluctuations may be presented as a strong argument in favor of RG results which after all do not seem to be artifacts of the approximations.

CHAPTER 4

RANDOM FIELD

4.1 INTRODUCTION

In the previous chapter we have shown how successful the exactly solvable model was in demonstrating same qualitative results as RG analysis. However, we must add an essential difference between the two approaches. The existence of first order phase transition through RG analysis was an intuitive prediction and not an explicit proof. The main success of the model was in proving such a first order transition initiated by fluctuations, even though it considers partially fluctuation interactions only. Indeed, in our model when these fluctuations were suppressed the jump of the first order becomes continuous as expected. We would like now to examine an effect of a random field on phase transitions within the context of the exactly solvable model. We do so using the replica method [35]. Suppose we have the following GLW free energy functional

$$H = \frac{1}{2}\int d^{d}x\left[\tau S^{2}(\boldsymbol{x}) + c(\nabla S(\boldsymbol{x}))^{2} + uS^{4}(\boldsymbol{x}) - hS(\boldsymbol{x}) - h(\boldsymbol{x})S(\boldsymbol{x})\right] \tag{4.1}$$

$S(\boldsymbol{x})$ is a continuous scalar order parameter, h is constant conjugate field and $h(\boldsymbol{x})$ is the random field which is time independent (that is frozen in). A physical system that the above functional may represent is a ferromagnet where each lattice point is occupied by a specific type of magnetic moment. Now if another type of magnetic moments, (of different strength), randomly replaces some of the original moments then we have a random system described by functional Eq. (4.1). Within the context of MF theory this random case does not exhibit any new results with respect to the pure case. On the other hand, RG predicts interesting changes such as a shifting in dimensionality by 2 where critical behavior occurs. As we see below the treatment of the problem using the model reveals a qualitative agreement with RG theory.

4.2 GENERAL RELATIONS

The free energy averaged over the random field has the form

$$-F = \int_{-\infty}^{\infty} \prod_{x} [dh(x)] P[h(x)] \ln Z[h(x)] , \tag{4.2}$$

where $P[h(x)]$ is a probability distribution function of $h(x)$. Trying to calculate F directly from Eq. (4.2) creates mathematical difficulties which are overcome when the replica method is used. Therefore, we replicate n times the partition function Z by defining an n-component vector $\varphi(\mathrm{x}) \equiv [S_1(x), \ldots , S_n(x)]$. Hence,

$$-F = \frac{\partial}{\partial n}\left(\int \prod_{x} [d^n \varphi(x)] \exp\left(-H_{eff}[\varphi(x)]\right)\right)\Big|_{n=0} , \tag{4.3}$$

where $H_{eff}[\varphi]$ is an effective free energy functional given by,

$$H_{eff}[\varphi(x)] \equiv \frac{1}{2}\int_{-\infty}^{\infty} d^d x \left[\tau\,|\varphi(x)|^2 + c(\nabla\varphi(x))^2 - Q[\varphi] + \sum_{i=1}^{n}\left[u\,\varphi_i^4(x) - h\,\varphi_i(x)\right]\right] \tag{4.4}$$

with

$$e^{Q[\varphi(x)]} \equiv \int_{-\infty}^{\infty} \prod_{x} [dh(x)] P[h(x)]\, e^{\frac{1}{2}\int_{-\infty}^{\infty} d^d x \sum_{i=1}^{n} h(x)\,\varphi_i(x)} \tag{4.5}$$

If we suppose that the probability $P[h]$ at site $\boldsymbol{x}$ is independent with that at site $\boldsymbol{x}'$ such that, $P\{h(\boldsymbol{x})\} = \Pi_x \rho\{h(\boldsymbol{x})\}$, then we have

$$Q[\boldsymbol{\varphi}] = \int d^d x \ln\left\{\int dh(\boldsymbol{x})\, \rho[h(\boldsymbol{x})] e^{\frac{1}{2}\sum_{i=1}^{n} h(\boldsymbol{x})\,\varphi_i(\boldsymbol{x})}\right\} \tag{4.6}$$

We also assume that the distribution function $\rho[h(\boldsymbol{x})]$ is a Gaussian one,

$$\rho[h(\boldsymbol{x})] = \frac{e^{\frac{-h^2(\boldsymbol{x})}{2B}}}{\sqrt{2\pi B}} \tag{4.7}$$

where B is a measure of the random field. The effective Hamiltonian then becomes

$$H_{eff} = \frac{1}{2}\int d^d x\left[\tau\,|\boldsymbol{\varphi}(\boldsymbol{x})|^2 + c\,(\nabla\boldsymbol{\varphi}(\boldsymbol{x}))^2 + \sum_{i=1}^{n}\left[u\,\varphi_i^4(\boldsymbol{x}) - h\,\varphi_i(\boldsymbol{x})\right] - B\left(\sum_{i=1}^{n}\varphi_i(\boldsymbol{x})\right)^2\right] \tag{4.8}$$

We consider the above free energy functional within the framework of the exactly solvable model which finally puts the partition function in the form

$$\begin{aligned} Z = \int\prod_{i=1}^{n}(D\varphi_{iq}\,dx_i\,dy_i)\exp\Bigg\{&-\frac{V}{2}\sum_{i=1}^{n}\left[\tau x_i + u x_i^2 - x_i y_i - \frac{h}{V}\varphi_{i0}\right] \\ &-\frac{1}{2}\sum_{i,q}^{n,\infty}\varphi_{iq}^2(y_i + cq^2 - B) + \frac{B}{2}\sum_{q}^{\infty}\left(\sum_{i=1,i\neq j}^{n}\varphi_{iq}\right)\left(\sum_{j=1,j\neq i}^{n}\varphi_{j-q}\right)\Bigg\} \end{aligned} \tag{4.9}$$

We notice that Eq. (4.9) is not diagonal with respect to components of the vector $\boldsymbol{\varphi}$. The nondiagonal part is caused by the strength of the random conjugate field. The pure φ^4 model is reproduced by suppressing the field, that is $B \to 0$. The results of the

φ^4 model studied within the context of the exactly solvable model have already been presented in chapter 2.

Let us now study the random field problem, with a nonzero B in Eq. (4.9). We define F to be the summation of the exponent of (4.9). To be able to proceed with the investigation of the critical behavior of the system we first must diagonalize the nxn matrix defined by the last two summations of F. After diagonalization we require that $y_i = y_j \equiv y$ since only this choice reproduces the pure φ^4 upon suppression of the random field. Explicitly, this is so because in the limit of $B \to 0$ the degeneracy of the eigenvalues of every other choice does not reduce to n-fold as expected from considerations of the simple φ^4 model treated within the context of the replica method.

Hence, when $y_i = y_j \equiv y$ the matrix has two distinct eigenvalues,

$$\begin{aligned}\lambda_1 &= -\frac{1}{2}(y + cq^2)\\ \lambda_2 &= -\frac{1}{2}(y + cq^2 - nB)\end{aligned} \tag{4.10}$$

The first one, λ_1, is $(n$-$1)$-fold degenerate. These eigenvalues can be used to diagonalize the nxn matrix of interest as well as transform φ_{i0} of term $h/2\sum\varphi_{i0}$ into a new order parameter say ψ_{i0}. An orthonormal basis for the solution space of the nxn matrix is generated by,

$$\sum_{i=1}^{n} \varphi_{i0} = \sqrt{n}\,\psi_{n0} \tag{4.11}$$

The above relationship makes it possible to completely replace φ_{i0} with ψ_{i0} in Eq. (4.9) and to perform all Gaussian integrals with respect to ψ_{iq}. Therefore, the partition function becomes

$$Z = \int \prod_{i=1}^{n} (dx_i)\, dy \exp\left[-\frac{V}{2}\sum_{i=1}^{n}\left[\tau x_i + u x_i^2 - y x_i\right] + \frac{(1-n)}{2}\sum_{q=0}\ln(y + cq^2) \right.$$
$$\left. - \frac{1}{2}\sum_{q=0}\ln(y + cq^2 - nB) + \frac{nVh^2}{2(y - nB)} \right] \tag{4.12}$$

and after treating the summations with respect to q and renormalize τ as in chapter 2, we derive

$$F(x_i,y,h) = \sum_{i=1}^{n}\left[t x_i + u x_i^2 - y x_i\right] - (1-n) f_d(y;c) + f_d(y - nB;c) - \frac{nh^2}{(y - nB)} \tag{4.13}$$

Using $\partial F/\partial x_i = 0$, we simplify the above equation by replacing x_i

$$F(y,h) = \sum_{i=1}^{n}\left[-\frac{t^2}{4u} - \frac{y^2}{4u} + \frac{yt}{2u} + f_d(y;c) - \frac{h^2}{(y - nB)} \right] - f_d(y;c) + f_d(y - nB;c) \tag{4.14}$$

and the saddle point of F is found by considering in addition,

$$\frac{\partial F(y,h)}{\partial y} = 0 \Rightarrow -\frac{ny}{2u} + \frac{nt}{2u} + (n-1) f_d'(y;c) + f_d'(y - nB;c) + \frac{nh^2}{(y - nB)^2} = 0 \tag{4.15}$$

The solution for y must then be substituted in $F(y,h)$ and the averaged value of the free energy is given by

$$F = \lim_{n \to 0} \frac{F(y,h)}{n} \tag{4.16}$$

An expression of an averaged order parameter Φ, at zero field is obtained from

$$\Phi = -\lim_{h \to 0} \frac{\partial F}{\partial h} = \lim_{h \to 0} \frac{h}{y(h)} \tag{4.17}$$

In order to have a non-trivial solution for Φ we must seek a solution for y of the form $y(h) = ah$ with a a constant to be determined from Eq. (4.15). It can be shown that when Eq. (4.15) is expanded in powers of n, for any d and up to order n, (which is sufficient since we consider the $n \to 0$ limit), the solution for $y(h)$ is independent of n. Respectively, for d not even (including non-integers) and d even the resulting equations are

$$-\frac{y}{2u} + \frac{t}{2u} + \frac{\kappa(c) y^{\frac{(d-2)}{2}}}{2} + \frac{h^2}{y^2} - \frac{\kappa(c)(d-2)B y^{\frac{(d-4)}{2}}}{4} = 0 \tag{4.18.a}$$

$$\begin{aligned} &-\frac{y}{2u} + \frac{t}{2u} + \frac{\mu(c)d}{2} y^{\frac{(d-2)}{2}} \ln y + \mu(c) y^{\frac{(d-2)}{2}} + \frac{h^2}{y^2} \\ &-\mu(c)B(d-2) y^{\frac{(d-4)}{2}} \left[\frac{1}{2} + \frac{d}{2(d-2)} + \frac{d}{4}\ln y\right] = 0 \end{aligned} \tag{4.18.b}$$

In addition, the averaged free energy becomes,

$$F = -\frac{t^2}{4u} - \frac{y^2}{4u} + \frac{yt}{2u} + f_d(y;c) - \frac{h^2}{y} - B f_d'(y;c) \tag{4.19}$$

This Eq. must be used to find a possible expression of an order parameter, Φ Eq.

(4.17). Notice that Eqs. (4.12-4.19) in the limit of $B \rightarrow 0$ reproduce the corresponding ones of the φ^4 model as expected since in this limit the random conjugate field vanishes.

4.3 RESULTS

From Eqs. (4.17) and (4.18) it is seen that for $d \leq 4$ no solution for Φ exists. Thus the random field, regardless of how weak it is, does not allow a phase transition for $d \leq 4$. When $d > 4$ we get the second order transition occurring for $t = 0$ as predicted in Ref. [29] using RG arguments. Furthermore, the critical exponent δ for the random d-dimensional system is exactly the same as that derived for a $(d - 2)$-dimensional pure one. That is, for the random $4 < d\ (=d' + 2) < 6$, and $d > 6$ system the critical exponent δ is equal to $(d' + 2)/(d' - 2)$, and 3 respectively. More explicitly, this result is obtained from Eqs. (4.18) after replacing y with h/Φ and then solving in terms of Φ for small h and with $t = 0$. In addition, the model derives from Eq. (4.18.b) the same logarithmic corrections implied by (2.15) for the random 6-dimensional system, and the simple φ^4 4-dimensional system . Therefore, we note how the upper marginal dimension where MF theory results hold has shifted from 4 in the pure case, to 6 in the random case. Moreover, the lower limit of destruction of a phase transition is 4 for the random case, where it used to be 2 for the pure case. Note that within the framework of the model the critical exponent β is equal to 1/2 whenever a phase transition is possible. Therefore, using the scaling relations, it is sufficient to know two of the six critical exponents in order to specify all of them. The crossover from critical to MF behavior in $4 < d < 6$ occurs when

$$h > \left[\frac{(d-2)}{4} B\,|\kappa(c)|\right]^{\frac{3}{(4-d)}} (2u)^{\frac{d}{2(4-d)}} \tag{4.20}$$

and it is obtained by requiring that in Eq. (4.18.a) the first term, which is responsible for MF behavior, is greater than the last one, which is responsible for critical behavior.

Since the dimensional crossover that includes a destruction of phase transitions by a random field for $d \leq 4$ occurs due to fluctuation interactions, in the case when fluctuations are suppressed in the limit of $c \to \infty$ we must recover the regular MF transitions. Indeed, in this limit Eqs. (4.18.a, 4.18.b) give the MF critical exponents and the second order phase transition is restored independently of dimensionality or the presence of random fields. Furthermore, inequality (4.20) is always true. Therefore, this test shows that the suppression of fluctuations produces a more dominant effect on the system in comparison to the effect of the random field. This is expected because when the MF theory is used for the study of the random system, no new results are derived in the picture of phase transitions.

4.4 CONCLUSION

In conclusion we have explicitly shown that an arbitrarily small random field forbids the occurrence of a phase transition for $d \leq 4$. The critical exponents of the random d-dimensional system agree with those of the (d - 2) pure one. Logarithmic corrections to the critical exponents were found for $d = 6$. The lower limit of dimension for which a phase transition is not occurring and the upper limit of dimension for which MF results hold have increased by 2. When the random field is suppressed all results are reduced to those of the pure system. The exactly solvable model's results qualitatively recover those obtained by RG analysis. MF finds no changes in the critical behavior of the random field problem.

CHAPTER 5

RANDOM TEMPERATURE

We will now examine the problem of nonmagnetic impurities that cause local fluctuations of critical temperature. Such type of disorder is often called "random temperature" and is described by the free energy functional

$$H = \frac{1}{2}\int d^d x\left[\tau_0 S^2(x) + \tau(x) S^2(x) + c(\nabla S(x))^2 + u S^4(x) - h S(x)\right] \tag{5.1}$$

where $\tau(\mathbf{x})$ is a random perturbation of local temperature. A physical system that the above functional may represent is a ferromagnet for which randomly selected magnetic moments are replaced by atoms with no net magnetic moment. This creates different regions in the sample with different phase transition temperatures, and the question we like to ask is what kind of effect this has on the entire phase transition picture. The random temperature is assumed to be a δ-correlated function of the form, $<\tau(\mathbf{x})\,\tau(\mathbf{x}')>$ $= B\delta(\mathbf{x} - \mathbf{x}')$ which implies no long-range interactions. More specifically, the δ-correlated function indicates that one non-magnetic atom does not feel another but all of them feel the order parameter. After following the steps of the replica method and treating the problem within the context of the exactly solvable model we derive,

$$F(x_i,y_i,h) = \sum_{i=1}^{n}\left[tx_i + ux_i^2 - x_i y_i + f_d(y_i;c) - \frac{h^2}{y_i}\right] - B\left(\sum_{i=1}^{n} x_i\right)^2 \tag{5.2}$$

Using the saddle point equations $\partial F/\partial x_i = 0$ and $\partial F/\partial y_i = 0$, we obtain y_i and x_i which are then used in Eq. (4.16) to find the averaged free energy. Having in mind that the only physical choice is $y_i = y_j \equiv y$, (see chapter 4), which implies $x_i = x_j \equiv x$ the saddle point equations give

$$t + 2uf_d'(y;c) + \frac{2uh^2}{y^2} - y - 2nBf_d'(y;c) - \frac{2nBh^2}{y^2} = 0 \tag{5.3}$$

Now by writing the solution of the above equation as $y = y_0 + n^\epsilon y_1$ with y_0 corresponding to the pure case, and having in mind that the order parameter is given by

$$\Phi = \lim_{h\to 0}\lim_{n\to 0} \frac{h}{(y_0 + n^\epsilon y_1)} = \lim_{h\to 0} \frac{h}{y_0} \tag{5.4}$$

we conclude that the random temperature system is identical with the pure φ^4 model. Alternatively, this can be seen by studying the averaged free energy which we argue is the same as in the pure case. This is so because any possible new contribution to the random temperature averaged free energy must come from the last term of Eq. (5.2). But this does not happen since in the limit $n \to 0$ of Eq. (4.16) this term vanishes. Consequently, the critical exponents of the random temperature problem are the same with those of the pure φ^4 model. This result is rather expected because as we saw in chapter 4 the critical exponent $\beta = 1/2$ which is the MF value. β is defined through the temperature dependance of the order parameter which can be expressed as $(T_c - T)^\beta$. In other words, the fact that β has a classical value may perhaps be interpreted as that within the framework of the model temperature dependent functions are less sensitive to fluctuations than field dependent ones. Therefore, when local fluctuations of temperature are considered through the second term of functional (5.1) no new changes result in the picture of phase transition with respect to the pure case which was studied in chapter 2. This is also the case in MF theory. Of course, the pure case studied by MF theory and the model present significant differences as was shown in chapter 2. RG theory finds that when the specific-heat exponent α is negative the fixed point of the pure case is still stable , that is the behavior of the pure and random cases are the same, and when α is positive the pure fixed point is unstable and a new fixed point appears, in other words, there is still a second order phase transition but with different critical exponents, [3,6].

CHAPTER 6

COUPLED ORDER PARAMETERS IN THE PRESENCE OF RANDOM FIELDS

6.1 INTRODUCTION

The influence of random fields is frequently used as a reason to account for qualitative differences between experimental results and theoretical predictions concerning a phase transition. Renormalization group (RG) theory has investigated such systems and in many cases found interesting results such as a dimensional reduction by 2 [3,26-32,34]. In some cases exactly solvable models were used to recover and/or clarify RG results [28, 55]. In this chapter we substantially generalize the study done in chapters 3 and 4. We are interested in the critical behavior of systems with two coupled scalar order parameters, which in addition are under the influence of two random fields. The study is done within the context of the exactly solvable model [40]. We explicitly prove that regardless of the presence of fluctuation interactions which is the reason for the existence of the first order phase transition in the pure case of two coupled order parameters [50], (see also chapter 3), the influence of a random field is the cause of the replacement of the first order by a second order phase transition. In addition, we prove a destruction of phase transitions by two random fields for $d \leq 4$ due to fluctuation interactions. The suppression of these fluctuations produce MF results independently of random fields or dimensionality.

6.2 GENERAL RELATIONS

The Ginzburg-Landau free energy functional of interest is

$$F[S_1,S_2] = \frac{1}{2}\int d^dx \left\{ \sum_{i=1}^{2}\left[\tau_i S_i^2(x) + c_i\left(\nabla S_i(x)\right)^2 + \frac{1}{4} g_i S_i^4(x)\right] + \frac{1}{2} w\, S_1^2(x)\, S_2^2(x)\right\} - \sum_{i=1}^{2}\left[h_i S_i(x) + h_i(x)\, S_i(x)\right], \tag{6.1}$$

where S_i is a scalar order parameter with h_i and $h_i(x)$ being constant and quenched random fields respectively. If $P_i[h_i(x)]$ is a probability distribution function of the random field $h_i(x)$ then, using the replica method [35], the averaged with respect to $h_i(x)$ free energy of the system is obtained from

$$\begin{aligned} -F &\equiv \left\langle \ln\left[\int \prod_{i=1}^{2} \left(DS_i(x)\right)\, \exp\left(-H[S_i(x)]\right)\right]\right\rangle \\ &= \frac{\partial}{\partial n}\left(\int \prod_{i=1}^{2}\left[D^n \varphi_i(x)\right] \exp\left(-H_{eff}[\boldsymbol{\varphi}_i(x)]\right)\right)\Big|_{n=0}, \end{aligned} \tag{6.2}$$

where φ_i (x) is an n-component vector, φ_i (x) $\equiv$ $[S_{i1}(x), \ldots., S_{in}(x)]$, and $H_{eff}\,[\varphi_i]$ is defined as

$$\begin{aligned} H_{eff}[\boldsymbol{\varphi}_i(x)] \equiv \frac{1}{2}\int_{-\infty}^{\infty} d^dx \Bigg[&\sum_{i=1}^{2}\left(\tau_i\, |\boldsymbol{\varphi}_i(x)|^2 + c_i\left(\nabla\boldsymbol{\varphi}_i(x)\right)^2 + \sum_{j=1}^{n}\left(\frac{g_i}{4}\varphi_{ij}^4(x) - h_i\,\varphi_{ij}(x)\right)\right) + \\ &\frac{w}{2}\sum_{j=1}^{n} \varphi_{1j}^2(x)\, \varphi_{2j}^2(x)\Bigg] - Q[\boldsymbol{\varphi}_i] \end{aligned} \tag{6.3}$$

with

$$Q[\varphi_i] = \int d^dx \ln\left\{\prod_{i=1}^{2}\left(\int dh_i(x)\,\rho_i[h_i(x)]\right) e^{\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{n} h_i(x)\,\varphi_{ij}(x)}\right\} \tag{6.4}$$

and where $P_i\{h(x)\} = \Pi_x \rho_i\{h(x)\}$. After choosing a Gaussian distribution function,

$$\rho_i[h_i(x)] = \frac{e^{\frac{-h_i^2(x)}{2B_i}}}{\sqrt{2\pi B_i}} \tag{6.5}$$

$H_{eff}\,[\varphi]$ takes the form

$$\begin{aligned} H_{eff}[\varphi_i(x)] \equiv \frac{1}{2}\int_{-\infty}^{\infty} d^dx \Bigg[\sum_{i=1}^{2}\left(\tau_i\,|\varphi_i(x)|^2 + c_i(\nabla\varphi_i(x))^2 + \sum_{j=1}^{n}\left(\frac{g_i}{4}\varphi_{ij}^4(x) - h_i\,\varphi_{ij}(x)\right)\right) + \\ \frac{w}{2}\sum_{j=1}^{n}\varphi_{1j}^2(x)\,\varphi_{2j}^2(x)\Bigg] - B_1\left(\sum_{j=1}^{n}\varphi_{1j}(x)\right)^2 - B_2\left(\sum_{j=1}^{n}\varphi_{2j}(x)\right)^2 \end{aligned} \tag{6.6}$$

The above effective free energy functional is now treated within the context of the exactly solvable model which reduces the quartic terms as follows [40],

$$\begin{aligned} \int d^dx\,\varphi_{ij}^4(x) &\rightarrow \frac{a_{ij}^2[\varphi_i]}{V}; \qquad a_{ij}[\varphi_{ij}] \equiv \int d^dx\,\varphi_{ij}^2 \\ \int d^dx\,\varphi_{1j}^2(x)\,\varphi_{2j}^2(x) &\rightarrow \frac{a_{1j}[\varphi_{1j}]\,a_{2j}[\varphi_{2j}]}{V} \end{aligned} \tag{6.7}$$

When this model was applied to functional Eq. (6.1) with zero random fields, it demonstrated a rich picture of phase transitions [50]. Explicitly, it proved the existence of fluctuation-induced first order phase transition. In this chapter we are interested to see how the same system behaves under the influence of the two random fields.

After the reduction Eq. (6.7) and the use of a transformation analogous to Hubbard-Stratonovich (2.3) the partition function becomes

$$Z = \int \prod_{i=1,j=1}^{2,n} (D\varphi_{iq} dx_{ij} dy_{ij}) \exp\left\{ -\frac{V}{2} \sum_{i=1,j=1}^{2,n} \left[\tau_i x_{ij} + \frac{g_i}{4} x_{ij}^2 + \frac{w}{4} x_{1j} x_{2j} - x_{ij} y_{ij} - \frac{h_i}{V} \varphi_{ij0} \right] \right.$$
$$\left. - \frac{1}{2} \sum_{i,j,q}^{2,n,\infty} |\varphi_{ijq}|^2 (y_{ij} + c_i q^2 - B_i) + \sum_{i=1,q}^{2,\infty} \sum_{j \neq j'}^{n} \frac{B_i}{2} \varphi_{ijq} \varphi_{ij'-q} \right\} \tag{6.8}$$

Functional integrals may be calculated after the diagonalization with respect to components of the vector φ_i. For a fixed i, (i=1 or i=2), we notice that only the choice with all y_{ij} equal to one another can reproduce the pure case ($B_1 \to 0$, $B_2 \to 0$). Integrals with respect to $y_{ij} \equiv y_i$ and x_{ij} may be performed using the steepest descend method, and the partition function may be obtained. In the thermodynamic limit $V \to 0$ the calculation is exact. Explicitly,

$$Z = \int \prod_{i=1}^{2} (Dx_{ij} dy_i) \exp\left[-\frac{V}{2} F(x_{ij}, y_i; h_1, h_2) \right] \tag{6.9}$$

with

$$F(x_{ij},y_i;h_1h_2) = \sum_{i=1}^{2}\left[\sum_{j=1}^{n}\left(t_i x_{ij} + \frac{g_i}{4}x_{ij}^2 - y_i x_{ij} + \frac{w}{4}x_{1j}x_{2j}\right) + (n-1)f_d(y_i;c_i) + f_d(y_i - nB_i;c_i) - \frac{nh_i^2}{y_i - nB_i}\right] \tag{6.10}$$

The normalized trial critical temperature t_i is $t_i = \tau_i + 1/2\,[\,g_i\theta(A;c_i) + w\theta(A;c_{i'\neq i})] \equiv (T - T_i)/T_i$. Expressions of the averaged free energy and order parameter φ_i at zero constant field h_i are given by

$$F = \lim_{n\to 0}\frac{1}{n}F(x_{ij},y_i;h_1,h_2) \tag{6.11}$$

and

$$\varphi_i = -\lim_{h_i\to 0}\frac{\partial F}{\partial h_i} = \lim_{h_i\to 0}\frac{h_i}{y_i(h_i)} \tag{6.12}$$

with x_{ij} and y_i obtained from equilibrium Eqs. $\partial F/\partial x_{ij}=0$ and $\partial F/\partial y_i=0$. From $\partial F/\partial x_{ij}=0$ it is derived that for a fixed i all n of the variables x_{ij} are equal to one another hence $x_{ij} \equiv x_i$. After eliminating x_i we obtain two Eqs. for y_1 and y_2

$$\frac{-2ng_{i'}}{\Delta}(y_i - t_i) - \frac{2nw}{\Delta}(t_{i'} - y_{i'}) + (n-1)f_d'(y_i;c_i) + f_d'(y_i - nB_i;c_i) + \frac{nh_i^2}{(y_i - nB_i)^2} = 0 \tag{6.13}$$

with $\Delta = g_1 g_2 - w^2$ and $i \neq i'$. When Eq. (6.13) is expanded in powers of n, for any d and up to order n, (which is sufficient since we consider the $n \to 0$ limit), the solution for $y_i(h_i)$ is independent of n. Respectively, for d not even (including non-integers) and d even the resulting equations are

$$\frac{-2g_{i'}y_i}{\Delta} + \frac{2g_{i'}t_i}{\Delta} - \frac{2wt_{i'}}{\Delta} + \frac{2wy_{i'}}{\Delta} + \begin{cases} \dfrac{d}{2}\kappa(c_i)y_i^{\frac{(d-2)}{2}} \\ \mu(c_i)y_i^{\frac{(d-2)}{2}}\left(1 + \dfrac{d}{2}\ln y_i\right) \end{cases}$$

$$+ \frac{h_i^2}{y_i^2} + \begin{cases} \dfrac{-d(d-2)\kappa(c_i)}{4}B_i y_i^{\frac{(d-4)}{2}} \\ -\mu(c_i)B_i y_i^{\frac{(d-4)}{2}}\left(\dfrac{d}{2} + \dfrac{(d-2)}{2} + \dfrac{d(d-2)}{4}\ln y_i\right) \end{cases} = 0 \quad \text{for} \begin{cases} d = \text{non-even} \\ d = \text{even} \end{cases} \tag{6.14}$$

6.3 RESULTS

The most interesting case is when $d=3$. The random case presents a completely different picture than the pure case studied in chapter 3. For example, when both B_1 and B_2 are not equal to zero it is derived from Eqs. (6.12) and (6.14) that when $h_i \to 0$ no solution exists for φ_i for $d \leq 4$. A phase transition recovers when the fluctuations are suppressed in the limit $c_i \to \infty$ even at the presence of random fields, see Eq. (6.14). In the pure case depending on the system, (defined by w and Δ), a phase transition into either one of phases 1 or 2 may occur, (when $w > 0$, $\Delta < 0$), or a phase transition into either one of the three low symmetry phases may occur, (for $\Delta > 0$). Specifically, the system of $w > 0$, $\Delta < 0$ presents a fluctuation-induced first order phase transition into either phase 1 or 2, see Eqs. (3.20, 3.22). In the random case the situation is quite different. That is, the mixed phase never realizes as long as at least one of the random fields is present, and when $B_i = 0$, (with $B_{i'} \neq 0$), only phase i occurs. For the last case we will show that the transition is of the second order, that is, the random field replaces the first order phase transition found in the pure case.

Say $B_1 = 0$, then using Eqs. (6.12) and (6.14), at zero constant fields, $h_i \to 0$, the temperature dependance of the order parameter φ_1 of phase 1 is obtained. Explicitly we derive

$$\varphi_{1\pm}^2 = \frac{2w|\kappa_2|}{g_1}\left[-\frac{|\kappa_2|\Delta}{2g_1} \pm \sqrt{\left(\frac{\kappa_2\Delta}{2g_1}\right)^2 + t_2 - \frac{wt_1}{g_1}}\right] - \frac{2t_1}{g_1}$$
$$\mp \frac{w|\kappa_2|B_2}{g_1\sqrt{\left(\frac{\kappa_2\Delta}{2g_1}\right)^2 + t_2 - \frac{wt_1}{g_1}}} \qquad (6.15)$$

The "plus" solution corresponds to the lowest free energy. In the limit $B_2 \to 0$, Eq. (6.15) reduces to the expression corresponding to the pure case, [50]. The most interesting question we would like to ask is whether Eq. (6.15) may imply a fluctuation-induced first order phase transition as it was for the pure case. Unlike the pure case however, we prove that for arbitrarily small B_2 Eq. (6.15) corresponds to second order phase transition only. To do so we think as follows: In general a characteristic feature of first order phase transitions is the presence of two saddle points in the diagram of the free energy vs the order parameter see figures (8, 10). The transition temperature corresponding to the first saddle point is given by equating the two solutions of the order parameter, $\varphi_{1+} = \varphi_{1-}$. Therefore, if Eq. (6.15) were to correspond to a first order phase transition the transition temperature would have been given by

$$\varphi_{1+} = \varphi_{1-} \Rightarrow \left(\frac{\kappa_2\Delta}{2g_1}\right)^2 + t_2 - \frac{wt_1}{g_1} - B_2 = 0 \qquad (6.16)$$

and the equilibrium order parameter evaluated at a point $(t_2,\ t_1)$ that satisfies simultaneously Eq. (6.16) and inequality $t_1 < (w\, \kappa_2^2\, |\Delta|\,)\, /\, (2\, g_1)$ by

$$\varphi_{1+}^2 = \frac{w\kappa_2^2 |\Delta|}{g_1^2} - \frac{2t_1}{g_1} \tag{6.17}$$

Note that condition $t_1 < (w\, \kappa_2^2\, |\Delta|\,)\, /\, (2\, g_1)$ guarantees the positiveness of the order parameter. But let us show that Eq. (6.17) does not correspond to a jump as expected from a first order phase transition, and that it merely is the value of the order parameter calculated at the point $(t_2,\, t_1)$, which corresponds to simply a temperature lower than an actual second-order-phase-transition temperature. If it did corresponded to a jump, then the order parameter Eq. (6.15) evaluated at a point $(t_2\, ,\, t_1 + \epsilon)$ where ϵ is positively and arbitrarily small, should have been imaginary. But this is not the case. Explicitly,

$$\varphi_{1+}^2(t_2\, , t_1 + \epsilon) = \left(-\frac{2\sqrt{2}w^2 |\kappa_2|}{g_1^2 \sqrt{B_2}} - \frac{2}{g_1} \right)\epsilon + \frac{w\kappa_2^2 |\Delta|}{g_1^2} - \frac{2t_1}{g_1} \tag{6.18}$$

and having in mind that the result of the last two terms is always a positive number, it can be inferred that for an arbitrarily but finitely small ϵ and a finite B_2, $\varphi_{1+}(t_2,\, t_1 + \epsilon)$ has a positive definite value. Hence, Eq. (6.16) does not correspond to a transition temperature of any kind, and indeed Eq. (6.15) is the temperature dependance of the order parameter of phase 1 undergoing a second order phase transition. Note how when $B_2 \to 0$, $\varphi_{1+}(t_2,\, t_1 + \epsilon)$ in Eq. (6.18) becomes imaginary which implies that in this limit Eq. (6.16) is indeed the first saddle point of a fluctuation-induced first order phase transition Eq. (3.20).

6.4 CONCLUSION

We show that, despite a fluctuation-induced first order phase transition proven to exist in the pure case of coupled order parameters in chapter 3, we see here how the existence of a random field replaces such a transition by a continuous one. In addition, unlike the pure case where depending on the system a phase transition into either one of the three low symmetry phases may be possible, here the result is different: for 3-dimensional systems if both random fields are present no phase transition is possible. Furthermore, if one field is present then the mixed phase is unstable and only phase 1 (or 2 depending what random field is zero) exhibits a phase transition of the second kind. Within MF theory the random case is similar to the pure one and within RG theory is still an open question.

CHAPTER 7

EXACT RENORMALIZATION GROUP EQUATION FREE OF REDUNDANT OPERATORS

7.1 INTRODUCTION

In the previous chapters we have explored critical phenomena using the exactly solvable model which takes into account fluctuation interactions partially. The fact that the model's results are in a qualitative agreement with RG theory is not only considered a success for the model but also for RG theory since the theory's results which are obtain through perturbation were never guaranteed to be correct. This is interpreted as a sign that RG is indeed an adequate theory for critical phenomena, and therefore, we use its fundamental formalism to derive an exact RG functional equation from which all known results of phase transitions should be derivable. The derivation of an exact RG equation is important for many reasons, for example, unlike exactly solvable models, all fluctuation interactions are considered, thus providing an alternative approach to the goal of obtaining well established results on critical phenomena. Even though exact RG equations were derived before as was mentioned in the introduction, the exact equation we derive here [56], has the important difference in that it does not contain redundant operators which have no physical meaning. Their exclusion from the transformation is therefore a necessary but at the same time a very difficult task, so an exact RG equation which does not contain any is always very pleasing.

7.2 FORMULATION OF THE PROBLEM

The Wilson functional RG equation for the Landau type Hamiltonians H can be symbolically written as $dH/dl = R\{H\}$, [18] where R is some nonlinear operator performing RG transformations and l signifies a transformation of the length scale. The fact that the correlation function is scale invariant at the critical point suggests that H must be so at criticality as well, that is $dH/dl = 0$. This is so because the construction of R is based on the procedure which changes the scale of H. Hence, the fixed-point Hamiltonian H^* at the critical point of a phase transition is determined by $R\{H^*\} = 0$. For a linear deviation from H^* the equation $dH/dl = R\{H\}$ gives

$$\frac{\partial \Delta H}{\partial l} = L \Delta H; \qquad L \equiv \frac{\delta R\{H\}}{\delta H} \tag{7.1}$$

Its solution can be expressed in terms of the eigenvectors O_λ of the operator L

$$\Delta H = \sum_\lambda \mu_\lambda e^{\lambda l} O_\lambda \tag{7.2}$$

where λ are the eigenvalues corresponding to the eigenvectors O_λ with μ_λ the expansion coefficients [54]. Depending on the sign of λ the eigenvectors are classified as relevant ($\lambda > 0$), irrelevant ($\lambda < 0$), and marginal ($\lambda = 0$). Critical behavior can be associated with positive λ only. Unfortunately however, for conventional RG transformation equations there exist many eigenvectors O_λ having their corresponding eigenvalues greater than zero, that is, $\lambda > 0$. Only the one defining the critical exponent ν is physically meaningful. The others should be treated as redundant with no physical meaning.

Wegner [38] suggested some criteria for distinguishing the physical operators. According to Wegner, those operators whose eigenvalues are dependent on a particular choice of the RG equation should be treated as redundant. These operators must be excluded from the RG procedure by imposing additional conditions. In this chapter we show that there is a formulation of the exact RG equation which leaves no room for additional conditions and therefore it must not contain any redundant operators.

7.3 DERIVATION OF RG EQUATION

Suppose we have the most general kind of Ginzburg-Landau-Wilson functional

$$H_I[\boldsymbol{\varphi}(q)] = \sum_{k=0}^{\infty} 2^{1-2k} \int_{q_1 \cdots q_{2k}} \sum_{\alpha_1,\ldots,\alpha_{2k}=1}^{n} \left[g_k^{\alpha_1,\ldots,\alpha_{2k}}(q_1,\ldots,q_{2k})\,(2\pi)^d\,\delta(\sum_{i=1}^{2k} q_i) \prod_{i=1}^{2k} \varphi^{\alpha_i}(q_i) \right] \tag{7.3}$$

where $\boldsymbol{\varphi}$ is an n-component vector and $\int_q \equiv \int d^dq/(2\pi)^d$. The non-local vertices g_k must have the symmetry

$$g^{\ldots\alpha_i\ldots\alpha_j\ldots}(\ldots,q_i,\ldots,q_j,\ldots) = g^{\ldots\alpha_j\ldots\alpha_i\ldots}(\ldots,q_j,\ldots,q_i,\ldots). \tag{7.4}$$

We choose to cut off all momentum integrals at an upper momentum Λ by adding to H_I

$$H_0[\boldsymbol{\varphi}] = \frac{1}{2}\int_q G_0^{-1}(q,\Lambda)\,|\boldsymbol{\varphi}(q)|^2, \tag{7.5}$$

where the propagator G_0 is defined by

$$G_0(q,\Lambda) = q^{-2} S(q^2/\Lambda^2). \tag{7.6}$$

Here $S(x)$ is a monotonic function with the properties $S(x=0)=1$ and $\lim_{x\to\infty} S(x)x^m = 0$ for any m. Assuming that the vertices $g_k(q_1,\ldots,q_{2k})$ do not diverge with increasing q_i then the term H_0 provides either a smooth cutoff, when $S(x)$ is a smooth function, or a sharp cutoff, when $S(x)$ is a step function.

We now perform the following two steps which are standard for the RG theory. First, apply a Kadanoff transformation to thin out the original Hamiltonian (that is, to decrease the number of degrees of freedom), by integrating those Fourier components $\varphi(q)$ corresponding to momenta within a spherical shell $\Lambda(1-\xi)<q<\Lambda$ in momentum space with $\xi \ll 1$. The Kadanoff transformation succeeds in bringing down the cutoff momentum to $\Lambda(1-\xi)$. Second, all the rest unintegrated Fourier components are relabeled. In other words, a scale transformation is applied so that the original cutoff momentum Λ is restored. To be able to proceed we write

$$Z = \int D\boldsymbol{\varphi} \exp(-H[\boldsymbol{\varphi}]) = Z_0 \langle \exp(-H_I[\boldsymbol{\varphi}]) \rangle_{0,\Lambda} \equiv Z_0 \langle w[\boldsymbol{\varphi}] \rangle_{0,\Lambda}, \tag{7.7}$$

and the averaging $<\ldots>_{0,\Lambda}$ is performed with respect to the Gaussian functional $H_0[\varphi]$ at a given value of Λ. Furthermore, if we define $\varphi(q) = \varphi_1(q) + \varphi_2(q)$ and $G_0(q, \Lambda) = G_{01}(q, \Lambda_1) + G_{02}(q, \Lambda_2)$ then it can be shown that

$$\begin{aligned} \langle w[\boldsymbol{\varphi}] \rangle_{0,\Lambda} &\equiv Z_0^{-1} \int D\boldsymbol{\varphi}\, w[\boldsymbol{\varphi}] \exp(-H_0[\boldsymbol{\varphi}]) \\ &= Z_{01}^{-1} Z_{02}^{-1} \int D\boldsymbol{\varphi}_1 D\boldsymbol{\varphi}_2\, w[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] \exp(-H_0[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]) \end{aligned} \tag{7.8}$$

where

$$Z_{0i} = \int D\boldsymbol{\varphi}_i \exp\left[-\frac{1}{2}\int_q G_{0i}^{-1}(q,\Lambda_i)\,|\boldsymbol{\varphi}_i(q)|^2\right];$$

$$H_0[\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2] = \frac{1}{2}\int_q G_{01}^{-1}(q,\Lambda_1)\,|\boldsymbol{\varphi}_1(q)|^2 + \frac{1}{2}\int_q G_{02}^{-1}(q,\Lambda_2)\,|\boldsymbol{\varphi}_2(q)|^2 \tag{7.9}$$

Now by choosing $G_{01}(q,\Lambda_1) = G_0(q,\Lambda(1-\xi))$ with $\xi \ll 1$, we make the function G_{02} of the order ξ,

$$G_{02}(q,\Lambda_2) = G_0(q,\Lambda) - G_{01}(q,\Lambda_1) \simeq \xi\Lambda\frac{\partial G_0(q,\Lambda)}{\partial\Lambda} \equiv 2\xi h(q)\ ,$$

$$h(q) = q^{-2}\Lambda^2\frac{dS(q^2/\Lambda)}{d\Lambda^2}\ , \tag{7.10}$$

as well as $\varphi_2(q)$ are the modes with momenta within a shell $\Lambda(1-\xi) < q < \Lambda$ which should be integrated out. To integrate the short wave modes and have the first step completed we expand $<w[\varphi_1,\varphi_2]>$ with respect to the small parameter ξ and retain first, nonvanishing terms only. Having in mind the Gaussian nature of $H_0[\varphi_1,\varphi_2]$ the completion of the integration yields

$$\langle w[\boldsymbol{\varphi}(q)]\rangle_{0,\Lambda} \simeq \left\langle \left(1 + \xi V\int_q h(q)\sum_\alpha \frac{\delta^2}{\delta\varphi^\alpha(q)\,\delta\varphi^\alpha(-q)}\right) w[\boldsymbol{\varphi}(q)]\right\rangle_{0,\Lambda(1-\xi)}, \tag{7.11}$$

Consequently, the right hand side of Eq. (7.11) contains effectively only modes with $q < \Lambda(1-\xi)$.

For the second step we must rescale the momentum to restore the original cutoff Λ through transformation $\boldsymbol{q} = \boldsymbol{q}'(1 - \xi)$. This rescaling changes $H_0[\Lambda]$ to $H_0[\Lambda(1-\xi)]$.

However we must restore this change because this is essential to the restoration of Λ, and we do so by transforming $\varphi(\boldsymbol{q})$

$$\begin{aligned}\varphi^{\alpha}(\boldsymbol{q}) &= \sum_{\beta} [\delta^{\alpha\beta} + \xi\epsilon^{\alpha\beta}(\boldsymbol{q})]\, \varphi^{\beta}(\boldsymbol{q}(1+\xi)) \\ &= \sum_{\beta}\left[\delta^{\alpha\beta} + \xi\left(\epsilon^{\alpha\beta}(\boldsymbol{q}) + \delta^{\alpha\beta}\boldsymbol{q}\cdot\frac{\partial}{\partial \boldsymbol{q}}\right)\right]\varphi^{\beta}(\boldsymbol{q}),\end{aligned} \tag{7.12}$$

where at present $\epsilon^{\alpha\beta}(\boldsymbol{q})$ is an arbitrary tensor with the only property $\epsilon^{\alpha\beta}(\boldsymbol{q}) = \epsilon^{\beta\alpha}(-\boldsymbol{q})$ which preserves the symmetry (7.4). The above transformation changes $<w[\varphi(\boldsymbol{q})]>_{0,\Lambda(1-\xi)}$ of Eq. (7.11), and after keeping terms of lowest order in ξ and remembering that $w[\varphi] = exp(-H_I[\varphi])$ we derive the RG equation for the functional H_I

$$\begin{aligned}\dot{H}_I[\varphi] &= Vd\frac{\partial H_I[\varphi]}{\partial V} + \frac{V}{2}\int_q\sum_{\alpha}\eta^{\alpha\alpha}(q) - \frac{1}{2}\int_q\sum_{\alpha,\beta}\eta^{\alpha\beta}(q)\, G_0^{-1}(q,\Lambda)\, \varphi^{\alpha}(q)\, \varphi^{\beta}(-q) \\ &+ \int_q\sum_{\alpha,\beta}\left[\left(\frac{d+2}{2}\delta^{\alpha\beta} - \frac{\eta^{\alpha\beta}(q)}{2}\right)\varphi^{\alpha}(q) + \delta^{\alpha\beta}\, q\cdot\frac{\partial\varphi^{\alpha}(q)}{\partial q}\right]\frac{\delta H_I[\varphi]}{\delta\varphi^{\beta}(q)} \\ &+ \int_q h(q)\sum_{\alpha}\left[\frac{\delta^2 H_I[\varphi]}{\delta\varphi^{\alpha}(q)\,\delta\varphi^{\alpha}(-q)} - \frac{\delta H_I[\varphi]}{\delta\varphi^{\alpha}(q)}\,\frac{\delta H_I[\varphi]}{\delta\varphi^{\alpha}(-q)}\right].\end{aligned} \tag{7.13}$$

Here we defined $\eta^{\alpha\beta}(\boldsymbol{q})$ as

$$\eta^{\alpha\beta}(q) = \delta^{\alpha\beta}(d+2) - 2\epsilon^{\alpha\beta}(-q). \tag{7.14}$$

This is the generalization (arbitrary symmetry) of the exact RG equation which has no redundant operators derived in [39]. Note that if we choose $\eta^{\alpha\beta}(\boldsymbol{q}) = 0$ then the resulting RG equation will be similar to the traditional ones, and as in those cases it will contain redundant operators. However a proper choice of the tensor $\eta^{\alpha\beta}$ can make the RG Eq. (7.13) free of redundant operators. To find this proper choice we think as follows. Explicitly, the RG transformation Eq. (7.13) generates different vertices $\boldsymbol{g}$. Some of the q-dependent part of this renormalization can be incorporated into G_0. This means that the cutoff Λ is affected which should however remain unchanged. To avoid so, we define $\eta^{\alpha\beta}(\boldsymbol{q})$ such that it cancels out the q-dependent renormalization of the vertex $g_1(\boldsymbol{q})$. In order to achieve this, we use the RG Eq. (7.13) to write explicitly the change of vertices $g_1^{\alpha\beta}$ corresponding to zeroth and first order in $\varphi^{\alpha}(\boldsymbol{q})\varphi^{\beta}(-\boldsymbol{q})$. Then we require that $g_1^{\alpha\beta}$ is momentum independent initially and must remain so after the transformation so that H_0 which controls the cutoff remains intact. This requirement finds a momentum dependent expression for tensor $\eta^{\alpha\beta}(\boldsymbol{q})$. To see all these, let us carry out the above series of steps in a relatively detailed manner. First, we extract an explicit equation for the vertex $\dot{g}_1(\boldsymbol{q})$ from Eq. (7.13),

$$\begin{aligned}\dot{g}_1^{\alpha\beta}(\boldsymbol{q}) &= -\eta^{\alpha\beta}(\boldsymbol{q})\, G_0^{-1}(q,\Lambda) + \sum_{\gamma}\left[2\delta^{\alpha\gamma} - \eta^{\alpha\gamma}(\boldsymbol{q}) - \delta^{\alpha\gamma}\boldsymbol{q}\cdot\frac{\partial}{\partial \boldsymbol{q}}\right] g_1^{\gamma\beta}(\boldsymbol{q}) \\ &+ \sum_{\gamma} Q^{\alpha\beta\gamma\gamma}(\boldsymbol{q}) - 2\sum_{\gamma} g_1^{\alpha\gamma}(\boldsymbol{q})\, g_1^{\gamma\beta}(\boldsymbol{q})\, h(q),\end{aligned} \tag{7.15}$$

where

$$Q^{\alpha\beta\gamma\delta}(\boldsymbol{q}) = 3\int_{p} h(p) g_2^{\alpha\beta\gamma\delta}(\boldsymbol{q}, -\boldsymbol{q}, \boldsymbol{p}, -\boldsymbol{p}). \tag{7.16}$$

We can now split Eq. (7.15) into two equations: one for $\dot{g}_{10}$, which is the momentum independent part of $g_1(q)$, and another for $g_1'(q) = g_1(q) - g_{10}$,

$$\dot{g}_{10}^{\alpha\beta} = \sum_\gamma [2\delta^{\alpha\gamma} - \eta^{\alpha\gamma}(0)] g_{10}^{\gamma\beta} + \sum_\gamma Q^{\alpha\beta\gamma\gamma}(0) - 2\sum_\gamma g_{10}^{\alpha\gamma} g_{10}^{\gamma\beta} h(0); \tag{7.17}$$

$$\begin{aligned} \dot{g}_1^{\prime\alpha\beta}(q) = & -\eta^{\alpha\beta}(q) G_0^{-1}(q,\Lambda) + \sum_\gamma \Bigg\{ \left[2\delta^{\alpha\gamma} - \eta^{\alpha\gamma}(q) - \delta^{\alpha\gamma} q \cdot \frac{\partial}{\partial q} \right] g_1^{\prime\gamma\beta}(q) \\ & - [\eta^{\alpha\gamma}(q) - \eta^{\alpha\gamma}(0)] g_{10}^{\gamma\beta} + Q^{\alpha\beta\gamma\gamma}(q) - Q^{\alpha\beta\gamma\gamma}(0) - 2 g_{10}^{\alpha\gamma} g_{10}^{\gamma\beta} [h(q) - h(0)] \\ & - 2\left[g_{10}^{\alpha\gamma} g_1^{\prime\gamma\beta}(q) + g_1^{\prime\alpha\gamma}(q) g_{10}^{\gamma\beta} + g_1^{\prime\alpha\gamma}(q) g_1^{\prime\gamma\beta}(q) \right] h(q) \Bigg\}. \end{aligned} \tag{7.18}$$

Using Eq. (7.18) we can define the function $\eta^{\alpha\beta}(q)$ such that the derivative of $g_1'(q)$ is equal to zero. This means that if the vertex $g_1(q)$ of the initial functional H_I is constant then a q-dependent part of this vertex will not be generated and the functional H_0 will be intact within the renormalization procedure. The requirement $g_1'(q) = 0$ implies that

$$\begin{aligned} \eta^{\alpha\beta}(q) = \eta^{\alpha\beta}(0) - \sum_\nu \Big\{ & \eta^{\alpha\nu}(0) G_0^{-1}(q,\Lambda) + \sum_\gamma \big[2[h(q) - h(0)] g_{10}^{\alpha\gamma} g_{10}^{\gamma\nu} \\ & - Q^{\alpha\nu\gamma\gamma}(q) + Q^{\alpha\nu\gamma\gamma}(0) \big] \Big\} \left[g_{10}^{\nu\beta} + G_0^{-1}(q,\Lambda) \delta^{\nu\beta} \right]^{-1}. \end{aligned} \tag{7.19}$$

Eq. (7.19) defines the momentum dependent part of the function $\eta^{\alpha\beta}(q)$. We still have to define n^2 components of the tensor $\eta^{\alpha\beta}(0)$. We can use these values to simplify the RG equations and clarify the physical meaning of $\eta^{\alpha\beta}$. To do this, let us diagonalize the vertex g_{10} in the initial functional H_I. This can always be done without loss of generality. The diagonal components of this tensor are trial critical temperatures for the corresponding components of the order parameter $\varphi(q)$. However, as one can see from Eq. (7.17), even if the non-diagonal part of the tensor $g_{10}^{\alpha\beta}$ does not exist in the initial functional, it will be generated within the renormalization process. We can use the arbitrariness of tensor $\eta^{\alpha\beta}$ to keep the tensor $g_{10}{}^{\alpha\beta}$ diagonal. In order to do this, let us split Eq. (7.17) into two separate equations, for diagonal and non-diagonal parts of the vertex $g_{10}{}^{\alpha\beta}$. Defining $g_{10}{}^{\alpha\beta} = \delta^{\alpha\beta}r^{\alpha} + (1-\delta^{\alpha\beta})r^{\alpha\beta}$, we have

$$\dot{r}^{\alpha} = (2-\eta^{\alpha})r^{\alpha} + Q^{\alpha}(0) - 2(r^{\alpha})^2 h(0); \tag{7.20}$$

$$\begin{aligned} \dot{r}^{\alpha\beta} &= (2-\eta^{\alpha})r^{\alpha\beta} - \sum_{\gamma}\tilde{\eta}^{\alpha\gamma}r^{\gamma\beta} + \tilde{Q}^{\alpha\beta}(0) \\ &- \tilde{\eta}^{\alpha\beta}r^{\beta} - 2\left[r^{\alpha}r^{\alpha\beta} + r^{\alpha\beta}r^{\beta} + \sum_{\gamma} r^{\alpha\gamma}r^{\gamma\beta}\right]h(0), \end{aligned} \tag{7.21}$$

where η^{α}, Q^{α} and $\tilde{\eta}^{\alpha\beta}$, $\tilde{Q}^{\alpha\beta}$ are diagonal and non-diagonal elements of tensors $\eta^{\alpha\beta}$ and $\sum_{\gamma}Q^{\alpha\beta\gamma\gamma}$, respectively,

$$\begin{aligned} \eta^{\alpha\beta}(0) &= \delta^{\alpha\beta}\eta^{\alpha} + (1-\delta^{\alpha\beta})\tilde{\eta}^{\alpha\beta}; \\ \sum_{\gamma} Q^{\alpha\beta\gamma\gamma}(q) &= \delta^{\alpha\beta}Q^{\alpha}(q) + (1-\delta^{\alpha\beta})\tilde{Q}^{\alpha\beta}(q). \end{aligned} \tag{7.22}$$

Now by choosing

$$\tilde{\eta}^{\alpha\beta} = \tilde{Q}^{\alpha\beta}/r^{\beta}, \tag{7.23}$$

we provide that if the initial functional does not contain non-diagonal parts of the vertex $\hat{g}_1$, then this vertex remains diagonal after the renormalization. If at last we require that the expansion of $\eta^{\alpha\alpha}(q)$ does not contain q^2 terms, then the following equation defines the diagonal part of the tensor $\eta^{\alpha\beta}(0)$

$$\eta^{\alpha} = \frac{d}{dq^2}[Q^{\alpha}(q) - 2h(q)(r^{\alpha})^2]_{q=0}. \qquad (7.24)$$

Function $\eta^{\alpha\beta}(q)$ is now completely defined and there is no more freedom in the exact RG Eq. (7.13), therefore, it must contain no redundant operators. The physical meaning of the function $\eta^{\alpha\beta}$ is suggested by the Eq. (7.20): at the stable fixed point of the functional (7.3), η^{α} is equal to the critical exponent η of the corresponding critical mode φ^{α}. Furthermore, one may consider the possibility of constructing a new perturbation theory using the Fisher exponents $\eta^{\alpha}(0)$. In fact this is the only parameter in the theory which is numerically small for most systems. For the isotropic GLW functional, such a perturbation procedure was already developed [25].

7.4 CONCLUSION

We have obtained the exact RG equation Eq. (7.13) for the GLW functional for a system with an arbitrary symmetry. This equation is expressed in terms of an arbitrary tensor $\eta^{\alpha\beta}(\mathbf{q})$ which when properly chosen, Eqs. (7.19, 7.23, 7.24), leaves no room for additional conditions and therefore the exact RG equation contains no redundant operators. Even though the developed procedure may be more cumbersome than the traditional Wilson approach [18], unlike the case of the traditional approach where the exact RG equations contain redundant operators, in our scheme such a problem does not exist since the proper choice of $\eta^{\alpha\beta}(\mathbf{q})$ leaves Eq. (7.13) free of redundant operators.

CHAPTER 8

SUMMARY

Let us highlight the results of each chapter of the thesis. In chapter 1 we discuss three different approaches to the field of critical phenomena at phase transitions. On the one hand, the rather easy mathematically MF theory is conceptually imprecise therefore in general gives incorrect results. Despite that, however, MF results are still appreciated for pedagogical and comparison purposes with results of other approaches. On the other hand, RG theory which is conceptually precise, at the same time is mathematically complex and the approximation methods that is using to get results very often do not give clean cut answers on the most interesting of systems. Unfortunately RG theory can determine only critical asymptotics. There are many predictions of RG theory that totally contradict a phase transition picture obtain from MF theory. These predictions, which are crucial for the understanding of the true critical behavior at a phase transition, cannot be substantiated within the framework of RG. Consequently, the development of exactly solvable models which conceptually are somewhere between the MF and RG theories rose as a natural necessity to shine light on fundamental disagreements between MF results and RG predictions. This idea motivated and challenged us to develop the exactly solvable model in chapter 2 in order to investigate important physical systems, pure and random, to discover and clarify their true behavior at a phase transition. The exactly solvable model can serve a double purpose: not only it can help create a better understanding of RG theory but it can also add to the results. In chapters 2 through 6 we apply the model successfully. In the last chapter we have a different direction in mind. Since after the use of the exactly solvable model we saw how RG predictions are true, we are more confident about the theory and, therefore, we apply its fundamental formalism to construct the exact RG functional partial differential equation of the most arbitrary symmetry Hamiltonian which contains no redundant operators.

The exactly solvable model is described in chapter 2. The model considers fluctuation interactions of equal and antiparallel momenta. In this thesis it was first applied in the φ^4 model. In that case it was shown how the exactly solvable model demonstrates critical behavior for $2 < d < 4$ and MF behavior for $d > 4$. MF theory finds the same behavior regardless of dimensionality. In addition the model finds logarithmic corrections for critical exponents at $d = 4$. At last, crossover effects from critical to MF are discussed. Since these results were in qualitative agreement with RG analysis we decided to apply the model in more complicated systems pure and random in chapters 3, 4, 5, and 6.

In chapter 3 the system of two interacting order parameters is studied. The model demonstrates many basic features of RG theory. Instead of a continuous order-disorder phase transition predicted by MF theory the model finds the first order phase transition induced by fluctuations. This phase transition occurs when $w > 0$ and $\Delta < 0$. Because the model is exactly solvable it is possible to calculate the jump of the order parameter at the transition point and also the temperature dependance of the free energy which undergoes transformations typical for first order phase transitions. This was shown in terms of graphs of the free energy vs the order parameter for various temperatures. When radii of interactions $c_i^{1/2}$ increase, order parameter jumps decrease and in the limit infinite range of interaction the first order phase transition disappears. In this limit the results of the model reduce to MF theory results as it should be. Since the model takes into account fluctuations only partially and nonetheless demonstrates results similar to the RG approach, it shows that such qualitative effects as first order phase transitions induced by fluctuations are not artifacts of RG theory but real ones.

In chapters 4, 5 and 6 the model is applied to random systems. In chapter 4 we consider the effect of a random quenched field on the critical behavior of a uniform d-dimensional system. Since the model is exactly solvable, we are able to explicitly show that the critical behavior of the random system is similar to the critical behavior of the pure system of $(d-2)$-dimension. Namely, for $d \leq 4$ the system is unstable with respect to a random field, *i.e.* even infinitesimally small field destroys phase transitions. For $4 < d < 6$ the random system has the same critical exponents as the pure one for $2 < d < 4$, at $d = 6$ logarithmic corrections to critical exponents of the random system appear and for $d > 6$ the random system demonstrates the mean field critical behavior. The crossover effects from d-dimensional behavior to $(d-2)$-dimensional one are discussed. It is found that the suppression of fluctuations dominate the existence of the random field, hence in this case the critical exponents are the MF ones even for a nonzero random field.

In chapter 5 we find that for the "random temperature" type of disorder the model demonstrates the same kind of critical behavior as MF theory.

In chapter 6 the system of chapter 3 is substantially generalized when the two interacting order parameters are now under the influence of two random fields with short-range spatial interactions. Using the replica method within the context of the exactly solvable model it is proven that when one of the random fields is on the fluctuation-induced first order phase transition shown to exist in the pure case in chapter 3, is now replaced by a second order. When both fields are on, then a phase transition does not occur for $d \leq 4$.

In chapter 7 the central ideas of RG theory have been employed to a general type Hamiltonian with an arbitrary symmetry. Here we derive an exact RG equation free of redundant operators. Such operators are of no physical meaning and must be transformed away in the standard approach.

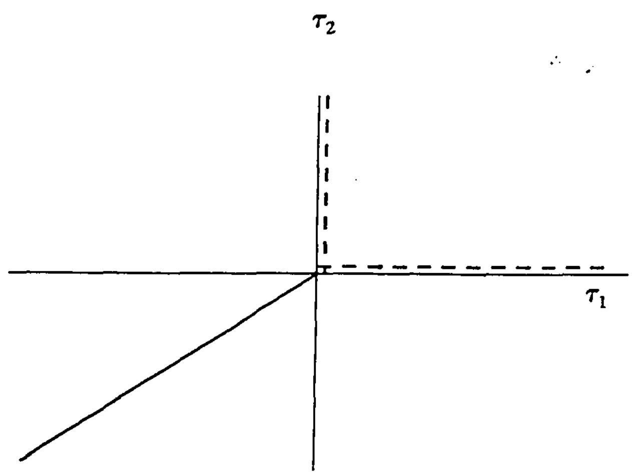

Figure 1: MF phase diagram with $\Delta < 0$, $w > 0$. Dashed lines $\tau_1=0$ and $\tau_2=0$ separate disorder from phases 1 and 2 respectively, and indicate a second order phase transition. The intersection point $\tau_1=\tau_2=0$ is a bicritical point. The coexistence of the two phases curve is the solid line $\tau_1\sqrt{g_2}=\tau_2\sqrt{g_1}$. A phase transition between the two phases through this line is of first order. Within the context of the exactly solvable model the bicritical point is replaced by a fluctuation-induced first order phase transition and figure 4 is produced.

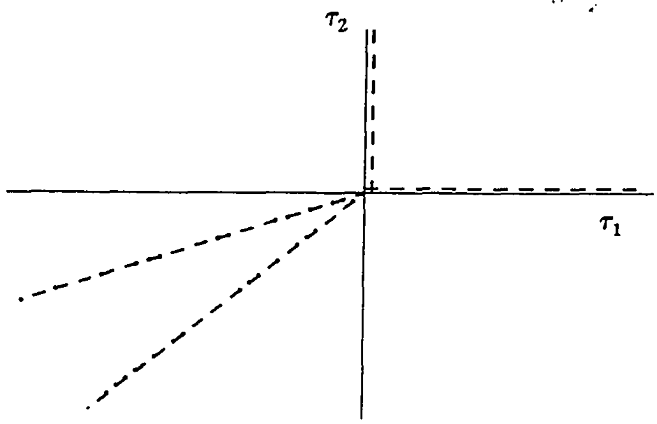

Figure 2: MF phase diagram with $\Delta > 0$, $w > 0$. The phase transition from disorder to phase 1 or phase 2 through line $\tau_1=0$ or $\tau_2=0$ respectively is of second order. Lines $w\ \tau_2=g_2\ \tau_1$ and $g_1\ \tau_2 = w\ \tau_1$ define the region where the mixed phase occurs, and a phase transition from phase 2 or 1 into the mixed phase through these lines is of second order respectively. The point where all lines intersect is a tetracritical point.

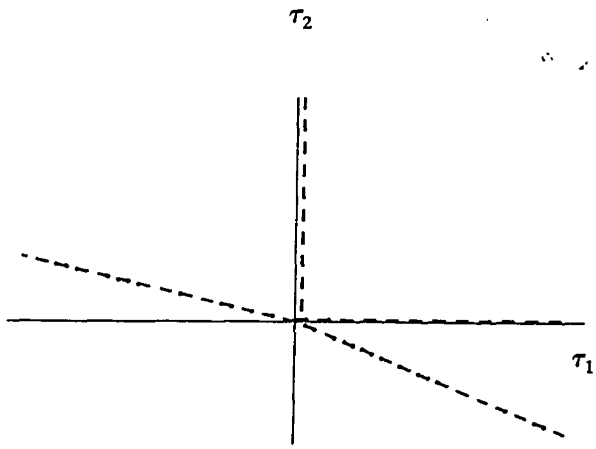

Figure 3: MF phase diagram with $\Delta > 0$, $w < 0$. The phase transition from disorder to phase 1 or phase 2 through line $\tau_1=0$ or $\tau_2=0$ respectively is of second order. Lines $w\ \tau_2=g_2\ \tau_1$ and $g_1\ \tau_2 = w\ \tau_1$ define the region where the mixed phase occurs, and a phase transition from phase 2 or 1 into the mixed phase through these lines is of second order respectively. The point where all lines intersect is a tetracritical point.

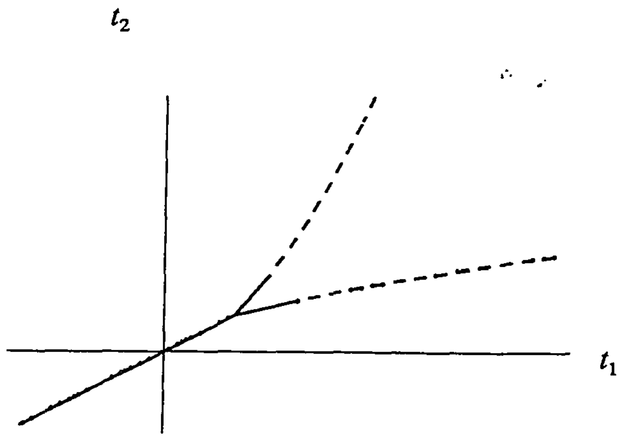

Figure 4: The phase diagram of $\Delta < 0$ and $w > 0$ obtained by the model. It shows how the plane (t_1,t_2) is separated into phase 1, phase 2, and the disorder phase. It is obtained by requiring that the expression of the order parameter is greater than zero. The dashed lines which separate disorder from phase 1 and 2 indicate a second kind order-disorder phase transition. These lines are parts of Eqs. $t_2=z(t_1)$ and $t_1=z(t_2)$ which are respectively the dashed lines that separate phase 1 from disorder and phase 2 from disorder. On the other hand, the two solid lines $t_2=z_1(t_1)$ and $t_1=z_2(t_2)$ separate disorder from phase 1 and disorder from phase 2 respectively. Entering phase 1 or 2 from disorder through these lines the phase transition is of the first order. From the interception point of these two solid lines starts another solid line which is the coexistence curve between phases 1 and 2 given by $F(\varphi_{1+}\neq 0, \varphi_2=0) = F(\varphi_1=0, \varphi_{2+}\neq 0)$. Due to the complexity of the coexistence equation, the coexistence line is obtained from numerical considerations. An order-order phase transition is of the first order. Upon suppression of fluctuations this diagram reduces to figure 1 as expected.

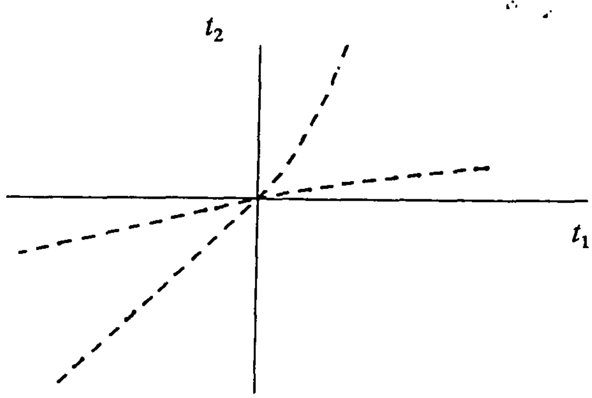

Figure 5: Model's phase diagram of $\Delta > 0$ and $w > 0$. The plane (t_1, t_2) is separated into the disorder phase, phase 1, phase 2 and the mixed phase. Phase transitions from disorder to order, or from order to order are of the second order. Eqs. $t_2=z(t_1)$ and $t_1=z(t_2)$ are respectively the dashed lines that separate phase 1 from disorder and phase 2 from disorder in the first quadrant. Lines $g_1t_2 = wt_1$ and $g_2t_1 = wt_2$ separate the mixed phase from phase 1 and 2 respectively, and phase transitions between these low symmetry phases is of the second order.

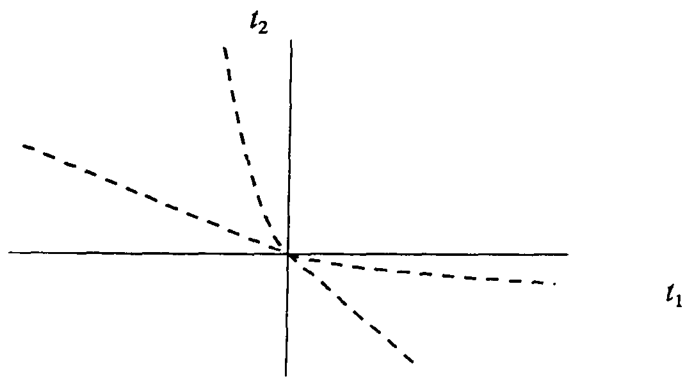

Figure 6: Phase diagram of systems with $\Delta > 0$ and $w < 0$. It has the same qualitative features as figure 5.

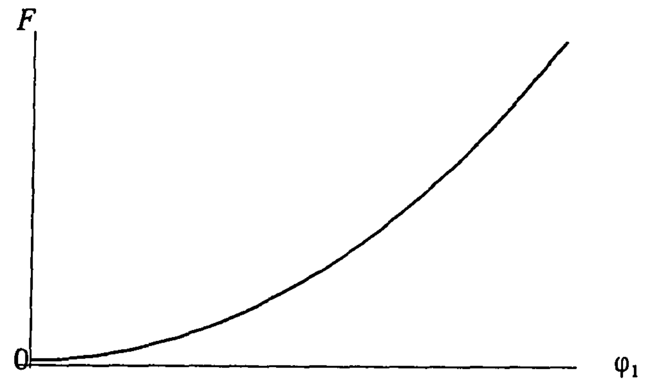

Figure 7: Phase diagram of the non-equilibrium free energy vs the order parameter φ_1, (with φ_2=0), at a temperature corresponding to the disorder phase. At φ_1 =0 the equilibrium free energy is the one that corresponds to the disorder phase.

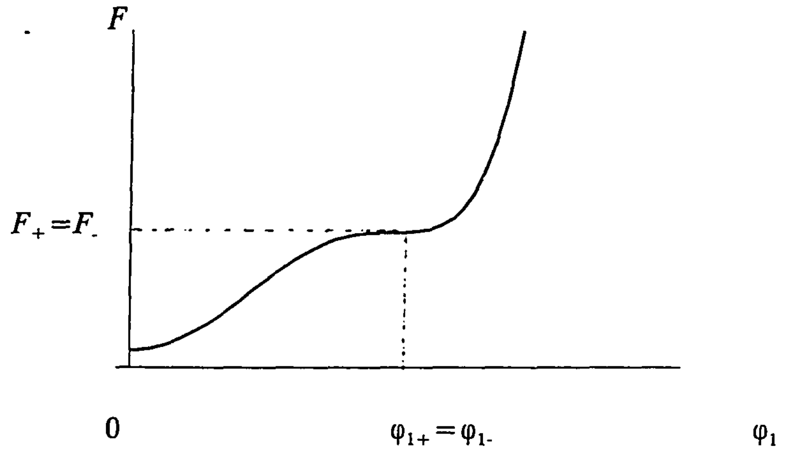

Figure 8: Non equilibrium free energy $F(\varphi_1 \neq 0, \varphi_2 = 0)$ vs φ_1 at critical temperature T_{cr2}. If the system is in phase 1 then at temperature T_{cr2} it makes a first order phase transition into disorder. The free energy of disorder is lower than the equilibrium free energy of phase 1. The saddle point is where $\varphi_{1+} = \varphi_{1-}$ and $F_+ = F_-$. If the system is originally in disorder at $T = T_{cr2}$ no phase transition occurs. In that case the transition from disorder to phase 1 occurs at $T = T_{cr3}$, figure 10, where the second saddle point exists. This phenomenon is known as hysteresis and is a characteristic of first order phase transitions. In other words, the transition temperatures are different upon heating or cooling.

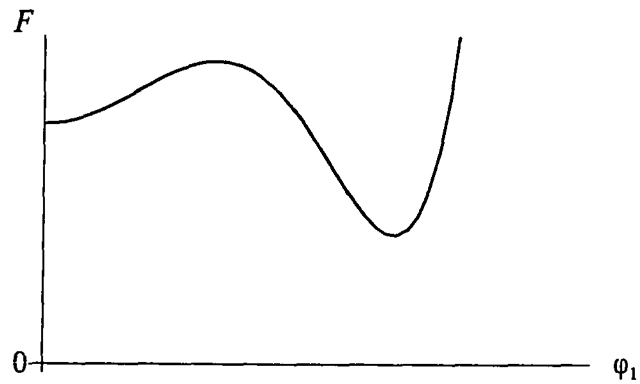

Figure 9: The phase diagram of the non-equilibrium free energy $F(\varphi_1 \neq 0, \varphi_2 = 0)$ vs φ_1 at temperature T such that $T_{cr1} < T < T_{cr2}$. T_{cr2} and T_{cr1} are the temperatures of the first and second saddle points respectively, (see also figures 8 and 10). The local maximum corresponds to the point (F_-, φ_{1-}), and right of this point the local minimum corresponds to the point (F_+, φ_{1+}).

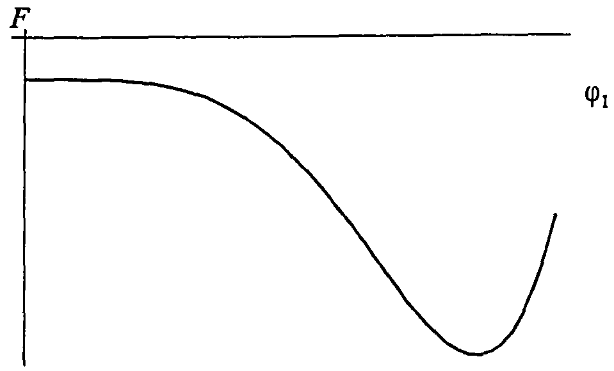

Figure 10: Non equilibrium free energy $F(\varphi_1 \neq 0, \varphi_2 = 0)$ vs φ_1 at critical temperature T_{cr1}. This is the second saddle point. It is at this temperature that a phase transition occurs from disorder into phase 1. The equilibrium free energy is F_+.

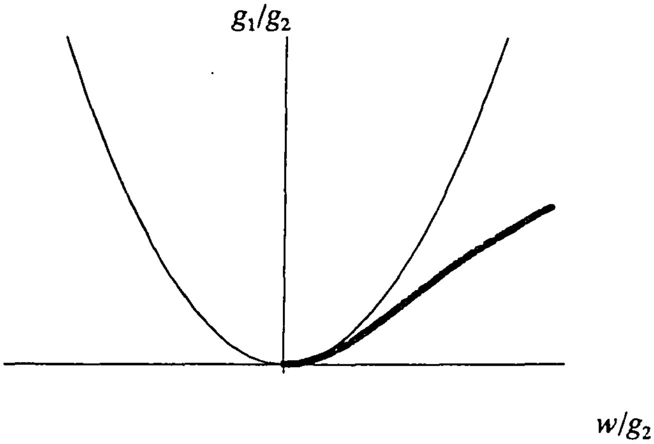

Figure 11: Phase diagram that shows how the parameter space is separated into regions that correspond to first and second order phase transition from disorder into phase 1. The parabola is $\Delta = 0$ and the thick line is Eq. (3.30). The region of first order transition is between Eq. (3.30) and the horizontal axis. The rest of the region in the first quadrant as well as the part of the second quadrant that $\Delta > 0$ is the region of second order transition. Upon suppression of fluctuations the region of first order transition disappears and the entire first quadrant becomes a region of second order transition.

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