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# CONJUGACY PROBLEMS FOR $R^k$ ACTIONS

by

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## INTRODUCTION

If $G$ is a Lie group and $V$ a $C^\infty$ manifold, then a $C^\infty$ action of $G$ on $V$ is a $C^\infty$ mapping $\varphi$ from $G \times V$ to $V$ such that for every $v$ in $V$ and $g_1$, $g_2$ in $G$,

$$\varphi ( g_1 g_2, v ) = \varphi ( g_1, \varphi ( g_2, v ))$$

and

$$\varphi ( e, v ) = v$$

where $e$ is the identity element of $G$. Here we will be considering the case where $G$ is $R^{n-1}$ and $V$ is a compact connected orientable n-manifold.

The orbit of a point $p$ of $V$ under $\varphi$ is

$$\mathcal{O}_\varphi (p) = \{ v \in V \mid v = \varphi ( g, p ), g \in G \} .$$

The orbit space of $\varphi$, denoted by $V/\varphi$, is the set of all orbits of $\varphi$ with the quotient topology. The action $\varphi$ is said to be free if whenever $\varphi ( g, v )$ is equal to $v$, $g$ is equal to the identity element of $G$. $\varphi$ is said to be locally free if all orbits of $\varphi$ have dimension $k$, where $k$ is the dimension of $G$. We will be concerned only with free and locally free actions.

In the study of linear transformations on a vector space, one is primarily concerned not with an individual transformation, but rather with its canonical form after choice of some particular basis. In the same way, in the case of actions on manifolds one is interested not in particular actions, but rather with equivalence classes of actions, which give rise to a canonical form. In trying to understand the global behaviour, or orbit structure, of actions of $G$ on

V, the natural equivalence relation is some type of conjugacy. Smale has surveyed results in this area in [7].

*Definition:* An action $\varphi$ of $G$ on $V$ is *differentiably (topologically) conjugate* to an action $\varphi'$ of $G$ on $V'$ if there is a diffeomorphism (homeomorphism)

$$f : V \to V'$$

such that

$$f(\varphi(g, v)) = \varphi'(g, f(v))$$

for all $g$ in $G$ and for all $v$ in $V$.

Differential conjugacy is a strong condition, since conjugating by a diffeomorphism preserves eigenvalues of the derivative at a fixed point. Thus this type of conjugacy would not admit small perturbations of the action. Topological conjugacy defines larger equivalence classes, and is in general more useful. In fact for discrete groups it is completely satisfactory.

However the study of actions of non-discrete groups requires a third notion, broader than the previous ones. This notion is also indicated by considerations of structural stability (that is, one would like structurally stable diffeomorphisms to be dense in Diff(V); this is only so if structural stability is defined in terms of the following notion.)

*Definition:* Two actions $\varphi$ and $\varphi'$ of a Lie group $G$ on manifolds $V$ and $V'$ respectively are *topologically equivalent* if there is a homeomorphism $f$ mapping $V$ to $V'$ such that for all $v$ in $V$,

$$f(\mathcal{O}_{\varphi}(v)) = \mathcal{O}_{\varphi'}(f(v)) .$$

(That is, $f$ induces a mapping $\bar{f}$ from the orbit space $V/\varphi$ to $V/\varphi'$ such that the following commutes:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"'] & V/\varphi \arrow[d, "\bar{f}"] \\ V' \arrow[r] & V'/\varphi' \end{tikzcd}$$

where the maps from $V$ to $V/\varphi$ and $V'$ to $V'/\varphi'$ are projections.)

A good deal is known about actions of compact groups (see [2] ). Two much studied cases of non-compact groups are when $G$ is equal to the integers (and the action corresponds to iterates of a diffeomorphism), and when $G$ is equal to the real numbers (and the action corresponds to a flow), see [7] . However, relatively little is known about actions of other non-compact groups.

We will concentrate on a situation which is modelled on non-singular actions of $R^{n-1}$ on the n-torus $T^n$.

In Part I we begin by summarizing some probably known facts about linear actions of $R^k$ on a torus $T^n$, that is, actions

$$\bar{\alpha} : R^k \times T^n \longrightarrow T^n$$

which are obtained as projections of linear actions

$$\alpha : R^k \times R^n \longrightarrow R^n$$

defined by

$$\alpha(r,x) = x + Ar$$

where $A$ is a non-singular $n$ by $k$ matrix. That is, if $\pi$ is the projection of $R^n$ onto $T^n$ (where $T^n$ is defined as $R^n/Z^n$, $Z^n$ being the integer lattice), and $\pi(x)$ equals $\bar{x}$, then $\bar{\alpha}$ is defined by

$$\bar{\alpha}(r,\bar{x}) = \pi(x + Ar) = \pi(\alpha(r,x)) .$$

We show that if $\alpha$ and $\beta$ are non-singular linear actions of $R^k$

and $R^{n-k}$ respectively on $R^n$ whose orbits are orthogonal, then the corresponding actions $\bar{\alpha}$ and $\bar{\beta}$ on $T^n$ are related; $\bar{\alpha}$ is free if and only if $\bar{\beta}$ is ergodic. We discuss a canonical form for a plane $\Omega$, its Plucker coordinates, and how these coordinates relate to properties such as freeness and ergodicity of a linear action on a torus with $\pi(\Omega)$ as an orbit.

In Part II we consider free actions $\varphi$ of $R^{n-1}$ on a compact orientable n-manifold $V$ which have no compact orbits. We show that in some sense most of these are equivalent to linear actions on the n-torus $T^n$. As in the case of diffeomorphisms of the 1-torus $S^1$, the sense in which they are linear (i.e. the type of equivalence; topologically conjugate or topologically equivalent) depends on the degree of irrationality of "rotation numbers". (See [8] .) Definition of these rotation numbers involves choice of an embedding of $S^1$ in $V$, denoted by $p_t$ where $t$ is in the quotient $R/Z$, which is transverse to the orbits of the action. The action on $V$ induces a pseudogroup of local diffeomorphisms of the embedded circle. It is known that there is a Riemannian metric on $V$ which is invariant under this pseudogroup ( [5] ); this gives rise to a group of rotations of $S^1$, which is generated by $n - 1$ real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ . (By rotations we mean maps which take $p_t$ to $p_{t+\lambda}$ for some $\lambda$.) For each $\lambda_i$, $i = 1, \ldots, n - 1$, there is defined a unique return function $r_i$ mapping $S^1$ to $R^{n-1}$, given by

$$\varphi(r_i(t), p_t) = p_{t+\lambda_i} \quad .$$

We show that if at least one of these rotation numbers $\lambda_i$ is not closely approximated by rational numbers, then it is possible to deform the original embedded circle in such a way that not only does the pseudogroup act as a group of rotations, but the return functions $r_i$ are constant for each i, $i = 1, \ldots, n - 1$. Such an action will then be topologically conjugate to a linear action on $T^n$. More precisely, the condition on the irrationality of the rotation number $\lambda_i$ is that there exist a positive real number C and an number $\gamma$ greater than 2 such that for any integers m and n,

$$| \lambda_i \pm \frac{m}{n} | > \frac{C}{n^\gamma}$$

(i.e. $\lambda_i$ satisfies a Liouville inequality).

However not all free actions on a compact orientable manifold with no compact orbits are topologically conjugate to linear actions. We describe an example (given in [8] ) of an action $\varphi$ of $R^1$ on $T^2$ such that there is no homeomorphism h of $T^2$ such that the action defined by $h(\varphi(r,h^{-1}(x))$ is linear.

In the case of actions which have no rotation numbers satisfying a Liouville inequality, we can however show topological equivalence to a linear action if we assume that there is some one-dimensional subspace L of $R^{n-1}$ such that the orbits of $\varphi$ restricted to this line are dense in a two-dimensional submanifold of V. This argument, in Part III, is by induction; we show that there is a basis $r^1, \ldots, r^j$ of $R^{n-1}$ such that $r^1$ generates L, and the closure of the orbits of the action $\varphi$ restricted to the span of the first j basis vectors has dimension $j + 1$ ; we show that each of these restricted actions is topologically equivalent to a linear action.

Part IV deals with the case of locally free actions $\varphi$ of $R^{n-1}$ on a compact n- manifold V with no compact orbits; we show that there is an action $\psi$ of $T^k \times R^{n-k-1}$ on $V$ for some $k$ such that the following diagram commutes:

$$\begin{tikzcd}
R^{n-1} \times V \arrow[r, "\varphi"] \arrow[d, "i"'] & V \\
R^k \times (R^{n-k-1} \times V) \arrow[d, "\pi \times Id"'] & \\
T^k \times R^{n-k-1} \times V \arrow[uur, "\psi"'] &
\end{tikzcd}$$

where $\pi$ is the projection of $R^k$ onto $T^k$ and $i$ is an isomorphism, and such that $\psi$ is a free action. That is, a locally free action can be written as a sum of a free action of $R^{n-k-1}$ and a free action of $T^k$ for some $k$.

## PART I. LINEAR CASE

Consider the n-torus $T^n$ as the quotient $R^n/Z^n$, where $Z^n$ is the integer lattice. Let $\pi$ be the projection map of $R^n$ onto $T^n$. In general, we will indicate elements of and actions on $R^n$ by symbols $x$, $\alpha$ ; corresponding elements of and actions on $T^n$ will be called $\overline{x}$, $\overline{\alpha}$ .

Let $\alpha$ be a non-singular linear action of $R^k$ on $R^n$, $k \leq n$ ; that is, $\alpha$ is a mapping

$$\alpha : R^k \times R^n \longrightarrow R^n$$

defined by

$$\alpha(r,x) = x + Ar$$

where $A$ is an $n$ by $k$ matrix of rank $k$. The orbit of $\alpha$ through 0 is just the k-plane $\Omega_A$ spanned by the $k$ column vectors of the matrix $A$; other orbits of $\alpha$ are translates of $\Omega_A$.

An action $\overline{\alpha}$ of $R^k$ on $T^n$ is defined from $\alpha$ by

$$\overline{\alpha}(r,\overline{x}) = \pi\alpha(r,x)$$

where $\overline{x} = \pi(x)$. We will call such actions $\overline{\alpha}$ on $T^n$ non-singular linear actions. (They are always locally free, since the dimension of the orbits of $\overline{\alpha}$ in $T^n$ is $k$.)

The action $\overline{\alpha}$ is free if for no non-zero $r$ in $R^k$ and $\overline{x}$ in $T^n$ is it true that $\overline{\alpha}(r,\overline{x}) = \overline{x}$ ; equivalently, if for no non-zero $r$ in $R^k$ is it true that $\overline{\alpha}(r,\overline{0}) = \overline{0}$ . This means that the orbit of $\alpha$ through 0 in $R^n$ contains no element of $Z^n$ except 0 (and hence no non-zero element of $Q^n$ where $Q$ is the rational numbers). Thus $\overline{\alpha}$ is free if and only if

$$\Omega_A \cap Q^n = \{0\} .$$

Consider the $n - k$ plane $\Omega_B$ orthogonal to $\Omega_A$. By picking a basis $B^1,\ldots,B^{n-k}$ for this plane, we can find a non-singular linear action $\beta$ of $R^{n-k}$ on $R^n$ given by

$$\beta(r,x) = x + Br$$

where $B$ is the non-singular $n$ by $n - k$ matrix with columns $B^1,\ldots,B^{n-k}$, which has $\Omega_B$ as its orbit through $0$. $\beta$ defines an action $\overline{\beta}$ of $R^{n-k}$ on $T^n$. In the rest of this section, $\Omega_A$ and $\Omega_B$ will represent orthogonal planes (of dimension $k$ and $n - k$ respectively), and $\overline{\alpha}$, $\overline{\beta}$ will be the corresponding actions of $R^k$ and $R^{n-k}$ respectively on $T^n$. We use the following Lemma to relate properties of $\overline{\alpha}$ and $\overline{\beta}$.

LEMMA 1.1. *Let* $B$ *be an* $n$ *by* $n - k$ *matrix, and let* $\mu$ *be a measure on* $T^n$ *invariant under the action* $\overline{\beta}$ *of* $R^{n-k}$ *on* $T^n$, *where*

$$\overline{\beta}(r,\overline{x}) = \pi(x + Br) .$$

*Then the following are equivalent:*

i. $\Omega_A \cap Z^n = \{0\}$ , *where* $\Omega_A$ *is the orthogonal complement of* $\Omega_B$.

ii. *Whenever a function* $f$ *in* $L^2(T^n,\mu)$ *satisfies* $f = U_t f$ *for every* $t$ *in* $R^{n-k}$, $f$ *is a constant function (where* $U_t f(x) = f(\beta(t,x))$), *i.e.* $\beta$ *is ergodic.*

Relating this to the above situation; $\beta$ is ergodic if and only if every orbit is dense, so the Lemma states that $\alpha$ is free if and only if $\beta$ has dense orbits, where $\alpha$ and $\beta$ are "orthogonal" actions.

*Proof:* To show that (i) implies (ii), we assume that (ii) is false; suppose that $f$ is a non-constant function in $L^2(T^n,\mu)$ such that for all $t$ in $R^{n-k}$, $f = U_t(f)$. If $f$ has the Fourier expansion

$$f(\bar{x}) \sim \sum_{N \in Z^n} a_N e^{2\pi i (N,x)}$$

(where ( , ) indicates the inner product), then

$$U_t f(\bar{x}) \sim \sum_{N \in Z^n} a_N e^{2\pi i (N,x)} e^{2\pi i (N,B(t))} .$$

Since $f$ is non-constant, there is some non-zero $M$ in $Z^n$ such that $a_M$ is not zero. We have then that for all $t$ in $R^{n-k}$,

$$a_M = a_M e^{2\pi i (M,B(t))} .$$

Thus for every $t$ in $R^{n-k}$, $(M,B(t))$ is an integer. But $(M,B(t))$ is a continuous integer-valued function of $t$ which assumes the value 0 for $t = 0$, hence is identically 0. Thus $M$ is a non-zero element of the intersection of $\Omega_A$ and $Z^n$, which contradicts (i). Thus (i) implies (ii).

To see that (ii) implies (i), assume (i) is false, that is there is a non-zero $M$ in $Z^n$ such that $(M,B(t)) = 0$ for all $t$ in $R^{n-k}$. Then if $f(\bar{x}) = e^{2\pi i (M,x)}$, we have $f = U_t f$ for all $t$ in $R^{n-k}$ which contradicts (ii). Thus (ii) implies (i).

We would like to generalize Lemma 1.1, to relate properties of $\bar{\alpha}$ and $\bar{\beta}$ where neither is necessarily free. To this end we need the concept of rational dimension.

Definition: The rational dimension of a set of real numbers is the dimension of their span when $R$ is viewed as a vector space over $Q$.

We will first consider the situation when $k$ is equal to 1.

PROPOSITION 1.2. *Let $\bar{\alpha}$ be a non-singular linear action of $R$ on $T^n$ defined by*

$$\overline{\alpha}(t,\overline{x}) = \pi(x + tv)$$

*where $\overline{x} = \pi(x)$ and $v = (v_1,\dots,v_n)$ is in $R^n$. Then $\alpha$ is free if and only if the rational dimension of the coordinates of $v$ is strictly greater than 1. In fact $\overline{\mathcal{O}}_{\overline{\alpha}}(\overline{x})$ , the closure of the orbit of $\overline{\alpha}$ through $\overline{x}$, is isomorphic to a torus $T^p$, where $p$ is the rational dimension of the coordinates of $v$.*

*Proof:* The first statement is easily seen by noting that $\overline{\alpha}$ is free if $Rv \cap Q^n = \{0\}$ , (since $\mathcal{O}_{\alpha}(0) = Rv$ ). So $\overline{\alpha}$ is free if and only if $v$ has two coordinates $v_i$ and $v_j \neq 0$ such that $v_i/v_j$ is not rational (that is, whenever $tv_i$ is a non-zero rational, $tv_j$ is not).

Let $p$ be the rational dimension of $\{v_1,\dots,v_n\}$. We can suppose that the basis of $R^n$ is such that $\{v_1,\dots,v_p\}$ are rationally independent. Thus for $p + 1 \leqq j \leqq n$ , $v_j$ can be written as a sum of integral multiples of $\{v_1,\dots,v_n\}$; that is, there exist integers $N_{j,i}$ for $p + 1 \leqq j \leqq n$ and $1 \leqq i \leqq p$ , such that

$$\begin{aligned} v_{p+1} &= N_{p+1,1}\, v_1 + \dots + N_{p+1,p}\, v_p \\ &\vdots \\ v_n &= N_{n,1}\, v_1 + \dots + N_{n,p}\, v_p \ . \end{aligned}$$

Consider the vectors $N^i$, $p + 1 \leqq i \leqq n$ , in $Z^n$ defined by

$$N^i = (N_{i,1} , \dots , N_{i,p} , 0 , \dots ,-1 , \dots , 0)$$

(where 1 is in the $i^{th}$ place). These vectors generate a subgroup of $Z^n$ of rank n-p and so (by the proof of the fundamental theorem of Abelian groups) there are vectors $N^1,\dots,N^p$ in $Z^n$ such that $\{N^1,\dots,N^n\}$ generate $Z^n$ and the $n$ by $n$ matrix with $i^{th}$ row $N^i$ has an inverse with all entries integers. $\{N^1,\dots,N^n\}$ form a basis of $R^n$; if we define an automorphism $I$ of $R^n$ by this matrix with respect to $\{e^1,\ldots,e^n\}$, (the standard basis of $R^n$), and extending by linearity, then $I$ projects to a diffeomorphism of $T^n$, and $I(v)$ has $0$ as its $j^{th}$ coordinate, for $p + 1 \leqq j \leqq n$. Thus $\overline{\mathcal{O}}_{\overline{\alpha}}(\overline{x})$ is contained in the isomorphic image of $T^p$, the p-torus generated by $\{e^1,\ldots,e^p\}$. We claim that in fact $\overline{\mathcal{O}}_{\overline{\alpha}}(\overline{x})$ is equal to this isomorphic image of $T^p$. Let $I(v) = (v_1',\ldots,v_p',0,\ldots,0)$; note that the rational dimension of $\{v_1',\ldots,v_p'\}$ is $p$. Let $\alpha'$ be the action of $R$ on $R^p$ determined by $(v_1',\ldots,v_p')$. We will use Lemma 1.1 to show that the orbits of $\overline{\alpha}'$ are dense in $T^p$ (and hence that the orbits of $\overline{\alpha}$ are dense in the isomorphic image of $T^p$). Suppose not; let $\beta'$ be an action of $R^{p-1}$ on $R^p$ with orbits orthogonal to those of $\alpha'$. By Lemma 1.1, $\overline{\beta}'$ is not free, and so there is an $M$ in $Z^n - \{0\}$ which is orthogonal to the line $R(v_1',\ldots,v_p')$; that is, $M \neq 0$ and $M\cdot(v_1',\ldots,v_p') = 0$. But this contradicts the fact that the rational dimension of $\{v_1',\ldots,v_p'\}$ is $p$. So $\overline{\mathcal{O}}_{\overline{\alpha}}(\overline{x})$ is isomorphic to $T^p$, and Proposition 1.2 is proved.

We will use another fact about rational dimension.

<u>PROPOSITION 1.3</u>: <u>If</u> $p$ <u>is the rational dimension of the coordinates of</u> $v = (v_1,\ldots,v_n)$ <u>in</u> $R^n$, <u>and if there are</u> $r$ R-<u>linearly independent</u> n-<u>vectors</u> $N^i$ <u>in</u> $Q^n$ <u>such that</u> $N^i\cdot v = 0$ <u>for</u> $1 \leqq i \leqq r$, <u>then</u> $p \leqq n - r$.

<u>Proof</u>: Consider the following diagram of Q-vector spaces;

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & Q^n \arrow[r, "e_v"] & R_Q \end{tikzcd}$$

where $R_Q$ denotes the reals viewed as a Q-vector space, $e_v(N) = N\cdot v$, and $K = \text{Kernel}\ (e_v)$. The Q-dimension of the image of $e_v$ is just the rational dimension of $\{v_1,\ldots,v_n\}$. Since

$$n = \dim_Q(\operatorname{Im} e_v) + \dim_Q(K)$$

we are done if we note that $\dim_Q(K) \geq \dim_R(K) \geq r$; this is because the $r$ vectors $N_i$ in $K$ are R-linearly independent which implies that they are Q-linearly independent.

We will also use the following fact:

PROPOSITION 1.4: *If $\Omega$ is a k-dimensional vector subspace of $R^n$, $k \geq 2$, then there is a $(k-1)$-dimensional vector subspace $\Gamma$ of $\Omega$ such that* $\overline{\pi(\Gamma)} = \overline{\pi(\Omega)}$.

*Proof*: We can assume that if $m$ is the dimension of $\overline{\pi(\Omega)}$, then $m$ is equal to $n$. (The argument is as in Proposition 1.2.) Suppose that the conclusion is false, i.e. suppose that every $(k-1)$-dimensional subspace $\Gamma$ of $\Omega$ has $\dim(\overline{\pi(\Gamma)}) < n$. Then by Lemma 1.1, if $\Gamma^\perp$ denotes the orthogonal complement of $\Gamma$, $\Gamma^\perp \cap Z^n \neq \{0\}$. We can thus define a map $\theta$ from the set of $(k-1)$-planes in $\Omega$ to $Z^n - \{0\}$ such that $\theta(\Gamma) \in \Gamma^\perp \cap Z^n$. $\theta$ must be one-to-one because if $\Gamma_1$ and $\Gamma_2$ are distinct $(k-1)$-planes in $\Omega$, together they span $\Omega$; if $\theta(\Gamma_1) = \theta(\Gamma_2) = N \neq 0$, then $N$ is orthogonal to $\Omega$ which contradicts the assumption that the dimension of $\overline{\pi(\Omega)}$ is equal to $n$, again by Lemma 1.1. But the existence of such a map $\theta$ is impossible, since $Z^n$ is countable, but the set of $(k-1)$-planes in $\Omega$ is uncountable. Thus Proposition 1.4 is proved.

We now consider non-singular linear actions $\bar{\alpha}$ of $R^k$ on $T^n$ which, though not necessarily free, must of course be locally free.

The following generalizes Lemma 1.1.

PROPOSITION 1.5: *Suppose that the non-singular linear action* $\bar{\alpha}$ *of* $R^k$ *on* $T^n$ *is defined from a linear action* $\alpha$ *of* $R^k$ *on* $R^n$, *and* $\Omega_A$ *is the* k-*plane in* $R^n$ *corresponding to the orbits of* $\alpha$. *Let* $\Omega_Z$ *be the smallest vector subspace of* $\Omega_A$ *containing* $\Omega_A \cap Z^n$, *and suppose that the dimension of* $\Omega_Z$ *is* r. *Let* $\Omega_B$ *be the orthogonal complement of* $\Omega_A$, *and let* s *be the dimension of* $\overline{\pi(\Omega_B)}$. *Then* $s = n - r$.

(Relating this to the notation of Lemma 1.1; condition (i), that $\Omega_A \cap Z^n = \{0\}$, is equivalent in this Proposition to $r = 0$. Condition (ii), which was essentially that the orbits of the action $\beta$ determined by B are dense, is equivalent to $s = n$. Since $s = n - r$, $r = 0$ if and only if $s = n$.)

*Proof:* Let $\{N^1,\dots,N^r\}$ be a basis for $\Omega_Z$, where $N^i \in Z^n$ for $i = 1,\dots,r$. Let $\Omega_C$ be such that $\Omega_A = \Omega_Z \oplus \Omega_C$. (Then a linear action $\bar{\gamma}$ of $R^{k-r}$ on $T^n$ which has $\pi(\Omega_C)$ as an orbit will be free, while $\pi(\Omega_Z)$ is just an embedding of $S^1 \times \dots \times S^1$ (r times) in $T^n$. The orbits of $\alpha$ are embeddings of $\underbrace{R^1 \times\dots\times R^1}_{k-r} \times \underbrace{S^1 \times\dots\times S^1}_{r}$ in $T^n$.

To see that $s \geqq n - r$, we apply Lemma 1.1 to actions represented by the planes $\Omega_C$ and $\Omega_B \oplus \Omega_Z$. The former is free, so by Lemma 1.1 the latter has dense orbits, or

$$\dim\left(\pi^{-1}(\overline{\pi(\Omega_B \oplus \Omega_Z)})\right) = n \quad .$$

Note that

$$\dim\left(\pi^{-1}(\overline{\pi(\Omega_Z)})\right) = \dim\left(\pi^{-1}(\pi(\Omega_Z))\right) = r \ .$$

Also

$$\dim\left(\pi^{-1}(\overline{\pi(\Omega_B \oplus \Omega_Z)})\right) \leqq \dim\left(\pi^{-1}(\overline{\pi\,\Omega_B})\right) + \dim\left(\pi^{-1}(\overline{\pi\,\Omega_Z})\right) .$$

Thus

$$s = \dim\,(\overline{\pi\,\Omega_B}) = \dim\left(\pi^{-1}(\overline{\pi\,\Omega_B})\right) \geqq n - r \quad .$$

Now to show that $s \leqq n - r$ ; by Proposition 1.4, let $v \in \Omega_B$ be such that $\dim\,(\overline{\pi(Rv)}) = s$ . By Proposition 1.2, the coordinates of $v$ have rational dimension $s$. Then by Proposition 1.3, since $v$ is orthogonal to the $r$ basis vectors $\{N^1,\ldots,N^r\}$ of $\Omega_Z$ which are in $Q^n$, $s \leqq n - r$. Thus $s = n - r$ and Proposition 1.5 is proved.

There is an obvious algebraic condition on an $n$ by $k$ matrix $A$ equivalent to the corresponding action $\alpha$ of $R^k$ on $T^n$ having dense orbits; namely that if $A^i$, $i = 1,\ldots,k$, are the column $n$-vectors of $A$, then there is a vector $v$ in the span (over $R$) of $\{A^1,\ldots,A^k\}$ such that the rational dimension of the coordinates of $v$ is $n$. A condition on $A$ equivalent to $\alpha$ being free is obtained as follows; let $D_i$ be the map of $R^k$ to $R$ given by

$$D_i(r) = B^i \cdot r$$

where $B^i$ is the $i^{th}$ row $k$-vector of $A$, $i = 1,\ldots,n$. Then $\alpha$ is free if and only if for no $r$ in $R^k - \{0\}$ is

$$Ar = \left( D_1(r),\ldots,D_n(r) \right) \in Q^n \quad ,$$

or equivalently, if

$$(1.1) \qquad \bigcap_{i=1}^{n} D_i^{-1}(Q) = \{0\} .$$

It would be desirable to have some other more convenient expression of a condition for an action either to be free or have dense orbits. Such a condition should depend only on the $k$-plane $\Omega_A$ in $R^n$ spanned by the vectors $A^i$, not on the choice of basis in $\Omega_A$. To this end, we now consider ways in which to coordinatize k-planes in $R^n$.

Let $\Sigma_k$ be the set of subsets of $\{1,\ldots,n\}$ with $k$ elements. Let $\{e^1,\ldots,e^n\}$ be the standard basis for $R^n$. If $\sigma \in \Sigma_k$, let

$$\omega_\sigma = e^{i(1)} \wedge \ldots \wedge e^{i(k)}$$

where $i(j)$ is in $\sigma$ for $1 \leqq j \leqq k$, $i(j) < i(j+1)$, and $\wedge$ is the usual exterior product. Any exterior product of $k$ n-vectors can be expressed as an R-linear combination of $\{\omega_\sigma \mid \sigma \in \Sigma_k\}$. In particular, if $A^1,\ldots,A^k$ are the column n-vectors of $A$ (and thus a basis of $\Omega_A$) then $\Omega_A$ corresponds to $A^1 \wedge \ldots \wedge A^k$ in the sense that a vector $v$ is in $\Omega_A$ if and only if $v \wedge A^1 \wedge \ldots \wedge A^k = 0$. A simple computation yields that

$$A^1 \wedge \ldots \wedge A^k = \sum_{\sigma \in \Sigma_k} a_\sigma \omega_\sigma$$

where the real number $a_\sigma$ is the determinant of the $k$ by $k$ submatrix of $A$ with rows corresponding to elements of $\sigma$. These $\binom{n}{k}$ numbers $a_\sigma$ are called the Plucker coordinates of $\Omega_A$. (Note that the dimension of the space of k-planes in $R^n$ is $k(n-k)$, so there are obviously relations on the Plucker coordinates of $\Omega_A$. In [1] there is a description of the method for finding conditions necessary for a collection of $\binom{n}{k}$ real numbers to be the Plucker coordinates of a k-plane.) Changing the basis of $\Omega_A$ induces a change in Plucker coordinates of multiplication by a constant.

A vector $v = (v_1,\ldots,v_n)$ in $R^n$ is an element of $\Omega_A$ if and only if

$$\begin{aligned} 0 &= v \wedge (A^1 \wedge \ldots \wedge A^k) \\ &= \left( \sum_{i=1}^{n} v_i e^i \right) \wedge \left( \sum_{\sigma \in \Sigma_k} a_\sigma \omega_\sigma \right) \end{aligned}$$

which is equivalent to

$$(1.2) \qquad \sum_{\tau\in\Sigma_{k+1}} c_\tau\, \omega_\tau = 0$$

where

$$c_\tau = \sum_{j\in\tau} (-1)^{f(\tau,j)} v_j a_{\tau(j)}$$

where $f(\tau,j)$ is 0 or 1 according to the position of $j$ in $\tau$, and $\tau(j)$ is $\tau - \{j\} \in \Sigma_k$ .

It is possible, by a change of basis in $\Omega_A$ and by a reordering of basis in $R^n$, to find a matrix $\bar{A}$ of the form

$$\bar{A} = \left|\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ x_{11} & x_{21} & \cdots & x_{k1} \\ \vdots & \vdots & & \vdots \\ x_{1(n-k)} & x_{2(n-k)} & \cdots & x_{k(n-k)} \end{array}\right|$$

where $\Omega_{\bar{A}} = \Omega_A$ . Then it is seen that

$$x_{ij} = \pm\ a_{\{1,\ldots,\hat{j},\ldots,k,k+i\}} = \pm\ a_\sigma$$

where $a_\sigma$ is a Plucker coordinate of $\Omega_{\bar{A}}$ .

Note that if $v = (v_1,\ldots,v_k)$, then

$$\bar{A}v = (v_1,\ldots,v_k,\ v\cdot X^1,\ldots,\ v\cdot X^{n-k})$$

where $X^i = (x_{i1},\ldots,x_{ik})$. Since the first $k$ coordinates of $\bar{A}v$ are the coordinates of $v$, in order for $\bar{A}v$ to be in $Q^n$, $v$ must be an element of $Q^k$. Let $C_i$ be the map of $Q^k$ to $R$ given by

$$C_i(N) = X^i\cdot N ,$$

for $i = 1,\ldots, n - k$. We see that a linear action $\alpha$ with $\Omega_A$ as an orbit defines a free action $\alpha$ of $R^k$ on $T^n$ if and only if

$$(1.3) \qquad \bigcap_{i=1}^{n-k} C_i^{-1}(Q) = \{0\} .$$

(This condition, although of the same type as 1.1, is somewhat more useful, since here there are $n - k$ (instead of $n$) terms in the intersection, and also since the maps $C_i$ are defined in terms of Plucker coordinates and are thus independent of choice of basis.)

One way in which condition 1.3 can be satisfied, of course, is by having $C_i^{-1}(Q) = 0$ for some $i$. That is, if for some $i$ the set $\{x_{i1}, \dots x_{ik}, 1\}$ has rational dimension $k + 1$, then $\alpha$ is free. This corresponds to saying that there is some $\tau$ in $\Sigma_{k+1}$ such that the Plucker coordinates $\{a_\sigma \mid \sigma \in \Sigma_k , \sigma \subset \tau\}$ have rational dimension $k + 1$. However this condition is not necessary for $\alpha$ to be free; for example, if

$$A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{vmatrix}$$

then the corresponding action $\bar{\alpha}$ of $R^2$ on $T^4$ is easily seen to be free, but neither $\{\sqrt{2}, 1, 1\}$ nor $\{-1, \sqrt{2}, 1\}$ has rational dimension 3 .

We now summarize the situation for a non-singular linear action of $R^{n-1}$ on $T^n$. There are $n = \binom{n}{n-1}$ Plucker coordinates for an $(n - 1)$-plane in $R^n$; denote them by $a_{\sigma(i)}$, $i = 1, \dots, n$, where $a_{\sigma(i)}$ is the determinant of $A$ minus its $i^{th}$ row. Note that by 1.2, a vector $v = (v_1, \dots, v_n)$ is in $\Omega_A$ if and only if

$$0 = v \wedge A^1 \wedge \dots \wedge A^k = \sum_{i=1}^{n} v_i a_{\sigma(i)} (-1)^{i+1} e^1 \wedge \dots \wedge e^n ,$$

or equivalently,

$$\sum_{i=1}^{n} v_i a_{\sigma(i)} (-1)^{i+1}$$

or $v \cdot b = 0$ , where

$$b = (a_{\sigma(1)}, -a_{\sigma(2)}, \ldots, \pm a_{\sigma(n)}) \quad .$$

Thus $b = (b_1, \ldots, b_n)$ is normal to $\Omega_A$ . We have the following equivalent statements (if $\overline{\beta}$ is the action of $R$ on $T^n$ corresponding to b);

i. $\overline{\beta}$ has dense orbits .

ii. rational dimension of $\{b_i \mid 1 \leqq i \leqq n\}$ is $n$ .

iii. rational dimension of $\{a_{\sigma(i)} \mid 1 \leqq i \leqq n\}$ is $n$ .

iv. $\overline{\alpha}$ is free.

Also the following are equivalent:

i. $\overline{\beta}$ is free.

ii. rational dimension of $\{b_i \mid 1 \leqq i \leqq n\} > 1$ .

iii. rational dimension of $\{a_{\sigma(i)} \mid 1 \leqq i \leqq n\} > 1$ .

iv. $\overline{\alpha}$ has dense orbits.

## PART II. ACTIONS CONJUGATE TO LINEAR ACTIONS

Let $\varphi$ be a locally free action of $R^{n-1}$ on a compact connected orientable n-manifold $V$ with no compact orbits. We begin by describing results of Sacksteder in [5].

First, it is known that since no orbit of $\varphi$ is compact, every orbit is dense in $V$ and the holonomy group of each orbit has at most two elements ([5], Theorems 8 and 9). Since $V$ is orientable and the action of $R^{n-1}$ orients the orbits of $\varphi$, the foliation induced on $V$ by $\varphi$ is orientable and the holonomy is thus zero.

One can find a $C^\infty$ embedding of the circle $S^1$ in $V$ which is transverse to the orbits of $\varphi$; in fact it can be assumed that there is a Riemannian metric on $V$ such that the intersection is orthogonal. The orbit of every point $p$ of $V$ must intersect this circle (because if we let $A$ be the union of the orbits meeting $S^1$, then by Theorem 4 of [6], either $A$ is equal to $V$, or there is an orbit in the boundary of $A$ with infinite holonomy.)

The action $\varphi$ induces a pseudogroup $\Gamma$ of local diffeomorphisms of $S^1$. Theorem 6 of [5] shows that there is a bundle-like metric on $V$ (that is, a Riemannian metric such that distance between nearby orbits measured along orthogonal trajectories is "locally" constant) which is obtained from the original metric in the following way. First it is shown that there is a metric on $S^1$ which is invariant under $\Gamma$. The new metric is constructed from this metric on $S^1$ and the original metric on tangent vectors to the orbits. The metric on $S^1$ need not be differentiable with respect to the original atlas; however, possibly by changing atlases, one obtains a smooth bundle-like metric on $V$ which when restricted to $S^1$ is invariant under the action of $\Gamma$. This means that we can assume $S^1$ to be parametrized as $p_t$, where $t$ is an element of the quotient $R/Z$, and each element of $\Gamma$ acts as a rotation. (Since $V$ is orientable, we can exclude the possibility of reflections.) That is, for any $f$ in $\Gamma$, there is a real number $\lambda$ such that for any $p_t$ in the domain of $f$,

$$f(p_t) = p_{t+\lambda} \ .$$

We now state a theorem concerning the free case.

*THEOREM 2.1. Let* $\varphi$ *be a free* $C^\infty$ *action of* $R^{n-1}$ *on a compact connected orientable n-manifold* $V$ *. (As described above, there is a bundle-like metric on* $V$ *, and* $S^1$ *embedded in* $V$ *such that* $S^1$ *intersects orbits of* $\varphi$ *orthogonally, and elements of the pseudo-group* $\Gamma$ *induced by* $\varphi$ *on* $S^1$ *act as rotations.) Suppose that there is a real number* $\lambda$ *such that for some* $f$ *in* $\Gamma$ *, and* $p_t$ *in the domain of* $f$,

$$f(p_t) = p_{t+\lambda} \ ,$$

*where* $\lambda$ *satisfies the Liouville inequality*

$$\left| \lambda \pm \frac{m}{n} \right| > \frac{C}{n^\gamma} \tag{2.1}$$

*for some fixed positive real number* $C$ *, fixed number* $\gamma$ *greater than 2, and for arbitrary* $m$ *and* $n$. *Then* $\varphi$ *is topologically conjugate to a linear action* $\bar{\alpha}$ *of* $R^{n-1}$ *on the n-torus* $T^n$.

(It is known that all irrational numbers except a set of transcendental numbers of measure 0 satisfy an inequality such as 2.1.)

The proof will involve a series of Lemmas (in which the conditions of Theorem 2.1 will be assumed).

*LEMMA 2.1.* *If* $\lambda$ *is such that for some* $f$ *in* $\Gamma$ *and some* $p_t$ *in the domain of* $f$, $f(p_t) = p_{t+\lambda}$ , *then there exists an* $\bar{f}$ *in* $\Gamma$ *such that* $\bar{f}$ *restricted to the domain of* $f$ *is the same as* $f$, *and for all* $p_t$ *in* $S^1$, $\bar{f}(p_t) = p_{t+\lambda}$ .

*Proof:* Consider an endpoint $p_t$ in $S^1$ of an open set of points

$$\{ p_s \mid s \in (t-\varepsilon,\ t) \text{ and } p_{s+\lambda} \in \mathcal{O}_\varphi(p_s) \} .$$

The properties of the metric on $V$ guarantee that the orbit through $p_t$ will intersect $S^1$ at the endpoint of the set

$$\{ p_{s+\lambda} \mid s \in (t-\varepsilon,\ t) \},$$

that is, at $p_{t+\lambda}$. Thus it can be shown that the set of $p_t$ such that $p_{t+\lambda}$ is an element of $\mathcal{O}_\varphi(p_t)$ is both open and closed, non-empty, hence all of $S^1$. So Lemma 2.1 is proved.

Thus given such a $\lambda$, when $\varphi$ is free we can define a unique "return function"

$$r_\lambda : S^1 \longrightarrow R^{n-1}$$

by

$$\varphi\,(\, r_\lambda(t),\ p_t \,) = p_{t+\lambda} .$$

The function $r_\lambda$ is differentiable since it can be described locally as the projection onto $R^{n-1}$ of the inverse image of $S^1$ under the differentiable map which sends a point $(r,s)$ in a neighborhood $U$ of $R^{n-1} \times R$ to $\varphi(r,p_s)$. Here $U$ is chosen to contain $W\times(s-\varepsilon,s+\varepsilon)$ for $W$ an open neighborhood in $R^{n-1}$ of the line determined by the origen and $r_\lambda(s)$ , and such that $U$ is mapped diffeomorphically into $V$. The next Lemma states that if one return function is constant, then they all are constant.

**LEMMA 2.2.** *Let* $S^1$ *be an embedding of a circle in* $V$ *as above, parametrized as* $p_t$*, where* $t$ *is in* $R/Z$*. Suppose that there is an irrational number* $\lambda$ *such that for all* $t$ *in* $R/Z$*,* $p_{t+\lambda}$ *is in* $\mathcal{O}_\varphi(p_t)$ *, and the return function* $r_\lambda(t)$ *is a constant* $r_\lambda$ *in* $R^{n-1}$*. Then for any other choice of* $\mu$ *such that* $p_{t+\mu}$ *is in* $\mathcal{O}_\varphi(p_t)$ *for every* $t$ *in* $R/Z$*, if we define* $r_\mu$ *by* $\varphi(r_\mu, p_0) = p_\mu$ *, we have*

$$\varphi(r_\mu, p_t) = p_{t+\mu} \quad ,$$

*that is, the return function* $r_\mu(t)$ *is also constant.*

**Proof:** Consider the closed curve $q_t$ defined by

$$q_{t+\mu} = \varphi(r_\mu, p_t)$$

where $t$ is an element of $R/Z$. Note that

$$q_\mu = \varphi(r_\mu, p_0) = p_\mu \ .$$

Also

$$\begin{aligned} \varphi(r_\lambda, q_{t+\mu}) &= \varphi(r_\lambda,\ \varphi(r_\mu, p_t)) \\ &= \varphi(r_\lambda + r_\mu,\ p_t) \\ &= \varphi(r_\mu,\ p_{t+\lambda}) \\ &= q_{t+\lambda+\mu} \ . \end{aligned}$$

Thus the curve $q_t$ is $r_\lambda$-invariant, and intersects the curve $p_t$ at $p_\mu$ which is equal to $q_\mu$. Since $\lambda$ is irrational, the projected image of the set

$$\{\ \mu + n\lambda \mid n = 1,2,\ldots\ \}$$

is dense in $R/Z$ , and so we can conclude that the curves are the same. That is, since

$$p_{\mu+n\lambda} = \varphi(nr_\lambda,\ p_\mu) = \varphi(nr_\lambda,\ q_\mu) = q_{\mu+n\lambda} \ ,$$

$p_t$ is also $r_\mu$-invariant, or $\varphi(r_\mu,\ p_t) = q_{t+\mu} = p_{t+\mu}$ , so Lemma 2.2 is proved.

For the proof of Theorem 2.1, we will show that there is an embedding of $S^1$ in $V$ with all return functions constant. To do this, we need the following Lemma.

*LEMMA 2.3.* *Let* $F$ *be a* $C^\infty$ *mapping from* $S^1$ *to* $R^{n-1}$ *such that* $\int_{S^1} F = 0$. *Define the map* $T$ *of the set of continuous maps from* $S^1$ *to* $R^{n-1}$ *to itself by* $Tg(t) = g(t + \lambda)$ , *for* $t \in R/Z$ . *If* $\lambda$ *satisfies the inequality* (2.1), *then there is a* $C^\infty$ *solution* $g$ *mapping* $S^1$ *to* $R^{n-1}$ *of the equation*

$$g - Tg = F . \tag{2.2}$$

*Proof:* The mapping $F$ is $C^\infty$ if and only if for each $k = 1, \ldots, n - 1$ and for all integers $s$ greater than or equal to $0$ ,

$$\sum_{j=-\infty}^{\infty} j^s \, |a_j^k| < \infty \tag{2.3}$$

where $a_j^k$ is the $j^{\text{th}}$ Fourier coefficient of the mapping $F^k$ from $S^1$ to $R$ , where $F$ is equal to $(F^1, \ldots, F^{n-1})$. ([3], page 26.)

Thus, if the equation (2.3) has a solution $g$ equal to $(g^1, \ldots, g^{n-1})$, where $g^k$ has the Fourier expansion

$$g^k(t) \sim \sum_{j=-\infty}^{\infty} b_j^k \, e^{2\pi i j t} ,$$

then a simple calculation yields that

$$b_j^k = \left( \frac{1}{2 \sin j\pi\lambda} \right) e^{\pi i(-j\lambda - \frac{1}{2})} \; a_j^k$$

and so

$$| \, b_j^k \, | = \left( \frac{1}{2 \, |\sin j\pi\lambda|} \right) | \, a_j^k \, | \, .$$

Since $\lambda$ satisfies 2.1, we have that for all $m$ and $j$ ,

$$\left| \, \lambda \pm \frac{m}{j} \, \right| > C/j^\gamma$$

and so

$$|\lambda j\pi \pm \pi m| > B/j^{\delta}$$

where $\delta$ is $\gamma - 1$ and $B$ is equal to $C\pi$. This implies that

$$|\sin \lambda j\pi| > |\sin (B/j^{\delta})|,$$

and so

$$|b_j^k| < \frac{|a_j^k|}{2|\sin(B/j^{\delta})|} = \frac{j^{\delta}|a_j^k|}{2B} \frac{B/j^{\delta}}{|\sin(B/j^{\delta})|} .$$

Thus

$$\sum_{j=-\infty}^{\infty} |b_j^k| < \frac{1}{2B} \sum_{j=-\infty}^{\infty} j^{\delta}|a_j^k| \frac{B/j^{\delta}}{|\sin(B/j^{\delta})|} . \tag{2.4}$$

If $|j^{\delta}|$ is sufficiently large, then

$$\frac{B/j^{\delta}}{\sin(B/j^{\delta})} < 2$$

and hence the series on the right in (2.4) is bounded, by (2.3). In fact, for all $s = 1, 2, \ldots$,

$$\sum_{j=-\infty}^{\infty} j^{s}|b_j^k| \leqq \frac{1}{2B} \sum_{j=-\infty}^{\infty} j^{s+\delta}|a_j^k| \frac{B/j^{\delta}}{|\sin(B/j^{\delta})|}$$

where the series on the right is bounded, as before. Thus $g$ as defined by $\{b_j^k\}$ is $C^{\infty}$ and a solution to (2.2), and Lemma 2.3 is proved.

<u>MAIN LEMMA 2.4.</u> <u>Suppose that there is a real number</u> $\lambda$ <u>such that for some</u> $f$ <u>in</u> $\Gamma$, $f(p_t) = p_{t+\lambda}$, <u>where</u> $\lambda$ <u>satisfies the Liouville inequality</u> (2.1). <u>Then there is an embedding of the circle</u> $q_t$ <u>in</u> $V$ <u>such that all return functions are constant.</u>

Proof: By Lemma 2.2 it suffices to find an embedding $q_t$ of the circle in V with just one return function $r_\lambda$ constant, provided that $\lambda$ is irrational. In some cases one can construct an embedding $\tilde{p}_t$ of the circle with constant $\lambda$-return function from the original embedding $p_t$ in the following way. If K is a mapping of $S^1$ to $R^{n-1}$, and we define $\tilde{p}_t$ by $\tilde{p}_t = \varphi(K(t), p_t)$, then the return functions $r_\lambda$ and $\tilde{r}_\lambda$ respectively corresponding to the rotation number $\lambda$ are related as follows;

$$\tilde{r}_\lambda(t) = r_\lambda(t) - K(t) + K(t+\lambda) \quad (2.5)$$

So if $\tilde{r}_\lambda$ is to be a constant C, one must have $K(t) - K(t+\lambda) = r_\lambda(t) - C$ ; or, differentiating,

$$K'(t) - K'(t+\lambda) = r'_\lambda(t) .$$

Since $\lambda$ satisfies the Liouville inequality (2.1), Lemma 2.3 guarantees the existence of a $C^\infty$ mapping K from $S^1$ to $R^{n-1}$ such that equation (2.5) is true. $\tilde{p}_t$ need not be an embedding; however if $\tilde{p}_{t_1} = \tilde{p}_{t_2}$, then for all integers n, $\tilde{p}_{t_1+n\lambda} = \tilde{p}_{t_2+n\lambda}$, so for all t, $\tilde{p}_t = \tilde{p}_{t+(t_2-t_1)}$. One can find an embedding as desired by considering $t_0 = \min\{t \mid p_t = p_0\}$ and representing $S^1$ as $R/Z \cdot t_0$. We can assume that $t_0 = 1$. Thus the Main Lemma is proved.

Proof of Theorem 2.1: We can assume that there is an embedding $p_t$ of R/Z in V with all return functions constant. Such constants form a subgroup of $R^{n-1}$. We can now define an action $\psi$ of $R^1$ on V in the following way; every point v in V can be expressed as $\varphi(r, p_t)$ for some t in R/Z, r in $R^{n-1}$. Let

$$\psi(s, \varphi(r, p_t)) = \varphi(r, p_{t+s}) .$$

In particular,

$$\psi(s, p_t) = p_{t+s} .$$

The action $\psi$ is well-defined, because suppose that

$$\varphi(r, p_t) = \varphi(r', p_{t'}) ,$$

or equivalently

$$p_{t'} = \varphi(r - r', p_t) .$$

This means that $r - r'$ is the return constant corresponding to the rotation number $t' - t$. We see that for every real number $s$ ,

$$\begin{aligned}\psi(s, \varphi(r, p_t)) &= \varphi(r, p_{t+s}) \\ &= \varphi(r - (r-r'), p_{t+(t'-t)+s}) \\ &= \varphi(r', p_{t'+s}) \\ &= \psi(s, \varphi(r', p_{t'})) .\end{aligned}$$

Clearly $\psi$ commutes with $\varphi$, and so we have an action $\Phi$ of $R^n$ on $V$ defined by

$$\Phi((r,s), v) = \psi(s, \varphi(r,v)) = \varphi(r, \psi(s,v))$$

for $(r,s)$ in $R^{n-1} \times R$.

The orbit of any point $v$ of $V$ under $\Phi$ is the whole of $V$; thus there is a homeomorphism

$$h : R^n/I_v \longrightarrow V$$

induced by the map

$$\bar{h} : R^n \longrightarrow V$$

given by

$$\bar{h}(r) = \Phi(r,v) .$$

Here $I_v$ denotes the isotropy subgroup of $v$ under $\Phi$ ; that is, the set of all $r$ in $R^n$ such that $\Phi(r,v)$ is equal to $v$. Note that if $e$ is the vector $(0,\ldots,0,1)$ in $R^n$, then $e$ is an element of $I_v$; also, if $r_\lambda$ in $R^{n-1}$ is a return constant for the action $\varphi$ corresponding to a rotation number $\lambda$, then if $\bar{r}_\lambda = (r_\lambda,0)$ in

$R^{n-1} \times R$ , then $\bar{r}_\lambda - \lambda e$ is also in $I_v$. All elements of $I_v$ are expressible as a sum of vectors of these two types.

In fact, it is known that the quotient of $R^n$ by a discrete subgroup is a compact $n$-manifold if and only if the subgroup is isomorphic to $Z^n$. This implies that $I_v$ is generated over the integers by $n$ vectors in $R^n$; we can choose a set of free generators $\{e, \bar{r}_{\lambda_1} - \lambda_1 e, \ldots , \bar{r}_{\lambda_{n-1}} - \lambda_{n-1} e\}$ where $\bar{r}_{\lambda_i} = (r_{\lambda_i}, 0)$ and $r_{\lambda_i}$ is the return constant in $R^{n-1}$ corresponding to the rotation number $\lambda_i$ . Since $I_v$ is discrete, this set is also linearly independent over $R$ (and hence the set $\{r_{\lambda_1}, \ldots, r_{\lambda_{n-1}}\}$ in $R^{n-1}$ is linearly independent).

There is thus an isomorphism

$$\bar{\ell} : Z^n \longrightarrow I_v$$

which can be extended to an isomorphism

$$\bar{\ell} : R^n \longrightarrow R^n .$$

This defines an isomorphism

$$\ell : R^n/Z^n \longrightarrow R^n/I_v ,$$

which respects the group action of $R^n$ on both spaces. Note that the action $\Phi$ on $V$ corresponds under the homeomorphism $h$ to the group action of $R^n$ on $R^n/I_v$ given by addition.

We have the following commuting diagram;

$$\begin{tikzcd}
R^n \arrow[r, "\bar{\ell}"] \arrow[d, "\pi"'] & R^n \arrow[r, "\bar{h}"] \arrow[d, "\pi_{I_v}"'] & V \arrow[d, "id"] \\
T^n = R^n/Z^n \arrow[r, "\ell"'] & R^n/I_v \arrow[r, "h"'] & V
\end{tikzcd}$$

If $r$ and $w$ are elements of $R^n$ and $\pi(w) = [w]$ (where $\pi$ is the projection of $R^n$ to $T^n = R^n/Z^n$), we have

$$\begin{aligned} \ell^{-1}h^{-1}\Phi(r,\ h\ell[w]) &= \ell^{-1}h^{-1}\Phi(r,\ \overline{h\ell w}) \\ &= \ell^{-1}h^{-1}\Phi(r,\ \Phi(\overline{\ell}w,\ v)) \\ &= [\overline{\ell}^{-1}\ \overline{h}^{-1}\Phi(r+\overline{\ell}w,\ v)] \\ &= [\overline{\ell}^{-1}(r+\overline{\ell}w] \\ &= [w] + [\overline{\ell}^{-1}r]\ , \end{aligned}$$

which means that $\Phi$ is conjugate (by $h\ell$ ) to a linear action. Since $\varphi(r,v)$ is equal to $\Phi((r,0),\ v)$ , we have

$$(h\ell)^{-1}\varphi(r,\ h\ell[w]) = [w] + [\overline{\ell}^{-1}(r,0)] = [w] + [A(r)]$$

where we let $A(r)$ be $\overline{\ell}^{-1}(r,0)$. Theorem 2.1 is now proved.

One cannot expect to prove a result like Theorem 2.1 without some condition on the degree of irrationality of a "rotation number". That is, there do exist free actions with no compact orbits for which there are no circles with constant return functions (hence actions which are not topologically conjugate to linear actions).

An example of such an action of $R^1$ on the 2-torus $T^2$ is given in [8]. It is seen that if $\lambda$ is an irrational number which can be well approximated by rationals (that is, $\lambda$ does not satisfy (2.1)), then there is a $C^\infty$ function $F$ mapping $S^1$ to the positive real numbers such that equation (2.2) has <u>no</u> $L^1$ solution $g$. The action is defined as follows; viewing $T^2$ as a unit square with sides identified, orbits are the lines of slope $\lambda$ , and parametrization is such that it takes time $\int_0^t F(s)\,ds$ for a point on the line $x = 0$ to return to this line.

## PART III. ACTIONS TOPOLOGICALLY EQUIVALENT TO LINEAR ACTIONS

We now prove a partial result for the case when no "rotation number" as defined in Part II satisfies a Liouville inequality (2.1). The following theorem gives a sufficient condition for an action to be topologically equivalent to a linear one.

*THEOREM 3.1. Let* $\varphi$ *be a free action of* $R^{n-1}$ *on a compact connected n-manifold* $V$ *. Suppose that there is an injection* $i_1$ *of* $R$ *into* $R^{n-1}$ *such that if* $\psi_1$ *is the action of* $R$ *on* $V$ *defined by*

$$\psi_1(t,v) = \varphi(i_1(t), v) \ , \tag{3.1}$$

*then the closure of the orbit of* $\psi_1$ *through* $v_0$ *in* $V$, $\overline{\mathcal{O}}_{\psi_1}(v_0)$ , *is a submanifold of dimension* 2. *Then there exists a homeomorphism*

$$f : V \longrightarrow T^n$$

*and a linear action*

$$\bar{\alpha} : R^{n-1} \times T^n \longrightarrow T^n$$

*such that* $f$ *induces a map* $\tilde{f}$ *on the orbit spaces,*

$$\tilde{f} : V/\varphi \longrightarrow T^n/\bar{\alpha} \ ;$$

*i.e.* $\varphi$ *is topologically equivalent to a linear action of* $R^{n-1}$ *on* $T^n$.

We first show that we can assume a stronger hypothesis.

*LEMMA 3.1. Let* $\varphi$ *be a free action of* $R^{n-1}$ *on a compact n-manifold* $V$. *Suppose that there is an injection* $i_1$ *of* $R$ *in* $R^{n-1}$ *such that the action* $\psi_1$ *defined by* (3.1) *is such that* $\overline{\mathcal{O}}_{\psi_1}(v_0)$ *is a two-dimensional submanifold of* $V$. *Then there are injections* $i_j$ *of* $R$ *in* $R^{n-1}$, $j = 2,\dots, n-1$, *such that*

$$R^{n-1} = i_1 R \oplus \dots \oplus i_{n-1} R \;,$$

*and if* $\psi_j$ *is the action of* $R^j$ *on* $V$ *defined by*

$$\psi_j((r_1,\dots,r_j),\ v) = \varphi\left(\sum_{s=1}^{j} i_s(r_s)\ ,\ v\right) \tag{3.2}$$

*then* $\overline{\mathcal{O}}_{\psi_j}(v_0)$ *has dimension* $j+1$ *for* $j = 1,\dots,n-1$.

*Proof:* First we prove that $\overline{\mathcal{O}}_{\psi_1}(v_0)$ is transverse to the action of $\varphi$. Since $\overline{\mathcal{O}}_{\psi_1}(v_0)$ is a compact two-dimensional manifold which admits a free action of $R^1$, there is a diffeomorphism $d$ mapping the 2-torus $T^2$ to $\overline{\mathcal{O}}_{\psi_1}(v_0)$. Assume that at a point $v$ in $d(T^2)$, $d(T^2)$ is not transverse to $\mathcal{O}_\varphi(v)$, i.e. the intersection of the tangent space of $d(T^2)$ and the tangent space of $\mathcal{O}_\varphi(v)$ contains a vector which is not tangent to $\mathcal{O}_{\psi_1}(v)$. Let $r$ in $R^{n-1}$ be such that the action $\tilde{\varphi}(t,-)$ of $R$ on $V$ defined by $\varphi(tr,-)$ has this vector as tangent to its orbit through $v$; that is, the orbit of $\tilde{\varphi}$ is tangent to $d(T^2)$ at $v$. Since $\psi_1$ and $\tilde{\varphi}$ commute, $\tilde{\varphi}$ is tangent to $d(T^2)$ at all points in $\mathcal{O}_{\psi_1}(v)$. But $\mathcal{O}_{\psi_1}(v)$ is dense in $d(T^2)$, and so $\tilde{\varphi}$ is tangent to $d(T^2)$ at all points of $d(T^2)$. This means that $\psi_1$ and $\tilde{\varphi}$ define an action of $R^2$ on $d(T^2)$ which must have a periodic orbit, contradicting the fact that $\varphi$ is free. Thus $\overline{\mathcal{O}}_{\psi_1}(v_0)$ is transverse to the action of $\varphi$.

The proof is by induction. We show that if $\psi_{j-1}$ is defined by the injections $i_1,\dots,i_{j-1}$ such that $\overline{\mathcal{O}}_{\psi_{j-1}}(v_0)$ is a submanifold of dimension $j$ transverse to the orbits of $\varphi$, then there exists an injection $i_j$ of $R$ in $R^{n-1}$ such that $\overline{\mathcal{O}}_{\psi_j}(v_0)$ is a submanifold of dimension $j+1$. This will be done by showing that there is an $r$ in $R^{n-1}$ such that $r$ is not an element of the $j-1$ plane $P_{j-1}$ spanned by the images of $i_1,\dots,i_{j-1}$, and such that $\varphi(r,v_0)$ is in

$\overline{\mathfrak{S}}_{\psi_{j-1}}(v_0)$. (Then we can let $i_j(t) = tr$, and the dimension of $\overline{\mathfrak{S}}_{\psi_j}(v_0)$ will be $j+1$, since $\overline{\mathfrak{S}}_{\psi_j}(v_0)$ can be viewed as the product space $[0,1] \times \overline{\mathfrak{S}}_{\psi_{j-1}}(v_0)$, where $\{0\} \times \overline{\mathfrak{S}}_{\psi_{j-1}}(v_0)$ is identified with $\{1\} \times \overline{\mathfrak{S}}_{\psi_{j-1}}(v_0)$ by the differentiable map $\varphi(r,-)$ when r is minimal.)

To show that such an r exists, let N be a normal vector to $P_{j-1}$ in $R^{n-1}$. Consider the half-planes

$$H_m = \{ r \in R^{n-1} \mid r \cdot N \geqq m \}$$

where m is an integer. Let

$$J_m = \{ \varphi(r,v_0) \mid r \in H_m \}$$

and let

$$H = \bigcap_{m=1}^{\infty} \overline{J}_m .$$

H is non-empty, because the manifold V is compact. We claim that H is saturated with respect to $\varphi$, that is, if $v \in H$, then $\varphi(r,v) \in H$ for every r in $R^{n-1}$. To prove this, observe that if $v \in H$, then

$$(3.3) \quad v = \lim_{m\to\infty}(v_m) = \lim_{m\to\infty} \varphi(t_m,v_0) , \quad t_m \in R^{n-1} , \ t_m \cdot N \geqq m .$$

Consider

$$\varphi(t,v) = \lim_{m\to\infty} \varphi(t,v_m) = \lim_{m\to\infty} \varphi(t + t_m, v_0) ;$$

we have

$$(t + t_m) \cdot N = t \cdot N + t_m \cdot N$$

where $t \cdot N$ is a constant $m_0$ and $t_m \cdot N \geqq m$. Thus

$$(t + t_m) \cdot N \geqq m + m_0$$

which means that $\varphi(t,v_m)$ is in $J_{m+m_0}$, and so $\varphi(t,v)$ is in H; so H is saturated with respect to $\varphi$.

Since $\overline{\mathfrak{S}}_{\psi_{j-1}}(v_0)$ is transverse to the orbits of $\varphi$, it contains an embedded circle which is transverse to the orbits of $\varphi$. As in the third paragraph of Part II, every orbit of $\varphi$ must meet this embedded circle, thus every $\varphi$-orbit must meet $\bar{\Theta}_{\psi_{j-1}}(v_0)$. Thus $H \cap \bar{\Theta}_{\psi_{j-1}}(v_0)$ contains an element $v$ as in (3.3). Since $\bar{\Theta}_{\psi_{j-1}}(v_0)$ is transverse to the orbits of $\varphi$, there is a neighborhood $U$ of $\bar{\Theta}_{\psi_{j-1}}(v_0)$ such that if $w$ is in $U$, then there is some bounded $r(w)$ in $R^{n-1}$ such that $\varphi(r(w), w)$ is in $\bar{\Theta}_{\psi_{j-1}}(v_0)$. Thus for sufficiently large $m$, $\varphi(t_m, v_0)$ is in $U$, and so $\varphi(r(t_m) + t_m, v_0)$ is in $\bar{\Theta}_{\psi_{j-1}}(v_0)$ where $r(t_m) + t_m$ is not in $P_{j-1}$. Lemma 3.1 is now proved.

We will need the following Lemma in the proof of Theorem 3.1.

LEMMA 3.2. *Let* $\bar{h}$ *be a homeomorphism of* $T^j$ *to itself, and let* $h$ *be a lifting of* $h$ *to* $R^j$ *; that is,*

$$\begin{tikzcd}
R^j \arrow[r, "h"] \arrow[d, "\pi"'] & R^j \arrow[d, "\pi"] \\
T^j \arrow[r, "\bar{h}"] & T^j
\end{tikzcd}$$

*commutes. Suppose that there are linearly independent vectors* $r^1, \ldots, r^{j-1}$ *in* $R^j$ *such that if* $P_{j-1}$ *is the span of* $\{r^1, \ldots, r^{j-1}\}$, *then* $P_{j-1} \cap Z^j = \{0\}$ *and* $\pi(P_{j-1})$ *is dense in* $T^j$, *and such that for each* $s$, $s = 1, \ldots, j-1$, $h$ *maps translates of the line* $Rr^s = \{ tr^s \mid t \in R \}$ *to such lines in an orientation-preserving way. Suppose also that* $h(P_{j-1})$ *is not equal to* $P_{j-1}$. *Then* $\bar{h}$ *is isotopic to the projection of a translation on* $R^j$.

*Proof*: Let $N$ be normal to $P_{j-1}$ in $R^j$, of unit length. Then we can write

$$h(v) = v + \rho(v)N + \tau(v)$$

where $\rho(v)$ is a real number, and $\tau(v)$ is in $P_{j-1}$. Since $h$ maps translates of $P_{j-1}$ to translates of $P_{j-1}$, $\rho(v)$ is constant on translates of $P_{j-1}$. Also, since $h$ projects to a map on $T^j$, so does $\rho$ ; thus $\rho$ is a continuous real-valued function which is constant on a dense subset of $T^j$, so $\rho(v)$ is constant. This constant is not zero since $h(P_{j-1})$ is not equal to $P_{j-1}$. (The value of $\rho(v)$ depends on the choice of lifting $h$.) Let $r^j$ be $\rho(v)N$ . Then

$$h(v) = v + r^j + \tau(v).$$

The vectors $r^1,\dots, r^j$ form a basis of $R^j$. Suppose that in terms of this basis, if $v$ is in $R^j$,

$$v = \sum_{i=1}^{j} v_i r^i = (v_1,\dots,v_j) .$$

Then there are real-valued functions $\tau_i$ on $R^{j-1}$, $i = 1,\dots, j-1$ , such that

$$\tau(v) = \sum_{i=1}^{j-1} \tau_i(v)r^i = (\tau_1(v),\dots, \tau_{j-1}(v), 0) .$$

We claim that in fact $\tau_i$ depends only on $v_i$ and $v_j$. For example, taking $i = 1$, (and again writing elements of $R^j$ in terms of the basis $\{r^1,\dots,r^j\}$) if $v = (v_1,\dots,v_j)$ and $v' = (v_1,v_2',\dots,v_{j-1}',v_j)$ agree in their first and $j^{th}$ entries, then we claim that $\tau_1(v) = \tau_1(v')$. To show this, consider the vectors

$$\begin{aligned} v = w^1 &= (v_1, v_2, v_3, \dots , v_{j-1}, v_j) \\ w^2 &= (v_1, v_2', v_3, \dots , v_{j-1}, v_j) \\ &\vdots \\ v' = w^{j-1} &= (v_1, v_2', v_3',\dots , v_{j-1}', v_j) \end{aligned}$$

Since $h$ maps lines parallel to the $i^{th}$ axis to lines parallel to the

$i^{th}$ axis for $i = 1, \ldots, j - 1$, $h$ maps points which differ only in their $i^{th}$ coordinate to points which differ only in their $i^{th}$ coordinate. Thus since $w^s$ and $w^{s+1}$ differ only in their $(s + 1)^{st}$ entry, for $s = 1, \ldots, j - 2$, $h(w^s)$ and $h(w^{s+1})$ differ only in their $(s + 1)^{st}$ entry, and so for some real number $z$

$$h(w^s) - h(w^{s+1}) = (0,\ldots,0,\underbrace{z,}_{s+1} 0,\ldots,0) \ ;$$

The right hand side is equal to

$$w^s + r^j + \tau(w^s) - w^{s+1} - r^j - \tau(w^{s+1})$$
$$= (0,\ldots,0,\underbrace{v_{s+1} - v'_{s+1}}_{s+1},0,\ldots,0) + \tau(w^s) - \tau(w^{s+1})$$

This means that $\tau_1(w^s) - \tau_1(w^{s+1}) = 0$ for $s = 1, \ldots, j - 2$, and so $\tau_1(w^1) = \tau_1(w^2) = \ldots = \tau_1(w^{j-1})$, and thus $\tau_1(v) = \tau_1(v')$. The argument is the same for each $i = 1, \ldots, j - 1$.

Thus we can write

$$\tau(v) = \sum_{i=1}^{j-1} \tau_i(v_i, v_j) r^i$$

where $\tau_i$ is a map from $R^2$ to $R$, and $v_i$ is the $i^{th}$ coordinate of $v$ in terms of the basis $\{r^1, \ldots, r^j\}$.

Consider the homotopy

$$H : [0,1] \times R^j \longrightarrow R^j$$

defined by

$$H_t(v) = v + r^j + t\ \tau(v) \ .$$

Note that $H_0(v) = v + r^j$, that is translation by $r^j$, and $H_1(v) = h(v)$. We claim that this homotopy of maps of $R^j$ projects to a homotopy of maps of $T^j$; that is, if $v$ is in $R^j$ and $M$ is in $Z^j$, then we must show that for all $t$ in $[0,1]$,

$$H_t(v) - H_t(v + M) \in Z^j$$

or equivalently,

$$v + r^j + t\ \tau(v) - v - M - r^j - t\ \tau(v+M) \in Z^j ,$$

or,

$$t(\tau(v) - \tau(v+M)) \in Z^j .$$

Since $H_1$ is just $h$, and $h$ projects to a homeomorphism $h$ of $T^j$, we know that $\tau(v) - \tau(v+M)$ is in $Z^j$; but by hypothesis, $P_{j-1} \cap Z^j = \{0\}$ , and so $\tau(v) - \tau(v+M)$ is in fact zero. Thus there is a map $\overline{H}$ such that

$$\begin{tikzcd}
{[0,1] \times R^j} \arrow[r, "H"] \arrow[d, "Id \times \pi"'] & R^j \arrow[d, "\pi"] \\
{[0,1] \times T^j} \arrow[r, "\overline{H}"] & T^j
\end{tikzcd}$$

commutes.

To show that $\overline{H}$ is an isotopy, we must show that for each $t$, $\overline{H}_t$ is a one-to-one map of $T^j$ onto $T^j$.

$T^j$ is partitioned by projections of translates of $P_{j-1}$. Since $\overline{h}$ is a homeomorphism of $T^j$ respecting this partition, $h$ induces a one-to-one onto map on the sets of the form $\pi(P_{j-1} + sr^j)$ for real numbers $s$. Notice that if

$$\overline{h}(\pi(P_{j-1}+ sr^j)) = \pi(P_{j-1} + s'r^j) ,$$

then since $h(v)$ and $H_t(v)$ differ only by an element of $P_{j-1}$,

$$\overline{H}_t(\pi(P_{j-1}+ sr^j)) = \pi(P_{j-1} + s'r^j)$$

and so $H_t$ also induces a one-to-one onto map on sets of the form $\pi(P_{j-1} + sr^j)$ . Thus in order to show that $H_t$ is a one-to-one map of $T^j$ onto $T^j$, it suffices to show that its restriction to a set of the form $\pi(P_{j-1} + sr^j)$ is one-to-one and onto.

Also, since the restriction of $\pi : R^j \longrightarrow T^j$ to any translate of $P_{j-1}$ is one-to-one, it suffices to show that $H_t$ is one-to-one and onto on any translate of $P_{j-1}$, that is (writing vectors in $R^j$ in terms of the basis $\{r^1, \ldots, r^j\}$) on any set of the form $\{(v_1, \ldots, v_j) \mid v_j \text{ is constant}\}$. In fact, since $H_t$ maps lines parallel to $Rr^s$ to such lines for any $s = 1, \ldots, j-1$, it is sufficient to show that the restriction of $H_t$ to some line

$$L = \{L(u) = (v_1, \ldots, u, \ldots, v_j) \mid v_i = \text{constant for } i \neq s,\ u \in R\}$$

is one-to-one and onto some other such line. However, on this line $L$,

$$H_1(L(u)) = L(u) + r^j + \tau_s(u, v_j) r^s = h(L(u)) .$$

Now by hypothesis $h$ is an orientation-preserving homeomorphism of $L$ onto $L + r^j = \{\ell + r^j \mid \ell \in L\}$; such homeomorphisms of a line correspond to monotonically increasing functions of $R$ onto $R$, hence the map

$$u \longrightarrow u + \tau_s(u, v_j)$$

is monotonically increasing from $R$ onto $R$. The map $H_t$ of $L$ to $L + r^j$ corresponds to the function

$$u \longrightarrow u + t\ \tau_s(u, v_j)$$

from $R$ to $R$. But monotonically increasing maps of $R$ onto $R$ are convex, and we can write

$$u + t\ \tau_s(u, v_j) = t(u + \tau_s(u, v_j)) + (1 - t)u$$

and so, since the identity function is one-to-one from $R$ onto $R$, $H_t$ is a homeomorphism of $L$ to $L + r^j$. Thus we have shown that $\bar{H}$ is an isotopy, and Lemma 3.2 is proved.

(In the course of the proof we showed that in fact the isotopy $\bar{H}_t$ from $\bar{h}$ to the projection of the translation by $r^j$ also has the following property; for any $t$ in $[0,1]$, $H_t$ maps translates of the line $Rr^s$ to such lines in an orientation-preserving way.)

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Let the action $\theta_j$ of $R$ on $V$ be defined by

$$\theta_j(t,v) = \varphi(i_j(t), v)$$

where $i_j$ for $j = 1,\ldots, n-1$ is as in Lemma 3.1. Note that $\theta_j$ defines an action of $R$ on $\overline{\mathcal{O}}_{\psi_k}(v_0)$ for any $k$ ; we will also call these actions $\theta_j$. The actions $\theta_j$ commute, and $\psi_j$ is equal to $\theta_1 \oplus \ldots \oplus \theta_j$.

Definition: We will say that an action $\psi$ of $R^m$ on $V$ is strongly topologically equivalent (s.t.e.) to an action $\psi'$ of $R^m$ on $V'$, where $\psi = \theta_1 \oplus \ldots \oplus \theta_m$ and $\psi' = \theta'_1 \oplus \ldots \oplus \theta'_m$ , if there exists a homeomorphism $f : V \to V'$ such that $f$ carries the orbits of the action $\theta_j$ of $R$ on $V$ to the orbits of the action $\theta'_j$ of $R$ on $V'$, for $j = 1,\ldots, n-1$. We also require that $f$ preserve the orientation on these lines given by the actions of $R$ . (Note that this implies that $f$ maps $\psi$-orbits to $\psi'$-orbits also, i.e. $\psi$ and $\psi'$ are topologically equivalent.)

The proof of the theorem is by induction; we will show that if $\{\psi_1,\ldots, \psi_{n-1}\}$ is as in (3.2), then $\varphi$ , which is equal to $\psi_{n-1}$, is s.t.e. to a linear action $\alpha_{n-1}$ of $R^{n-1}$ on $T^n$ by showing that for every $j = 1,\ldots, n-1$, the action

$$\psi_j : R^j \times \overline{\mathcal{O}}_{\psi_j}(v_0) \longrightarrow \overline{\mathcal{O}}_{\psi_j}(v_0)$$

is s.t.e. to a linear action $\alpha_j$ of $R^j$ on $T^{j+1}$ .

For $j$ equal to 2, $\bar{\mathcal{O}}_{\psi_{j-1}}(v_0)$ is a compact two-dimensional manifold. It is known ([8]) that there is a homeomorphism $f_2$ of $\bar{\mathcal{O}}_{\psi_1}(v_0)$ to $T^2$ and a linear action $\alpha_1$ of $R$ on $T^2$ given by

$$\alpha_1(r,\bar{x}) = \bar{x} + r\, r^1$$

such that $f_2$ carries $\psi_1$ orbits to $\alpha_1$ orbits and orientation is preserved, i.e. $\psi_1$ is s.t.e. to $\alpha_1$.

Now suppose that the action

$$\psi_{j-1} : R^{j-1} \times \bar{\mathcal{O}}_{\psi_{j-1}}(v_0) \longrightarrow \bar{\mathcal{O}}_{\psi_{j-1}}(v_0)$$

is s.t.e. to a linear action $\alpha_{j-1}$ of $R^{j-1}$ on $T^j$ by a homeomorphism $f_j : \bar{\mathcal{O}}_{\psi_{j-1}}(v_0) \longrightarrow T^j$. We will show that $\psi_j$ is s.t.e. to some linear action $\alpha_j$ of $R^j$ on $T^{j+1}$.

Since the actions $\theta_k$, $k = 1,\ldots, n-1$, commute, $\theta_j$ maps $\psi_{j-1}$ orbits to $\psi_{j-1}$ orbits; similarly for their closures. There is a real number $t$ such that $t \neq 0$ and

$$\theta_j(t,v) \in \bar{\mathcal{O}}_{\psi_{j-1}}(v) .$$

(If no such $t$ exists, then one has a contradiction to Theorem 2 of [6].) Since $V$ is compact, there is a minimal such $t$ ; we can assume that the minimal such $t$ is 1.

Let the map $h_j$ of $T^j$ to itself be defined by

$$h_j(\bar{x}) = f_j\theta_j(1, f_j^{-1}(\bar{x})),$$

that is,

$$\begin{tikzcd} \bar{\mathcal{O}}_{\psi_{j-1}}(v_0) \arrow[r, "\theta_j(1,-)"] \arrow[d, "f_j"'] & \bar{\mathcal{O}}_{\psi_{j-1}}(v_0) \arrow[d, "f_j"] \\ T^j \arrow[r, "h_j"] & T^j \end{tikzcd}$$

commutes. $h_j$ is a homeomorphism since it has a continuous inverse.

Let $([0,1] \times T^j)/\bar{h}_j^{-1}$ denote the space $[0,1] \times T^j$ with the points $(0,\bar{x})$ and $(1,\bar{h}_j^{-1}(\bar{x}))$ identified, with the quotient topology. We can define a homeomorphism

$$g : ([0,1] \times T^j)/\bar{h}_j^{-1} \longrightarrow \overline{\Theta}_{\psi_j}(v_0)$$

as follows:

$$g(t,\bar{x}) = \theta_j(t,\ f_j^{-1}(\bar{x})) \quad .$$

Note that

$$\begin{aligned} g(0,\bar{x}) &= \theta_j(0,\ f_j^{-1}(\bar{x})) = f_j^{-1}(\bar{x}) \\ &= \theta_j(1,\ f_j^{-1}\bar{h}_j^{-1}(\bar{x})) \\ &= g(1,\ \bar{h}_j^{-1}(\bar{x})), \end{aligned}$$

so $g$ is well defined. We wish to show that $([0,1] \times T^j)/\bar{h}_j^{-1}$ is homeomorphic to $T^{j+1}$ in a particular way. We begin by showing that $\bar{h}_j$ is isotopic to a translation. Consider a lifting $h_j$ of $\bar{h}_j$ ;

$$\begin{tikzcd} R^j \arrow[r, "h_j"] \arrow[d, "\pi"'] & R^j \arrow[d, "\pi"] \\ T^j \arrow[r, "\bar{h}_j"] & T^j \end{tikzcd}$$

Since $\psi_{j-1}$ is s.t.e. to some linear action $\alpha_{j-1}$ (and the actions $\theta_s$ of $R$ on $V$ commute), $h_j$ maps translates of the $(j - 1)$-plane $P_{j-1}$, which is equal to the span of the vectors $r^1, \ldots, r^{j-1}$ in $R^n$ , to translates of $P_{j-1}$. In fact, $h_j$ maps translates of the line $Rr^s$ to translates of $Rr^s$ for each $s = 1, \ldots, j - 1$ . Orientation of these lines is preserved by $h_j$, because the orientation of the lines $f_j^{-1}(Rr^s)$ is preserved by $\theta_j(1,-)$ (since $\theta_j$ commutes with the actions $\theta_s$, $s = 1, \ldots, j - 1$), and conjugation of $\theta_j(1,-)$ by $f_j$ does not change this property. Note also that $h_j(P_{j-1})$ is not equal to $P_{j-1}$, since $\varphi$ is free. (That is, if for some $w$ in $P_{j-1}$ and $x$ in $R^j$, $h_j(x) = x + w$, then $\theta_j(1, f_j^{-1}x)$ would equal $f_j^{-1}(x + w)$, but $f_j^{-1}(x)$ and $f_j^{-1}(x + w)$ lie on the same $\psi_{j-1}$ orbit.) Lemma 3.2 is thus applicable. There is an isotopy $\overline{H}_t$ of maps from $T^j$ to $T^j$ such that $\overline{H}_1$ is $\overline{h}_j$, and $\overline{H}_0(\overline{x}) = \pi(x + r^j)$, where $r^j$ is some fixed element of $R^j$.

Finally, we must define a homeomorphism $f_{j+1}$ from $\overline{\Theta}_{\psi_j}(v_0)$ to $T^{j+1}$. We have a homeomorphism

$$g^{-1} : \overline{\Theta}_{\psi_j}(v_0) \longrightarrow ([0,1] \times T^j)/\,\overline{h}_j^{\,-1}$$

where the action on the image of $g^{-1}$ induced by $\psi_j$ is s.t.e. to the action defined by $\overline{\alpha}_{j-1}$ on $T^j$ and addition in the $[0,1]$ direction. Let the map $S$ of $R^j$ to $R^j$ be defined by $S(x) = x - r^j$. We now define a map

$$K : ([0,1] \times T^j) / \,\overline{h}_j^{\,-1} \longrightarrow ([0,1] \times T^j) / S$$

by

$$K(t, \overline{x}) = (t, \overline{H}_t(\overline{x})) .$$

Note that

$$\begin{aligned} K(0,\overline{x}) &= (0, \overline{H}_0(\overline{x})) = (0, S(\overline{x})) = (1, \overline{x}) \\ &= (1, \overline{h}_j\overline{h}_j^{\,-1}(\overline{x})) = (1, \overline{H}_1(\overline{h}_j^{\,-1}\,\overline{x})) \\ &= K(1, \overline{h}_j^{\,-1}(\overline{x})) \quad . \end{aligned}$$

Thus $K$ is well defined and continuous; since it has a continuous inverse, $K$ is a homeomorphism.

The map $Kg^{-1}$ from $\overline{\Theta}_{\psi_j}(v_0)$ to $([0,1] \times T^j) / S$ induces an action on the quotient space from the action of $\psi_j$ on $\overline{\Theta}_{\psi_j}(v_0)$ ; the induced action will be s.t.e. to the action given by addition in the $[0,1]$ direction, and again by $\alpha_{j-1}$ on $T^j$, since $H_t$

respects the lines $\pi(Rr^s)$ in $T^j$.

Finally, we can define a homeomorphism L,

$$L : ([0,1] \times T^j) / S \longrightarrow ([0,1] \times T^j) / id = T^{j+1}$$

given by

$$L(t, \bar{x}) = (t, \bar{x} + t\bar{r}^j) \quad .$$

Note that

$$L(0, \bar{x}) = (0, \bar{x}) = (1, \bar{x}) = (1, \bar{x} - \bar{r}^j + \bar{r}^j)$$
$$= L(1, \bar{x} - \bar{r}^j) = L(1, S(\bar{x})) \quad ,$$

so L is well-defined. Here the induced action will be s.t.e. to the action $\alpha_j$ given by $\alpha_{j-1}$ on $T^j$ and the linear action in the $(1, r^j)$ direction.

Thus we have the composite map $LKg^{-1}$ from $\bar{\mathfrak{S}}_{\psi_j}(v_0)$ to $T^{j+1}$ such that the orbits of the action $\psi_j$ of $R^j$ on $\bar{\mathfrak{S}}_{\psi_j}(v_0)$ correspond (in the required way) to the orbits of the linear action $\alpha_j$ of $R^j$ on $T^{j+1}$, and so $\psi_j$ is s.t.e. to $\alpha_j$ . So the induction step is completed, and Theorem 3.1 is proved.

## PART IV. LOCALLY FREE ACTIONS

We include a theorem indicating how the previous results give considerable information about locally free actions with no compact orbits.

**THEOREM 4.1:** *Suppose that* $\varphi$ *is a locally free action of* $R^{n-1}$ *on a compact connected* $n$*-manifold with no compact orbits. Then*

i. *The isotropy group* $I_p$ *of a point* $p$ *in* $V$ *is a constant group* $I$ , *isomorphic to* $Z^k$ *for some* $k < n-1$, *and*

ii. *There is a free action* $\psi$ *of* $T^k \times R^{n-k-1}$ *on* $V$ *such that the following diagram commutes*

$$\begin{tikzcd}
R^{n-1} \times V \arrow[r, "\varphi"] \arrow[d] & V \\
(T^k \times R^{n-k-1}) \times V \arrow[ur, "\psi"'] &
\end{tikzcd}$$

*(where the map from* $R^{n-1}$ *to* $T^k \times R^{n-k-1}$ *is given by the projection of* $R^k$ *to* $T^k$ *on the first* $k$ *factors of* $R^{n-1}$*).*

Proof: To prove (i), observe that the isotropy group is constant on each orbit of $\varphi$ ; since each orbit is dense, we need to show that if a sequence $v_i$ in the orbit of some point $v_0$ is such that $v_i \to v$ in $V$ , then the isotropy group of $v_i$, $I_{v_i}$ , is a subset of $I_v$. But by continuity of $\varphi$ , if $r$ is in $I_{v_i}$ , then $\varphi(r,v_i) = v_i$ converges to $v$, and so $\varphi(r,v) = v$, that is, $r$ is in $I_v$.

For (ii), consider $R^{n-1}/ I$ , where $I$ is the (constant) isotropy group. $R^{n-1}/ I$ is isomorphic to $T^k \times R^{n-k-1}$ for some $k$, and we can find an isomorphic image of $R^{n-1}$ such that the projection map $\pi$ is as above. $\psi$ is defined as $\psi(t,s,v) = \varphi(\pi^{-1}(t,s), v)$ , which is well-defined. Suppose that $\psi(t,s,v) = \psi(t',s',v)$. Then

$$\varphi(\pi^{-1}(t - t', s - s'), v) = v \quad ,$$

and so $\pi^{-1}(t - t', s - s')$ is contained in $I$ ; thus in $T^k \times R^{n-k-1}$ we have that $(t - t', s - s') = [0]$ . Thus $t - t' = [0]$ and $s - s' = 0$, and so $\psi$ is free and Theorem 4.1 is proved.

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# AUTOBIOGRAPHICAL STATEMENT

I was born in Cambridge, England on August 31, 1944. My family first came to the United States in 1949; we moved frequently, living in the Boston area, Washington D.C., returning to England for two years, then Long Island, New York City, and again the Boston area. I went to an assortment of schools, both public and private, day and boarding. I was in high school at the Rudolf Steiner School in New York, and the Putney School in Putney, Vermont. I attended Radcliffe College for two years and received my B.A. with a major in Mathematics from Barnard College in June, 1965. In that year I began study at the Graduate Division of the City University of New York.