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PRESENTATIONS OF SOLVMANIFOLDS

by

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CONTENTS

CHAPTER I

Introduction

This paper is a study of the presentations of compact solvmanifolds. A solvmanifold is the homogeneous space of a connected solvable Lie group. Let the expression *Solvable Lie group* mean a simply connected, connected solvable Lie group. A coset representation $\underline{S}/\underline{C}$ of a solvmanifold M is called a *presentation* if $\underline{S}$ is a Solvable Lie group, $\underline{C}$ contains no non-trivial subgroup normal in $\underline{S}$, and, if $\underline{C}_0$ denotes the identity component of $\underline{C}$, $\underline{C}_0$ is contained in the commutator subgroup of $\underline{S}$.

The interest in the presentations of compact solvmanifolds arises from the question of when does a Solvable Liè group contain a discrete uniform subgroup. By the work of H. C. Wang [9] and L. Auslander [1] not every fundamental group of a compact solvmanifold can be imbedded as a discrete uniform subgroup of a Solvable Lie group. Hence, there exist compact solvmanifolds M that can only be represented as homogeneous spaces of Solvable Lie groups $\underline{S}$ with the dimension of $\underline{S}$ strictly greater than that of M.

If we call the dimension of $\underline{S}$, the dimension of the presentation $\underline{S}/\underline{C}$ of the solvmanifold M , then the main theorem of the paper is that: unless it is the circle, M has presentations of arbitrarily high dimension.

In the last part of the paper we also obtain a characterization of all presentations of the torus of dimension greater than 1 .

CHAPTER II

Preliminaries

1. Conventions and notation:

A Lie group $\underline{G}$ will be denoted by an italic capital. When $\underline{G}$ is connected, the corresponding Lie algebra G, will be denoted by the some unitalicized capital letter. All Lie groups and Lie algebras will be real.

The expression Solvable (resp. Nilpotent) Lie group will mean a simply connected, connected, solvable (resp. nilpotent) Lie group. If $\underline{G}$ is a Lie group then we write $\underline{G}_0$ for its identity component. A solvable Lie algebra will be finite dimensional. The expression one-parameter subgroup will be **abbreviated** by 1-PS.

We will use the following symbols:

$\mathbb{R}$ = the real numbers;

$\mathbb{Z}$ = the integers;

$\mathbb{C}$ = the complex numbers;

$\mathbb{T}^n$ = the n-dimensional torus.

2. A note on the proofs:

In our proofs we will freely and implicitly make use of the equivalence of the category of simply connected

connected Lie groups withe the category of finite dimensional Lie algebras.

For example, we will extend our notational convention thus: if G is a finite dimensional Lie algebra, we write $\underline{G}$ for the corresponding simply connected, connected Lie group.

3. Presentations of solvmanifolds:

Definition 1: A solvmanifold (resp. nilmanifold) is a homogeneous $\underline{G}$-manifold, where $\underline{G}$ is a solvable (resp. nilpotent) connected Lie group.

Proposition 1(Mostow [7]): A solvmanifold M can be represented as a coser space $\underline{S}/\underline{C}$ where

(i) $\underline{S}$ is a Solvable Lie group;

(ii) $\underline{C}$ contains no non-trivial connected subgroup that is normal in $\underline{S}$;

(iii) $\underline{C}_0 \subset [\underline{S},\underline{S}]$.

Definition 2: A pair $(\underline{S},\underline{C})$ satisfying (i), (ii) and (iii) with M homeomorphic to $\underline{S}/\underline{C}$ will be called a presentation of the solvmanifold M. The dimension of the presentation $(\underline{S},\underline{C})$ is the dimension of $\underline{S}$. We will call the group $\underline{C}$ the kernel of the presentation $(\underline{S},\underline{C})$.

4. The semi-simple splitting:

The concept of the semi-simple splitting of a Lie group was first introduced by L. Auslander (see L. Auslander and L. W. Green [4], L. Auslander and C. C. Moore [5], and L. Auslander and J. Brezin [3]). We quote the following material to introduce the necessary terminology. For the relevant properties we shall refer to the sources mentioned above.

Theorem 1(L. Auslander): Let $\underline{S}$ be a Solvable Lie group. Then there exists a unique Nilpotent Lie group, $\underline{N}_s$, and a unique connected abelian semi-simple group, $\underline{T}_s$, of automorphisms of $\underline{N}_s$ such that

(i) $\underline{S} \subset \underline{T}_s \cdot \underline{N}_s$.

(ii) Let p: $\underline{T}_s \cdot \underline{N}_s \longrightarrow \underline{N}_s$, be the projection. Then p restricted to $\underline{S}$ is a homeomorphism of $\underline{S}$ onto $\underline{N}_s$.

(iii) $\underline{S}$ and $\underline{N}_s$ generate $\underline{T}_s \cdot \underline{N}_s$.

Definition 1(L. Auslander): Given a Solvable Lie group $\underline{S}$, we call $\underline{T}_s \cdot \underline{N}_s$ of Theorem 1 the semi-simple splitting of $\underline{S}$ and denote it by $\underline{S}_s$. $\underline{T}_s$ is called the semi-simple part of $\underline{S}$ and $\underline{S}_s$.

CHAPTER III

Existence of Presentations of Solvmanifolds

1. Introduction:

For a compact solvmanifold M, all presentations with a discrete kernel have the same dimension (see L. Auslander and M. Auslander [2], pp. 934-935). It is the purpose of this chapter to show that M, unless it is a circle, has presentations of arbitrarily high dimension.

By G. D. Mostow [7], H. C. Wang [9], and L. Auslander [1], we have that compact solvmanifolds are characterized by algebraic properties of their fundamental groups. So, in keeping with the spirit of this work, we shall construct presentations of M from its fundamental group.

The method of proof is a modification of the construction used by L. Auslander in [1] to show that every fundamental group of a solvmanifold is the fundamental group of a compact solvmanifold.

In this chapter we will restrict our discussion to solvmanifolds whose fundamental groups are not nilpotent. This is necessary because of the technical device in section 3. However, in chapter IV we shall fill this gap by using slightly different methods.

Let G be a fundamental group of a solvmanifold. Then by H. C. Wang [9], G can be imbedded as a discrete uniform subgroup of a simply connected, solvable Lie group with a finite number of components. By L. Auslander [1], there is a compact solvmanifold M with $\pi_1(M) = G$. This is proved by constructing a presentation of M from an "enlargement" of the group given by Wang's result. We will show that this "enlargement" can be modified so as to produce a presentation of dimension greater than any given integer.

It may happen that G can be imbedded as a discrete uniform subgroup of a Solvable Lie group. Then a further modification of the construction in [1] is needed.

2. Free Lie algebras:

Let X be a set. Then we have the free Lie algebra on X, L(X). L(X) is a graded algebra and we write $L^n(X)$ for the homogeneous component consisting of elements of length n.

When card(X) is finite, $L^n(X)$ is a finite dimensional vector space for all n. We have the important property that there is an injection $\underline{\underline{GL}}(L^1(X))$ into $\underline{\underline{Aut}}(L(X))$ of non-singular linear maps in $L^1(X)$ to automorphisms of L(X). Furthermore if L is a Lie algebra having X for a basis then L is the image of L(X).

3. An automorphism:

Lemma 1: Let $\underline{S}$ be a Solvable Lie group, $\underline{N}$ being its nil-radical. Suppose that $\underline{S} \neq \underline{N}$. Then there exists a non-trivial semi-simple automorphism f of $\underline{S}$ inducing the identity on $\underline{S}/\underline{N}$ such that f is either positive (all proper values are positive) or f is orthogonal.

Proof: The semi-simple part of $\underline{S}$, $\underline{T}_s$, is an abelian groups of semi-simple automorphisms of $\underline{S}$ inducing the identity on $\underline{S}/\underline{N}$ [5]. If $\underline{T}_s$ were trivial then $\underline{S}$ would be nilpotent.

To finish the proof we remark that the group of automorphisms of $\underline{S}$ is algebraic and so, contains the positive and orthogonal parts of its elements. And since the projection of $\underline{S}$ onto $\underline{S}/\underline{N}$ is rational, if f induces the identity on $\underline{S}/\underline{N}$, then so do its positive and orthogonal parts.

QED.

4. A construction:

The two theorems in this section are very similar. They are used in section 5, however, to apply to the very distinct cases of Wang's result; so we have kept them separate.

Theorem 2: Let F be a non-trivial, finitely generated, abelian group of orthogonal automorphisms of the solvable Lie algebra S. Let N be the nil-radical of S. And suppose F induces the identity on S/N. Then for any $n \in \mathbb{Z}$ there is a 5-tuple $(\bar{S}, \bar{\pi}, \bar{F}, \bar{T}, n)$ such that

(i) $\bar{S}$ is a solvable Lie algebra;

(ii) $\bar{\pi}: \bar{S} \longrightarrow S$ is an epimorphism;

(iii) $\bar{T}$ is a torus group of automorphisms of $\bar{S}$, inducing the identity on $\bar{S}/\bar{N}$ where $\bar{N}$ is the nil-radical of $\bar{S}$;

(iv) $\bar{F}$ is a subgroup of $\bar{T}$, isomorphic to F, and induces F on S through $\bar{\pi}$;

(v) the dimension of ker $\bar{\pi} > n$;

(vi) if $f \in F$ is of finite order, then there is a closed 1-PS of $\bar{T}$, $\bar{f}(t)$, such that $\bar{f}(1)$ induces f on S and ker $\bar{\pi}$ contains no non-null $\bar{f}(t)$-invariant subspaces.

Proof: Since F is semi-simple and abelian and induces $1_{S/N}$ on S/N, there is a complement K of N in S such that $F/_K = 1_K$.

We now decompose $F/_N$: since F is orthogonal and abelian N is the direct sum of 1-dimensional and 2-dimensional F-invariant subspaces.

Let,

U_1 be the sum of the 1-dimensional F-invariant subspaces on which F is the identity;

U_{-1} be the sum of the remaining 1-dimensional F-invariant subspaces.

Now define the following equivalence relation on the 2-dimensional F-invariant subspaces, D_i, $i = 1, \ldots, g$. Let $D_i \sim D_j$ if for all $f \in F$ either

(a) f induces the same positive angle of rotation in D_i and D_j

or

(b) f induces the same negative angle of rotation in D_i and D_j.

Partition the interval $[1, g] \subset \mathbb{Z}$ into J_k, $k = 1, \ldots, s$, so that $D_i \sim D_j$ if and only if $i \in J_k$ and $j \in J_k$. Reorient the orthonormal bases in equivalent D_i, if necessary, so that in the above only condition (a) holds.

Let $W_k = \bigoplus_{i \in J_k} D_i$.

Let,

$x_1, \ldots, x_p$ be a basis of K,
$x_1^*, \ldots, x_q^*$ be a basis of U_1,
$y_1, \ldots, y_r$ be a basis of U_{-1} and
$(z_i^k, \bar{z}_i^k)$ a properly oriented orthonormal basis of D_i for $i \in J_k$.

The remainder of the proof falls into two parts: that the decomposition above is compatible with the action of K, $ad_S(K)$ and that the proper enlargement can be constructed.

<u>Lemma</u> 1: $ad_S(K)$ has the following properties:

(i) $ad_S(K)(K) \subset U_1$;

(ii) $ad_S(K)(U_1) \subset U_1$;

(iii) $ad_S(K)(U_{-1}) \subset U_{-1}$;

(iv) $ad_S(K)(W_k) \subset W_k$ for $k = 1, \ldots, s$;

(v) if $x \in K$,

$$ad_S(x)(D_i) = 0 \text{ or } D_j \text{ for } i \in J_k \text{ and } j \in J_k ;$$

(vi) if L_1 and L_2 are linear transformations of D_i onto D_j resulting from the composition of elements in $ad_S(K)$, then they induce the same orientation on D_j from D_i.

Proof:

(i) Let $f \in F$. Then, for $\bar{x} \in K$,

$$(f \circ ad_S(x))(\bar{x}) = f([x, \bar{x}]) = [x, \bar{x}].$$

(ii) For $u \in U_1$ we have $f([x, u]) = [x, u]$.

(iii) If $u \in U_{-1}$ there is an $f \in F$ such that $f(u) = -u$. Hence

$$f([x, u]) = [x, f(u)]$$
$$= -[x, u].$$

(iv) and (v) Let $(z_i^k, \bar{z}_i^k)$ be the previously chosen basis of D_i. Then there is an $f \in F$ such that

$$f(z_i^k) = \cos\theta\, z_i^k + \sin\theta\, \bar{z}_i^k ,$$

$$f(\bar{z}_i^k) = -\sin\theta\, z_i^k + \cos\theta\, \bar{z}_i^k ,$$

with $0 \neq |\cos\theta| \neq 1$. But

$$f([x, z_i^k]) = \cos\theta\, [x, z_i^k] + \sin\theta\, [x, \bar{z}_i^k] ,$$

$$f([x, \bar{z}_i^k]) = -\sin\theta\, [x, z_i^k] + \cos\theta\, [x, \bar{z}_i^k] .$$

Since f is linear, if $[x, z_i^k] = 0$, then $\sin\theta [x, \bar{z}_i^k] = 0$. But $\sin\theta \neq 0$. Hence $[x, \bar{z}_i^k] = 0$.

A routine computation shows that $ad(x)(z_i^k)$ and

$ad(x)(\bar{z}_i^k)$ are independent if they are not 0. Hence, $ad(x)(D_i)$ is a 2-dimensional invariant subspace or the null space. Since f is represented by a proper rotation relative to the basis $([x, z_i^k], [x, \bar{z}_i^k])$,

$$ad_S(x)(D_i) = D_j \quad \text{for} \quad i \in J_k \quad \text{and} \quad j \in J_k .$$

This also shows that $([x, z_i^k], [x, \bar{z}_i^k])$ is an orthonormal basis compatibly oriented with $(z_j^k, \bar{z}_j^k)$.

QED.

Define the 1-PS, T_k, of $\underline{\underline{GL}}(W_k)$ by

$$T_k(t)(z_i^k) = \cos 2\pi t \; z_i^k + \sin 2\pi t \; \bar{z}_i^k \;;$$

$$T_k(t)(\bar{z}_i^k) = -\sin 2\pi t \; z_i^k + \cos 2\pi t \; \bar{z}_i^k \;.$$

Lemma 2: $ad_S(x)/_{W_k}$ and $T_k(t)$ commute for all $x \in K$, and $t \in \mathbb{R}$.

Proof: If $ad(x)(D_i) = 0$, the lemma is clear. If $ad(x)(D_i) = D_j$ then by lemma 1, $([x, z_i^k], [x, \bar{z}_i^k])$ is an orthonormal basis compatibly oriented with respect to $(z_j^k, \bar{z}_j^k)$. Therefore $T_k(t)$ induces the same linear transformation relative to the basis $([x, z_i^k], [x, \bar{z}_i^k])$ of D_j as it does relative to the basis $(z_i^k, \bar{z}_i^k)$ of D_i.

QED.

The proof now consists of two cases according as $U_{-1} = 0$ or $\neq 0$.

Case I: $U_{-1} \neq 0$.

Let V^j be a 2m-dimensional vector snaes and suppose $\theta_j(t)$ is a compact 1-PS of $GL(V^j)$. Suppose further that V^j has no 1-dimensional $\theta_j(t)$-invariant subspaces and that the parameter is chosen so that $\theta_j(1) = 1_{V^j}$ and $\theta_j(\frac{1}{2}) = -1_{V^j}$. Let $e_1^j, \ldots, e_{2m}^j$ be a basis of V^j such that $e_2^j, \ldots, e_{2m}^j$ spans a subspace containing no non-null $\theta_j(t)$-invariant subspace. Define the monomorphism

$$\varphi : U_{-1} \longrightarrow \bigoplus_{j=1}^{r} V^j = V \text{ by}$$

$$y_j \longmapsto e_1^j, \quad j=1, \ldots, r.$$

For each j, extend $\theta_j(t)$ to a 1-PS of $GL(V)$ by requiring that it act as the identity off V^j.

Let, $\theta: \mathbb{R}^r \longrightarrow \underline{\underline{GL}}(V)$ by defining

$$\theta(t_1, \ldots, t_r) = \theta_1(t_1)\cdot \ldots \cdot\theta_r(t_r) .$$

Let $\theta(\mathbb{R}^r) = T$. Let $E = \{x_i, x_j^*, e_k^{\ell}, z_\lambda^\mu, \bar{z}_\lambda^\mu$

$i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$; $\ell = 1, 2, \ldots, s$;

$k = 1, 2, \ldots, 2m$; $\mu = 1, 2, \ldots, r$; $\lambda \in J_\mu$.

Consider the free Lie algebra on E, $L(E)$. Extend T, T_i, to non-singular linear transformations on $L^1(E)$ by requiring that they act as the identity on the other basis elements. Then extend them to automorphism groups of $L(E)$, T^* and T_i^*. Let $T^{\#}$ be the group generated by T^* and T_i^*, $i = 1, 2, \ldots, s$. Define the projection $\pi^{\#}: L(E) \longrightarrow S$ by sending corresponding symbols to each other and letting

$$\pi^{\#}(e_k^{\ell}) = \begin{cases} y_\ell & \text{when } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

<u>Lemma</u> 3: There is a subgroup, $F^{\#}$, of $T^{\#}$ inducing F on S.

<u>Proof</u>: Define the following maps on F :

(a) $ch_j : F \longrightarrow \mathbb{Z}/2\mathbb{Z}$ by $f(y_j) = ch_j(f)y_j$;

(b) $\overline{ch}_j(f) = 1$ if $ch_j(f) = 1$ and

$\overline{ch}_j(f) = \frac{1}{2}$ if $ch_j(f) = -1$;

(c) $CH_k : F \longrightarrow \mathbb{R}/2\mathbb{Z}$ where $CH_k(f)$ is the angle of the rotation (mod 2π) induced by f in W_k .

Let $f \in F$ and define $f^{\#} \in T^{\#}$ by

$$f^{\#} = T^*(\overline{ch}_1(f),\ldots,\overline{ch}_r(f))\cdot$$

$$\cdot T_1^*(CH_1(f))\cdot \ldots \cdot T_s^*(CH_s(f)) .$$

Define the monomorphism $F \longrightarrow T^{\#}$ by $f \longmapsto f^{\#}$ and let $F^{\#}$ be the image of F . It remains only to show that $\ker \pi^{\#}$ is $F^{\#}$-invariant. But this is clear since $F^{\#}$ is a scalar on V .

QED.

Define the ideal I in $L(E)$ generated by the following words:

1) if N is k^*-step nilpotent, then every word of length k^*+1 not containing any x_i ;

2) if in S ,

$$[x_i, x_j] = \sum_k a_{ijk} x_k^* ,$$

then

$$[x_i, x_j] - \sum a_{ijk} x_k^* ;$$

3) if in S ,

$$[x_i, x_j^*] = \sum a_{ijk}^* x_k^* ,$$

then

$$[x_i, x_j^*] - \sum a_{ijk}^* x_k^* ;$$

4) if in S ,

$$[x_i, y_j] = \sum b_{ijk} y_k ,$$

then

$$[x_i, e_{\pm}^{j}] - \sum b_{ijk} e_{\pm}^{k}$$

5) if in S ,

$$[x_\ell, z_i^j] = c(\ell, i) z^j_{d(\ell, i)} + \bar{c}(\ell, i)\bar{z}^j_{d(\ell, i)}$$

and

$$[x_\ell, \bar{z}_i^j] = -\bar{c}(\ell, i) z^j_{d(\ell, i)} + c(\ell, i)\bar{z}^j_{d(\ell, i)} ,$$

then

$$[x_\ell, z_i^j] - (c(\ell, i) z^j_{d(\ell, i)} + \bar{c}(\ell, i)\bar{z}^j_{d(\ell, j)})$$

and

$$[x_\ell, \bar{z}_i^j] - (-\bar{c}(\ell, i) z^j_{d(\ell, i)} + c(\ell, i)\bar{z}^j_{d(\ell, i)}) .$$

<u>We claim</u> I <u>has the following properties</u>:

<u>Claim</u> A: I is $T^\#$-invariant.

Since $T^\#$ consists of automorphisms of $L(E)$ we need only check that $T^\#$ takes the generators of I into I. Since $T^\#$ is abelian, we can check for T^* and T_i^* separately. For T_i^* the relations 1),2),3) and

4) follow by definition; lemma 2 implies 5). For T^* 1), 2), 3) and 5) follow by definition, so it remains to show that T^* takes generators of the form 4) into I.

Let $t^* \in T^*$ and suppose $t^*(e_i^j) = \sum M_{ik} e_k^j$. Then,

$$t^*([x_i, e_1^j]) = [x_i, t^*(e_1^j)]$$
$$= [x_i, \sum_A M_{1A} e_A^j]$$
$$= \sum_A M_{1A} [x_i, e_A^j].$$

On the otherhand,

$$t^*(\sum b_{ijk} e_1^k) = \sum b_{ijk} t^*(e_1^k)$$
$$= \sum b_{ijk} \sum M_{1A} e_A^k.$$

Hence,

$$t^*([x_i, e_1^j] - \sum b_{ijk} e_1^k) =$$
$$= \sum M_{1A}([x_i, e_A^j] - \sum b_{ijk} e_A^k),$$

which is in I.

<u>Claim</u> B: $\pi^{\#}(I) = 0$.

Again we need only check on generators. For 1) it follows since any word not containg an x_i is carried by $\pi^{\#}$ into N . For 2), 3), and 5), it is clear. Since for $j \neq 1$, $\pi^{\#}(e^i_j) = 0$, we have it for 4).

Claim C: I is of finite codimension in L(E) .

Let $\bar{S} = L(E)/I$. $[\bar{S},\bar{S}]$ consists of the images under the canonical projection of words containing no x_i . Hence $[\bar{S},\bar{S}]$ is a finite dinensional k*-step nilpotent Lie algebra.

$\bar{S}/[\bar{S},\bar{S}]$ is the image of $L(E)/[L(E),L(E)]$ and hence is finite dimensional.

From claims A, B, and C, it follows that $\bar{S}$ is a finite dimensional solvable Lie algebra with $T^{\#}$ inducing a torus group $\bar{T}$ of automorphisms of $\bar{S}$. Write $\bar{\pi}$ for the induced epimorphism $\bar{S} \longrightarrow S$.

We are almost done, but it may happen that $\ker \bar{\pi}$ contains non-null $\bar{T}$-invariant subspaces. So we propose to factor out a maximal $\bar{T}$-invariant subspace of $\ker \bar{\pi}$ and in order to do this we need the following lemmas.

Lemma 4: (with the above notation) The maximal $\bar{T}$-invariant subspace of ker $\bar{\pi}$ has codimension (in ker $\bar{\pi}$) of at most $r(2m-1)$.

Proof: Let V be the subspace of $L^1(E)$ spanned by the e_i^j . Then by construction V is $T^\#$-invariant. Since $\ker\pi^\# \cap V$ is the subspace spanned by e_i^j for $i \geq 2$, by construction $\ker \pi^\# \cap V$ contains no non-null $T^\#$-invariant subspaces. But, since V is $T^\#$-invariant, any $T^\#$-invariant subspace contained in $\ker \pi^\#$ has codimension $\geq r(2m-1)$.

Consider the projection $p: L(E) \longrightarrow L(E)/I$. Since I is contained in $\ker \pi^\#$, p induces a map of $\ker \pi^\#$ onto $\ker \bar{\pi}$. But I is $T^\#$-invariant and so, any $\bar{T}$-invariant subspace of $\ker \bar{\pi}$ is pulled back by p to a $T^\#$-invariant subspace of $\ker \pi^\#$ containing I . This proves the lemma.

QED.

Lemma 5: The maximal $\bar{T}$-invariant subspace contained in in $\ker \bar{\pi}$ is an ideal in $\bar{S}$.

<u>Proof</u>: Let Q be the maximal $\overline{T}$-invariant subspace of $\ker \bar{\pi}$. Let $I(Q)$ be the ideal generated by Q in $\overline{S}$. Since $\ker \bar{\pi}$ is an ideal, $I(Q)$ is contained in $\ker \bar{\pi}$. But since $\overline{T}$ consists of automorphisms of $\overline{S}$, $I(Q)$ is $\overline{T}$-invariant. Hence $I(Q) = Q$, since Q is maximal.

QED.

To finish off Case I of the theorem, we factor out by Q and relable appropriately.

<u>Case</u> II: $U_{-1} = 0$.

Suppose without loss of generality that

$$W_1 = \bigoplus_{i=1}^{n(1)} D_i \neq 0 .$$

Let V^j, $\theta_j(t)$, $e_1^j, \ldots, e_{2m}^j$ be as in Case I.
Consider $V^j \otimes \mathbb{C}$ with the $\mathbb{R}$-basis $e_1^j \otimes 1, \ldots, e_{2m}^j \otimes 1$, $e_1^j \otimes i, \ldots, e_{2m}^j \otimes i$. Relable these elements $e_1^j, \ldots, e_{2m}^j$, $\bar{e}_1^j, \ldots, \bar{e}_{2m}^j$, respectively. Let,

$$R_j(t, \bar{t}) = \exp(2\pi i t)(\theta_j(\bar{t}) \otimes 1) .$$

Now let,

$$V = \bigoplus_{j=1}^{n(1)} V^j \otimes \mathbb{C}$$

and inject W_1 into V by

$$z_j^1 \longmapsto e_1^j \ ;$$

$$\bar{z}_j^1 \longmapsto \bar{e}_1^j \ .$$

Extend R_j to a transformation in $\underline{\underline{GL}}(V)$ by requiring it be the identity off $V^j \otimes \mathbb{C}$. And let

$$\theta\colon \ \mathbb{R}^{2n(1)} \longrightarrow \underline{\underline{GL}}(V)$$

be defined by

$$\theta(t_1, \ldots, t_{n(1)}, \bar{t}_1, \ldots, \bar{t}_{n(1)}) =$$

$$= R_1(t_1, \bar{t}_1) \cdot \ \ldots \ \cdot R_{n(1)}(t_{n(1)}, \bar{t}_{n(1)}) \ .$$

Let $T_1 = \theta(\mathbb{R}^{2n(1)})$ and define T_i for $i > 1$ as in Case I.

Let $E = \{x_i, x_j^*, z_{r^*}^{m^*}, \bar{z}_{r^*}^{m^*}, e_{s^*}^{n^*}, \bar{e}_{s^*}^{n^*} | i = 1, 2, \ldots, p;$ $j = 1, 2, \ldots, q;$ $(m^*, r^*) = (2, 1), \ldots, (2, n(2)), \ldots, (s, 1), \ldots, (s, n(s));$ $n^* = 1, 2, \ldots, n(1);$ $s^* = 1, 2, \ldots, 2m\}$.

Consider the free Lie algebra on E, $L(E)$. Extend T_i to groups of non-singular linear transformations of $L^1(E)$ by requiring that they act as the identity on the other basis elements. Then extend them to subgroups of $\underline{\underline{\text{Aut}}}(L(E))$, T_i^*. Let $T^{\#}$ be the group generated by the T_i^*. Define the epimorphism

$\pi^{\#}: L(E) \longrightarrow S$, by sending corresponding symbols to each other and letting

$$\pi^{\#}(e_{s^*}^{n^*}) = \begin{cases} z_{n^*}^{1} & \text{for } s^* = 1 \\ 0 & \text{otherwise;} \end{cases}$$

$$\pi^{\#}(\bar{e}_{s^*}^{n^*}) = \begin{cases} \bar{z}_{n^*}^{1} & \text{for } s^* = 1 \\ 0 & \text{otherwise.} \end{cases}$$

<u>Lemma</u> 3': There is a subgroup, $F^{\#}$, of $T^{\#}$ inducing F on S.

<u>Proof</u>: Let CH_i be defined as in lemma 3. Then define $F \longrightarrow T^{\#}$ by

$$f^{\#} = T_1^*(CH_1(f), \ldots, CH_1(f), 1, \ldots, 1) \cdot$$
$$\cdot T_2^*(CH_2(f)) \cdot \ldots \cdot T_s^*(CH_s(f)) .$$

The rest of the proof is the same as in lemma 3.

QED.

Define the ideal I as in Case I except in 5) delete the case $j = 1$ and substitute for 4) the following:

4') if in S,

$$[x_{\bar{i}}, z_i^1] = c(\bar{i}, i) z^1_{d(\bar{i}, i)} + \bar{c}(\bar{i}, i) \bar{z}^1_{d(\bar{i}, i)}$$

and

$$[x_{\bar{i}}, \bar{z}_i^1] = -\bar{c}(\bar{i}, i) z^1_{d(\bar{i}, i)} + c(\bar{i}, i) \bar{z}^1_{d(\bar{i},)} ,$$

then

$$[x_{\underline{1}},\ e^i_j] - (c(\underline{1},\ i)e_j^{d(\underline{1},\ i)} + \bar{c}(\underline{1},\ i)\bar{e}_j^{d(\underline{1},\ i)})$$

and

$$[x_{\underline{1}},\ \bar{e}^i_j] - (-\bar{c}(\underline{1},\ i)e_j^{d(\underline{1},i)} + c(\underline{1},\ i)\bar{e}_j^{d(\underline{1},\ i)}) .$$

We now must check whether I has the desired properties in this case. The only claim where there is any new problem is claim A where we have to check whether I is invariant under T_1^*. This is simply a computation which for the sake of completeness we will show.

<u>Lemma</u> 6: T_1 acting on V can be written

$$T_1(e^i_j) = \sum A_{jk}e^i_k + \sum B_{jk}\bar{e}^i_k ,$$

$$T_1(\bar{e}^i_j) = - \sum B_{jk}e^i_k + \sum A_{jk}\bar{e}^i_k .$$

<u>Proof</u>: Suppose on V^j , $\theta_j(e^j_i) = \sum M_{ik}e^j_k$. Then,

$$R_j(t, \bar{t})(e_i^j \otimes 1) = \exp(2\pi i t)(\sum M_{ik}(\bar{t}) e_k^j \otimes 1)$$

$$= \sum (\cos 2\pi t \, M_{ik}(\bar{t}) e_k^j \otimes 1 +$$

$$+ \sin 2\pi t \, M_{ik}(\bar{t}) e_k^j \otimes i)$$

$$= \sum \cos 2\pi t \, M_{ik}(\bar{t}) e_k^j +$$

$$+ \sum \sin 2\pi t \, M_{ik}(\bar{t}) \bar{e}_k^j .$$

Similarly,

$$R_j(t, \bar{t})(e_i^j \otimes i) = - \sum \sin 2\pi t \, M_{ik}(\bar{t}) e_k^j +$$

$$+ \sum \cos 2\pi t \, M_{ik}(\bar{t}) \bar{e}_k^j .$$

QED.

So suppressing parameters we have:

$$T_1([x_{\underline{1}}, e_j^i]) = [x_{\underline{1}}, T_1(e_j^i)]$$

$$= [x_{\underline{1}}, \sum A_{jk} e_k^i + \sum B_{jk} \bar{e}_k^i]$$

$$= \sum A_{jk} [x_{\underline{1}}, e_k^i] + \sum B_{jk} [x_{\underline{1}}, \bar{e}_k^i].$$

Also,

$$T_1(c(\ell, i)e_j^{d(\ell, i)} + \bar{c}(\ell, i)\bar{e}_j^{d(\ell, i)})$$

$$= c(\ell, i)(\sum A_{jk}e_k^{d(\ell, i)} + \sum B_{jk}\bar{e}_k^{d(\ell, i)})$$

$$+ \bar{c}(\ell, i)(-\sum B_{jk}e_k^{d(\ell, i)} + \sum A_{jk}\bar{e}_k^{d(\ell, i)}) .$$

Hence,

$$T_1([x_\ell, e_j^i] - c(\ell, i)e_j^{d(\ell, i)} - \bar{c}(\ell, i)\bar{e}_j^{d(\ell, i)})$$

$$= \sum A_{jk}([x_\ell, e_k^i] - c(\ell, i)e_k^{d(\ell, i)} -$$

$$- \bar{c}(\ell, i)\bar{e}_k^{d(\ell, i)})$$

$$+ \sum B_{jk}([x_\ell, \bar{e}_k^i] + \bar{c}(\ell, i)e_k^{d(\ell, i)} -$$

$$- c(\ell, i)\bar{e}_k^{d(\ell, i)}) ,$$

which is in I . Similarly for

$$T_1([x_\ell, \bar{e}_j^i] + \bar{c}(\ell, i)e_j^{d(\ell, i)} - c(\ell, i)\bar{e}_k^{d(\ell, i)}) .$$

Hence I is $T^{\#}$-invariant.

For the rest, Case I works word for word except that the codimension of the maximal $\bar{T}$-invariant subspace of $\ker \bar{\pi} \geq 2(2m-1)$.

All that is needed to finish the proof of theorem 2 is the following lemma:

Lemma 7: (with the preceding notation) Let $f \in F$ be of finite order. Then there is a closed 1-PS of $\overline{T}$, $\overline{f}(t)$, such that $\overline{f}(1)$ induces f on S and $\ker \overline{\pi}$ contains no non-null $\overline{f}(t)$-invariant subspaces.

Proof: In Case I, the "characters" $CH_i(f)$ will be rational, since f is of finite order. Hence, the 1-PS

$$f^{\#}(t) = T^*(t\overline{ch}_1(f),\ldots, t\overline{ch}_r(f))\cdot$$

$$\cdot T_1^*(tCH_1(f))\cdot \ldots \cdot T_s^*(tCH_s(f))$$

will induce a compact 1-PS of $\overline{T}$. The remainder of the statement follows from the fact that the induced 1-PS will act by $0_j(t)$ on V^j and the construction.

In Case II, take

$$f^{\#}(t) = T_1^*(tCH_1(f),\ldots, tCH_1(f), t,\ldots, t)\cdot$$

$$\cdot T_2^*(tCH_2(f))\cdot \ldots \cdot T_s^*(tCH_s(f)) .$$

QED.

QED.

<u>Definition</u> 1: We call the 5-tuple given by Theorem 2, $(\overline{S}, \overline{\pi}, \overline{F}, \overline{T}, n)$, a <u>presentation enlargement of degree</u> $> n$ <u>of the pair</u> (F, S).

<u>Theorem</u> 3: Given a solvable Lie algebra S, which is not nilpotent, and any integer n, there exists a presentation enlargement of degree $> n$ of the pair $(1_S, S)$.

<u>Proof</u>: We shall use the automorphism f given by lemma 1 in section 3 to decompose S. The proof falls into two cases depending on whether f is positive or orthogonal.

<u>Case</u> I: f is orthogonal.

Apply theorem 2 to the group generated by f. For the group $\overline{T}$ take the 1-PS given by (vi) inducing 1_S on S.

<u>Case</u> II: f is positive.

The proof of this Case is very nearly identical with that of Case I in theorem 2. We assume all notation from theorem 2 is still in force unless it is re-defined.

f induces $1_{S/N}$ on S/N. Since f is semi-simple, we can find a complement to N in S such that $f/_K = 1_K$. We then decompose N into f-invariant subspaces:

$$N = \bigoplus_i U_i, \quad \text{where if } u \in U_i, \text{ then}$$
$$f(u) = p_i^* u .$$

Let,

$x_1, \ldots, x_p$ be a basis of K,

$x_1^*, \ldots, x_q^*$ be a basis of U_0, with $p_0^* = 1$

$y_{i\,1}, \ldots, y_{i\,n(i)}$ be a basis of U_i,

for $i = 1, 2, \ldots, r$.

Now this decomposition is consistent with the action of K, $ad_S(K)$. Since we have,

<u>Lemma</u> 1': $ad_S(K)$ has the following properties:

(i) $ad_S(K)(K) \subset U_0$;

(ii) $ad_S(K)(U_i) \subset U_i$.

Proof: Compute as in the proof of lemma 1, using the fact that f is an automorphism.

QED.

Since f is non-trivial there is a $p_i^* \neq 1$. So we can assume, without loss of generality that $p_1^* \neq 1$. Inject U_1 into

$$V = \bigoplus_{j=1}^{n(1)} V^j$$

by the fixed monomorphism φ defined by

$$\varphi(y_{1\,j}) = e_1^j, \qquad j = 1, 2, \ldots, n(1).$$

For each j, extend $\theta_j(t)$ to a 1-PS of $GL(V)$ by requiring that it act as the identity off V^j. Let,

$$\theta(t) = \theta_1(t) \cdot \ldots \cdot \theta_{n(1)}(t).$$

Let $E = \{x_i, x_j^*, y_{k\,l}, e_{s^*}^{n^*} \mid i = 1, 2, \ldots, p;$ $j = 1, 2, \ldots, q;$ $(k, l) = (2, 1), \ldots, (2, n(2)), \ldots, (r, 1), \ldots, (r, n(r));$ $s^* = 1, 2, \ldots, 2m;$ $n^* = 1, 2, \ldots, n(1).\}$.

Extend $\theta(t)$ to a non-singular linear transformation of $L^1(E)$ by requiring it to be the identity on x_i, x_j^*, $y_{k\ 1}$. This extends to a group of automorphisms of $L(E)$, $T^{\#}(t)$.

We also have an epimorphism $\pi^{\#}: L(E) \longrightarrow S$, taking all the appropriate symbols onto themselves and such that

$$\pi^{\#}(e_1^j) = y_{1\ j} \ ;$$

$$\pi^{\#}(e_{s*}^j) = 0 \quad \text{for} \quad s* = 2, \ldots, 2m \ .$$

Define the ideal I as in Case I of theorem 2 but with the following changes:

4) becomes

4'') if in S,

$$[x_i, y_{1\ j}] = \sum b_{i1jk} y_{1\ k}$$

then,

$$[x_i, e_1^j] - \sum b_{i1jk} e_1^k \ .$$

And 5) becomes

5") if in S for $j > 1$,

$$[x_i, y_{j\ k}] = \sum_{l} b_{ijkl} y_{jl}$$

then

$$[x_i, y_{j\ k}] - \sum_{l} b_{ijkl} y_{jl} .$$

The proof, <u>mutatis mutandis</u>, of Case I of theorem 2 now works here. More explicitly, here we have $T^{\#} = T^*$, and in the proof of claim A on p. 19, we must substitute for b_{ijk}, b_{iljk}.

QED.

5. The existence theorem:

Let M be a solvmanifold whose fundamental group, $_1(M) = G$, is not nilpotent. In this section we will show how the constructions in section 4 can be used to produce presentations of M of arbitrarily high dimension. The whole of this discussion is modeled on sections 1 and 2 of [1].

By Wang [9], G can be imbedded as a discrete uniform subgroup in a simply connected, solvable Lie group

$\underline{S}$ with finitely many components, such that $[\underline{S}, \underline{S}] \subset \underline{S}_0$. By Mostow [8], $\underline{S} = F \cdot \underline{S}_0$, where F is a finite group. So F is abelian.

Let $\underline{N}$ be the nil-radical of $\underline{S}_0$. Then $G_0 = \underline{S}_0 \cap G$ is a discrete uniform subgroup of $\underline{S}_0$. $\underline{N} \cap G$ is a discrete uniform subgroup of $\underline{N}$ and $\underline{N}/\underline{N} \cap G$ is a free abelian group on, say s generators.

Further by Wang's construction we have F induces the identity on $\underline{S}_0/\underline{N}$ and if $\underline{N} \cap G$ and $\theta_1, \ldots, \theta_s$ generate G there are positive integers $m_1, \ldots, m_s$ such that $\theta_i^{m_i}$ lie on 1-PS's of $\underline{S}_0$.

Let $\overline{\underline{S}}$, $\underline{S}$ be Solvable Lie groups and $\bar{\pi}: \overline{\underline{S}} \longrightarrow \underline{S}$ be an epimorphism. Suppose G is a discrete uniform subgroup of $\underline{S}$, then by [1] and [7] $\bar{\pi}^{-1}(G)$ is uniform in $\overline{\underline{S}}$ and $\underline{S}/G$ and $\overline{\underline{S}}/\bar{\pi}^{-1}(G)$ are homeomorphic. Further if $\overline{\underline{N}}$ is the nil-radical of $\overline{\underline{S}}$, then $\ker \bar{\pi} \subset \overline{\underline{N}}$ and $\bar{\pi}^{-1}(G) \cap \overline{\underline{N}}$ is uniform in $\overline{\underline{N}}$.

<u>Theorem</u> 4: (with the same notation) M has presentations of arbitrarily high dimension.

<u>Proof</u>: There are two cases depending on the Wang construction. Let $\pi_1(M)$ be imbedded as a discrete uniform subgroup in $\underline{S} = F\cdot\underline{S}_0$.

<u>Case</u> I: $F = 1_S$.

Let $(\overline{S}, \overline{\pi}, 1_{\overline{S}}, \overline{T}, n)$ be a presentation enlargement of degree $> n$ of the pair $(1_S, S)$. Let $h_1,\ldots,h_s$ be pre-images of $\theta_1,\ldots, \theta_s$ in $\underline{\overline{S}}$. Then $h_1,\ldots, h_s$ and $\underline{\overline{\pi}}^{-1}(G) \cap \underline{\overline{N}}$ generate $\underline{\overline{\pi}}^{-1}(G)$.

By the preceding there exist $x_1,\ldots,x_s$ in S such that $\exp(x_i) = \theta_i^{m_i}$. So there exist X_i in $\overline{S}$ such that $\exp(X_i) = h_i^{m_i}$ (mod ker $\underline{\overline{\pi}}$) .

Since $\overline{T}$ is compact and induces the identity on $\overline{S}/\overline{N}$ we can choose a complement K to $\overline{N}$ in $\overline{S}$ such that $\overline{T}/_K = 1_K$. Hence we can choose $Y_i \in \overline{S}$ such that $\overline{T}(Y_i) = Y_i$ and $Y_i = X_i$ (mod $\overline{N}$) .

Let $p\colon \underline{\overline{S}} \longrightarrow \underline{\overline{S}}/\underline{\overline{N}} = \mathbb{R}^s$ denote the canonical projection. Then, if $p(h_i) = v_i$, $p(h_i^{m_i}) = m_i v_i$. But then $p(\exp((1/m_i)Y_i) = v_i$.

Hence, $\exp((1/m_i)Y_i) = h_i \pmod{\underline{\overline{N}}}$. Now let, $\underline{S}^{\#} = \underline{\overline{T}} \cdot \underline{\overline{S}}$. Since $\overline{T}$ is a compact 1-PS of $\underline{\underline{\mathrm{Aut}}}(\underline{\overline{S}})$ we can choose H in the Lie algebra of $\overline{T}$ such that $\exp(H) = 1_{\underline{\overline{S}}}$ and $\exp(tH) = \overline{T}(t)$. Now consider the subalgebra S^* of $S^{\#}$ generated by $\overline{N}$ and $(H + (1/m_1)Y_1, (1/m_2)Y_2, \ldots, (1/m_s)Y_s$.

Let $\underline{S}^*$ be the connected subgroup of $\underline{S}^{\#}$ whose Lie algebra is S^* (temporarily violating our notational convention). We must now show that $\underline{S}^*$ is closed and simply connected. $\underline{S}^* \supset \underline{\overline{N}}$ and since $\underline{\overline{N}}$ is characteristic in $\underline{\overline{S}}$, $\underline{\overline{N}}$ is closed. $\underline{S}^*/\underline{\overline{N}}$ is a vector subgroup of the abelian Lie group $\overline{T} \cdot \underline{\overline{S}}/\underline{\overline{N}}$ and hence is simply connected. Since $\underline{\overline{N}}$ is simply connected, so is $\underline{S}^*$. Similarly for closed. Hence $\underline{\overline{\pi}}^{-1}(G) \cap \underline{S}^*$ is closed in $\underline{S}^*$.

But $\underline{S}^* \supset \underline{\overline{\pi}}^{-1}(G)$, since $\underline{S}^* \supset \underline{\overline{N}}$ and contains $\exp(H + (1/m_1)Y_1)$, $\exp((1/m_i)Y_i)$ for $i = 2, \ldots, s$. Since Y_i is left fixed by $\overline{T}$, $[H, Y_1] = 0$.

Hence,

$$\exp(H + (1/m_1)Y_1) = \exp(H)\exp((1/m_1)Y_1)$$
$$= \exp((1/m_1)Y_1) .$$

Therefore $h_i \in \underline{S}^*$ for $i = 1, \ldots, s$.

Now $\underline{S}^*/\bar{\pi}^{-1}(G)$ is compact, since $\bar{\pi}^{-1}(G) \cap \underline{\bar{N}}$ is uniform in $\underline{\bar{N}}$ and the images of $h_1, \ldots, h_s$ in $\underline{S}^*/\underline{\bar{N}}$ generate a uniform lattice. Hence $\underline{S}^*/\bar{\pi}^{-1}(G)$ is a fiber bundle with compact base and compact fiber.

It remains only to show that $(\underline{S}^*, \bar{\pi}^{-1}(G))$ is a presentation. Now $\ker \bar{\pi} \subset \underline{\bar{N}}$. Since G is discrete, $(\bar{\pi}^{-1}(G))_0 = \ker \bar{\pi}$. By construction $\ker \bar{\pi}$ contains no non-null $\bar{T}$-invariant subspaces. Since $[H, Y_1] = 0$ and $\mathrm{ad}(Y_1)(\ker \bar{\pi}) \subset \ker \bar{\pi}$, if $\ker \bar{\pi}$ had a non-null ideal in S^*, it would have a non-null $\bar{T}$-invariant subspace. Hence we have that $\ker \bar{\pi}$ contains no connected subgroups normal in $\underline{S}^*$. But $\dim(\underline{S}^*, \bar{\pi}^{-1}(G)) > n$ and n was arbitrarily chosen.

Case II: $F \neq 1_S$.

Let $(\overline{S}_0, \bar{\pi}, \overline{F}, \overline{T}, n)$ be a presentation enlargement of degree $> n$ of the pair (F, S_0) . Let $\underline{\overline{S}} = \overline{F} \cdot \underline{\overline{S}}_0$. We extend $\bar{\pi}$ to an epimorphism of $\underline{\overline{S}}$ onto $\underline{S}$, $\bar{\pi}_*$ by defining

$$\bar{\pi}_*(\overline{f}, \overline{s}) = (i(\overline{f}), \bar{\pi}(\overline{s})) ,$$

where $\overline{f} \in \overline{F}$ and $i(\overline{f})$ is the image of $\overline{f}$ in F . Hence $\ker \bar{\pi}_* = \ker \bar{\pi}$. And so, $\ker \bar{\pi}_*$ is contained in $\underline{\overline{N}}$, the nil-radical of $\underline{\overline{S}}_0$. Consider in $\bar{\pi}_*^{-1}(G)$ pre-images $h_1, \ldots, h_s$ of $\theta_1, \ldots, \theta_s$ in G . Since $\bar{\pi}_*^{-1}(G)/\bar{\pi}_*^{-1}(G) \cap \underline{\overline{N}}$ is free abelian on s generators, $\bar{\pi}_*^{-1}(G) \cap \underline{\overline{N}}$ and $h_1, \ldots, h_s$ generate $\bar{\pi}_*^{-1}(G)$.

Again by the preceding, there exist $x_1, \ldots, x_s$ in S_0 such that $\exp(x_i) = 0_i^{m_i}$. Since $\bar{\pi}$ is an epimorphism of $\underline{\overline{S}}_0$ onto $\underline{S}_0$ there are $X_i \in \overline{S}_0$ such that $\bar{\pi}$, the induced mapping of $\overline{S}_0$ onto S_0 carries X_i onto x_i .

$$\text{So} \quad \bar{\pi}(\exp(X_i)) = \theta_i^{m_i} \quad \text{and therefore,}$$

$$\exp(X_i) = h_i^{m_i} \pmod{\ker \bar{\pi}} .$$

Since $\overline{T}$ is compact and induces the identity on $\overline{S}_0/\overline{N}$, we can choose a complement K to $\overline{N}$ in $\overline{S}_0$ such that $\overline{T}/_K = 1_K$. Hence, since $X_i \notin \overline{N}$, we can choose $Y_i \in \overline{S}_0$ such that $\overline{T}(Y_i) = Y_i$ and $Y_i = X_i \pmod{\overline{N}}$.

Since $\underline{\overline{N}}$ is characteristic in $\underline{\overline{S}}_0$, it is normal in $\underline{\overline{S}}$. Let $p\colon \underline{\overline{S}} \longrightarrow \underline{\overline{S}}/\underline{\overline{N}}$. Then $\underline{\overline{S}}/\underline{\overline{N}} = \overline{F} \oplus \mathbb{R}^s$. Let $h_i = (\bar{f}_i, s_i)$ and $p(h_i) = (\bar{f}_i, v_i)$. Let $\exp((1/m_i)Y_i) = (0, r_i)$. Now $p(\exp(Y_i)) = p(0, r_i^{m_i})$ $= (0, m_i v_i)$. But $p(h_i^{m_i}) = (\bar{f}_i^{m_i}, m_i v_i) = (0, m_i v_i)$. Hence $p(\exp((1/m_i)Y_i) = (0, v_i)$. Therefore,

$$\bar{f}_i(\exp((1/m_i)Y_i) = h_i \pmod{\underline{\overline{N}}} .$$

Let $\underline{S}^{\#} = \overline{T}\cdot\underline{\overline{S}}$. Choose t_i in the Lie algebra of $\overline{T}$ so that $\exp(t_i) = \bar{f}_i$. By theorem 2 this can be done in such a way that $\ker \bar{\pi}$ contains no non-null subspaces invariant under $\exp(rt_i)$ for $r \in \mathbb{R}$.

Consider the subalgebra S^* of $S^{\#}$ generated by $\overline{N}$ and $(t_i + (1/m_i)Y_i)$. Let $\underline{S}^*$ be the connected subgroup of $\underline{S}^{\#}$ whose Lie algebra is S^* . $\underline{S}^*$ is simply connected and closed for exactly the same reasons as in Case I.

Since Y_i is left fixed by $\overline{T}$, $[t_i, Y_i] = 0$ and hence,

$$\exp(t_i + (1/m_i)Y_i) = \exp(t_i)\exp((1/m_i)Y_i)$$

$$= h_i \pmod{\underline{\overline{N}}}.$$

As in Case I, $\underline{\overline{\pi}}_*^{-1}(G)$ is uniform in $\underline{S}^*$. Then just as in Case I, $\ker \underline{\overline{\pi}}_* = \ker \underline{\overline{\pi}}$ contains no non-trivial connected subgroup normal in $\underline{S}^*$.

QED.

CHAPTER IV

Presentations of Nilmanifolds

1. Introduction and preliminaries:

In this chapter we fill the gap left in chapter III; we will show that a compact solvmanifold with nilpotent fundamental group has presentations of arbitrarily high dimension. Moreover, we will be able to prove, using the semi-simple splitting, a uniqueness theorem for presentations of tori.

To fix notation: if X is a finite set let $N_k(X)$ be the free k-step nilpotent Lie algebra on X. Then $N_k(X)$ is finite dimensional. We also want the following notation: let V be the subspace spanned by X then we also denote $N_k(X)$ by $N_k(V)$.

Recall that for any nilpotent Lie algebra N, N is generated by any complement to $[N, N]$. Further, if A is an abelian group of semi-simple automorphisms of N, A preserves a complement to $[N, N]$ and hence, the action of A is determined by its induced action on $N/[N, N]$.

<u>Proposition</u> 1: Let $(\underline{S}, \underline{C})$ be a presentation of a solvmanifold with $\pi_1(\underline{S}/\underline{C})$ k-step nilpotent, then $\underline{C}$ is k-step nilpotent.

<u>Proof</u>: Let $\underline{S}_s = \underline{T}_s \cdot \underline{N}_s$ be the semi-simple splitting of $\underline{S}$. Then by the proof of Theorem I.2, p. 56 of [4], $\underline{C}$ $\underline{N}_s$. Since the projection p restricted to $\underline{S}$ is a homeomorphism, $\underline{C}$ is a closed uniform subgroup of $\underline{N}_s$. Hence $\underline{N}_s$ is the Lie hull of $\underline{C}$, $\underline{N}(\underline{C})$. But $\underline{N}(\underline{C}^j) = \underline{N}(\underline{C})^j$. Since $\underline{C}/\underline{C}_0$ is k-step nilpotent $\underline{C}^k \subset \underline{C}_0$. Hence $\underline{N}_s^k \subset \underline{C}_0$. But $\underline{N}_s^k$ is characteristic and connected in $\underline{N}_s$ and hence, is normal and connected in $\underline{S}$. Since $(\underline{S}, \underline{C})$ is a presentation, $\underline{N}_s^k = 1$ and $\underline{C}^k = 1$.

QED.

We shall need the following proposition from [4] .

<u>Proposition</u> 2: (L. Auslander) Let $(\underline{S}, \underline{C})$ be as in proposition 1, then $\underline{T}_s$ is compact.

2. Tori:

Theorem 5: Let $(\underline{S}, \underline{C})$ be a presentation of the torus, $\mathbb{T}^n$ then,

(i) $\underline{S} = \mathbb{R}^s \cdot \mathbb{R}^t$;

(ii) $\mathbb{R}^s$ acts compactly;

(iii) $s+t \geq n$, $s \leq n$, and

(a) if $s+t = n$, then $\underline{C}$ is discrete;

(b) if $s+t > n$, then $s < n$ and we have further conditions on the action of $\mathbb{R}^s$, ad $\mathbb{R}^s$: let $\mathbb{R}^t = U \oplus D$, with U the sum of the 1-dimensional ad $\mathbb{R}^s$-invariant subspaces and D the sum of the 2-dimensional ad $\mathbb{R}^s$-invariant subspaces and let their dimensions be t_1 and t_2 , respectively, then if $t_1 = 0$, $t+s-n < t_2$ and if $t_1 \neq 0$, $t+s-n \leq t_2$.

For any set of allowable conditions above there exists a presentation; so, in particular, there are presentations of arbitrarily high dimension for any torus, $\mathbb{T}^n$, with $n \neq 1$.

Proof: The case $s+t = n$ is proved by L. Auslander and

M. Auslander in [2]. Our method will be to reduce the other cases to this case.

Sufficiency: In D we can choose a basis $e_1, \ldots, e_{2m}$ such that the subspace spanned by $e_{i+1}, \ldots, e_{2m}$, for $i \geqslant 1$ contains no non-null subspaces which are ad $\mathbb{R}^s$-invariant. Since ad $\mathbb{R}^s$ is compact, ker ad $= \mathbb{Z}^s$. Let $\mathbb{Z}^{t_1}$ be a uniform lattice in U. Now if $t_1 = 0$, then we must take the e_j as above, but if $t_1 \neq 0$ we can take a copy of the e_j in a subspace oblique to D and U and let $i = 0$.

Now, we let

$$\underline{C} = \mathbb{Z}^s \times \mathbb{Z}^{t_1} \times \mathbb{Z}e_1 \times \ldots \times \mathbb{Z}e_i \times$$

$$\times \mathbb{R}e_{i+1} \times \ldots \times \mathbb{R}e_{2m}$$

and hence, $\pi_1(\underline{S}/\underline{C}) = \mathbb{Z}^{s+t_1+i}$. And any connected subgroup of $\underline{C}$ normal in $\underline{S}$ is an ad $\mathbb{R}^s$-invariant subspace of $\mathbb{R}e_{i+1} \times \ldots \times \mathbb{R}e_{2m}$ and hence is null.

Necessity: Let $(\underline{S}, \underline{C})$ be a presentation of $\mathbb{T}^n$, then by proposition 1 in section 1, $\underline{S}_s = \underline{T}_s \cdot \mathbb{R}^a$ and $\underline{C}$ is closed and uniform in $\mathbb{R}^a$. Hence $\underline{C} = \mathbb{Z}^n \times \mathbb{R}^b$. And therefore $\underline{S}$ acts on $\mathbb{T}^{n+b}$ with discrete isotropy group. So by [2], $\underline{S} = \mathbb{R}^s \cdot \mathbb{R}^t$ with $s+t = n+b$, and $\mathrm{ad}\,\mathbb{R}^s$ is compact.

The remainder of the proof consists of computations to show that the restrictions on the exponents s and t hold.

(i) If $b = 0$, then $\underline{C}$ is discrete and $s+t = n$;

(ii) if $b > 0$, then $s+t > n$ and $\mathrm{ad}\,\mathbb{R}^s$ is non-trivial, for otherwise $\mathbb{R}^b$ would be normal in $\underline{S}$. But if $\mathrm{ad}\,\mathbb{R}^s$ is non-trivial, $\mathbb{R}^t$ is the nil-radical of $\underline{S}$ (see lemma 1, below). Then we have $\mathbb{R}^b \subset \mathbb{R}^t$. Hence $s < n$. Now, if $t_1 = 0$, $t = t_2$ and $b < t$. And if $t_1 > 0$, then we can have $b \leq t_2$.

QED.

<u>Lemma</u> 1: If $\mathrm{ad}\,\mathbb{R}^s \neq 1$, then $\mathbb{R}^t$ is the nil-radical of $\mathbb{R}^s \cdot \mathbb{R}^t$.

<u>Proof</u>: We write $(x, y)\cdot(\bar{x}, \bar{y}) = (x+\bar{x}, \mathrm{ad}(x)(\bar{y})+y)$.

Then we have

$$\overbrace{\left[\ldots \left[(x, y), \left[(x, y), (0, \bar{y})\right]\right] \ldots \right]}^{k} =$$

$$(0, (-1)^{k+1}(1_{\mathbb{R}^t} - ad(x))^k(\bar{y})) .$$

So (x, y) is in the nil-radical only if $ad(x)$ is unipotent, but since $ad\,\mathbb{R}^s$ is compact, $ad(x)$ is semi-simple.

QED.

3. <u>k-step nilpotent fundamental group</u>:

Let G be a k-step nilpotent fundamental group of a solvmanifold, with $k \neq 1$. Then by Malcev [6] G can be imbedded as a discrete uniform subgroup of a k-step Nilpotent Lie group, $\underline{N}(G) = \underline{N}$. Let $M = \underline{N}/G$. Let $\dim \underline{N}/[\underline{N}, \underline{N}] = n$. In [6], we also find that the image of G under the projection $\underline{N} \longrightarrow \underline{N}/\ \underline{N}, \underline{N}$ is discrete and uniform.

<u>Theorem</u> 6: (with the above notation) There exist presentations of M of arbitrarily high dimension.

<u>Proof</u>: Let $0_j(t)$ and $e_1^j, \ldots, e_{2m}^j$ be as in chapter III.

Since j will $= 1$, suppress it in the notation. Let $E = \{z_1, \ldots, z_{n-1}, e_1, \ldots, e_{2m}\}$. Extend $\theta(t)$ to a 1-PS of automorphisms, $\theta^{\#}(t)$, of $N_k(E)$ by requiring $\theta^{\#}(t)(z_i) = z_i$.

Now let K be a complement to $[N, N]$ in N and suppose $c_1, \ldots, c_n$ is a basis of K. Define an epimorphism $\pi^{\#}: N_k(E) \longrightarrow N$ by $\pi^{\#}(z_i) = c_i$ for $i = 1, 2, \ldots, n-1$, and $\pi^{\#}(e_1) = c_n$.

Let W be the subspace spanned by $z_1, \ldots, z_{n-1}, e_1, \ldots, e_{2m}$ in $N_k(E)$ and let V be the subspace spanned by $e_1, \ldots, e_{2m}$ in $N_k(E)$. Then since V is $\theta^{\#}(t)$-invariant and $\ker \pi^{\#} \cap V = \text{span}(e_2, \ldots, e_{2m})$ we have by the same argument as in theorem 2 that the maximal $\theta^{\#}$-invariant subspace of $\ker \pi^{\#}$ has codimension of at most $(2m-1)$. We factor out by this ideal Q to obtain $\bar{\pi}: N_k(E) = \bar{N} \longrightarrow N$, an epimorphism, with $\ker \bar{\pi}$ of dimension $>(2m-1)$. Also, we obtain a compact 1-PS of automorphisms of $\bar{N}$, $\bar{\theta}(t)$, such that $\ker \bar{\pi}$ contains no non-null $\bar{\theta}$-invariant subspaces.

Proceding as in the proof of theorem 2, we use this "enlargement" to construct a presentation of M. In fact since exponential is an analytic manifold isomorphism of N onto $\underline{N}$, we can give the group explicitly.

Let $\bar{z}_i$ and $\bar{e}_i$ denote the images of z_i and e_i in $\bar{N}$, respectively. Since $\theta^{\#}(t)$ fixes z_1, $\bar{\theta}(t)$ fixes $\bar{z}_1$. Identify $\bar{\theta}(t)$ with the 1-PS of automorphisms it induces on the group: $\exp(\mathrm{span}(\bar{z}_2, \ldots, \bar{z}_{n-1}, \bar{e}_1, \ldots, \bar{e}_{2m}, [\bar{N}, \bar{N}])) = \underline{H}$.

Then if,

$$\underline{S} = (\bar{\theta}(t) + \exp(tz_1)) \cdot \underline{H} \quad \text{for} \quad t \in \mathbb{R},$$

$(\underline{S}, \bar{\pi}^{-1}(G))$ is a presentation of M.

QED.

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