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Identification, parameter restrictions and equivalence in factor analysis models

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IDENTIFICATION, PARAMETER RESTRICTIONS AND EQUIVALENCE IN FACTOR ANALYSIS MODELS

by

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FIGURE SYMBOLS GLOSSARY

y: Observed measure

λ: First-order factor loading

ϵ: Residual variance for an observed variable

η: First-order factor

ζ: Unique variance for a first-order factor

γ: Second-order factor loading

ξ: Second-order factor

ω: Covariation between first-order factors

CHAPTER 1

INTRODUCTION AND FACTOR ANALYSIS OVERVIEW

Factor analysis is a popular data analysis methodology that summarizes a large quantity of observed variables with a smaller number of unobserved variables. It is used in fields such as education, psychology, sociology, and marketing, with variables representing aptitude, intelligence, personality, values, and attitudes. A variety of commonly used factor analysis models will be employed to examine model identification, parameter restrictions leading to a quasi-hierarchy of models, and model equivalence.

There are numerous ways in which to represent the variety of factor analytic techniques. A brief description of factor analysis will be provided, including exploratory, higher-order, and confirmatory models. This is followed by a general examination of the theory and models to be investigated. Next will be a discussion of some of the principal theoretical/ analytical contributions to factor analytic methods, in approximately temporal order (in a separate chapter). Finally, the area of study will be discussed.

Exploratory Factor Analysis

Exploratory factor analysis, as the name indicates, is concerned with the preliminary investigation of a collection of observed measures. A goal is to find a smaller number of unobserved variables (q) than observed variables (p) that purport to describe the area being measured. In the absence of a theory guiding this type of examination, it is successful to the extent that the number of simplifying latent variables (factors) are much fewer than the quantity of observed measures.

The basic relationship between observed and unobserved variables is the following:

$$y = \Lambda \eta + \epsilon \tag{1}$$

In this equation y is a (p x 1) vector of observed variables, η is a (q x 1) vector of factors, Λ is a (p x q) matrix of factor loadings to be estimated, and ϵ is a (p x 1) vector of unique or residual variables.

The following assumptions are applied to the relationship depicted in Equation 1: the expected values of observed variables, factors and residuals are all equal to zero (i.e., $E(y) = 0$, $E(\eta) = 0$, $E(\epsilon) = 0$), and the covariances between the factors and residuals are equal to zero (i.e., $E(\eta\epsilon') = E(\epsilon\eta') = 0$).

Multiplying each side of Equation 1 by its transpose and taking the expected values of the result yields the following relationship between the (p x p variable) population covariance matrix (Σ) and the model parameters:

$$\begin{aligned} \Sigma = yy' &= E(\Lambda\, \eta + \epsilon)\ (\Lambda\, \eta + \epsilon)' \\ &= E(\Lambda\, \eta + \epsilon)\ (\eta'\ \Lambda' + \epsilon') \\ &= E(\Lambda\, \eta\, \eta'\, \Lambda' + \epsilon\, \eta'\, \Lambda' + \Lambda\, \eta\, \epsilon' + \epsilon\, \epsilon') \\ &= \Lambda\, \Omega\, \Lambda' + \Theta_\epsilon \qquad (2) \end{aligned}$$

where Ω is the (q x q) covariance matrix between the factors, and Θ_ϵ is $E(\epsilon\ \epsilon')$, the (p x p) matrix of residual variances and covariances. Θ_ϵ is usually a diagonal matrix.

The discussion that follows examines the prior factor analysis solution in the context of Euclidean space. For purposes of illustration one may consider a model with two factors. The axes in our coordinate space would correspond to the (two) factors in the model, with variables as the data points, and factor loadings as the coordinates. The positioning of these axes reflects a geometric representation of the degree of association present between the factors. Specifically, the cosine of the angle between these axes is the correlation between the factors. For instance, if the axes are perpendicular there is no correlation between the factors. Such a situation is referred to as an orthogonal solution. This occurs when Ω is a diagonal matrix. If however the axes are not perpendicular, the factors are correlated. This is referred to as an oblique solution, and occurs when the off-diagonal elements of Ω are non-zero.

It is often the case that factor-analytic solutions are not unique. That is, the axes may be re-positioned relative to the variables, which are at fixed locations in the factor space. This shifting of axes, referred to as rotation, is a characteristic of the indeterminacy of an exploratory factor analysis solution. (Some procedures for dealing with this are discussed in a later section).

Second-Order Factor Analysis

In the case of an oblique solution, one can think of examining the (first-order) factor correlations for the presence of an underlying, simplified, factor structure. While the analysis would not proceed in this fashion, such a process is akin to obtaining a second-order factor analytic structure in terms of the observed variables.

In the case of a second-order factor model the following equation relates the first-order factors (η) to the second-order factors (ξ):

$$\eta = \Gamma \xi + \zeta \tag{3}$$

In this equation Γ is a (q x r) matrix of the loadings of the first-order factors on the second-order factors, ξ is a (r x 1) vector of second-order factors, and ζ is a (q x 1) vector of unique variables for the first-order factors. The following assumptions are usually made: the expected values of the observed variables, first-and-second order factors, residual and unique variables are all equal to zero (i.e., $E(y) = 0$, $E(\eta) = 0$, $E(\xi) = 0$, $E(\epsilon) = 0$, $E(\zeta) = 0$). It is also assumed that the covariance between the second-order factors and the unique variables are equal to zero (i.e., $E(\xi\ \zeta') = 0$).

The resulting equation expressing the population covariance matrix as a function of the estimated parameters is

$$\Sigma = \Lambda\ (\Gamma\ \Phi\ \Gamma' + \Psi)\ \Lambda' + \theta_\epsilon \quad (4)$$

$$= \Lambda\ \Gamma\ \Phi\ \Gamma'\ \Lambda' + \Lambda\ \Psi\ \Lambda' + \theta_\epsilon \quad (5)$$

where Φ is the $(r \times r)$ covariance matrix of the second-order factors and Ψ, usually a diagonal matrix, is the $(q \times q)$ covariance matrix of the first-order residuals.

A comparison between equations two and four reveals the following equality:

$$\Sigma = \Lambda\ \Omega\ \Lambda' + \Theta_\epsilon = \Lambda\ (\Gamma\ \Phi\ \Gamma' + \Psi)\ \Lambda' + \Theta_\epsilon$$

where it is observed that the covariance matrix of the first-order factors (Ω) is equal to $\Gamma\ \Phi\ \Gamma' + \Psi$.

Confirmatory Factor Analysis

Drawing upon (germane) theoretical or other relevant information for the data used in the (aforementioned) analyses, various constraints (and structure) can be placed upon the parameters of a factor analysis model. The resulting model can be examined as a test of the theory in question, by virtue of how well the model fits the observed data. When factor analysis is conducted in this fashion, the resulting models can be considered to have been constructed with the use of confirmatory factor analysis. In this situation, the primary objective is the development of a simple model that is plausible in terms of fit to the observed data.

The fit of a model can be assessed by comparing the difference between the observed data covariance matrix (S) with the one that is produced by the model being tested (Σ). In general, smaller residual matrices ($S - \Sigma$) are evidence of better fitting models.

There are various measures that examine the fit of a model to the data. One of the most popular is the chi-square goodness-of-fit test. Under the assumption of multivariate normality, the chi-square statistic tests the appropriateness of the model against the hypothesis that Σ is unconstrained. This chi-square test has degrees of freedom (df) equal to $(1/2(p)(p+1) - t)$, where p is the number of observed variables and t is the number of parameters estimated by the model.

Less formally, a comparison of the chi-square value to the degrees of freedom for the model determines the plausibility of the model. Large chi-square values relative to the degrees of freedom indicate a model that is not consistent with the data. Conversely, chi-square values that are close to or less than the degrees of freedom are indicative of a model that fits the data.

It is the use of a model that has constraints (providing structure) placed upon its parameters that most readily differentiates exploratory from confirmatory factor analysis. It is possible to conduct first-order or higher-order confirmatory factor analysis.

Model Identification and Parameter Restrictions

The primary comparisons to be examined in this work will deal with the identification of, and relationships among, selected second- and first-order factor analysis models. Identification deals with the ability to solve for unknown parameters in the model in terms of the elements of the known covariance matrix. The way to examine whether a model is identified is to determine that each of the parameters in the model is identified. When all model parameters are identified, then the model is said to be identified (except in the case of empirical under-identification).

In both exploratory and confirmatory factor analysis, some restrictions are necessary (but not sufficient) to identify the model by fixing the scale of each factor. This is usually done by constraining the variance of the factors, typically to one, or by fixing one factor loading for each factor to one.

In order to ascertain whether a model is identified, one could proceed in the following manner: (1) diagram the model, (2) write equations for the observed variables in terms of the parameters to be estimated, (3) multiply each equation by iteself, and each other equation, one at a time, (4) take expected values of the resulting equations, and (5) solve for the parameters of these equations in terms of the observed variances and covariances.

While there is not an exhaustive list of necessary and sufficient identification conditions, certain rules can be applied in the process of determining if a model is identified. If the number of unknown elements exceeds the number of observed elements, then the model is not identified. In these situations, one may wish to consider parameter restrictions on the model in an effort to identify the model. For example, consider a one-factor model with two observed variables. This model can be represented as shown in Figure 1.

In this and subsequent drawings, circles depict unobserved, or latent variables, and squares represent observed variables. The factor loadings are represented by λ's, and the residual terms by ϵ's.

The equations for this model, obtained by an application of Equation 1, are as follows:

$$y_1 = \lambda_1 \eta + \epsilon_1$$

$$y_2 = \lambda_2 \eta + \epsilon_2$$

It is assumed that ϵ_1 and ϵ_2 are uncorrelated, and that the variance of η is equal to one. There are four unknowns in this model (λ_1, λ_2, $\Theta_{\epsilon\,1}$,$\Theta_{\epsilon\,2}$), and only three known covariance elements. As there are more unknown parameters to be solved for than known elements, this model is not identified. If one were willing to make the restriction either that the two factor loadings were equal ($\lambda_1 = \lambda_2 = \lambda$) or the two residual variances were equal ($\Theta_{\epsilon\,1} = \Theta_{\epsilon\,2} = \Theta_\epsilon$), then there would be three unknowns. In this case, the model would be just identified. The proof of this can be seen by solving for the three parameters in terms of the observed variances and covariances. The following relationships exist between the observed terms and the estimated parameters based upon conducting steps three and four (mentioned earlier) to the equations for y_1, and y_2 with the assumption that the factor loadings are equal:

$$\sigma_{11} = \lambda^2 + \Theta_{\epsilon\,1}$$

$$\sigma_{12} = \lambda^2$$

$$\sigma_{22} = \lambda^2 + \Theta_{\epsilon\,2}$$

Therefore, $\lambda = \sigma_{12}{}^{1/2}$ permitting $\Theta_{\epsilon\,1}$ and $\Theta_{\epsilon\,2}$ to be obtained by subtraction:

$$\Theta_{\epsilon\,1} = \sigma_{11} - \sigma_{12}$$

$$\Theta_{\epsilon\,2} = \sigma_{22} - \sigma_{12}$$

If, however, both restrictions were employed, then the resulting model would have two unknowns and would be over-identified. In this situation, there is more than one way to solve for some of the unknown parameters, thereby providing for a test of the fit of the model to the data. Either of these last two situations may be theoretically unreasonable and hence preclude efforts at identification.

If, instead of making these restrictions, a third observed variable were added, the model would have six parameters to be estimated and six known values (three variances and three covariances). Such a model would be just-identified. The proof for the identification of this model starts by writing the equations for the observed variables in terms of the model parameters. This results in following three equations:

$$y_1 = \lambda_1 \eta + \epsilon_1$$

$$y_2 = \lambda_2 \eta + \epsilon_2$$

$$y_3 = \lambda_3 \eta + \epsilon_3$$

Multiplying each equation by itself, and then by each of the other two equations, and taking the resulting expected values produces the following six equations:

$$\sigma_{11} = \lambda_1^2 + \Theta_{\epsilon\,1}$$

$$\sigma_{22} = \lambda_2^2 + \Theta_{\epsilon\,2}$$

$$\sigma_{33} = \lambda_3^2 + \Theta_{\epsilon\,3}$$

$$\sigma_{12} = \lambda_1 \lambda_2$$

$$\sigma_{13} = \lambda_1 \lambda_3$$

$$\sigma_{23} = \lambda_2 \lambda_3$$

The three equations representing the covariance terms are used to solve for the λ elements. Dividing the two equations that a particular λ appears in, and then multiplying by the remaining equation yields the following parameter equalities:

$$\lambda_1^2 = (\sigma_{12}\ \sigma_{13})\ /\ \sigma_{23}$$
$$\lambda_2^2 = (\sigma_{12}\ \sigma_{23})\ /\ \sigma_{13}$$
$$\lambda_3^2 = (\sigma_{13}\ \sigma_{23})\ /\ \sigma_{12}$$

Taking the square root of each of these equations obtains the values for the three λ terms. Performing the appropriate subtraction, provides a solution for the three residual terms. Specifically, the following equalities result:

$$\Theta_{\epsilon\,1} = \sigma_{11} - (\sigma_{12}\ \sigma_{13})/\sigma_{23}$$
$$\Theta_{\epsilon\,2} = \sigma_{22} - (\sigma_{12}\ \sigma_{23})/\sigma_{13}$$
$$\Theta_{\epsilon\,3} = \sigma_{33} - (\sigma_{13}\ \sigma_{23})/\sigma_{12}$$

Adding a fourth observed variable to the previous model would result in ten covariance elements with eight parameters to be solved for. This model would be over-identified. If, however, these four variables were part of a two-factor model with two variables loading each factor, a model such as that depicted in Figure 2 would result. This model is not identified. However, if the two factors were permitted to correlate, and the factor variances are scaled to equal one, the model is identified.

Certain factor analysis models can be shown to be equivalent or else to be more- (or less-) restricted forms of one another. Two models are said to be equivalent when there is a one-to-one transformation between their respective estimated parameters. For two models to be equivalent they must estimate the same number of parameters; however, this is not a sufficient condition for equivalence to occur. (This will be shown in a later section).

Model Descriptions

Next, the following four often-used factor analysis models will be briefly discussed: one-factor, second-order, group-factor, and bi-factor. Issues pertaining to identification, equivalence, and restricted forms of these models, where applicable, will be considered in a later section.

For purposes of illustration the four factor analysis models will be presented in terms of six observed variables. A data set such as this has (6)(7)/2 = 21 variances/covariances. Models with additional variables will be considered in a later section.

The one-factor model, shown in Figure 3, has each variable represented by a loading on the (common) factor and a residual term. These residual terms are uncorrelated. The model is made identified by constraining the variance of the factor to one. For the six observed variables depicted, 12 parameters are estimated (6 factor loadings [λ's] and 6 residual variances [θ_ϵ's]) resulting in 21-12 = 9 degrees-of-freedom. One possible second-order model, shown in Figure 4, has one second-order factor measured by three first-order factors, each of which has two observed variables per factor. This model has a residual variable associated with each observed variable, as well as a unique variable for each of the first-order factors. The model is made identified by fixing the second-order factor variance to one and constraining one loading for each first-order factor to one. This results in a model with 15 estimated parameters and six (21-15) degrees-of-freedom.

The corresponding group-factor model, depicted in Figure 5, is composed of three correlated factors each measured by two observed variables. The model is made identified by fixing the factor variances to one. This produces a model with 15 parameters and six degrees-of-freedom.

The corresponding bi-factor model, shown in Figure 6, can be thought of as a group-factor model in conjunction with a general factor that is uncorrelated with the group factors. This model, with all factor variances constrained equal to one, has 21 parameters but is not identified with only six variables.

Parameter restrictions (and relaxations) will be considered for these four basic factor analysis models. The consequences of certain parameter specifications can have far-reaching results. These can range from making identified a previously unidentified model, to producing two different model-types that are equivalent. In certain situations where equivalence between models occurs, algebraic proofs will be provided. It will also be shown how a partially ordered chain exists relating the resulting models in certain situations.

CHAPTER 2

REVIEW OF PRINCIPAL THEORETICAL/ANALYTICAL FACTOR ANALYTIC CONTRIBUTIONS

Charles Spearman (1904), as a result of his examination of human mental abilities, and his opposition to the averaging techniques favored by Binet at the time, put forward his two-factor theory of intelligence. What Spearman referred to as his two-factor theory would currently be referred to as a one-factor model: each observed variable has a loading on a common factor plus a unique, or residual, term.

Spearman's model can be expressed by the following equation:

$$x_i - \lambda_1 g + s_i$$

where x_i is an observed variable *i*, g is a general factor, s_i is a specific factor for variable i, and λ_i is the factor loading of variable *i* on g. For each observed variable *i*, g is uncorrelated with s, and for a group of tests "every s is independent of every other one, unless the two operations are closely similar" (Spearman, 1927, p. 413).

Spearman and Wynn Jones (1950), in support of Spearman's theory of intelligence, present a cursory examination of some of the factor analytic theories and methods of the day including those of Holzinger, Hotelling and Kelley, and Thurstone. However, Spearman apparently did not find this body of work to be helpful, although he had previously noted (Spearman, 1904) that some of the problems involved in the investigation of his theory would be removed "on the sole condition of adequate methodics" (p. 159). Spearman and Wynn Jones present Holzinger's bi-factor method as most resembling Spearman's model in that they both contain a general factor. The bi-factor model also has group factors that are non-overlapping and uncorrelated, both with themselves and with the general factor. Also containing a general factor is the principal axes method of Hotelling and Kelley. This model also contains other factors, each of which is comprised of roughly equal numbers of positive and negative factor loadings. Lastly, Thurstone's multiple factor method is presented as "complicated": "For this [method] contains no [italics in original] general factor, but many positive and overlapping ones, as also many zeros [i.e., zero factor loadings]" (p. 41).

Godfrey Thomson (1956) presented an alternative explanation to Spearman's single common factor. Thomson referred to his position as 'sampling theory'. This was a physiologically based view of the mind that was founded upon bonds within the brain, or some sub-pools of the brain. The correlation present between two intelligence tests was theorized to be due to the overlapping of bonds utilized by each of these tests. Thomson argued that one of the outgrowths of his theory was that the amount of specific variance associated with each test under consideration was lower than when compared to Spearman's two-factor theory.

Another writer from the British school, Philip Vernon (1950), preferred a hierarchical theory in which g is extracted first, followed by group factors, both major and minor, with specific factors accounting for the remaining variance. Vernon disagreed with Spearman's (general) insistence that there is only g and specific factors (as well as Thurstone's method of multiple-factor analysis in which g is not present).

Vernon (1950) reported an analysis of 13 tests administered to 1,000 British Army recruits. A solution calling for five factors was obtained. This solution, hierarchical in nature, extracted g as the first factor. Each test had a positive factor loading on this factor. Factors two and three were interpreted as two major group factors. These were referred to as the v:ed and k:m factors, with the notation being representative of verbal-numerical-educational in the case of the former, and practical-mechanical-spatial-physical for the latter. In this particular study, the v:ed factor was found to be hierarchical in structure to two minor group factors (v [verbal] and n [number]). Vernon pointed out that when there is a sufficient quantity of tests, both group factors subdivide into minor group factors.

Figure 7 depicts how the factor loading matrix might look for Vernon's analysis, based on factor loadings he reported (1950, p. 23). (An X in the figure is indicative of a reported factor loading.)

Vernon is in keeping with another member of the British school, Cyril Burt. Burt's simple summation method (1939; 1944) of factor analysis, as a precursor to Thurstone's multiple factor analysis, allowed a solution with a general factor and group factors. While retaining a general factor like Spearman, Vernon and Burt argued for the presence of additional group factors.

Unlike the British school of factor analysts, Thurstone's multiple-factor analysis is not concerned with extracting g as the first factor. In fact, as Thurstone (1969b) states:

> Instead of asking whether the central intellective factor g can be demonstrated in any set of correlations, with or without disturbing group factors, we ask how many factors are indicated by the correlations without restriction as to whether they are general or group factors. (p. vi)

His theory of multiple-factor analysis was not concerned with isolating g; it "was developed especially for the purpose of isolating primary traits" (Thurstone, 1969a, p.10). Yet, he noted that the methodology and theory are "so general that they will probably be found useful as well in solving other problems in the biological and in the social sciences" (Thurstone, 1969a, p 1).

Conceptually, multiple-factor analysis is conducted in two steps: (1) extracting factors from a correlation matrix of observed variables, and (2) rotating this solution to the primary axes. Thurstone believed it essential for rotation to occur prior to interpretation. Thurstone presented the notion of simple structure or simple configuration as a criterion for rotating a factor reference frame with regard to the factor loadings. In order to achieve simple structure, one seeks to maximize the number of zero (or near-zero) factor loadings. This is done by simultaneously minimizing the number of non-zero values for both tests and factors; that is, minimizing the non-zero entries across the rows and vertically down the columns in the rotated factor matrix.

Thurstone's position towards the type of rotation changed over time from initially favoring an orthogonal solution to an acceptance of correlated factors. His early position (Thurstone, 1969a) stated that "unless there is good evidence to the contrary, it is probably psychologically preferable to have a set of orthogonal factors" (p. 73). Thurstone's advocacy of oblique factor structure is evidenced by the statement (Thurstone, 1969b), "it is my conviction that this restriction [orthogonal reference frame] should not be imposed if we are looking for meaningful parameters" (p. vii). He went on to say that it is likely that one may find from the analyses of correlated primary factors "a second-order general factor, determined from correlated primaries, [that] may turn out to be Spearman's general intellective factor g" (p. viii).

Thurstone (and Thurstone, 1968) provide an example of the use of multiple factor analysis in an examination of mental abilities. They analyzed the records of 710 eighth-grade children (out of a total sample of 1,154) who were administered and completed a battery of 60 cognitive performance tests. Ten factors were rotated obliquely to simple structure.

The naming and interpretation of the ten factors obtained in the analysis generally employed variables with factor loadings of at least .40. Similarly, factor loadings with an absolute value no larger than .20 were omitted for purposes of naming and interpretation. (These conventions were also employed in a similar analysis reported by Thurstone [1969a]; however, in this situation the factors were rotated to an orthogonal solution.) Seven of the ten primary factors are given an interpretation, with six of them being "rather more definite" (Thurstone & Thurstone, 1968, p. 24). The correlations between these six factors were examined and a single second-order factor obtained. This general intellective factor, resembling Spearman's g, "makes its appearance, not as a separate factor, but as a factor inherent in the primaries and their correlations" (Thurstone & Thurstone, 1968, p.26).

Thurstone (1969a) also suggested the possibility of using second-order factor analysis in a confirmatory manner as a means for examining three theories, Spearman's two-factor theory, Thomson's sampling theory, and Thurstone's multiple-factor theory.

A simplified representation of the basic factor analytic models advanced by Spearman, Vernon, and Thurstone is depicted in Figure 8, adapted from Vernon (1950, p. 18). The factor loadings for the different types of factors (general, group, multiple, and specific) are indicated for each model by the first letter of the three individuals' surname. (That is, an S depicts a factor loading in accordance with Spearman's model, while V and T represent factor loadings germane to the models of Vernon and Thurstone respectively.) In this figure some similarities and dissimilarities between models can be observed. For instance, both Spearman and Vernon called for the presence of g, while Thurstone did not. There is also some similarity between Vernon's major group factors and Thurstone's primary factors.

Cattell (1950), in addition to discussing a variety of factor analytic approaches based on three elements (tests, individuals, and occasions), mentioned the multidisciplinary acceptance and use of factor analytic methods. Areas mentioned included psychology, sociology, education, economics, and political science.

Cattell also discussed what might be considered a preliminary notion of confirmatory factor analysis. His discussion suggested that an initial factor analysis could guide the factor identification process. This would be followed by a second study that included the tests with the highest factor loadings. The results of a second factor analysis would be to produce the same factors but with increased factor loadings.

Cattell (1952) spoke of the need to employ oblique factor solutions, as they "present the better application of the scientific principle of parsimony" (p. 123). An outgrowth of a correlated factor solution is that further explanation and learning can be sought in the form of second-(and higher-) order factor analysis.

Up until this point, most of the factor analytic work was conducted with first-order factors. With a few exceptions and sparse or crude applications (Carroll, 1941; Thurstone, 1969a; 1969b; Thurstone & Thurstone, 1968), second- (and higher-) order factor analysis began receiving more detailed attention about 30 years ago.

As part of an examination of his theory of crystallized and fluid intelligence, Cattell (1963) describes his use of second- and third-order factor analysis. Employing a sample of 277 individuals measured on 44 ability and personality variables, Cattell's analysis yielded 22 first-order factors, eight second-order factors, and four third-order factors. Another study (Horn & Cattell, 1966) in support of Cattell's theory employed different observed variables, but proceeded to obtain a similar second-order solution as interpreted by the authors. Both of these works received criticism, particularly the earlier study, by Humphreys (1966). Humphreys argued that in both studies there were methodological flaws. Specifically he cited low intercorrelations between the observed variables in the earlier work, and the extraction of too many first-order factors in both instances.

Gorsuch (1966) in a reanalysis of a study by Sassenrath (1964), utilized higher-order factor analysis in the area of test anxiety. Sassenrath had obtained seven orthogonal first-order factors from 34 observed variables. Gorsuch found evidence to support two underlying second-order factors and one third-order factor.

Higher-order factor analysis of personality dimensions have also been documented. In particular, Howarth (1976) examined the 57 items of the Eysenck Personality Inventory (EPI). In his analysis he derived 12 first-order factors, four second-order factors, and two third-order factors. Based on this higher-order factor analytic work Howarth suggested users be more critical in their acceptance of the EPI extraversion scale.

Partly as a test of the results obtained by Howarth (1976), Walkey and Green (1981) conducted a first-order factor analysis and compared the results with those obtained by Howarth. This study while seemingly confirmatory in nature relied exclusively on exploratory techniques. A similar situation is observed in a study reported by Horn and Stankov (1982) in which they performed a second-order factor analysis employing auditory and visual abilities of intelligence. However they reject the notion of confirmatory modeling citing its "capitalization on the chance," and that "such capitalization is likely to produce non-replicable results" (p. 179).

Huba, Segal, and Singer (1977) performed exploratory factor analyses on 28 scales of the Imaginal Processes Inventory, a battery measuring attitudes towards daydreaming and related mental style. They obtained eight first-order factors and three second-order factors that were interpreted and viewed as theoretically consistent.

Confirmatory second-order factor analysis as an area has seen its emergence and growth in terms of documented studies during approximately the last 15 years. Much of the growth in this area is undoubtedly related to the availability of commercially obtainable computer software that simplifies the analytical task of the analyses. Three popularly available examples of such software are EQS (Bentler, 1986), RAM (McArdle and McDonald, 1984), and LISREL (Joreskog & Sorbom, 1986).

The areas represented by confirmatory second-order factor analysis, like those of exploratory factor analysis, range from intelligence (Undheim & Gustafsson, 1987) and intellectual abilities (Weeks, 1980) to attitudes about health (Newcomb & Bentler, 1987) and job satisfaction (Goffin & Jackson, 1988), to personality (Marsh & Richards, 1987) and self-concept (Marsh & Hocevar, 1985).

Joreskog (1971) proposed that a second-order factor analysis model could be employed as a means of investigating the variance decomposition of observed tests in situations lacking parallel testing forms. Specifically, he proposed the ability to separate the unique variance into the component elements of specific and error variance. An application of this concept was demonstrated and expanded upon by Rindskopf and Rose (1988).

Weeks (1980) examined data for six ability tests and two achievement tests in a longitudinal re-analysis of a study by

Olsson and Bergman (1977). Weeks found a second-order factor underlying the four previously-determined first-order factors. His longitudinal second-order model, however, does not fit the data as well as the original structural-equation model of Olsson and Bergman.

Marsh and Hocevar (1985) employed confirmatory factor analytic methods in an examination of higher-order factor structure and factorial invariance across groups. They found support for seven first-order factors of self-concept among 28 measured variables for each group. Similarly they concluded that a higher-order model with three second-order factors and one third-order factor offered a more parsimonious explanation for each group. However in the subsequent factorial invariance analyses across groups, invariance was limited to the factor loadings of the first-order model. Even this finding can be questioned, as it was by Marsh and Hocevar. Based on the chi-square test, they equivocate about whether factorial invariance exists across groups.

Marsh and Richards (1987) examined the factorial structure of Rotter's Internal-External (I-E) scale with the use of confirmatory second-order factor analysis. They derived an acceptable-fitting model with five first-order factors and one second-order factor. This was interpreted as an indication that Rotter's IE construct may exist as a higher-order factor.

In an examination of cognitive abilities among three age-groups, Undheim and Gustafsson (1987) utilized confirmatory higher-order factor analytic methods. They examined a variety of second- and third-order models in support of Cattell's factors representing fluid (Gf) and crystallized (Gc) intelligence. While they encountered difficulties in obtaining acceptable models, Undheim and Gustafsson contend that "none of the studies caused rejection of the hypothesis of a perfect relationship between g [general intelligence] and Gf" (p. 169).

Newcomb and Bentler (1987) conducted a confirmatory factor analysis on measures of health status and health service utilization. Their analyses of 15 self-report indicators yielded a model with four first-order factors and one second-order factor. In an effort to fit this model it was necessary to correlate the second-order factor to one of the first-order factors, constrain a negative unique term for an observed measure to zero, and to permit correlations among nine unique estimates.

Newcomb and Bentler were forthright in their mention of something that they did that is often done in confirmatory factor analysis: Prior to the confirmatory analysis, exploratory factor analysis was performed on the aggregate data set without benefit of a hold-out, or validation, sample to determine the first-order model structure.

Goffin and Jackson (1988) examined the factor structure of a 42-item job satisfaction scale (Index of Organizational Reactions). They conducted confirmatory factor analysis at both the first- and second-order. Based on a variety of model-fit indices, the best first-order model consisted of eight correlated factors. At the second-order, two correlated higher-order factors are added to the structure obtained at the first-order. In so doing, Goffin and Jackson have specified a second-order factor model that contains both correlated first-order factors and factor-loadings for these factors on the second-order factors.

Gerbing and Anderson (1984) discuss the reformation of first-order factor models that have correlated unique terms for the observed variables into second-order factor models. They discussed a special case wherein the first-order and second-order models evidenced equivalent fit to the data. Gerbing and Anderson referred to this situation as "direct correspondence" (p. 576). They incorrectly stated that direct correspondence occurs "whenever correlated error terms are added for all pairs of indicators in a first-order factor structure which would be indicators of the same first-order factor within a correspondent second-order structure" (p. 576). This misstatement has been previously identified by Rindskopf and Rose (1988), wherein they also discussed the concept of discriminability (of fit) between factor analysis models.

Rindskopf and Rose (1988) provided an application of Joreskog's (1971) mention of the use of second-order factor analysis for variance decomposition. Utilizing a second-order factor analysis model they obtained estimates for the common, specific, and error variance components of observed variables.

These three components were obtained from the matrix elements presented earlier as Equation 5. Specifically this equation, which depicts the covariance matrix of the observed measures in terms of the model parameters, presents the common variance as equal to the appropriate matrix elements of $\Lambda \Gamma \Phi \Gamma' \Lambda'$. Similarly the specific variance is equal to elements from $\Lambda \Psi \Lambda'$, and the error variance estimates come from elements of Θ_ϵ.

Employing the three variance components, Rindskopf and Rose derived estimates of reliability and measure validity. The reliability of an observed measure, in terms of a second-order factor model, was operationalized as the sum of a measures common and specific variance elements divided by its total variance. Measure validity, defined as how well an observed variable measures a second-order factor, was operationalized as a variables common variance divided by its total variance. Rindskopf and Rose also calculated method validity; that is how well first-order factors, thought of as methods of measurement, measure second-order factors. This was operationalized as the proportion of variation in a first-order factor ($\Gamma \Phi \Gamma' + \Psi$) due to variation in a second-order factor ($\Gamma \Phi \Gamma'$). For a specific first-order factor it is the appropriate elements of $\Gamma \Phi \Gamma'$ divided by $\Gamma \Phi \Gamma' + \Psi$.

General issues of identification for certain factor analytic models, and various parameter restrictions applied to those models, were discussed by Rindskopf and Rose. They referred to the concept of equivalence of fit between factor analysis models as discriminability. Empirical examples were employed as a means of demonstration; however, algebraic proofs of model identification and equivalence were not presented.

Lee and Hershberger (1990) building on the work of Stelzl (1986) considered a priori methods of determining when models are equivalent. This work is limited to the structural part of a model. Lee and Hershberger generate equivalent models through the use of what they refer to as the replacing rule. This rule can be applied to a grouping of variables referred to as a focal block when this block is recursive, and when the relationship between the focal block and the block immediately preceding it and the block succeeding it are recursive as well. Lee and Hershberger refer to this as limited block-recursiveness.

Within a focal block that is limited block-recursive, equivalent models are generated "through the replacement of direct paths with residual correlations, through the replacement of residual correlations with direct paths, or through the inversion of path directions" (p. 313).

However in discussing criteria by which to select between equivalent models Lee and Hershberger "do not rule out the possibility that equivalent models can have different numbers of parameters" (p. 333). This view of model equivalence implies that equal fitting models are equivalent regardless of the number of parameters, rather than being more-(or less-) constrained models.

Luijben (1991) considers necessary and sufficient conditions for (local) equivalence generated within the context of nested models, where two new models are created by adding a parameter to a baseline model.

CHAPTER 3

METHODS

The current work examines issues of identification, parameter restrictions, and equivalence among four basic types of factor analysis models: one-factor, second-order, group-factor, and bi-factor. The comparisons will be conducted in two settings: one with six observed measures, the other with nine.

The initial issue to be examined is the identification of each of the four factor analysis models. Algebraic proofs will be provided that models are identified models; in such situations, solutions will be presented for the model parameters in terms of the elements of the population variance-covariance matrix. Guidelines useful in establishing model identification, primarily for factor loadings, which are also generally the parameters to be solved first, are: (1) multiplying two equations containing the parameter to be solved, and dividing this product by the covariance expressions which includes the parameters in the product not immediately being solved, and (2) substituting solutions of parameters which have been identified into equations containing parameters to be identified and solving by substitution. This latter guideline is particularly useful when identifying residual, or unique, variances.

In situations where a model is not identified, efforts will be made to find minimal identifying restrictions. The obvious situation of a model which has more parameters than variances and covariances is not identified. Models are also not identified which have a subset of equations, that are the only equations in which certain parameters appear, and these equations are fewer in number than the quantity of parameters to be solved. While occurrence of either of these situations is sufficient for a model not to be identified, rules are not available to prove a model is not identified.

Model identification in the case of six observed measures will be considered first. It is expected that the one-factor, second-order and group-factor models will be identified, but that the bi-factor model will not be identified. A first step towards identifying the bi-factor model will be to restrict, to zero, the correlations for the group-factors. While this restriction removes three parameters from the model it is expected to still have unidentified elements. Next, each pair of group factor loadings will be constrained equal. This results in a model with 15 estimated parameters, the same as the second-order and group-factor models, and is expected to be identified. The one-factor model has 12 estimated parameters.

For nine observed measures there is reason to believe that the one-factor, second-order, and group-factor models will be identified, while the bi-factor model will not. However, restricting the value of the group-factor correlations to zero is expected to identify the model. This results in a model with 27 estimated parameters. The second-order and group-factor models each have 21 parameters, while the one-factor model has 18.

Parameter restrictions and relaxations will be explored for the four models wherein one model will be examined as being possibly more-(or less-) restricted than another. A quasi-hierarchy of models is expected to be demonstrated for models with both six and nine observed measures. For instance, if the factor loadings of the general factor are restricted to equal zero for the bi-factor model, the group-factor model results. This shows the group-factor model to be a special case of the bi-factor model. The second-order model is seen to be a special case of the group-factor model where a structure has been placed on the group-factor correlations, in the form of a second-order factor. This nesting of models will also show the one-factor model to be a special case of the second-order model when the residual terms for the first-order factors are constrained to equal zero.

Based on various parameter restrictions and relaxations, a partially ordered chain is expected between the models. This ordering is expected to show the one-factor model as the most restricted of the four basic models, while the bi-factor model is the least restricted.

In certain special cases, parameter restrictions in a model may lead to models which are equivalent, instead of more-restricted. In instances where equivalence between models can be demonstrated, algebraic representations will be provided for the elements of the variance/covariance matrix in terms of the respective models' parameters.

For instance, in the case of six observed measures, if the one-factor model were to be considered as consisting of three pairs of measures with each pair having a correlated unique variable term, a one-factor model with correlated errors results. It is expected to show that this model is equivalent to the second-order model. This correlated error model is also expected to be demonstrated as equivalent to the bi-factor model with uncorrelated group-factors and equal group-factor loadings. It is also expected to show the equivalence between the second-order and group-factor models.

In the case of nine observed measures, a one-factor model, with correlated error terms within the three triads of measures is not expected to produce a first-order correlated error model equivalent to the second-order model (as in the case of six observed measures). The first-order correlated error model is expected to be demonstrated as equivalent to the bi-factor model with uncorrelated group factors. Also in the case of nine measures, it is expected to show the equivalence between the second-order and group-factor models.

As a result of the equivalence, or non-discriminability, between models the choice of model acceptance would be made on the basis of theoretical plausibility or parsimony, rather than a comparison between model goodness-of-fit test statistics.

IDENTIFICATION

The following general steps will be followed to demonstrate the identification of the four types of factor analysis models. Expressions are written which represent each of the (n) observed measures in terms of the model parameters. These equations form the basis for obtaining representations of the variance and (nonredundant) covariance elements of the observed covariance matrix. A model is identified when all of the model parameters can be represented in terms of the covariance elements.

6 Observed Measures

One-Factor Model

The first model to be examined is the one-factor model. (Refer to Figure 3 for a pictorial representation). The equations for the observed measures in terms of the model parameters follow.

$$y_1 - \lambda_1 \eta + \epsilon_1 \quad (1)$$

$$y_2 - \lambda_2 \eta + \epsilon_2 \quad (2)$$

$$y_3 - \lambda_3 \eta + \epsilon_3 \quad (3)$$

$$y_4 - \lambda_4 \eta + \epsilon_4 \quad (4)$$

$$y_5 = \lambda_5 \eta + \epsilon_5 \quad (5)$$

$$y_6 = \lambda_6 \eta + \epsilon_6 \quad (6)$$

The following are assumptions for the one-factor model: the expected values of the observed measures, factor, and residuals are all equal to zero (i.e., $E(y_i) = 0$, $E(\eta) = 0$, $E(\epsilon_i) = 0$ for $i = 1, 2, \ldots 6$), the factor variance equals one (i.e., $Var(\eta) = 1.0$), and the covariances between the factor and residuals are equal to zero (i.e., $Cov(\eta\epsilon_i) = 0$), as are the covariances between the residuals (i.e., $Cov(\epsilon_i\epsilon_j) = 0, i \neq j$).

Representations for the covariance elements are obtained in the usual manner: Each equation is multiplied by itself and all other equations in turn. The expected value is taken for each of the [n(n+1)]/2 resulting nonredundant equations. This results in representations for the observed covariance elements in terms of the model parameters. They are as follows for the one-factor model.

$$\sigma_{11} = \lambda_1^2 + \theta_{\epsilon_{11}} \quad (7)$$

$$\sigma_{22} = \lambda_2^2 + \theta_{\epsilon_{22}} \quad (8)$$

$$\sigma_{33} = \lambda_3^2 + \theta_{\epsilon_{33}} \quad (9)$$

$$\sigma_{44} = \lambda_4^2 + \theta_{\epsilon_{44}} \quad (10)$$

$$\sigma_{55} = \lambda_5^2 + \theta_{\epsilon_{55}} \quad (11)$$

$$\sigma_{66} = \lambda_6^2 + \theta_{\epsilon 66} \tag{12}$$

$$\sigma_{12} = \lambda_1 \lambda_2 \tag{13}$$

$$\sigma_{13} = \lambda_1 \lambda_3 \tag{14}$$

$$\sigma_{14} = \lambda_1 \lambda_4 \tag{15}$$

$$\sigma_{15} = \lambda_1 \lambda_5 \tag{16}$$

$$\sigma_{16} = \lambda_1 \lambda_6 \tag{17}$$

$$\sigma_{23} = \lambda_2 \lambda_3 \tag{18}$$

$$\sigma_{24} = \lambda_2 \lambda_4 \tag{19}$$

$$\sigma_{25} = \lambda_2 \lambda_5 \tag{20}$$

$$\sigma_{26} = \lambda_2 \lambda_6 \tag{21}$$

$$\sigma_{34} = \lambda_3 \lambda_4 \tag{22}$$

$$\sigma_{35} = \lambda_3 \lambda_5 \tag{23}$$

$$\sigma_{36} = \lambda_3 \lambda_6 \tag{24}$$

$$\sigma_{45} = \lambda_4 \lambda_5 \tag{25}$$

$$\sigma_{46} = \lambda_4 \lambda_6 \tag{26}$$

$$\sigma_{56} = \lambda_5 \lambda_6 \tag{27}$$

These 21 covariance element expressions are now used to solve for the 12 model parameters. Solutions for the λ's will be done first.

$$\lambda_1^2 = \frac{\sigma_{12}\,\sigma_{13}}{\sigma_{23}}$$

$$\lambda_2^2 = \frac{\sigma_{12}\,\sigma_{23}}{\sigma_{13}}$$

$$\lambda_3^2 = \frac{\sigma_{13}\ \sigma_{23}}{\sigma_{12}}$$

$$\lambda_4^2 = \frac{\sigma_{14}\ \sigma_{24}}{\sigma_{12}}$$

$$\lambda_5^2 = \frac{\sigma_{15}\ \sigma_{25}}{\sigma_{12}}$$

$$\lambda_6^2 = \frac{\sigma_{16}\ \sigma_{26}}{\sigma_{12}}$$

The residual variances are obtained by subtraction in Equations 7-11. For example from Equation 7:

$$\theta_{\epsilon_{11}} = \sigma_{11} - \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

Similar subtractions would be conducted for the remaining five residual variances.

As all twelve model parameters are identified, the one-factor model with six-observed measures is identified.

One-Factor Model With Correlated Errors

The next model to be examined is the one-factor model with correlated errors within each of the three pairs of observed measures. (That is, for example, Cov $(\epsilon_1 \epsilon_2) = \theta_{\epsilon 12}$.) The equations for the observed measures for this model are the same as for the prior model. Eighteen of the twenty-one variance and covariance expressions remain unchanged. The following equalities reflect the impact due to the correlated errors.

$$\sigma_{12} = \lambda_1 \lambda_2 + \theta_{\epsilon_{12}} \quad (28)$$

$$\sigma_{34} = \lambda_3 \lambda_4 + \theta_{\epsilon_{34}} \quad (29)$$

$$\sigma_{56} = \lambda_5 \lambda_6 + \theta_{\epsilon_{56}} \quad (30)$$

The correlated residual terms bring the total number of parameters to be estimated by the model to fifteen (6-λ's; 9- θ_{ϵ}'s).

Again the λ's are solved first.

$$\lambda_1^2 = \frac{\sigma_{14}\ \sigma_{16}}{\sigma_{46}}$$

$$\lambda_2^2 = \frac{\sigma_{24}\ \sigma_{26}}{\sigma_{46}}$$

$$\lambda_3^2 = \frac{\sigma_{13}\ \sigma_{35}}{\sigma_{15}}$$

$$\lambda_4^2 = \frac{\sigma_{14}\ \sigma_{45}}{\sigma_{15}}$$

$$\lambda_5^2 = \frac{\sigma_{15}\ \sigma_{45}}{\sigma_{14}}$$

$$\lambda_6^2 = \frac{\sigma_{16}\ \sigma_{46}}{\sigma_{14}}$$

As in the first model the θ_ϵ parameters can be solved by conducting the appropriate subtraction: Here it is necessary to solve nine parameters (using Equations 7-12 and 28-30). For example solutions based on Equations 7 and 28 follow (respectively).

$$\theta_{\epsilon_{11}} = \sigma_{11} - \frac{\sigma_{14}\ \sigma_{16}}{\sigma_{46}}$$

$$\theta_{\epsilon_{12}} = \sigma_{12} - \left[\frac{\sigma_{14}\ \sigma_{16}}{\sigma_{46}}\right]^{1/2} \left[\frac{\sigma_{24}\ \sigma_{26}}{\sigma_{46}}\right]^{1/2}$$

The seven remaining θ_ϵ parameters would be obtained by similar subtractions, thereby identifying the one-factor model with correlated errors.

Second-Order Model

Next we turn to the second-order factor analysis model. (Refer to Figure 4 for a pictorial representation.) The following assumptions are made: the expected values of the observed measures, first- and second-order factors, and residual and unique variables are all equal to zero (i.e., E(y) = 0, E(n) = 0, E(ξ) = 0, E(ξ) = 0). The covariances between the second-order factors and the unique measures are equal to zero (i.e., E($\xi\zeta'$) = 0). Additional identifying restrictions are also placed on the model: the variance of the second-order factor equals one (i.e., Var(ξ) = 1.0), and one factor loading for each first-order factor equals one.

The equations for the six observed measures follow:

$$y_1 = 1 * \eta_1 + \epsilon_1$$

$$y_2 = \lambda_1 \eta_1 + \epsilon_2$$

$$y_3 = 1 * \eta_2 + \epsilon_3$$

$$y_4 = \lambda_2 \eta_2 + \epsilon_4$$

$$y_5 = 1 * \eta_3 + \epsilon_5$$

$$y_6 = \lambda_3 \eta_3 + \epsilon_6$$

The equations for the three first-order factors follow:

$$\eta_1 = \gamma_1 \xi + \zeta_1$$

$$\eta_2 = \gamma_2 \xi + \zeta_2$$

$$\eta_3 = \gamma_3 \xi + \zeta_3$$

The Var (η_i), (i = 1, 2, 3), = $\gamma_i^2 + \psi_i$, because the Var (ξ) = 1.0.

To solve for σ_{11}, multiply the equation for y_1 by its transpose, and take the resulting expected value, producing:

$$\sigma_{11} = Var\ \eta_1 + \theta_{e_{11}}$$

By substitution this becomes:

$$\sigma_{11} = \gamma_1^2 + \psi_1 + \theta_{e_{11}}. \qquad (31)$$

In similar fashion the other five variances and fifteen covariances can be represented as follows.

$$\sigma_{22} = \lambda_1^2\ (\gamma_1^2 + \psi_1) + \theta_{e_{22}} \qquad (32)$$

$$\sigma_{33} = \gamma_2^2 + \psi_2 + \theta_{e_{33}} \qquad (33)$$

$$\sigma_{44} = \lambda_2^2\ (\gamma_2^2 + \psi_2) + \theta_{e_{44}} \qquad (34)$$

$$\sigma_{55} = \gamma_3^2 + \psi_3 + \theta_{e_{55}} \qquad (35)$$

$$\sigma_{66} = \lambda_3^2\ (\gamma_3^2 + \psi_3) + \theta_{e_{66}} \qquad (36)$$

$$\sigma_{12} = \lambda_1\ (\gamma_1^2 + \psi_1) \qquad (37)$$

$$\sigma_{13} = \gamma_1\ \gamma_2 \qquad (38)$$

$$\sigma_{14} = \lambda_2 \gamma_1 \gamma_2 \tag{39}$$

$$\sigma_{15} = \gamma_1 \gamma_3 \tag{40}$$

$$\sigma_{16} = \lambda_3 \gamma_1 \gamma_3 \tag{41}$$

$$\sigma_{23} = \lambda_1 \gamma_1 \gamma_2 \tag{42}$$

$$\sigma_{24} = \lambda_1 \lambda_2 \gamma_1 \gamma_2 \tag{43}$$

$$\sigma_{25} = \lambda_1 \gamma_1 \gamma_3 \tag{44}$$

$$\sigma_{26} = \lambda_1 \lambda_3 \gamma_1 \gamma_3 \tag{45}$$

$$\sigma_{34} = \lambda_2 (\gamma_2^2 + \psi_2) \tag{46}$$

$$\sigma_{35} = \gamma_2 \gamma_3 \tag{47}$$

$$\sigma_{36} = \lambda_3 \gamma_2 \gamma_3 \tag{48}$$

$$\sigma_{45} = \lambda_2 \gamma_2 \gamma_3 \tag{49}$$

$$\sigma_{46} = \lambda_2 \lambda_3 \gamma_2 \gamma_3 \tag{50}$$

$$\sigma_{56} = \lambda_3 (\gamma_3^2 + \psi_3) \tag{51}$$

Solving first for the three λ parameters results in the following:

$$\lambda_1 = \frac{\sigma_{23}}{\sigma_{13}}$$

$$\lambda_2 = \frac{\sigma_{14}}{\sigma_{13}}$$

$$\lambda_3 = \frac{\sigma_{16}}{\sigma_{15}}$$

Solving for the three γ parameters the following equalities result:

$$\gamma_1^2 = \frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}}$$

$$\gamma_2^2 = \frac{\sigma_{13}\ \sigma_{35}}{\sigma_{15}}$$

$$\gamma_3^2 = \frac{\sigma_{15}\ \sigma_{35}}{\sigma_{13}}$$

With estimates for these 6 parameters, the three ψ parameters can be obtained by subtraction and division in Equations 37, 46, and 51.

$$\psi_1 = \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}} - \frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}}$$

$$\psi_2 = \frac{\sigma_{13}\ \sigma_{34}}{\sigma_{14}} - \frac{\sigma_{13}\ \sigma_{35}}{\sigma_{15}}$$

$$\psi_3 = \frac{\sigma_{15}\ \sigma_{56}}{\sigma_{16}} - \frac{\sigma_{15}\ \sigma_{35}}{\sigma_{13}}$$

With solutions for these nine parameters, the six residual variances (θ_{ϵ}) in Equations 31-36 can be obtained straightforwardly by subtraction. For example, from Equation 31:

$$\theta_{\epsilon_{11}} = \sigma_{11} - \frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}} - \left[\frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}} - \frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}}\right].$$

As all fifteen parameters of the second-order factor analysis model are identified the model is identified.

Group-Factor Model

The next model to be examined for identification is the group-factor model. (Refer to Figure 5 for a pictorial representation.) The equations for the observed variables of this model are as follows.

$$y_1 = \lambda_1\ \eta_1 + \epsilon_1$$
$$y_2 = \lambda_2\ \eta_1 + \epsilon_2$$
$$y_3 = \lambda_3\ \eta_2 + \epsilon_3$$
$$y_4 = \lambda_4\ \eta_2 + \epsilon_4$$
$$y_5 = \lambda_5\ \eta_3 + \epsilon_5$$
$$y_6 = \lambda_6\ \eta_3 + \epsilon_6$$

This model has correlated factors with the factor variances constrained to a value of one (i.e., $\mathrm{Var}(n_j) = 1.0$ for j = 1, 2, 3). The variance and covariance expressions follow.

$$\sigma_{11} = \lambda_1^2 + \theta_{\epsilon_{11}} \quad (52)$$

$$\sigma_{22} = \lambda_2^2 + \theta_{\epsilon_{22}} \quad (53)$$

$$\sigma_{33} = \lambda_3^2 + \theta_{\epsilon_{33}} \quad (54)$$

$$\sigma_{44} = \lambda_4^2 + \theta_{\epsilon_{44}} \quad (55)$$

$$\sigma_{55} = \lambda_5^2 + \theta_{\epsilon_{55}} \quad (56)$$

$$\sigma_{66} = \lambda_6^2 + \theta_{\epsilon_{66}} \quad (57)$$

$$\sigma_{12} = \lambda_1 \lambda_2 \quad (58)$$

$$\sigma_{13} = \lambda_1 \lambda_3 \omega_{12} \quad (59)$$

$$\sigma_{14} = \lambda_1 \lambda_4 \omega_{12} \quad (60)$$

$$\sigma_{15} = \lambda_1 \lambda_5 \omega_{13} \quad (61)$$

$$\sigma_{16} = \lambda_1 \lambda_6 \omega_{13} \quad (62)$$

$$\sigma_{23} = \lambda_2 \lambda_3 \omega_{12} \quad (63)$$

$$\sigma_{24} = \lambda_2 \lambda_4 \omega_{12} \quad (64)$$

$$\sigma_{25} = \lambda_2 \lambda_5 \omega_{13} \quad (65)$$

$$\sigma_{26} = \lambda_2 \lambda_6 \omega_{13} \quad (66)$$

$$\sigma_{34} = \lambda_3 \lambda_4 \quad (67)$$

$$\sigma_{35} = \lambda_3 \lambda_5 \omega_{23} \quad (68)$$

$$\sigma_{36} = \lambda_3 \lambda_6 \omega_{23} \quad (69)$$

$$\sigma_{45} = \lambda_4 \lambda_5 \omega_{23} \quad (70)$$

$$\sigma_{46} = \lambda_4 \lambda_6 \omega_{23} \quad (71)$$

$$\sigma_{56} = \lambda_5 \lambda_6 \quad (72)$$

As in the prior models the λ's are solved first. This results in the following equalities.

$$\lambda_1^2 = \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

$$\lambda_2^2 = \frac{\sigma_{12}\ \sigma_{23}}{\sigma_{13}}$$

$$\lambda_3^2 = \frac{\sigma_{13}\ \sigma_{34}}{\sigma_{14}}$$

$$\lambda_4^2 = \frac{\sigma_{14}\ \sigma_{34}}{\sigma_{13}}$$

$$\lambda_5^2 = \frac{\sigma_{15}\ \sigma_{56}}{\sigma_{16}}$$

$$\lambda_6^2 = \frac{\sigma_{16}\ \sigma_{56}}{\sigma_{15}}$$

The residual variances are obtained next by subtraction in Equations 52-57. For example from Equation 52:

$$\theta_{\epsilon_{11}} = \sigma_{11} - \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

The five remaining $\theta_{\epsilon's}$ would be obtained in similar fashion.

The ω terms can be solved most directly by substituting λ solutions in the appropriate equations and simplifying the resulting expression. The following results from Equation 59:

$$\omega_{12} = \frac{\sigma_{13}}{\lambda_1\ \lambda_3}$$

$$= \left[\frac{\sigma_{14}\ \sigma_{23}}{\sigma_{12}\ \sigma_{34}}\right]^{1/2}$$

The two remaining factor correlation solutions follow (from Equations 61 and 68):

$$\omega_{23} = \left[\frac{\sigma_{14}\ \sigma_{16}\ \sigma_{35}^2}{\sigma_{13}\ \sigma_{15}\ \sigma_{34}\ \sigma_{56}}\right]^{1/2}$$

$$\omega_{13} = \left[\frac{\sigma_{15}^2 \ \sigma_{16} \ \sigma_{23}}{\sigma_{12} \ \sigma_{13} \ \sigma_{15} \ \sigma_{56}} \right]^{1/2}$$

Bi-Factor Model

The next model to be examined for identification is the bi-factor model. (Refer to Figure 6 for a pictorial representation.) The equations for the observed variables in terms of the unobserved variables are as follows.

$$y_1 = \lambda_{11} \eta_1 + \lambda_{14} \eta_4 + \epsilon_1$$
$$y_2 = \lambda_{21} \eta_1 + \lambda_{24} \eta_4 + \epsilon_2$$
$$y_3 = \lambda_{32} \eta_2 + \lambda_{34} \eta_4 + \epsilon_3$$
$$y_4 = \lambda_{42} \eta_2 + \lambda_{44} \eta_4 + \epsilon_4$$
$$y_5 = \lambda_{53} \eta_3 + \lambda_{54} \eta_4 + \epsilon_5$$
$$y_6 = \lambda_{63} \eta_3 + \lambda_{64} \eta_4 + \epsilon_6$$

This model has correlated group-factors, and all factor variances constrained to a value of one (i.e., $Var(n_j) = 1.0$ for $j = 1, \ldots, 4$).

The variance and covariance expressions for this model follow.

$$\sigma_{11} = \lambda_{11}^2 + \lambda_{14}^2 + \theta_{\epsilon_{11}} \quad (73)$$
$$\sigma_{22} = \lambda_{21}^2 + \lambda_{24}^2 + \theta_{\epsilon_{22}} \quad (74)$$
$$\sigma_{33} = \lambda_{32}^2 + \lambda_{34}^2 + \theta_{\epsilon_{33}} \quad (75)$$

$$\sigma_{44} = \lambda_{42}^2 + \lambda_{44}^2 + \theta_{\epsilon_{44}} \quad (76)$$

$$\sigma_{55} = \lambda_{53}^2 + \lambda_{54}^2 + \theta_{\epsilon_{55}} \quad (77)$$

$$\sigma_{66} = \lambda_{63}^2 + \lambda_{64}^2 + \theta_{\epsilon_{66}} \quad (78)$$

$$\sigma_{12} = \lambda_{11}\,\lambda_{21} + \lambda_{14}\,\lambda_{24} \quad (79)$$

$$\sigma_{13} = \lambda_{11}\,\lambda_{32}\,\omega_{12} + \lambda_{14}\,\lambda_{34} \quad (80)$$

$$\sigma_{14} = \lambda_{11}\,\lambda_{42}\,\omega_{12} + \lambda_{14}\,\lambda_{44} \quad (81)$$

$$\sigma_{15} = \lambda_{11}\,\lambda_{53}\,\omega_{13} + \lambda_{14}\,\lambda_{54} \quad (82)$$

$$\sigma_{16} = \lambda_{11}\,\lambda_{63}\,\omega_{13} + \lambda_{14}\,\lambda_{64} \quad (83)$$

$$\sigma_{23} = \lambda_{21}\,\lambda_{32}\,\omega_{12} + \lambda_{24}\,\lambda_{34} \quad (84)$$

$$\sigma_{24} = \lambda_{21}\,\lambda_{42}\,\omega_{12} + \lambda_{24}\,\lambda_{44} \quad (85)$$

$$\sigma_{25} = \lambda_{21}\,\lambda_{53}\,\omega_{13} + \lambda_{24}\,\lambda_{54} \quad (86)$$

$$\sigma_{26} = \lambda_{21}\,\lambda_{63}\,\omega_{13} + \lambda_{24}\,\lambda_{64} \quad (87)$$

$$\sigma_{34} = \lambda_{32}\,\lambda_{42} + \lambda_{34}\,\lambda_{44} \quad (88)$$

$$\sigma_{35} = \lambda_{32}\,\lambda_{53}\,\omega_{23} + \lambda_{34}\,\lambda_{54} \quad (89)$$

$$\sigma_{36} = \lambda_{32}\,\lambda_{63}\,\omega_{23} + \lambda_{34}\,\lambda_{64} \quad (90)$$

$$\sigma_{45} = \lambda_{42}\,\lambda_{53}\,\omega_{23} + \lambda_{44}\,\lambda_{54} \quad (91)$$

$$\sigma_{46} = \lambda_{42}\,\lambda_{63}\,\omega_{23} + \lambda_{44}\,\lambda_{64} \quad (92)$$

$$\sigma_{56} = \lambda_{53}\,\lambda_{63} + \lambda_{54}\,\lambda_{64} \quad (93)$$

The first general-factor loading can be isolated using Equations 80, 82, and 89, as follows:

$$\lambda_{14}^2 = \frac{(\sigma_{13} - \lambda_{11}\,\lambda_{32}\,\omega_{12})\;(\sigma_{15} - \lambda_{11}\,\lambda_{53}\,\omega_{13})}{(\sigma_{35} - \lambda_{32}\,\lambda_{53}\,\omega_{23})}$$

Alternatively, λ_{14} could have been isolated using Equations 80, 83, and 90, resulting in the following representation.

$$\lambda_{14}^2 = \frac{(\sigma_{13} - \lambda_{11}\,\lambda_{32}\,\omega_{12})\ (\sigma_{16} - \lambda_{11}\,\lambda_{63}\,\omega_{13})}{(\sigma_{36} - \lambda_{32}\,\lambda_{63}\,\omega_{23})}$$

Although these are equivalent representations for λ_{14}, setting them equal and simplifying algebraically does not produce a solution for any parameter. Similar expressions result for the other five general-factor loadings.

While it does not appear that any of these parameters can be solved, a satisfactory manner of establishing this could not be found. (It can be noted however that a restricted form of this model [the bi-factor model with group-factor correlations constrained equal to zero] is also not identified.) It would, however, appear useful to constrain (to zero) the factor correlations.

Bi-Factor Model With Group-Factor Correlations Constrained Equal To Zero

Constraining the three group-factor correlations to a value of zero has the effect of removing the ω elements and the two group-factor λ's in Equations 80-87 and 89-92.

With the model in this form the first general-factor loading is now represented as follows:

$$\lambda_{14}^2 = \frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}}$$

The other five general-factor loadings are similarly obtained.

We proceed next to identify the group-factor loadings. In the current form each of the (six) group-factor loadings appear in two equations: Once as a product (in Equations 79, 88, and 93), and separately as one of three summed components (in Equations 73-78). For instance λ_{11} appears only in two equations (73 and 79). After substituting values for the previously identified parameters λ_{14} and λ_{24} in these equations, there remains three parameters to be solved (λ_{11}, λ_{21}, and $\theta_{\epsilon 11}$), thereby revealing a subset of the model which consists of a greater number of parameters than equations. This is sufficient to demonstrate the model to be non-identified.

These parameters cannot be uniquely solved, but the three group-factor loading product equalities ($\lambda_{11} * \lambda_{21}$; $\lambda_{32} * \lambda_{42}$; $\lambda_{53} * \lambda_{63}$) in Equations 79, 88, and 93 indicate that a restricted form of this model might be identified.

<u>Bi-Factor Model With Group-Factor Correlations Constrained Equal to Zero And Constrained Equal Pairwise Group-Factor Loadings</u>

The next step towards model identification is to constrain equal the group-factor loadings from the same group-factor (i.e., $\lambda_{11} = \lambda_{21}$; $\lambda_{32} = \lambda_{42}$; $\lambda_{53} = \lambda_{63}$). As with the prior restriction, this restriction can in certain instances be justified from a theoretical perspective.

The variance and covariance expressions for the model in this form follow:

$$\sigma_{11} = \lambda_{11}^2 + \lambda_{14}^2 + \theta_{\epsilon_{11}} \quad (94)$$

$$\sigma_{22} = \lambda_{11}^2 + \lambda_{24}^2 + \theta_{\epsilon_{22}} \quad (95)$$

$$\sigma_{33} = \lambda_{32}^2 + \lambda_{34}^2 + \theta_{\epsilon_{33}} \quad (96)$$

$$\sigma_{44} = \lambda_{32}^2 + \lambda_{44}^2 + \theta_{\epsilon_{44}} \quad (97)$$

$$\sigma_{55} = \lambda_{53}^2 + \lambda_{54}^2 + \theta_{\epsilon_{55}} \quad (98)$$

$$\sigma_{66} = \lambda_{53}^2 + \lambda_{64}^2 + \theta_{\epsilon_{66}} \quad (99)$$

$$\sigma_{12} = \lambda_{11}^2 + \lambda_{14}\ \lambda_{24} \quad (100)$$

$$\sigma_{13} = \lambda_{14}\ \lambda_{34} \quad (101)$$

$$\sigma_{14} = \lambda_{14}\ \lambda_{44} \quad (102)$$

$$\sigma_{15} = \lambda_{14}\ \lambda_{54} \quad (103)$$

$$\sigma_{16} = \lambda_{14}\ \lambda_{64} \quad (104)$$

$$\sigma_{23} = \lambda_{24}\ \lambda_{34} \quad (105)$$

$$\sigma_{24} = \lambda_{24}\ \lambda_{44} \quad (106)$$

$$\sigma_{25} = \lambda_{24}\ \lambda_{54} \quad (107)$$

$$\sigma_{26} = \lambda_{24}\ \lambda_{64} \quad (108)$$

$$\sigma_{34} = \lambda_{32}^2 + \lambda_{34}\ \lambda_{44} \quad (109)$$

$$\sigma_{35} = \lambda_{34}\ \lambda_{54} \quad (110)$$

$$\sigma_{36} = \lambda_{34}\ \lambda_{64} \quad (111)$$

$$\sigma_{45} = \lambda_{44}\ \lambda_{54} \quad (112)$$

$$\sigma_{46} = \lambda_{44}\ \lambda_{64} \quad (113)$$

$$\sigma_{56} = \lambda_{53}^2 + \lambda_{54}\,\lambda_{64} \tag{114}$$

Solutions will be done first for the general-factor loadings.

$$\lambda_{14}^2 = \frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}}$$

$$\lambda_{24}^2 = \frac{\sigma_{23}\ \sigma_{25}}{\sigma_{35}}$$

$$\lambda_{34}^2 = \frac{\sigma_{13}\ \sigma_{35}}{\sigma_{15}}$$

$$\lambda_{44}^2 = \frac{\sigma_{14}\ \sigma_{45}}{\sigma_{15}}$$

$$\lambda_{54}^2 = \frac{\sigma_{15}\ \sigma_{35}}{\sigma_{13}}$$

$$\lambda_{64}^2 = \frac{\sigma_{16}\ \sigma_{46}}{\sigma_{14}}$$

The three group-factor λ's are obtained next by substitution in Equations 100, 109, and 114.

$$\lambda_{11}^2 = \sigma_{12} - \left[\frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}}\right]^{1/2} \left[\frac{\sigma_{23}\ \sigma_{25}}{\sigma_{35}}\right]^{1/2}$$

$$\lambda_{32}^2 = \sigma_{34} - \left[\frac{\sigma_{13}\ \sigma_{35}}{\sigma_{15}}\right]^{1/2} \left[\frac{\sigma_{14}\ \sigma_{45}}{\sigma_{15}}\right]^{1/2}$$

$$\lambda_{53}^2 = \sigma_{56} - \left[\frac{\sigma_{15}\ \sigma_{35}}{\sigma_{13}}\right]^{1/2} \left[\frac{\sigma_{16}\ \sigma_{46}}{\sigma_{14}}\right]^{1/2}$$

Solutions for the six θ_ϵ parameters are obtained by subtraction in Equations 94 to 99. For instance, the solution resulting from Equation 94 follows.

$$\theta_{\epsilon_{11}} = \sigma_{11} - \left[\sigma_{12} - \left[\frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}}\right]^{1/2} \left[\frac{\sigma_{23}\ \sigma_{25}}{\sigma_{35}}\right]^{1/2}\right] - \frac{\sigma_{13}\ \sigma_{15}}{\sigma_{35}}$$

As all 15 model parameters are identified, the doubly-constrained bi-factor model is identified.

9 Observed Measures

This section examines the identification of the four factor-analytic models when there are nine observed measures. Unless otherwise stated, the assumptions previously specified for each model are applicable here as well.

One-Factor Model

First to be looked at is the one-factor model. The equations for the nine observed measures and the 45 parameter expressions follow.

$y_1 = \lambda_1 \eta + \epsilon_1$

$y_2 = \lambda_2 \eta + \epsilon_2$

$y_3 = \lambda_3 \eta + \epsilon_3$

$y_4 = \lambda_4 \eta + \epsilon_4$

$y_5 = \lambda_5 \eta + \epsilon_5$

$y_6 = \lambda_6 \eta + \epsilon_6$

$y_7 = \lambda_7 \eta + \epsilon_7$

$y_8 = \lambda_8 \eta + \epsilon_8$

$y_9 = \lambda_9 \eta + \epsilon_9$

$$\sigma_{11} = \lambda_1^2 + \theta_{\epsilon_{11}} \qquad (115)$$

$$\sigma_{22} = \lambda_2^2 + \theta_{\epsilon_{22}} \qquad (116)$$

$$\sigma_{33} = \lambda_3^2 + \theta_{\epsilon_{33}} \qquad (117)$$

$$\sigma_{44} = \lambda_4^2 + \theta_{e_{44}} \qquad (118)$$

$$\sigma_{55} = \lambda_5^2 + \theta_{e_{55}} \qquad (119)$$

$$\sigma_{66} = \lambda_6^2 + \theta_{e_{66}} \qquad (120)$$

$$\sigma_{77} = \lambda_7^2 + \theta_{e_{77}} \qquad (121)$$

$$\sigma_{88} = \lambda_8^2 + \theta_{e_{88}} \qquad (122)$$

$$\sigma_{99} = \lambda_9^2 + \theta_{e_{99}} \qquad (123)$$

$$\sigma_{12} = \lambda_1 \lambda_2 \qquad (124)$$

$$\sigma_{13} = \lambda_1 \lambda_3 \qquad (125)$$

$$\sigma_{14} = \lambda_1 \lambda_4 \qquad (126)$$

$$\sigma_{15} = \lambda_1 \lambda_5 \qquad (127)$$

$$\sigma_{16} = \lambda_1 \lambda_6 \qquad (128)$$

$$\sigma_{17} = \lambda_1 \lambda_7 \qquad (129)$$

$$\sigma_{18} = \lambda_1 \lambda_8 \qquad (130)$$

$$\sigma_{19} = \lambda_1 \lambda_9 \qquad (131)$$

$$\sigma_{23} = \lambda_2 \lambda_3 \qquad (132)$$

$$\sigma_{24} = \lambda_2 \lambda_4 \qquad (133)$$

$$\sigma_{25} = \lambda_2 \lambda_5 \qquad (134)$$

$$\sigma_{26} = \lambda_2 \lambda_6 \qquad (135)$$

$$\sigma_{27} = \lambda_2 \lambda_7 \qquad (136)$$

$$\sigma_{28} = \lambda_2 \lambda_8 \qquad (137)$$

$$\sigma_{29} = \lambda_2 \lambda_9 \qquad (138)$$

$$\sigma_{34} = \lambda_3 \lambda_4 \qquad (139)$$

$$\sigma_{35} = \lambda_3 \lambda_5 \tag{140}$$

$$\sigma_{36} = \lambda_3 \lambda_6 \tag{141}$$

$$\sigma_{37} = \lambda_3 \lambda_7 \tag{142}$$

$$\sigma_{38} = \lambda_3 \lambda_8 \tag{143}$$

$$\sigma_{39} = \lambda_3 \lambda_9 \tag{144}$$

$$\sigma_{45} = \lambda_4 \lambda_5 \tag{145}$$

$$\sigma_{46} = \lambda_4 \lambda_6 \tag{146}$$

$$\sigma_{47} = \lambda_4 \lambda_7 \tag{147}$$

$$\sigma_{48} = \lambda_4 \lambda_8 \tag{148}$$

$$\sigma_{49} = \lambda_4 \lambda_9 \tag{149}$$

$$\sigma_{56} = \lambda_5 \lambda_6 \tag{150}$$

$$\sigma_{57} = \lambda_5 \lambda_7 \tag{151}$$

$$\sigma_{58} = \lambda_5 \lambda_8 \tag{152}$$

$$\sigma_{59} = \lambda_5 \lambda_9 \tag{153}$$

$$\sigma_{67} = \lambda_6 \lambda_7 \tag{154}$$

$$\sigma_{68} = \lambda_6 \lambda_8 \tag{155}$$

$$\sigma_{69} = \lambda_6 \lambda_9 \tag{156}$$

$$\sigma_{78} = \lambda_7 \lambda_8 \tag{157}$$

$$\sigma_{79} = \lambda_7 \lambda_9 \tag{158}$$

$$\sigma_{89} = \lambda_8 \lambda_9 \tag{159}$$

Again, the λ elements will be solved first.

$$\lambda_1^2 = \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

$$\lambda_2^2 = \frac{\sigma_{12}\ \sigma_{23}}{\sigma_{13}}$$

$$\lambda_3^2 = \frac{\sigma_{13}\ \sigma_{23}}{\sigma_{12}}$$

$$\lambda_4^2 = \frac{\sigma_{14}\ \sigma_{24}}{\sigma_{12}}$$

$$\lambda_5^2 = \frac{\sigma_{15}\ \sigma_{25}}{\sigma_{12}}$$

$$\lambda_6^2 = \frac{\sigma_{16}\ \sigma_{26}}{\sigma_{12}}$$

$$\lambda_7^2 = \frac{\sigma_{17}\ \sigma_{27}}{\sigma_{12}}$$

$$\lambda_8^2 = \frac{\sigma_{18}\ \sigma_{28}}{\sigma_{12}}$$

$$\lambda_9^2 = \frac{\sigma_{19}\ \sigma_{29}}{\sigma_{12}}$$

The solutions for the unique variances are obtained by subtraction in Equations 115 to 123. For instance from Equation 115:

$$\theta_{e_{11}} = \sigma_{11} - \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

The other eight variances follow.

$$\theta_{e_{22}} = \sigma_{22} - \frac{\sigma_{12}\ \sigma_{23}}{\sigma_{13}}$$

$$\theta_{e_{33}} = \sigma_{33} - \frac{\sigma_{13}\ \sigma_{23}}{\sigma_{12}}$$

$$\theta_{e_{44}} = \sigma_{44} - \frac{\sigma_{14}\ \sigma_{24}}{\sigma_{12}}$$

$$\theta_{e_{55}} = \sigma_{55} - \frac{\sigma_{15}\ \sigma_{25}}{\sigma_{12}}$$

$$\theta_{e_{66}} = \sigma_{66} - \frac{\sigma_{16}\ \sigma_{26}}{\sigma_{12}}$$

$$\theta_{e_{77}} = \sigma_{77} - \frac{\sigma_{17}\ \sigma_{27}}{\sigma_{12}}$$

$$\theta_{e_{88}} = \sigma_{88} - \frac{\sigma_{18}\ \sigma_{28}}{\sigma_{12}}$$

$$\theta_{e_{99}} = \sigma_{99} - \frac{\sigma_{19}\ \sigma_{29}}{\sigma_{12}}$$

As all eighteen parameters are identified, the one-factor model is identified.

<u>One-Factor Model With Correlated Errors</u>

This model has three correlated error terms within each of the first, second, and third triads of observed measures. All nine equations for the observed measures are identical to the prior model. Following are the nine covariance expressions which differ from that model (124-125, 132, 145-146, 150, and 157-159).

$$\sigma_{12} = \lambda_1 \lambda_2 + \theta_{e_{12}}$$

$$\sigma_{13} = \lambda_1 \lambda_3 + \theta_{e_{13}}$$

$$\sigma_{23} = \lambda_2 \lambda_2 + \theta_{e_{23}}$$

$$\sigma_{45} = \lambda_4 \lambda_5 + \theta_{e_{45}}$$

$$\sigma_{46} = \lambda_4 \lambda_6 + \theta_{e_{46}}$$

$$\sigma_{56} = \lambda_5 \lambda_6 + \theta_{e_{56}}$$

$$\sigma_{78} = \lambda_7 \lambda_8 + \theta_{e_{78}}$$

$$\sigma_{79} = \lambda_7 \lambda_9 + \theta_{e_{79}}$$

$$\sigma_{89} = \lambda_8 \lambda_9 + \theta_{e_{89}}$$

Again the λ elements will be solved first.

$$\lambda_1^2 = \frac{\sigma_{14} \sigma_{17}}{\sigma_{47}}$$

$$\lambda_2^2 = \frac{\sigma_{24} \sigma_{27}}{\sigma_{47}}$$

$$\lambda_3^2 = \frac{\sigma_{34} \sigma_{37}}{\sigma_{47}}$$

$$\lambda_4^2 = \frac{\sigma_{14} \sigma_{47}}{\sigma_{17}}$$

$$\lambda_5^2 = \frac{\sigma_{15} \sigma_{57}}{\sigma_{17}}$$

$$\lambda_6^2 = \frac{\sigma_{16} \sigma_{67}}{\sigma_{17}}$$

$$\lambda_7^2 = \frac{\sigma_{17} \sigma_{47}}{\sigma_{14}}$$

$$\lambda_8^2 = \frac{\sigma_{18} \sigma_{48}}{\sigma_{14}}$$

$$\lambda_9^2 = \frac{\sigma_{19}\ \sigma_{49}}{\sigma_{14}}$$

The solutions for the nine unique variances and nine covariances follow.

$$\theta_{e_{11}} = \sigma_{11} - \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}$$

$$\theta_{e_{22}} = \sigma_{22} - \frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}$$

$$\theta_{e_{33}} = \sigma_{33} - \frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}$$

$$\theta_{e_{44}} = \sigma_{44} - \frac{\sigma_{14}\ \sigma_{47}}{\sigma_{17}}$$

$$\theta_{e_{55}} = \sigma_{55} - \frac{\sigma_{15}\ \sigma_{57}}{\sigma_{17}}$$

$$\theta_{e_{66}} = \sigma_{66} - \frac{\sigma_{16}\ \sigma_{67}}{\sigma_{17}}$$

$$\theta_{e_{77}} = \sigma_{77} - \frac{\sigma_{17}\ \sigma_{47}}{\sigma_{14}}$$

$$\theta_{e_{88}} = \sigma_{88} - \frac{\sigma_{18}\ \sigma_{48}}{\sigma_{14}}$$

$$\theta_{e_{99}} = \sigma_{99} - \frac{\sigma_{19}\ \sigma_{49}}{\sigma_{14}}$$

$$\theta_{e_{12}} = \sigma_{12} - \left[\frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right]^{1/2} \left[\frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}\right]^{1/2}$$

$$\theta_{e_{13}} = \sigma_{13} - \left[\frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right]^{1/2} \left[\frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}\right]^{1/2}$$

$$\theta_{e_{23}} = \sigma_{23} - \left[\frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}\right]^{1/2} \left[\frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}\right]^{1/2}$$

$$\theta_{\epsilon_{45}} = \sigma_{45} - \left[\frac{\sigma_{14}\ \sigma_{47}}{\sigma_{17}}\right]^{1/2} \left[\frac{\sigma_{15}\ \sigma_{57}}{\sigma_{17}}\right]^{1/2}$$

$$\theta_{\epsilon_{46}} = \sigma_{46} - \left[\frac{\sigma_{14}\ \sigma_{47}}{\sigma_{17}}\right]^{1/2} \left[\frac{\sigma_{16}\ \sigma_{67}}{\sigma_{17}}\right]^{1/2}$$

$$\theta_{\epsilon_{56}} = \sigma_{56} - \left[\frac{\sigma_{15}\ \sigma_{57}}{\sigma_{17}}\right]^{1/2} \left[\frac{\sigma_{16}\ \sigma_{67}}{\sigma_{17}}\right]^{1/2}$$

$$\theta_{\epsilon_{78}} = \sigma_{78} - \left[\frac{\sigma_{17}\ \sigma_{47}}{\sigma_{14}}\right]^{1/2} \left[\frac{\sigma_{18}\ \sigma_{48}}{\sigma_{14}}\right]^{1/2}$$

$$\theta_{\epsilon_{79}} = \sigma_{79} - \left[\frac{\sigma_{17}\ \sigma_{47}}{\sigma_{14}}\right]^{1/2} \left[\frac{\sigma_{19}\ \sigma_{49}}{\sigma_{14}}\right]^{1/2}$$

$$\theta_{\epsilon_{89}} = \sigma_{89} - \left[\frac{\sigma_{18}\ \sigma_{48}}{\sigma_{14}}\right]^{1/2} \left[\frac{\sigma_{19}\ \sigma_{49}}{\sigma_{14}}\right]^{1/2}$$

As all 27 parameters are identified, the one-factor model with correlated errors is identified.

Second-Order Model

The second-order model is the next to be identified. The equations for the observed measures follow.

$$y_1 = 1 * \eta_1 + \epsilon_1$$

$$y_2 = \lambda_1\ \eta_1 + \epsilon_2$$

$$y_3 = \lambda_2\ \eta_1 + \epsilon_3$$

$$y_4 = 1 * \eta_2 + \epsilon_4$$

$$y_5 = \lambda_3\ \eta_2 + \epsilon_5$$

$$y_6 = \lambda_4\ \eta_2 + \epsilon_6$$

$$y_7 = 1 * \eta_3 + \epsilon_7$$

$y_8 = \lambda_5 \eta_3 + \epsilon_8$

$y_9 = \lambda_6 \eta_3 + \epsilon_9$

The assumptions and restrictions of the second-order model with six measures are applied. The variance and covariance elements follow.

$$\sigma_{11} = \gamma_1^2 + \psi_1 + \theta_{\epsilon_{11}} \quad (160)$$

$$\sigma_{22} = \lambda_1^2 (\gamma_1^2 + \psi_1) + \theta_{\epsilon_{22}} \quad (161)$$

$$\sigma_{33} = \lambda_2^2 (\gamma_1^2 + \psi_1) + \theta_{\epsilon_{33}} \quad (162)$$

$$\sigma_{44} = \gamma_2^2 + \psi_2 + \theta_{\epsilon_{44}} \quad (163)$$

$$\sigma_{55} = \lambda_3^2 (\gamma_2^2 + \psi_2) + \theta_{\epsilon_{55}} \quad (164)$$

$$\sigma_{66} = \lambda_4^2 (\gamma_2^2 + \psi_2) + \theta_{\epsilon_{66}} \quad (165)$$

$$\sigma_{77} = \gamma_3^2 + \psi_3 + \theta_{\epsilon_{77}} \quad (166)$$

$$\sigma_{88} = \lambda_5^2 (\gamma_3^2 + \psi_3) + \theta_{\epsilon_{88}} \quad (167)$$

$$\sigma_{99} = \lambda_6^2 (\gamma_3^2 + \psi_3) + \theta_{\epsilon_{99}} \quad (168)$$

$$\sigma_{12} = \lambda_1 (\gamma_1^2 + \psi_1) \quad (169)$$

$$\sigma_{13} = \lambda_2 (\gamma_1^2 + \psi_1) \quad (170)$$

$$\sigma_{14} = \gamma_1 \gamma_2 \quad (171)$$

$$\sigma_{15} = \lambda_3 \gamma_1 \gamma_2 \quad (172)$$

$$\sigma_{16} = \lambda_4 \gamma_1 \gamma_2 \quad (173)$$

$$\sigma_{17} = \gamma_1 \gamma_3 \quad (174)$$

$$\sigma_{18} = \lambda_5 \gamma_1 \gamma_3 \quad (175)$$

$$\sigma_{19} = \lambda_6 \gamma_1 \gamma_3 \quad (176)$$

$$\sigma_{23} = \lambda_1 \lambda_2 (\gamma_1^2 + \psi_1) \quad (177)$$

$$\sigma_{24} = \lambda_1 \gamma_1 \gamma_2 \quad (178)$$

$$\sigma_{25} = \lambda_1 \lambda_3 \gamma_1 \gamma_2 \quad (179)$$

$$\sigma_{26} = \lambda_1 \lambda_4 \gamma_1 \gamma_2 \quad (180)$$

$$\sigma_{27} = \lambda_1 \gamma_1 \gamma_3 \quad (181)$$

$$\sigma_{28} = \lambda_1 \lambda_5 \gamma_1 \gamma_3 \quad (182)$$

$$\sigma_{29} = \lambda_1 \lambda_6 \gamma_1 \gamma_3 \quad (183)$$

$$\sigma_{34} = \lambda_2 \gamma_1 \gamma_2 \quad (184)$$

$$\sigma_{35} = \lambda_2 \lambda_3 \gamma_1 \gamma_2 \quad (185)$$

$$\sigma_{36} = \lambda_2 \lambda_4 \gamma_1 \gamma_2 \quad (186)$$

$$\sigma_{37} = \lambda_2 \gamma_1 \gamma_3 \quad (187)$$

$$\sigma_{38} = \lambda_2 \lambda_5 \gamma_1 \gamma_3 \quad (188)$$

$$\sigma_{39} = \lambda_2 \lambda_6 \gamma_1 \gamma_3 \quad (189)$$

$$\sigma_{45} = \lambda_3 (\gamma_2^2 + \psi_2) \quad (190)$$

$$\sigma_{46} = \lambda_4 (\gamma_2^2 + \psi_2) \quad (191)$$

$$\sigma_{47} = \gamma_2 \gamma_3 \quad (192)$$

$$\sigma_{48} = \lambda_5 \gamma_2 \gamma_3 \quad (193)$$

$$\sigma_{49} = \lambda_6 \gamma_2 \gamma_3 \quad (194)$$

$$\sigma_{56} = \lambda_3 \lambda_4 (\gamma_2^2 + \psi_2) \quad (195)$$

$$\sigma_{57} = \lambda_3 \gamma_2 \gamma_3 \quad (196)$$

$$\sigma_{58} = \lambda_3 \lambda_5 \gamma_2 \gamma_3 \quad (197)$$

$$\sigma_{59} = \lambda_3 \lambda_6 \gamma_2 \gamma_3 \quad (198)$$

$$\sigma_{67} = \lambda_4 \gamma_2 \gamma_3 \quad (199)$$

$$\sigma_{68} = \lambda_4 \lambda_5 \gamma_2 \gamma_3 \tag{200}$$

$$\sigma_{69} = \lambda_4 \lambda_6 \gamma_2 \gamma_3 \tag{201}$$

$$\sigma_{78} = \lambda_5 (\gamma_3^2 + \psi_3) \tag{202}$$

$$\sigma_{79} = \lambda_6 (\gamma_3^2 + \psi_3) \tag{203}$$

$$\sigma_{89} = \lambda_5 \lambda_6 (\gamma_3^2 + \psi_3) \tag{204}$$

Again the λ parameters will be solved first.

$$\lambda_1 = \frac{\sigma_{24}}{\sigma_{14}}$$

$$\lambda_2 = \frac{\sigma_{34}}{\sigma_{14}}$$

$$\lambda_3 = \frac{\sigma_{15}}{\sigma_{14}}$$

$$\lambda_4 = \frac{\sigma_{16}}{\sigma_{14}}$$

$$\lambda_5 = \frac{\sigma_{18}}{\sigma_{17}}$$

$$\lambda_6 = \frac{\sigma_{19}}{\sigma_{17}}$$

Next are representations for the γ 's.

$$\gamma_1^2 = \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}$$

$$\gamma_2^2 = \frac{\sigma_{14}\ \sigma_{47}}{\sigma_{17}}$$

$$\gamma_3^2 = \frac{\sigma_{17}\ \sigma_{47}}{\sigma_{14}}$$

The ψ's are solved next, substituting the λ and γ solutions into equations resulting from the appropriate divisions and subtractions.

$$\psi_1 = \frac{\sigma_{12}\ \sigma_{14}}{\sigma_{24}} - \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}$$

$$\psi_2 = \frac{\sigma_{14}\ \sigma_{45}}{\sigma_{15}} - \frac{\sigma_{14}\ \sigma_{47}}{\sigma_{17}}$$

$$\psi_3 = \frac{\sigma_{17}\ \sigma_{78}}{\sigma_{18}} - \frac{\sigma_{17}\ \sigma_{47}}{\sigma_{14}}$$

The θ_ϵ parameters are solved for by subtraction in Equations 160-168. As an example from Equation 160:

$$\theta_{e_{11}} = \sigma_{11} - \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}} - \left[\frac{\sigma_{12}\ \sigma_{14}}{\sigma_{24}} - \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right]$$

The remaining θ_ϵ parameters are solved by similar subtraction and substitution, thereby identifying the second-order model.

Group-Factor Model

The group-factor model is considered next. First the equations for the observed measures are written, followed by the resulting expressions for the variance and covariance elements.

$$y_1 = \lambda_1 \eta_1 + \epsilon_1$$

$$y_2 = \lambda_2 \eta_1 + \epsilon_2$$

$$y_3 = \lambda_3 \eta_1 + \epsilon_3$$

$$y_4 = \lambda_4 \eta_2 + \epsilon_4$$

$$y_5 = \lambda_5 \eta_2 + \epsilon_5$$

$$y_6 = \lambda_6 \eta_2 + \epsilon_6$$

$$y_7 = \lambda_7 \eta_3 + \epsilon_7$$

$$y_8 = \lambda_8 \eta_3 + \epsilon_8$$

$$y_9 = \lambda_9 \eta_3 + \epsilon_9$$

$$\sigma_{11} = \lambda_1^2 + \theta_{\epsilon_{11}} \qquad (205)$$

$$\sigma_{22} = \lambda_2^2 + \theta_{\epsilon_{22}} \qquad (206)$$

$$\sigma_{33} = \lambda_3^2 + \theta_{\epsilon_{33}} \qquad (207)$$

$$\sigma_{44} = \lambda_4^2 + \theta_{\epsilon_{44}} \qquad (208)$$

$$\sigma_{55} = \lambda_5^2 + \theta_{\epsilon_{55}} \qquad (209)$$

$$\sigma_{66} = \lambda_6^2 + \theta_{\epsilon_{66}} \qquad (210)$$

$$\sigma_{77} = \lambda_7^2 + \theta_{\epsilon_{77}} \qquad (211)$$

$$\sigma_{88} = \lambda_8^2 + \theta_{\epsilon_{88}} \quad (212)$$

$$\sigma_{99} = \lambda_9^2 + \theta_{\epsilon_{99}} \quad (213)$$

$$\sigma_{12} = \lambda_1 \lambda_2 \quad (214)$$

$$\sigma_{13} = \lambda_1 \lambda_3 \quad (215)$$

$$\sigma_{14} = \lambda_1 \lambda_4 \omega_{12} \quad (216)$$

$$\sigma_{15} = \lambda_1 \lambda_5 \omega_{12} \quad (217)$$

$$\sigma_{16} = \lambda_1 \lambda_6 \omega_{12} \quad (218)$$

$$\sigma_{17} = \lambda_1 \lambda_7 \omega_{13} \quad (219)$$

$$\sigma_{18} = \lambda_1 \lambda_8 \omega_{13} \quad (220)$$

$$\sigma_{19} = \lambda_1 \lambda_9 \omega_{13} \quad (221)$$

$$\sigma_{23} = \lambda_2 \lambda_3 \quad (222)$$

$$\sigma_{24} = \lambda_2 \lambda_4 \omega_{12} \quad (223)$$

$$\sigma_{25} = \lambda_2 \lambda_5 \omega_{12} \quad (224)$$

$$\sigma_{26} = \lambda_2 \lambda_6 \omega_{12} \quad (225)$$

$$\sigma_{27} = \lambda_2 \lambda_7 \omega_{13} \quad (226)$$

$$\sigma_{28} = \lambda_2 \lambda_8 \omega_{13} \quad (227)$$

$$\sigma_{29} = \lambda_2 \lambda_9 \omega_{13} \quad (228)$$

$$\sigma_{34} = \lambda_3 \lambda_4 \omega_{12} \quad (229)$$

$$\sigma_{35} = \lambda_3 \lambda_5 \omega_{12} \quad (230)$$

$$\sigma_{36} = \lambda_3 \lambda_6 \omega_{12} \quad (231)$$

$$\sigma_{37} = \lambda_3 \lambda_7 \omega_{13} \quad (232)$$

$$\sigma_{38} = \lambda_3 \lambda_8 \omega_{13} \quad (233)$$

$$\sigma_{39} = \lambda_3 \lambda_9 \omega_{13} \quad (234)$$

$$\sigma_{45} = \lambda_4 \lambda_5 \tag{235}$$

$$\sigma_{46} = \lambda_4 \lambda_6 \tag{236}$$

$$\sigma_{47} = \lambda_4 \lambda_7 \omega_{23} \tag{237}$$

$$\sigma_{48} = \lambda_4 \lambda_8 \omega_{23} \tag{238}$$

$$\sigma_{49} = \lambda_4 \lambda_9 \omega_{23} \tag{239}$$

$$\sigma_{56} = \lambda_5 \lambda_6 \tag{240}$$

$$\sigma_{57} = \lambda_5 \lambda_7 \omega_{23} \tag{241}$$

$$\sigma_{58} = \lambda_5 \lambda_8 \omega_{23} \tag{242}$$

$$\sigma_{59} = \lambda_5 \lambda_9 \omega_{23} \tag{243}$$

$$\sigma_{67} = \lambda_6 \lambda_7 \omega_{23} \tag{244}$$

$$\sigma_{68} = \lambda_6 \lambda_8 \omega_{23} \tag{245}$$

$$\sigma_{69} = \lambda_6 \lambda_9 \omega_{23} \tag{246}$$

$$\sigma_{78} = \lambda_7 \lambda_8 \tag{247}$$

$$\sigma_{79} = \lambda_7 \lambda_9 \tag{248}$$

$$\sigma_{89} = \lambda_8 \lambda_9 \tag{249}$$

Again the λ's are solved first.

$$\lambda_1^2 = \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

$$\lambda_2^2 = \frac{\sigma_{12}\ \sigma_{23}}{\sigma_{13}}$$

$$\lambda_3^2 = \frac{\sigma_{13}\ \sigma_{23}}{\sigma_{12}}$$

$$\lambda_4^2 = \frac{\sigma_{45}\ \sigma_{46}}{\sigma_{56}}$$

$$\lambda_5^2 = \frac{\sigma_{45}\ \sigma_{56}}{\sigma_{46}}$$

$$\lambda_6^2 = \frac{\sigma_{46}\ \sigma_{56}}{\sigma_{45}}$$

$$\lambda_7^2 = \frac{\sigma_{78}\ \sigma_{79}}{\sigma_{89}}$$

$$\lambda_8^2 = \frac{\sigma_{78}\ \sigma_{89}}{\sigma_{79}}$$

$$\lambda_9^2 = \frac{\sigma_{79}\ \sigma_{89}}{\sigma_{78}}$$

The ω parameters are solved by division in Equations 216, 220, and 238.

$$\omega_{12} = \left[\frac{\sigma_{14}^2\ \sigma_{23}\ \sigma_{56}}{\sigma_{12}\ \sigma_{13}\ \sigma_{45}\ \sigma_{46}}\right]^{1/2}$$

$$\omega_{13} = \left[\frac{\sigma_{17}^2\ \sigma_{23}\ \sigma_{89}}{\sigma_{12}\ \sigma_{13}\ \sigma_{78}\ \sigma_{79}}\right]^{1/2}$$

$$\omega_{23} = \left[\frac{\sigma_{47}^{2}\ \sigma_{56}\ \sigma_{89}}{\sigma_{45}\ \sigma_{46}\ \sigma_{78}\ \sigma_{79}}\right]^{1/2}$$

The θ_e 's are solved for by subtraction, and then substituting for the appropriate, previously solved for, λ element.

$$\theta_{e_{11}} = \sigma_{11} - \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

$$\theta_{e_{22}} = \sigma_{22} - \frac{\sigma_{12}\ \sigma_{23}}{\sigma_{13}}$$

$$\theta_{e_{33}} = \sigma_{33} - \frac{\sigma_{13}\ \sigma_{23}}{\sigma_{12}}$$

$$\theta_{e_{44}} = \sigma_{44} - \frac{\sigma_{45}\ \sigma_{46}}{\sigma_{56}}$$

$$\theta_{e_{55}} = \sigma_{55} - \frac{\sigma_{45}\ \sigma_{56}}{\sigma_{46}}$$

$$\theta_{e_{66}} = \sigma_{66} - \frac{\sigma_{46}\ \sigma_{56}}{\sigma_{45}}$$

$$\theta_{e_{77}} = \sigma_{77} - \frac{\sigma_{78}\ \sigma_{79}}{\sigma_{89}}$$

$$\theta_{e_{88}} = \sigma_{88} - \frac{\sigma_{78}\ \sigma_{89}}{\sigma_{79}}$$

$$\theta_{e_{99}} = \sigma_{99} - \frac{\sigma_{79}\ \sigma_{89}}{\sigma_{78}}$$

As all 21 parameters in the model have solutions, the group-factor model is identified.

Bi-Factor Model

The next model to be identified is the bi-factor model. The equations for the observed measures follow.

$$y_1 = \lambda_{11}\,\eta_1 + \lambda_{14}\,\eta_4 + \epsilon_1$$

$$y_2 = \lambda_{21}\,\eta_1 + \lambda_{24}\,\eta_4 + \epsilon_2$$

$$y_3 = \lambda_{31}\,\eta_1 + \lambda_{34}\,\eta_4 + \epsilon_3$$

$$y_4 = \lambda_{42}\,\eta_2 + \lambda_{44}\,\eta_4 + \epsilon_4$$

$$y_5 = \lambda_{52}\,\eta_2 + \lambda_{54}\,\eta_4 + \epsilon_5$$

$$y_6 = \lambda_{62}\,\eta_2 + \lambda_{64}\,\eta_4 + \epsilon_6$$

$$y_7 = \lambda_{73}\,\eta_3 + \lambda_{74}\,\eta_4 + \epsilon_7$$

$$y_8 = \lambda_{83}\,\eta_3 + \lambda_{84}\,\eta_4 + \epsilon_8$$

$$y_9 = \lambda_{93}\,\eta_3 + \lambda_{94}\,\eta_4 + \epsilon_9$$

Expressions for the variances and covariances are presented next.

$$\sigma_{11} = \lambda_{11}^2 + \lambda_{14}^2 + \theta_{\epsilon_{11}} \qquad (250)$$

$$\sigma_{22} = \lambda_{21}^2 + \lambda_{24}^2 + \theta_{\epsilon_{22}} \qquad (251)$$

$$\sigma_{33} = \lambda_{31}^2 + \lambda_{34}^2 + \theta_{\epsilon_{33}} \qquad (252)$$

$$\sigma_{44} = \lambda_{42}^2 + \lambda_{44}^2 + \theta_{\epsilon_{44}} \qquad (253)$$

$$\sigma_{55} = \lambda_{52}^2 + \lambda_{54}^2 + \theta_{\epsilon_{55}} \qquad (254)$$

$$\sigma_{66} = \lambda_{62}^2 + \lambda_{64}^2 + \theta_{\epsilon_{66}} \qquad (255)$$

$$\sigma_{77} = \lambda_{73}^2 + \lambda_{74}^2 + \theta_{\epsilon_{77}} \qquad (256)$$

$$\sigma_{88} = \lambda_{83}{}^2 + \lambda_{84}{}^2 + \theta_{\epsilon 88} \quad (257)$$

$$\sigma_{99} = \lambda_{93}{}^2 + \lambda_{94}{}^2 + \theta_{\epsilon 99} \quad (258)$$

$$\sigma_{12} = \lambda_{11}\ \lambda_{21} + \lambda_{14}\ \lambda_{24} \quad (259)$$

$$\sigma_{13} = \lambda_{11}\ \lambda_{31} + \lambda_{14}\ \lambda_{34} \quad (260)$$

$$\sigma_{14} = \lambda_{11}\ \lambda_{42}\ \Omega_{12} + \lambda_{14}\ \lambda_{44} \quad (261)$$

$$\sigma_{15} = \lambda_{11}\ \lambda_{52}\ \Omega_{12} + \lambda_{14}\ \lambda_{54} \quad (262)$$

$$\sigma_{16} = \lambda_{11}\ \lambda_{62}\ \Omega_{12} + \lambda_{14}\ \lambda_{64} \quad (263)$$

$$\sigma_{17} = \lambda_{11}\ \lambda_{73}\ \Omega_{13} + \lambda_{14}\ \lambda_{74} \quad (264)$$

$$\sigma_{18} = \lambda_{11}\ \lambda_{83}\ \Omega_{13} + \lambda_{14}\ \lambda_{84} \quad (265)$$

$$\sigma_{19} = \lambda_{11}\ \lambda_{93}\ \Omega_{13} + \lambda_{14}\ \lambda_{94} \quad (266)$$

$$\sigma_{23} = \lambda_{21}\ \lambda_{31} + \lambda_{24}\ \lambda_{34} \quad (267)$$

$$\sigma_{24} = \lambda_{21}\ \lambda_{42}\ \Omega_{12} + \lambda_{24}\ \lambda_{44} \quad (268)$$

$$\sigma_{25} = \lambda_{21}\ \lambda_{52}\ \Omega_{12} + \lambda_{24}\ \lambda_{54} \quad (269)$$

$$\sigma_{26} = \lambda_{21}\ \lambda_{62}\ \Omega_{12} + \lambda_{24}\ \lambda_{64} \quad (270)$$

$$\sigma_{27} = \lambda_{21}\ \lambda_{73}\ \Omega_{13} + \lambda_{24}\ \lambda_{74} \quad (271)$$

$$\sigma_{28} = \lambda_{21}\ \lambda_{83}\ \Omega_{13} + \lambda_{24}\ \lambda_{84} \quad (272)$$

$$\sigma_{29} = \lambda_{21}\ \lambda_{93}\ \Omega_{13} + \lambda_{24}\ \lambda_{94} \quad (273)$$

$$\sigma_{34} = \lambda_{31}\ \lambda_{42}\ \Omega_{12} + \lambda_{34}\ \lambda_{44} \quad (274)$$

$$\sigma_{35} = \lambda_{31}\ \lambda_{52}\ \Omega_{12} + \lambda_{34}\ \lambda_{54} \quad (275)$$

$$\sigma_{36} = \lambda_{31}\ \lambda_{62}\ \Omega_{12} + \lambda_{34}\ \lambda_{64} \quad (276)$$

$$\sigma_{37} = \lambda_{31}\ \lambda_{73}\ \Omega_{13} + \lambda_{34}\ \lambda_{74} \quad (277)$$

$$\sigma_{38} = \lambda_{31}\ \lambda_{83}\ \Omega_{13} + \lambda_{34}\ \lambda_{84} \quad (278)$$

$$\sigma_{39} = \lambda_{31}\ \lambda_{93}\ \Omega_{13} + \lambda_{34}\ \lambda_{94} \quad (279)$$

$$\sigma_{45} = \lambda_{42}\,\lambda_{52} + \lambda_{44}\,\lambda_{54} \tag{280}$$

$$\sigma_{46} = \lambda_{42}\,\lambda_{62} + \lambda_{44}\,\lambda_{64} \tag{281}$$

$$\sigma_{47} = \lambda_{42}\,\lambda_{73}\,\Omega_{23} + \lambda_{44}\,\lambda_{74} \tag{282}$$

$$\sigma_{48} = \lambda_{42}\,\lambda_{83}\,\Omega_{23} + \lambda_{44}\,\lambda_{84} \tag{283}$$

$$\sigma_{49} = \lambda_{42}\,\lambda_{93}\,\Omega_{23} + \lambda_{44}\,\lambda_{94} \tag{284}$$

$$\sigma_{56} = \lambda_{52}\,\lambda_{62} + \lambda_{54}\,\lambda_{64} \tag{285}$$

$$\sigma_{57} = \lambda_{52}\,\lambda_{73}\,\Omega_{23} + \lambda_{54}\,\lambda_{74} \tag{286}$$

$$\sigma_{58} = \lambda_{52}\,\lambda_{83}\,\Omega_{23} + \lambda_{54}\,\lambda_{84} \tag{287}$$

$$\sigma_{59} = \lambda_{52}\,\lambda_{93}\,\Omega_{23} + \lambda_{54}\,\lambda_{94} \tag{288}$$

$$\sigma_{67} = \lambda_{62}\,\lambda_{73}\,\Omega_{23} + \lambda_{64}\,\lambda_{74} \tag{289}$$

$$\sigma_{68} = \lambda_{62}\,\lambda_{83}\,\Omega_{23} + \lambda_{64}\,\lambda_{84} \tag{290}$$

$$\sigma_{69} = \lambda_{62}\,\lambda_{93}\,\Omega_{23} + \lambda_{64}\,\lambda_{94} \tag{291}$$

$$\sigma_{78} = \lambda_{73}\,\lambda_{83} + \lambda_{74}\,\lambda_{84} \tag{292}$$

$$\sigma_{79} = \lambda_{73}\,\lambda_{93} + \lambda_{74}\,\lambda_{94} \tag{293}$$

$$\sigma_{89} = \lambda_{83}\,\lambda_{93} + \lambda_{84}\,\lambda_{94} \tag{294}$$

If a solution for the group-factor loadings is attempted first, as it was for the bi-factor model with six measures, the following expression results (from Equations 259, 260, and 267).

$$\lambda_{11}^{2} = \frac{(\sigma_{12} - \lambda_{14}\,\lambda_{24})\;(\sigma_{13} - \lambda_{14}\,\lambda_{34})}{(\sigma_{23} - \lambda_{24}\,\lambda_{34})}$$

A solution for one of the general-factor loadings using Equations 261, 264, and 282 follows.

$$\lambda_{14}^2 = \frac{(\sigma_{14} - \lambda_{11}\,\lambda_{42}\,\Omega_{12})\ (\sigma_{17} - \lambda_{11}\,\lambda_{73}\,\Omega_{13})}{(\sigma_{47} - \lambda_{42}\,\lambda_{73}\,\Omega_{23})}$$

Parameter solutions can not be obtained, as was the case with six observed measures, but a satisfactory manner of proving the model to be unidentified is not available.

Bi-Factor Model With Group-Factor Correlations Constrained Equal To Zero

In an attempt to identify the model, the group-factor correlations are constrained to zero as was done with six observed measures. The variance and covariance expressions for the restricted model follow.

$$\sigma_{11} = \lambda_{11}^2 + \lambda_{14}^2 + \theta_{\epsilon_{11}} \qquad (295)$$

$$\sigma_{22} = \lambda_{21}^2 + \lambda_{24}^2 + \theta_{\epsilon_{22}} \qquad (296)$$

$$\sigma_{33} = \lambda_{31}^2 + \lambda_{34}^2 + \theta_{\epsilon_{33}} \qquad (297)$$

$$\sigma_{44} = \lambda_{42}^2 + \lambda_{44}^2 + \theta_{\epsilon_{44}} \qquad (298)$$

$$\sigma_{55} = \lambda_{52}^2 + \lambda_{54}^2 + \theta_{\epsilon_{55}} \qquad (299)$$

$$\sigma_{66} = \lambda_{62}^2 + \lambda_{64}^2 + \theta_{\epsilon_{66}} \qquad (300)$$

$$\sigma_{77} = \lambda_{73}^2 + \lambda_{74}^2 + \theta_{\epsilon_{77}} \qquad (301)$$

$$\sigma_{88} = \lambda_{83}^2 + \lambda_{84}^2 + \theta_{\epsilon_{88}} \qquad (302)$$

$$\sigma_{99} = \lambda_{93}^{2} + \lambda_{94}^{2} + \theta_{\epsilon_{99}} \quad (303)$$

$$\sigma_{12} = \lambda_{11} \lambda_{21} + \lambda_{14} \lambda_{24} \quad (304)$$

$$\sigma_{13} = \lambda_{11} \lambda_{31} + \lambda_{14} \lambda_{34} \quad (305)$$

$$\sigma_{14} = \lambda_{14} \lambda_{44} \quad (306)$$

$$\sigma_{15} = \lambda_{14} \lambda_{54} \quad (307)$$

$$\sigma_{16} = \lambda_{14} \lambda_{64} \quad (308)$$

$$\sigma_{17} = \lambda_{14} \lambda_{74} \quad (309)$$

$$\sigma_{18} = \lambda_{14} \lambda_{84} \quad (310)$$

$$\sigma_{19} = \lambda_{14} \lambda_{94} \quad (311)$$

$$\sigma_{23} = \lambda_{21} \lambda_{31} + \lambda_{24} \lambda_{34} \quad (312)$$

$$\sigma_{24} = \lambda_{24} \lambda_{44} \quad (313)$$

$$\sigma_{25} = \lambda_{24} \lambda_{54} \quad (314)$$

$$\sigma_{26} = \lambda_{24} \lambda_{64} \quad (315)$$

$$\sigma_{27} = \lambda_{24} \lambda_{74} \quad (316)$$

$$\sigma_{28} = \lambda_{24} \lambda_{84} \quad (317)$$

$$\sigma_{29} = \lambda_{24} \lambda_{94} \quad (318)$$

$$\sigma_{34} = \lambda_{34} \lambda_{44} \quad (319)$$

$$\sigma_{35} = \lambda_{34} \lambda_{54} \quad (320)$$

$$\sigma_{36} = \lambda_{34} \lambda_{64} \quad (321)$$

$$\sigma_{37} = \lambda_{34} \lambda_{74} \quad (322)$$

$$\sigma_{38} = \lambda_{34} \lambda_{84} \quad (323)$$

$$\sigma_{39} = \lambda_{34} \lambda_{94} \quad (324)$$

$$\sigma_{45} = \lambda_{42} \lambda_{52} + \lambda_{44} \lambda_{54} \quad (325)$$

$$\sigma_{46} = \lambda_{42}\,\lambda_{62} + \lambda_{44}\,\lambda_{64} \tag{326}$$

$$\sigma_{47} = \lambda_{44}\,\lambda_{74} \tag{327}$$

$$\sigma_{48} = \lambda_{44}\,\lambda_{84} \tag{328}$$

$$\sigma_{49} = \lambda_{44}\,\lambda_{94} \tag{329}$$

$$\sigma_{56} = \lambda_{52}\,\lambda_{62} + \lambda_{54}\,\lambda_{64} \tag{330}$$

$$\sigma_{57} = \lambda_{54}\,\lambda_{74} \tag{331}$$

$$\sigma_{58} = \lambda_{54}\,\lambda_{84} \tag{332}$$

$$\sigma_{59} = \lambda_{54}\,\lambda_{94} \tag{333}$$

$$\sigma_{67} = \lambda_{64}\,\lambda_{74} \tag{334}$$

$$\sigma_{68} = \lambda_{64}\,\lambda_{84} \tag{335}$$

$$\sigma_{69} = \lambda_{64}\,\lambda_{94} \tag{336}$$

$$\sigma_{78} = \lambda_{73}\,\lambda_{83} + \lambda_{74}\,\lambda_{84} \tag{337}$$

$$\sigma_{79} = \lambda_{73}\,\lambda_{93} + \lambda_{74}\,\lambda_{94} \tag{338}$$

$$\sigma_{89} = \lambda_{83}\,\lambda_{93} + \lambda_{84}\,\lambda_{94} \tag{339}$$

The general-factor λ's are solved first.

$$\lambda_{14}^{2} = \frac{\sigma_{14}\,\sigma_{17}}{\sigma_{47}}$$

$$\lambda_{24}^{2} = \frac{\sigma_{24}\,\sigma_{27}}{\sigma_{47}}$$

$$\lambda_{34}^{2} = \frac{\sigma_{34}\,\sigma_{37}}{\sigma_{47}}$$

$$\lambda_{44}^{2} = \frac{\sigma_{24}\,\sigma_{47}}{\sigma_{27}}$$

$$\lambda_{54}^2 = \frac{\sigma_{25}\ \sigma_{57}}{\sigma_{27}}$$

$$\lambda_{64}^2 = \frac{\sigma_{26}\ \sigma_{67}}{\sigma_{27}}$$

$$\lambda_{74}^2 = \frac{\sigma_{27}\ \sigma_{47}}{\sigma_{24}}$$

$$\lambda_{84}^2 = \frac{\sigma_{28}\ \sigma_{48}}{\sigma_{24}}$$

$$\lambda_{94}^2 = \frac{\sigma_{29}\ \sigma_{49}}{\sigma_{24}}$$

The first group-factor λ's are solved using Equations 304, 305, and 312.

$$\lambda_{11}^2 = \frac{(\sigma_{12} - \lambda_{14}\ \lambda_{24})\ (\sigma_{13} - \lambda_{14}\ \lambda_{34})}{(\sigma_{23} - \lambda_{24}\ \lambda_{34})}$$

$$= \frac{\left[\sigma_{12} - \left(\frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}\right)^{1/2}\right]\left[\sigma_{13} - \left(\frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}\right)^{1/2}\right]}{\left[\sigma_{23} - \left(\frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}\right)^{1/2}\right]}$$

If the expressions within these three pair of brackets are assigned letter values the solution becomes:

$$\lambda_{11}^2 = \frac{[A]\ \ [B]}{[C]}$$

As such the shorthand expressions for λ_{21} and λ_{31} follow.

$$\lambda_{21}^{2} = \frac{[A]\ [C]}{[B]}$$

$$\lambda_{31}^{2} = \frac{[B]\ [C]}{[A]}$$

Similar solutions for λ_{42}, λ_{52}, λ_{62} are obtained for the second group-factor using Equations 325, 326, and 330. Similarly, solutions for the third group factor λ's (λ_{73}, λ_{83}, λ_{93}) are obtained from Equations 337-339.

The residual variances are obtained by subtraction in Equations 295-303. For instance from Equation 295:

$$\theta_{\epsilon_{11}} = \sigma_{11} - \frac{\left[\sigma_{12} - \left(\frac{\sigma_{14}\sigma_{17}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{24}\sigma_{27}}{\sigma_{47}}\right)^{1/2}\right]\left[\sigma_{13} - \left(\frac{\sigma_{14}\sigma_{17}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{34}\sigma_{37}}{\sigma_{47}}\right)^{1/2}\right]}{\left[\sigma_{23} - \left(\frac{\sigma_{24}\sigma_{27}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{34}\sigma_{37}}{\sigma_{47}}\right)^{1/2}\right]}$$

$$- \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}$$

As all 27 parameters are identified the bi-factor model with group-factor correlations constrained to zero is identified.

While it is not necessary for identification (as was the case with six observed measures), a model with the additional constraint of equal λ's within each of the group-factors would also be identified with nine observed measures. The expressions for this doubly-constrained bi-factor model would differ from the prior model for six variance equations (296-297, 299-300, and 302-303) and nine covariance equations (304-305, 312, 325-326, 330, and 337-339). The new expressions would be as follows.

$$\sigma_{22} = \lambda_{11}^2 + \lambda_{14}^2 + \theta_{e_{11}}$$

$$\sigma_{33} = \lambda_{11}^2 + \lambda_{24}^2 + \theta_{e_{22}}$$

$$\sigma_{55} = \lambda_{42}^2 + \lambda_{54}^2 + \theta_{e_{55}}$$

$$\sigma_{66} = \lambda_{42}^2 + \lambda_{64}^2 + \theta_{e_{66}}$$

$$\sigma_{88} = \lambda_{73}^2 + \lambda_{84}^2 + \theta_{e_{88}}$$

$$\sigma_{99} = \lambda_{73}^2 + \lambda_{94}^2 + \theta_{e_{99}}$$

$$\sigma_{12} = \lambda_{11}^2 + \lambda_{14}\ \lambda_{24}$$

$$\sigma_{13} = \lambda_{11}^2 + \lambda_{14}\ \lambda_{34}$$

$$\sigma_{23} = \lambda_{11}^2 + \lambda_{24}\ \lambda_{34}$$

$$\sigma_{45} = \lambda_{42}^2 + \lambda_{44}\ \lambda_{54}$$

$$\sigma_{46} = \lambda_{42}^2 + \lambda_{44}\ \lambda_{64}$$

$$\sigma_{56} = \lambda_{42}^2 + \lambda_{54}\ \lambda_{64}$$

$$\sigma_{78} = \lambda_{73}^2 + \lambda_{74}\ \lambda_{84}$$

$$\sigma_{79} = \lambda_{73}^2 + \lambda_{74}\ \lambda_{94}$$

$$\sigma_{89} = \lambda_{73}^2 + \lambda_{84}\ \lambda_{94}$$

The general-factors are solved first.

$$\lambda_{14}^2 = \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}$$

$$\lambda_{24}^2 = \frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}$$

$$\lambda_{34}^2 = \frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}$$

$$\lambda_{44}^2 = \frac{\sigma_{14}\ \sigma_{47}}{\sigma_{17}}$$

$$\lambda_{54}^2 = \frac{\sigma_{15}\ \sigma_{57}}{\sigma_{17}}$$

$$\lambda_{64}^2 = \frac{\sigma_{16}\ \sigma_{67}}{\sigma_{17}}$$

$$\lambda_{74}^2 = \frac{\sigma_{17}\ \sigma_{67}}{\sigma_{16}}$$

$$\lambda_{84}^2 = \frac{\sigma_{18}\ \sigma_{58}}{\sigma_{15}}$$

$$\lambda_{94}^2 = \frac{\sigma_{19}\ \sigma_{59}}{\sigma_{15}}$$

The twelve remaining parameters (λ_{11}, λ_{42}, λ_{73}, $\theta_{\epsilon\ 11}$ - $\theta_{\epsilon\ 99}$) are solved by substitution and simplification as was done for the six measure doubly-constrained bi-factor model.

For example, the solutions for λ_{11} and $\theta_{\epsilon\ 11}$ are as follows:

$$\lambda_{11}^2 = \sigma_{12} - \lambda_{14}\ \lambda_{24}$$

$$= \sigma_{12} - \left[\frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right]^{1/2} \left[\frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}\right]^{1/2}$$

$$\theta_{\epsilon_{11}} = \sigma_{11} - \lambda_{11}^2 - \lambda_{14}^2$$

$$= \sigma_{11} - \left[\sigma_{12} - \left[\frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right]^{1/2} \left[\frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}\right]^{1/2}\right] - \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}$$

12 Observed Measures

One-Factor Model

The equations for this one-factor model are an extension of the six and nine measure models. The general form of this equation is:

$$y_i = \lambda_i \, \eta + \epsilon_i \qquad i = 1, 2 \ldots 12$$

Representations for the variances and covariances have the following elements in addition to those of Equations 115-159.

$$\sigma_{10,10} = \lambda_{10}^2 + \theta_{\epsilon_{10,10}} \tag{340}$$

$$\sigma_{11,11} = \lambda_{11}^2 + \theta_{\epsilon_{11,11}} \tag{341}$$

$$\sigma_{12,12} = \lambda_{12}^2 + \theta_{\epsilon_{12,12}} \tag{342}$$

$$\sigma_{1,10} = \lambda_1 \, \lambda_{10} \tag{343}$$

$$\sigma_{1,11} = \lambda_1 \, \lambda_{11} \tag{344}$$

$$\sigma_{1,12} = \lambda_1 \, \lambda_{12} \tag{345}$$

$$\sigma_{2,10} = \lambda_2 \, \lambda_{10} \tag{346}$$

$$\sigma_{2,11} = \lambda_2 \, \lambda_{11} \tag{347}$$

$$\sigma_{2,12} = \lambda_2 \, \lambda_{12} \tag{348}$$

$$\sigma_{3,10} = \lambda_3 \, \lambda_{10} \tag{349}$$

$$\sigma_{3,11} = \lambda_3 \, \lambda_{11} \tag{350}$$

$$\sigma_{3,12} = \lambda_3 \, \lambda_{12} \tag{351}$$

$$\sigma_{4,10} = \lambda_4 \, \lambda_{10} \tag{352}$$

$$\sigma_{4,11} = \lambda_4 \lambda_{11} \quad (353)$$

$$\sigma_{4,12} = \lambda_4 \lambda_{12} \quad (354)$$

$$\sigma_{5,10} = \lambda_5 \lambda_{10} \quad (355)$$

$$\sigma_{5,11} = \lambda_5 \lambda_{11} \quad (356)$$

$$\sigma_{5,12} = \lambda_5 \lambda_{12} \quad (357)$$

$$\sigma_{6,10} = \lambda_6 \lambda_{10} \quad (358)$$

$$\sigma_{6,11} = \lambda_6 \lambda_{11} \quad (359)$$

$$\sigma_{6,12} = \lambda_6 \lambda_{12} \quad (360)$$

$$\sigma_{7,10} = \lambda_7 \lambda_{10} \quad (361)$$

$$\sigma_{7,11} = \lambda_7 \lambda_{11} \quad (362)$$

$$\sigma_{7,12} = \lambda_7 \lambda_{12} \quad (363)$$

$$\sigma_{8,10} = \lambda_8 \lambda_{10} \quad (364)$$

$$\sigma_{8,11} = \lambda_8 \lambda_{11} \quad (365)$$

$$\sigma_{8,12} = \lambda_8 \lambda_{12} \quad (366)$$

$$\sigma_{9,10} = \lambda_9 \lambda_{10} \quad (367)$$

$$\sigma_{9,11} = \lambda_9 \lambda_{11} \quad (368)$$

$$\sigma_{9,12} = \lambda_9 \lambda_{12} \quad (369)$$

$$\sigma_{10,11} = \lambda_{10} \lambda_{11} \quad (370)$$

$$\sigma_{10,12} = \lambda_{10} \lambda_{12} \quad (371)$$

$$\sigma_{11,12} = \lambda_{11} \lambda_{12} \quad (372)$$

As this model has been fully identified previously with both six and nine measures, only those six incremental parameters (λ_{10}, λ_{11}, λ_{12}, $\theta_{\epsilon 10,10}$, $\theta_{\epsilon 11,11}$, $\theta_{\epsilon 12,12}$) which result from the additional observed measures will be presented here to show that the model is identified.

$$\lambda_{10}^2 = \frac{\sigma_{10,11}\ \sigma_{10,12}}{\sigma_{11,12}}$$

$$\lambda_{11}^2 = \frac{\sigma_{10,11}\ \sigma_{11,12}}{\sigma_{10,12}}$$

$$\lambda_{12}^2 = \frac{\sigma_{10,12}\ \sigma_{11,12}}{\sigma_{10,11}}$$

$$\theta_{\epsilon_{10,10}} = \sigma_{10,10} - \frac{\sigma_{10,11}\ \sigma_{10,12}}{\sigma_{11,12}}$$

$$\theta_{\epsilon_{11,11}} = \sigma_{11,11} - \frac{\sigma_{10,11}\ \sigma_{11,12}}{\sigma_{10,12}}$$

$$\theta_{\epsilon_{12,12}} = \sigma_{12,12} - \frac{\sigma_{10,12}\ \sigma_{11,12}}{\sigma_{10,11}}$$

Second-Order Model

The second-order model is examined for identification next. In this instance the model will have four, as opposed to three, first-order factors. There remains one second-order factor. The equations for this model follow. (The assumptions for a second-order model stated earlier apply here as well.)

$$y_1 = 1 * \eta_1 + \epsilon_1$$

$$y_2 = \lambda_1\ \eta_1 + \epsilon_2$$

$y_3 = \lambda_2 \eta_1 + \epsilon_3$

$y_4 = 1 * \eta_2 + \epsilon_4$

$y_5 = \lambda_3 \eta_2 + \epsilon_5$

$y_6 = \lambda_4 \eta_2 + \epsilon_6$

$y_7 = 1 * \eta_3 + \epsilon_7$

$y_8 = \lambda_5 \eta_3 + \epsilon_8$

$y_9 = \lambda_6 \eta_3 + \epsilon_9$

$y_{10} = 1 * \eta_4 + \epsilon_{10}$

$y_{11} = \lambda_7 \eta_4 + \epsilon_{11}$

$y_{12} = \lambda_8 \eta_4 + \epsilon_{12}$

The variances and covariances presented earlier as Equations 160-204 would be identical here. The following expressions are the result of the additional three observed measures and the one first-order factor.

$$\sigma_{10,10} = \gamma_4^2 + \psi_4 + \theta_{\epsilon_{10,10}} \quad (373)$$

$$\sigma_{11,11} = \lambda_7^2 (\gamma_4^2 + \psi_4) + \theta_{\epsilon_{11,11}} \quad (374)$$

$$\sigma_{12,12} = \lambda_8^2 (\gamma_4^2 + \psi_4) + \theta_{\epsilon_{12,12}} \quad (375)$$

$$\sigma_{1,10} = \gamma_1 \gamma_4 \quad (376)$$

$$\sigma_{1,11} = \lambda_7 \gamma_1 \gamma_4 \quad (377)$$

$$\sigma_{1,12} = \lambda_8 \gamma_1 \gamma_4 \quad (378)$$

$$\sigma_{2,10} = \lambda_1 \gamma_1 \gamma_4 \quad (379)$$

$$\sigma_{2,11} = \lambda_1 \lambda_7 \gamma_1 \gamma_4 \quad (380)$$

$$\sigma_{2,12} = \lambda_1 \lambda_8 \gamma_1 \gamma_4 \quad (381)$$

$$\sigma_{3,10} = \lambda_2 \gamma_1 \gamma_4 \quad (382)$$

$$\sigma_{3,11} = \lambda_2 \lambda_7 \gamma_1 \gamma_4 \quad (383)$$

$$\sigma_{3,12} = \lambda_2 \lambda_8 \gamma_1 \gamma_4 \quad (384)$$

$$\sigma_{4,10} = \gamma_2 \gamma_4 \quad (385)$$

$$\sigma_{4,11} = \lambda_7 \gamma_2 \gamma_4 \quad (386)$$

$$\sigma_{4,12} = \lambda_8 \gamma_2 \gamma_4 \quad (387)$$

$$\sigma_{5,10} = \lambda_3 \gamma_2 \gamma_4 \quad (388)$$

$$\sigma_{5,11} = \lambda_3 \lambda_7 \gamma_2 \gamma_4 \quad (389)$$

$$\sigma_{5,12} = \lambda_3 \lambda_8 \gamma_2 \gamma_4 \quad (390)$$

$$\sigma_{6,10} = \lambda_4 \gamma_2 \gamma_4 \quad (391)$$

$$\sigma_{6,11} = \lambda_4 \lambda_7 \gamma_2 \gamma_4 \quad (392)$$

$$\sigma_{6,12} = \lambda_4 \lambda_8 \gamma_2 \gamma_4 \quad (393)$$

$$\sigma_{7,10} = \gamma_3 \gamma_4 \quad (394)$$

$$\sigma_{7,11} = \lambda_7 \gamma_3 \gamma_4 \quad (395)$$

$$\sigma_{7,12} = \lambda_8 \gamma_3 \gamma_4 \quad (396)$$

$$\sigma_{8,10} = \lambda_5 \gamma_3 \gamma_4 \quad (397)$$

$$\sigma_{8,11} = \lambda_5 \lambda_7 \gamma_3 \gamma_4 \quad (398)$$

$$\sigma_{8,12} = \lambda_5 \lambda_8 \gamma_3 \gamma_4 \quad (399)$$

$$\sigma_{9,10} = \lambda_6 \gamma_3 \gamma_4 \quad (400)$$

$$\sigma_{9,11} = \lambda_6 \lambda_7 \gamma_3 \gamma_4 \quad (401)$$

$$\sigma_{9,12} = \lambda_6 \lambda_8 \gamma_3 \gamma_4 \quad (402)$$

$$\sigma_{10,11} = \lambda_7 \, (\gamma_4^2 + \psi_4) \quad (403)$$

$$\sigma_{10,12} = \lambda_8 \ (\gamma_4^2 + \psi_4) \tag{404}$$

$$\sigma_{11,12} = \lambda_7 \ \lambda_8 \ (\gamma_4^2 + \psi_4) \tag{405}$$

Solutions for the 21 model parameters in Equations 160-204 (λ_1 - λ_6, γ_1 - γ_3, ψ_1 - ψ_3, and $\theta_{\epsilon 11}$ - $\theta_{\epsilon 99}$) with nine observed measures would be identified here. The seven incremental parameters (λ_7, λ_8, γ_4, ψ_4, $\theta_{\epsilon 10,10}$, $\theta_{\epsilon 11,11}$, $\theta_{\epsilon 12,12}$) resulting from the additional three observed measures are solved next.

The first-order factor loadings are solved first using Equations 376-378.

$$\lambda_7 = \frac{\sigma_{1,11}}{\sigma_{1,10}}$$

$$\lambda_8 = \frac{\sigma_{1,12}}{\sigma_{1,10}}$$

The second-order factor loading is solved using Equations 171, 376, and 385.

$$\gamma_4^2 = \frac{\sigma_{1,10} \ \sigma_{4,10}}{\sigma_{14}}$$

The unique variance is solved by substitution in Equation 403.

$$\psi_4 = \frac{\sigma_{10,11} \ \sigma_{1,10}}{\sigma_{1,11}} - \frac{\sigma_{1,10} \ \sigma_{4,10}}{\sigma_{14}}$$

The three new residual variances are solved using Equations 373-375. For example from Equation 373:

$$\theta_{\epsilon_{10,10}} = \sigma_{10,10} - \frac{\sigma_{1,10}\ \sigma_{4,10}}{\sigma_{14}} - \left[\frac{\sigma_{10,11}\ \sigma_{1,10}}{\sigma_{1,11}} - \frac{\sigma_{1,10}\ \sigma_{4,10}}{\sigma_{14}}\right]$$

As these seven parameters are identified, this second-order model is identified.

Group-Factor Model

This model has four correlated-factors, and factor variances equal to a value of one (i.e., $Var(\eta_j) = 1.0$ for $j = 1,..,4$).

Equations for the group-factor model follow.

$$y_1 = \lambda_1\ \eta_1 + \epsilon_1$$

$$y_2 = \lambda_2\ \eta_1 + \epsilon_2$$

$$y_3 = \lambda_3\ \eta_1 + \epsilon_3$$

$$y_4 = \lambda_4\ \eta_2 + \epsilon_4$$

$$y_5 = \lambda_5\ \eta_2 + \epsilon_5$$

$$y_6 = \lambda_6\ \eta_2 + \epsilon_6$$

$$y_7 = \lambda_7\ \eta_3 + \epsilon_7$$

$$y_8 = \lambda_8\ \eta_3 + \epsilon_8$$

$$y_9 = \lambda_9\ \eta_3 + \epsilon_9$$

$$y_{10} = \lambda_{10}\ \eta_4 + \epsilon_{10}$$

$$y_{11} = \lambda_{11}\ \eta_4 + \epsilon_{11}$$

$$y_{12} = \lambda_{12}\ \eta_4 + \epsilon_{12}$$

Representations for the variances and covariances follow.

$$\sigma_{11} = \lambda_1^2 + \theta_{\epsilon_{11}} \quad (406)$$

$$\sigma_{22} = \lambda_2^2 + \theta_{\epsilon_{22}} \quad (407)$$

$$\sigma_{33} = \lambda_3^2 + \theta_{\epsilon_{33}} \quad (408)$$

$$\sigma_{44} = \lambda_4^2 + \theta_{\epsilon_{44}} \quad (409)$$

$$\sigma_{55} = \lambda_5^2 + \theta_{\epsilon_{55}} \quad (410)$$

$$\sigma_{66} = \lambda_6^2 + \theta_{\epsilon_{66}} \quad (411)$$

$$\sigma_{77} = \lambda_7^2 + \theta_{\epsilon_{77}} \quad (412)$$

$$\sigma_{88} = \lambda_8^2 + \theta_{\epsilon_{88}} \quad (413)$$

$$\sigma_{99} = \lambda_9^2 + \theta_{\epsilon_{99}} \quad (414)$$

$$\sigma_{10,10} = \lambda_{10}^2 + \theta_{\epsilon_{10,10}} \quad (415)$$

$$\sigma_{11,11} = \lambda_{11}^2 + \theta_{\epsilon_{11,11}} \quad (416)$$

$$\sigma_{12,12} = \lambda_{12}^2 + \theta_{\epsilon_{12,12}} \quad (417)$$

$$\sigma_{12} = \lambda_1 \lambda_2 \quad (418)$$

$$\sigma_{13} = \lambda_1 \lambda_3 \quad (419)$$

$$\sigma_{14} = \lambda_1 \lambda_4 \omega_{12} \quad (420)$$

$$\sigma_{15} = \lambda_1 \lambda_5 \omega_{12} \quad (421)$$

$$\sigma_{16} = \lambda_1 \lambda_6 \omega_{12} \quad (422)$$

$$\sigma_{17} = \lambda_1 \lambda_7 \omega_{13} \quad (423)$$

$$\sigma_{18} = \lambda_1 \lambda_8 \omega_{13} \quad (424)$$

$$\sigma_{19} = \lambda_1 \lambda_9 \omega_{13} \quad (425)$$

$$\sigma_{1,10} = \lambda_1 \lambda_{10} \omega_{14} \quad (426)$$

$$\sigma_{1,11} = \lambda_1 \lambda_{11} \omega_{14} \tag{427}$$

$$\sigma_{1,12} = \lambda_1 \lambda_{12} \omega_{14} \tag{428}$$

$$\sigma_{23} = \lambda_2 \lambda_3 \tag{429}$$

$$\sigma_{24} = \lambda_2 \lambda_4 \omega_{12} \tag{430}$$

$$\sigma_{25} = \lambda_2 \lambda_5 \omega_{12} \tag{431}$$

$$\sigma_{26} = \lambda_2 \lambda_6 \omega_{12} \tag{432}$$

$$\sigma_{27} = \lambda_2 \lambda_7 \omega_{13} \tag{433}$$

$$\sigma_{28} = \lambda_2 \lambda_8 \omega_{13} \tag{434}$$

$$\sigma_{29} = \lambda_2 \lambda_9 \omega_{13} \tag{435}$$

$$\sigma_{2,10} = \lambda_2 \lambda_{10} \omega_{14} \tag{436}$$

$$\sigma_{2,11} = \lambda_2 \lambda_{11} \omega_{14} \tag{437}$$

$$\sigma_{2,12} = \lambda_2 \lambda_{12} \omega_{14} \tag{438}$$

$$\sigma_{34} = \lambda_3 \lambda_4 \omega_{12} \tag{439}$$

$$\sigma_{35} = \lambda_3 \lambda_5 \omega_{12} \tag{440}$$

$$\sigma_{36} = \lambda_3 \lambda_6 \omega_{12} \tag{441}$$

$$\sigma_{37} = \lambda_3 \lambda_7 \omega_{13} \tag{442}$$

$$\sigma_{38} = \lambda_3 \lambda_8 \omega_{13} \tag{443}$$

$$\sigma_{39} = \lambda_3 \lambda_9 \omega_{13} \tag{444}$$

$$\sigma_{3,10} = \lambda_3 \lambda_{10} \omega_{14} \tag{445}$$

$$\sigma_{3,11} = \lambda_3 \lambda_{11} \omega_{14} \tag{446}$$

$$\sigma_{3,12} = \lambda_3 \lambda_{12} \omega_{14} \tag{447}$$

$$\sigma_{45} = \lambda_4 \lambda_5 \tag{448}$$

$$\sigma_{46} = \lambda_4 \lambda_6 \tag{449}$$

$$\sigma_{47} = \lambda_4 \lambda_7 \omega_{23} \tag{450}$$

$$\sigma_{48} = \lambda_4 \lambda_8 \omega_{23} \tag{451}$$

$$\sigma_{49} = \lambda_4 \lambda_9 \omega_{23} \tag{452}$$

$$\sigma_{4,10} = \lambda_4 \lambda_{10} \omega_{24} \tag{453}$$

$$\sigma_{4,11} = \lambda_4 \lambda_{11} \omega_{24} \tag{454}$$

$$\sigma_{4,12} = \lambda_4 \lambda_{12} \omega_{24} \tag{455}$$

$$\sigma_{56} = \lambda_5 \lambda_6 \tag{456}$$

$$\sigma_{57} = \lambda_5 \lambda_7 \omega_{23} \tag{457}$$

$$\sigma_{58} = \lambda_5 \lambda_8 \omega_{23} \tag{458}$$

$$\sigma_{59} = \lambda_5 \lambda_9 \omega_{23} \tag{459}$$

$$\sigma_{5,10} = \lambda_5 \lambda_{10} \omega_{24} \tag{460}$$

$$\sigma_{5,11} = \lambda_5 \lambda_{11} \omega_{24} \tag{461}$$

$$\sigma_{5,12} = \lambda_5 \lambda_{12} \omega_{24} \tag{462}$$

$$\sigma_{67} = \lambda_6 \lambda_7 \omega_{23} \tag{463}$$

$$\sigma_{68} = \lambda_6 \lambda_8 \omega_{23} \tag{464}$$

$$\sigma_{69} = \lambda_6 \lambda_9 \omega_{23} \tag{465}$$

$$\sigma_{6,10} = \lambda_6 \lambda_{10} \omega_{24} \tag{466}$$

$$\sigma_{6,11} = \lambda_6 \lambda_{11} \omega_{24} \tag{467}$$

$$\sigma_{6,12} = \lambda_6 \lambda_{12} \omega_{24} \tag{468}$$

$$\sigma_{78} = \lambda_7 \lambda_8 \tag{469}$$

$$\sigma_{79} = \lambda_7 \lambda_9 \tag{470}$$

$$\sigma_{7,10} = \lambda_7 \lambda_{10} \omega_{34} \tag{471}$$

$$\sigma_{7,11} = \lambda_7 \lambda_{11} \omega_{34} \tag{472}$$

$$\sigma_{7,12} = \lambda_7 \lambda_{12} \omega_{34} \quad (473)$$

$$\sigma_{89} = \lambda_8 \lambda_9 \quad (474)$$

$$\sigma_{8,10} = \lambda_8 \lambda_{10} \omega_{34} \quad (475)$$

$$\sigma_{8,11} = \lambda_8 \lambda_{11} \omega_{34} \quad (476)$$

$$\sigma_{8,12} = \lambda_8 \lambda_{12} \omega_{34} \quad (477)$$

$$\sigma_{9,10} = \lambda_9 \lambda_{10} \omega_{34} \quad (478)$$

$$\sigma_{9,11} = \lambda_9 \lambda_{11} \omega_{34} \quad (479)$$

$$\sigma_{9,12} = \lambda_9 \lambda_{12} \omega_{34} \quad (480)$$

$$\sigma_{10,11} = \lambda_{10} \lambda_{11} \quad (481)$$

$$\sigma_{10,12} = \lambda_{10} \lambda_{12} \quad (482)$$

$$\sigma_{11,12} = \lambda_{11} \lambda_{12} \quad (483)$$

Again the λ's are solved first, as follows:

$$\lambda_1^2 = \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

$$\lambda_2^2 = \frac{\sigma_{12}\ \sigma_{23}}{\sigma_{13}}$$

$$\lambda_3^2 = \frac{\sigma_{13}\ \sigma_{23}}{\sigma_{12}}$$

$$\lambda_4^2 = \frac{\sigma_{45}\ \sigma_{46}}{\sigma_{56}}$$

$$\lambda_5^2 = \frac{\sigma_{45}\ \sigma_{56}}{\sigma_{46}}$$

$$\lambda_6^2 = \frac{\sigma_{46}\ \sigma_{56}}{\sigma_{45}}$$

$$\lambda_7^2 = \frac{\sigma_{78}\ \sigma_{79}}{\sigma_{89}}$$

$$\lambda_8^2 = \frac{\sigma_{78}\ \sigma_{89}}{\sigma_{79}}$$

$$\lambda_9^2 = \frac{\sigma_{79}\ \sigma_{89}}{\sigma_{78}}$$

$$\lambda_{10}^2 = \frac{\sigma_{10,11}\ \sigma_{10,12}}{\sigma_{11,12}}$$

$$\lambda_{11}^2 = \frac{\sigma_{10,11}\ \sigma_{11,12}}{\sigma_{10,12}}$$

$$\lambda_{12}^2 = \frac{\sigma_{10,12}\ \sigma_{11,12}}{\sigma_{10,11}}$$

Solutions for the six ω's can be done by division and substitution (for the appropriate λ's) in Equations 420, 423, 426, 450, 453, and 471.

$$\omega_{12} = \left[\frac{\sigma_{14}^2\ \sigma_{23}\ \sigma_{56}}{\sigma_{12}\ \sigma_{13}\ \sigma_{45}\ \sigma_{46}}\right]^{1/2}$$

$$\omega_{13} = \left[\frac{\sigma_{17}^2\ \sigma_{23}\ \sigma_{89}}{\sigma_{12}\ \sigma_{13}\ \sigma_{78}\ \sigma_{79}}\right]^{1/2}$$

$$\omega_{14} = \left[\frac{\sigma_{1,10}{}^2\ \sigma_{23}\ \sigma_{11,12}}{\sigma_{12}\ \sigma_{13}\ \sigma_{10,11}\ \sigma_{10,12}}\right]^{1/2}$$

$$\omega_{23} = \left[\frac{\sigma_{47}{}^2\ \sigma_{56}\ \sigma_{89}}{\sigma_{45}\ \sigma_{46}\ \sigma_{78}\ \sigma_{79}}\right]^{1/2}$$

$$\omega_{24} = \left[\frac{\sigma_{49}{}^2\ \sigma_{56}\ \sigma_{78}}{\sigma_{45}\ \sigma_{46}\ \sigma_{79}\ \sigma_{89}}\right]^{1/2}$$

$$\omega_{34} = \left[\frac{\sigma_{7,10}{}^2\ \sigma_{89}\ \sigma_{11,12}}{\sigma_{78}\ \sigma_{79}\ \sigma_{10,11}\ \sigma_{10,12}}\right]^{1/2}$$

Solutions for the $\theta_{\epsilon's}$ can be done by subtraction and substitution in Equations 406-417. For example, from Equation 406:

$$\theta_{\epsilon_{11}} = \sigma_{11} - \frac{\sigma_{12}\ \sigma_{13}}{\sigma_{23}}$$

As all 30 parameters are identified the group-factor model is identified.

Bi-Factor Model

This model has correlated group-factors, and all factor variances equal to a value of one (i.e., $Var(\eta_j) = 1.0$ for $j = 1, \ldots, 5$). The equations for the observed measures follow.

$$y_1 = \lambda_{11}\,\eta_1 + \lambda_{15}\,\eta_5 + \epsilon_1$$

$$y_2 = \lambda_{21}\,\eta_1 + \lambda_{25}\,\eta_5 + \epsilon_2$$

$$y_3 = \lambda_{31}\,\eta_1 + \lambda_{35}\,\eta_5 + \epsilon_3$$

$$y_4 = \lambda_{42}\,\eta_2 + \lambda_{45}\,\eta_5 + \epsilon_4$$

$$y_5 = \lambda_{52}\,\eta_2 + \lambda_{55}\,\eta_5 + \epsilon_5$$

$$y_6 = \lambda_{62}\,\eta_2 + \lambda_{65}\,\eta_5 + \epsilon_6$$

$$y_7 = \lambda_{73}\,\eta_3 + \lambda_{75}\,\eta_5 + \epsilon_7$$

$$y_8 = \lambda_{83}\,\eta_3 + \lambda_{85}\,\eta_5 + \epsilon_8$$

$$y_9 = \lambda_{93}\,\eta_3 + \lambda_{95}\,\eta_5 + \epsilon_9$$

$$y_{10} = \lambda_{10,4}\,\eta_4 + \lambda_{10,5}\,\eta_5 + \epsilon_{10}$$

$$y_{11} = \lambda_{11,4}\,\eta_4 + \lambda_{11,5}\,\eta_5 + \epsilon_{11}$$

$$y_{12} = \lambda_{12,4}\,\eta_4 + \lambda_{12,5}\,\eta_5 + \epsilon_{12}$$

Representations for the variances and covariances follow.

$$\sigma_{11} = \lambda_{11}^2 + \lambda_{15}^2 + \theta_{\epsilon_{11}} \tag{484}$$

$$\sigma_{22} = \lambda_{21}^2 + \lambda_{25}^2 + \theta_{\epsilon_{22}} \tag{485}$$

$$\sigma_{33} = \lambda_{31}^2 + \lambda_{35}^2 + \theta_{\epsilon_{33}} \tag{486}$$

$$\sigma_{44} = \lambda_{42}^2 + \lambda_{45}^2 + \theta_{\epsilon_{44}} \tag{487}$$

$$\sigma_{55} = \lambda_{52}^2 + \lambda_{55}^2 + \theta_{\epsilon_{55}} \tag{488}$$

$$\sigma_{66} = \lambda_{62}^2 + \lambda_{65}^2 + \theta_{\epsilon_{66}} \tag{489}$$

$$\sigma_{77} = \lambda_{73}^2 + \lambda_{75}^2 + \theta_{\epsilon_{77}} \tag{490}$$

$$\sigma_{88} = \lambda_{83}^2 + \lambda_{85}^2 + \theta_{\epsilon_{88}} \tag{491}$$

$$\sigma_{99} = \lambda_{93}^2 + \lambda_{95}^2 + \theta_{\epsilon_{99}} \tag{492}$$

$$\sigma_{10,10} = \lambda_{10,4}^2 + \lambda_{10,5}^2 + \theta_{\epsilon_{10,10}} \quad (493)$$

$$\sigma_{11,11} = \lambda_{11,4}^2 + \lambda_{11,5}^2 + \theta_{\epsilon_{11,11}} \quad (494)$$

$$\sigma_{12,12} = \lambda_{12,4}^2 + \lambda_{12,5}^2 + \theta_{\epsilon_{12,12}} \quad (495)$$

$$\sigma_{12} = \lambda_{11}\,\lambda_{21} + \lambda_{15}\,\lambda_{25} \quad (496)$$

$$\sigma_{13} = \lambda_{11}\,\lambda_{31} + \lambda_{15}\,\lambda_{35} \quad (497)$$

$$\sigma_{14} = \lambda_{11}\,\lambda_{42}\,\omega_{12} + \lambda_{15}\,\lambda_{45} \quad (498)$$

$$\sigma_{15} = \lambda_{11}\,\lambda_{52}\,\omega_{12} + \lambda_{15}\,\lambda_{55} \quad (499)$$

$$\sigma_{16} = \lambda_{11}\,\lambda_{62}\,\omega_{12} + \lambda_{15}\,\lambda_{65} \quad (500)$$

$$\sigma_{17} = \lambda_{11}\,\lambda_{73}\,\omega_{13} + \lambda_{15}\,\lambda_{75} \quad (501)$$

$$\sigma_{18} = \lambda_{11}\,\lambda_{83}\,\omega_{13} + \lambda_{15}\,\lambda_{85} \quad (502)$$

$$\sigma_{19} = \lambda_{11}\,\lambda_{93}\,\omega_{13} + \lambda_{15}\,\lambda_{95} \quad (503)$$

$$\sigma_{1,10} = \lambda_{11}\,\lambda_{10,4}\,\omega_{14} + \lambda_{15}\,\lambda_{10,5} \quad (504)$$

$$\sigma_{1,11} = \lambda_{11}\,\lambda_{11,4}\,\omega_{14} + \lambda_{15}\,\lambda_{11,5} \quad (505)$$

$$\sigma_{1,12} = \lambda_{11}\,\lambda_{12,4}\,\omega_{14} + \lambda_{15}\,\lambda_{12,5} \quad (506)$$

$$\sigma_{23} = \lambda_{21}\,\lambda_{31} + \lambda_{25}\,\lambda_{35} \quad (507)$$

$$\sigma_{24} = \lambda_{21}\,\lambda_{42}\,\omega_{12} + \lambda_{25}\,\lambda_{45} \quad (508)$$

$$\sigma_{25} = \lambda_{21}\,\lambda_{52}\,\omega_{12} + \lambda_{25}\,\lambda_{55} \quad (509)$$

$$\sigma_{26} = \lambda_{21}\,\lambda_{62}\,\omega_{12} + \lambda_{25}\,\lambda_{65} \quad (510)$$

$$\sigma_{27} = \lambda_{21}\,\lambda_{73}\,\omega_{13} + \lambda_{25}\,\lambda_{75} \quad (511)$$

$$\sigma_{28} = \lambda_{21}\,\lambda_{83}\,\omega_{13} + \lambda_{25}\,\lambda_{85} \quad (512)$$

$$\sigma_{29} = \lambda_{21}\,\lambda_{93}\,\omega_{13} + \lambda_{25}\,\lambda_{95} \quad (513)$$

$$\sigma_{2,10} = \lambda_{21}\,\lambda_{10,4}\,\omega_{14} + \lambda_{25}\,\lambda_{10,5} \quad (514)$$

$$\sigma_{2,11} = \lambda_{21}\,\lambda_{11,4}\,\omega_{14} + \lambda_{25}\,\lambda_{11,5} \quad (515)$$

$$\sigma_{2,12} = \lambda_{21}\ \lambda_{12,4}\ \omega_{14} + \lambda_{25}\ \lambda_{12,5} \tag{516}$$

$$\sigma_{34} = \lambda_{31}\ \lambda_{42}\ \omega_{12} + \lambda_{35}\ \lambda_{45} \tag{517}$$

$$\sigma_{35} = \lambda_{31}\ \lambda_{52}\ \omega_{12} + \lambda_{35}\ \lambda_{55} \tag{518}$$

$$\sigma_{36} = \lambda_{31}\ \lambda_{62}\ \omega_{12} + \lambda_{35}\ \lambda_{65} \tag{519}$$

$$\sigma_{37} = \lambda_{31}\ \lambda_{73}\ \omega_{13} + \lambda_{35}\ \lambda_{75} \tag{520}$$

$$\sigma_{38} = \lambda_{31}\ \lambda_{83}\ \omega_{13} + \lambda_{35}\ \lambda_{85} \tag{521}$$

$$\sigma_{39} = \lambda_{31}\ \lambda_{93}\ \omega_{13} + \lambda_{35}\ \lambda_{95} \tag{522}$$

$$\sigma_{3,10} = \lambda_{31}\ \lambda_{10,4}\ \omega_{14} + \lambda_{35}\ \lambda_{10,5} \tag{523}$$

$$\sigma_{3,11} = \lambda_{31}\ \lambda_{11,4}\ \omega_{14} + \lambda_{35}\ \lambda_{11,5} \tag{524}$$

$$\sigma_{3,12} = \lambda_{31}\ \lambda_{12,4}\ \omega_{14} + \lambda_{35}\ \lambda_{12,5} \tag{525}$$

$$\sigma_{45} = \lambda_{42}\ \lambda_{52} + \lambda_{45}\ \lambda_{55} \tag{526}$$

$$\sigma_{46} = \lambda_{42}\ \lambda_{62} + \lambda_{45}\ \lambda_{65} \tag{527}$$

$$\sigma_{47} = \lambda_{42}\ \lambda_{73}\ \omega_{23} + \lambda_{45}\ \lambda_{75} \tag{528}$$

$$\sigma_{48} = \lambda_{42}\ \lambda_{83}\ \omega_{23} + \lambda_{45}\ \lambda_{85} \tag{529}$$

$$\sigma_{49} = \lambda_{42}\ \lambda_{93}\ \omega_{23} + \lambda_{45}\ \lambda_{95} \tag{530}$$

$$\sigma_{4,10} = \lambda_{42}\ \lambda_{10,4}\ \omega_{24} + \lambda_{45}\ \lambda_{10,5} \tag{531}$$

$$\sigma_{4,11} = \lambda_{42}\ \lambda_{11,4}\ \omega_{24} + \lambda_{45}\ \lambda_{11,5} \tag{532}$$

$$\sigma_{4,12} = \lambda_{42}\ \lambda_{12,4}\ \omega_{24} + \lambda_{45}\ \lambda_{12,5} \tag{533}$$

$$\sigma_{56} = \lambda_{52}\ \lambda_{62} + \lambda_{55}\ \lambda_{65} \tag{534}$$

$$\sigma_{57} = \lambda_{52}\ \lambda_{73}\ \omega_{23} + \lambda_{55}\ \lambda_{75} \tag{535}$$

$$\sigma_{58} = \lambda_{52}\ \lambda_{83}\ \omega_{23} + \lambda_{55}\ \lambda_{85} \tag{536}$$

$$\sigma_{59} = \lambda_{52}\ \lambda_{93}\ \omega_{23} + \lambda_{55}\ \lambda_{95} \tag{537}$$

$$\sigma_{5,10} = \lambda_{52}\ \lambda_{10,4}\ \omega_{24} + \lambda_{55}\ \lambda_{10,5} \tag{538}$$

$$\sigma_{5,11} = \lambda_{52}\,\lambda_{11,4}\,\omega_{24} + \lambda_{55}\,\lambda_{11,5} \tag{539}$$

$$\sigma_{5,12} = \lambda_{52}\,\lambda_{12,4}\,\omega_{24} + \lambda_{55}\,\lambda_{12,5} \tag{540}$$

$$\sigma_{67} = \lambda_{62}\,\lambda_{73}\,\omega_{23} + \lambda_{65}\,\lambda_{75} \tag{541}$$

$$\sigma_{68} = \lambda_{62}\,\lambda_{83}\,\omega_{23} + \lambda_{65}\,\lambda_{85} \tag{542}$$

$$\sigma_{69} = \lambda_{62}\,\lambda_{93}\,\omega_{23} + \lambda_{65}\,\lambda_{95} \tag{543}$$

$$\sigma_{6,10} = \lambda_{62}\,\lambda_{10,4}\,\omega_{24} + \lambda_{65}\,\lambda_{10,5} \tag{544}$$

$$\sigma_{6,11} = \lambda_{62}\,\lambda_{11,4}\,\omega_{24} + \lambda_{65}\,\lambda_{11,5} \tag{545}$$

$$\sigma_{6,12} = \lambda_{62}\,\lambda_{12,4}\,\omega_{24} + \lambda_{65}\,\lambda_{12,5} \tag{546}$$

$$\sigma_{78} = \lambda_{73}\,\lambda_{83} + \lambda_{75}\,\lambda_{85} \tag{547}$$

$$\sigma_{79} = \lambda_{73}\,\lambda_{93} + \lambda_{75}\,\lambda_{95} \tag{548}$$

$$\sigma_{7,10} = \lambda_{73}\,\lambda_{10,4}\,\omega_{34} + \lambda_{75}\,\lambda_{10,5} \tag{549}$$

$$\sigma_{7,11} = \lambda_{73}\,\lambda_{11,4}\,\omega_{34} + \lambda_{75}\,\lambda_{11,5} \tag{550}$$

$$\sigma_{7,12} = \lambda_{73}\,\lambda_{12,4}\,\omega_{34} + \lambda_{75}\,\lambda_{12,5} \tag{551}$$

$$\sigma_{89} = \lambda_{83}\,\lambda_{93} + \lambda_{85}\,\lambda_{95} \tag{552}$$

$$\sigma_{8,10} = \lambda_{83}\,\lambda_{10,4}\,\omega_{34} + \lambda_{85}\,\lambda_{10,5} \tag{553}$$

$$\sigma_{8,11} = \lambda_{83}\,\lambda_{11,4}\,\omega_{34} + \lambda_{85}\,\lambda_{11,5} \tag{554}$$

$$\sigma_{8,12} = \lambda_{83}\,\lambda_{12,4}\,\omega_{34} + \lambda_{85}\,\lambda_{12,5} \tag{555}$$

$$\sigma_{9,10} = \lambda_{93}\,\lambda_{10,4}\,\omega_{34} + \lambda_{95}\,\lambda_{10,5} \tag{556}$$

$$\sigma_{9,11} = \lambda_{93}\,\lambda_{11,4}\,\omega_{34} + \lambda_{95}\,\lambda_{11,5} \tag{557}$$

$$\sigma_{9,12} = \lambda_{93}\,\lambda_{12,4}\,\omega_{34} + \lambda_{95}\,\lambda_{12,5} \tag{558}$$

$$\sigma_{10,11} = \lambda_{10,4}\,\lambda_{11,4} + \lambda_{10,5}\,\lambda_{11,5} \tag{559}$$

$$\sigma_{10,12} = \lambda_{10,4}\,\lambda_{12,4} + \lambda_{10,5}\,\lambda_{12,5} \tag{560}$$

$$\sigma_{11,12} = \lambda_{11,4}\,\lambda_{12,4} + \lambda_{11,5}\,\lambda_{12,5} \tag{561}$$

The first general-factor loading can be isolated using Equations 498, 501, and 528 as follows.

$$\lambda_{15}^2 = \frac{(\sigma_{14} - \lambda_{11}\,\lambda_{42}\,\omega_{12})\ (\sigma_{17} - \lambda_{11}\,\lambda_{73}\,\omega_{13})}{(\sigma_{47} - \lambda_{42}\,\lambda_{73}\,\omega_{23})}$$

As was the situation with both the six and nine measure models, these parameters can't be solved. Again it would appear useful to constrain (to zero) the group-factor correlations.

<u>Bi-Factor Model With Group-Factor Correlations Constrained Equal To Zero</u>

The bi-factor model with the group-factor correlations constrained to a value of zero has the same equations and variance expressions as the bi-factor model. Equations which have ω terms in them are affected in that a ω element and the λ's by which they are pre-multiplied are removed from the equations. This occurs for several equations: 498-506, 508-525, 528-533, 535-546, 549-551, and 553-558. For example, Equation 498 becomes:

$$\sigma_{14} = \lambda_{15}\,\lambda_{45} \qquad (562)$$

The other equations previously mentioned would be similarly affected. As this effect has been fully demonstrated for the six and nine measure constrained bi-factor model it will not be done here.

Solutions will be done for the general-factor λ's first.

$$\lambda_{15}^2 = \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}$$

$$\lambda_{25}^2 = \frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}$$

$$\lambda_{35}^2 = \frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}$$

$$\lambda_{45}^2 = \frac{\sigma_{47}\ \sigma_{4,10}}{\sigma_{7,10}}$$

$$\lambda_{55}^2 = \frac{\sigma_{57}\ \sigma_{5,10}}{\sigma_{7,10}}$$

$$\lambda_{65}^2 = \frac{\sigma_{67}\ \sigma_{6,10}}{\sigma_{7,10}}$$

$$\lambda_{75}^2 = \frac{\sigma_{17}\ \sigma_{7,10}}{\sigma_{1,10}}$$

$$\lambda_{85}^2 = \frac{\sigma_{18}\ \sigma_{8,10}}{\sigma_{1,10}}$$

$$\lambda_{95}^2 = \frac{\sigma_{19}\ \sigma_{9,10}}{\sigma_{1,10}}$$

$$\lambda_{10,5}^2 = \frac{\sigma_{1,10}\ \sigma_{4,10}}{\sigma_{14}}$$

$$\lambda_{11,5}^2 = \frac{\sigma_{1,11}\ \sigma_{4,11}}{\sigma_{14}}$$

$$\lambda_{12,5}^2 = \frac{\sigma_{1,12}\ \sigma_{4,12}}{\sigma_{14}}$$

The three λ's for the first group-factor are solved next using Equations 496, 497 and 507.

$$\lambda_{11}^2 = \frac{(\sigma_{12} - \lambda_{15}\ \lambda_{25})\ (\sigma_{13} - \lambda_{15}\ \lambda_{35})}{(\sigma_{23} - \lambda_{25}\ \lambda_{35})}$$

$$= \frac{\left[\sigma_{12} - \left(\frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}\right)^{1/2}\right]\left[\sigma_{13} - \left(\frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}\right)^{1/2}\right]}{\left[\sigma_{23} - \left(\frac{\sigma_{24}\ \sigma_{27}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{34}\ \sigma_{37}}{\sigma_{47}}\right)^{1/2}\right]}$$

The other two λ's would be obtained in similar fashion.

The factor loadings for the remaining three group-factors are solved likewise using the following three groups of equations: Equations 526, 527 and 534 for the second group-factor loadings; Equations 547, 548, and 552 for the third group-factor loadings; and Equations 559-561 for the fourth group-factor loadings.

The θ_ϵ's are solved last by subtraction and substitution in Equations 484-495. For example, from Equation 484 the solution for $\theta_{\epsilon_{11}}$ follows.

$$\theta_{\epsilon_{11}} = \sigma_{11} - \frac{\left[\sigma_{12} - \left(\frac{\sigma_{14}\sigma_{17}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{24}\sigma_{27}}{\sigma_{47}}\right)^{1/2}\right]\left[\sigma_{13} - \left(\frac{\sigma_{14}\sigma_{17}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{34}\sigma_{37}}{\sigma_{47}}\right)^{1/2}\right]}{\left[\sigma_{23} - \left(\frac{\sigma_{24}\sigma_{27}}{\sigma_{47}}\right)^{1/2}\left(\frac{\sigma_{34}\sigma_{37}}{\sigma_{47}}\right)^{1/2}\right]}$$

$$- \frac{\sigma_{14}\ \sigma_{17}}{\sigma_{47}}$$

As all 36 parameters are identified the bi-factor model with group-factor correlations constrained to a value of zero is identified. A model with the additional constraint of equal λ's within each of the group-factors would also be identified with 12 observed measures, as was the case with both six and nine measures.

EQUIVALENCE

The following general steps will be followed to demonstrate instances of equivalence between the four types of factor analysis models. Models previously shown to be identified form the base for examination. Those models comprised of the same number of parameters for either six- or nine-measures form the final means for selecting possibly equivalent models.

Initially it was expected to find a one-to-one transformation between the estimated parameters for the models under consideration. This was true in only limited instances. For example, with six observed measures each factor loading of the one-factor model with correlated errors is equal to a general-factor loading in the doubly-constrained bi-factor model (i.e., $\lambda_i = \lambda_{i4}$; i = 1, 2, ... 6). The group-factor loadings from the latter model are equal to correlated residual elements from the former model (i.e., $\lambda_{11}{}^2 - \theta_{\epsilon12}$; $\lambda_{32}{}^2 - \theta_{\epsilon34}$; $\lambda_{53}{}^2 - \theta_{\epsilon56}$). However, the residual variances are not the same. For example, the first variance, $\theta_{\epsilon11}$, from the one-factor model with correlated errors is equal to $\sigma_{11} - \lambda_1{}^2$, while it equals $\sigma_{11} - \lambda_{11}{}^2 - \lambda_{14}{}^2$ for the doubly-constrained bi-factor model. As it was previously shown that $\lambda_1 = \lambda_{14}$, these expressions would be equal only when λ_{11} is equal to zero.

Models that have identical restrictions between the observed variances and covariances are equivalent models. A non-redundant set of restrictions between the observed variances and covariances, equal in number to the models' degrees of freedom, was considered next. Attempts to use that method were unsuccessful. That method involves solving non-linear equations, which generally include a sum of products. No general methodology is available to solve equations such as these.

The last approach began by considering the expressions from the variance/covariance equations of each selected model to determine if multiple solutions for parameters exist. That is, does more than one combination of model variances and covariances, used previously to identify the models, serve as a solution for each parameter. Instances where more than one combination of covariances represent a parameter are referred to as over-identifying expressions. These over-identifying expressions, being equivalent solutions for a particular parameter, are simplified algebraically resulting in equalities. Many of these equations are of the form $\sigma_{i\,j}\,\sigma_{k\,l} = \sigma_{i\,l}\,\sigma_{k\,j}$ (i, j, k, l = 1, 2, ... n; i ≠ j ≠ k ≠ l, n = 6, 9), but others are more complicated. For each model, these restrictions are then examined for all parameters with over-identifying solutions for instances of repeated equalities. All non-repeated equalities are obtained for each model. These non-repeated equalities represent the restrictions, in the form of covariance equalities, to be met by each model in order to establish equivalence. These equalities must exist for each model, either directly or by some manner of algebraic manipulation of the equalities which are present for a given model.

6 Observed Measures

In the case of six observed measures the following four models are examined for equivalence: one-factor with correlated errors, second-order, group-factor, and doubly-constrained bi-factor. Each of these models has 15 parameters, and hence six degrees-of-freedom.

One-Factor Model With Correlated Errors

For this model the six factor loadings (λ_1 - λ_6) have over-identifying expressions. These expressions and the resulting equalities follow for each parameter.

$$\lambda_1^2 = \frac{\sigma_{13}\sigma_{15}}{\sigma_{35}} = \frac{\sigma_{13}\sigma_{16}}{\sigma_{36}} = \frac{\sigma_{14}\sigma_{15}}{\sigma_{45}} = \frac{\sigma_{14}\sigma_{16}}{\sigma_{46}}$$

$$\sigma_{13}\ \sigma_{45} = \sigma_{14}\ \sigma_{35} \tag{562}$$

$$\sigma_{13}\ \sigma_{46} = \sigma_{14}\ \sigma_{36} \tag{563}$$

$$\sigma_{15}\ \sigma_{36} = \sigma_{16}\ \sigma_{35} \tag{564}$$

$$\sigma_{15}\ \sigma_{46} = \sigma_{16}\ \sigma_{45} \tag{565}$$

$$\lambda_2{}^2 = \frac{\sigma_{23}\sigma_{25}}{\sigma_{35}} = \frac{\sigma_{23}\sigma_{26}}{\sigma_{36}} = \frac{\sigma_{24}\sigma_{25}}{\sigma_{45}} = \frac{\sigma_{24}\sigma_{26}}{\sigma_{46}}$$

$$\sigma_{23}\ \sigma_{45} = \sigma_{24}\ \sigma_{35} \tag{566}$$

$$\sigma_{23}\ \sigma_{46} = \sigma_{24}\ \sigma_{36} \tag{567}$$

$$\sigma_{25}\ \sigma_{36} = \sigma_{26}\ \sigma_{35} \tag{568}$$

$$\sigma_{25}\ \sigma_{46} = \sigma_{26}\ \sigma_{45} \tag{569}$$

$$\lambda_3{}^2 = \frac{\sigma_{13}\sigma_{35}}{\sigma_{15}} = \frac{\sigma_{13}\sigma_{36}}{\sigma_{16}} = \frac{\sigma_{23}\sigma_{35}}{\sigma_{25}} = \frac{\sigma_{23}\sigma_{36}}{\sigma_{26}}$$

$$\sigma_{13}\ \sigma_{25} = \sigma_{15}\ \sigma_{23} \tag{570}$$

$$\sigma_{13}\ \sigma_{26} = \sigma_{16}\ \sigma_{23} \tag{571}$$

$$\sigma_{15}\ \sigma_{36} = \sigma_{16}\ \sigma_{35} \tag{572}$$

$$\sigma_{25}\ \sigma_{36} = \sigma_{26}\ \sigma_{35} \tag{573}$$

$$\lambda_4{}^2 = \frac{\sigma_{14}\sigma_{45}}{\sigma_{15}} = \frac{\sigma_{14}\sigma_{46}}{\sigma_{16}} = \frac{\sigma_{24}\sigma_{45}}{\sigma_{25}} = \frac{\sigma_{24}\sigma_{46}}{\sigma_{26}}$$

$$\sigma_{14}\ \sigma_{25} = \sigma_{15}\ \sigma_{24} \tag{574}$$

$$\sigma_{14}\ \sigma_{26} = \sigma_{16}\ \sigma_{24} \tag{575}$$

$$\sigma_{15}\ \sigma_{46} = \sigma_{16}\ \sigma_{45} \tag{576}$$

$$\sigma_{25}\ \sigma_{46} = \sigma_{26}\ \sigma_{45} \tag{577}$$

$$\lambda_5{}^2 = \frac{\sigma_{15}\sigma_{35}}{\sigma_{13}} = \frac{\sigma_{15}\sigma_{45}}{\sigma_{14}} = \frac{\sigma_{25}\sigma_{35}}{\sigma_{23}} = \frac{\sigma_{25}\sigma_{45}}{\sigma_{24}}$$

$$\sigma_{13}\ \sigma_{25} = \sigma_{15}\ \sigma_{23} \tag{578}$$

$$\sigma_{13}\ \sigma_{45} = \sigma_{14}\ \sigma_{35} \tag{579}$$

$$\sigma_{14}\ \sigma_{25} = \sigma_{15}\ \sigma_{24} \tag{580}$$

$$\sigma_{23}\ \sigma_{45} = \sigma_{24}\ \sigma_{35} \tag{581}$$

$$\lambda_6{}^2 = \frac{\sigma_{16}\sigma_{36}}{\sigma_{13}} = \frac{\sigma_{16}\sigma_{46}}{\sigma_{14}} = \frac{\sigma_{26}\sigma_{36}}{\sigma_{23}} = \frac{\sigma_{26}\sigma_{46}}{\sigma_{24}}$$

$$\sigma_{13}\ \sigma_{26} = \sigma_{16}\ \sigma_{23} \tag{582}$$

$$\sigma_{13}\ \sigma_{46} = \sigma_{14}\ \sigma_{36} \tag{583}$$

$$\sigma_{14}\ \sigma_{26} = \sigma_{16}\ \sigma_{24} \tag{584}$$

$$\sigma_{23}\ \sigma_{46} = \sigma_{24}\ \sigma_{36} \tag{585}$$

An examination of these 24 equalities reveals that each one appears twice. The twelve non-repeated expressions to be retained are 562-571, and 574-575.

As such, not all of the equations produced will be necessary to examine model equivalence.

Second-Order Model

The second-order model has six parameters (λ_1 - λ_3, γ_1 - γ_3) with over-identifying expressions. The twelve non-repeated equalities from the one-factor model with correlated errors occur here and in the two remaining models. They are not re-written in these sections. There are, however, three additional non-repeated equalities which occur in this (and the group-factor) model.

$$\lambda_1 = \frac{\sigma_{23}}{\sigma_{13}} = \frac{\sigma_{24}}{\sigma_{14}} = \frac{\sigma_{25}}{\sigma_{15}} = \frac{\sigma_{26}}{\sigma_{16}}$$

$$\sigma_{13}\,\sigma_{24} = \sigma_{14}\,\sigma_{23} \qquad (586)$$

$$\sigma_{15}\,\sigma_{26} = \sigma_{16}\,\sigma_{25} \qquad (587)$$

$$\lambda_2 = \frac{\sigma_{14}}{\sigma_{13}} = \frac{\sigma_{24}}{\sigma_{23}} = \frac{\sigma_{45}}{\sigma_{35}} = \frac{\sigma_{46}}{\sigma_{36}}$$

$$\sigma_{35}\,\sigma_{46} = \sigma_{36}\,\sigma_{45} \qquad (588)$$

$$\lambda_3 = \frac{\sigma_{16}}{\sigma_{15}} = \frac{\sigma_{26}}{\sigma_{25}} = \frac{\sigma_{36}}{\sigma_{35}} = \frac{\sigma_{46}}{\sigma_{45}}$$

$$\gamma_1^2 = \frac{\sigma_{13}\sigma_{15}}{\sigma_{35}} = \frac{\sigma_{13}\sigma_{16}}{\sigma_{36}} = \frac{\sigma_{14}\sigma_{16}}{\sigma_{46}}$$

$$\gamma_2^2 = \frac{\sigma_{13}\sigma_{35}}{\sigma_{15}} = \frac{\sigma_{13}\sigma_{36}}{\sigma_{16}} = \frac{\sigma_{23}\sigma_{36}}{\sigma_{26}}$$

$$\gamma_3^2 = \frac{\sigma_{15}\sigma_{35}}{\sigma_{13}} = \frac{\sigma_{15}\sigma_{45}}{\sigma_{14}} = \frac{\sigma_{25}\sigma_{45}}{\sigma_{24}}$$

<u>Group-Factor Model</u>

There are nine parameters (λ_1 - λ_6, ω_{12}, ω_{13}, ω_{23}) which have over-identifying expressions for the group-factor model. The non-repeated equalities appear for this and the three other models of this section in tabular form as Appendix A.

$$\lambda_1^2 = \frac{\sigma_{12}\sigma_{13}}{\sigma_{23}} = \frac{\sigma_{12}\sigma_{14}}{\sigma_{24}} = \frac{\sigma_{12}\sigma_{15}}{\sigma_{25}} = \frac{\sigma_{12}\sigma_{16}}{\sigma_{26}}$$

$$\lambda_2^{\,2} = \frac{\sigma_{12}\sigma_{23}}{\sigma_{13}} = \frac{\sigma_{12}\sigma_{24}}{\sigma_{14}} = \frac{\sigma_{12}\sigma_{25}}{\sigma_{15}} = \frac{\sigma_{12}\sigma_{26}}{\sigma_{16}}$$

$$\lambda_3^{\,2} = \frac{\sigma_{13}\sigma_{34}}{\sigma_{14}} = \frac{\sigma_{23}\sigma_{34}}{\sigma_{24}} = \frac{\sigma_{34}\sigma_{35}}{\sigma_{45}} = \frac{\sigma_{34}\sigma_{36}}{\sigma_{46}}$$

$$\lambda_4^{\,2} = \frac{\sigma_{14}\sigma_{34}}{\sigma_{13}} = \frac{\sigma_{24}\sigma_{34}}{\sigma_{23}} = \frac{\sigma_{34}\sigma_{45}}{\sigma_{35}} = \frac{\sigma_{34}\sigma_{46}}{\sigma_{36}}$$

$$\lambda_5^{\,2} = \frac{\sigma_{15}\sigma_{56}}{\sigma_{16}} = \frac{\sigma_{25}\sigma_{56}}{\sigma_{26}} = \frac{\sigma_{35}\sigma_{56}}{\sigma_{36}} = \frac{\sigma_{45}\sigma_{56}}{\sigma_{46}}$$

$$\lambda_6^{\,2} = \frac{\sigma_{16}\sigma_{56}}{\sigma_{15}} = \frac{\sigma_{26}\sigma_{56}}{\sigma_{25}} = \frac{\sigma_{36}\sigma_{56}}{\sigma_{35}} = \frac{\sigma_{46}\sigma_{56}}{\sigma_{45}}$$

$$\omega_{12} = \frac{\sigma_{13}}{\left(\frac{\sigma_{12}\sigma_{13}}{\sigma_{23}}\right)^{1/2}\left(\frac{\sigma_{13}\sigma_{34}}{\sigma_{14}}\right)^{1/2}} = \frac{\sigma_{14}}{\left(\frac{\sigma_{12}\sigma_{13}}{\sigma_{23}}\right)^{1/2}\left(\frac{\sigma_{14}\sigma_{34}}{\sigma_{13}}\right)^{1/2}}$$

$$= \frac{\sigma_{23}}{\left(\frac{\sigma_{12}\sigma_{23}}{\sigma_{13}}\right)^{1/2}\left(\frac{\sigma_{13}\sigma_{34}}{\sigma_{14}}\right)^{1/2}} = \frac{\sigma_{24}}{\left(\frac{\sigma_{12}\sigma_{23}}{\sigma_{12}}\right)^{1/2}\left(\frac{\sigma_{14}\sigma_{34}}{\sigma_{13}}\right)^{1/2}}$$

$$\omega_{13} = \frac{\sigma_{15}}{\left(\frac{\sigma_{12}\sigma_{13}}{\sigma_{23}}\right)^{1/2}\left(\frac{\sigma_{15}\sigma_{56}}{\sigma_{16}}\right)^{1/2}} = \frac{\sigma_{16}}{\left(\frac{\sigma_{12}\sigma_{13}}{\sigma_{23}}\right)^{1/2}\left(\frac{\sigma_{16}\sigma_{56}}{\sigma_{15}}\right)^{1/2}}$$

$$= \frac{\sigma_{25}}{\left(\frac{\sigma_{12}\sigma_{23}}{\sigma_{13}}\right)^{1/2}\left(\frac{\sigma_{15}\sigma_{56}}{\sigma_{16}}\right)^{1/2}} = \frac{\sigma_{26}}{\left(\frac{\sigma_{12}\sigma_{23}}{\sigma_{13}}\right)^{1/2}\left(\frac{\sigma_{16}\sigma_{56}}{\sigma_{15}}\right)^{1/2}}$$

$$\omega_{23} = \frac{\sigma_{35}}{\left(\frac{\sigma_{13}\sigma_{34}}{\sigma_{14}}\right)^{1/2}\left(\frac{\sigma_{15}\sigma_{56}}{\sigma_{16}}\right)^{1/2}} = \frac{\sigma_{36}}{\left(\frac{\sigma_{13}\sigma_{34}}{\sigma_{14}}\right)^{1/2}\left(\frac{\sigma_{16}\sigma_{56}}{\sigma_{15}}\right)^{1/2}}$$

$$= \frac{\sigma_{45}}{\left(\frac{\sigma_{14}\sigma_{34}}{\sigma_{13}}\right)^{1/2}\left(\frac{\sigma_{15}\sigma_{56}}{\sigma_{16}}\right)^{1/2}} = \frac{\sigma_{46}}{\left(\frac{\sigma_{14}\sigma_{34}}{\sigma_{13}}\right)^{1/2}\left(\frac{\sigma_{16}\sigma_{56}}{\sigma_{15}}\right)^{1/2}}$$

Doubly Constrained Bi-Factor Model

The six general-factor loadings have over-identifying expressions for the doubly-constrained bi-factor model.

$$\lambda_{14}^2 = \frac{\sigma_{13}\sigma_{15}}{\sigma_{35}} = \frac{\sigma_{13}\sigma_{16}}{\sigma_{36}} = \frac{\sigma_{14}\sigma_{15}}{\sigma_{45}} = \frac{\sigma_{14}\sigma_{16}}{\sigma_{46}}$$

$$\lambda_{24}^2 = \frac{\sigma_{23}\sigma_{25}}{\sigma_{35}} = \frac{\sigma_{23}\sigma_{26}}{\sigma_{36}} = \frac{\sigma_{24}\sigma_{25}}{\sigma_{45}} = \frac{\sigma_{24}\sigma_{26}}{\sigma_{46}}$$

$$\lambda_{34}^2 = \frac{\sigma_{13}\sigma_{35}}{\sigma_{15}} = \frac{\sigma_{13}\sigma_{36}}{\sigma_{16}} = \frac{\sigma_{23}\sigma_{35}}{\sigma_{25}} = \frac{\sigma_{23}\sigma_{36}}{\sigma_{26}}$$

$$\lambda_{44}^2 = \frac{\sigma_{14}\sigma_{45}}{\sigma_{15}} = \frac{\sigma_{14}\sigma_{46}}{\sigma_{16}} = \frac{\sigma_{24}\sigma_{45}}{\sigma_{25}} = \frac{\sigma_{24}\sigma_{46}}{\sigma_{26}}$$

$$\lambda_{54}^2 = \frac{\sigma_{15}\sigma_{35}}{\sigma_{13}} = \frac{\sigma_{15}\sigma_{45}}{\sigma_{14}} = \frac{\sigma_{25}\sigma_{35}}{\sigma_{23}} = \frac{\sigma_{25}\sigma_{45}}{\sigma_{24}}$$

$$\lambda_{64}^2 = \frac{\sigma_{16}\sigma_{36}}{\sigma_{13}} = \frac{\sigma_{16}\sigma_{46}}{\sigma_{14}} = \frac{\sigma_{26}\sigma_{36}}{\sigma_{23}} = \frac{\sigma_{26}\sigma_{46}}{\sigma_{24}}$$

Referring to Appendix A, the presence of an "X" indicates that that particular equality expression exists (directly) for that model. The absence of an X requires some algebraic solution using the restrictions which are present. For instance, the second-order and group-factor models share the same 15 equalities thereby establishing their equivalence. The one-factor model with correlated errors and the doubly constrained bi-factor model each contain the same 12 restrictions thereby establishing their equivalence. The three missing restrictions for these two models can be obtained straightforwardly by division involving the 12 equalities which are present. This results in establishing that the four models are equivalent when there are six observed measures.

9 Observed Measures

In the case of nine-observed measures there are two different instances of equivalence. The first is that between the one-factor model with correlated errors and the bi-factor model with uncorrelated group factors. The second instance is that between the second-order and group-factor models. These latter models each have 21 parameters, while the former two are each comprised of 27 parameters.

One-Factor Model With Correlated Errors

This model has nine parameters (λ_1 - λ_9) which have over-identifying expressions. The parameters and the expressions follow. These expressions produce 81 non-repeated equalities which are listed in Appendix B.

$$\lambda_1^2 = \frac{\sigma_{14}\sigma_{17}}{\sigma_{47}} = \frac{\sigma_{14}\sigma_{18}}{\sigma_{48}} = \frac{\sigma_{14}\sigma_{19}}{\sigma_{49}} = \frac{\sigma_{15}\sigma_{17}}{\sigma_{57}}$$

$$= \frac{\sigma_{15}\sigma_{18}}{\sigma_{58}} = \frac{\sigma_{15}\sigma_{19}}{\sigma_{59}} = \frac{\sigma_{16}\sigma_{17}}{\sigma_{67}} = \frac{\sigma_{16}\sigma_{18}}{\sigma_{68}} = \frac{\sigma_{16}\sigma_{19}}{\sigma_{69}}$$

$$\lambda_2^2 = \frac{\sigma_{24}\sigma_{27}}{\sigma_{47}} = \frac{\sigma_{24}\sigma_{28}}{\sigma_{48}} = \frac{\sigma_{24}\sigma_{29}}{\sigma_{49}} = \frac{\sigma_{25}\sigma_{27}}{\sigma_{57}}$$

$$= \frac{\sigma_{25}\sigma_{28}}{\sigma_{58}} = \frac{\sigma_{25}\sigma_{29}}{\sigma_{59}} = \frac{\sigma_{26}\sigma_{27}}{\sigma_{67}} = \frac{\sigma_{26}\sigma_{28}}{\sigma_{68}} = \frac{\sigma_{26}\sigma_{29}}{\sigma_{69}}$$

$$\lambda_3^2 = \frac{\sigma_{34}\sigma_{37}}{\sigma_{47}} = \frac{\sigma_{34}\sigma_{38}}{\sigma_{48}} = \frac{\sigma_{34}\sigma_{39}}{\sigma_{49}} = \frac{\sigma_{35}\sigma_{37}}{\sigma_{57}}$$

$$= \frac{\sigma_{35}\sigma_{38}}{\sigma_{58}} = \frac{\sigma_{35}\sigma_{39}}{\sigma_{59}} = \frac{\sigma_{36}\sigma_{37}}{\sigma_{67}} = \frac{\sigma_{36}\sigma_{38}}{\sigma_{68}} = \frac{\sigma_{36}\sigma_{39}}{\sigma_{69}}$$

$$\lambda_4^2 = \frac{\sigma_{14}\sigma_{47}}{\sigma_{17}} = \frac{\sigma_{14}\sigma_{48}}{\sigma_{18}} = \frac{\sigma_{14}\sigma_{49}}{\sigma_{19}} = \frac{\sigma_{24}\sigma_{47}}{\sigma_{27}}$$

$$= \frac{\sigma_{24}\sigma_{48}}{\sigma_{28}} = \frac{\sigma_{24}\sigma_{49}}{\sigma_{29}} = \frac{\sigma_{34}\sigma_{47}}{\sigma_{37}} = \frac{\sigma_{34}\sigma_{48}}{\sigma_{38}} = \frac{\sigma_{34}\sigma_{49}}{\sigma_{39}}$$

$$\lambda_5{}^2 - \frac{\sigma_{15}\sigma_{57}}{\sigma_{17}} - \frac{\sigma_{15}\sigma_{58}}{\sigma_{18}} - \frac{\sigma_{15}\sigma_{59}}{\sigma_{19}} - \frac{\sigma_{25}\sigma_{57}}{\sigma_{27}}$$

$$- \frac{\sigma_{25}\sigma_{58}}{\sigma_{28}} - \frac{\sigma_{25}\sigma_{59}}{\sigma_{29}} - \frac{\sigma_{35}\sigma_{57}}{\sigma_{37}} - \frac{\sigma_{35}\sigma_{58}}{\sigma_{38}} - \frac{\sigma_{35}\sigma_{59}}{\sigma_{39}}$$

$$\lambda_6{}^2 - \frac{\sigma_{16}\sigma_{67}}{\sigma_{17}} - \frac{\sigma_{16}\sigma_{68}}{\sigma_{18}} - \frac{\sigma_{16}\sigma_{69}}{\sigma_{19}} - \frac{\sigma_{26}\sigma_{67}}{\sigma_{27}}$$

$$- \frac{\sigma_{26}\sigma_{68}}{\sigma_{28}} - \frac{\sigma_{26}\sigma_{69}}{\sigma_{29}} - \frac{\sigma_{36}\sigma_{67}}{\sigma_{37}} - \frac{\sigma_{36}\sigma_{68}}{\sigma_{38}} - \frac{\sigma_{36}\sigma_{69}}{\sigma_{39}}$$

$$\lambda_7{}^2 - \frac{\sigma_{17}\sigma_{47}}{\sigma_{14}} - \frac{\sigma_{17}\sigma_{57}}{\sigma_{15}} - \frac{\sigma_{17}\sigma_{67}}{\sigma_{16}} - \frac{\sigma_{27}\sigma_{47}}{\sigma_{24}}$$

$$- \frac{\sigma_{27}\sigma_{57}}{\sigma_{25}} - \frac{\sigma_{27}\sigma_{67}}{\sigma_{26}} - \frac{\sigma_{37}\sigma_{47}}{\sigma_{34}} - \frac{\sigma_{37}\sigma_{57}}{\sigma_{35}} - \frac{\sigma_{37}\sigma_{67}}{\sigma_{36}}$$

$$\lambda_8{}^2 - \frac{\sigma_{18}\sigma_{48}}{\sigma_{14}} - \frac{\sigma_{18}\sigma_{58}}{\sigma_{15}} - \frac{\sigma_{18}\sigma_{68}}{\sigma_{16}} - \frac{\sigma_{28}\sigma_{48}}{\sigma_{24}}$$

$$- \frac{\sigma_{28}\sigma_{58}}{\sigma_{25}} - \frac{\sigma_{28}\sigma_{68}}{\sigma_{26}} - \frac{\sigma_{38}\sigma_{48}}{\sigma_{34}} - \frac{\sigma_{38}\sigma_{58}}{\sigma_{35}} - \frac{\sigma_{38}\sigma_{68}}{\sigma_{36}}$$

$$\lambda_9^2 = \frac{\sigma_{19}\sigma_{49}}{\sigma_{14}} = \frac{\sigma_{19}\sigma_{59}}{\sigma_{15}} = \frac{\sigma_{19}\sigma_{69}}{\sigma_{16}} = \frac{\sigma_{29}\sigma_{49}}{\sigma_{24}}$$

$$= \frac{\sigma_{29}\sigma_{59}}{\sigma_{25}} = \frac{\sigma_{29}\sigma_{69}}{\sigma_{26}} = \frac{\sigma_{39}\sigma_{49}}{\sigma_{34}} = \frac{\sigma_{39}\sigma_{59}}{\sigma_{35}} = \frac{\sigma_{39}\sigma_{69}}{\sigma_{36}}$$

Bi-Factor Model With Uncorrelated Group Factors

This model has nine parameters (λ_{14} - λ_{94}) with over-identifying expressions. The expressions are the same as those of the one-factor model with correlated errors. Expressions associated with the first factor loading (λ_1) in the one-factor model are the same as those for the first general-factor loading (λ_{14}) in the current model.

Consequently the number of expressions and resulting non-repeated equalities are identical in this model with those of the one-factor model with correlated errors. Reference to Appendix B reveals that each equality listed is present (indicated by an "X") in both models, thereby demonstrating the models to be equivalent.

Second-Order Model

The second-order model has 12 parameters (λ_1 - λ_6, γ_1 - γ_3, ψ_1 -

ψ_3) with over-identifying expressions. These expressions follow, with the 132 resulting equalities listed in Appendix C.

$$\lambda_1 = \frac{\sigma_{23}}{\sigma_{13}} = \frac{\sigma_{24}}{\sigma_{14}} = \frac{\sigma_{25}}{\sigma_{15}} = \frac{\sigma_{26}}{\sigma_{16}} = \frac{\sigma_{27}}{\sigma_{17}} = \frac{\sigma_{28}}{\sigma_{18}} = \frac{\sigma_{29}}{\sigma_{19}}$$

$$\lambda_2 = \frac{\sigma_{23}}{\sigma_{12}} = \frac{\sigma_{34}}{\sigma_{14}} = \frac{\sigma_{35}}{\sigma_{15}} = \frac{\sigma_{36}}{\sigma_{16}} = \frac{\sigma_{37}}{\sigma_{17}} = \frac{\sigma_{38}}{\sigma_{18}} = \frac{\sigma_{39}}{\sigma_{19}}$$

$$\lambda_3 = \frac{\sigma_{15}}{\sigma_{14}} = \frac{\sigma_{25}}{\sigma_{24}} = \frac{\sigma_{35}}{\sigma_{34}} = \frac{\sigma_{56}}{\sigma_{46}} = \frac{\sigma_{57}}{\sigma_{47}} = \frac{\sigma_{58}}{\sigma_{48}} = \frac{\sigma_{59}}{\sigma_{49}}$$

$$\lambda_4 = \frac{\sigma_{16}}{\sigma_{14}} = \frac{\sigma_{26}}{\sigma_{24}} = \frac{\sigma_{36}}{\sigma_{34}} = \frac{\sigma_{56}}{\sigma_{45}} = \frac{\sigma_{67}}{\sigma_{47}} = \frac{\sigma_{68}}{\sigma_{48}} = \frac{\sigma_{69}}{\sigma_{49}}$$

$$\lambda_5 = \frac{\sigma_{18}}{\sigma_{17}} = \frac{\sigma_{28}}{\sigma_{27}} = \frac{\sigma_{38}}{\sigma_{37}} = \frac{\sigma_{48}}{\sigma_{47}} = \frac{\sigma_{58}}{\sigma_{57}} = \frac{\sigma_{68}}{\sigma_{67}} = \frac{\sigma_{89}}{\sigma_{79}}$$

$$\lambda_6 = \frac{\sigma_{19}}{\sigma_{17}} = \frac{\sigma_{29}}{\sigma_{27}} = \frac{\sigma_{39}}{\sigma_{37}} = \frac{\sigma_{49}}{\sigma_{47}} = \frac{\sigma_{59}}{\sigma_{57}} = \frac{\sigma_{69}}{\sigma_{67}} = \frac{\sigma_{89}}{\sigma_{78}}$$

$$\gamma_1^2 = \frac{\sigma_{14}\sigma_{17}}{\sigma_{47}} = \frac{\sigma_{14}\sigma_{18}}{\sigma_{48}} = \frac{\sigma_{14}\sigma_{19}}{\sigma_{49}} = \frac{\sigma_{15}\sigma_{17}}{\sigma_{57}}$$

$$= \frac{\sigma_{15}\sigma_{18}}{\sigma_{58}} = \frac{\sigma_{15}\sigma_{19}}{\sigma_{59}} = \frac{\sigma_{16}\sigma_{17}}{\sigma_{67}} = \frac{\sigma_{16}\sigma_{18}}{\sigma_{68}} = \frac{\sigma_{16}\sigma_{19}}{\sigma_{69}}$$

$$\gamma_2^2 = \frac{\sigma_{14}\sigma_{47}}{\sigma_{17}} = \frac{\sigma_{14}\sigma_{48}}{\sigma_{18}} = \frac{\sigma_{14}\sigma_{49}}{\sigma_{19}} = \frac{\sigma_{24}\sigma_{47}}{\sigma_{27}}$$

$$= \frac{\sigma_{24}\sigma_{48}}{\sigma_{28}} = \frac{\sigma_{24}\sigma_{49}}{\sigma_{29}} = \frac{\sigma_{34}\sigma_{47}}{\sigma_{37}} = \frac{\sigma_{34}\sigma_{48}}{\sigma_{38}} = \frac{\sigma_{34}\sigma_{49}}{\sigma_{39}}$$

$$\gamma_3^2 = \frac{\sigma_{17}\sigma_{47}}{\sigma_{14}} = \frac{\sigma_{17}\sigma_{57}}{\sigma_{15}} = \frac{\sigma_{17}\sigma_{67}}{\sigma_{16}} = \frac{\sigma_{27}\sigma_{47}}{\sigma_{24}}$$

$$= \frac{\sigma_{27}\sigma_{57}}{\sigma_{25}} = \frac{\sigma_{27}\sigma_{67}}{\sigma_{26}} = \frac{\sigma_{37}\sigma_{47}}{\sigma_{34}} = \frac{\sigma_{37}\sigma_{57}}{\sigma_{35}} = \frac{\sigma_{37}\sigma_{67}}{\sigma_{36}}$$

$$\psi_1 = \frac{\sigma_{12}\sigma_{14}}{\sigma_{24}} - \frac{\sigma_{14}\sigma_{17}}{\sigma_{47}} = \frac{\sigma_{13}\sigma_{14}}{\sigma_{34}} - \frac{\sigma_{14}\sigma_{17}}{\sigma_{47}} = \frac{\sigma_{14}^2\sigma_{23}}{\sigma_{24}\sigma_{34}} - \frac{\sigma_{14}\sigma_{17}}{\sigma_{47}}$$

$$\psi_2 = \frac{\sigma_{14}\sigma_{45}}{\sigma_{15}} = \frac{\sigma_{14}\sigma_{47}}{\sigma_{17}} = \frac{\sigma_{14}\sigma_{46}}{\sigma_{16}} = \frac{\sigma_{14}\sigma_{47}}{\sigma_{17}} = \frac{\sigma_{14}^2\sigma_{56}}{\sigma_{15}\sigma_{16}} = \frac{\sigma_{14}\sigma_{47}}{\sigma_{17}}$$

$$\psi_3 = \frac{\sigma_{17}\sigma_{78}}{\sigma_{18}} = \frac{\sigma_{17}\sigma_{47}}{\sigma_{14}} = \frac{\sigma_{17}\sigma_{79}}{\sigma_{19}} = \frac{\sigma_{17}\sigma_{47}}{\sigma_{14}} = \frac{\sigma_{17}^2\sigma_{89}}{\sigma_{18}\sigma_{19}} = \frac{\sigma_{17}\sigma_{47}}{\sigma_{14}}$$

<u>Group-Factor Model</u>

The group-factor model has 12 parameters ($\lambda_1 - \lambda_9$, ω_{12}, ω_{13}, ω_{23}) which have over-identifying expressions. These expressions follow. They result in 162 equalities which are listed in Appendix C.

$$\lambda_1^2 = \frac{\sigma_{12}\sigma_{13}}{\sigma_{23}} = \frac{\sigma_{12}\sigma_{14}}{\sigma_{24}} = \frac{\sigma_{12}\sigma_{15}}{\sigma_{25}} = \frac{\sigma_{12}\sigma_{16}}{\sigma_{26}} = \frac{\sigma_{12}\sigma_{17}}{\sigma_{27}}$$

$$= \frac{\sigma_{12}\sigma_{18}}{\sigma_{28}} = \frac{\sigma_{12}\sigma_{19}}{\sigma_{29}} = \frac{\sigma_{13}\sigma_{14}}{\sigma_{34}} = \frac{\sigma_{13}\sigma_{15}}{\sigma_{35}} = \frac{\sigma_{13}\sigma_{16}}{\sigma_{36}}$$

$$= \frac{\sigma_{13}\sigma_{17}}{\sigma_{37}} = \frac{\sigma_{13}\sigma_{18}}{\sigma_{38}} = \frac{\sigma_{13}\sigma_{19}}{\sigma_{39}}$$

$$\lambda_2{}^2 = \frac{\sigma_{12}\sigma_{23}}{\sigma_{13}} = \frac{\sigma_{12}\sigma_{24}}{\sigma_{14}} = \frac{\sigma_{12}\sigma_{25}}{\sigma_{15}} = \frac{\sigma_{12}\sigma_{26}}{\sigma_{16}} = \frac{\sigma_{12}\sigma_{27}}{\sigma_{17}}$$

$$= \frac{\sigma_{12}\sigma_{28}}{\sigma_{18}} = \frac{\sigma_{12}\sigma_{29}}{\sigma_{19}} = \frac{\sigma_{23}\sigma_{24}}{\sigma_{34}} = \frac{\sigma_{23}\sigma_{25}}{\sigma_{35}} = \frac{\sigma_{23}\sigma_{26}}{\sigma_{36}}$$

$$= \frac{\sigma_{23}\sigma_{27}}{\sigma_{37}} = \frac{\sigma_{23}\sigma_{28}}{\sigma_{38}} = \frac{\sigma_{23}\sigma_{29}}{\sigma_{39}}$$

$$\lambda_3{}^2 = \frac{\sigma_{13}\sigma_{23}}{\sigma_{12}} = \frac{\sigma_{13}\sigma_{34}}{\sigma_{14}} = \frac{\sigma_{13}\sigma_{35}}{\sigma_{15}} = \frac{\sigma_{13}\sigma_{36}}{\sigma_{16}} = \frac{\sigma_{13}\sigma_{37}}{\sigma_{17}}$$

$$= \frac{\sigma_{13}\sigma_{38}}{\sigma_{18}} = \frac{\sigma_{13}\sigma_{39}}{\sigma_{19}} = \frac{\sigma_{23}\sigma_{34}}{\sigma_{24}} = \frac{\sigma_{23}\sigma_{35}}{\sigma_{25}} = \frac{\sigma_{23}\sigma_{36}}{\sigma_{26}}$$

$$= \frac{\sigma_{23}\sigma_{37}}{\sigma_{27}} = \frac{\sigma_{23}\sigma_{38}}{\sigma_{28}} = \frac{\sigma_{23}\sigma_{39}}{\sigma_{29}}$$

$$\lambda_4{}^2 = \frac{\sigma_{45}\sigma_{46}}{\sigma_{56}} = \frac{\sigma_{45}\sigma_{14}}{\sigma_{15}} = \frac{\sigma_{45}\sigma_{24}}{\sigma_{25}} = \frac{\sigma_{45}\sigma_{34}}{\sigma_{35}} = \frac{\sigma_{45}\sigma_{47}}{\sigma_{57}}$$

$$= \frac{\sigma_{45}\sigma_{48}}{\sigma_{58}} = \frac{\sigma_{45}\sigma_{49}}{\sigma_{59}} = \frac{\sigma_{46}\sigma_{14}}{\sigma_{16}} = \frac{\sigma_{46}\sigma_{24}}{\sigma_{26}} = \frac{\sigma_{46}\sigma_{34}}{\sigma_{36}}$$

$$- \frac{\sigma_{46}\sigma_{47}}{\sigma_{67}} - \frac{\sigma_{46}\sigma_{48}}{\sigma_{68}} - \frac{\sigma_{46}\sigma_{49}}{\sigma_{69}}$$

$$\lambda_5{}^2 - \frac{\sigma_{45}\sigma_{56}}{\sigma_{46}} - \frac{\sigma_{15}\sigma_{45}}{\sigma_{14}} - \frac{\sigma_{25}\sigma_{45}}{\sigma_{24}} - \frac{\sigma_{35}\sigma_{45}}{\sigma_{34}} - \frac{\sigma_{45}\sigma_{57}}{\sigma_{47}}$$

$$- \frac{\sigma_{45}\sigma_{58}}{\sigma_{48}} - \frac{\sigma_{45}\sigma_{59}}{\sigma_{49}} - \frac{\sigma_{15}\sigma_{56}}{\sigma_{16}} - \frac{\sigma_{25}\sigma_{56}}{\sigma_{26}} - \frac{\sigma_{35}\sigma_{56}}{\sigma_{36}}$$

$$- \frac{\sigma_{56}\sigma_{57}}{\sigma_{67}} - \frac{\sigma_{56}\sigma_{58}}{\sigma_{68}} - \frac{\sigma_{56}\sigma_{59}}{\sigma_{69}}$$

$$\lambda_6{}^2 - \frac{\sigma_{46}\sigma_{56}}{\sigma_{45}} - \frac{\sigma_{16}\sigma_{46}}{\sigma_{14}} - \frac{\sigma_{26}\sigma_{46}}{\sigma_{24}} - \frac{\sigma_{36}\sigma_{46}}{\sigma_{34}} - \frac{\sigma_{46}\sigma_{67}}{\sigma_{47}}$$

$$- \frac{\sigma_{46}\sigma_{68}}{\sigma_{48}} - \frac{\sigma_{46}\sigma_{69}}{\sigma_{49}} - \frac{\sigma_{16}\sigma_{56}}{\sigma_{15}} - \frac{\sigma_{26}\sigma_{56}}{\sigma_{25}} - \frac{\sigma_{36}\sigma_{56}}{\sigma_{35}}$$

$$- \frac{\sigma_{56}\sigma_{67}}{\sigma_{57}} - \frac{\sigma_{56}\sigma_{68}}{\sigma_{58}} - \frac{\sigma_{56}\sigma_{69}}{\sigma_{59}}$$

$$\lambda_7{}^2 - \frac{\sigma_{78}\sigma_{79}}{\sigma_{89}} - \frac{\sigma_{17}\sigma_{78}}{\sigma_{18}} - \frac{\sigma_{27}\sigma_{78}}{\sigma_{28}} - \frac{\sigma_{37}\sigma_{78}}{\sigma_{38}} - \frac{\sigma_{47}\sigma_{78}}{\sigma_{48}}$$

$$- \frac{\sigma_{57}\sigma_{78}}{\sigma_{58}} - \frac{\sigma_{67}\sigma_{78}}{\sigma_{68}} - \frac{\sigma_{17}\sigma_{79}}{\sigma_{19}} - \frac{\sigma_{27}\sigma_{79}}{\sigma_{29}} - \frac{\sigma_{37}\sigma_{79}}{\sigma_{39}}$$

$$- \frac{\sigma_{47}\sigma_{79}}{\sigma_{49}} - \frac{\sigma_{57}\sigma_{79}}{\sigma_{59}} - \frac{\sigma_{67}\sigma_{79}}{\sigma_{69}}$$

$$\lambda_8^{\,2} - \frac{\sigma_{78}\sigma_{89}}{\sigma_{79}} - \frac{\sigma_{18}\sigma_{78}}{\sigma_{17}} - \frac{\sigma_{28}\sigma_{78}}{\sigma_{27}} - \frac{\sigma_{38}\sigma_{78}}{\sigma_{37}} - \frac{\sigma_{48}\sigma_{78}}{\sigma_{47}}$$

$$- \frac{\sigma_{58}\sigma_{78}}{\sigma_{57}} - \frac{\sigma_{68}\sigma_{78}}{\sigma_{67}} - \frac{\sigma_{18}\sigma_{89}}{\sigma_{19}} - \frac{\sigma_{28}\sigma_{89}}{\sigma_{29}} - \frac{\sigma_{38}\sigma_{89}}{\sigma_{39}}$$

$$- \frac{\sigma_{48}\sigma_{89}}{\sigma_{49}} - \frac{\sigma_{58}\sigma_{89}}{\sigma_{59}} - \frac{\sigma_{68}\sigma_{89}}{\sigma_{69}}$$

$$\lambda_9^{\,2} - \frac{\sigma_{79}\sigma_{89}}{\sigma_{78}} - \frac{\sigma_{19}\sigma_{79}}{\sigma_{17}} - \frac{\sigma_{29}\sigma_{79}}{\sigma_{27}} - \frac{\sigma_{39}\sigma_{79}}{\sigma_{37}} - \frac{\sigma_{49}\sigma_{79}}{\sigma_{47}}$$

$$- \frac{\sigma_{59}\sigma_{79}}{\sigma_{57}} - \frac{\sigma_{69}\sigma_{79}}{\sigma_{67}} - \frac{\sigma_{19}\sigma_{89}}{\sigma_{18}} - \frac{\sigma_{29}\sigma_{89}}{\sigma_{28}} - \frac{\sigma_{39}\sigma_{89}}{\sigma_{38}}$$

$$- \frac{\sigma_{49}\sigma_{89}}{\sigma_{48}} - \frac{\sigma_{59}\sigma_{89}}{\sigma_{58}} - \frac{\sigma_{69}\sigma_{89}}{\sigma_{68}}$$

$$\omega_{12}^2 = \frac{\sigma_{14}^2\sigma_{23}\sigma_{56}}{\sigma_{12}\sigma_{13}\sigma_{45}\sigma_{46}} - \frac{\sigma_{15}^2\sigma_{23}\sigma_{46}}{\sigma_{12}\sigma_{13}\sigma_{45}\sigma_{56}} - \frac{\sigma_{16}^2\sigma_{23}\sigma_{45}}{\sigma_{12}\sigma_{13}\sigma_{46}\sigma_{56}}$$

$$- \frac{\sigma_{13}\sigma_{24}^2\sigma_{56}}{\sigma_{12}\sigma_{23}\sigma_{45}\sigma_{46}} - \frac{\sigma_{13}\sigma_{25}^2\sigma_{46}}{\sigma_{12}\sigma_{23}\sigma_{45}\sigma_{56}} - \frac{\sigma_{13}\sigma_{26}^2\sigma_{45}}{\sigma_{12}\sigma_{23}\sigma_{46}\sigma_{56}}$$

$$- \frac{\sigma_{12}\sigma_{34}^2\sigma_{56}}{\sigma_{13}\sigma_{23}\sigma_{45}\sigma_{46}} - \frac{\sigma_{12}\sigma_{35}^2\sigma_{46}}{\sigma_{13}\sigma_{23}\sigma_{45}\sigma_{56}} - \frac{\sigma_{12}\sigma_{36}^2\sigma_{45}}{\sigma_{13}\sigma_{23}\sigma_{46}\sigma_{56}}$$

$$\omega_{13}^2 = \frac{\sigma_{17}^2\sigma_{23}\sigma_{89}}{\sigma_{12}\sigma_{13}\sigma_{78}\sigma_{79}} - \frac{\sigma_{18}^2\sigma_{23}\sigma_{79}}{\sigma_{12}\sigma_{13}\sigma_{78}\sigma_{89}} - \frac{\sigma_{19}^2\sigma_{23}\sigma_{78}}{\sigma_{12}\sigma_{13}\sigma_{79}\sigma_{89}}$$

$$- \frac{\sigma_{13}\sigma_{27}^2\sigma_{89}}{\sigma_{12}\sigma_{23}\sigma_{78}\sigma_{79}} - \frac{\sigma_{13}\sigma_{28}^2\sigma_{79}}{\sigma_{12}\sigma_{23}\sigma_{78}\sigma_{89}} - \frac{\sigma_{13}\sigma_{29}^2\sigma_{78}}{\sigma_{12}\sigma_{23}\sigma_{79}\sigma_{89}}$$

$$- \frac{\sigma_{12}\sigma_{37}^2\sigma_{89}}{\sigma_{13}\sigma_{23}\sigma_{78}\sigma_{79}} - \frac{\sigma_{12}\sigma_{38}^2\sigma_{79}}{\sigma_{13}\sigma_{23}\sigma_{78}\sigma_{89}} - \frac{\sigma_{12}\sigma_{39}^2\sigma_{78}}{\sigma_{13}\sigma_{23}\sigma_{79}\sigma_{89}}$$

$$\omega_{23}^2 = \frac{\sigma_{47}^2\sigma_{56}\sigma_{89}}{\sigma_{45}\sigma_{46}\sigma_{78}\sigma_{79}} - \frac{\sigma_{48}^2\sigma_{56}\sigma_{79}}{\sigma_{45}\sigma_{46}\sigma_{78}\sigma_{89}} - \frac{\sigma_{49}^2\sigma_{56}\sigma_{78}}{\sigma_{45}\sigma_{46}\sigma_{79}\sigma_{89}}$$

$$= \frac{\sigma_{46}\sigma_{57}{}^2\sigma_{89}}{\sigma_{45}\sigma_{56}\sigma_{78}\sigma_{79}} = \frac{\sigma_{46}\sigma_{58}{}^2\sigma_{79}}{\sigma_{45}\sigma_{56}\sigma_{78}\sigma_{89}} = \frac{\sigma_{46}\sigma_{59}{}^2\sigma_{78}}{\sigma_{45}\sigma_{56}\sigma_{79}\sigma_{89}}$$

$$= \frac{\sigma_{45}\sigma_{67}{}^2\sigma_{89}}{\sigma_{46}\sigma_{56}\sigma_{78}\sigma_{79}} = \frac{\sigma_{45}\sigma_{68}{}^2\sigma_{79}}{\sigma_{46}\sigma_{56}\sigma_{78}\sigma_{89}} = \frac{\sigma_{45}\sigma_{69}{}^2\sigma_{78}}{\sigma_{46}\sigma_{56}\sigma_{79}\sigma_{89}}$$

Doubly-Constrained Bi-Factor Model

This model has 12 parameters (λ_{14} - λ_{94}, λ_{11}, λ_{42}, λ_{73}) that have over-identifying expressions. The expressions for the general-factor loadings (λ_{14} - λ_{94}) are the same as in the bi-factor model with uncorrelated group-factors (i.e. singly-constrained bi-factor model). As they were described in an earlier section they will not be presented again. The over-identifying expressions for the group-factor loadings follow.

$$\lambda_{11}{}^2 = \sigma_{12} - \lambda_{14}\lambda_{24} = \sigma_{13} - \lambda_{14}\lambda_{34} = \sigma_{23} - \lambda_{24}\lambda_{34}$$

$$\lambda_{42}{}^2 = \sigma_{45} - \lambda_{44}\lambda_{54} = \sigma_{46} - \lambda_{44}\lambda_{64} = \sigma_{56} - \lambda_{54}\lambda_{64}$$

$$\lambda_{73}{}^2 = \sigma_{78} - \lambda_{74}\lambda_{84} = \sigma_{79} - \lambda_{74}\lambda_{94} = \sigma_{89} - \lambda_{84}\lambda_{94}$$

Efforts to simplify these expressions either by substitution for one of the general-factor loadings within one of the three group-factor solutions, or by solving across two of these expressions does not produce any additional equalities from those found in the singly-constrained bi-factor model, in spite of these three additional over-identified parameters.

The resulting equalities for the group-factor and second-order models are found in Appendix C. The second-order model has 132 of the 162 restrictions present in the group-factor model.

The 30 restrictions not listed for the second-order model are derived algebraically by either substitution or division. For example, the first equality needed ($\sigma_{12}\ \sigma_{35} = \sigma_{13}\ \sigma_{25}$) is obtained by substitution involving Equation numbers C.004 and C.014. This is indicated in the appendix as S:004,014, where S indicates a restriction obtained by substitution involving Appendix (C) equations. Instances where division of two equalities was used is indicated by instances where two Appendix C equations are listed and separated by a diagonal line.

As all 30 restrictions were obtained, the second-order model and group-factor model are equivalent for nine observed measures.

QUASI-HIERARCHY

The factor analytic models considered here can be viewed hierarchically, that is, with certain models being more general than others. The one-factor model is generally the most restricted while the bi-factor model is least restricted. Located between these extremes would be the second-order and group-factor models. Going from the least restricted to the most restricted model involves increasing parameter restrictions. Models can be thought of as nested when restrictions placed on a (less restricted) model produce a model which is more restricted. In this sense the models can be thought of as nested. However, in certain instances models may be equivalent instead of more- or less-restricted. This is most likely to occur in instances when there are few observed measures per factor, or few first-order factors per second-order factor.

For instance, beginning with the bi-factor model and constraining each of the general-factor loadings to a value of zero results in the group-factor model. Additionally, the second-order factor structure of the second-order model represents restrictions on the correlations between the group-factors of the group-factor model. The one-factor model can be thought of as a special case of the second-order model where the unique variances of the first-order factors are constrained to equal zero.

Additional relationships between models occur as some of these models are constrained to be more-or less-restricted. These relationships will be considered for both six and nine observed measures.

6 Observed Measures

The one-factor model, comprised of 12 parameters, is the most restricted of the six observed measure models. The one-factor model with correlated errors, while obviously less restricted than the one-factor model is equivalent to the second-order, group-factor, and doubly-constrained bi-factor models. (Appendix A shows the models to be equivalent.) These four equivalent models, in particular the second-order and group-factor, demonstrate that in certain instances, models may be equivalent instead of more-or less-restricted. (Refer to Figure 9 for a pictorial representation.)

9 Observed Measures

There are six identified models under consideration with nine observed measures. The one-factor model, with 18 parameters, is the most restricted. The one-factor model with correlated errors and the bi-factor model with uncorrelated group-factors, each having 27 parameters, are both equivalent (refer to Appendix B), and the least restricted models.

The doubly-constrained bi-factor model is more restricted than the two prior models by virtue of having six-fewer parameters, resulting from the equality constraints placed on the group-factor loadings.

As was the case with six observed measures, the second-order model and group-factor model are equivalent (see Appendix C) instead of the second-order model being more restricted than the group-factor model.

The restrictions present in Appendix B (one-factor model with correlated errors and bi-factor model with uncorrelated group factors) are a subset of those appearing in Appendix C (second-order and group-factor models). Figure 10 presents an ordered listing of the quasi-hierarchy existing for these models with nine observed measures.

With both six and nine observed measures, the second-order and group-factor models are equivalent instead of the second-order model being more restricted than the group-factor model. Referring to the identification of 12 measure models it can be seen that the two models would not be equivalent. In this case the second-order model has three observed measures for each of four first-order factors, and a total of 28 parameters, while the group-factor model, consisting of four group-factors, each measured by three observed measures, has 30 parameters. In this instance the number of observed measures per factor, and the number of first-order factors per second-order factor are large enough for the second-order model to be more restricted than the group-factor model. However, had these 12 measure models been comprised of three first-order factors (still with one second-order factor for the second-order model), and three group-factors each measured by four observed measures, again, there would be the same number of parameters (27) in the models.

CHAPTER 4

DISCUSSION

Following a discussion and review of factor analysis, the current work considered issues of model identification, equivalence, and hierarchy for four commonly used factor analysis models (with differing numbers of observed measures).

The four models investigated were the one-factor, second-order, group-factor, and bi-factor. Each of these models was examined to determine if it was identified in three scenarios: six, nine, and twelve observed measures.

A model is defined as identified when all of its parameters can be represented in terms of the elements from the observed measures' (population) variance/covariance matrix. The process of identification proceeded with the following steps: (1) diagramming the basic model, (2) writing equations for the observed measures in terms of the parameters of the model, (3) multiplying each equation by itself, and each other equation, one at a time, (4) taking the expected values of these products, and (5) representing each parameter in terms of variances and covariances.

Following these steps, the one-factor, second-order, and group-factor models were shown to be identified in all three observed measure scenarios. The bi-factor model was not identified in any of these scenarios.

Parameter restrictions were employed in an attempt to identify the bi-factor model. With six observed measures two sets of constraints (uncorrelated group-factors and equal factor loadings for each of the three group-factors) were necessary to identify the model. With both nine and twelve observed measures it was only necessary to constrain the group factor correlations to equal zero to identify the bi-factor model. The additional restriction of equal factor loadings for each group-factor was also shown for completeness.

Parameter relaxations were also considered for the one-factor model. The residuals for each triad of observed measures were allowed to correlate. The resulting one-factor model with correlated errors was shown to be identified.

While conditions were discussed under which a model would not be identified, proofs were unable to be generated, except for instances when there exist more parameters to be solved for than observed variance/covariance expressions to solve with. Future work in this area is necessary to develop procedures for demonstrating that a model is unidentified. Empirically it is possible to determine if a model is identified in a relatively simplified manner. One such approach involves fitting the model in question to a data set that contains the same number of variables as there are observed measures in the model. Obtaining estimates for the model parameters and standard errors would reveal the model to be identified. In much the same way it would be desirable to derive streamlined theoretical procedures for establishing model identification so that unidentified models could be made explicit prior to their being included in the design or analysis of a study, or from being part of a theory generation or testing process.

Next, model equivalence was examined for both the six and nine observed measure scenarios. Proving that two models are equivalent was a more involved process than anticipated. It was initially expected that a one-to-one transformation between the respective estimated parameters of two models would be an appropriate means of demonstrating equivalence. This was not the case; while there were instances where these relationships could be observed, they could not be obtained for all the necessary relationships.

Equivalence was also considered in the context of relationships between the observed variances and covariances wherein a (small) number of identical, non-redundant restrictions exist that would be equal to the models' degrees of freedom. The search for these non-redundant expressions proved elusive. It was the combination of all these non-redundant relationships between models that were necessary to prove equivalence.

Model equivalence was demonstrated by establishing that models contained identical restrictions on relationships among their variance and covariance elements. This was done for each model under consideration by examining the identifying representations for each parameter to determine instances of multiple solutions. Instances where this occurred were presented. Algebraic simplification of these over-identifying relationships produced a set of restrictions for each model. Models were examined to determine whether they were comprised of the same restrictions. These restrictions occur either directly, as seen by listing all the restrictions between models being examined for equivalence, or indirectly, involving some algebraic manipulation of the observed restrictions. Models containing identical restrictions are equivalent.

With six observed measures the following models are all equivalent: one-factor with correlated errors, second-order, group-factor, and doubly-constrained bi-factor. When there are nine observed measures, there are two pairs of equivalent models: the second-order and group-factor, and the one-factor with correlated errors and bi-factor with uncorrelated group-factors. The number of over-identifying restrictions produced for the four equivalent six-measure models is 15; This increases to 81 for the nine-measure doubly-constrained bi-factor and the correlated error one-factor models. The value doubles (to 162) for the second-order and group-factor models. Aside from being comprised of a somewhat unwieldy volume of relationships, the opportunity for error is a factor to be considered for models with more observed measures.

A general solution or simplified procedure for establishing model equivalence would be of assistance in extending the investigation of equivalent models. These inquiries could consider situations with additional observed measures. For instance, with 12 observed-measures, the second-order model with one second-order factor and three first-order factors, and the group-factor model with three-group factors have the same number of parameters (27). It is not known whether they are equivalent, as with six- and nine-observed measures.

Investigations could also be conducted within models. Preliminary work with the structural aspect of a path model (Stelzl, 1986; Lee and Hershberger, 1990), or nested forms of a baseline structural equations model have been considered (Luijben, 1991).

Future work could also be extended to consider models other than those considered here. For instance, in the case of four observed measures, a two-factor model is equivalent to a simplex model.

Next the ordered-structure, or hierarchy, existing between these models was examined, wherein restrictions on one model generate the next model on the hierarchy. The bi-factor model is the least restricted of these four models, while one-factor model is the most restricted.

Restricting the general-factor loadings to equal zero show the group-factor model to be a special case of the bi-factor model. Also, the second-order factor structure in the second-order model can be thought of as placing a structure on the factor correlations of the group factor model, thereby revealing the second-order model to be a special case of the group-factor model. In a similar manner the one-factor model can be thought of as a special case of the second-order model where the unique variances of the first-order factors are constrained to equal zero.

For six and nine observed measures certain models are equivalent instead of more restricted. In both scenarios the second-order model is equivalent to the group-factor model. With twelve observed measures, models comprised of four (first-order and group-) factors are not equivalent, but, rather the second-order model is more restricted than the group-factor model.

Parameter restrictions and relaxations can be used to: (1) make previously unidentified models identified, (2) produce equivalent models, and (3) create a partially-ordered chain between models.

FIGURE 1

ONE-FACTOR MODEL; TWO OBSERVED MEASURES

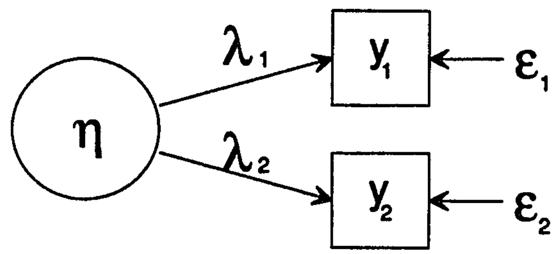


FIGURE 2

TWO-FACTOR MODEL; FOUR OBSERVED MEASURES

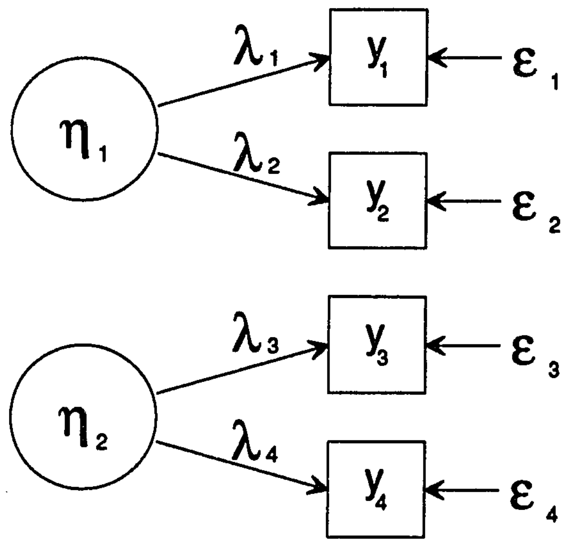


FIGURE 3

ONE-FACTOR MODEL; SIX OBSERVED MEASURES

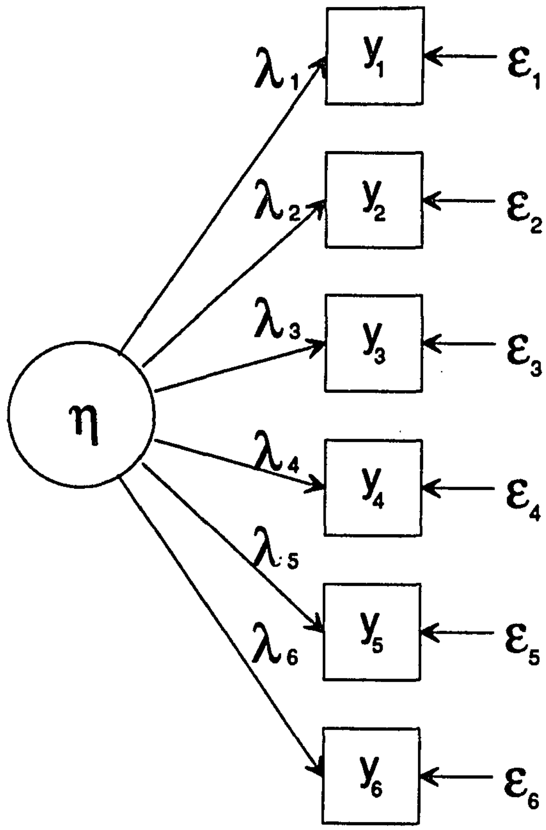


FIGURE 4

SECOND-ORDER FACTOR MODEL; THREE FIRST-ORDER FACTORS; SIX OBSERVED MEASURES

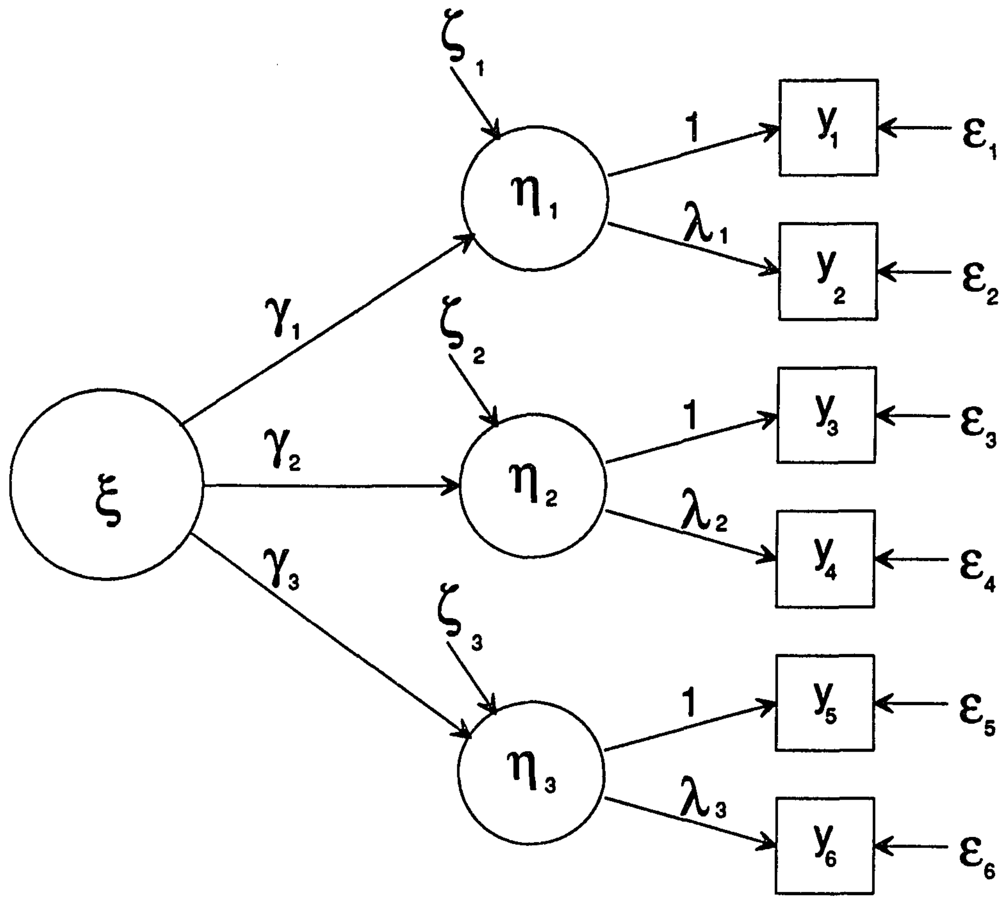


FIGURE 5

GROUP-FACTOR MODEL; SIX OBSERVED MEASURES

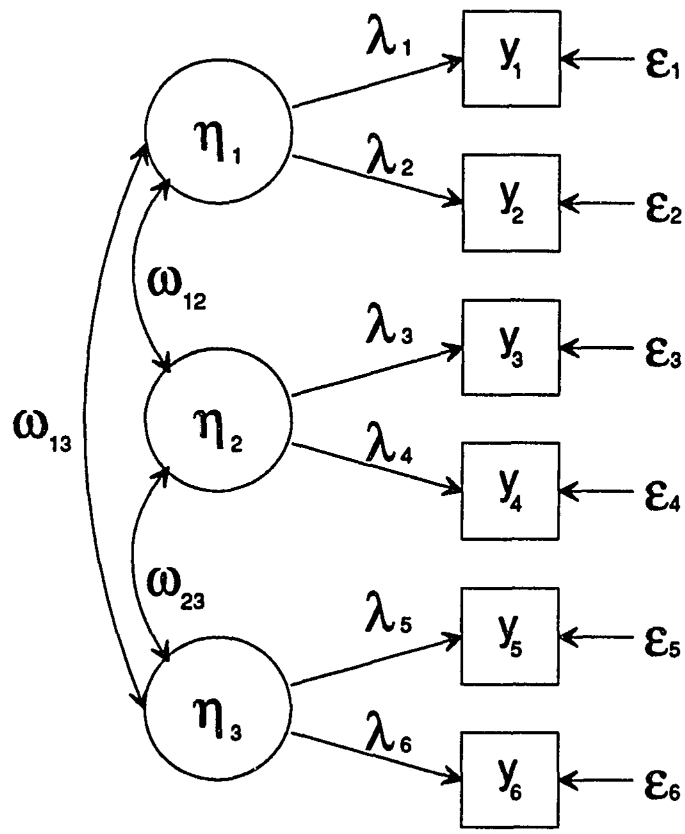


FIGURE 6

BI-FACTOR MODEL; SIX OBSERVED MEASURES

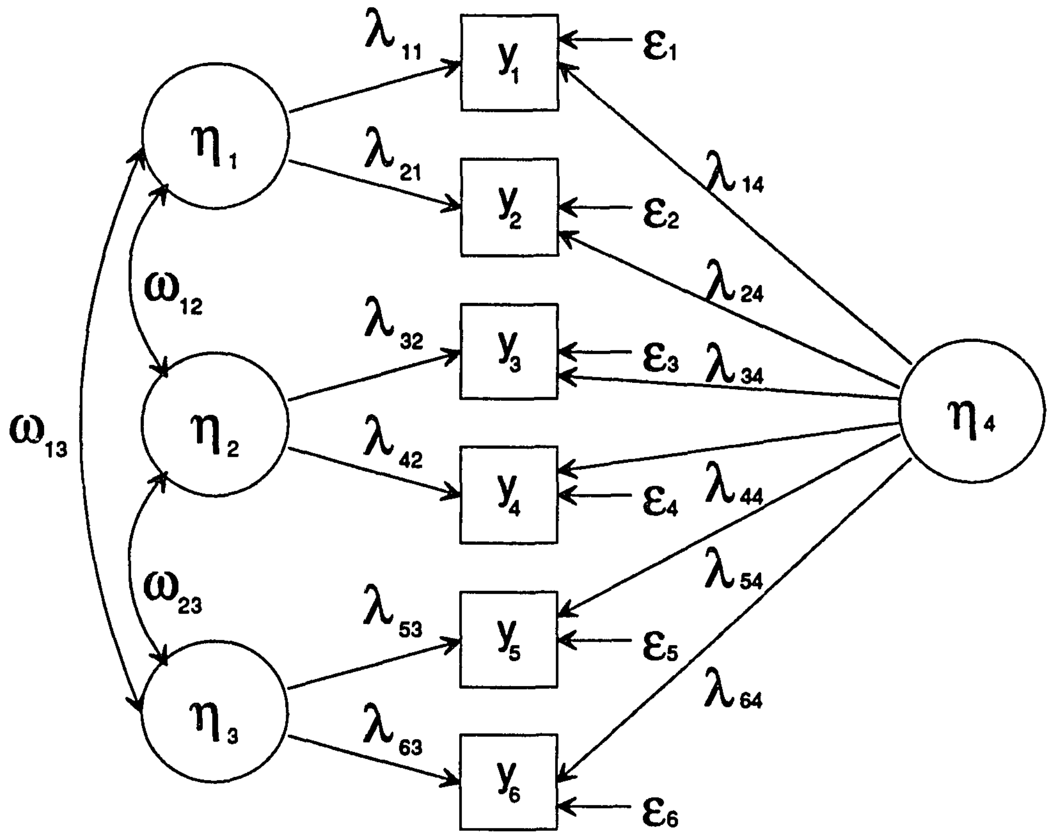


FIGURE 7

FACTOR-LOADING MATRIX FOR VERNON (1950)

	General Factor	Major Group Factors		Minor Group Factors	
Test	g	k:m	v:ed	v	n
1	X	X			
2	X				
3	X	X			
4	X	X			
5	X	X			
6	X	X			
7	X		X	X	
8	X		X	X	
9	X		X	X	
10	X		X	X	
11	X		X		X
12	X		X		X
13	X		X		X

X = Factor Loading

FIGURE 8

COMPARISON OF FACTOR MODELS FOR SPEARMAN, VERNON, AND THURSTONE

	General Factor	Group-Factors			Multiple-Factors				Specific Factors
Test	g	A	B	C	A	B	C	D	
1	S,V	V			T				S,V,T
2	S,V	V			T			T	S,V,T
3	S,V	V			T	T			S,V,T
4	S,V	V			T				S,V,T
5	S,V		V	V		T	T	T	S,V,T
6	S,V		V		T	T			S,V,T
7	S,V		V			T			S,V,T
8	S,V			V			T	T	S,V,T
9	S,V			V			T		S,V,T
10	S,V			V			T		S,V,T

S - Spearman

V - Vernon

T - Thurstone

FIGURE 9

QUASI-HIERARCHY: SIX OBSERVED MEASURES

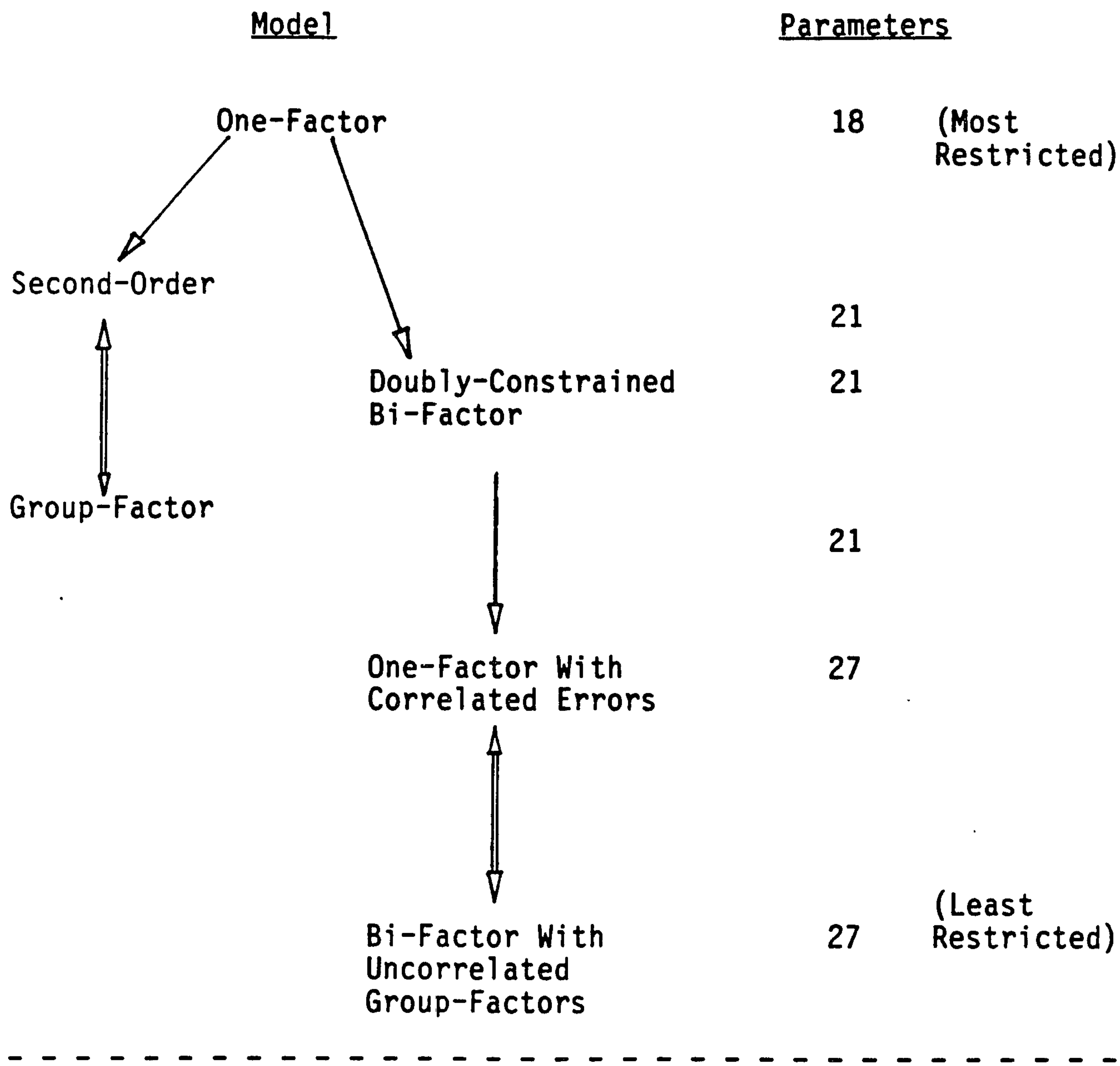
Model	Parameters	
One-Factor	12	(Most Restricted)
↓		
Second-Order	15	
↕		
Group-Factor	15	
↕		
One-Factor With Correlated Errors	15	
↕		
Doubly-Constrained Bi-Factor	15	(Least Restricted)

- -

<══> Equivalent Models

FIGURE 10

QUASI-HIERARCHY: NINE OBSERVED MEASURES



<====> Equivalent Models

Appendix A

Non-Repeated Equality Expressions: 6 Measures

Expression	Equation Number	Model: One-Factor w/Correlated Errors	Second-Order	Group-Factor	Doubly-Constrained Bi-Factor
$\sigma_{13}\ \sigma_{24} = \sigma_{14}\ \sigma_{23}$	586	562/566	X	X	562/566
$\sigma_{13}\ \sigma_{25} = \sigma_{15}\ \sigma_{23}$	570	X	X	X	X
$\sigma_{13}\ \sigma_{26} = \sigma_{16}\ \sigma_{23}$	571	X	X	X	X
$\sigma_{13}\ \sigma_{45} = \sigma_{14}\ \sigma_{35}$	562	X	X	X	X
$\sigma_{13}\ \sigma_{46} = \sigma_{14}\ \sigma_{36}$	563	X	X	X	X
$\sigma_{14}\ \sigma_{25} = \sigma_{15}\ \sigma_{24}$	574	X	X	X	X
$\sigma_{14}\ \sigma_{26} = \sigma_{16}\ \sigma_{24}$	575	X	X	X	X
$\sigma_{15}\ \sigma_{26} = \sigma_{16}\ \sigma_{25}$	587	574/575	X	X	574/575
$\sigma_{15}\ \sigma_{36} = \sigma_{16}\ \sigma_{35}$	564	X	X	X	X
$\sigma_{15}\ \sigma_{46} = \sigma_{16}\ \sigma_{45}$	565	X	X	X	X
$\sigma_{23}\ \sigma_{45} = \sigma_{24}\ \sigma_{35}$	566	X	X	X	X
$\sigma_{23}\ \sigma_{46} = \sigma_{24}\ \sigma_{36}$	567	X	X	X	X
$\sigma_{25}\ \sigma_{36} = \sigma_{26}\ \sigma_{35}$	568	X	X	X	X
$\sigma_{25}\ \sigma_{46} = \sigma_{26}\ \sigma_{45}$	569	X	X	X	X
$\sigma_{35}\ \sigma_{46} = \sigma_{36}\ \sigma_{45}$	588	562/563	X	X	562/563

Appendix B

Non-Repeated Equality Expressions: 9 Measures

Expression	Equation Number	Model: One-Factor w/Correlated Errors	Model: Bi-Factor w/Correlated Group Factors
$\sigma_{14}\ \sigma_{27} = \sigma_{17}\ \sigma_{24}$	B.01	X	X
$\sigma_{14}\ \sigma_{28} = \sigma_{18}\ \sigma_{24}$	B.02	X	X
$\sigma_{14}\ \sigma_{29} = \sigma_{19}\ \sigma_{24}$	B.03	X	X
$\sigma_{14}\ \sigma_{37} = \sigma_{17}\ \sigma_{34}$	B.04	X	X
$\sigma_{14}\ \sigma_{38} = \sigma_{18}\ \sigma_{34}$	B.05	X	X
$\sigma_{14}\ \sigma_{39} = \sigma_{19}\ \sigma_{34}$	B.06	X	X
$\sigma_{14}\ \sigma_{57} = \sigma_{15}\ \sigma_{47}$	B.07	X	X
$\sigma_{14}\ \sigma_{58} = \sigma_{15}\ \sigma_{48}$	B.08	X	X
$\sigma_{14}\ \sigma_{59} = \sigma_{15}\ \sigma_{49}$	B.09	X	X
$\sigma_{14}\ \sigma_{67} = \sigma_{16}\ \sigma_{47}$	B.10	X	X
$\sigma_{14}\ \sigma_{68} = \sigma_{16}\ \sigma_{48}$	B.11	X	X
$\sigma_{14}\ \sigma_{69} = \sigma_{16}\ \sigma_{49}$	B.12	X	X
$\sigma_{15}\ \sigma_{27} = \sigma_{17}\ \sigma_{25}$	B.13	X	X
$\sigma_{15}\ \sigma_{28} = \sigma_{18}\ \sigma_{25}$	B.14	X	X
$\sigma_{15}\ \sigma_{29} = \sigma_{19}\ \sigma_{25}$	B.15	X	X
$\sigma_{15}\ \sigma_{37} = \sigma_{17}\ \sigma_{35}$	B.16	X	X
$\sigma_{15}\ \sigma_{38} = \sigma_{18}\ \sigma_{35}$	B.17	X	X
$\sigma_{15}\ \sigma_{39} = \sigma_{19}\ \sigma_{35}$	B.18	X	X
$\sigma_{15}\ \sigma_{67} = \sigma_{16}\ \sigma_{57}$	B.19	X	X

Expression	Equation Number	Model: One-Factor w/Correlated Errors	Model: Bi-Factor w/Correlated Group Factors
$\sigma_{15}\ \sigma_{68} - \sigma_{16}\ \sigma_{58}$	B.20	X	X
$\sigma_{15}\ \sigma_{69} - \sigma_{16}\ \sigma_{59}$	B.21	X	X
$\sigma_{16}\ \sigma_{27} - \sigma_{17}\ \sigma_{26}$	B.22	X	X
$\sigma_{16}\ \sigma_{28} - \sigma_{18}\ \sigma_{26}$	B.23	X	X
$\sigma_{16}\ \sigma_{29} - \sigma_{19}\ \sigma_{26}$	B.24	X	X
$\sigma_{16}\ \sigma_{37} - \sigma_{17}\ \sigma_{36}$	B.25	X	X
$\sigma_{16}\ \sigma_{38} - \sigma_{18}\ \sigma_{36}$	B.26	X	X
$\sigma_{16}\ \sigma_{39} - \sigma_{19}\ \sigma_{36}$	B.27	X	X
$\sigma_{17}\ \sigma_{48} - \sigma_{18}\ \sigma_{47}$	B.28	X	X
$\sigma_{17}\ \sigma_{49} - \sigma_{19}\ \sigma_{47}$	B.29	X	X
$\sigma_{17}\ \sigma_{58} - \sigma_{18}\ \sigma_{57}$	B.30	X	X
$\sigma_{17}\ \sigma_{59} - \sigma_{19}\ \sigma_{57}$	B.31	X	X
$\sigma_{17}\ \sigma_{68} - \sigma_{18}\ \sigma_{67}$	B.32	X	X
$\sigma_{17}\ \sigma_{69} - \sigma_{19}\ \sigma_{67}$	B.33	X	X
$\sigma_{18}\ \sigma_{49} - \sigma_{19}\ \sigma_{48}$	B.34	X	X
$\sigma_{18}\ \sigma_{59} - \sigma_{19}\ \sigma_{58}$	B.35	X	X
$\sigma_{18}\ \sigma_{69} - \sigma_{19}\ \sigma_{68}$	B.36	X	X
$\sigma_{24}\ \sigma_{37} - \sigma_{27}\ \sigma_{34}$	B.37	X	X
$\sigma_{24}\ \sigma_{38} - \sigma_{28}\ \sigma_{34}$	B.38	X	X
$\sigma_{24}\ \sigma_{39} - \sigma_{29}\ \sigma_{34}$	B.39	X	X
$\sigma_{24}\ \sigma_{57} - \sigma_{25}\ \sigma_{47}$	B.40	X	X

Expression	Equation Number	Model: One-Factor w/Correlated Errors	Model: Bi-Factor w/Correlated Group Factors
$\sigma_{24}\ \sigma_{58} - \sigma_{25}\ \sigma_{48}$	B.41	X	X
$\sigma_{24}\ \sigma_{59} - \sigma_{25}\ \sigma_{49}$	B.42	X	X
$\sigma_{24}\ \sigma_{67} - \sigma_{26}\ \sigma_{47}$	B.43	X	X
$\sigma_{24}\ \sigma_{68} - \sigma_{26}\ \sigma_{48}$	B.44	X	X
$\sigma_{24}\ \sigma_{69} - \sigma_{26}\ \sigma_{49}$	B.45	X	X
$\sigma_{25}\ \sigma_{37} - \sigma_{27}\ \sigma_{35}$	B.46	X	X
$\sigma_{25}\ \sigma_{38} - \sigma_{28}\ \sigma_{35}$	B.47	X	X
$\sigma_{25}\ \sigma_{39} - \sigma_{29}\ \sigma_{35}$	B.48	X	X
$\sigma_{25}\ \sigma_{67} - \sigma_{26}\ \sigma_{57}$	B.49	X	X
$\sigma_{25}\ \sigma_{68} - \sigma_{26}\ \sigma_{58}$	B.50	X	X
$\sigma_{25}\ \sigma_{69} - \sigma_{26}\ \sigma_{59}$	B.51	X	X
$\sigma_{26}\ \sigma_{37} - \sigma_{27}\ \sigma_{36}$	B.52	X	X
$\sigma_{26}\ \sigma_{38} - \sigma_{28}\ \sigma_{36}$	B.53	X	X
$\sigma_{26}\ \sigma_{39} - \sigma_{29}\ \sigma_{36}$	B.54	X	X
$\sigma_{27}\ \sigma_{48} - \sigma_{28}\ \sigma_{47}$	B.55	X	X
$\sigma_{27}\ \sigma_{49} - \sigma_{29}\ \sigma_{47}$	B.56	X	X
$\sigma_{27}\ \sigma_{58} - \sigma_{28}\ \sigma_{57}$	B.57	X	X
$\sigma_{27}\ \sigma_{59} - \sigma_{29}\ \sigma_{57}$	B.58	X	X
$\sigma_{27}\ \sigma_{68} - \sigma_{28}\ \sigma_{67}$	B.59	X	X
$\sigma_{27}\ \sigma_{69} - \sigma_{29}\ \sigma_{67}$	B.60	X	X
$\sigma_{28}\ \sigma_{49} - \sigma_{29}\ \sigma_{48}$	B.61	X	X

Expression	Equation Number	Model: One-Factor w/Correlated Errors	Model: Bi-Factor w/Correlated Group Factors
$\sigma_{28}\ \sigma_{59} - \sigma_{29}\ \sigma_{58}$	B.62	X	X
$\sigma_{28}\ \sigma_{69} - \sigma_{29}\ \sigma_{68}$	B.63	X	X
$\sigma_{34}\ \sigma_{57} - \sigma_{35}\ \sigma_{47}$	B.64	X	X
$\sigma_{34}\ \sigma_{58} - \sigma_{35}\ \sigma_{48}$	B.65	X	X
$\sigma_{34}\ \sigma_{59} - \sigma_{35}\ \sigma_{49}$	B.66	X	X
$\sigma_{34}\ \sigma_{67} - \sigma_{36}\ \sigma_{47}$	B.67	X	X
$\sigma_{34}\ \sigma_{68} - \sigma_{36}\ \sigma_{48}$	B.68	X	X
$\sigma_{34}\ \sigma_{69} - \sigma_{36}\ \sigma_{49}$	B.69	X	X
$\sigma_{35}\ \sigma_{67} - \sigma_{36}\ \sigma_{57}$	B.70	X	X
$\sigma_{35}\ \sigma_{68} - \sigma_{36}\ \sigma_{58}$	B.71	X	X
$\sigma_{35}\ \sigma_{69} - \sigma_{36}\ \sigma_{59}$	B.72	X	X
$\sigma_{37}\ \sigma_{48} - \sigma_{38}\ \sigma_{47}$	B.73	X	X
$\sigma_{37}\ \sigma_{49} - \sigma_{39}\ \sigma_{47}$	B.74	X	X
$\sigma_{37}\ \sigma_{58} - \sigma_{38}\ \sigma_{57}$	B.75	X	X
$\sigma_{37}\ \sigma_{59} - \sigma_{39}\ \sigma_{57}$	B.76	X	X
$\sigma_{37}\ \sigma_{68} - \sigma_{38}\ \sigma_{67}$	B.77	X	X
$\sigma_{37}\ \sigma_{69} - \sigma_{39}\ \sigma_{67}$	B.78	X	X
$\sigma_{38}\ \sigma_{49} - \sigma_{39}\ \sigma_{48}$	B.79	X	X
$\sigma_{38}\ \sigma_{59} - \sigma_{39}\ \sigma_{58}$	B.80	X	X
$\sigma_{38}\ \sigma_{69} - \sigma_{39}\ \sigma_{68}$	B.81	X	X

Appendix C

Non-Repeated Equality Expressions: 9 Measures

Expression	Model: Equation Number	Model: Second-Order	Model: Group Factor
$\sigma_{12}\ \sigma_{34} - \sigma_{13}\ \sigma_{24}$	C.001	X	X
$\sigma_{12}\ \sigma_{34} - \sigma_{14}\ \sigma_{23}$	C.002	X	X
$\sigma_{12}\ \sigma_{35} - \sigma_{13}\ \sigma_{25}$	C.003	S:004,014	X
$\sigma_{12}\ \sigma_{35} - \sigma_{15}\ \sigma_{23}$	C.004	X	X
$\sigma_{12}\ \sigma_{36} - \sigma_{13}\ \sigma_{26}$	C.005	S:006,015	X
$\sigma_{12}\ \sigma_{36} - \sigma_{16}\ \sigma_{23}$	C.006	X	X
$\sigma_{12}\ \sigma_{37} - \sigma_{13}\ \sigma_{27}$	C.007	S:008,016	X
$\sigma_{12}\ \sigma_{37} - \sigma_{17}\ \sigma_{23}$	C.008	X	X
$\sigma_{12}\ \sigma_{38} - \sigma_{13}\ \sigma_{28}$	C.009	S:010,017	X
$\sigma_{12}\ \sigma_{38} - \sigma_{18}\ \sigma_{23}$	C.010	X	X
$\sigma_{12}\ \sigma_{39} - \sigma_{13}\ \sigma_{29}$	C.011	S:012,018	X
$\sigma_{12}\ \sigma_{39} - \sigma_{19}\ \sigma_{23}$	C.012	X	X
$\sigma_{13}\ \sigma_{24} - \sigma_{14}\ \sigma_{23}$	C.013	X	X
$\sigma_{13}\ \sigma_{25} - \sigma_{15}\ \sigma_{23}$	C.014	X	X
$\sigma_{13}\ \sigma_{26} - \sigma_{16}\ \sigma_{23}$	C.015	X	X
$\sigma_{13}\ \sigma_{27} - \sigma_{17}\ \sigma_{23}$	C.016	X	X
$\sigma_{13}\ \sigma_{28} - \sigma_{18}\ \sigma_{23}$	C.017	X	X
$\sigma_{13}\ \sigma_{29} - \sigma_{19}\ \sigma_{23}$	C.018	X	X
$\sigma_{14}\ \sigma_{25} - \sigma_{15}\ \sigma_{24}$	C.019	X	X

Expression	Model: Equation Number	Second-Order	Group Factor
$\sigma_{14}\,\sigma_{26} - \sigma_{16}\,\sigma_{24}$	C.020	X	X
$\sigma_{14}\,\sigma_{27} - \sigma_{17}\,\sigma_{24}$	C.021	X	X
$\sigma_{14}\,\sigma_{28} - \sigma_{18}\,\sigma_{24}$	C.022	X	X
$\sigma_{14}\,\sigma_{29} - \sigma_{19}\,\sigma_{24}$	C.023	X	X
$\sigma_{14}\,\sigma_{35} - \sigma_{15}\,\sigma_{34}$	C.024	X	X
$\sigma_{14}\,\sigma_{36} - \sigma_{16}\,\sigma_{34}$	C.025	X	X
$\sigma_{14}\,\sigma_{37} - \sigma_{17}\,\sigma_{34}$	C.026	X	X
$\sigma_{14}\,\sigma_{38} - \sigma_{18}\,\sigma_{34}$	C.027	X	X
$\sigma_{14}\,\sigma_{39} - \sigma_{19}\,\sigma_{34}$	C.028	X	X
$\sigma_{14}\,\sigma_{56} - \sigma_{15}\,\sigma_{46}$	C.029	X	X
$\sigma_{14}\,\sigma_{56} - \sigma_{16}\,\sigma_{45}$	C.030	X	X
$\sigma_{14}\,\sigma_{57} - \sigma_{15}\,\sigma_{47}$	C.031	X	X
$\sigma_{14}\,\sigma_{58} - \sigma_{15}\,\sigma_{48}$	C.032	X	X
$\sigma_{14}\,\sigma_{59} - \sigma_{15}\,\sigma_{49}$	C.033	X	X
$\sigma_{14}\,\sigma_{67} - \sigma_{16}\,\sigma_{47}$	C.034	X	X
$\sigma_{14}\,\sigma_{68} - \sigma_{16}\,\sigma_{48}$	C.035	X	X
$\sigma_{14}\,\sigma_{69} - \sigma_{16}\,\sigma_{49}$	C.036	X	X
$\sigma_{15}\,\sigma_{26} - \sigma_{16}\,\sigma_{25}$	C.037	X	X
$\sigma_{15}\,\sigma_{27} - \sigma_{17}\,\sigma_{25}$	C.038	X	X
$\sigma_{15}\,\sigma_{28} - \sigma_{18}\,\sigma_{25}$	C.039	X	X
$\sigma_{15}\,\sigma_{29} - \sigma_{19}\,\sigma_{25}$	C.040	X	X

Expression	Equation Number	Model: Second-Order	Model: Group Factor
$\sigma_{15}\ \sigma_{36} - \sigma_{16}\ \sigma_{35}$	C.041	X	X
$\sigma_{15}\ \sigma_{37} - \sigma_{17}\ \sigma_{35}$	C.042	X	X
$\sigma_{15}\ \sigma_{38} - \sigma_{18}\ \sigma_{35}$	C.043	X	X
$\sigma_{15}\ \sigma_{39} - \sigma_{19}\ \sigma_{35}$	C.044	X	X
$\sigma_{15}\ \sigma_{46} - \sigma_{16}\ \sigma_{45}$	C.045	X	X
$\sigma_{15}\ \sigma_{67} - \sigma_{16}\ \sigma_{57}$	C.046	X	X
$\sigma_{15}\ \sigma_{68} - \sigma_{16}\ \sigma_{58}$	C.047	X	X
$\sigma_{15}\ \sigma_{69} - \sigma_{16}\ \sigma_{59}$	C.048	X	X
$\sigma_{16}\ \sigma_{27} - \sigma_{17}\ \sigma_{26}$	C.049	X	X
$\sigma_{16}\ \sigma_{28} - \sigma_{18}\ \sigma_{26}$	C.050	X	X
$\sigma_{16}\ \sigma_{29} - \sigma_{19}\ \sigma_{26}$	C.051	X	X
$\sigma_{16}\ \sigma_{37} - \sigma_{17}\ \sigma_{36}$	C.052	X	X
$\sigma_{16}\ \sigma_{38} - \sigma_{18}\ \sigma_{36}$	C.053	X	X
$\sigma_{16}\ \sigma_{39} - \sigma_{19}\ \sigma_{36}$	C.054	X	X
$\sigma_{17}\ \sigma_{28} - \sigma_{18}\ \sigma_{27}$	C.055	X	X
$\sigma_{17}\ \sigma_{29} - \sigma_{19}\ \sigma_{27}$	C.056	X	X
$\sigma_{17}\ \sigma_{38} - \sigma_{18}\ \sigma_{37}$	C.057	X	X
$\sigma_{17}\ \sigma_{39} - \sigma_{19}\ \sigma_{37}$	C.058	X	X
$\sigma_{17}\ \sigma_{48} - \sigma_{18}\ \sigma_{47}$	C.059	X	X
$\sigma_{17}\ \sigma_{49} - \sigma_{19}\ \sigma_{47}$	C.060	X	X
$\sigma_{17}\ \sigma_{58} - \sigma_{18}\ \sigma_{57}$	C.061	X	X

Expression	Equation Number	Model: Second-Order	Model: Group Factor
$\sigma_{17}\ \sigma_{59} - \sigma_{19}\ \sigma_{57}$	C.062	X	X
$\sigma_{17}\ \sigma_{68} - \sigma_{18}\ \sigma_{67}$	C.063	X	X
$\sigma_{17}\ \sigma_{69} - \sigma_{19}\ \sigma_{67}$	C.064	X	X
$\sigma_{17}\ \sigma_{89} - \sigma_{18}\ \sigma_{79}$	C.065	X	X
$\sigma_{17}\ \sigma_{89} - \sigma_{19}\ \sigma_{78}$	C.066	X	X
$\sigma_{18}\ \sigma_{29} - \sigma_{19}\ \sigma_{28}$	C.067	X	X
$\sigma_{18}\ \sigma_{39} - \sigma_{19}\ \sigma_{38}$	C.068	X	X
$\sigma_{18}\ \sigma_{49} - \sigma_{19}\ \sigma_{48}$	C.069	X	X
$\sigma_{18}\ \sigma_{59} - \sigma_{19}\ \sigma_{58}$	C.070	X	X
$\sigma_{18}\ \sigma_{69} - \sigma_{19}\ \sigma_{68}$	C.071	X	X
$\sigma_{18}\ \sigma_{79} - \sigma_{19}\ \sigma_{78}$	C.072	X	X
$\sigma_{24}\ \sigma_{35} - \sigma_{25}\ \sigma_{34}$	C.073	X	X
$\sigma_{24}\ \sigma_{36} - \sigma_{26}\ \sigma_{34}$	C.074	X	X
$\sigma_{24}\ \sigma_{37} - \sigma_{27}\ \sigma_{34}$	C.075	X	X
$\sigma_{24}\ \sigma_{38} - \sigma_{28}\ \sigma_{34}$	C.076	X	X
$\sigma_{24}\ \sigma_{39} - \sigma_{29}\ \sigma_{34}$	C.077	X	X
$\sigma_{24}\ \sigma_{56} - \sigma_{25}\ \sigma_{46}$	C.078	X	X
$\sigma_{24}\ \sigma_{56} - \sigma_{26}\ \sigma_{45}$	C.079	X	X
$\sigma_{24}\ \sigma_{57} - \sigma_{25}\ \sigma_{47}$	C.080	X	X
$\sigma_{24}\ \sigma_{58} - \sigma_{25}\ \sigma_{48}$	C.081	X	X
$\sigma_{24}\ \sigma_{59} - \sigma_{25}\ \sigma_{49}$	C.082	X	X

Expression	Model: Equation Number	Second-Order	Group Factor
$\sigma_{24}\,\sigma_{67} - \sigma_{26}\,\sigma_{47}$	C.083	X	X
$\sigma_{24}\,\sigma_{68} - \sigma_{26}\,\sigma_{48}$	C.084	X	X
$\sigma_{24}\,\sigma_{69} - \sigma_{26}\,\sigma_{49}$	C.085	X	X
$\sigma_{25}\,\sigma_{36} - \sigma_{26}\,\sigma_{35}$	C.086	C.073/C.074	X
$\sigma_{25}\,\sigma_{37} - \sigma_{27}\,\sigma_{35}$	C.087	X	X
$\sigma_{25}\,\sigma_{38} - \sigma_{28}\,\sigma_{35}$	C.088	C.073/C.076	X
$\sigma_{25}\,\sigma_{39} - \sigma_{29}\,\sigma_{35}$	C.089	C.073/C.077	X
$\sigma_{25}\,\sigma_{46} - \sigma_{26}\,\sigma_{45}$	C.090	S:073,074	X
$\sigma_{25}\,\sigma_{67} - \sigma_{26}\,\sigma_{57}$	C.091	X	X
$\sigma_{25}\,\sigma_{68} - \sigma_{26}\,\sigma_{58}$	C.092	C.081/C.084	X
$\sigma_{25}\,\sigma_{69} - \sigma_{26}\,\sigma_{59}$	C.093	C.082/C.085	X
$\sigma_{26}\,\sigma_{37} - \sigma_{27}\,\sigma_{36}$	C.094	X	X
$\sigma_{26}\,\sigma_{38} - \sigma_{28}\,\sigma_{36}$	C.095	C.074/C.076	X
$\sigma_{26}\,\sigma_{39} - \sigma_{29}\,\sigma_{36}$	C.096	C.074/C.077	X
$\sigma_{27}\,\sigma_{38} - \sigma_{28}\,\sigma_{37}$	C.097	X	X
$\sigma_{27}\,\sigma_{39} - \sigma_{29}\,\sigma_{37}$	C.098	X	X
$\sigma_{27}\,\sigma_{48} - \sigma_{28}\,\sigma_{47}$	C.099	X	X
$\sigma_{27}\,\sigma_{49} - \sigma_{29}\,\sigma_{47}$	C.100	X	X
$\sigma_{27}\,\sigma_{58} - \sigma_{28}\,\sigma_{57}$	C.101	X	X
$\sigma_{27}\,\sigma_{59} - \sigma_{29}\,\sigma_{57}$	C.102	X	X
$\sigma_{27}\,\sigma_{68} - \sigma_{28}\,\sigma_{67}$	C.103	X	X

Expression	Model: Equation Number	Second-Order	Group Factor
$\sigma_{27}\sigma_{69} - \sigma_{29}\sigma_{67}$	C.104	X	X
$\sigma_{27}\sigma_{89} - \sigma_{28}\sigma_{79}$	C.105	X	X
$\sigma_{27}\sigma_{89} - \sigma_{29}\sigma_{78}$	C.106	X	X
$\sigma_{28}\sigma_{39} - \sigma_{29}\sigma_{38}$	C.107	C.076/C.077	X
$\sigma_{28}\sigma_{49} - \sigma_{29}\sigma_{48}$	C.108	X	X
$\sigma_{28}\sigma_{59} - \sigma_{29}\sigma_{58}$	C.109	C.101/C.102	X
$\sigma_{28}\sigma_{69} - \sigma_{29}\sigma_{68}$	C.110	C.103/C.104	X
$\sigma_{28}\sigma_{79} - \sigma_{29}\sigma_{78}$	C.111	S:105,106	X
$\sigma_{34}\sigma_{56} - \sigma_{35}\sigma_{46}$	C.112	X	X
$\sigma_{34}\sigma_{56} - \sigma_{36}\sigma_{45}$	C.113	X	X
$\sigma_{34}\sigma_{57} - \sigma_{35}\sigma_{47}$	C.114	X	X
$\sigma_{34}\sigma_{58} - \sigma_{35}\sigma_{48}$	C.115	X	X
$\sigma_{34}\sigma_{59} - \sigma_{35}\sigma_{49}$	C.116	X	X
$\sigma_{34}\sigma_{67} - \sigma_{36}\sigma_{47}$	C.117	X	X
$\sigma_{34}\sigma_{68} - \sigma_{36}\sigma_{48}$	C.118	X	X
$\sigma_{34}\sigma_{69} - \sigma_{36}\sigma_{49}$	C.119	X	X
$\sigma_{35}\sigma_{46} - \sigma_{36}\sigma_{45}$	C.120	S:112,113	X
$\sigma_{35}\sigma_{67} - \sigma_{36}\sigma_{57}$	C.121	X	X
$\sigma_{35}\sigma_{68} - \sigma_{36}\sigma_{58}$	C.122	C.115/C.118	X
$\sigma_{35}\sigma_{69} - \sigma_{36}\sigma_{59}$	C.123	C.116/C.119	X
$\sigma_{37}\sigma_{48} - \sigma_{38}\sigma_{47}$	C.124	X	X

Expression	Model:		
	Equation Number	Second-Order	Group Factor
$\sigma_{37}\ \sigma_{49} - \sigma_{39}\ \sigma_{47}$	C.125	X	X
$\sigma_{37}\ \sigma_{58} - \sigma_{38}\ \sigma_{57}$	C.126	X	X
$\sigma_{37}\ \sigma_{59} - \sigma_{39}\ \sigma_{57}$	C.127	X	X
$\sigma_{37}\ \sigma_{68} - \sigma_{38}\ \sigma_{67}$	C.128	X	X
$\sigma_{37}\ \sigma_{69} - \sigma_{39}\ \sigma_{67}$	C.129	X	X
$\sigma_{37}\ \sigma_{89} - \sigma_{38}\ \sigma_{79}$	C.130	X	X
$\sigma_{37}\ \sigma_{89} - \sigma_{39}\ \sigma_{78}$	C.131	X	X
$\sigma_{38}\ \sigma_{49} - \sigma_{39}\ \sigma_{48}$	C.132	X	X
$\sigma_{38}\ \sigma_{59} - \sigma_{39}\ \sigma_{58}$	C.133	C.126/C.127	X
$\sigma_{38}\ \sigma_{69} - \sigma_{39}\ \sigma_{68}$	C.134	C.128/C.129	X
$\sigma_{38}\ \sigma_{79} - \sigma_{39}\ \sigma_{78}$	C.135	S:130,131	X
$\sigma_{45}\ \sigma_{67} - \sigma_{46}\ \sigma_{57}$	C.136	S:137,142	X
$\sigma_{45}\ \sigma_{67} - \sigma_{47}\ \sigma_{56}$	C.137	X	X
$\sigma_{45}\ \sigma_{68} - \sigma_{46}\ \sigma_{58}$	C.138	S:139,143	X
$\sigma_{45}\ \sigma_{68} - \sigma_{48}\ \sigma_{56}$	C.139	X	X
$\sigma_{45}\ \sigma_{69} - \sigma_{46}\ \sigma_{59}$	C.140	S:141,144	X
$\sigma_{45}\ \sigma_{69} - \sigma_{49}\ \sigma_{56}$	C.141	X	X
$\sigma_{46}\ \sigma_{57} - \sigma_{47}\ \sigma_{56}$	C.142	X	X
$\sigma_{46}\ \sigma_{58} - \sigma_{48}\ \sigma_{56}$	C.143	X	X
$\sigma_{46}\ \sigma_{59} - \sigma_{49}\ \sigma_{56}$	C.144	X	X
$\sigma_{47}\ \sigma_{58} - \sigma_{48}\ \sigma_{57}$	C.145	X	X

Expression	Model: Equation Number	Second-Order	Group Factor
$\sigma_{47}\ \sigma_{59} - \sigma_{49}\ \sigma_{57}$	C.146	X	X
$\sigma_{47}\ \sigma_{68} - \sigma_{48}\ \sigma_{67}$	C.147	X	X
$\sigma_{47}\ \sigma_{69} - \sigma_{49}\ \sigma_{67}$	C.148	X	X
$\sigma_{47}\ \sigma_{89} - \sigma_{48}\ \sigma_{79}$	C.149	X	X
$\sigma_{47}\ \sigma_{89} - \sigma_{49}\ \sigma_{78}$	C.150	X	X
$\sigma_{48}\ \sigma_{59} - \sigma_{49}\ \sigma_{58}$	C.151	X	X
$\sigma_{48}\ \sigma_{69} - \sigma_{49}\ \sigma_{68}$	C.152	X	X
$\sigma_{48}\ \sigma_{79} - \sigma_{49}\ \sigma_{78}$	C.153	S:149,150	X
$\sigma_{57}\ \sigma_{68} - \sigma_{58}\ \sigma_{67}$	C.154	X	X
$\sigma_{57}\ \sigma_{69} - \sigma_{59}\ \sigma_{67}$	C.155	X	X
$\sigma_{57}\ \sigma_{89} - \sigma_{58}\ \sigma_{79}$	C.156	X	X
$\sigma_{57}\ \sigma_{89} - \sigma_{59}\ \sigma_{78}$	C.157	X	X
$\sigma_{58}\ \sigma_{69} - \sigma_{59}\ \sigma_{68}$	C.158	C.154/C.155	X
$\sigma_{58}\ \sigma_{79} - \sigma_{59}\ \sigma_{78}$	C.159	S:156,157	X
$\sigma_{67}\ \sigma_{89} - \sigma_{68}\ \sigma_{79}$	C.160	X	X
$\sigma_{67}\ \sigma_{89} - \sigma_{69}\ \sigma_{78}$	C.161	X	X
$\sigma_{68}\ \sigma_{79} - \sigma_{69}\ \sigma_{78}$	C.162	S:160,161	X

Reference List

Bentler, P. M. (1986). _Theory and implementation of EQS: A structural equations program_. Los Angeles: BMDP Statistical Software, Inc.

Burt, C. (1939). The relations of educational abilities. _British Journal of Educational Psychology_, _9_, 45-71.

Burt, C. (1944). Mental abilities and mental factors. _British Journal of Educational Psychology_, _14_, 85-94.

Carroll, J. B. (1941). A factor analysis of verbal abilities. _Psychometrika_, _6_, 279-307.

Cattell, R. B. (1952). _Factor analysis: An introduction and manual for the psychologist and social scientist_. New York: Harper.

Cattell, R. B. (1963). Theory of fluid and crystallized intelligence: A critical experiment. _Journal of Educational Psychology_, _54_, 1-22.

Gerbing, D. W. & Anderson, J. C. (1984). On the meaning of within-factor correlated measurement errors. _Journal of Consumer Research_, _11_, 572-580.

Goffin, R. D. & Jackson, D. N. (1988). The structural validity of the Index of Organizational Reactions. _Multivariate Behavioral Research_, _23_, 327-347.

Gorsuch, R. L. (1966). The general factor in the Test Anxiety Questionnaire. _Psychological Reports_, _19_, 308.

Horn, J. L. & Cattell, R. B. (1966). Refinement and test of the theory of fluid and crystallized general intelligences. Journal of Educational Psychology, 57, 253-270.

Horn, J. L. & Stankov, L. (1982). Auditory and visual factors of intelligence. Intelligence, 6, 165-185.

Howarth, E. (1976). A psychometric investigation of Eysenck's Personality Inventory. Journal of Personality Assessment, 40, 173-185.

Huba, G. J. & Segal, B. & Singer, J. L. (1977). Consistency of daydreaming styles across samples of college males and female drug and alcohol users. Journal of Abnormal Psychology 86, 99-102.

Humphreys, L. G. (1967). Critique of Cattell's "Theory of fluid and crystallized intelligence: A Critical Experiment." Journal of Educational Psychology, 58, 129-136.

Joreskog, K. G. (1971). Statistical analysis of sets of congeneric tests. Psychometrika, 36, 109-133.

Joreskog, K. G. & Sorbom, D. (1986). LISREL VI: Users guide (4th ed.). Mooresville, IN: Scientific Software, Inc.

Lee, S. & Hershberger, S. (1990). A simple rule for generating equivalent models in covariance structure modeling. Multivariate Behavioral Research, 25, 313-334.

Luijben, T.C.W. (1991). Equivalent models in covariance structure analysis. Psychometrika, 56, 653-665.

Marsh, H. W. & Hocevar, D. (1985). Application of confirmatory factor analysis to the study of self-concept: First- and higher order factor models and their invariance across groups. Psychological Bulletin, 97, 562-582.

Marsh, H. W. & Richards, G. E. (1987). The multidimensionality of the Rotter I-E scale and its higher-order structure: An application of confirmatory factor analysis. Multivariate Behavioral Research, 22, 39-69.

McArdle, J. H. & McDonald, R. P. (1984). Some algebraic properties of the Reticular Action Model for moment structures. British Journal of Mathematical and Statistical Psychology, 37, 234-251.

Newcomb, M. D. & Bentler, P. M. (1987). Self-reports methods of assessing health status and health service utilization: A hierarchical confirmatory analysis. Multivariate Behavioral Research, 22, 415-436.

Olsson, V. & Bergman, L. R. (1977). A longitudinal factor model for studying change in ability structure. Multivariate Behavioral Research, 22, 221.242.

Rindskopf, D. & Rose, T. (1988). Some theory and applications of confirmatory second-order factor analysis. Multivariate Behavioral Research, 23, 51-67.

Sassenrath, J. M. (1964). A factor analysis of rating-scale items on the Test Anxiety Questionnaire. Journal of Consulting Psychology, 28, 371-377.

Spearman, C. (1904). General intelligence, objectively determined and measured. American Journal of Psychology, 15, 201-293.

Spearman, C. (1927). The abilities of man. New York: MacMillan.

Spearman, C. & Wynn Jones, L. (1950). Human ability. London: MacMillan.

Stelzl, I. (1986). Changing a causal hypothesis without changing the fit: Some rules for generating equivalent path models. Multivariate Behavioral Research, 21, 309-331.

Thomson, G. H. (1956). The factorial analysis of human ability (5th ed.). Cambridge, England: Riverside Press, 1951; reprint (page references are to reprint edition).

Thurstone, L. L. (1969a). Primary mental abilities. Chicago: University of Chicago Press, 1938; reprint (page references are to reprint edition).

Thurstone, L. L. (1969b). Multiple-factor analysis. Chicago: University of Chicago Press, 1947; reprint (page references are to reprint edition).

Thurstone, L. L. & Thurstone, T. G. (1968). Factorial studies of intelligence. Psychometric Monographs, 2 (Psychometric Society), 1941; reprint, Chicago: University of Chicago Press (page references are to reprint edition).

Undheim, J. O. & Gustafsson, J.-E. (1987). The hierarchical organization of cognitive abilities: restoring general intelligence through the use of linear structural relations (LISREL). Multivariate Behavioral Research, 22, 149-171.

Vernon, P.E. (1950). The structure of human abilities. London: Methuen.

Walkey, F. H. & Green, D. E. (1981). The structure of the Eysenck Personality Inventory: A comparison between simple and more complex analyses of a multiple scale questionnaire. Multivariate Behavioral Research, 16, 361-372.

Weeks, D. G. (1980). A second-order longitudinal model of ability structure. Multivariate Behavioral Research, 15, 353-365.