

INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.



University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700 800/521-0600

Order Number 9417518

**The creeping motion of a body between two parallel planes with
application to osmosis in biological membranes**

Yu, Zhihai, Ph.D.

City University of New York, 1993

U·M·I

**300 N. Zeeb Rd.
Ann Arbor, MI 48106**

+

THE CREEPING MOTION OF A BODY BETWEEN TWO PARALLEL PLANES
WITH APPLICATION TO OSMOSIS IN BIOLOGICAL MEMBRANES

By
Zhihai Yu

A dissertation submitted to the Graduate Faculty in
Engineering in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
The City University of New York

1993

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

1/11/94

Date

Peter Ganatos

Professor Peter Ganatos

Chair of Examining Committee

1/11/94

Date

Gerard G. Lowen

Professor Gerard G. Lowen

Executive Officer

Professor Peter Ganatos - Advisor

Dean Zeev Dagan

Professor Charles Maldarelli

Professor Ali Sadegh

Professor Sheldon Weinbaum

Professor Anthony Rosato (Dept. of M.E., NJIT)

Supervisory Committee

The City University of New York

ABSTRACT

THE CREEPING MOTION OF A BODY BETWEEN TWO PARALLEL PLANES WITH APPLICATION TO OSMOSIS IN BIOLOGICAL MEMBRANES

By

Zhihai Yu

Advisor: Professor Peter Ganatos

A strong interaction theory for the hydrodynamic interaction of an arbitrarily shaped particle in a channel is presented and the solutions are applied to a model for osmosis in biological membranes.

Part one of the dissertation contains a derivation of the fundamental solution for a point force between two parallel planes, which analytically satisfies the no-slip conditions on the planes. This fundamental solution is approximated in closed form with a maximum relative error of 1.2%. The closed form of the solution is an important feature which makes it computationally feasible to apply the boundary integral method to the problem of an arbitrarily shaped rigid body moving in an arbitrary direction between two parallel planes.

The second part of the thesis contains an intensive study of the motion of a spheroidal particle between two parallel walls under various Stokes flow conditions using the boundary integral method with the fundamental solution developed in part one. The accuracy of the solution technique is first tested by comparison with the existing strong

interaction theory for the motion of a sphere moving between two plane walls, the asymptotic solutions for a spheroidal particle between two parallel walls at large particle-wall spacings and the strong interaction solutions for the motion of a spheroidal particle adjacent to a single planar boundary. Good agreement was obtained with all existing solutions in the range where the existing theories are valid. Numerical results are then presented for the force and torque acting on a spheroidal particle in a channel as a function of wall spacing, particle position, orientation (including azimuthal orientation) and aspect ratio.

The last part of the dissertation, which is the primary motive of the present research, contains an application of the theory to predict values of the reflection coefficient and the diffusive permeability for osmosis in biological membranes. For the first time the hydrodynamic theory used to predict these coefficients includes the effect of the shape and orientation of the solute particle.

ACKNOWLEDGEMENTS

I wish to thank my mentor Professor Peter Ganatos for his support and guidance during the course of this research and the help in the spirit from my wife, Fangming. I also wish to thank all members of the Supervisory Committee for their help and encouragement.

This work was supported in part by the PSC-CUNY Research Award Program.

TABLE OF CONTENTS

Chapter 1	Introduction	1
Chapter 2	The fundamental solution between two infinite parallel planes	2
2.1	Introduction	2
2.2	Formulation of a point force between two infinite parallel planes	4
2.2.1	Stokeslet pointing perpendicular to the planes	6
2.2.2	Stokeslet pointing parallel to the planes	11
2.2.3	Approximation to the stokeslet between two parallel planes	19
2.4	Concluding remarks	28
Appendix A		30
References		75
Chapter 3	The motion of an arbitrarily shaped rigid body between two parallel walls	76
3.1	Introduction	76
3.2	Formulation for an arbitrary body between two parallel walls	78
3.3	Comparison with existing solutions for a particle near one wall or between two parallel walls	86
3.3.1	Solutions for a sphere between two parallel walls	86
3.3.2	Solutions for a spheroid near one wall	88

3.3.3 Solutions for a spheroid between two walls with both the semi-major axis and the velocity parallel to the walls	89
3.4 Solutions for a spheroid at an arbitrary position with an orientation of arbitrary α and $\gamma=0^\circ$ between two parallel walls	93
3.5 Solutions for a spheroid at an arbitrary position with an orientation of arbitrary α and γ between two parallel walls	140
3.6 Concluding remarks	140
Appendix A	145
References	146
Chapter 4 Osmotic flow in an intercellular cleft	147
4.1 Introduction	147
4.2 Osmotic phenomena in an intercellular cleft	148
4.2.1 Formulation of the osmotic flow in the cleft	148
4.2.2 Numerical values of the reflection coefficient and diffusive permeability	151
4.3 Concluding remarks	161
References	162

List of Tables

Table 1 a,b,c Curve fitting constants and comparison between the exact and their approximate expressions for equations (2.34)	22
Table 2 a Comparison of the correction terms of the stokeslet components at certain pairs of points. The accelerated solution is used as the exact one.	26
Table 2 b Comparison of the correction terms of the stokeslet components at one pair of points. Blake's solution is used as the exact one.	27
Table 3 Resistance coefficient F_3^h for a sphere moving perpendicular to the walls at various positions s .	87
Table 4 Resistance coefficients for a sphere rigidly held in Couette flow with the upper wall moving or in Poiseuille flow at position $s=0.25$.	87
TABLE 5 Comparison of values of resistance coefficients for an oblate spheroid near one wall with aspect ratio $\epsilon=0.5$ at various particle-to-wall spacings H_0 .	90
TABLE 6 Comparison of values of resistance coefficients for a prolate spheroid between two parallel walls with a aspect ratio $\epsilon=0.5$ at position $s=0.25$.	91
TABLE 7 Comparison of values of resistance coefficients for an oblate spheroid between two parallel walls with a aspect ratio $\epsilon=0.5$ at position $s=0.25$.	92
Table 8 a,b Curve fitting constants for the resistance coefficients F_1^h , F_1^p in equation (4.13) for a sphere.	154
Table 9 a,b,c,d Curve fitting constants for the resistance coefficients F_1^h , F_1^p in equation (4.13) for a spheroid with $\epsilon=0.1$.	155
Table 10 a,b,c,d Curve fitting constants for the resistance coefficients F_1^h , F_1^p in equation (4.13) for a spheroid with $\epsilon=0.5$.	160
Table 11 Comparison of values of the diffusive permeability ω and reflection coefficient σ for a spherical solute particle in a channel.	165
Table 12 The theoretical prediction of the diffusive permeability ω and reflection coefficient σ for a spheroidal solute particle in a channel.	167

List of Figures

Figure 1. Geometric configuration of a stokeslet at point Y between two parallel walls. 5

Figure 2. Geometry of a body of revolution between two parallel walls. The body position parameter s is defined as the ratio H_0/H . 80

Figure 3. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 96

Figure 4. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 98

Figure 5. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 99

Figure 6. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rotating about an axis parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 100

Figure 7. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 101

Figure 8. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 103

Figure 9. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$.

(a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 104

Figure 10. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, due to translation perpendicular to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 106

Figure 11. The force component parallel to the walls acting on an oblate or a prolate spheroid an having aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 107

Figure 12. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 108

Figure 13. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 109

Figure 14. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 111

Figure 15. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 112

Figure 16. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 113

Figure 17. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rotating about an axis parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 114

Figure 18. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 115

Figure 19. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 116

Figure 20. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 117

Figure 21. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation perpendicular to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 118

Figure 22. The torque acting on an oblate or a prolate spheroid an having aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation perpendicular to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 119

Figure 23. The force component parallel to the walls acting on an oblate or a prolate spheroid an having aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 120

Figure 24. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 121

Figure 25. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 122

Figure 26. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 124

Figure 27. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 125

Figure 28. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 126

Figure 29. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rotating about an axis parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 127

Figure 30. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 128

Figure 31. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 129

Figure 32. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in Couette flow with the upper wall moving. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$, ----, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 130

Figure 33. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, due to translation perpendicular to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 131

Figure 34. The force component parallel to the walls acting on an oblate or a prolate

spheroid an having aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 132

Figure 35. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 133

Figure 36. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$ 134

Figure 37. The force component parallel to the walls acting on an oblate spheroid having an aspect ratio $\epsilon=0.5$ with its symmetry axis oriented at (a) $\alpha=0^\circ$ and $\gamma=45^\circ$, (b) $\alpha=45^\circ$ and $\gamma=45^\circ$, (c) $\alpha=0^\circ$ and $\gamma=90^\circ$, (d) $\alpha=45^\circ$ and $\gamma=90^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. 141

Figure 38. The force component parallel to the walls acting on a prolate spheroid having an aspect ratio $\epsilon=0.5$ with its symmetry axis oriented at (a) $\alpha=0^\circ$ and $\gamma=45^\circ$, (b) $\alpha=45^\circ$ and $\gamma=45^\circ$, (c) $\alpha=0^\circ$ and $\gamma=90^\circ$, (d) $\alpha=45^\circ$ and $\gamma=90^\circ$, due to translation parallel to the walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. 142

Figure 39. The force component parallel to the walls acting on an oblate spheroid having an aspect ratio $\epsilon=0.5$ with its symmetry axis oriented at (a) $\alpha=0^\circ$ and $\gamma=45^\circ$, (b) $\alpha=45^\circ$ and $\gamma=45^\circ$, (c) $\alpha=0^\circ$ and $\gamma=90^\circ$, (d) $\alpha=45^\circ$ and $\gamma=90^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. 143

Figure 40. The force component parallel to the walls acting on a prolate spheroid having an aspect ratio $\epsilon=0.5$ with its symmetry axis oriented at (a) $\alpha=0^\circ$ and $\gamma=45^\circ$, (b) $\alpha=45^\circ$ and $\gamma=45^\circ$, (c) $\alpha=0^\circ$ and $\gamma=90^\circ$, (d) $\alpha=45^\circ$ and $\gamma=90^\circ$, rigidly held in Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. 144

Figure 41. The resistance coefficients as a function of α (when $\gamma=0^\circ$) for an oblate or a prolate spheroid having an aspect ratio $\epsilon=0.1$ or 0.5 at position $s=0.25$, $H=4.4$ or $H=12.0$. (a) F_1^h , (b) F_1^p , (c) F_3^h , (d) F_3^p . _____, oblate; -----, prolate. 135

Figure 42. The resistance coefficients as a function of α (when $\gamma=0^\circ$) for an oblate or a prolate spheroid having an aspect ratio $\epsilon=0.1$ or 0.5 at position $s=0.25$, $H=4.4$ or $H=12.0$. (a) T_2^r , (b) T_2^h , (c) T_2^s , (d) T_2^b . _____, oblate; -----, prolate. 136

Figure 43. The resistance coefficients as a function of the aspect ratio ϵ for an oblate or a prolate spheroid at position $s=0.25$, $H=4.4$ or $H=12.0$, and oriented at $\alpha=18^\circ$ or 45° and $\gamma=0^\circ$. (a) F_1^i , (b) F_1^s , (c) F_3^i , (d) F_1^p . —, oblate; -----, prolate. 138

Figure 44. The resistance coefficients as a function of the aspect ratio ϵ for an oblate or a prolate spheroid at position $s=0.25$, $H=4.4$ or $H=12.0$, and oriented at $\alpha=18^\circ$ or 45° and $\gamma=0^\circ$. (a) F_3^i , (b) T_2^r , (c) F_3^p , (d) T_2^p . —, oblate; -----, prolate. 139

Chapter 1 Introduction

The motion of an arbitrarily shaped rigid body immersed in a viscous fluid bounded by two infinite parallel planes is studied in this thesis. The formulation is based on the boundary integral method. In chapter 2, a closed-form (approximate) fundamental solution for a point force between two parallel planes is developed which is ideally suitable for use with the boundary integral method. In chapter 3, an intensive study of the hydrodynamic interaction of a spheroidal particle between two parallel walls is presented. Finally, in chapter 4, these new solutions are used in the study of osmotic phenomena in an intercellular cleft. Theoretical values of the reflection coefficient and diffusive permeability are obtained, which include the effect of the shape and the orientation of the solute particle.

Chapter 2 The fundamental solution between two infinite parallel planes

2.1 Introduction

Stokes flows containing suspended particles in the presence of confining boundaries are of great interest and practical importance. Several methods have been used to treat such problems, including the method of reflections, the boundary collocation method, the finite element method and the boundary integral method. Among these methods the boundary integral method has a greater versatility than any of the other methods. It was first applied to the treatment of three-dimensional unbounded Stokes flow problems by Youngren & Acrivos (1975). It was also applied to treating Stokes flow problems with deformable interfaces by Rallison & Acrivos (1978). A recent problem treated with this method is the axisymmetric motion of a spherical bubble with a surfactant cap rising out of a hole in a planar wall by Dagan et al (1988). An essential ingredient of the boundary integral method for Stokes flow problems is the availability of a proper fundamental solution or Green's function for a given flow geometry.

In general there are two kinds of fundamental solutions suitable for the treatment of Stokes flow problems, the free space stokeslet solution and the fundamental solution which describes a point force in the presence of confining boundaries on which the no-slip boundary conditions are analytically satisfied. The free space stokeslet, which is very simple, requires the distribution of the point force over the confining boundaries which are infinite in most cases and its application with the boundary integral method requires numerical integration to be performed over these surfaces which is very time consuming.

On other hand the fundamental solution which analytically satisfies the no-slip conditions along the confining boundaries, which is generally more complicated, only requires performing the integration along the surface of the particle which is always finite. This feature of the solution provides a significant saving of computational effort. Recent studies utilizing this type fundamental solution include the work of Hsu & Ganatos (1989) and Wu et al (1992). The fundamental solutions derived thus far for bounded flow geometries include the solution for a point force (stokeslet) near a planar wall by Blake (1971), between two infinite parallel planes by Liron & Mochon (1976), inside an infinite cylindrical tube by Liron & Shahar (1978) and in a region bounded by a circular orifice in an infinite plane by Miyazaki & Hasimoto (1984). Of these fundamental solutions, Blake's solution, which has the simplest form, does not involve an infinite series or infinite integral. This closed form solution has provided the key ingredient to the study of the motion of a particle adjacent to a planar wall with the boundary integral method.

The present research will develop an alternate approach to derive the fundamental solution which analytically satisfies the no-slip conditions along the infinite confining boundaries. This alternate approach will construct the fundamental solution in the form of a free space stokeslet plus the finite correction terms needed to satisfy the no-slip conditions on the confining boundaries. The solution for a stokeslet between two infinite parallel planes will be developed as a demonstration of this approach.

2.2 Formulation for a point force between two infinite parallel planes

In this section, the fundamental solution will be developed for the creeping flow between two parallel planes. As mentioned in introduction, the feasibility of the boundary integral method depends on the simplicity of the solution. The solution for the flow between two infinite parallel planes has been developed by Liron (1976), which is based on Blake's (1971) theory of reflection images. It has the form of infinite integral or infinite series, and both forms converge slowly as the field point $\mathbf{X}$ and stokeslet point $\mathbf{Y}$ approach each other, where the point $\mathbf{X}$ and $\mathbf{Y}$ are shown in Figure 1. The ability to compute the velocity field in the vicinity of the stokeslet efficiently is the crucial feature required by the boundary integral method because the majority of the field points $\mathbf{X}$ are within the distance a , the characteristic radius of the particle, from the stokeslet points $\mathbf{Y}$. Thus Liron's solution is not suitable for the application with the boundary integral method and an alternate form needs to be developed.

Consider a point force (stokeslet) in a viscous incompressible fluid between two parallel planes separated a distance H apart as shown in Figure 1. The origin of the cartesian coordinate system (x_1, x_2, x_3) is fixed on the lower plane, and the origin of the cartesian coordinate system (x, y, z) is located at the point $(0, 0, b)$ in the system (x_1, x_2, x_3) . The stokeslet is situated at the origin of the coordinate system (x, y, z) , and the fundamental solution can be represented by

$$\mathbf{U}_i^{\mathbf{k}} = \mathbf{V}_i^{\mathbf{k}} + \mathbf{W}_i^{\mathbf{k}} \quad (2.1)$$

where $\mathbf{V}_i^{\mathbf{k}}(\mathbf{X}, \mathbf{Y})$ is the well known free space stokeslet solution and $\mathbf{W}_i^{\mathbf{k}}(\mathbf{X}, \mathbf{Y})$ is called

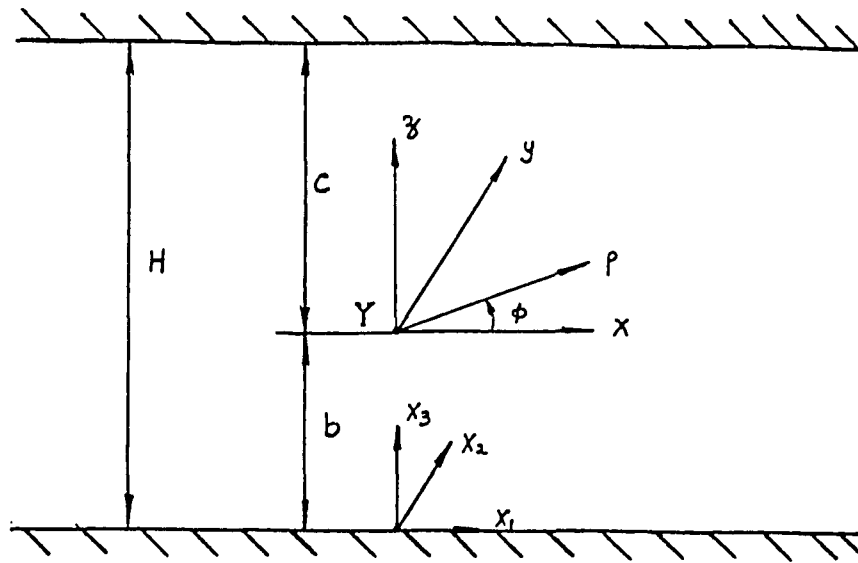


Figure 1. Geometric configuration of a stokeslet at point Y between two parallel walls.

the supplementary solution, which contains the correction terms needed to make the fundamental solution $U_i^k(\mathbf{X}, \mathbf{Y})$ analytically satisfy the no-slip conditions on the planes.

For simplicity we will separately consider two cases, one where the stokeslet points perpendicular to the planes and the other where the stokeslet points parallel to the planes. Owing to the linearity of the governing equations and boundary conditions, the general solution for an arbitrarily oriented stokeslet can be obtained by a simple linear superposition of these two basic solutions.

2.2.1 Stokeslet pointing perpendicular to the planes

We consider a stokeslet located at the origin of the coordinate system (x, y, z) in Figure 1, which points in the direction of the z -axis. The governing equations for the flow are

$$\mu \nabla^2 \mathbf{U} = \nabla \mathbf{p}, \quad \nabla \cdot \mathbf{U} = 0 \quad (2.2 \text{ a,b})$$

Owing to the axisymmetric nature of the flow, it is convenient to work with the stream function ψ , which analytically satisfies equation (2.2 b). The radial and axial components of the fluid velocity in the coordinate system (ρ, ϕ, z) shown in Figure 1 are related to the stream function ψ by

$$U_\rho = -\frac{1}{\rho} \frac{\partial \psi}{\partial z}, \quad U_z = \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} \quad (2.3 \text{ a,b})$$

and the governing equations (2.2) can be replaced by

$$\mathbf{D}^2(\mathbf{D}^2\psi)=0 \quad (2.4)$$

where $\mathbf{D}^2 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}$ is the axisymmetric Stokesian operator. For the stokeslet

pointing perpendicular to the planes, the stream function is linearly composed of two parts as proposed in equation (2.1)

$$\psi = \psi_v + \psi_w \quad (2.5)$$

The part ψ_v , which is the contribution of the free space stokeslet, can be compactly expressed in cylindrical coordinates by

$$\psi_v = \frac{1}{8\pi\mu} \left[\frac{\rho^2}{(\rho^2 + z^2)^{1/2}} \right] \quad (2.6)$$

The part ψ_w represents the contribution of the confining boundaries, which is the sum of all the separable solutions of (2.4) in terms of cylindrical coordinates to produce finite velocities everywhere within the confining boundaries

$$\psi_w = \int_0^\infty [A(t)\exp(tz) + B(t)\exp(-tz) + C(t)tz \exp(tz) + D(t)tz\exp(-tz)] \rho J_1(\rho t) dt \quad (2.7)$$

The functions $A(t)$ - $D(t)$ in above equation are unknown functions of the separation variable t , which will be determined by satisfying the no-slip boundary conditions along the two parallel planes. The total velocity field due to the stokeslet pointing perpendicular to the planes is given by

$$U_\rho = \frac{1}{4\pi\mu} \left(\frac{3}{\rho} I_3 \left[\frac{z}{(\rho^2+z^2)^{1/2}} \right] - \frac{2}{(\rho^2+z^2)^{1/2}} \frac{z}{\rho} I_2 \left[\frac{z}{(\rho^2+z^2)^{1/2}} \right] \right) + \int_0^\infty E(t,z) t J_1(t\rho) dt \quad (2.8 \text{ a})$$

$$U_z = \frac{1}{4\pi\mu} \left(\frac{2}{(\rho^2+z^2)^{1/2}} I_2 \left[\frac{z}{(\rho^2+z^2)^{1/2}} \right] + \frac{1}{(\rho^2+z^2)^{1/2}} P_2 \left[\frac{z}{(\rho^2+z^2)^{1/2}} \right] \right) + \int_0^\infty F(t,z) t J_0(t\rho) dt \quad (2.8 \text{ b})$$

where

$$E(t,z) = \frac{-A(t) \exp(tz) + B(t) \exp(-tz) - C(t)}{(1+tz) \exp(tz) - D(t)(1-tz) \exp(-tz)} \quad (2.9 \text{ a})$$

$$F(t,z) = \frac{A(t) \exp(tz) + B(t) \exp(-tz) + C(t)}{tz \exp(tz) + D(t)tz \exp(-tz)} \quad (2.9 \text{ b})$$

Here J_n , I_n , P_n are Bessel functions of the first kind, modified Bessel functions of the first kind and Legendre functions of order n respectively.

Applying the no-slip boundary conditions along the two planes (see Figure 1), i.e. $U_\rho = U_z = 0$ at $z_1 = -b$ and $z_2 = c$ in (2.8 a,b), and inverting the Hankel transforms yields the unknown functions $E(t,z)$ and $F(t,z)$ at $z = z_j$, with $j=1, 2$.

$$E(t,z_j) = \frac{-1}{4\pi\mu} \int_0^\infty \left(3 I_3 \left[\frac{z_j}{(\alpha^2+z_j^2)^{1/2}} \right] - \frac{2z_j}{(\alpha^2+z_j^2)^{1/2}} I_2 \left[\frac{z_j}{(\alpha^2+z_j^2)^{1/2}} \right] \right) J_1(\alpha t) d\alpha \quad (2.10 \text{ a})$$

$$F(t,z_j) = \frac{-1}{4\pi\mu} \int_0^\infty \left(\frac{2\alpha}{(\alpha^2+z_j^2)^{1/2}} I_2 \left[\frac{z_j}{(\alpha^2+z_j^2)^{1/2}} \right] + \frac{\alpha}{(\alpha^2+z_j^2)^{1/2}} P_2 \left[\frac{z_j}{(\alpha^2+z_j^2)^{1/2}} \right] \right) J_0(\alpha t) d\alpha \quad (2.10 \text{ b})$$

The integrals involved in above equations are evaluated analytically with the following formulas which are found in Gradshteyn & Ryzhik (1980)

$$\int_0^{\infty} \frac{1}{(\alpha^2 + z_j^2)^{1/2}} I_2\left[\frac{z_j}{(\alpha^2 + z_j^2)^{1/2}}\right] J_1(\alpha t) d\alpha = \frac{1}{2} \exp(-t|z_j|) \quad (2.11 \text{ a})$$

$$\int_0^{\infty} I_3\left[\frac{z_j}{(\alpha^2 + z_j^2)^{1/2}}\right] J_1(\alpha t) d\alpha = \frac{z_j}{2} \exp(-t|z_j|) \quad (2.11 \text{ b})$$

$$\int_0^{\infty} \frac{\alpha}{(\alpha^2 + z_j^2)^{1/2}} I_2\left[\frac{z_j}{(\alpha^2 + z_j^2)^{1/2}}\right] J_0(\alpha t) d\alpha = \frac{(1 - t|z_j|)}{2t} \exp(-t|z_j|) \quad (2.11 \text{ c})$$

$$\int_0^{\infty} \frac{\alpha}{(\alpha^2 + z_j^2)^{1/2}} P_2\left[\frac{z_j}{(\alpha^2 + z_j^2)^{1/2}}\right] J_0(\alpha t) d\alpha = \frac{(3t|z_j| - 1)}{2t} \exp(-t|z_j|) \quad (2.11 \text{ d})$$

The evaluations of (2.10 a,b) yield the expressions for $E(t, z_j)$ and $F(t, z_j)$

$$E(t, z_j) = \frac{-1}{4\pi\mu} \frac{z_j}{2} \exp(-t|z_j|) \quad (2.12 \text{ a})$$

$$F(t, z_j) = \frac{-1}{4\pi\mu} \frac{1}{2t} (1 + t|z_j|) \exp(-t|z_j|) \quad (2.12 \text{ b})$$

In order to obtain the functions $E(t, z)$ and $F(t, z)$, the functions $E(t, z_j)$ and $F(t, z_j)$ given by (2.12) are substituted into (2.9 a,b). This yields four simultaneous linear algebraic equations which may be solved for the unknown functions $A(t)$ - $D(t)$. Introducing the obtained functions $A(t)$ - $D(t)$ into (2.9 a,b) yields the desired expressions for the functions $E(t, z)$ and $F(t, z)$ which are valid at any arbitrary value of z between two walls.

Therefore the velocity field for a stokeslet pointing perpendicular to the planes (2.8 a,b) is furnished in terms of a single infinite integral

$$U_{\rho} = \frac{1}{4\pi\mu} \left(\frac{3}{\rho} I_3 \left[\frac{z}{(\rho^2+z^2)^{1/2}} \right] - \frac{2}{(\rho^2+z^2)^{1/2}} \frac{z}{\rho} I_2 \left[\frac{z}{(\rho^2+z^2)^{1/2}} \right] + \right. \\ \left. \int_0^{\infty} \left[G_3(\sigma, \eta) \frac{b}{2} \exp(-tb) + G_3(\eta, \sigma) \frac{c}{2} \exp(-tc) \right. \right. \\ \left. \left. + G_1(\sigma, \eta) \frac{1}{2t} (tb+1) \exp(-tb) - G_1(\eta, \sigma) \right. \right. \\ \left. \left. \frac{1}{2t} (tc+1) \exp(-tc) \right] t J_1(t\rho) dt \right) \quad (2.13 \text{ a})$$

$$U_z = \frac{1}{4\pi\mu} \left(\frac{2}{(\rho^2+z^2)^{1/2}} I_2 \left[\frac{z}{(\rho^2+z^2)^{1/2}} \right] + \frac{1}{(\rho^2+z^2)^{1/2}} P_2 \left[\frac{z}{(\rho^2+z^2)^{1/2}} \right] + \right. \\ \left. \int_0^{\infty} \left[-G_2(\sigma, \eta) \frac{b}{2} \exp(-tb) - G_2(\eta, \sigma) \frac{c}{2} \exp(-tc) \right. \right. \\ \left. \left. - G_4(\sigma, \eta) \frac{1}{2t} (tb+1) \exp(-tb) + G_4(\eta, \sigma) \right. \right. \\ \left. \left. \frac{1}{2t} (tc+1) \exp(-tc) \right] t J_0(t\rho) dt \right) \quad (2.13 \text{ b})$$

where the functions G_i with dummy variables ξ, θ are defined as the following

$$G_{1,2}(\xi, \theta) = (\omega \theta \sinh \xi \pm \xi \sinh \omega \sinh \theta) / (\sinh^2 \omega - \omega^2) \quad (2.14 \text{ a})$$

$$G_{3,4}(\xi, \theta) = [(\omega \theta \cosh \xi - \sinh \theta \sinh \omega) \pm (\omega \xi \sinh \xi - \cosh \theta \sinh \omega)] / (\sinh^2 \omega - \omega^2) \quad (2.14 \text{ b})$$

$$\sigma = t(z+b), \eta = t(z-c), \omega = t(b+c) \quad (2.15 \text{ a,b,c})$$

In above equations the subscripts 1, 3 and 2, 4 refer to the plus and minus signs respectively, and b, c are the wall positions as shown in Figure 1.

2.2.2 Stokeslet pointing parallel to the planes

Again consider a stokeslet located at the origin of the coordinate system (x,y,z) in Figure 1, but pointing in the direction along the x-axis. The governing equations for the flow field are given by (2.2). However the flow is no longer axisymmetric and the stream function equation (2.4) cannot be used. The velocity field for the free space stokeslet pointing along the x-axis is given in cartesian coordinates by

$$V_x = \frac{1}{8\pi\mu} \left(\frac{1}{r} + \frac{x^2}{r^3} \right) \quad (2.16 \text{ a})$$

$$V_y = \frac{1}{8\pi\mu} \left(\frac{xy}{r^3} \right), \quad V_z = \frac{1}{8\pi\mu} \left(\frac{xz}{r^3} \right) \quad (2.16 \text{ b,c})$$

where

$$r^2 = x^2 + y^2 + z^2 \quad (2.16 \text{ d})$$

The supplementary solution W which is defined in (2.1) represents the contributions of the confining boundaries and its cartesian components are given by

$$W_x = \int_0^{\infty} \int_0^{\infty} F_1(s,t,z) \cos(sx) \cos(ty) ds dt \quad (2.17 \text{ a})$$

$$W_y = \int_0^{\infty} \int_0^{\infty} F_2(s,t,z) \sin(sx) \sin(ty) ds dt \quad (2.17 \text{ b})$$

$$W_z = \int_0^{\infty} \int_0^{\infty} F_3(s,t,z) \sin(sx) \cos(ty) ds dt \quad (2.17 \text{ c})$$

where the functions $F_1(s,t,z)$, $F_2(s,t,z)$ and $F_3(s,t,z)$ consist of a linear combination of six

unknown functions of the separation variables s and t

$$F_1(s,t,z) = \left(A_1(s,t) \left(1 + \frac{s^2}{q} z \right) - A_2(s,t) \frac{st}{q} z - A_3(s,t) sz \right) \exp(qz) \\ + \left(B_1(s,t) \left(1 - \frac{s^2}{q} z \right) + B_2(s,t) \frac{st}{q} z - B_3(s,t) sz \right) \exp(-qz) \quad (2.18 \text{ a})$$

$$F_2(s,t,z) = \left(-A_1(s,t) \frac{st}{q} z + A_2(s,t) \left(1 + \frac{t^2}{q} z \right) + A_3(s,t) tz \right) \exp(qz) \\ + \left(B_1(s,t) \frac{st}{q} z + B_2(s,t) \left(1 - \frac{t^2}{q} z \right) + B_3(s,t) tz \right) \exp(-qz) \quad (2.18 \text{ b})$$

$$F_3(s,t,z) = [A_1(s,t)sz - A_2(s,t)tz + A_3(s,t)(1 - qz)] \exp(qz) \\ + [B_1(s,t)sz - B_2(s,t)tz + B_3(s,t)(1 + qz)] \exp(-qz) \quad (2.18 \text{ c})$$

where

$$q^2 = s^2 + t^2 \quad (2.18 \text{ d})$$

Applying the no-slip boundary conditions along the two planes, i.e. $U_\rho = U_z = 0$ at $z_1 = -b$ and $z_2 = c$ in (2.1), and inverting the double Fourier transform yield the unknown functions $F_1(s,t,z)$, $F_2(s,t,z)$ and $F_3(s,t,z)$ at $z = z_j$, with $j=1, 2$.

$$F_1(s,t,z_j) = \frac{1}{8\pi\mu} \int_0^\infty \int_0^\infty \left(\frac{1}{r} + \frac{x^2}{r^2} \right) \cos(sx) \cos(ty) dx dy \quad (2.19 \text{ a})$$

$$F_2(s,t,z_j) = \frac{1}{8\pi\mu} \int_0^\infty \int_0^\infty \left(\frac{xy}{r^2} \right) \sin(sx) \sin(ty) dx dy \quad (2.19 \text{ b})$$

$$F_2(s,t,z_j) = \frac{1}{8\pi\mu} \int_0^\infty \int_0^\infty \left(\frac{xz_j}{r^2} \right) \sin(sx) \cos(ty) dx dy \quad (2.19 \text{ c})$$

where

$$r = (x^2 + y^2 + z_j^2)^{1/2} \quad (2.19 \text{ d})$$

Like the situation in the perpendicular case, the double integrations involved in $F_1(s,t,z_j)$, $F_2(s,t,z_j)$ and $F_3(s,t,z_j)$ are performed analytically with the formulas found in Gradshteyn & Ryzhik (1980). The following formulas are used for the evaluations of the double integrals

$$\int_0^\infty \int_0^\infty \frac{1}{(x^2 + y^2 + z_j^2)^{1/2}} \cos(sx) \cos(ty) dx dy = \left(\frac{\pi}{2} \frac{|z_j|}{q} \right)^{1/2} K_{-1/2}(q|z_j|) \quad (2.20 \text{ a})$$

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{x^2}{(x^2 + y^2 + z_j^2)^{3/2}} \cos(sx) \cos(ty) dx dy \\ = \left(\frac{\pi}{2} \frac{|z_j|}{q} \right)^{1/2} \left(K_{-1/2}(q|z_j|) - \frac{s z_j^2}{(q|z_j|)^{3/2}} K_{-3/2}(q|z_j|) \right) \end{aligned} \quad (2.20 \text{ b})$$

$$\int_0^\infty \int_0^\infty \frac{xy}{(x^2 + y^2 + z_j^2)^{3/2}} \sin(sx) \sin(ty) dx dy = \left(\frac{\pi}{2|z_j|} \right)^{1/2} \left(\frac{st}{q^{3/2}} \right) K_{-3/2}(q|z_j|) \quad (2.20 \text{ c})$$

$$\int_0^\infty \int_0^\infty \frac{x}{(x^2 + y^2 + z_j^2)^{3/2}} \sin(sx) \cos(ty) dx dy = \left(\frac{\pi}{2} \frac{|z_j|}{q} \right)^{1/2} s K_{-1/2}(q|z_j|) \quad (2.20 \text{ d})$$

The expansions of modified Bessel functions of the second kind appearing in (2.20) may be evaluated using the expressions

$$K_{-1/2}(q|z_j|) = \left(\frac{\pi}{2q|z_j|} \right)^{1/2} \exp(-q|z_j|) \quad (2.20 \text{ e})$$

$$K_{-3/2}(q|z_j|) = \left(\frac{\pi}{2q|z_j|} \right)^{1/2} \exp(-q|z_j|) \left(1 + \frac{1}{q|z_j|} \right) \quad (2.20 \text{ f})$$

Thus (2.19) may be written as

$$F_1(s, t, z_j) = \frac{-1}{4\pi\mu} \frac{\exp(-q|z_j|)}{q^2} \left(2q - \frac{s^2}{(q|z_j|)^{1/2}} \left(|z_j| + \frac{1}{q} \right) \right) \quad (2.21 \text{ a})$$

$$F_2(s, t, z_j) = \frac{-1}{4\pi\mu} \frac{st}{q^2} \exp(-q|z_j|) \left(|z_j| + \frac{1}{q} \right) \quad (2.21 \text{ b})$$

$$F_3(s, t, z_j) = \frac{-1}{4\pi\mu} \frac{sz_j}{q} \exp(-q|z_j|) \quad (2.21 \text{ c})$$

Substituting (2.21) into (2.18) evaluated at $z=z_j$, $j=1, 2$ results in six simultaneous linear algebraic equations which may be solved for the six unknown functions $A_1(s, t)$ - $A_3(s, t)$ and $B_1(s, t)$ - $B_3(s, t)$. Then substituting the obtained functions $A_1(s, t)$ - $A_3(s, t)$ and $B_1(s, t)$ - $B_3(s, t)$ back into (2.18) yields the desired expressions for the functions $F_1(s, t, z)$, $F_2(s, t, z)$ and $F_3(s, t, z)$ which are valid at any arbitrary value of z between two walls. The final expressions are

$$\begin{aligned}
F_1(s,t,z) = & \frac{\sinh \eta^*}{\sinh \omega^*} \frac{2 \exp(-qb)}{q\pi} \left(2 - \frac{s^2}{q(qb)^{1/2}} (b + \frac{1}{q}) \right) - \frac{\sinh \sigma^*}{\sinh \omega^*} \frac{2}{q\pi} \\
& \exp(-qc) \left(2 - \frac{s^2}{q(qc)^{1/2}} (c + \frac{1}{q}) \right) - G_5(\sigma^*, \eta^*) \frac{2s^2}{q^4\pi} \exp(-qb) \\
& \left(2q - \left(\frac{s^2}{(qb)^{1/2}} + t^2 \right) (b + \frac{1}{q}) \right) + G_5(\eta^*, \sigma^*) \frac{2s^2}{q^4\pi} \exp(-qc) \\
& \left(2q - \left(\frac{s^2}{(qc)^{1/2}} + t^2 \right) (c + \frac{1}{q}) \right) + G_1(\sigma^*, \eta^*) \frac{2s^2b}{q^2\pi} \exp(-qb) \\
& + G_1(\eta^*, \sigma^*) \frac{2s^2c}{q^2\pi} \exp(-qc)
\end{aligned} \tag{2.22 a}$$

$$\begin{aligned}
F_2(s,t,z) = & G_5(\sigma^*, \eta^*) \frac{2st}{q^3\pi} \exp(-qb) \left(2q - \left(\frac{s^2}{(qb)^{1/2}} - s^2 \right) (b + \frac{1}{q}) \right) \\
& - G_5(\eta^*, \sigma^*) \frac{2st}{q^3\pi} \exp(-qc) \left(2q - \left(\frac{s^2}{(qc)^{1/2}} - s^2 \right) (c + \frac{1}{q}) \right) \\
& - G_3(\sigma^*, \eta^*) \frac{2st}{q^2\pi} (b + \frac{1}{q}) \exp(-qb) + G_3(\sigma^*, \eta^*) \frac{2st}{q^2\pi} \\
& (c + \frac{1}{q}) \exp(-qc) - G_1(\sigma^*, \eta^*) \frac{2stb}{q^2\pi} \exp(-qb) \\
& - G_1(\eta^*, \sigma^*) \frac{2stc}{q^2\pi} \exp(-qc)
\end{aligned} \tag{2.22 b}$$

$$\begin{aligned}
F_3(s,t,z) = & G_2(\sigma^*, \eta^*) \frac{2s^2}{q^3\pi} \exp(-qb) \left(-2q + \left(\frac{s^2}{(qb)^{1/2}} - t^2 \right) (b + \frac{1}{q}) \right) \\
& + G_2(\eta^*, \sigma^*) \frac{2s^2}{q^3\pi} \exp(-qc) \left(2q - \left(\frac{s^2}{(qc)^{1/2}} + t^2 \right) (c + \frac{1}{q}) \right) \\
& + G_4(\sigma^*, \eta^*) \frac{2sb}{q\pi} \exp(-qb) + G_4(\eta^*, \sigma^*) \frac{2sc}{q\pi} \exp(-qc)
\end{aligned} \tag{2.22 c}$$

where

$$G_5(\xi, \theta) = (\sinh \omega^* [\omega^* \sinh \xi - \xi \sinh \omega^* \cosh \theta] + [\omega^* \theta \sinh \omega^* \cosh \xi - \omega^{*2} \sinh \theta]) / \sinh \omega^* (\sinh^2 \omega^* - \omega^{*2}) \tag{2.23}$$

$$\sigma^* = q(z+b), \quad \eta^* = q(z-c), \quad \omega^* = q(b+c) \tag{2.24 a,b,c}$$

In above equations ξ, θ are dummy variables and the functions $G_1(\xi, \theta) - G_4(\xi, \theta)$ are defined previously in (2.14) with ω replaced by ω^* .

After introducing (2.22) into (2.17) we have the supplemental solution for the velocity field due to the stokeslet pointing in the direction of x-axis in terms of a double infinite integral. By transforming the integrals of (2.17 a,b,c) into the cylindrical coordinates (ρ, ϕ, z) , the integration in the azimuthal direction (ϕ) can be performed analytically. With the transformations

$$s = q \cos \phi, \quad t = q \sin \phi \quad (2.25 \text{ a,b})$$

equations (2.17 a,b,c) become

$$W_x = \int_0^\infty \int_0^{\pi/2} q F_1(q, \phi, z) \cos(qx \cos \phi) \cos(qy \sin \phi) d\phi dq \quad (2.26 \text{ a})$$

$$W_y = \int_0^\infty \int_0^{\pi/2} q F_2(q, \phi, z) \sin(qx \cos \phi) \sin(qy \sin \phi) d\phi dq \quad (2.26 \text{ b})$$

$$W_z = \int_0^\infty \int_0^{\pi/2} q F_3(q, \phi, z) \sin(qx \cos \phi) \cos(qy \sin \phi) d\phi dq \quad (2.26 \text{ c})$$

Using the following formulas which are also found in Gradshteyn & Ryzhik (1980) the above integrals can be evaluated in the ϕ -direction analytically

$$\int_0^{\pi/2} \cos(a_1 \cos \phi) \cos(a_2 \sin \phi) d\phi = \frac{\pi}{2} J_0(a_3) \quad (2.27 \text{ a})$$

$$\int_0^{\pi/2} \cos^2 \phi \cos(a_1 \cos \phi) \cos(a_2 \sin \phi) d\phi = \frac{a_1^2 \pi}{2a_3^2} \left(J_0(a_3) + \frac{(a_2^2 - a_1^2)}{a_1^2 a_3} J_1(a_3) \right) \quad (2.27 \text{ b})$$

$$\int_0^{\pi/2} \cos \phi \sin \phi \sin(a_1 \cos \phi) \sin(a_2 \sin \phi) d\phi = -\frac{a_1 a_2 \pi}{2a_3^2} \left(J_0(a_3) + \frac{2}{a_3} J_1(a_3) \right) \quad (2.27 \text{ c})$$

$$\int_0^{\pi/2} \cos \phi \sin(a_1 \cos \phi) \cos(a_2 \sin \phi) d\phi = \frac{a_1 \pi}{2a_3} J_1(a_3) \quad (2.27 \text{ d})$$

where a_1 and a_2 are positive constants and $a_3 = (a_1^2 + a_2^2)^{1/2}$. Thus the velocity field due to the stokeslet pointing in the direction of x-axis is expressed in terms of a single infinite integral as was done for the perpendicular case. The final expressions are

$$U_x = \frac{1}{8\pi\mu} \left[\left(\frac{1}{r} + \frac{x^2}{r^3} \right) + \int_0^\infty \left(\left[2 \frac{\sinh \eta^*}{\sinh \omega^*} J_0(q\rho) + \left[-\frac{\sinh \eta^*}{\sinh \omega^*} (1+qb) + G_3(\sigma^*, \eta^*)(qb-1) + G_1(\sigma^*, \eta^*)(qb) \right] \frac{K_1}{\rho^2} \right] \exp(-qb) + \left[-2 \frac{\sinh \sigma^*}{\sinh \omega^*} J_0(q\rho) + \left[\frac{\sinh \sigma^*}{\sinh \omega^*} (1+qc) - G_3(\eta^*, \sigma^*)(qc-1) + G_1(\eta^*, \sigma^*)(qc) \right] \frac{K_1}{\rho^2} \right] \exp(-qc) \right) dq \right] \quad (2.28 \text{ a})$$

$$U_y = \frac{1}{8\pi\mu} \left[\frac{xy}{r^3} + \int_0^\infty \left(\left[-G_3(\sigma^*, \eta^*) 2xy + G_3(\sigma^*, \eta^*) xy(1+qb) + G_1(\sigma^*, \eta^*) xy(qb) \right] \frac{K_2}{\rho^2} \exp(-qb) + \left[G_3(\eta^*, \sigma^*) 2xy - G_3(\eta^*, \sigma^*) xy(1+qc) + G_1(\eta^*, \sigma^*) xy(qc) \right] \frac{K_2}{\rho^2} \exp(-qc) \right) dq \right] \quad (2.28 \text{ b})$$

$$U_z = \frac{1}{8\pi\mu} \left[\frac{xz}{r^3} + \int_0^\infty \left[G_2(\sigma^*, \eta^*)(qb-1) + G_4(\sigma^*, \eta^*)(qb) \right] \frac{x}{\rho} J_1(q\rho) \exp(-qb) - \right. \\ \left. [G_2(\eta^*, \sigma^*)(qc-1) - G_4(\eta^*, \sigma^*)(qc)] \frac{x}{\rho} J_1(q\rho) \exp(-qc) \right] dq \quad (2.28 \text{ c})$$

where

$$K_1 = x^2 J_0(q\rho) + (y^2 - x^2) \frac{J_1(q\rho)}{q\rho} \quad (2.29 \text{ a})$$

$$K_2 = J_0(q\rho) - 2 \frac{J_1(q\rho)}{q\rho} \quad (2.29 \text{ b})$$

The supplemental solution of the velocity field due to the stokeslet pointing in the direction of y-axis can be easily obtained by transformation of the velocity components and the coordinate system.

$$W_x^*(x, y, z) = -W_y(-y, x, z) \quad (2.30 \text{ a})$$

$$W_y^*(x, y, z) = W_x(-y, x, z) \quad (2.30 \text{ b})$$

$$W_z^*(x, y, z) = W_z(-y, x, z) \quad (2.30 \text{ c})$$

Here W^* represents the velocity field due to a stokeslet pointing in the direction of y-axis and W represents the velocity field when the stokeslet points along the x-axis. Thus the velocity field due to a stokeslet pointing in y-direction also can be expressed in the desired form. Therefore the entire velocity field due to a stokeslet pointing parallel to the planes can be described completely by equations (2.28 a,b,c) and 2.30 (a,b,c).

2.2.3 Approximation to the stokeslet between two planes

Since both solutions for a stokeslet perpendicular and parallel to the planes are in the form of an infinite integral they still need to be simplified in order to reduce the excessive CPU time required for the multiple numerical evaluation. It is therefore necessary to accelerate the speed of convergence of the infinite integrals appearing in the supplemental solutions or to find a way to evaluate them analytically. The major difficulty preventing the analytical evaluation of the integrals comes from the denominator of the integrands which consist of the following three combinations of hyperbolic functions

$$\frac{1}{(\sinh^2 x - x^2) \sinh x} \quad (2.31 \text{ a})$$

$$\frac{1}{\sinh^2 x - x^2}, \quad \frac{1}{\sinh x} \quad (2.31 \text{ b,c})$$

However their asymptotic expansions, which are simple exponential functions, make it possible to evaluate the asymptotic forms of the integrands analytically. Then the following scheme is applied to the infinite integral.

$$\int_0^\infty F(x) dx = \int_0^\infty [F(x) - F_\infty(x)] dx + \int_0^\infty F_\infty(x) dx \quad (2.32)$$

where $F_\infty(x)$ is the asymptotic form of $F(x)$. The integral on the far right of (2.32) is evaluated analytically, and the integral to its left must still be evaluated numerically but its rate of convergence is faster than the original integral on the far left of (2.32). With this scheme (2.28 a,b,c) become

$$U_x = \frac{1}{8\pi\mu} \left[\left(\frac{1}{r} + \frac{x^2}{r^3} \right) + \int_0^\infty [W_x - W_{x,\infty}] dq - \int_0^\infty W_{x,\infty} dq \right] \quad (2.33 \text{ a})$$

$$U_y = \frac{1}{8\pi\mu} \left[\left(\frac{xy}{r^3} \right) + \int_0^\infty [W_y - W_{y,\infty}] dq - \int_0^\infty W_{y,\infty} dq \right] \quad (2.33 \text{ b})$$

$$U_z = \frac{1}{8\pi\mu} \left[\left(\frac{xz}{r^3} \right) + \int_0^\infty [W_z - W_{z,\infty}] dq - \int_0^\infty W_{z,\infty} dq \right] \quad (2.33 \text{ c})$$

Due to their excessive length, the expressions of $W_{x,\infty}$, $W_{y,\infty}$ and $W_{z,\infty}$ and their integrations from zero to infinity are not listed here but can be provided upon request. It should be noted that the integrands of the numerical evaluation in (2.33 a,b,c) must be computed by their Taylor series for small values x and by their asymptotic expressions for large values x to avoid the large round-off errors and machine overflows respectively. The fundamental solution expressed in this form is referred to as the accelerated solution or the exact solution.

Another approach for speeding up evaluation of the supplemental solution is to curve fit the denominators (2.31) with simpler functions which make analytic evaluation of the integrals possible. This can be accomplished using products of monomials and exponentials as the basis functions. After a number of trials, the following forms were found which meet the required form with the smallest errors.

$$\frac{x^5}{(\sinh^2 x - x^2) \sinh x} = \frac{\exp(-3x)(3+8x^5) + \exp(-4x)}{(A_1 x + B_1 x^2 + C_1 x^3 + D_1 x^4)} \quad (2.34 \text{ a})$$

$$\frac{x^4}{\sinh^2 x - x^2} = \frac{\exp(-2x)(3+4x^4) + \exp(-3x)}{(A_2 x + B_2 x^2 + C_2 x^3)} \quad (2.34 \text{ b})$$

$$\frac{x}{\sinh x} = \frac{\exp(-x)(1+2x) + \exp(-1.25x)}{(A_3 x + B_3 x^2 + C_3 x^3)} \quad (2.34 \text{ c})$$

The first terms without the unknown constants A_i , B_i , C_i , D_i on the right-hand side of (2.34) consist of the leading terms of the Taylor's expansions and the asymptotic expansions of the functions to be approximated. The terms which contain those unknown constants on the right-hand side of (2.34) are the proper products of monomials and exponentials required to offset the errors produced by the first terms for the entire region. Using the least-squares method, the fitting constants were obtained and the maximum relative error of these approximate expressions is 1.3%, as shown in Table 1. By substituting the approximate forms for the denominators in the integrands, all the infinite integrals are reduced to one of the following forms and can be evaluated analytically with the formulas found in Gradshteyn & Ryzhik (1980).

$$\int_0^{\infty} \exp(-\alpha x) J_n(\beta x) x^{m+1} dx = (-1)^{m+1} \frac{d^{m+1}}{d\alpha^{m+1}} \left[\frac{1}{(\alpha^2 + \beta^2)^{1/2}} \right] \quad (2.35)$$

$n=0, m \geq -1; \quad n=1, m \geq -2$

Table 1 a,b,c Curve fitting constants and comparison between the exact and their approximate expressions for equations (2.34)

$$A_1 = 8.548222 \quad A_2 = 6.805372 \quad A_3 = -1.107158$$

$$B_1 = 26.853589 \quad B_2 = 6.276871 \quad B_3 = -0.288413$$

$$C_1 = 10.290846 \quad C_2 = 20.877834 \quad C_3 = -0.456605$$

$$D_1 = 46.536141$$

$$f_1(x) = x / \sinh(x), \quad p_1(x) = \text{approximation.}$$

$$f_2(x) = x^4 / (\sinh(x)^2 - x^2), \quad p_2(x) = \text{approximation.}$$

$$f_3(x) = x^5 / [(\sinh(x)^2 - x^2) \sinh(x)], \quad p_3(x) = \text{approximation.}$$

(a)

x	$f_1(x)$	$p_1(x)$	$\frac{ f_1(x) - p_1(x) }{f_1(x)} \times 100\%$
0.03	0.9998E+00	0.9991E+00	0.07%
0.30	0.9851E+00	0.9835E+00	0.16%
0.90	0.8767E+00	0.8779E+00	0.14%
9.00	0.2221E-02	0.2220E-02	0.05%
15.0	0.9177E-05	0.9179E-05	0.02%
27.0	0.1015E-09	0.1027E-09	1.18%
30.0	0.5615E-11	0.5687E-11	1.28%
36.0	0.1670E-13	0.1691E-13	1.25%
42.0	0.4829E-16	0.4885E-16	1.16%
51.0	0.7237E-20	0.7307E-20	0.97%
54.0	0.3815E-21	0.3850E-21	0.92%

(b)

x	$f_2(x)$	$p_2(x)$	$\frac{ f_2(x)-p_2(x) }{f_2(x)} \times 100\%$
0.03	0.2996E+01	0.3017E+01	0.60%
0.30	0.2964E+01	0.2953E+01	0.37%
0.90	0.2692E+01	0.2706E+01	0.52%
9.00	0.3997E-03	0.3998E-03	0.02%
15.0	0.1895E-07	0.1895E-07	0.00%
27.0	0.7509E-17	0.7509E-17	0.00%

(c)

x	$f_3(x)$	$p_3(x)$	$\frac{ f_3(x)-p_3(x) }{f_3(x)} \times 100\%$
0.03	0.2999E+01	0.2991E+01	0.27%
0.30	0.2920E+01	0.2925E+01	0.17%
0.90	0.2360E+01	0.2363E+01	0.12%
9.00	0.8878E-06	0.8879E-06	0.01%
15.0	0.1605E-12	0.1605E-12	0.00%
27.0	0.7622E-27	0.7622E-27	0.00%

For the stokeslet pointing perpendicular to the planes

$$W_p = 0.5 [b*RG_3(1,2,2) - c*RG_3(2,2,2) + b*RG_1(1,2,2) + RG_1(1,2,1) - c*RG_1(2,2,2) - RG_1(2,2,1)] \quad (2.36 \text{ a})$$

$$W_z = -0.5 [b*RG_2(1,1,2) + c*RG_2(2,1,2) + b*RG_4(1,1,2) + RG_4(1,1,1) + c*RG_4(2,1,2) + RG_4(2,1,1)] \quad (2.36 \text{ b})$$

For the stokeslet pointing parallel to the planes

$$\begin{aligned} W_x = & \left[\left(\frac{r_1}{\rho} \right)^2 - 2 \right] [RG_5(1,1,1) + RG_5(2,1,1)] + \left(\frac{r_1}{\rho} \right)^2 [b*RG_5(1,1,2) + c*RG_5(2,1,2)] \\ & + \frac{r_2^2 - r_1^2}{\rho^3} [RG_5(1,2,3) + b*RG_5(1,2,1) + RG_5(2,2,3) + c*RG_5(2,2,1)] \\ & + \left(\frac{r_1}{\rho} \right)^2 [b*RG_6(1,1,2) - RG_6(1,1,1) + c*RG_6(2,1,2) - RG_6(2,1,1)] \\ & + \frac{r_2^2 - r_1^2}{\rho^3} [b*RG_6(1,2,1) - RG_6(1,2,3) + c*RG_6(2,2,1) - RG_6(2,2,3)] \\ & + \left(\frac{r_1}{\rho} \right)^2 [b*RG_1(1,1,2) + c*RG_1(2,1,2)] + \frac{r_2^2 - r_1^2}{\rho^3} [b*RG_1(1,2,1) + c*RG_1(2,2,1)] \end{aligned} \quad (2.37 \text{ a})$$

$$\begin{aligned} W_y = & -\frac{2r_1r_2}{\rho^2} [RG_6(1,1,1) + RG_6(2,1,1) - [RG_6(1,2,3) + RG_6(2,2,3)] \frac{2}{\rho}] \\ & + \frac{r_1r_2}{\rho^2} [RG_3(1,1,1) + b*RG_3(1,1,2) + RG_3(2,1,1) + c*RG_3(2,1,2)] \\ & - \frac{2r_1r_2}{\rho^3} [RG_3(1,2,3) + b*RG_3(1,2,1) + RG_3(2,2,3) + c*RG_3(2,2,1)] \\ & + \frac{r_1r_2}{\rho^2} [RG_1(1,1,2) + RG_1(2,1,2) - [RG_1(1,2,1) + RG_1(2,2,1)] \frac{2}{\rho}] \end{aligned} \quad (2.37 \text{ b})$$

$$W_z = \frac{r_1}{\rho} [b * RG_2(1,2,2) - RG_2(1,2,1) - c * RG_2(2,2,2) + RG_2(2,2,1)] \\ + \frac{r_1}{\rho} [b * RG_4(1,2,2) - c * RG_4(2,2,2)] \quad (2.37 \text{ c})$$

$$\rho = (r_1^2 + r_2^2)^{1/2} \quad (2.37 \text{ d})$$

where r_1 and r_2 are the distances between the stokeslet and the field point along the directions of x-axis and y-axis respectively. The expressions of $RG_n(i, j, k)$, where $n=1, 2, 3, 4, 5, 6$ and $i, j=1, 2; k=1, 2, 3$, are the analytical evaluations of the involved infinite integrals. $RG_n(1, j, k)$ are listed in Appendix A. $RG_n(2, j, k)$ can be obtained by replacing b by c , c by b and z by $-z$ in $RG_n(1, j, k)$. The fundamental solution obtained in this way is referred to as the approximate solution. The approximate solution provides almost the same results as the accelerated one at a small fraction of the computational effort. Table 2a presents a comparison of the correction terms between these two solutions at some pairs of random points, where the accelerated one serves as the exact solution. The results show that the maximum relative error is 1.2%. The maximum error occurs for the components with directions of both force and velocity parallel to the planes when the second wall is moved far away. Table 2b presents a comparison of the correction terms between the approximate solution and Blake's (1971) solution and shows that the maximum error is also 1.2% when the stokeslet is far removed from one of the walls.

Table 2 a Comparison of the correction terms of the stokeslet components at certain pairs of points, where the accelerated solution is used as the exact one.

(a) Correction term of the accelerated solution.

(b) Correction term of the approximate solution.

		$x_1, x_2, x_3 / y_1, y_2, y_3$	$x_1, x_2, x_3 / y_1, y_2, y_3$	$x_1, x_2, x_3 / y_1, y_2, y_3$
		x: 0.01,0.03,0.02	x: 0.01,0.03,0.5	x: 0.01,0.03,1.2
		y: 0.0,0.0,0.01	y: 0.0,0.0,0.3	y: 0.0,0.0,1.0
W_{11}	a	-0.70741E+01	-0.48138E+00	-0.26934E+00
	b	-0.72037E+01	-0.48705E+00	-0.26951E+00
W_{12}	a	-0.33336E+00	0.60154E-04	0.73792E-05
	b	-0.33363E+00	0.60104E-04	0.73911E-05
W_{13}	a	0.27008E+00	0.44525E-02	0.24693E-03
	b	0.27008E+00	0.44546E-02	0.24762E-03
W_{21}	a	-0.33363E+00	0.60154E-04	0.73792E-05
	b	-0.33363E+00	0.60104E-04	0.73911E-05
W_{22}	a	-0.79538E+01	-0.48123E+00	-0.26932E+00
	b	-0.80987E+01	-0.48690E+00	-0.26949E+00
W_{23}	a	0.81026E+00	0.13357E-01	0.74079E-03
	b	0.81026E+00	0.13364E-01	0.74286E-03
W_{31}	a	-0.87381E+00	-0.64201E-02	-0.42540E-03
	b	-0.87381E+00	-0.64245E-02	-0.42634E-03
W_{32}	a	-0.26214E+01	-0.19260E-01	-0.12762E-02
	b	-0.26214E+01	-0.19273E-01	-0.12790E-02
W_{33}	a	-0.89605E+01	-0.91708E+00	-0.39522E+00
	b	-0.89619E+01	-0.91814E+00	-0.39558E+00

Table 2 b Comparison of the correction terms of the stokeslet components at one pair of points, where the Blake's solution is used as the exact one.

(a) Correction term of the Blake's solution.

(b) Correction term of the approximate solution.

		x: 0.01,0.03,0.02; y: 0.0,0.0,0.01
W_{11}	a	-0.70741E+01
	b	-0.72037E+01
W_{12}	a	-0.33336E+00
	b	-0.33363E+00
W_{13}	a	0.27008E+00
	b	0.27008E+00
W_{21}	a	-0.33363E+00
	b	-0.33363E+00
W_{22}	a	-0.79538E+01
	b	-0.80987E+01
W_{23}	a	0.81026E+00
	b	0.81026E+00
W_{31}	a	-0.87381E+00
	b	-0.87381E+00
W_{32}	a	-0.26214E+01
	b	-0.26214E+01
W_{33}	a	-0.89605E+01
	b	-0.89619E+01

Both forms of the solution are also convenient in form, (the free space stokeslet + the finite correction terms). This feature brings a tremendous simplification for handling the singular elements during the application of the boundary integral method because the analytical evaluation procedure of the free space stokeslet over the singular elements developed by Youngren & Acrivos (1975) can be used directly. Since the approximate solution is in closed form it will be used in the next chapter as the fundamental solution to treat the motion of an arbitrary body between two walls with the boundary integral method.

2.3 Concluding remarks

The discussion in section 2.2 not only provides the fundamental solution between two parallel planes in closed form with a maximum relative error of 1.2% but also illustrates a general approach which can be used to develop the fundamental solution in geometries containing infinite confining boundaries. In this approach the fundamental solution is written in the form of a free space stokeslet plus the finite correction terms, which is a very convenient form for the application of the boundary integral method. From the point of view of this form, the singular terms are contained in the free space stokeslet and are the same regardless of the boundary geometry. The supplemental solution or the correction terms in (2.1) usually can be written in the form of an infinite integral or infinite series, which is used to cancel the disturbance produced by the free space stokeslet and felt by the infinite confining boundaries. Since the free space stokeslet is very simple in any coordinate system, the inverting transform, which must be performed

to determine the unknown functions of the separation variables in the supplemental solution, usually can be evaluated analytically. The next crucial procedure is to analytically evaluate the integral or infinite series of the supplemental solution, which may include some approximations as in the present chapter, and to get desired closed form of the fundamental solution. Although such a scheme cannot be specified in general, the fundamental solution inside an infinite cylinder is a likely candidate for treatment by this approach. Finally the accuracy of the fundamental solution obtained by this approach only depends on accuracy of the approximations used in the procedure.

Appendix A

This appendix contains a listing of the RG_n (i, j, k) functions appearing in (2.36-37)

$$\begin{aligned}
 RG_1(1,1,2) = & \left\{ 2(b+c)(z-c) \cdot \frac{3(2b+2c-z) \cdot [2(2b+2c-z)^2 - 3p^2]}{[(2b+2c-z)^2 + p^2]^{7/2}} - (z+b) \cdot \right. \\
 & \frac{[2(2b+2c-z)^2 - p^2]}{[(2b+2c-z)^2 + p^2]^{5/2}} - 2(b+c)(z-c) \cdot \frac{3(4b+2c+z) \cdot [2(4b+2c+z)^2 - 3p^2]}{[(4b+2c+z)^2 + p^2]^{7/2}} \\
 & - (z+b) \frac{[2(4b+2c+z)^2 - p^2]}{[(4b+2c+z)^2 + p^2]^{5/2}} + (z+b) \cdot \left. \frac{[2(2b+2c-z)^2 - p^2]}{[(2b+2c-z)^2 + p^2]^{5/2}} \right\} + \left\{ - \frac{3(z-c)}{2(b+c)^3} \cdot \ln \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2} \right] \right. \\
 & - \frac{3(z+b)}{4(b+c)^4} \left[(2b+2c-z) \cdot \ln \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2} \right] \cdot \sqrt{(2b+2c-z)^2 + p^2} \right. \\
 & + \frac{3(z-c)}{2(b+c)^3} \cdot \ln \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2} \right] - \frac{3(z+b)}{4(b+c)^4} \cdot \left. \left[(4b+2c+z) \cdot \right. \right. \\
 & \ln \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2} \right] - \sqrt{(4b+2c+z)^2 + p^2} \left. \right] + \frac{3(z+b)}{4(b+c)^4} \left[(2b+2c-z) \cdot \right. \\
 & \ln \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2} \right] - \sqrt{(2b+2c-z)^2 + p^2} \left. \right] + (4b+4c-z) \cdot \ln \left[(4b+4c-z) + \right. \\
 & \left. \sqrt{(4b+4c-z)^2 + p^2} \right] - \sqrt{(4b+4c-z)^2 + p^2} \left. \right\} + \left\{ \frac{A_2(z-c)}{2(b+c)^3} \cdot \frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} \right. \\
 & + \frac{A_2(z+b)}{4(b+c)^3} \cdot \ln \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2} \right] - \frac{A_2(z+b)}{4(b+c)^3} \cdot \left[\ln \left[(3b+ \right. \right. \\
 & \left. \left. 3c+z) + \sqrt{(3b+3c+z)^2 + p^2} \right] + \ln \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2} \right] \right] \left. \right\} + \\
 & \left\{ \frac{B_2(z-c)}{2(b+c)} \cdot \frac{(3b+3c-z)}{[(3b+3c-z)^2 + p^2]^{3/2}} - \frac{B_2(z+b)}{4(b+c)^2} \cdot \frac{1}{[(3b+3c-z)^2 + p^2]^{1/2}} - \right. \\
 & \frac{B_2(z-c)}{2(b+c)} \cdot \frac{(5b+3c+z)}{[(5b+3c+z)^2 + p^2]^{3/2}} - \frac{B_2(z+b)}{4(b+c)^2} \cdot \frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} + \frac{B_2(z+b)}{4(b+c)^2} \\
 & \cdot \left. \left[\frac{1}{[(3b+3c+z)^2 + p^2]^{1/2}} + \frac{1}{[(5b+5c-z)^2 + p^2]^{1/2}} \right] \right\} + \left\{ \frac{C_2(z-c)}{2} \cdot \frac{2(3b+3c-z)^2 - p^2}{[(3b+3c-z)^2 + p^2]^{5/2}} \right. \\
 & - \frac{C_2(z+b)}{4(b+c)} \cdot \frac{(3b+3c-z)}{[(3b+3c-z)^2 + p^2]^{3/2}} - \frac{C_2(z-c)}{2} \cdot \frac{2(5b+3c+z)^2 - p^2}{[(5b+3c+z)^2 + p^2]^{5/2}} - \\
 & \left. \frac{C_2(z+b)}{4(b+c)} \cdot \frac{(5b+3c+z)}{[(5b+3c+z)^2 + p^2]^{3/2}} + \frac{C_2(z+b)}{4(b+c)} \left[\frac{(3b+3c+z)}{[(3b+3c+z)^2 + p^2]^{3/2}} + \frac{(5b+5c-z)}{[(5b+5c-z)^2 + p^2]^{3/2}} \right] \right\} \\
 & \quad \quad \quad (A-1)
 \end{aligned}$$

$$\begin{aligned}
R_{G_4}(1,1,1) = & \left\{ 2(b+c)(2-c) \left[\frac{2(2b+2c-z)^2 - p^2}{[(2b+2c-z)^2 + p^2]^{5/2}} + \frac{2(4b+2c+z)^2 - p^2}{[(4b+2c+z)^2 + p^2]^{5/2}} \right] \right. \\
& + \left[\frac{1}{[(2b+z)^2 + p^2]^{1/2}} + \frac{1}{[(4b+4c-z)^2 + p^2]^{1/2}} - \frac{1}{[(2b+2c-z)^2 + p^2]^{1/2}} - \right. \\
& \left. \left. \frac{1}{[(4b+2c+z)^2 + p^2]^{1/2}} \right] + (b+2c-z) \cdot \left[\frac{(4b+2c+z)}{[(4b+2c+z)^2 + p^2]^{3/2}} - \frac{(2b+2c-z)}{[(2b+2c-z)^2 + p^2]^{3/2}} \right] \right. \\
& + (2+b) \cdot \left[\frac{(2b+z)}{[(2b+z)^2 + p^2]^{3/2}} - \frac{(4b+4c-z)}{[(4b+4c-z)^2 + p^2]^{3/2}} \right] + \left\{ \frac{3(2-c)}{2(b+c)^3} [(2b+2c-z) \cdot \right. \\
& \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2}] - \sqrt{(2b+2c-z)^2 + p^2} + (4b+2c+z) \cdot \ln[\\
& (4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2}] - \sqrt{(4b+2c+z)^2 + p^2} + \frac{3}{4(b+c)^4} \left[\left(\frac{(2b+z)^3}{6} - \right. \right. \\
& \left. \left. \frac{p^2}{4} (2b+z) \right) \cdot \ln[(2b+z) + \sqrt{(2b+z)^2 + p^2}] + \left[\frac{p^2}{4} - \frac{(2b+z)^2}{6} \right] \cdot \sqrt{(2b+z)^2 + p^2} \right. \\
& - \frac{5}{36} [(2b+z)^2 + p^2]^{3/2} \left[\frac{(4b+4c-z)^3}{6} - \frac{p^2(4b+4c-z)}{4} \right] \cdot \ln[(4b+4c-z) + \\
& \sqrt{(4b+4c-z)^2 + p^2}] + \left[\frac{p^2}{4} - \frac{(4b+4c-z)^2}{6} \right] \cdot \sqrt{(4b+4c-z)^2 + p^2} - \frac{5}{36} [(4b+4c \\
& - z)^2 + p^2]^{3/2} - \left[\frac{(2b+2c-z)^3}{6} - \frac{p^2(2b+2c-z)}{4} \right] \cdot \ln[(2b+2c-z) + \\
& \sqrt{(2b+2c-z)^2 + p^2}] - \left[\frac{p^2}{4} - \frac{(2b+2c-z)^2}{6} \right] \cdot \sqrt{(2b+2c-z)^2 + p^2} + \frac{5}{36} [(2b+ \\
& 2c-z)^2 + p^2]^{3/2} \left[\frac{(4b+2c+z)^3}{6} - \frac{p^2(4b+2c+z)}{4} \right] \cdot \ln[(4b+2c+z) + \\
& \sqrt{(4b+2c+z)^2 + p^2}] - \left[\frac{p^2}{4} - \frac{(4b+2c+z)^2}{6} \right] \cdot \sqrt{(4b+2c+z)^2 + p^2} + \frac{5}{36} [(4b+ \\
& 2c+z)^2 + p^2]^{3/2} \left. \right\} + \frac{3(2+b)}{4(b+c)^4} \left[\left[\frac{p^2}{4} - \frac{(2b+z)^2}{6} \right] \cdot \ln[(2b+z) + \sqrt{(2b+z)^2 + p^2}] \right. \\
& + \frac{3}{4} (2b+z) \cdot \sqrt{(2b+z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(4b+4c-z)^2}{6} \right] \cdot \ln[(4b+4c-z) + \\
& \sqrt{(4b+4c-z)^2 + p^2}] - \frac{3}{4} (4b+4c-z) \cdot \sqrt{(4b+4c-z)^2 + p^2} + \frac{3(2b+2c-z)}{4(b+c)^4} \left[\left[\frac{p^2}{4} - \right. \right. \\
& \left. \left. \frac{(4b+2c+z)^2}{6} \right] \cdot \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2}] + \frac{3}{4} (4b+2c+z) \cdot \right. \\
& \sqrt{(4b+2c+z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(2b+2c-z)^2}{6} \right] \cdot \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2}] - \\
& \left. \left. \frac{3}{4} (2b+2c-z) \cdot \sqrt{(2b+2c-z)^2 + p^2} \right] \right\} + \left\{ \frac{A_2(2-c)}{2(b+c)^3} [-\ln[(3b+3c-z) + \right. \right. \\
& \left. \left. \sqrt{(3b+3c-z)^2 + p^2}] - \ln[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] \right] + \frac{A_2}{4(b+c)^3} \left[\frac{p^2}{4} - \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left. \frac{(3b+c+2)^2}{2} \right] \cdot \ln \left[(3b+c+2) + \sqrt{(3b+c+2)^2 + p^2} \right] + \frac{3}{4} (3b+c+2) \cdot \sqrt{(3b+c+2)^2 + p^2} \\
& + \left[\frac{p^2}{4} - \frac{(5b+5c-2)^2}{2} \right] \cdot \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2} \right] + \frac{3}{4} (5b+5c-2) \\
& \cdot \sqrt{(5b+5c-2)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(3b+3c-2)^2}{2} \right] \cdot \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] \\
& - \frac{3}{4} (3b+3c-2) \cdot \sqrt{(3b+3c-2)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(5b+3c+2)^2}{2} \right] \cdot \ln \left[(5b+3c+2) + \right. \\
& \left. \sqrt{(5b+3c+2)^2 + p^2} \right] - \frac{3}{4} (5b+3c+2) \cdot \sqrt{(5b+3c+2)^2 + p^2} + \frac{A_2(2+b)}{4(b+c)^3} \left[\right. \\
& (3b+c+2) \cdot \ln \left[(3b+c+2) + \sqrt{(3b+c+2)^2 + p^2} \right] - \sqrt{(3b+c+2)^2 + p^2} - (5b+5c- \\
& 2) \cdot \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2} \right] + \sqrt{(5b+5c-2)^2 + p^2} + \frac{A_2(b+2c-2)}{4(b+c)^3} \cdot \\
& \left. \left[(5b+3c+2) \cdot \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] - \sqrt{(5b+3c+2)^2 + p^2} - (3b+3c-2) \cdot \right. \right. \\
& \left. \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] + \sqrt{(3b+3c-2)^2 + p^2} \right] \left. \right\} + \left\{ \frac{B_2(2-c)}{2(b+c)} \cdot \right. \\
& \left[\frac{1}{[(3b+3c-2)^2 + p^2]^{\frac{1}{2}}} + \frac{1}{[(5b+3c+2)^2 + p^2]^{\frac{1}{2}}} \right] + \frac{B_2}{4(b+c)^2} \left[(3b+c+2) \cdot \ln \left[(3b+c+2) \right. \right. \\
& \left. \left. + \sqrt{(3b+c+2)^2 + p^2} \right] - \sqrt{(3b+c+2)^2 + p^2} + (5b+5c-2) \cdot \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2} \right] \right. \\
& \left. - \sqrt{(5b+5c-2)^2 + p^2} - (3b+3c-2) \cdot \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] + \right. \\
& \left. \sqrt{(3b+3c-2)^2 + p^2} - (5b+3c+2) \cdot \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] + \right. \\
& \left. \sqrt{(5b+3c+2)^2 + p^2} \right] + \frac{B_2(2+b)}{4(b+c)^2} \left[-\ln \left[(3b+c+2) + \sqrt{(3b+c+2)^2 + p^2} \right] + \ln \left[(5b+ \right. \\
& \left. 5c-2) + \sqrt{(5b+5c-2)^2 + p^2} \right] \right] + \frac{B_2(b+2c-2)}{4(b+c)^2} \left[-\ln \left[(5b+3c+2) + \right. \right. \\
& \left. \left. \sqrt{(5b+3c+2)^2 + p^2} \right] + \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] \right] \left. \right\} + \left\{ \frac{C_2(2-c)}{2} \left[\right. \right. \\
& \frac{(3b+3c-2)}{[(3b+3c-2)^2 + p^2]^{\frac{3}{2}}} + \frac{(5b+3c+2)}{[(5b+3c+2)^2 + p^2]^{\frac{3}{2}}} \left. \right] + \frac{C_2}{4(b+c)} \left[-\ln \left[(3b+c+2) + \right. \right. \\
& \left. \left. \sqrt{(3b+c+2)^2 + p^2} \right] - \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2} \right] + \ln \left[(3b+3c-2) + \right. \right. \\
& \left. \left. \sqrt{(3b+3c-2)^2 + p^2} \right] + \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] \right] + \frac{C_2(2+b)}{4(b+c)} \left[\right. \\
& \left. \frac{1}{[(3b+c+2)^2 + p^2]^{\frac{1}{2}}} - \frac{1}{[(5b+5c-2)^2 + p^2]^{\frac{1}{2}}} \right] + \frac{C_2(b+2c-2)}{4(b+c)} \left[\frac{1}{[(5b+3c+2)^2 + p^2]^{\frac{1}{2}}} \right.
\end{aligned}$$

$$\begin{aligned}
& - \left[\frac{(3b+3c-z)^2 \rho^2}{2} \right]^{1/2} \Big\} , , \quad (A-2) \\
& R G_4(1,1,2) = \left\{ 2(4c)(z-c) \cdot \left[\frac{3(2b+2c-z) \cdot [2(2b+2c-z)-3\rho^2]}{(2b+2c-z)^2 + \rho^2} \right]^{1/2} + \right. \\
& \quad \left[\frac{3(4b+2c+z) \cdot [2(4b+2c+z)-3\rho^2]}{(4b+2c+z)^2 + \rho^2} \right]^{1/2} + \left[\frac{(2b+z)}{(2b+2c-z)^2 + \rho^2} \right]^{3/2} + \frac{(4b+4c-z)}{[(4b+4c-z)^2 + \rho^2]^{3/2}} \\
& \quad - \frac{(2b+2c-z)}{(2b+2c-z)^2 + \rho^2} \cdot \frac{(4b+2c+z)}{[(4b+2c+z)^2 + \rho^2]^{3/2}} + \frac{(b+2c-z)}{[(4b+2c+z)^2 + \rho^2]^{3/2}} \cdot \frac{[2(4b+2c+z)-\rho^2]}{[(4b+2c+z)^2 + \rho^2]^{5/2}} \\
& \quad - \frac{2(2b+2c-z)^2 - \rho^2}{[(2b+z)^2 + \rho^2]^{5/2}} + (z+c) \cdot \left[\frac{2(2b+z)^2 - \rho^2}{[(2b+z)^2 + \rho^2]^{5/2}} - \frac{2(4b+4c-z)^2 - \rho^2}{[(4b+4c-z)^2 + \rho^2]^{5/2}} \right] + \\
& \quad \left\{ \frac{3(2c-c)}{2(b+c)^3} \left[-1/h \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + \rho^2} \right] - 1/h \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + \rho^2} \right] \right] \right. \\
& \quad + \frac{3}{4(b+c)^4} \left[\left\{ \frac{\rho^2}{4} - \frac{(2b+z)^2}{2} \right\} \cdot 1/h \left[(2b+z) + \sqrt{(2b+z)^2 + \rho^2} \right] + \frac{3}{4} (2b+z) \sqrt{(2b+z)^2 + \rho^2} \right. \\
& \quad \left. \left. + \left\{ \frac{\rho^2}{4} - \frac{(4b+4c-z)^2}{2} \right\} \cdot 1/h \left[(4b+4c-z) + \sqrt{(4b+4c-z)^2 + \rho^2} \right] + \frac{3}{4} (4b+4c-z) \cdot \right. \right. \\
& \quad \left. \left. \sqrt{(4b+4c-z)^2 + \rho^2} - \left\{ \frac{\rho^2}{4} - \frac{(2b+2c-z)^2}{2} \right\} \cdot 1/h \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + \rho^2} \right] - \right. \right. \\
& \quad \left. \left. \frac{3}{4} (2b+2c-z) \cdot \sqrt{(2b+2c-z)^2 + \rho^2} - \left\{ \frac{\rho^2}{4} - \frac{(4b+2c+z)^2}{2} \right\} \cdot 1/h \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + \rho^2} \right] + \right. \right. \\
& \quad \left. \left. \sqrt{(4b+2c+z)^2 + \rho^2} \right] - \frac{3}{4} (4b+2c+z) \cdot \sqrt{(4b+2c+z)^2 + \rho^2} + \frac{3(2b+b)}{4(b+c)^4} (2b+z) \cdot \right. \\
& \quad \left. 1/h \left[(2b+z) + \sqrt{(2b+z)^2 + \rho^2} \right] - \sqrt{(2b+z)^2 + \rho^2} - (4b+4c-z) \cdot 1/h \left[(4b+4c-z) + \sqrt{(4b+4c-z)^2 + \rho^2} \right] \right. \\
& \quad \left. \sqrt{(4b+4c-z)^2 + \rho^2} + \sqrt{(4b+4c-z)^2 + \rho^2} \right] + \frac{3(b+2c-z)}{4(b+c)^4} \left[(4b+2c+z) \cdot 1/h \left[\right. \right. \\
& \quad \left. \left. (4b+2c+z) + \sqrt{(4b+2c+z)^2 + \rho^2} \right] - \sqrt{(4b+2c+z)^2 + \rho^2} - (2b+2c-z) \cdot 1/h \left[(2b+2c \right. \right. \\
& \quad \left. \left. - z) + \sqrt{(2b+2c-z)^2 + \rho^2} \right] + \sqrt{(2b+2c-z)^2 + \rho^2} \right] + \left\{ \frac{A_2(z-c)}{2(b+c)^2} \left[\frac{1}{[(3b+3c-z)^2 + \rho^2]^{3/2}} \right. \right. \\
& \quad \left. \left. + \frac{1}{[(5b+3c+z)^2 + \rho^2]^{3/2}} \right] + \frac{A_2}{4(b+c)^3} \left[(3b+4c+z) \cdot 1/h \left[(3b+4c+z) + \sqrt{(3b+4c+z)^2 + \rho^2} \right] - \right. \right. \\
& \quad \left. \left. \sqrt{(3b+4c+z)^2 + \rho^2} + (5b+5c-z) \cdot 1/h \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + \rho^2} \right] - \right. \right. \\
& \quad \left. \left. \sqrt{(5b+5c-z)^2 + \rho^2} - (3b+3c-z) \cdot 1/h \left[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + \rho^2} \right] + \right. \right. \\
& \quad \left. \left. \sqrt{(3b+3c-z)^2 + \rho^2} - (5b+3c+z) \cdot 1/h \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + \rho^2} \right] + \right. \right. \\
& \quad \left. \left. \sqrt{(5b+3c+z)^2 + \rho^2} + \frac{A_2(2b+b)}{4(b+c)^3} \left[-1/h \left[(3b+c+z) + \sqrt{(3b+c+z)^2 + \rho^2} \right] + \right. \right. \\
& \quad \left. \left. \sqrt{(3b+c+z)^2 + \rho^2} \right] + \right.
\end{aligned}$$

$$\begin{aligned}
& \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] + \frac{A_2(b+2c-z)}{4(b+c)^3} \left[-\ln[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] \right. \\
& \left. + \ln[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] \right] + \left\{ \frac{B_2(z-c)}{2(b+c)} \left[\frac{(3b+3c-z)}{[(3b+3c-z)^2 + p^2]^{3/2}} + \frac{(5b+3c+z)}{[(5b+3c+z)^2 + p^2]^{3/2}} \right] \right. \\
& \left. + \frac{B_2}{4(b+c)^2} \left[-\ln[(3b+c+z) + \sqrt{(3b+c+z)^2 + p^2}] - \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] + \ln[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] \right. \right. \\
& \left. \left. + \ln[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] \right] + \frac{B_2(z+b)}{4(b+c)^2} \left[\frac{1}{[(3b+c+z)^2 + p^2]^{1/2}} - \frac{1}{[(5b+5c-z)^2 + p^2]^{1/2}} \right] \right. \\
& \left. + \frac{B_2(b+2c-z)}{4(b+c)^2} \left[\frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} - \frac{1}{[(3b+3c-z)^2 + p^2]^{1/2}} \right] \right\} + \left\{ \frac{C_2(z-c)}{2} \left[\frac{2(3b+3c-z)^2 - p^2}{[(3b+3c-z)^2 + p^2]^{5/2}} + \frac{2(5b+3c+z)^2 - p^2}{[(5b+3c+z)^2 + p^2]^{5/2}} \right] \right. \\
& \left. + \frac{C_2}{4(b+c)} \left[\frac{1}{[(3b+c+z)^2 + p^2]^{1/2}} + \frac{1}{[(5b+5c-z)^2 + p^2]^{1/2}} - \frac{1}{[(3b+3c-z)^2 + p^2]^{1/2}} - \frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} \right] \right. \\
& \left. + \frac{C_2(z+b)}{4(b+c)} \left[\frac{(3b+c+z)}{[(3b+c+z)^2 + p^2]^{3/2}} - \frac{(5b+5c-z)}{[(5b+5c-z)^2 + p^2]^{3/2}} \right] \right. \\
& \left. + \frac{C_2(b+2c-z)}{4(b+c)} \left[\frac{(5b+3c+z)}{[(5b+3c+z)^2 + p^2]^{3/2}} - \frac{(3b+3c-z)}{[(3b+3c-z)^2 + p^2]^{3/2}} \right] \right\} // (A-3) \\
& R(G_3(1, 2, 2)) = \left\{ 2(b+c)(z-c) \left[\frac{[12(2b+2c-z)^2 - 3p^2] \cdot f}{[(2b+2c-z)^2 + p^2]^{7/2}} + \frac{[12(4b+2c+z)^2 - 3p^2] \cdot f}{[(4b+2c+z)^2 + p^2]^{7/2}} \right] \right. \\
& \left. + \left[\frac{f}{[(2b+z)^2 + p^2]^{3/2}} + \frac{f}{[(4b+4c-z)^2 + p^2]^{3/2}} - \frac{f}{[(2b+2c-z)^2 + p^2]^{3/2}} - \frac{f}{[(4b+2c+z)^2 + p^2]^{3/2}} \right] \right. \\
& \left. + 2(b+c) \cdot \left[\frac{3(2b+2c-z) \cdot f}{[(2b+2c-z)^2 + p^2]^{5/2}} - \frac{3(4b+2c+z) \cdot f}{[(4b+2c+z)^2 + p^2]^{5/2}} \right] \right. \\
& \left. - (z+b) \cdot \left[\frac{3(2b+2c-z) \cdot f}{[(2b+2c-z)^2 + p^2]^{5/2}} - \frac{3(4b+2c+z) \cdot f}{[(4b+2c+z)^2 + p^2]^{5/2}} + \frac{3 \cdot (2b+z) \cdot f}{[(2b+z)^2 + p^2]^{5/2}} - \frac{3(4b+4c-z) \cdot f}{[(4b+4c-z)^2 + p^2]^{5/2}} \right] \right\} + \left\{ \frac{3(z-c)}{2(b+c)^3} \left[-\frac{(2b+2c-z)}{f} + \frac{1}{f} \sqrt{(2b+2c-z)^2 + p^2} \right. \right. \\
& \left. \left. - \frac{(4b+2c+z)}{f} + \frac{1}{f} \sqrt{(4b+2c+z)^2 + p^2} \right] + \frac{3}{4(b+c)^4} \left[\frac{f}{2} (2b+z) \cdot \ln[(2b+z) + \sqrt{(2b+z)^2 + p^2}] \right. \right. \\
& \left. \left. - \frac{f}{2} \sqrt{(2b+z)^2 + p^2} - \frac{(2b+z)^3}{6 \cdot f} + \frac{[(2b+z)^2 + p^2]^{3/2}}{6 \cdot f} + \frac{f}{2} (4b+4c-z) \cdot \ln[(4b+4c-z) + \sqrt{(4b+4c-z)^2 + p^2}] \right. \right. \\
& \left. \left. - \frac{f}{2} \sqrt{(4b+4c-z)^2 + p^2} - \frac{(4b+4c-z)^3}{6 \cdot f} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{[(4b+4c-2)^2+p^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(2b+2c-2) \cdot \frac{1}{h} \left[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2} \right] + \\
& \frac{p}{2} \sqrt{(2b+2c-2)^2+p^2} + \frac{(2b+2c-2)^3}{6 \cdot p} - \frac{[(2b+2c-2)^2+p^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(4b+2c+2) \cdot \\
& \frac{1}{h} \left[(4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2} \right] + \frac{p}{2} \sqrt{(4b+2c+2)^2+p^2} + \frac{(4b+2c+2)^3}{6 \cdot p} - \\
& \frac{[(4b+2c+2)^2+p^2]^{3/2}}{6 \cdot p} \Bigg] + \frac{3}{2(b+c)^3} \left[- \frac{(2b+2c-2)}{2 \cdot p} \sqrt{(2b+2c-2)^2+p^2} - \frac{p}{2} \cdot \right. \\
& \left. \frac{1}{h} \left[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2} \right] + \frac{(2b+2c-2)}{2 \cdot p} + \frac{(4b+2c+2)}{2 \cdot p} \cdot \right. \\
& \left. \sqrt{(4b+2c+2)^2+p^2} + \frac{p}{2} \frac{1}{h} \left[(4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2} \right] - \frac{(4b+2c+2)^2}{2 \cdot p} \right] \\
& - \frac{3(2+b)}{4(b+c)^4} \left[- \frac{(2b+2c-2)}{2 \cdot p} \sqrt{(2b+2c-2)^2+p^2} - \frac{p}{2} \frac{1}{h} \left[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2} \right] \right. \\
& \left. + \frac{(2b+2c-2)^2}{2 \cdot p} + \frac{(4b+2c+2)}{2 \cdot p} \sqrt{(4b+2c+2)^2+p^2} + \frac{p}{2} \frac{1}{h} \left[(4b+2c+2) + \right. \right. \\
& \left. \left. \sqrt{(4b+2c+2)^2+p^2} \right] - \frac{(4b+2c+2)^2}{2 \cdot p} - \frac{(2b+2)}{2 \cdot p} \sqrt{(2b+2)^2+p^2} - \frac{p}{2} \frac{1}{h} \left[(2b+2) + \right. \right. \\
& \left. \left. \sqrt{(2b+2)^2+p^2} \right] + \frac{(2b+2)^2}{2 \cdot p} + \frac{(4b+4c-2)}{2 \cdot p} \sqrt{(4b+4c-2)^2+p^2} + \frac{p}{2} \frac{1}{h} \left[(4b+ \right. \right. \\
& \left. \left. 4c+2) + \sqrt{(4b+4c-2)^2+p^2} \right] - \frac{(4b+4c-2)^2}{2 \cdot p} \Bigg] + \left\{ \frac{A_2(2-c)}{2(b+c)^2} \left[\right. \right. \\
& \left. \frac{\sqrt{(3b+3c-2)^2+p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2+p^2]^{1/2}} + \frac{\sqrt{(5b+3c+2)^2+p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2+p^2]^{1/2}} \right\} + \frac{A_2}{4(b+c)^3} \cdot \\
& \left[- \frac{(3b+c+2)}{2 \cdot p} \sqrt{(3b+c+2)^2+p^2} - \frac{p}{2} \frac{1}{h} \left[(3b+c+2) + \sqrt{(3b+c+2)^2+p^2} \right] + \frac{(3b+c+2)^2}{2 \cdot p} \right. \\
& \left. - \frac{(5b+5c-2)}{2 \cdot p} \sqrt{(5b+5c-2)^2+p^2} - \frac{p}{2} \frac{1}{h} \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2+p^2} \right] + \right. \\
& \left. \frac{(5b+5c-2)^2}{2 \cdot p} + \frac{(3b+3c-2)}{2 \cdot p} \sqrt{(3b+3c-2)^2+p^2} + \frac{p}{2} \frac{1}{h} \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2+p^2} \right] \right. \\
& \left. - \frac{(3b+3c-2)^2}{2 \cdot p} + \frac{(5b+3c+2)}{2 \cdot p} \sqrt{(5b+3c+2)^2+p^2} + \frac{p}{2} \frac{1}{h} \left[(5b+3c+2) + \right. \right. \\
& \left. \left. \sqrt{(5b+3c+2)^2+p^2} \right] - \frac{(5b+3c+2)^2}{2 \cdot p} \right] + \frac{A_2}{2(b+c)^2} \left[- \frac{(3b+3c-2)}{p} + \frac{\sqrt{(3b+3c-2)^2+p^2}}{p} \right. \\
& \left. + \frac{(5b+3c+2)}{p} - \frac{1}{p} \sqrt{(5b+3c+2)^2+p^2} \right] - \frac{A_2(2+b)}{4(b+c)^3} \left[- \frac{(3b+3c-2)}{p} + \frac{\sqrt{(3b+3c-2)^2+p^2}}{p} \right. \\
& \left. + \frac{(5b+3c+2)}{p} - \frac{1}{p} \sqrt{(5b+3c+2)^2+p^2} \right] - \frac{(3b+c+2)}{p} + \frac{1}{p} \sqrt{(3b+c+2)^2+p^2} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{(5b+5c-z)}{p} - \frac{1}{p} \sqrt{(5b+5c-z)^2 + p^2} \Big] \Big\} + \Big\{ \frac{B_2(2-c)}{2(b+c)} \Big[\frac{p}{[(3b+3c-z)^2 + p^2]^{3/2}} \\
& + \frac{p}{[(5b+3c+z)^2 + p^2]^{3/2}} \Big] + \frac{B_2}{4(b+c)^2} \Big[-\frac{(3b+c+z)}{p} + \frac{1}{p} \sqrt{(3b+c+z)^2 + p^2} - \\
& \frac{(5b+5c-z)}{p} + \frac{1}{p} \sqrt{(5b+5c-z)^2 + p^2} + \frac{(3b+3c-z)}{p} - \frac{1}{p} \sqrt{(3b+3c-z)^2 + p^2} + \\
& \frac{(5b+3c+z)}{p} - \frac{1}{p} \sqrt{(5b+3c+z)^2 + p^2} \Big] + \frac{B_2}{2(b+c)} \Big[\frac{\sqrt{(3b+3c-z)^2 + p^2} - (3b+3c-z)}{[(3b+3c-z)^2 + p^2]^{1/2}} \cdot p \\
& - \frac{\sqrt{(5b+3c+z)^2 + p^2} - (5b+3c+z)}{p \cdot [(5b+3c+z)^2 + p^2]^{1/2}} \Big] - \frac{B_2(2+b)}{4(b+c)^2} \Big[\frac{\sqrt{(3b+3c-z)^2 + p^2} - (3b+3c-z)}{p \cdot [(3b+3c-z)^2 + p^2]^{1/2}} \\
& - \frac{\sqrt{(5b+3c+z)^2 + p^2} - (5b+3c+z)}{p \cdot [(5b+3c+z)^2 + p^2]^{1/2}} + \frac{\sqrt{(3b+c+z)^2 + p^2} - (3b+c+z)}{p \cdot [(3b+c+z)^2 + p^2]^{1/2}} - \\
& \frac{\sqrt{(5b+5c-z)^2 + p^2} - (5b+5c-z)}{p \cdot [(5b+5c-z)^2 + p^2]^{1/2}} \Big] \Big\} + \Big\{ \frac{C_2(2-c)}{2} \Big[\frac{3b+3c-z}{[(3b+3c-z)^2 + p^2]^{5/2}} \cdot p \\
& + \frac{3(5b+3c+z) \cdot p}{[(5b+3c+z)^2 + p^2]^{5/2}} \Big] + \frac{C_2}{4(b+c)} \Big[\frac{\sqrt{(3b+c+z)^2 + p^2} - (3b+c+z)}{p \cdot [(3b+c+z)^2 + p^2]^{1/2}} + \\
& \frac{\sqrt{(5b+5c-z)^2 + p^2} - (5b+5c-z)}{p \cdot [(5b+5c-z)^2 + p^2]^{1/2}} - \frac{\sqrt{(3b+3c-z)^2 + p^2} - (3b+3c-z)}{p \cdot [(3b+3c-z)^2 + p^2]^{1/2}} - \\
& \frac{\sqrt{(5b+3c+z)^2 + p^2} - (5b+3c+z)}{p \cdot [(5b+3c+z)^2 + p^2]^{1/2}} \Big] + \frac{C_2}{2} \Big[\frac{p}{[(3b+3c-z)^2 + p^2]^{3/2}} - \\
& \frac{p}{[(5b+3c+z)^2 + p^2]^{3/2}} \Big] - \frac{C_2(2+b)}{4(b+c)} \Big[\frac{p}{[(3b+3c-z)^2 + p^2]^{3/2}} \cdot \frac{p}{[(5b+3c+z)^2 + p^2]^{3/2}} \\
& + \frac{p}{[(3b+c+z)^2 + p^2]^{3/2}} - \frac{p}{[(5b+5c-z)^2 + p^2]^{3/2}} \Big] \Big\} \Big\} \quad (A-4)
\end{aligned}$$

$$\begin{aligned}
R_{G_1}(1, 2, 1) &= \Big\{ 2(b+c)(2-c) \Big[\frac{3(2b+2c-z) \cdot p}{[(2b+2c-z)^2 + p^2]^{5/2}} - \frac{3(4b+2c+z) \cdot p}{[(4b+2c+z)^2 + p^2]^{5/2}} \Big] \\
& + (2+b) \Big[\frac{p}{[(2b+2c-z)^2 + p^2]^{3/2}} + \frac{p}{[(4b+2c+z)^2 + p^2]^{3/2}} - \frac{p}{[(2b+z)^2 + p^2]^{3/2}} \Big] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(5b+5c-2)}{p} - \frac{1}{p} \sqrt{(5b+5c-2)^2 + p^2} \Big] \Big\} + \Big\{ \frac{C_2(z-c)}{2} \Big[\frac{p}{[(3b+3c-2)^2 + p^2]^{3/2}} \\
& - \frac{p}{[(5b+3c+2)^2 + p^2]^{3/2}} \Big] + \frac{C_2(2+b)}{4(b+c)} \Big[\frac{\sqrt{(3b+3c-2)^2 + p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2 + p^2]^{1/2}} + \\
& \frac{\sqrt{(5b+3c+2)^2 + p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2 + p^2]^{1/2}} - \frac{\sqrt{(3b+c+2)^2 + p^2} - (3b+c+2)}{p \cdot [(3b+c+2)^2 + p^2]^{1/2}} - \\
& \frac{\sqrt{(5b+5c-2)^2 + p^2} - (5b+5c-2)}{p \cdot [(5b+5c-2)^2 + p^2]^{1/2}} \Big] \Big\} \quad (A-5)
\end{aligned}$$

$$\begin{aligned}
RG_1(1, 2, 2) &= \Big\{ 2(b+c)(z-c) \Big[\frac{p \cdot [12(2b+2c-2)^2 - 3p^2]}{[(2b+2c-2)^2 + p^2]^{7/2}} - \\
& \frac{p \cdot [12(4b+2c+2)^2 - 3p^2]}{[(4b+2c+2)^2 + p^2]^{7/2}} \Big] + (2+b) \cdot \Big[\frac{3(2b+2c-2) \cdot p}{[(2b+2c-2)^2 + p^2]^{5/2}} + \\
& \frac{3(4b+2c+2) \cdot p}{[(4b+2c+2)^2 + p^2]^{5/2}} - \frac{3(2b+2) \cdot p}{[(2b+2)^2 + p^2]^{5/2}} - \frac{3(4b+4c-2) \cdot p}{[(4b+4c-2)^2 + p^2]^{5/2}} \Big] \Big\} + \\
& \frac{3(z-c)}{2(b+c)^3} \Big[- \frac{(2b+2c-2)}{p} + \frac{1}{p} \sqrt{(2b+2c-2)^2 + p^2} + \frac{(4b+2c+2)}{p} - \frac{1}{p} \cdot \\
& \sqrt{(4b+2c+2)^2 + p^2} \Big] + \frac{3(2+b)}{4(b+c)^4} \Big[- \frac{(2b+2c-2)}{2 \cdot p} \sqrt{(2b+2c-2)^2 + p^2} - \frac{p}{2} \cdot \ln \Big[\\
& (2b+2c-2) + \sqrt{(2b+2c-2)^2 + p^2} \Big] + \frac{(2b+2c-2)^2}{2 \cdot p} - \frac{(4b+2c+2)}{2 \cdot p} \sqrt{(4b+2c+2)^2 + p^2} \\
& - \frac{p}{2} \ln \Big[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + p^2} \Big] + \frac{(4b+2c+2)^2}{2 \cdot p} + \frac{(2b+2)}{2 \cdot p} \sqrt{(2b+2)^2 + p^2} \\
& + \frac{p}{2} \ln \Big[(2b+2) + \sqrt{(2b+2)^2 + p^2} \Big] - \frac{(2b+2)^2}{2 \cdot p} + \frac{(4b+4c-2)}{2 \cdot p} \sqrt{(4b+4c-2)^2 + p^2} \\
& + \frac{p}{2} \ln \Big[(4b+4c-2) + \sqrt{(4b+4c-2)^2 + p^2} \Big] - \frac{(4b+4c-2)^2}{2 \cdot p} \Big] \Big\} + \frac{A_2(z-c)}{2(b+c)^2} \Big[\\
& \frac{\sqrt{(3b+3c-2)^2 + p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2 + p^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2 + p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2 + p^2]^{1/2}} \Big] + \frac{A_2(2+b)}{4(b+c)} \\
& \Big[- \frac{(3b+3c-2)}{p} + \frac{1}{p} \sqrt{(3b+3c-2)^2 + p^2} - \frac{(5b+3c+2)}{p} + \frac{1}{p} \sqrt{(5b+3c+2)^2 + p^2} +
\end{aligned}$$

$$\begin{aligned}
& - \frac{p}{[(4b+4c-2)^2 + p^2]^{3/2}} \Big] \Big\} + \left\{ \frac{3(2-c)}{2(b+c)^3} \left[- \frac{(2b+2c-2)}{2 \cdot p} \sqrt{(2b+2c-2)^2 + p^2} - \frac{p}{2} \cdot \right. \right. \\
& \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2 + p^2}] + \frac{(2b+2c-2)^2}{2 \cdot p} + \frac{(4b+2c+2)}{2 \cdot p} \cdot \\
& \sqrt{(4b+2c+2)^2 + p^2} + \frac{p}{2} \cdot \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + p^2}] - \frac{(4b+2c+2)^2}{2 \cdot p} \Big] + \\
& \frac{3(2+b)}{4(b+c)^4} \left[\frac{p}{2} (2b+2c-2) \cdot \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2 + p^2}] - \frac{p}{2} \cdot \right. \\
& \sqrt{(2b+2c-2)^2 + p^2} - \frac{(2b+2c-2)^3}{6 \cdot p} + \frac{[(2b+2c-2)^2 + p^2]^{3/2}}{6 \cdot p} + \frac{p}{2} (4b+2c+2) \cdot \\
& \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + p^2}] - \frac{p}{2} \sqrt{(4b+2c+2)^2 + p^2} - \frac{(4b+2c+2)^3}{6 \cdot p} + \\
& \frac{[(4b+2c+2)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (2b+2) \cdot \ln[(2b+2) + \sqrt{(2b+2)^2 + p^2}] + \frac{p}{2} \cdot \\
& \sqrt{(2b+2)^2 + p^2} + \frac{(2b+2)^3}{6 \cdot p} - \frac{[(2b+2)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (4b+4c-2) \cdot \ln[\\
& (4b+4c-2) + \sqrt{(4b+4c-2)^2 + p^2}] + \frac{p}{2} \sqrt{(4b+4c-2)^2 + p^2} + \frac{(4b+4c-2)^3}{6 \cdot p} - \\
& \frac{[(4b+4c-2)^2 + p^2]^{3/2}}{6 \cdot p} \Big] \Big\} + \left\{ \frac{A_2(2-c)}{2(b+c)^2} \left[- \frac{(3b+3c-2)}{p} + \frac{1}{p} \sqrt{(3b+3c-2)^2 + p^2} \right. \right. \\
& + \frac{(5b+3c+2)}{p} - \frac{1}{p} \sqrt{(5b+3c+2)^2 + p^2} \Big] + \frac{A_2(2+b)}{4(b+c)^3} \left[- \frac{(3b+3c-2)}{2 \cdot p} \cdot \right. \\
& \sqrt{(3b+3c-2)^2 + p^2} - \frac{p}{2} \ln[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2}] + \frac{(3b+3c-2)^2}{2 \cdot p} \\
& - \frac{(5b+3c+2)}{2 \cdot p} \sqrt{(5b+3c+2)^2 + p^2} - \frac{p}{2} \cdot \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] \\
& + \frac{(5b+3c+2)}{2 \cdot p} + \frac{(3b+3c+2)}{2 \cdot p} \sqrt{(3b+3c+2)^2 + p^2} + \frac{p}{2} \ln[(3b+3c+2) + \sqrt{(3b+3c+2)^2 + p^2}] \\
& - \frac{(3b+3c+2)^2}{2 \cdot p} + \frac{(5b+5c-2)}{2 \cdot p} \sqrt{(5b+5c-2)^2 + p^2} + \frac{p}{2} \ln[(5b+5c-2) + \\
& \sqrt{(5b+5c-2)^2 + p^2}] - \frac{(5b+5c-2)^2}{2 \cdot p} \Big] \Big\} + \left\{ \frac{B_2(2-c)}{2(b+c)} \left[\sqrt{(3b+3c-2)^2 + p^2} - (3b+3c-2) \right. \right. \\
& - \frac{\sqrt{(5b+3c+2)^2 + p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2 + p^2]^{1/2}} \Big] + \frac{B_2(2+b)}{4(b+c)^2} \left[- \frac{(3b+3c-2)}{p} + \frac{1}{p} \sqrt{(3b+3c-2)^2 + p^2} \right. \\
& - \frac{(5b+3c+2)}{p} + \frac{1}{p} \sqrt{(5b+3c+2)^2 + p^2} + \frac{(3b+3c+2)}{p} - \frac{1}{p} \sqrt{(3b+3c+2)^2 + p^2} +
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{(3b+c+z)}{p} - \frac{1}{p} \sqrt{(3b+c+z)^2 + p^2} + \frac{(5b+5c-z)}{p} - \frac{1}{p} \sqrt{(5b+5c-z)^2 + p^2} \right\} + \\
& \left\{ \frac{B_2(z-c)}{2(b+c)} \left[\frac{p}{[(3b+3c-z)^2 + p^2]^{3/2}} - \frac{p}{[(5b+3c+z)^2 + p^2]^{3/2}} \right] + \frac{B_2(z+b)}{4(b+c)^2} \left[\right. \right. \\
& \frac{\sqrt{(3b+3c-z)^2 + p^2} - (3b+3c-z)}{p \cdot [(3b+3c-z)^2 + p^2]^{1/2}} + \frac{\sqrt{(5b+3c+z)^2 + p^2} - (5b+3c+z)}{p \cdot [(5b+3c+z)^2 + p^2]^{1/2}} - \\
& \left. \left. \frac{\sqrt{(3b+c+z)^2 + p^2} - (3b+c-z)}{p \cdot [(3b+c+z)^2 + p^2]^{1/2}} - \frac{\sqrt{(5b+5c-z)^2 + p^2} - (5b+5c-z)}{p \cdot [(5b+5c-z)^2 + p^2]^{1/2}} \right] \right\} + \\
& \left\{ \frac{C_2(z-c)}{2} \left[\frac{3(3b+3c-z) \cdot p}{[(3b+3c-z)^2 + p^2]^{5/2}} - \frac{3(5b+3c+z) \cdot p}{[(5b+3c+z)^2 + p^2]^{5/2}} \right] + \frac{C_2(z+b)}{4(b+c)} \right. \\
& \left[\frac{p}{[(3b+3c-z)^2 + p^2]^{3/2}} + \frac{p}{[(5b+3c+z)^2 + p^2]^{3/2}} - \frac{p}{[(3b+c+z)^2 + p^2]^{3/2}} \right. \\
& \left. \left. - \frac{p}{[(5b+5c-z)^2 + p^2]^{3/2}} \right] \right\}, \quad (A-6)
\end{aligned}$$

$$\begin{aligned}
RG_5(1,1,1) &= \left\{ 2 \cdot \left[\frac{1}{[(2b+2c-z)^2 + p^2]^{1/2}} - \frac{1}{[(2b+z)^2 + p^2]^{1/2}} \right] + \frac{1}{(b+c)} \right. \\
& \left. [-\ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2}] + \ln[(2b+z) + \sqrt{(2b+z)^2 + p^2}]] \right\} + \\
& \left\{ A_3 \cdot \left[\frac{1}{[(3b+3c-z)^2 + p^2]^{1/2}} - \frac{1}{[(3b+c+z)^2 + p^2]^{1/2}} \right] + B_3(b+c) \left[\frac{3b+3c-z}{[(3b+3c-z)^2 + p^2]^{3/2}} \right. \right. \\
& \left. \left. - \frac{3b+c+z}{[(3b+c+z)^2 + p^2]^{3/2}} \right] + C_3(b+c) \cdot \left[\frac{2(3b+3c-z)^2 - p^2}{[(3b+3c-z)^2 + p^2]^{5/2}} - \frac{2(3b+c+z)^2 - p^2}{[(3b+c+z)^2 + p^2]^{5/2}} \right] \right\} \\
& \quad (A-7)
\end{aligned}$$

$$\begin{aligned}
RG_5(1,1,2) &= \left\{ 2 \cdot \left[\frac{2b+2c-z}{[(2b+2c-z)^2 + p^2]^{3/2}} - \frac{2b+z}{[(2b+z)^2 + p^2]^{3/2}} \right] + \frac{1}{(b+c)} \right. \\
& \left[\frac{1}{[(2b+2c-z)^2 + p^2]^{1/2}} - \frac{1}{[(2b+z)^2 + p^2]^{1/2}} \right] \right\} + \left\{ A_3 \left[\frac{3b+3c-z}{[(3b+3c-z)^2 + p^2]^{3/2}} - \right. \right. \\
& \left. \frac{3b+c+z}{[(3b+c+z)^2 + p^2]^{3/2}} \right] + B_3(b+c) \left[\frac{2(3b+3c-z)^2 - p^2}{[(3b+3c-z)^2 + p^2]^{5/2}} - \frac{2(3b+c+z)^2 - p^2}{[(3b+c+z)^2 + p^2]^{5/2}} \right] \right\}
\end{aligned}$$

$$C_3(b+c)^2 \left\{ \frac{3(3b+3c-2)[2(3b+3c-2)^2-3p^2]}{[(3b+3c-2)^2+p^2]^{3/2}} - \frac{3(3b+c+2) \cdot [2(3b+c+2)^2-3p^2]}{[(3b+c+2)^2+p^2]^{3/2}} \right\} // (A-8)$$

$$\begin{aligned} R_{G_5}(1, 2, 3) = & \left\{ 2 \left[-\frac{(2b+2c-2)}{p} + \frac{1}{p} \sqrt{(2b+2c-2)^2+p^2} + \frac{2b+2}{p} - \frac{1}{p} \cdot \right. \right. \\ & \left. \left. \sqrt{(2b+2)^2+p^2} \right] + \frac{1}{(b+c)} \left[-\frac{(2b+2c-2)}{2 \cdot p} \sqrt{(2b+2c-2)^2+p^2} - \frac{p}{2} \ln \left[\frac{(2b+2c-2)}{2 \cdot p} \right. \right. \right. \\ & \left. \left. + \sqrt{(2b+2c-2)^2+p^2} \right] + \frac{(2b+2c-2)^2}{2 \cdot p} + \frac{(2b+2)}{2 \cdot p} \sqrt{(2b+2)^2+p^2} + \frac{p}{2} \cdot \ln \left[\right. \right. \\ & \left. \left. (2b+2) + \sqrt{(2b+2)^2+p^2} \right] - \frac{(2b+2)^2}{2 \cdot p} \right] \right\} + \left\{ A_3 \left[-\frac{(3b+3c-2)}{p} + \frac{1}{p} \cdot \right. \right. \\ & \left. \left. \sqrt{(3b+3c-2)^2+p^2} + \frac{(3b+c+2)}{p} - \frac{1}{p} \sqrt{(3b+c+2)^2+p^2} \right] + B_3(b+c) \cdot \left[\right. \right. \\ & \left. \left. \frac{\sqrt{(3b+3c-2)^2+p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2+p^2]^{1/2}} - \frac{\sqrt{(3b+c+2)^2+p^2} - (3b+c+2)}{p \cdot [(3b+c+2)^2+p^2]^{1/2}} \right] + C_3(b+c)^3 \cdot \left[\right. \right. \\ & \left. \left. \frac{p}{[(3b+3c-2)^2+p^2]^{3/2}} - \frac{p}{[(3b+c+2)^2+p^2]^{3/2}} \right] \right\} // (A-9) \end{aligned}$$

$$\begin{aligned} R_{G_5}(1, 2, 1) = & \left\{ 2 \cdot \left[\frac{\sqrt{(2b+2c-2)^2+p^2} - (2b+2c-2)}{p \cdot [(2b+2c-2)^2+p^2]^{1/2}} - \right. \right. \\ & \left. \left. \frac{\sqrt{(2b+2)^2+p^2} - (2b+2)}{p \cdot [(2b+2)^2+p^2]^{1/2}} \right] + \frac{1}{(b+c)} \left[-\frac{(2b+2c-2)}{p} + \frac{1}{p} \sqrt{(2b+2c-2)^2+p^2} + \right. \right. \\ & \left. \left. \frac{(2b+2)}{p} - \frac{1}{p} \sqrt{(2b+2)^2+p^2} \right] \right\} + \left\{ A_3 \cdot \left[\frac{\sqrt{(3b+3c-2)^2+p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2+p^2]^{1/2}} - \right. \right. \\ & \left. \left. \frac{\sqrt{(3b+c+2)^2+p^2} - (3b+c+2)}{p \cdot [(3b+c+2)^2+p^2]^{1/2}} \right] + B_3(b+c) \left[\frac{p}{[(3b+3c-2)^2+p^2]} - \frac{p}{[(3b+c+2)^2+p^2]} \right] \right. \\ & \left. + C_3(b+c)^2 \left[\frac{3(3b+3c-2) \cdot p}{[(3b+3c-2)^2+p^2]^{5/2}} - \frac{3(3b+c+2) \cdot p}{[(3b+c+2)^2+p^2]^{5/2}} \right] \right\} // (A-10) \end{aligned}$$

$$\begin{aligned}
RG_6(1, 1, 2) = & \left\{ 2(b+c) \cdot \left[\frac{2(2b+2c-2)^2 - p^2}{[(2b+2c-2)^2 + p^2]^{5/2}} + \frac{2(6b+4c+2)^2 - p^2}{[(6b+4c+2)^2 + p^2]^{5/2}} \right. \right. \\
& \frac{2(4b+2c+2)^2 - p^2}{[(4b+2c+2)^2 + p^2]^{5/2}} - \left. \frac{2(4b+4c-2)^2 - p^2}{[(4b+4c-2)^2 + p^2]^{5/2}} \right] + (2+b) \left[\frac{2[2(4b+4c-2)^2 - p^2]}{[(4b+4c-2)^2 + p^2]^{5/2}} \right. \\
& + \frac{2[2(4b+2c+2)^2 - p^2]}{[(4b+2c+2)^2 + p^2]^{5/2}} - \frac{2(2b+2c-2)^2 - p^2}{[(2b+2c-2)^2 + p^2]^{5/2}} - \left. \frac{2(6b+4c+2)^2 - p^2}{[(6b+4c+2)^2 + p^2]^{5/2}} \right. \\
& \left. \frac{2(2b+2)^2 - p^2}{[(2b+2)^2 + p^2]^{5/2}} - \frac{2(6b+6c-2)^2 - p^2}{[(6b+6c-2)^2 + p^2]^{5/2}} \right] + 2(b+c)(2-c) \cdot \left[\right. \\
& \frac{3(2b+2c-2)[2(2b+2c-2)^2 - 3p^2]}{[(2b+2c-2)^2 + p^2]^{7/2}} - \frac{3(6b+4c+2)[2(6b+4c+2)^2 - 3p^2]}{[(6b+4c+2)^2 + p^2]^{7/2}} \\
& + \frac{3(4b+2c+2)[2(4b+2c+2)^2 - 3p^2]}{[(4b+2c+2)^2 + p^2]^{7/2}} - \left. \frac{3(4b+4c-2)[2(4b+4c-2)^2 - 3p^2]}{[(4b+4c-2)^2 + p^2]^{7/2}} \right] \\
& - 4(b+c)^2 \left[\frac{3(4b+4c-2)[2(4b+4c-2)^2 - 3p^2]}{[(4b+4c-2)^2 + p^2]^{7/2}} - \frac{3(4b+2c+2)[2(4b+2c+2)^2 - 3p^2]}{[(4b+2c+2)^2 + p^2]^{7/2}} \right] \\
& + \left\{ \frac{3}{4(b+c)} \left[\left(\frac{p^2}{4} - \frac{(2b+2c-2)^2}{2} \right) \cdot \ln \left[(2b+2c-2) + \sqrt{(2b+2c-2)^2 + p^2} \right] + \frac{3}{4} \cdot \right. \right. \\
& (2b+2c-2) \cdot \sqrt{(2b+2c-2)^2 + p^2} + \left. \left(\frac{p^2}{4} - \frac{(6b+4c+2)^2}{2} \right) \cdot \ln \left[(6b+4c+2) + \right. \right. \\
& \left. \left. \sqrt{(6b+4c+2)^2 + p^2} \right] + \frac{3}{4} (6b+4c+2) \cdot \sqrt{(6b+4c+2)^2 + p^2} - \left(\frac{p^2}{4} - \frac{(4b+2c+2)^2}{2} \right) \cdot \right. \\
& \left. \ln \left[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + p^2} \right] - \frac{3}{4} (4b+2c+2) \cdot \sqrt{(4b+2c+2)^2 + p^2} - \right. \\
& \left. \left(\frac{p^2}{4} - \frac{(4b+4c-2)^2}{2} \right) \cdot \ln \left[(4b+4c-2) + \sqrt{(4b+4c-2)^2 + p^2} \right] - \frac{3}{4} (4b+4c-2) \cdot \right. \\
& \left. \sqrt{(4b+4c-2)^2 + p^2} \right] + \frac{3(2+b)}{8(b+c)^5} \left[2 \cdot \left(\frac{p^2}{4} - \frac{(4b+4c-2)^2}{2} \right) \cdot \ln \left[(4b+4c-2) + \right. \right. \\
& \left. \left. \sqrt{(4b+4c-2)^2 + p^2} \right] + 2 \cdot \frac{3}{4} (4b+4c-2) \cdot \sqrt{(4b+4c-2)^2 + p^2} + 2 \cdot \left(\frac{p^2}{4} - \frac{(4b+2c+2)^2}{2} \right) \cdot \right. \\
& \left. \ln \left[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + p^2} \right] + 2 \cdot \frac{3}{4} (4b+2c+2) \cdot \right. \\
& \left. \sqrt{(4b+2c+2)^2 + p^2} - \left(\frac{p^2}{4} - \frac{(2b+2c-2)^2}{2} \right) \cdot \ln \left[(2b+2c-2) + \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \sqrt{(2b+2c-z)^2+p^2} - \frac{3}{4}(2b+2c-z) \cdot \sqrt{(2b+2c-z)^2+p^2} - \left[\frac{p^2}{4} - \frac{(6b+4c+z)^2}{2} \right] \\
& \ln \left[(6b+4c+z) + \sqrt{(6b+4c+z)^2+p^2} \right] - \frac{3}{4}(6b+4c+z) \cdot \sqrt{(6b+4c+z)^2+p^2} \\
& - \left[\frac{p^2}{4} - \frac{(2b+z)^2}{2} \right] \cdot \ln \left[(2b+z) + \sqrt{(2b+z)^2+p^2} \right] - \frac{3}{4}(2b+z) \cdot \sqrt{(2b+z)^2+p^2} - \\
& \left[\frac{p^2}{4} - \frac{(6b+6c-z)^2}{2} \right] \cdot \ln \left[(6b+6c-z) + \sqrt{(6b+6c-z)^2+p^2} \right] - \frac{3}{4}(6b+6c-z) \cdot \\
& \sqrt{(6b+6c-z)^2+p^2} + \frac{3(2-c)}{4(b+c)^4} \left\{ (2b+2c-z) \cdot \ln \left[(2b+2c-z) + \right. \right. \\
& \left. \left. \sqrt{(2b+2c-z)^2+p^2} \right] - \left[\sqrt{(2b+2c-z)^2+p^2} \right] - (6b+4c+z) \cdot \ln \left[\right. \right. \\
& \left. \left. (6b+4c+z) + \sqrt{(6b+4c+z)^2+p^2} \right] + \sqrt{(6b+4c+z)^2+p^2} + (4b+2c+z) \cdot \right. \\
& \left. \ln \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2} \right] - \sqrt{(4b+2c+z)^2+p^2} - (4b+4c-z) \cdot \right. \\
& \left. \ln \left[(4b+4c-z) + \sqrt{(4b+4c-z)^2+p^2} \right] + \sqrt{(4b+4c-z)^2+p^2} \right\} - \frac{3}{2(b+c)^3} \left\{ \right. \\
& \left. (4b+4c-z) \cdot \ln \left[(4b+4c-z) + \sqrt{(4b+4c-z)^2+p^2} \right] - \sqrt{(4b+4c-z)^2+p^2} \right. \\
& \left. - (4b+2c+z) \cdot \ln \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2} \right] + \sqrt{(4b+2c+z)^2+p^2} \right\} \\
& + \left\{ \frac{A_1}{4(b+c)^3} \left[(3b+3c-z) \cdot \ln \left[(3b+3c-z) + \sqrt{(3b+3c-z)^2+p^2} \right] - \right. \right. \\
& \left. \left. \sqrt{(3b+3c-z)^2+p^2} + (7b+5c+z) \cdot \ln \left[(7b+5c+z) + \sqrt{(7b+5c+z)^2+p^2} \right] \right. \right. \\
& \left. \left. - \sqrt{(7b+5c+z)^2+p^2} - (5b+3c+z) \cdot \ln \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2+p^2} \right] \right. \right. \\
& \left. \left. + \sqrt{(5b+3c+z)^2+p^2} - (5b+5c-z) \cdot \ln \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2+p^2} \right] \right. \right. \\
& \left. \left. + \sqrt{(5b+5c-z)^2+p^2} \right] + \frac{A_1(2+b)}{8(b+c)^4} \left[2 \left[(5b+5c-z) \cdot \ln \left[(5b+5c-z) + \right. \right. \right. \right. \\
& \left. \left. \sqrt{(5b+5c-z)^2+p^2} \right] - 2 \sqrt{(5b+5c-z)^2+p^2} + 2 \cdot (5b+3c+z) \cdot \ln \left[\right. \right. \\
& \left. \left. (5b+3c+z) + \sqrt{(5b+3c+z)^2+p^2} \right] - 2 \sqrt{(5b+3c+z)^2+p^2} - (3b+3c-z) \cdot \ln \left[\right. \right. \\
& \left. \left. (3b+3c-z) + \sqrt{(3b+3c-z)^2+p^2} \right] + \sqrt{(3b+3c-z)^2+p^2} - (7b+5c+z) \cdot \right. \\
& \left. \ln \left[(7b+5c+z) + \sqrt{(7b+5c+z)^2+p^2} \right] + \sqrt{(7b+5c+z)^2+p^2} - \right.
\end{aligned}$$

$$\begin{aligned}
& (3b+c+2) \cdot \frac{1}{h} \left[(3b+c+2) + \sqrt{(3b+c+2)^2 + f^2} \right] + \sqrt{(3b+c+2)^2 + f^2} - \\
& (7b+7c-2) \cdot \frac{1}{h} \left[(7b+7c-2) + \sqrt{(7b+7c-2)^2 + f^2} \right] + \sqrt{(7b+7c-2)^2 + f^2} \\
& + \frac{A_1(2-c)}{4(b+c)^3} \left[-\frac{1}{h} \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + f^2} \right] + \frac{1}{h} \left[(7b+5c+2) + \sqrt{(7b+5c+2)^2 + f^2} \right] \right. \\
& - \frac{1}{h} \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + f^2} \right] + \frac{1}{h} \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + f^2} \right] \\
& \left. - \frac{A_1}{2(b+c)^2} \left[-\frac{1}{h} \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + f^2} \right] + \frac{1}{h} \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + f^2} \right] \right] \right\} \\
& + \left\{ \frac{B_1}{4(b+c)^2} \left[-\frac{1}{h} \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + f^2} \right] - \frac{1}{h} \left[(7b+5c+2) + \sqrt{(7b+5c+2)^2 + f^2} \right] \right. \right. \\
& + \frac{1}{h} \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + f^2} \right] + \frac{1}{h} \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + f^2} \right] \\
& \left. \left. + \frac{B_1(2+b)}{8(b+c)^3} \left[-2 \frac{1}{h} \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + f^2} \right] - 2 \cdot \frac{1}{h} \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + f^2} \right] \right. \right. \right. \\
& + \frac{1}{h} \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + f^2} \right] + \frac{1}{h} \left[(7b+5c+2) + \sqrt{(7b+5c+2)^2 + f^2} \right] \\
& + \frac{1}{h} \left[(3b+c+2) + \sqrt{(3b+c+2)^2 + f^2} \right] + \frac{1}{h} \left[(7b+7c-2) + \sqrt{(7b+7c-2)^2 + f^2} \right] \\
& \left. + \frac{B_1(2-c)}{4(b+c)^2} \left[\frac{1}{[(3b+3c-2)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(7b+5c+2)^2 + f^2]^{\frac{1}{2}}} + \frac{1}{[(5b+3c+2)^2 + f^2]^{\frac{1}{2}}} \right. \right. \\
& \left. \left. - \frac{1}{[(5b+5c-2)^2 + f^2]^{\frac{1}{2}}} \right] - \frac{B_1}{2(b+c)} \left[\frac{1}{[(5b+5c-2)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(5b+3c+2)^2 + f^2]^{\frac{1}{2}}} \right] \right\} \\
& + \left\{ \frac{C_1}{4(b+c)} \left[\frac{1}{[(3b+3c-2)^2 + f^2]^{\frac{1}{2}}} + \frac{1}{[(7b+5c+2)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(5b+3c+2)^2 + f^2]^{\frac{1}{2}}} \right. \right. \\
& \left. \left. - \frac{1}{[(5b+5c-2)^2 + f^2]^{\frac{1}{2}}} \right] + \frac{C_1(2+b)}{8(b+c)^2} \left[2 \frac{1}{[(5b+5c-2)^2 + f^2]^{\frac{1}{2}}} + \frac{2}{[(5b+3c+2)^2 + f^2]^{\frac{1}{2}}} \right. \right. \\
& \left. \left. - \frac{1}{[(3b+3c-2)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(7b+5c+2)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(3b+c+2)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(7b+7c-2)^2 + f^2]^{\frac{1}{2}}} \right] \right. \\
& \left. + \frac{C_1(2-c)}{4(b+c)} \left[\frac{3b+3c-2}{[(3b+3c-2)^2 + f^2]^{\frac{3}{2}}} - \frac{7b+5c+2}{[(7b+5c+2)^2 + f^2]^{\frac{3}{2}}} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{5b+3c+z}{[(5b+3c+z)^2+p^2]^{3/2}} - \frac{5b+5c-z}{[(5b+5c-z)^2+p^2]^{3/2}} \Big] - \frac{C_1}{2} \Big[\frac{5b+5c-z}{[(5b+5c-z)^2+p^2]^{3/2}} \\
& - \frac{5b+3c+z}{[(5b+3c+z)^2+p^2]^{3/2}} \Big] \Big\} + \Big\{ \frac{D_1}{4} \Big[\frac{3b+3c-z}{[(3b+3c-z)^2+p^2]^{3/2}} + \frac{7b+5c+z}{[(7b+5c+z)^2+p^2]^{3/2}} \\
& - \frac{5b+3c+z}{[(5b+3c+z)^2+p^2]^{3/2}} - \frac{5b+5c-z}{[(5b+5c-z)^2+p^2]^{3/2}} \Big] + \frac{D_1(2+b)}{8(b+c)} \Big[\frac{2(5b+5c-z)}{[(5b+5c-z)^2+p^2]^{3/2}} \\
& + \frac{2(5b+3c+z)}{[(5b+3c+z)^2+p^2]^{3/2}} - \frac{3b+3c-z}{[(3b+3c-z)^2+p^2]^{3/2}} - \frac{7b+5c+z}{[(7b+5c+z)^2+p^2]^{3/2}} - \\
& \frac{3b+c+z}{[(3b+c+z)^2+p^2]^{3/2}} - \frac{7b+7c-z}{[(7b+7c-z)^2+p^2]^{3/2}} \Big] + \frac{D_1(2-c)}{4} \Big[\\
& \frac{2(3b+3c-z)^2-p^2}{[(3b+3c-z)^2+p^2]^{5/2}} - \frac{2(7b+5c+z)^2-p^2}{[(7b+5c+z)^2+p^2]^{5/2}} + \frac{2(5b+3c+z)^2-p^2}{[(5b+3c+z)^2+p^2]^{5/2}} - \\
& \frac{2(5b+5c-z)^2-p^2}{[(5b+5c-z)^2+p^2]^{5/2}} \Big] - \frac{D_1(b+c)}{2} \Big[\frac{2(5b+5c-z)^2-p^2}{[(5b+5c-z)^2+p^2]^{5/2}} - \\
& \frac{2(5b+3c+z)^2-p^2}{[(5b+3c+z)^2+p^2]^{5/2}} \Big] \Big\} \quad (A-11)
\end{aligned}$$

$$\begin{aligned}
R_{f_6}(1, 2, 1) &= \Big\{ 2(b+c) \Big[\frac{p}{[(2b+2c-z)^2+p^2]^{3/2}} + \frac{p}{[(6b+4c+z)^2+p^2]^{3/2}} \\
& - \frac{p}{[(4b+2c+z)^2+p^2]^{3/2}} - \frac{p}{[(4b+4c-z)^2+p^2]^{3/2}} \Big] + (2+b) \Big[\frac{2 \cdot p}{[(4b+4c-z)^2+p^2]^{3/2}} \\
& + \frac{2 \cdot p}{[(4b+2c+z)^2+p^2]^{3/2}} - \frac{p}{[(2b+2c-z)^2+p^2]^{3/2}} - \frac{p}{[(6b+4c+z)^2+p^2]^{3/2}} - \\
& \frac{p}{[(2b+z)^2+p^2]^{3/2}} - \frac{p}{[(6b+6c-z)^2+p^2]^{3/2}} \Big] + 2(b+c)(2-c) \Big[\\
& \frac{3p \cdot (2b+2c-z)}{[(2b+2c-z)^2+p^2]^{5/2}} - \frac{3p \cdot (6b+4c+z)}{[(6b+4c+z)^2+p^2]^{5/2}} + \frac{3p \cdot (4b+2c+z)}{[(4b+2c+z)^2+p^2]^{5/2}}
\end{aligned}$$

$$\begin{aligned}
& - \frac{3p \cdot (4b+4c-z)}{[(4b+4c-z)^2+p^2]^{5/2}} - 4(b+c)^2 \left[\frac{3p \cdot (4b+4c-z)}{[(4b+4c-z)^2+p^2]^{5/2}} - \right. \\
& \left. \frac{3p \cdot (4b+2c+z)}{[(4b+2c+z)^2+p^2]^{5/2}} \right] + \left\{ \frac{3}{4(b+c)} \left[\left[\frac{p^3}{16} - \frac{p}{4} (2b+2c-z)^2 \right] \cdot \ln \right. \right. \\
& \left. \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2+p^2} \right] + \frac{5 \cdot p}{16} (2b+2c-z) \cdot \sqrt{(2b+2c-z)^2+p^2} - \right. \\
& \left. \frac{1}{24 \cdot p} [(2b+2c-z) \{ (2b+2c-z)^2+p^2 \}^{3/2} - (2b+2c-z)^4] + \left[\frac{p^3}{16} - \frac{p}{4} \cdot \right. \right. \\
& \left. \left. (6b+4c+z)^2 \right] \cdot \ln [(6b+4c+z) + \sqrt{(6b+4c+z)^2+p^2}] + \frac{5 \cdot p}{16} (6b+4c+z) \cdot \right. \\
& \left. \sqrt{(6b+4c+z)^2+p^2} - \frac{1}{24 \cdot p} [(6b+4c+z) \{ (6b+4c+z)^2+p^2 \}^{3/2} - (6b+4c+z)^4] \right. \\
& \left. - \left[\frac{p^3}{16} - \frac{p}{4} (4b+2c+z)^2 \right] \cdot \ln [(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2}] - \frac{5 \cdot p}{16} \cdot \right. \\
& \left. (4b+2c+z) \cdot \sqrt{(4b+2c+z)^2+p^2} + \frac{1}{24 \cdot p} [(4b+2c+z) \cdot \{ (4b+2c+z)^2+p^2 \}^{3/2} \right. \\
& \left. - (4b+2c+z)^4] - \left[\frac{p^3}{16} - \frac{p}{4} (4b+4c-z)^2 \right] \cdot \ln [(4b+4c-z) + \right. \\
& \left. \sqrt{(4b+4c-z)^2+p^2}] - \frac{5 \cdot p}{16} (4b+4c-z) \cdot \sqrt{(4b+4c-z)^2+p^2} + \frac{(4b+4c-z)}{24 \cdot p} \left[\right. \\
& \left. [(4b+4c-z)^2+p^2]^{3/2} - (4b+4c-z)^3 \right] \left. \right\} + \frac{3(2+b)}{8(b+c)^5} \left[2 \cdot \left[\frac{p^3}{16} - \frac{p}{4} (4b+4c-z)^2 \right] \right. \\
& \cdot \ln [(4b+4c-z) + \sqrt{(4b+4c-z)^2+p^2}] + 2 \cdot \frac{5 \cdot p}{16} (4b+4c-z) \cdot \sqrt{(4b+4c-z)^2+p^2} \\
& - 2 \cdot \frac{(4b+4c-z)}{24 \cdot p} \left[[(4b+4c-z)^2+p^2]^{3/2} - (4b+4c-z)^3 \right] + 2 \left[\frac{p^3}{16} - \frac{p}{4} \cdot \right. \\
& \left. (4b+2c+z)^2 \right] \cdot \ln [(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2}] + 2 \cdot \frac{5 \cdot p}{16} (4b+2c+z) \cdot \sqrt{(4b+2c+z)^2+p^2} \\
& - 2 \cdot \frac{(4b+2c+z)}{24 \cdot p} \left[[(4b+2c+z)^2+p^2]^{3/2} - (4b+2c+z)^3 \right] - \\
& \left[\frac{p^3}{16} - \frac{p}{4} (2b+2c-z)^2 \right] \cdot \ln [(2b+2c-z) + \sqrt{(2b+2c-z)^2+p^2}] - \frac{5 \cdot p}{16} \cdot \\
& (2b+2c-z) \cdot \sqrt{(2b+2c-z)^2+p^2} + \frac{(2b+2c-z)}{24 \cdot p} \left[[(2b+2c-z)^2+p^2]^{3/2} - \right. \\
& \left. (2b+2c-z)^3 \right] - \left[\frac{p^3}{16} - \frac{p}{4} (6b+4c+z)^2 \right] \cdot \ln [(6b+4c+z) + \sqrt{(6b+4c+z)^2+p^2}] \\
& - \frac{5 \cdot p}{16} (6b+4c+z) \cdot \sqrt{(6b+4c+z)^2+p^2} + \frac{(6b+4c+z)}{24 \cdot p} \left[[(6b+4c+z)^2+p^2]^{3/2} - \right.
\end{aligned}$$

$$\begin{aligned}
& (6b+4c+z)^3 \Big] - \left[\frac{p^3}{16} - \frac{p}{4} (2b+z)^2 \right] \cdot \ln \left[(2b+z) + \sqrt{(2b+z)^2 + p^2} \right] - \\
& \frac{5 \cdot p}{16} (2b+z) \sqrt{(2b+z)^2 + p^2} + \frac{(2b+z)}{24 \cdot p} \left[\left[(2b+z)^2 + p^2 \right]^{3/2} - (2b+z)^3 \right] - \left[\frac{p^3}{16} \right. \\
& \left. - \frac{p}{4} (6b+6c-2)^2 \right] \cdot \ln \left[(6b+6c-2) + \sqrt{(6b+6c-2)^2 + p^2} \right] - \frac{5 \cdot p}{16} (6b+6c-2) \\
& \sqrt{(6b+6c-2)^2 + p^2} + \frac{(6b+6c-2)}{24 \cdot p} \left[\left[(6b+6c-2)^2 + p^2 \right]^{3/2} - (6b+6c-2)^3 \right] + \\
& \frac{3(2-c)}{4(b+c)} \left\{ \frac{p}{2} \left[(2b+2c-2) \cdot \ln \left[(2b+2c-2) + \sqrt{(2b+2c-2)^2 + p^2} \right] - \right. \right. \\
& \left. \left. \sqrt{(2b+2c-2)^2 + p^2} \right] - \frac{1}{6 \cdot p} \left[(2b+2c-2)^3 - \left[(2b+2c-2)^2 + p^2 \right]^{3/2} \right] - \frac{p}{2} \cdot \right. \\
& \left. \left[(6b+4c+z) \cdot \ln \left[(6b+4c+z) + \sqrt{(6b+4c+z)^2 + p^2} \right] - \sqrt{(6b+4c+z)^2 + p^2} \right] \right. \\
& \left. + \frac{1}{6 \cdot p} \left[(6b+4c+z)^3 - \left[(6b+4c+z)^2 + p^2 \right]^{3/2} \right] + \frac{p}{2} \left[(4b+2c+z) \cdot \ln \right. \right. \\
& \left. \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2} \right] - \sqrt{(4b+2c+z)^2 + p^2} \right] - \frac{1}{6 \cdot p} \left[\right. \\
& \left. (4b+2c+z)^3 - \left[(4b+2c+z)^2 + p^2 \right]^{3/2} \right] - \frac{p}{2} \left[(4b+4c-2) \cdot \ln \left[\right. \right. \\
& \left. \left. (4b+4c-2) + \sqrt{(4b+4c-2)^2 + p^2} \right] - \sqrt{(4b+4c-2)^2 + p^2} \right] + \frac{1}{6 \cdot p} \left[\right. \\
& \left. (4b+4c-2)^3 - \left[(4b+4c-2)^2 + p^2 \right]^{3/2} \right] \Big] - \frac{3}{2(b+c)^3} \left[\frac{p}{2} \left[(4b+4c-2) \cdot \right. \right. \\
& \left. \ln \left[(4b+4c-2) + \sqrt{(4b+4c-2)^2 + p^2} \right] - \sqrt{(4b+4c-2)^2 + p^2} \right] - \frac{1}{6 \cdot p} \left[\right. \\
& \left. (4b+4c-2)^3 - \left[(4b+4c-2)^2 + p^2 \right]^{3/2} \right] - \frac{p}{2} \left[(4b+2c+z) \cdot \ln \left[(4b+2c+z) \right. \right. \\
& \left. \left. + \sqrt{(4b+2c+z)^2 + p^2} \right] - \sqrt{(4b+2c+z)^2 + p^2} \right] + \frac{1}{6 \cdot p} \left[(4b+2c+z)^3 - \right. \\
& \left. \left[(4b+2c+z)^2 + p^2 \right]^{3/2} \right] \Big] \Big\} + \left\{ \frac{A_1}{4(b+c)^3} \left[\frac{p}{2} \left[(3b+3c-2) \cdot \ln \left[(3b+3c-2) + \right. \right. \right. \right. \\
& \left. \left. \sqrt{(3b+3c-2)^2 + p^2} \right] - \sqrt{(3b+3c-2)^2 + p^2} \right] - \frac{1}{6 \cdot p} \left[(3b+3c-2)^3 - \left[(3b+3c-2)^2 \right. \right. \right. \\
& \left. \left. + p^2 \right]^{3/2} \right] + \frac{p}{2} \left[(7b+5c+z) \cdot \ln \left[(7b+5c+z) + \sqrt{(7b+5c+z)^2 + p^2} \right] - \right. \\
& \left. \sqrt{(7b+5c+z)^2 + p^2} \right] - \frac{1}{6 \cdot p} \left[(7b+5c+z)^3 - \left[(7b+5c+z)^2 + p^2 \right]^{3/2} \right] - \frac{p}{2}
\end{aligned}$$

$$\begin{aligned}
& [(5b+3c+z) \cdot \ln[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] - \sqrt{(5b+3c+z)^2 + p^2}] + \\
& \frac{1}{6 \cdot p} [(5b+3c+z)^3 - [(5b+3c+z)^2 + p^2]^{3/2}] - \frac{p}{2} [(5b+5c-z) \cdot \ln [\\
& (5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] - \sqrt{(5b+5c-z)^2 + p^2}] + \frac{1}{6 \cdot p} [(5b+5c-z)^3 - \\
& [(5b+5c-z)^2 + p^2]^{3/2}] + \frac{A_1(z+b)}{8(b+c)^4} [p \cdot [(5b+5c-z) \cdot \ln[(5b+5c-z) + \\
& \sqrt{(5b+5c-z)^2 + p^2}] - \sqrt{(5b+5c-z)^2 + p^2}] - \frac{2}{6 \cdot p} [(5b+5c-z)^3 - \\
& [(5b+5c-z)^2 + p^2]^{3/2}] + p \cdot [(5b+3c+z) \cdot \ln[(5b+3c+z) + \\
& \sqrt{(5b+3c+z)^2 + p^2}] - \sqrt{(5b+3c+z)^2 + p^2}] - \frac{2}{6 \cdot p} [(5b+3c+z)^3 - \\
& [(5b+3c+z)^2 + p^2]^{3/2}] - \frac{p}{2} [(3b+3c-z) \cdot \ln[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] - \\
& \sqrt{(3b+3c-z)^2 + p^2}] + \frac{1}{6 \cdot p} [(3b+3c-z)^3 - [(3b+3c-z)^2 + p^2]^{3/2}] - \frac{p}{2} [\\
& (7b+5c+z) \cdot \ln[(7b+5c+z) + \sqrt{(7b+5c+z)^2 + p^2}] - \sqrt{(7b+5c+z)^2 + p^2}] \\
& + \frac{1}{6 \cdot p} [(7b+5c+z)^3 - [(7b+5c+z)^2 + p^2]^{3/2}] - \frac{p}{2} [(3b+c+z) \cdot \ln [\\
& (3b+c+z) + \sqrt{(3b+c+z)^2 + p^2}] - \sqrt{(3b+c+z)^2 + p^2}] + \frac{1}{6 \cdot p} [(3b+c+z)^3 - \\
& [(3b+c+z)^2 + p^2]^{3/2}] - \frac{p}{2} [(7b+7c-z) \cdot \ln[(7b+7c-z) + \sqrt{(7b+7c-z)^2 + p^2}] - \\
& \sqrt{(7b+7c-z)^2 + p^2}] + \frac{1}{6 \cdot p} [(7b+7c-z)^3 - [(7b+7c-z)^2 + p^2]^{3/2}] + \frac{A_1(z-c)}{4(b+c)^3} \\
& [-\frac{(3b+3c-z)}{2 \cdot p} [\sqrt{(3b+3c-z)^2 + p^2} - (3b+3c-z)] - \frac{p}{2} \ln[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] \\
& + \frac{(7b+5c+z)}{2 \cdot p} [\sqrt{(7b+5c+z)^2 + p^2} - (7b+5c+z)] + \frac{p}{2} \ln[(7b+5c+z) + \sqrt{(7b+5c+z)^2 + p^2}] \\
& + \frac{p}{2} \ln[(7b+5c+z) + \sqrt{(7b+5c+z)^2 + p^2}] - \frac{(5b+3c+z)}{2 \cdot p} [\sqrt{(5b+3c+z)^2 + p^2} - \\
& (5b+3c+z)] - \frac{p}{2} \ln[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] + \frac{(5b+5c-z)}{2 \cdot p} \cdot [\\
& \sqrt{(5b+5c-z)^2 + p^2} - (5b+5c-z)] + \frac{p}{2} \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] \\
& - \frac{A_1}{2(b+c)^2} [-\frac{(5b+5c-z)}{2 \cdot p} [\sqrt{(5b+5c-z)^2 + p^2} - (5b+5c-z)] -
\end{aligned}$$

$$\begin{aligned}
& \frac{p}{2} \ln [(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] + \frac{(5b+3c+z)}{2 \cdot p} [\sqrt{(5b+3c+z)^2 + p^2} \\
& - (5b+3c+z)] + \frac{p}{2} \ln [(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] \Big\} + \left\{ \frac{B_1}{4(b+c)^2} \cdot \right. \\
& \left[-\frac{(3b+3c-z)}{2 \cdot p} [\sqrt{(3b+3c-z)^2 + p^2} - (3b+3c-z)] - \frac{p}{2} \ln [(3b+3c-z) + \right. \\
& \left. \sqrt{(3b+3c-z)^2 + p^2}] - \frac{(7b+5c+z)}{2 \cdot p} [\sqrt{(7b+5c+z)^2 + p^2} - (7b+5c+z)] - \right. \\
& \left. \frac{p}{2} \ln [(7b+5c+z) + \sqrt{(7b+5c+z)^2 + p^2}] + \frac{(5b+3c+z)}{2 \cdot p} [\sqrt{(5b+3c+z)^2 + p^2} - \right. \\
& \left. (5b+3c+z)] + \frac{p}{2} \ln [(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] + \frac{(5b+5c-z)}{2 \cdot p} \right. \\
& \left. \cdot [\sqrt{(5b+5c-z)^2 + p^2} - (5b+5c-z)] + \frac{p}{2} \ln [(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] \right\} \\
& + \frac{B_1(z+b)}{8(b+c)^3} \left[-\frac{(5b+5c-z)}{p} [\sqrt{(5b+5c-z)^2 + p^2} - (5b+5c-z)] - p \cdot \ln \right. \\
& \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2} \right] - \frac{(5b+3c+z)}{p} [\sqrt{(5b+3c+z)^2 + p^2} - \\
& (5b+3c+z)] - p \cdot \ln [(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] + \frac{(3b+3c-z)}{2 \cdot p} \\
& [\sqrt{(3b+3c-z)^2 + p^2} - (3b+3c-z)] + \frac{p}{2} \ln [(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] \\
& + \frac{(7b+5c+z)}{2 \cdot p} [\sqrt{(7b+5c+z)^2 + p^2} - (7b+5c+z)] + \frac{p}{2} \ln [(7b+5c+z) + \\
& \sqrt{(7b+5c+z)^2 + p^2}] + \frac{(5b+c+z)}{2 \cdot p} [\sqrt{(5b+c+z)^2 + p^2} - (5b+c+z)] + \frac{p}{2} \cdot \\
& \ln [(5b+c+z) + \sqrt{(5b+c+z)^2 + p^2}] + \frac{(7b+7c-z)}{2 \cdot p} [\sqrt{(7b+7c-z)^2 + p^2} - \\
& (7b+7c-z)] + \frac{p}{2} \ln [(7b+7c-z) + \sqrt{(7b+7c-z)^2 + p^2}] \Big\} + \frac{B_1(z-c)}{4(b+c)^2} \left[\right. \\
& -\frac{1}{p} [(3b+3c-z) - \sqrt{(3b+3c-z)^2 + p^2}] + \frac{1}{p} [(7b+5c+z) + \sqrt{(7b+5c+z)^2 + p^2}] \\
& -\frac{1}{p} [(5b+3c+z) - \sqrt{(5b+3c+z)^2 + p^2}] + \frac{1}{p} [(5b+5c-z) - \sqrt{(5b+5c-z)^2 + p^2}] \\
& \left. -\frac{B_1}{2(b+c)} \left[-\frac{1}{p} [(5b+5c-z) - \sqrt{(5b+5c-z)^2 + p^2}] + \frac{1}{p} [(5b+3c+z) - \sqrt{(5b+3c+z)^2 + p^2}] \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{C_1}{4(b+c)} \left[-\frac{1}{p} [(3b+3c-2) - \sqrt{(3b+3c-2)^2 + p^2}] - \frac{1}{p} [(7b+5c+2) - \sqrt{(7b+5c+2)^2 + p^2}] \right. \right. \\
& + \frac{1}{p} [(5b+3c+2) - \sqrt{(5b+3c+2)^2 + p^2}] + \frac{1}{p} [(5b+5c-2) - \sqrt{(5b+5c-2)^2 + p^2}] \left. \right] + \frac{C_1(2+b)}{8(b+c)^2} \\
& \cdot \left[\frac{-2}{p} [(5b+5c-2) - \sqrt{(5b+5c-2)^2 + p^2}] - \frac{2}{p} [(5b+3c+2) - \sqrt{(5b+3c+2)^2 + p^2}] + \right. \\
& \frac{1}{p} [(3b+3c-2) - \sqrt{(3b+3c-2)^2 + p^2}] + \frac{1}{p} [(7b+5c+2) - \sqrt{(7b+5c+2)^2 + p^2}] + \frac{1}{p} \cdot \\
& \left. [(3b+5c+2) - \sqrt{(3b+5c+2)^2 + p^2}] + \frac{1}{p} [(7b+7c-2) - \sqrt{(7b+7c-2)^2 + p^2}] \right] + \\
& \frac{C_1(2-c)}{4(b+c)} \left[\frac{\sqrt{(3b+3c-2)^2 + p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2 + p^2]^{1/2}} - \frac{\sqrt{(7b+5c+2)^2 + p^2} - (7b+5c+2)}{p \cdot [(7b+5c+2)^2 + p^2]^{1/2}} + \right. \\
& \frac{\sqrt{(5b+3c+2)^2 + p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2 + p^2]^{1/2}} - \frac{\sqrt{(5b+5c-2)^2 + p^2} - (5b+5c-2)}{p \cdot [(5b+5c-2)^2 + p^2]^{1/2}} \left. \right] - \frac{C_1}{2} \cdot \\
& \left\{ \frac{\sqrt{(5b+5c-2)^2 + p^2} - (5b+5c-2)}{p \cdot [(5b+5c-2)^2 + p^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2 + p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2 + p^2]^{1/2}} \right\} + \left\{ \frac{D_1}{4} \right. \\
& \left. \left[\frac{\sqrt{(3b+3c-2)^2 + p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2 + p^2]^{1/2}} + \frac{\sqrt{(7b+5c+2)^2 + p^2} - (7b+5c+2)}{p \cdot [(7b+5c+2)^2 + p^2]^{1/2}} - \right. \right. \\
& \frac{\sqrt{(5b+3c+2)^2 + p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2 + p^2]^{1/2}} - \frac{\sqrt{(5b+5c-2)^2 + p^2} - (5b+5c-2)}{p \cdot [(5b+5c-2)^2 + p^2]^{1/2}} \left. \right] + \frac{D_1(2+b)}{8(b+c)} \\
& \left[2 \frac{[\sqrt{(5b+5c-2)^2 + p^2} - (5b+5c-2)]}{p \cdot [(5b+5c-2)^2 + p^2]^{1/2}} + \frac{2[\sqrt{(5b+3c+2)^2 + p^2} - (5b+3c+2)]}{p \cdot [(5b+3c+2)^2 + p^2]^{1/2}} - \right. \\
& \frac{\sqrt{(3b+3c-2)^2 + p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2 + p^2]^{1/2}} - \frac{\sqrt{(7b+5c+2)^2 + p^2} - (7b+5c+2)}{p \cdot [(7b+5c+2)^2 + p^2]^{1/2}} - \\
& \frac{\sqrt{(3b+5c+2)^2 + p^2} - (3b+5c+2)}{p \cdot [(3b+5c+2)^2 + p^2]^{1/2}} - \frac{\sqrt{(7b+7c-2)^2 + p^2} - (7b+7c-2)}{p \cdot [(7b+7c-2)^2 + p^2]^{1/2}} \left. \right] + \frac{D_1(2-c)}{4} \\
& \left[\frac{p}{[(3b+3c-2)^2 + p^2]^{3/2}} - \frac{p}{[(7b+5c+2)^2 + p^2]^{3/2}} + \frac{p}{[(5b+3c+2)^2 + p^2]^{3/2}} - \frac{p}{[(5b+5c-2)^2 + p^2]^{3/2}} \right. \\
& \left. \right] - \frac{D_1(b+c)}{2} \left[\frac{p}{[(5b+5c-2)^2 + p^2]^{3/2}} - \frac{p}{[(5b+3c+2)^2 + p^2]^{3/2}} \right] \Big\} \quad (A-12)
\end{aligned}$$

$$\begin{aligned}
& - \left[\frac{(5b+5c-z)^3}{6} - \frac{p^2}{4}(5b+5c-z) \right] \cdot \ln \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2} \right] - \left[\frac{p^2}{4} - \frac{(5b+5c-z)^2}{6} \right] \\
& \left[\sqrt{(5b+5c-z)^2 + p^2} \right] + \frac{5}{36} \left[(5b+5c-z)^2 + p^2 \right]^{\frac{3}{2}} + \frac{3(z-c)}{4(5+c)} \left[\left[\frac{p^2}{4} - \frac{(6b+2c-z)^2}{2} \right] \cdot \right. \\
& \ln \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2} \right] + \frac{3}{4} (2b+2c-z) \sqrt{(2b+2c-z)^2 + p^2} \left. \right] - \left[\frac{p^2}{4} - \frac{(6b+2c-z)^2}{2} \right] \cdot \ln \left[(6b+2c-z) + \sqrt{(6b+2c-z)^2 + p^2} \right] - \frac{3}{4} (6b+2c-z) \sqrt{(6b+2c-z)^2 + p^2} \\
& + \left[\frac{p^2}{4} - \frac{(4b+2c+z)^2}{2} \right] \cdot \ln \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2} \right] + \frac{3}{4} (4b+2c+z) \cdot \\
& \sqrt{(4b+2c+z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(4b+4c-z)^2}{2} \right] \cdot \ln \left[(4b+4c-z) + \sqrt{(4b+4c-z)^2 + p^2} \right] - \frac{3}{4} \cdot \\
& (4b+4c-z) \sqrt{(4b+4c-z)^2 + p^2} - \frac{3}{2(5+c)^3} \left[\left[\frac{p^2}{4} - \frac{(4b+4c-z)^2}{2} \right] \cdot \ln \left[\right. \right. \\
& (4b+4c-z) + \sqrt{(4b+4c-z)^2 + p^2} \left. \right] + \frac{3}{4} (4b+4c-z) \sqrt{(4b+4c-z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(4b+2c+z)^2}{2} \right] \cdot \ln \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2} \right] - \frac{3}{4} (4b+2c+z) \cdot \\
& \sqrt{(4b+2c+z)^2 + p^2} \left. \right] + \left\{ \frac{A_1}{4(5+c)^3} \left[\left[\frac{p^2}{4} - \frac{(3b+3c-z)^2}{2} \right] \cdot \ln \left[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2} \right] + \frac{3}{4} (3b+3c-z) \sqrt{(3b+3c-z)^2 + p^2} + \left[\frac{p^2}{4} - \frac{(7b+5c+z)^2}{2} \right] \cdot \ln \left[(7b+5c+z) + \sqrt{(7b+5c+z)^2 + p^2} \right] + \frac{3}{4} (7b+5c+z) \sqrt{(7b+5c+z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(5b+3c+z)^2}{2} \right] \cdot \ln \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2} \right] - \frac{3}{4} (5b+3c+z) \sqrt{(5b+3c+z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(5b+5c-z)^2}{2} \right] \cdot \ln \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2} \right] - \frac{3}{4} (5b+5c-z) \cdot \sqrt{(5b+5c-z)^2 + p^2} + \frac{A_1(2+b)}{8(5+c)^4} \left\{ 2 \left[\frac{p^2}{4} - \frac{(5b+5c-z)^2}{2} \right] \cdot \ln \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2} \right] + 2 \cdot \frac{3}{4} (5b+5c-z) \sqrt{(5b+5c-z)^2 + p^2} + 2 \left[\frac{p^2}{4} - \frac{(5b+3c+z)^2}{2} \right] \cdot \ln \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2} \right] + 2 \cdot \frac{3}{4} (5b+3c+z) \sqrt{(5b+3c+z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(2b+2c-z)^2}{2} \right] \cdot \ln \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2} \right] - \frac{3}{4} (2b+2c-z) \sqrt{(2b+2c-z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(8b+6c+z)^2}{2} \right] \cdot \ln \left[(8b+6c+z) + \sqrt{(8b+6c+z)^2 + p^2} \right] - \frac{3}{4} (8b+6c+z) \cdot \sqrt{(8b+6c+z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(4b+2c+z)^2}{2} \right] \cdot \ln \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2} \right] - \frac{3}{4} (4b+2c+z) \cdot \sqrt{(4b+2c+z)^2 + p^2} - \left[\frac{p^2}{4} - \frac{(6b+6c-z)^2}{2} \right] \cdot \ln \left[(6b+6c-z) + \sqrt{(6b+6c-z)^2 + p^2} \right] - \frac{3}{4} (6b+6c-z) \cdot \sqrt{(6b+6c-z)^2 + p^2} \right\} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sqrt{(6b+6c-2)^2+f^2} - \frac{3}{4}(6b+6c-2)\sqrt{(6b+6c-2)^2+f^2} + \frac{A_1(2-c)}{4(b+c)^2} \left[(3b+3c-2) \cdot \right. \\
& \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2+f^2} \right] - \sqrt{(3b+3c-2)^2+f^2} - (7b+5c+2) / \ln \left[(7b+5c+2) + \right. \\
& \left. \sqrt{(7b+5c+2)^2+f^2} \right] + \sqrt{(7b+5c+2)^2+f^2} + (5b+3c+2) / \ln \left[(5b+3c+2) + \right. \\
& \left. \sqrt{(5b+3c+2)^2+f^2} \right] - \sqrt{(5b+3c+2)^2+f^2} - (5b+5c-2) / \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2} \right] \\
& \left. + \sqrt{(5b+5c-2)^2+f^2} \right] - \frac{A_1}{2(b+c)^2} \left[(5b+5c-2) \cdot \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2} \right] - \right. \\
& \left. \sqrt{(5b+5c-2)^2+f^2} - (5b+3c+2) \cdot \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2+f^2} \right] \right] + \left\{ \frac{B_1}{4(b+c)^2} \cdot \right. \\
& \left[(3b+3c-2) / \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2+f^2} \right] - \sqrt{(3b+3c-2)^2+f^2} + (7b+5c+2) \cdot \right. \\
& \ln \left[(7b+5c+2) + \sqrt{(7b+5c+2)^2+f^2} \right] - \sqrt{(7b+5c+2)^2+f^2} - (5b+3c+2) / \ln \left[\right. \\
& \left. (5b+3c+2) + \sqrt{(5b+3c+2)^2+f^2} \right] + \sqrt{(5b+3c+2)^2+f^2} - (5b+5c-2) / \ln \left[\right. \\
& \left. (5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2} \right] + \frac{B_1(2+b)}{8(b+c)^3} \left[2(5b+5c-2) / \ln \left[(5b+5c-2) + \right. \right. \\
& \left. \left. \sqrt{(5b+5c-2)^2+f^2} \right] - 2\sqrt{(5b+5c-2)^2+f^2} + 2(5b+3c+2) / \ln \left[(5b+3c+2) + \right. \right. \\
& \left. \left. \sqrt{(5b+3c+2)^2+f^2} \right] - 2\sqrt{(5b+3c+2)^2+f^2} - (2b+2c-2) / \ln \left[(2b+2c-2) + \right. \right. \\
& \left. \left. \sqrt{(2b+2c-2)^2+f^2} \right] + \sqrt{(2b+2c-2)^2+f^2} - (8b+6c+2) / \ln \left[(8b+6c+2) + \right. \right. \\
& \left. \left. \sqrt{(8b+6c+2)^2+f^2} \right] + \sqrt{(8b+6c+2)^2+f^2} - (4b+2c+2) / \ln \left[(4b+2c+2) + \right. \right. \\
& \left. \left. \sqrt{(4b+2c+2)^2+f^2} \right] + \sqrt{(4b+2c+2)^2+f^2} - (6b+6c-2) / \ln \left[(6b+6c-2) + \right. \right. \\
& \left. \left. \sqrt{(6b+6c-2)^2+f^2} \right] + \sqrt{(6b+6c-2)^2+f^2} \right] + \frac{B_1(2-c)}{4(b+c)^2} \left[- \ln \left[(3b+3c-2) + \right. \right. \\
& \left. \left. \sqrt{(3b+3c-2)^2+f^2} \right] + \ln \left[(7b+5c+2) + \sqrt{(7b+5c+2)^2+f^2} \right] - \ln \left[(5b+3c+2) + \right. \right. \\
& \left. \left. \sqrt{(5b+3c+2)^2+f^2} \right] + \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2} \right] \right] - \frac{B_1}{2(b+c)} \left[- \ln \left[\right. \right. \\
& \left. (5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2} \right] + \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2+f^2} \right] \left. \right] + \left\{ \frac{C_1}{4(b+c)} \cdot \right. \\
& \left[- \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2+f^2} \right] - \ln \left[(7b+5c+2) + \sqrt{(7b+5c+2)^2+f^2} \right] + \ln \left[\right. \right. \\
& \left. (5b+3c+2) + \sqrt{(5b+3c+2)^2+f^2} \right] + \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2} \right] \left. \right] + \frac{C_1(2+b)}{8(b+c)^2} \cdot \\
& \left[-2 \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2} \right] - 2 \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2+f^2} \right] + \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{h}[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2}] + \frac{1}{h}[(8b+6c+z) + \sqrt{(8b+6c+z)^2 + p^2}] + \frac{1}{h}[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2}] + \frac{1}{h}[(6b+6c-z) + \sqrt{(6b+6c-z)^2 + p^2}] + \\
& \frac{C_1(2-c)}{4(b+c)} \left\{ \frac{1}{[(3b+3c-z)^2 + p^2]^{1/2}} - \frac{1}{[(7b+5c+z)^2 + p^2]^{1/2}} + \frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} \right. \\
& \left. - \frac{1}{[(5b+5c-z)^2 + p^2]^{1/2}} \right\} - \frac{C_1}{2} \left\{ \frac{1}{[(8b+5c-z)^2 + p^2]^{1/2}} - \frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} \right\} \\
& + \left\{ \frac{D_1}{4} \left[\frac{1}{[(3b+3c-z)^2 + p^2]^{1/2}} + \frac{1}{[(7b+5c+z)^2 + p^2]^{1/2}} - \frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} \right. \right. \\
& \left. \left. - \frac{1}{[(5b+5c-z)^2 + p^2]^{1/2}} \right] + \frac{D_1(2+b)}{8(b+c)} \left[\frac{2}{[(5b+5c-z)^2 + p^2]^{1/2}} + \frac{2}{[(5b+3c+z)^2 + p^2]^{1/2}} \right. \right. \\
& \left. \left. - \frac{1}{[(2b+2c-z)^2 + p^2]^{1/2}} - \frac{1}{[(8b+6c+z)^2 + p^2]^{1/2}} - \frac{1}{[(4b+2c+z)^2 + p^2]^{1/2}} - \frac{1}{[(6b+6c-z)^2 + p^2]^{1/2}} \right] \right\} + \frac{D_1(2-c)}{4} \left[\frac{3b+3c-z}{[(3b+3c-z)^2 + p^2]^{3/2}} - \frac{7b+5c+z}{[(7b+5c+z)^2 + p^2]^{3/2}} \right. \\
& + \frac{5b+3c+z}{[(5b+3c+z)^2 + p^2]^{3/2}} - \frac{5b+5c-z}{[(5b+5c-z)^2 + p^2]^{3/2}} \left. \right] - \frac{D_1(5+c)}{2} \left[\frac{5b+5c-z}{[(5b+5c-z)^2 + p^2]^{3/2}} - \frac{5b+3c+z}{[(5b+3c+z)^2 + p^2]^{3/2}} \right] \Big\}, \quad (A-13)
\end{aligned}$$

$$\begin{aligned}
R(t_6(1,2,3)) = & \left\{ 2(b+c) \left[\frac{\sqrt{(2b+2c-z)^2 + p^2} - (2b+2c-z)}{p \cdot [(2b+2c-z)^2 + p^2]^{1/2}} + \frac{\sqrt{(6b+4c+z)^2 + p^2} - (6b+4c+z)}{p \cdot [(6b+4c+z)^2 + p^2]^{1/2}} \right. \right. \\
& - \frac{\sqrt{(4b+2c+z)^2 + p^2} - (4b+2c+z)}{p \cdot [(4b+2c+z)^2 + p^2]^{1/2}} - \frac{\sqrt{(4b+4c-z)^2 + p^2} - (4b+4c-z)}{p \cdot [(4b+4c-z)^2 + p^2]^{1/2}} \left. \right] + (2+b) \cdot \left[\right. \\
& \frac{2 \left[\sqrt{(4b+4c-z)^2 + p^2} - (4b+4c-z) \right]}{p \cdot [(4b+4c-z)^2 + p^2]^{1/2}} + \frac{2 \left[\sqrt{(4b+2c+z)^2 + p^2} - (4b+2c+z) \right]}{p \cdot [(4b+2c+z)^2 + p^2]^{1/2}} - \\
& \frac{\sqrt{(b+c-z)^2 + p^2} - (b+c-z)}{p \cdot [(b+c-z)^2 + p^2]^{1/2}} - \frac{\sqrt{(7b+5c+z)^2 + p^2} - (7b+5c+z)}{p \cdot [(7b+5c+z)^2 + p^2]^{1/2}} - \frac{\sqrt{(3b+c+z)^2 + p^2} - (3b+c+z)}{p \cdot [(3b+c+z)^2 + p^2]^{1/2}} \\
& \left. - \frac{\sqrt{(5b+5c-z)^2 + p^2} - (5b+5c-z)}{p \cdot [(5b+5c-z)^2 + p^2]^{1/2}} \right] + 2(b+c)(2-c) \left[\frac{p}{[(2b+2c-z)^2 + p^2]^{3/2}} - \frac{p}{[(6b+4c+z)^2 + p^2]^{3/2}} + \frac{p}{[(4b+2c+z)^2 + p^2]^{3/2}} - \frac{p}{[(4b+4c-z)^2 + p^2]^{3/2}} \right] - 4(b+c)^2.
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{p}{[(4b+4c-2)^2+p^2]^{3/2}} - \frac{p}{[(4b+2c+2)^2+p^2]^{3/2}} \right\} + \left\{ \frac{3}{4(6+c)} \left[\left(\frac{p}{12} (2b+4c-2) \right)^3 - \right. \right. \\
& \left. \frac{p^3}{16} (2b+2c-2) \right] \cdot \ln \left[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2} \right] + \left[\frac{p^3}{16} - \frac{p}{12} (2b+2c-2) \right] \cdot \\
& \sqrt{(2b+2c-2)^2+p^2} - \frac{7 \cdot p}{18 \times 8} [(4b+2c+2)^2+p^2]^{3/2} + \frac{[(2b+2c-2)^2+p^2]^{5/2}}{5 \times 24 \cdot p} - \frac{(2b+2c-2)^5}{5 \times 24 \cdot p} \\
& + \left[\frac{p}{12} (6b+4c+2) - \frac{p^3}{16} (6b+4c+2) \right] \cdot \ln \left[(6b+4c+2) + \sqrt{(6b+4c+2)^2+p^2} \right] + \\
& \left[\frac{p^3}{16} - \frac{p}{12} (6b+4c+2) \right] \cdot \sqrt{(6b+4c+2)^2+p^2} - \frac{7 \cdot p}{18 \times 8} [(6b+4c+2)^2+p^2]^{3/2} + \\
& \frac{[(6b+4c+2)^2+p^2]^{5/2}}{5 \times 24 \cdot p} - \frac{(6b+4c+2)^5}{5 \times 24 \cdot p} - \left[\frac{p}{12} (4b+2c+2) - \frac{p^3}{16} (4b+2c+2) \right] \cdot \\
& \ln \left[(4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2} \right] - \left[\frac{p^3}{16} - \frac{p}{12} (4b+2c+2) \right] \cdot \sqrt{(4b+2c+2)^2+p^2} + \\
& \frac{7 \cdot p}{18 \times 8} [(4b+2c+2)^2+p^2]^{3/2} - \frac{[(4b+2c+2)^2+p^2]^{5/2}}{5 \times 24 \cdot p} + \frac{(4b+2c+2)^5}{5 \times 24 \cdot p} - \left[\frac{p}{12} (4b+4c-2) - \right. \\
& \left. \frac{p^3}{16} (4b+4c-2) \right] \cdot \ln \left[(4b+4c-2) + \sqrt{(4b+4c-2)^2+p^2} \right] - \left[\frac{p^3}{16} - \frac{p}{12} (4b+4c-2) \right] \cdot \\
& \sqrt{(4b+4c-2)^2+p^2} + \frac{7 \cdot p}{18 \times 8} [(4b+4c-2)^2+p^2]^{3/2} - \frac{[(4b+4c-2)^2+p^2]^{5/2}}{5 \times 24 \cdot p} + \\
& \frac{(4b+4c-2)^5}{5 \times 24 \cdot p} \left. \right] + \frac{3(2+b)}{8(5+c)^5} \left[2 \left[\frac{p}{12} (4b+4c-2) - \frac{p^3}{16} (4b+4c-2) \right] \cdot \ln \left[(4b+4c-2) + \right. \right. \\
& \left. \sqrt{(4b+4c-2)^2+p^2} \right] + 2 \left[\frac{p^3}{16} - \frac{p}{12} (4b+4c-2) \right] \cdot \sqrt{(4b+4c-2)^2+p^2} - 2 \times \frac{7 \cdot p}{18 \times 8} \left[\right. \\
& \left. (4b+4c-2)^2+p^2 \right]^{3/2} + 2 \times \frac{[(4b+4c-2)^2+p^2]^{5/2}}{5 \times 24 \cdot p} - \frac{2 \times (4b+4c-2)^5}{5 \times 24 \cdot p} + 2 \cdot \left[\frac{p}{12} (4b+2c+2) - \right. \\
& \left. \frac{p^3}{16} (4b+2c+2) \right] \cdot \ln \left[(4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2} \right] + 2 \times \left[\frac{p^3}{16} - \frac{p}{12} (4b+2c+2) \right] \cdot \\
& \sqrt{(4b+2c+2)^2+p^2} - 2 \times \frac{7 \cdot p}{18 \times 8} [(4b+2c+2)^2+p^2]^{3/2} + 2 \times \frac{[(4b+2c+2)^2+p^2]^{5/2}}{5 \times 24 \cdot p} - 2 \times \\
& \frac{(4b+2c+2)^5}{5 \times 24 \cdot p} - \left[\frac{p}{12} (b+c-2) - \frac{p^3}{16} (b+c-2) \right] \cdot \ln \left[(b+c-2) + \sqrt{(b+c-2)^2+p^2} \right] - \\
& \left[\frac{p^3}{16} - \frac{p}{12} (b+c-2) \right] \cdot \sqrt{(b+c-2)^2+p^2} + \frac{7 \cdot p}{18 \times 8} [(b+c-2)^2+p^2]^{3/2} - \frac{[(b+c-2)^2+p^2]^{5/2}}{5 \times 24 \cdot p} \\
& + \frac{(b+c-2)^5}{5 \times 24 \cdot p} - \left[\frac{p}{12} (7b+5c+2) - \frac{p^3}{16} (7b+5c+2) \right] \cdot \ln \left[(7b+5c+2) + \sqrt{(7b+5c+2)^2+p^2} \right] \\
& - \left[\frac{p^3}{16} - \frac{p}{12} (7b+5c+2) \right] \cdot \sqrt{(7b+5c+2)^2+p^2} + \frac{7 \cdot p}{18 \times 8} [(7b+5c+2)^2+p^2]^{3/2} - \\
& \frac{[(7b+5c+2)^2+p^2]^{5/2}}{5 \times 24 \cdot p} + \frac{(7b+5c+2)^5}{5 \times 24 \cdot p} - \left[\frac{p}{12} (3b+c+2) - \frac{p^3}{16} (3b+c+2) \right] \cdot
\end{aligned}$$

$$\begin{aligned}
& \ln[(3b+c+2)+\sqrt{(3b+c+2)^2+f^2}] - \left[\frac{p^3}{16} - \frac{p}{12}(3b+c+2)^2\right] \cdot \sqrt{(3b+c+2)^2+f^2} + \frac{7 \cdot p}{18 \times 8} \\
& [(3b+c+2)^2+f^2]^{3/2} - \frac{[(3b+c+2)^2+f^2]^{5/2}}{5 \times 24 \cdot p} + \frac{(3b+c+2)^5}{5 \times 24 \cdot p} - \left[\frac{p}{12}(5b+5c-2)^3 - \frac{p^3}{16} \cdot \right. \\
& (5b+5c-2)] \cdot \ln[(5b+5c-2)+\sqrt{(5b+5c-2)^2+f^2}] - \left[\frac{p^3}{16} - \frac{p}{12}(5b+5c-2)^2\right] \cdot \\
& \sqrt{(5b+5c-2)^2+f^2} + \frac{7 \cdot p}{18 \times 8} [(5b+5c-2)^2+f^2]^{3/2} - \frac{[(5b+5c-2)^2+f^2]^{5/2}}{5 \times 24 \cdot p} + \frac{(5b+5c-2)^5}{5 \times 24 \cdot p} \\
& + \frac{3(2-c)}{4(1+b+c)} \left\{ \left[\frac{p^3}{16} - \frac{p}{4}(2b+2c-2)^2\right] \cdot \ln[(2b+2c-2)+\sqrt{(2b+2c-2)^2+f^2}] + \frac{5 \cdot p}{16} \cdot \right. \\
& (2b+2c-2) \sqrt{(2b+2c-2)^2+f^2} - \frac{(2b+2c-2)}{24 \cdot p} [(2b+2c-2)^2+f^2]^{3/2} - \frac{(2b+2c-2)^5}{24 \cdot p} \\
& - \left[\frac{p^3}{16} - \frac{p}{4}(6b+4c+2)^2\right] \cdot \ln[(6b+4c+2)+\sqrt{(6b+4c+2)^2+f^2}] - \frac{5 \cdot p}{16} (6b+4c+2) \cdot \\
& \sqrt{(6b+4c+2)^2+f^2} + \frac{(6b+4c+2)}{24 \cdot p} [(6b+4c+2)^2+f^2]^{3/2} - \frac{(6b+4c+2)^5}{24 \cdot p} + \left[\frac{p^3}{16} - \frac{p}{4} \cdot \right. \\
& (4b+2c+2)^2] \cdot \ln[(4b+2c+2)+\sqrt{(4b+2c+2)^2+f^2}] + \frac{5 \cdot p}{16} (4b+2c+2) \cdot \\
& \sqrt{(4b+2c+2)^2+f^2} - \frac{(4b+2c+2)}{24 \cdot p} [(4b+2c+2)^2+f^2]^{3/2} + \frac{(4b+2c+2)^5}{24 \cdot p} - \left[\frac{p^3}{16} - \right. \\
& \frac{p}{4}(4b+4c-2)^2] \cdot \ln[(4b+4c-2)+\sqrt{(4b+4c-2)^2+f^2}] - \frac{5 \cdot p}{16} (4b+4c-2) \sqrt{(4b+4c-2)^2+f^2} \\
& + \frac{(4b+4c-2)}{24 \cdot p} [(4b+4c-2)^2+f^2]^{3/2} - \frac{(4b+4c-2)^5}{24 \cdot p} \left. \right\} - \frac{3}{2(1+b+c)^3} \left\{ \left[\frac{p^3}{16} - \frac{p}{4}(4b+4c-2)^2\right] \cdot \right. \\
& \ln[(4b+4c-2)+\sqrt{(4b+4c-2)^2+f^2}] + \frac{5 \cdot p}{16} (4b+4c-2) \sqrt{(4b+4c-2)^2+f^2} - \frac{(4b+4c-2)}{24 \cdot p} \\
& [(4b+4c-2)^2+f^2]^{3/2} + \frac{(4b+4c-2)^5}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4}(4b+2c+2)^2\right] \cdot \ln[(4b+2c+2)+ \\
& \sqrt{(4b+2c+2)^2+f^2}] - \frac{5 \cdot p}{16} (4b+2c+2) \sqrt{(4b+2c+2)^2+f^2} + \frac{(4b+2c+2)}{24 \cdot p} [(4b+2c+2)^2 \\
& + f^2]^{3/2} - \frac{(4b+2c+2)^5}{24 \cdot p} \left. \right\} + \left\{ \frac{A_1}{4(1+b+c)^3} \left[\left[\frac{p^3}{16} - \frac{p}{4}(3b+3c-2)^2\right] \cdot \ln[(3b+3c-2) \right. \right. \\
& + \sqrt{(3b+3c-2)^2+f^2}] + \frac{5 \cdot p}{16} (3b+3c-2) \sqrt{(3b+3c-2)^2+f^2} - \frac{(3b+3c-2)}{24 \cdot p} [(3b+3c-2)^2+f^2]^{3/2} \\
& + \frac{(3b+3c-2)^5}{24 \cdot p} + \left[\frac{p^3}{16} - \frac{p}{4}(7b+5c+2)^2\right] \cdot \ln[(7b+5c+2)+\sqrt{(7b+5c+2)^2+f^2}] + \\
& \frac{5 \cdot p}{16} (7b+5c+2) \sqrt{(7b+5c+2)^2+f^2} - \frac{(7b+5c+2)}{24 \cdot p} [(7b+5c+2)^2+f^2]^{3/2} + \\
& \frac{(7b+5c+2)^5}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4}(5b+3c+2)^2\right] \cdot \ln[(5b+3c+2)+\sqrt{(5b+3c+2)^2+f^2}] - \\
& \frac{5 \cdot p}{16} (5b+3c+2) \sqrt{(5b+3c+2)^2+f^2} + \frac{(5b+3c+2)}{24 \cdot p} [(5b+3c+2)^2+f^2]^{3/2} - \\
& \frac{(5b+3c+2)^5}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4}(5b+5c-2)^2\right] \ln[(5b+5c-2)+\sqrt{(5b+5c-2)^2+f^2}] \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& - \frac{5 \cdot p}{16} (5b+5c-2) \cdot \sqrt{(5b+5c-2)^2 + p^2} + \frac{(5b+5c-2)}{24 \cdot p} [(5b+5c-2)^2 + p^2]^{3/2} \\
& \frac{(5b+5c-2)^4}{24 \cdot p} \Big] + \frac{A_1(2+b)}{8(b+c)^4} \Big[2 \cdot \left[\frac{p^3}{16} - \frac{p}{4} (5b+5c-2)^2 \right] \cdot \ln[(5b+5c-2) + \\
& \sqrt{(5b+5c-2)^2 + p^2}] + 2 \times \frac{5 \cdot p}{16} (5b+5c-2) \sqrt{(5b+5c-2)^2 + p^2} - 2 \times \frac{(5b+5c-2)}{24 \cdot p} \\
& [(5b+5c-2)^2 + p^2]^{3/2} + 2 \times \frac{(5b+5c-2)^4}{24 \cdot p} + 2 \cdot \left[\frac{p^3}{16} - \frac{p}{4} (5b+3c+2)^2 \right] \cdot \ln[\\
& (5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] + 2 \times \frac{5 \cdot p}{16} (5b+3c+2) \sqrt{(5b+3c+2)^2 + p^2} - 2 \times \\
& \frac{(5b+3c+2)}{24 \cdot p} [(5b+3c+2)^2 + p^2]^{3/2} + 2 \times \frac{(5b+3c+2)^4}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4} (2b+2c-2)^2 \right] \\
& \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2 + p^2}] - \frac{5 \cdot p}{16} (2b+2c-2) \sqrt{(2b+2c-2)^2 + p^2} + \\
& \frac{(2b+2c-2)}{24 \cdot p} [(2b+2c-2)^2 + p^2]^{3/2} - \frac{(2b+2c-2)^4}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4} (8b+6c+2)^2 \right] \cdot \\
& \ln[(8b+6c+2) + \sqrt{(8b+6c+2)^2 + p^2}] - \frac{5 \cdot p}{16} (8b+6c+2) \sqrt{(8b+6c+2)^2 + p^2} + \\
& \frac{(8b+6c+2)}{24 \cdot p} [(8b+6c+2)^2 + p^2]^{3/2} - \frac{(8b+6c+2)^4}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4} (4b+2c+2)^2 \right] \cdot \\
& \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + p^2}] - \frac{5 \cdot p}{16} (4b+2c+2) \sqrt{(4b+2c+2)^2 + p^2} + \\
& \frac{(4b+2c+2)}{24 \cdot p} [(4b+2c+2)^2 + p^2]^{3/2} - \frac{(4b+2c+2)^4}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4} (6b+6c-2)^2 \right] \cdot \\
& \ln[(6b+6c-2) + \sqrt{(6b+6c-2)^2 + p^2}] - \frac{5 \cdot p}{16} (6b+6c-2) \sqrt{(6b+6c-2)^2 + p^2} + \\
& \frac{(6b+6c-2)}{24 \cdot p} [(6b+6c-2)^2 + p^2]^{3/2} - \frac{(6b+6c-2)^4}{24 \cdot p} \Big] + \frac{A_1(2-c)}{4(b+c)^3} \left[\frac{p}{2} (3b+3c-2) \cdot \right. \\
& \ln[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2}] - \frac{p}{2} \sqrt{(3b+3c-2)^2 + p^2} - \frac{(3b+3c-2)^3}{6 \cdot p} + \\
& \frac{[(3b+3c-2)^2 - p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} \cdot (7b+5c+2) \ln[(7b+5c+2) + \sqrt{(7b+5c+2)^2 + p^2}] + \\
& \frac{p}{2} \sqrt{(7b+5c+2)^2 + p^2} + \frac{(7b+5c+2)^3}{6 \cdot p} - \frac{[(7b+5c+2)^2 + p^2]^{3/2}}{6 \cdot p} + \frac{p}{2} (5b+3c+2) \cdot \\
& \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] - \frac{p}{2} \sqrt{(5b+3c+2)^2 + p^2} - \frac{(5b+3c+2)^3}{6 \cdot p} + \\
& \frac{[(5b+3c+2)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (5b+5c-2) \cdot \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2}] \\
& + \frac{p}{2} \sqrt{(5b+5c-2)^2 + p^2} + \frac{(5b+5c-2)^3}{6 \cdot p} - \frac{[(5b+5c-2)^2 + p^2]^{3/2}}{6 \cdot p} \Big] - \frac{A_1}{2(b+c)^2} \cdot \\
& \left[\frac{p}{2} (5b+5c-2) \cdot \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2}] - \frac{p}{2} \sqrt{(5b+5c-2)^2 + p^2} - \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{(5b+5c-2)^3}{6 \cdot p} + \frac{[(5b+5c-2)^2+f^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(5b+3c+2) \cdot \ln[(5b+3c+2) + \\
& \sqrt{(5b+3c+2)^2+f^2}] + \frac{p}{2} \sqrt{(5b+3c+2)^2+f^2} + \frac{(5b+3c+2)^3}{6 \cdot p} - \frac{[(5b+3c+2)^2+f^2]^{3/2}}{6 \cdot p} \Bigg\} + \\
& \Bigg\{ \frac{B_1}{4(b+c)^2} \left[\frac{p}{2}(3b+3c-2) \cdot \ln[(3b+3c-2) + \sqrt{(3b+3c-2)^2+f^2}] - \frac{p}{2} \sqrt{(3b+3c-2)^2+f^2} - \right. \\
& \frac{(3b+3c-2)^3}{6 \cdot p} + \frac{[(3b+3c-2)^2+f^2]^{3/2}}{6 \cdot p} + \frac{p}{2}(7b+5c+2) \cdot \ln[(7b+5c+2) + \sqrt{(7b+5c+2)^2+f^2}] \\
& \left. - \frac{p}{2} \sqrt{(7b+5c+2)^2+f^2} - \frac{(7b+5c+2)^3}{6 \cdot p} + \frac{[(7b+5c+2)^2+f^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(5b+3c+2) \cdot \right. \\
& \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2+f^2}] + \frac{p}{2} \sqrt{(5b+3c+2)^2+f^2} + \frac{(5b+3c+2)^3}{6 \cdot p} \\
& \left. - \frac{[(5b+3c+2)^2+f^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(5b+5c-2) \cdot \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2}] + \frac{p}{2} \cdot \right. \\
& \left. \sqrt{(5b+5c-2)^2+f^2} + \frac{(5b+5c-2)^3}{6 \cdot p} - \frac{[(5b+5c-2)^2+f^2]^{3/2}}{6 \cdot p} \right] + \frac{B_1(2+b)}{8(b+c)^3} \left[2 \times \frac{p}{2} \cdot \right. \\
& (5b+5c-2) \cdot \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2+f^2}] - 2 \times \frac{p}{2} \sqrt{(5b+5c-2)^2+f^2} - \\
& 2 \times \frac{(5b+5c-2)^3}{6 \cdot p} + 2 \times \frac{[(5b+5c-2)^2+f^2]^{3/2}}{6 \cdot p} + 2 \times \frac{p}{2}(5b+3c+2) \cdot \ln[(5b+3c+2) \\
& + \sqrt{(5b+3c+2)^2+f^2}] - 2 \times \frac{p}{2} \sqrt{(5b+3c+2)^2+f^2} - 2 \times \frac{(5b+3c+2)^3}{6 \cdot p} + 2 \times \frac{[(5b+3c+2)^2+f^2]^{3/2}}{6 \cdot p} \\
& \left. - \frac{p}{2}(2b+2c-2) \cdot \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2+f^2}] + \frac{p}{2} \sqrt{(2b+2c-2)^2+f^2} + \right. \\
& \left. \frac{(2b+2c-2)^3}{6 \cdot p} - \frac{[(2b+2c-2)^2+f^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(8b+6c+2) \cdot \ln[(8b+6c+2) + \sqrt{(8b+6c+2)^2+f^2}] \right. \\
& \left. + \frac{p}{2} \sqrt{(8b+6c+2)^2+f^2} + \frac{(8b+6c+2)^3}{6 \cdot p} - \frac{[(8b+6c+2)^2+f^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(4b+2c+2) \cdot \right. \\
& \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2+f^2}] + \frac{p}{2} \sqrt{(4b+2c+2)^2+f^2} + \frac{(4b+2c+2)^3}{6 \cdot p} - \\
& \left. \frac{[(4b+2c+2)^2+f^2]^{3/2}}{6 \cdot p} - \frac{p}{3}(6b+6c-2) \cdot \ln[(6b+6c-2) + \sqrt{(6b+6c-2)^2+f^2}] + \frac{p}{2} \right. \\
& \left. \sqrt{(6b+6c-2)^2+f^2} + \frac{(6b+6c-2)^3}{6 \cdot p} - \frac{[(6b+6c-2)^2+f^2]^{3/2}}{6 \cdot p} \right] + \frac{B_1(2-c)}{4(b+c)^2} \left[- \right. \\
& \frac{(3b+3c-2)}{2 \cdot p} \cdot \sqrt{(3b+3c-2)^2+f^2} - \frac{p}{2} \ln[(3b+3c-2) + \sqrt{(3b+3c-2)^2+f^2}] + \frac{(3b+3c-2)^2}{2 \cdot p} \\
& + \frac{(7b+5c+2)}{2 \cdot p} \sqrt{(7b+5c+2)^2+f^2} + \frac{p}{2} \ln[(7b+5c+2) + \sqrt{(7b+5c+2)^2+f^2}] - \\
& \frac{(7b+5c+2)^2}{2 \cdot p} - \frac{(8b+3c+2)}{2 \cdot p} \sqrt{(5b+3c+2)^2+f^2} - \frac{p}{2} \cdot \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2+f^2}] \\
& \left. + \frac{(5b+3c+2)^2}{2 \cdot p} + \frac{(5b+5c-2)}{2 \cdot p} \sqrt{(5b+5c-2)^2+f^2} + \frac{p}{2} \cdot \ln[(5b+5c-2) + \right.
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{(5b+5c-2)^2}{2 \cdot p} \right] - \frac{B_1}{2(b+c)} \left[-\frac{(5b+5c-2)}{2 \cdot p} \sqrt{(5b+5c-2)^2 + p^2} \right. \\
& - \frac{f}{2} \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2}] + \frac{(5b+5c-2)^2}{2 \cdot p} + \frac{(5b+3c+2)}{2 \cdot p} \sqrt{(5b+3c+2)^2 + p^2} \\
& + \frac{f}{2} \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] - \frac{(5b+3c+2)^2}{2 \cdot p} \left. \right\} + \left\{ \frac{C_1}{4(b+c)} \left[-\frac{(3b+3c-2)}{2 \cdot p} \right. \right. \\
& - \sqrt{(3b+3c-2)^2 + p^2} - \frac{f}{2} \ln[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2}] + \frac{(3b+3c-2)^2}{2 \cdot p} - \frac{(7b+5c+2)}{2 \cdot p} \\
& \sqrt{(7b+5c+2)^2 + p^2} - \frac{f}{2} \ln[(7b+5c+2) + \sqrt{(7b+5c+2)^2 + p^2}] + \frac{(7b+5c+2)^2}{2 \cdot p} + \\
& \frac{(5b+3c+2)}{2 \cdot p} \sqrt{(5b+3c+2)^2 + p^2} + \frac{f}{2} \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] - \\
& \frac{(5b+3c+2)^2}{2 \cdot p} + \frac{(5b+5c-2)}{2 \cdot p} \sqrt{(5b+5c-2)^2 + p^2} + \frac{f}{2} \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2}] - \\
& \frac{(5b+5c-2)^2}{2 \cdot p} \left. \right\} + \frac{C_1(2-c)}{8(b+c)^2} \left[-2 \times \frac{(5b+5c-2)}{2 \cdot p} \sqrt{(5b+5c-2)^2 + p^2} - 2 \times \frac{f}{2} \ln[\right. \\
& (5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2}] + 2 \times \frac{(5b+5c-2)^2}{2 \cdot p} - 2 \times \frac{(5b+3c+2)}{2 \cdot p} \sqrt{(5b+3c+2)^2 + p^2} \\
& - 2 \times \frac{f}{2} \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] + 2 \times \frac{(5b+3c+2)^2}{2 \cdot p} + \frac{(2b+2c+2)}{2 \cdot p} \\
& \sqrt{(2b+2c+2)^2 + p^2} + \frac{f}{2} \ln[(2b+2c+2) + \sqrt{(2b+2c+2)^2 + p^2}] - \frac{(2b+2c+2)^2}{2 \cdot p} + \\
& \frac{(8b+6c+2)}{2 \cdot p} \sqrt{(8b+6c+2)^2 + p^2} + \frac{f}{2} \ln[(8b+6c+2) + \sqrt{(8b+6c+2)^2 + p^2}] - \\
& \frac{(8b+6c+2)^2}{2 \cdot p} + \frac{(4b+2c+2)}{2 \cdot p} \sqrt{(4b+2c+2)^2 + p^2} + \frac{f}{2} \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + p^2}] \\
& - \frac{(4b+2c+2)^2}{2 \cdot p} + \frac{(6b+6c-2)}{2 \cdot p} \sqrt{(6b+6c-2)^2 + p^2} + \frac{f}{2} \ln[(6b+6c-2) + \\
& \sqrt{(6b+6c-2)^2 + p^2}] - \frac{(6b+6c-2)^2}{2 \cdot p} \left. \right] + \frac{C_1(2-c)}{4(b+c)} \left[-\frac{(3b+3c-2)}{p} + \frac{\sqrt{(3b+3c-2)^2 + p^2}}{p} \right. \\
& + \frac{(7b+5c+2)}{p} - \frac{\sqrt{(7b+5c+2)^2 + p^2}}{p} - \frac{(5b+3c+2)}{p} + \frac{\sqrt{(5b+3c+2)^2 + p^2}}{p} + \frac{(5b+5c-2)}{p} \\
& - \frac{\sqrt{(5b+5c-2)^2 + p^2}}{p} \left. \right] - \frac{C_1}{2} \left[-\frac{(5b+5c-2)}{p} + \frac{\sqrt{(5b+5c-2)^2 + p^2}}{p} + \frac{(5b+3c+2)}{p} \right. \\
& - \frac{\sqrt{(5b+3c+2)^2 + p^2}}{p} \left. \right\} + \left\{ \frac{D_1}{4} \left[-\frac{(3b+3c-2)}{p} + \frac{\sqrt{(3b+3c-2)^2 + p^2}}{p} - \frac{(7b+5c+2)}{p} + \right. \right. \\
& \frac{\sqrt{(7b+5c+2)^2 + p^2}}{p} + \frac{(5b+3c+2)}{p} - \frac{\sqrt{(5b+3c+2)^2 + p^2}}{p} + \frac{(5b+5c-2)}{p} - \frac{\sqrt{(5b+5c-2)^2 + p^2}}{p} \left. \right] \\
& + \frac{D_1(2+b)}{8(b+c)} \left[-\frac{2(5b+5c-2)}{p} + \frac{2\sqrt{(5b+5c-2)^2 + p^2}}{p} - \frac{2(5b+3c+2)}{p} + \frac{2\sqrt{(5b+3c+2)^2 + p^2}}{p} \right. \\
& + \frac{(2b+2c-2)}{p} - \frac{\sqrt{(2b+2c-2)^2 + p^2}}{p} + \frac{(8b+6c+2)}{p} - \frac{\sqrt{(8b+6c+2)^2 + p^2}}{p} + \frac{(4b+2c+2)}{p}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\sqrt{(4b+2c+2)^2 + f^2}}{p} + \frac{(6b+6c-2)}{p} - \frac{\sqrt{(6b+6c-2)^2 + f^2}}{p} \Big] + \frac{D_1(2-c)}{4} \Big[\\
& \frac{\sqrt{(3b+3c-2)^2 + f^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2 + f^2]^{1/2}} - \frac{\sqrt{(7b+5c+2)^2 + f^2} - (7b+5c+2)}{p \cdot [(7b+5c+2)^2 + f^2]^{1/2}} + \\
& \frac{\sqrt{(5b+3c+2)^2 + f^2} - (5b+3c-2)}{p \cdot [(5b+3c+2)^2 + f^2]^{1/2}} - \frac{\sqrt{(5b+5c-2)^2 + f^2} - (5b+5c-2)}{p \cdot [(5b+5c-2)^2 + f^2]^{1/2}} \Big] - \frac{D_1(b+c)}{2} \\
& \left\{ \frac{\sqrt{(5b+5c-2)^2 + f^2} - (5b+5c-2)}{p \cdot [(5b+5c-2)^2 + f^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2 + f^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2 + f^2]^{1/2}} \right\} \quad (A-14)
\end{aligned}$$

$$\begin{aligned}
RG_1(1,1,2) = & \left\{ 2(b+c)(2-c) \left[\frac{3(2b+2c-2) \cdot [2(2b+2c-2)^2 - 3f^2]}{[(2b+2c-2)^2 + f^2]^{7/2}} - \right. \right. \\
& \left. \frac{3(4b+4c+2) \cdot [2(4b+4c+2)^2 - 3f^2]}{[(4b+4c+2)^2 + f^2]^{7/2}} \right] + (2+b) \cdot \left[\frac{2(2b+2c-2)^2 - f^2}{[(2b+2c-2)^2 + f^2]^{5/2}} + \right. \\
& \left. \frac{2(4b+4c+2)^2 - f^2}{[(4b+4c+2)^2 + f^2]^{5/2}} - \frac{2(2b+2)^2 - f^2}{[(2b+2)^2 + f^2]^{5/2}} - \frac{2(4b+4c-2)^2 - f^2}{[(4b+4c-2)^2 + f^2]^{5/2}} \right] \Big\} + \\
& \left\{ \frac{3(2-c)}{2(b+c)^3} \left[-\ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2 + f^2}] + \ln[(4b+4c+2) + \sqrt{(4b+4c+2)^2 + f^2}] \right] \right. \\
& + \frac{3(2+b)}{4(b+c)^4} \left[(2b+2c-2) \cdot \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2 + f^2}] - \sqrt{(2b+2c-2)^2 + f^2} \right. \\
& + (4b+4c+2) \cdot \ln[(4b+4c+2) + \sqrt{(4b+4c+2)^2 + f^2}] - \sqrt{(4b+4c+2)^2 + f^2} - \\
& (2b+2) \cdot \ln[(2b+2) + \sqrt{(2b+2)^2 + f^2}] + \sqrt{(2b+2)^2 + f^2} - (4b+4c-2) \cdot \ln[\\
& (4b+4c-2) + \sqrt{(4b+4c-2)^2 + f^2}] + \sqrt{(4b+4c-2)^2 + f^2} \Big] \Big\} + \left\{ \frac{A_1(2-c)}{2(b+c)^2} \cdot \right. \\
& \left[\frac{1}{[(3b+3c-2)^2 + f^2]^{1/2}} - \frac{1}{[(5b+3c+2)^2 + f^2]^{1/2}} \right] + \frac{A_1(2+b)}{4(b+c)^3} \left[-\ln[(3b+3c-2) + \right. \\
& \left. \sqrt{(3b+3c-2)^2 + f^2}] - \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + f^2}] + \ln[(3b+3c+2) + \right. \\
& \left. \sqrt{(3b+3c+2)^2 + f^2}] + \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + f^2}] \right] \Big\} + \left\{ \frac{B_1(2-c)}{2(b+c)} \left[\right. \right. \\
& \left. \frac{3b+3c-2}{[(3b+3c-2)^2 + f^2]^{3/2}} - \frac{5b+3c+2}{[(5b+3c+2)^2 + f^2]^{3/2}} \right] + \frac{B_1(2+b)}{4(b+c)^2} \left[\frac{1}{[(3b+3c-2)^2 + f^2]^{1/2}} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{[(5b+3c+2)^2+p^2]^{1/2}} - \frac{1}{[(3b+c+2)^2+p^2]^{1/2}} - \frac{1}{[(5b+5c-2)^2+p^2]^{1/2}} \Big\} + \\
& \left\{ \frac{C_0(2-c)}{2} \left[\frac{2(3b+3c-2)^2-p^2}{[(3b+3c-2)^2+p^2]^{3/2}} - \frac{2(3b+3c+2)^2-p^2}{[(3b+3c+2)^2+p^2]^{3/2}} \right] + \frac{C_1(2+b)}{4(b+c)} \right. \\
& \left[\frac{3b+3c-2}{[(3b+3c-2)^2+p^2]^{3/2}} + \frac{5b+3c+2}{[(5b+3c+2)^2+p^2]^{3/2}} - \frac{3b+c+2}{[(3b+c+2)^2+p^2]^{3/2}} \right. \\
& \left. \left. - \frac{5b+5c-2}{[(5b+5c-2)^2+p^2]^{3/2}} \right] \right\}, \quad (A-15)
\end{aligned}$$

$$\begin{aligned}
R_{G_3}(1,1,1) = & \left\{ 2(b+c)(2-c) \left[\frac{2(2b+2c-2)^2-p^2}{[(2b+2c-2)^2+p^2]^{3/2}} + \frac{2(4b+2c+2)^2-p^2}{[(4b+2c+2)^2+p^2]^{3/2}} \right] + \right. \\
& \left[\frac{1}{[(2b+2)^2+p^2]^{1/2}} + \frac{1}{[(4b+4c-2)^2+p^2]^{1/2}} - \frac{1}{[(2b+2c-2)^2+p^2]^{1/2}} - \frac{1}{[(4b+2c+2)^2+p^2]^{1/2}} \right] + \\
& 2(b+c) \left[\frac{2b+2c-2}{[(2b+2c-2)^2+p^2]^{3/2}} - \frac{4b+2c+2}{[(4b+2c+2)^2+p^2]^{3/2}} \right] - (2+b) \cdot \frac{(2b+2c-2)}{[(2b+2c-2)^2+p^2]^{3/2}} \\
& - \frac{4b+2c+2}{[(4b+2c+2)^2+p^2]^{3/2}} + \frac{2b+2}{[(2b+2)^2+p^2]^{3/2}} - \frac{4b+4c-2}{[(4b+4c-2)^2+p^2]^{3/2}} \Big\} + \left\{ \frac{3(2-c)}{2(b+c)^3} \cdot \right. \\
& [(2b+2c-2) \cdot \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2}] - \sqrt{(2b+2c-2)^2+p^2} + (4b+2c+2) \cdot \\
& \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2}] - \sqrt{(4b+2c+2)^2+p^2}] + \frac{3}{4(b+c)^4} \left[\left(\frac{2b+2}{6} \right)^3 - \frac{p^2}{4} \cdot \right. \\
& (2b+2) \cdot \ln[(2b+2) + \sqrt{(2b+2)^2+p^2}] + \left[\frac{p^2}{4} - \frac{(2b+2)^2}{6} \right] \cdot \sqrt{(2b+2)^2+p^2} - \frac{5}{36} \cdot \\
& [(2b+2)^2+p^2]^{3/2} + \left[\frac{p^2}{4} - \frac{(4b+4c-2)^2}{6} \right] \cdot \ln[(4b+4c-2) + \sqrt{(4b+4c-2)^2+p^2}] + \\
& \sqrt{(4b+4c-2)^2+p^2} + \left[\frac{p^2}{4} - \frac{(4b+4c-2)^2}{6} \right] \cdot \sqrt{(4b+4c-2)^2+p^2} - \frac{5}{36} [(4b+4c-2)^2+p^2]^{3/2} \\
& - \left[\frac{(2b+2c-2)^3}{6} - \frac{p^2}{4} (2b+2c-2) \right] \cdot \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2}] - \left[\frac{p^2}{4} - \frac{(2b+2c-2)^2}{6} \right] \cdot \\
& \sqrt{(2b+2c-2)^2+p^2} + \frac{5}{36} [(2b+2c-2)^2+p^2]^{3/2} - \left[\frac{(4b+2c+2)^3}{6} - \frac{p^2}{4} (4b+2c+2) \right] \cdot \\
& \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2}] - \left[\frac{p^2}{4} - \frac{(4b+2c+2)^2}{6} \right] \cdot \sqrt{(4b+2c+2)^2+p^2} + \frac{5}{36} \cdot \\
& [(4b+2c+2)^2+p^2]^{3/2} + \frac{3}{2(b+c)^3} \left[\left[\frac{p^2}{4} - \frac{(2b+2c-2)^2}{2} \right] \cdot \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2}] + \right. \\
& \sqrt{(2b+2c-2)^2+p^2} + \frac{3}{4} (2b+2c-2) \cdot \sqrt{(2b+2c-2)^2+p^2} - \left[\frac{p^2}{4} - \frac{(4b+2c+2)^2}{2} \right] \cdot \ln[\\
& (4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2}] - \frac{3}{4} (4b+2c+2) \cdot \sqrt{(4b+2c+2)^2+p^2} \Big] - \frac{3(2+b)}{4(b+c)^4}.
\end{aligned}$$

$$\begin{aligned}
& \left[\left(\frac{p^2}{4} - \frac{(2b+2c-2)^2}{2} \right) \cdot \ln \left[(2b+2c-2) + \sqrt{(2b+2c-2)^2 + p^2} \right] + \frac{3}{4} (2b+2c-2) \cdot \right. \\
& \sqrt{(2b+2c-2)^2 + p^2} - \left. \left(\frac{p^2}{4} - \frac{(4b+2c+2)^2}{2} \right) \cdot \ln \left[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + p^2} \right] - \right. \\
& \frac{3}{4} (4b+2c+2) \cdot \sqrt{(4b+2c+2)^2 + p^2} + \left. \left(\frac{p^2}{4} - \frac{(2b+2)^2}{2} \right) \cdot \ln \left[(2b+2) + \sqrt{(2b+2)^2 + p^2} \right] \right. \\
& + \frac{3}{4} (2b+2) \cdot \sqrt{(2b+2)^2 + p^2} - \left. \left(\frac{p^2}{4} - \frac{(4b+4c-2)^2}{2} \right) \cdot \ln \left[(4b+4c-2) + \sqrt{(4b+4c-2)^2 + p^2} \right] \right. \\
& - \frac{3}{4} (4b+4c-2) \cdot \sqrt{(4b+4c-2)^2 + p^2} \left. \right\} + \left\{ \frac{A_2(2-c)}{2(b+c)^2} \left[-\ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] \right. \right. \\
& - \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] \left. \right] + \frac{A_2}{4(b+c)^3} \left[\left(\frac{p^2}{4} - \frac{(3b+c+2)^2}{2} \right) \cdot \ln \left[(3b+c+2) + \right. \right. \\
& \sqrt{(3b+c+2)^2 + p^2} \left. \right] + \frac{3}{4} (3b+c+2) \cdot \sqrt{(3b+c+2)^2 + p^2} + \left. \left(\frac{p^2}{4} - \frac{(5b+5c-2)^2}{2} \right) \cdot \ln \left[(5b+5c-2) + \right. \right. \\
& \sqrt{(5b+5c-2)^2 + p^2} \left. \right] + \frac{3}{4} (5b+5c-2) \cdot \sqrt{(5b+5c-2)^2 + p^2} - \left. \left(\frac{p^2}{4} - \frac{(3b+3c-2)^2}{2} \right) \cdot \ln \left[\right. \right. \\
& (3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \left. \right] - \frac{3}{4} (3b+3c-2) \cdot \sqrt{(3b+3c-2)^2 + p^2} - \left. \left(\frac{p^2}{4} - \right. \right. \\
& \left. \left. \frac{(5b+3c+2)^2}{2} \right) \cdot \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] - \frac{3}{4} (5b+3c+2) \cdot \sqrt{(5b+3c+2)^2 + p^2} \right. \\
& + \frac{A_2}{2(b+c)^2} \left[(3b+3c-2) \cdot \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] - \sqrt{(3b+3c-2)^2 + p^2} - \right. \\
& (5b+3c+2) \cdot \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] + \sqrt{(5b+3c+2)^2 + p^2} \left. \right] - \frac{A_2(2+b)}{4(b+c)^3} \\
& \left[(3b+3c-2) \cdot \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] - \sqrt{(3b+3c-2)^2 + p^2} - (5b+3c+2) \cdot \right. \\
& \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] + \sqrt{(5b+3c+2)^2 + p^2} + (3b+c+2) \cdot \ln \left[\right. \\
& (3b+c+2) + \sqrt{(3b+c+2)^2 + p^2} \left. \right] - \sqrt{(3b+c+2)^2 + p^2} - (5b+5c-2) \cdot \ln \left[(5b+5c-2) + \right. \\
& \sqrt{(5b+5c-2)^2 + p^2} \left. \right] + \sqrt{(5b+5c-2)^2 + p^2} \left. \right] \left. \right\} + \left\{ \frac{B_2(2-c)}{2(b+c)} \left[\frac{1}{[(3b+3c-2)^2 + p^2]^{\frac{1}{2}}} \right. \right. \\
& + \frac{1}{[(5b+3c+2)^2 + p^2]^{\frac{1}{2}}} \left. \right] + \frac{B_2}{4(b+c)^2} \left[(3b+c+2) \cdot \ln \left[(3b+c+2) + \sqrt{(3b+c+2)^2 + p^2} \right] - \right. \\
& \sqrt{(3b+c+2)^2 + p^2} + (5b+5c-2) \cdot \ln \left[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2} \right] - \sqrt{(5b+5c-2)^2 + p^2} \\
& - (3b+3c-2) \cdot \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] + \sqrt{(3b+3c-2)^2 + p^2} - (5b+3c+2) \cdot \\
& \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] + \sqrt{(5b+3c+2)^2 + p^2} \left. \right] + \frac{B_2}{2(b+c)} \left[-\ln \left[\right. \right. \\
& (3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \left. \right] + \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] \left. \right] - \frac{B_2(2+b)}{4(b+c)^2} \\
& \left[-\ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2} \right] + \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2} \right] - \right.
\end{aligned}$$

$$\begin{aligned}
& \ln[(3b+c+z) + \sqrt{(3b+c+z)^2 + p^2}] + \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] \Big\} + \\
& \left\{ \frac{C_2(z-c)}{2} \left[\frac{3b+3c-z}{[(3b+3c-z)^2 + p^2]^{3/2}} + \frac{5b+3c+z}{[(5b+3c+z)^2 + p^2]^{3/2}} \right] + \frac{C_2}{4(b+c)} \cdot \left[- \right. \right. \\
& \ln[(3b+c+z) + \sqrt{(3b+c+z)^2 + p^2}] - \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] + \ln[\\
& (3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] + \ln[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] \Big] + \\
& \frac{C_2}{2} \left[\frac{1}{[(3b+3c-z)^2 + p^2]^{1/2}} - \frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} - \frac{C_2(z+b)}{4(b+c)} \cdot \left[\frac{1}{[(3b+3c-z)^2 + p^2]^{1/2}} \right. \right. \\
& \left. \left. - \frac{1}{[(5b+3c+z)^2 + p^2]^{1/2}} + \frac{1}{[(3b+c+z)^2 + p^2]^{1/2}} - \frac{1}{[(5b+5c-z)^2 + p^2]^{1/2}} \right] \right\} \quad (A-16)
\end{aligned}$$

$$\begin{aligned}
RG_3(1, 1, 2) = & \left\{ 2(b+c)(z-c) \left[\frac{3(2b+2c-z) \cdot [2(2b+2c-z)^2 - 3p^2]}{[(2b+2c-z)^2 + p^2]^{7/2}} + \right. \right. \\
& \frac{3(4b+2c+z) \cdot [2(4b+2c+z)^2 - 3p^2]}{[(4b+2c+z)^2 + p^2]^{7/2}} \Big] + \left[\frac{(2b+z)}{[(2b+z)^2 + p^2]^{3/2}} + \frac{(4b+4c-z)}{[(4b+4c-z)^2 + p^2]^{3/2}} \right. \\
& \left. - \frac{2b+2c-z}{[(2b+2c-z)^2 + p^2]^{3/2}} - \frac{4b+2c+z}{[(4b+2c+z)^2 + p^2]^{3/2}} \right] + 2(b+c) \cdot \left[\frac{2(2b+2c-z)^2 - p^2}{[(2b+2c-z)^2 + p^2]^{5/2}} - \right. \\
& \frac{2(4b+2c+z)^2 - p^2}{[(4b+2c+z)^2 + p^2]^{5/2}} \Big] - (z+b) \left[\frac{2(2b+2c-z)^2 - p^2}{[(2b+2c-z)^2 + p^2]^{5/2}} - \frac{2(4b+2c+z)^2 - p^2}{[(4b+2c+z)^2 + p^2]^{5/2}} \right. \\
& \left. + \frac{2(2b+z)^2 - p^2}{[(2b+z)^2 + p^2]^{5/2}} - \frac{2(4b+4c-z)^2 - p^2}{[(4b+4c-z)^2 + p^2]^{5/2}} \right] \Big\} + \left\{ \frac{3(z-c)}{2(b+c)^3} \left[- \ln[(2b+2c-z) + \right. \right. \\
& \sqrt{(2b+2c-z)^2 + p^2}] - \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2}] \Big] + \frac{3}{4(b+c)^4} \cdot \left[\left(\frac{p^2}{4} - \frac{(2b+z)^2}{2} \right) \cdot \right. \\
& \ln[(2b+z) + \sqrt{(2b+z)^2 + p^2}] + \frac{3}{4}(2b+z) \sqrt{(2b+z)^2 + p^2} + \left(\frac{p^2}{4} - \frac{(4b+4c-z)^2}{2} \right) \cdot \ln[\\
& (4b+4c-z) + \sqrt{(4b+4c-z)^2 + p^2}] + \frac{3}{4}(4b+4c-z) \cdot \sqrt{(4b+4c-z)^2 + p^2} - \left(\frac{p^2}{4} - \right. \\
& \left. \frac{(2b+2c-z)^2}{2} \right) \cdot \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2}] - \frac{3}{4}(2b+2c-z) \cdot \sqrt{(2b+2c-z)^2 + p^2} \\
& \left. - \left(\frac{p^2}{4} - \frac{(4b+2c+z)^2}{2} \right) \cdot \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2}] - \frac{3}{4}(4b+2c+z) \cdot \right. \\
& \left. \sqrt{(4b+2c+z)^2 + p^2} \right] + \frac{3}{2(b+c)^3} \left[(2b+2c-z) \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2}] - \right. \\
& \left. \sqrt{(2b+2c-z)^2 + p^2} - (4b+2c+z) \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + p^2}] - \sqrt{(4b+2c+z)^2 + p^2} \right] \\
& \left. - \frac{3(z+b)}{4(b+c)^4} \left[(2b+2c-z) \cdot \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + p^2}] - \sqrt{(2b+2c-z)^2 + p^2} - \right. \right.
\end{aligned}$$

$$\begin{aligned}
& (4b+2c+z) \cdot \frac{1}{h} \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + f^2} \right] + \sqrt{(4b+2c+z)^2 + f^2} + (2b+z) \cdot \\
& \frac{1}{h} \left[(2b+z) + \sqrt{(2b+z)^2 + f^2} \right] - \sqrt{(2b+z)^2 + f^2} - (4b+4c-z) \cdot \frac{1}{h} \left[(4b+4c-z) + \right. \\
& \left. \sqrt{(4b+4c-z)^2 + f^2} \right] + \sqrt{(4b+4c-z)^2 + f^2} \left. \right\} + \left\{ \frac{A_2(2-c)}{2(b+c)^2} \left[\frac{1}{[(3b+3c-z)^2 + f^2]^{\frac{1}{2}}} + \right. \right. \\
& \left. \left. \frac{1}{[(5b+3c+z)^2 + f^2]^{\frac{1}{2}}} \right] + \frac{A_2}{4(b+c)^3} \left[(3b+c+z) \cdot \frac{1}{h} \left[(3b+c+z) + \sqrt{(3b+c+z)^2 + f^2} \right] \right. \right. \\
& \left. \left. + \sqrt{(3b+c+z)^2 + f^2} + (5b+5c-z) \cdot \frac{1}{h} \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + f^2} \right] - \right. \right. \\
& \left. \left. \sqrt{(5b+5c-z)^2 + f^2} - (3b+3c-z) \cdot \frac{1}{h} \left[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + f^2} \right] + \sqrt{(3b+3c-z)^2 + f^2} \right. \right. \\
& \left. \left. + (5b+3c+z) \cdot \frac{1}{h} \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + f^2} \right] + \sqrt{(5b+3c+z)^2 + f^2} \right] + \frac{A_2}{2(b+c)^2} \cdot \right. \\
& \left. \left[-\frac{1}{h} \left[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + f^2} \right] + \frac{1}{h} \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + f^2} \right] \right] - \right. \\
& \left. \frac{A_2(2+b)}{4(b+c)^3} \left[-\frac{1}{h} \left[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + f^2} \right] + \frac{1}{h} \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + f^2} \right] \right] \right\} + \\
& \left\{ \frac{B_2(2-c)}{2(b+c)} \left[\frac{3b+3c-z}{[(3b+3c-z)^2 + f^2]^{\frac{3}{2}}} + \frac{5b+3c+z}{[(5b+3c+z)^2 + f^2]^{\frac{3}{2}}} \right] + \frac{B_2}{4(b+c)^2} \left[-\frac{1}{h} \left[\right. \right. \right. \\
& \left. \left. (3b+c+z) + \sqrt{(3b+c+z)^2 + f^2} \right] - \frac{1}{h} \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + f^2} \right] + \frac{1}{h} \left[\right. \right. \\
& \left. \left. (3b+3c-z) + \sqrt{(3b+3c-z)^2 + f^2} \right] + \frac{1}{h} \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + f^2} \right] \right] + \frac{B_2}{2(b+c)} \cdot \\
& \left[\frac{1}{[(3b+3c-z)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(5b+3c+z)^2 + f^2]^{\frac{1}{2}}} \right] - \frac{B_2(2+b)}{4(b+c)^2} \left[\frac{1}{[(3b+3c-z)^2 + f^2]^{\frac{1}{2}}} \right. \\
& \left. - \frac{1}{[(5b+3c+z)^2 + f^2]^{\frac{1}{2}}} + \frac{1}{[(3b+c+z)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(5b+5c-z)^2 + f^2]^{\frac{1}{2}}} \right] \left. \right\} + \left\{ \frac{C_2(2-c)}{2} \right. \\
& \left[\frac{2(3b+3c-z)^2 - f^2}{[(3b+3c-z)^2 + f^2]^{\frac{5}{2}}} + \frac{2(5b+3c+z)^2 - f^2}{[(5b+3c+z)^2 + f^2]^{\frac{5}{2}}} \right] + \frac{C_2}{4(b+c)} \left[\frac{1}{[(3b+c+z)^2 + f^2]^{\frac{1}{2}}} \right. \\
& \left. + \frac{1}{[(5b+5c-z)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(3b+3c-z)^2 + f^2]^{\frac{1}{2}}} - \frac{1}{[(5b+3c+z)^2 + f^2]^{\frac{1}{2}}} \right] + \frac{C_2}{2} \left[\right. \\
& \left. \frac{3b+3c-z}{[(3b+3c-z)^2 + f^2]^{\frac{3}{2}}} - \frac{5b+3c+z}{[(5b+3c+z)^2 + f^2]^{\frac{3}{2}}} \right] - \frac{C_2(2+b)}{4(b+c)} \left[\frac{3b+3c-z}{[(3b+3c-z)^2 + f^2]^{\frac{3}{2}}} \right. \\
& \left. - \frac{5b+3c+z}{[(5b+3c+z)^2 + f^2]^{\frac{3}{2}}} + \frac{3b+c+z}{[(3b+c+z)^2 + f^2]^{\frac{3}{2}}} - \frac{5b+5c-z}{[(5b+5c-z)^2 + f^2]^{\frac{3}{2}}} \right] \left. \right\} (A-17)
\end{aligned}$$

$$\begin{aligned}
RG_3(1,2,3) = & \left\{ 2(b+c)(z-c) \cdot \left[\frac{3(2b+2c-z) \cdot p}{[(2b+2c-z)^2+p^2]^{5/2}} + \frac{3(4b+2c+z) \cdot p}{[(4b+2c+z)^2+p^2]^{5/2}} \right] + \right. \\
& \left[\frac{\sqrt{(2b+2c-z)^2+p^2} - (2b+2c-z)}{p \cdot [(2b+2c-z)^2+p^2]^{3/2}} + \frac{\sqrt{(4b+2c+z)^2+p^2} - (4b+2c+z)}{p \cdot [(4b+2c+z)^2+p^2]^{3/2}} - \right. \\
& \left. \frac{\sqrt{(2b+2c-z)^2+p^2} - (2b+2c-z)}{p \cdot [(2b+2c-z)^2+p^2]^{3/2}} - \frac{\sqrt{(4b+2c+z)^2+p^2} - (4b+2c+z)}{p \cdot [(4b+2c+z)^2+p^2]^{3/2}} \right] + 2(b+c) \cdot \left\{ \right. \\
& \frac{p}{[(2b+2c-z)^2+p^2]^{3/2}} - \frac{p}{[(4b+2c+z)^2+p^2]^{3/2}} \left. \right\} - (z+b) \left[\frac{p}{[(2b+2c-z)^2+p^2]^{3/2}} \right. \\
& \left. - \frac{p}{[(4b+2c+z)^2+p^2]^{3/2}} + \frac{p}{[(2b+2c-z)^2+p^2]^{3/2}} - \frac{p}{[(4b+2c+z)^2+p^2]^{3/2}} \right] \left. \right\} + \left\{ \right. \\
& \frac{3(z-c)}{2(b+c)^3} \left[- \frac{(2b+2c-z)}{2 \cdot p} \sqrt{(2b+2c-z)^2+p^2} - \frac{p}{2} \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2+p^2}] \right. \\
& + \frac{(2b+2c-z)}{2 \cdot p} - \frac{(4b+2c+z)}{2 \cdot p} \sqrt{(4b+2c+z)^2+p^2} - \frac{p}{2} \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2}] \\
& + \frac{(4b+2c+z)}{2 \cdot p} \left. \right] + \frac{3}{4(b+c)^4} \left[\left(\frac{p^3}{16} - \frac{p}{4} (2b+2c)^2 \right) \cdot \ln[(2b+2c) + \sqrt{(2b+2c)^2+p^2}] \right. \\
& + \frac{5p}{16} (2b+2c) \sqrt{(2b+2c)^2+p^2} - \frac{(2b+2c)}{24 \cdot p} [(2b+2c)^2+p^2]^{3/2} + \frac{(2b+2c)^4}{24 \cdot p} \\
& + \left[\frac{p^3}{16} - \frac{p}{4} (4b+2c-z)^2 \right] \cdot \ln[(4b+2c-z) + \sqrt{(4b+2c-z)^2+p^2}] + \frac{5p}{16} (4b+2c-z) \cdot \sqrt{(4b+2c-z)^2+p^2} \\
& - \frac{(4b+2c-z)}{24 \cdot p} [(4b+2c-z)^2+p^2]^{3/2} + \frac{(4b+2c-z)^4}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4} (4b+2c+z)^2 \right] \cdot \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2}] \\
& - \frac{5p}{16} (4b+2c+z) \sqrt{(4b+2c+z)^2+p^2} + \frac{(4b+2c+z)}{24 \cdot p} [(4b+2c+z)^2+p^2]^{3/2} - \frac{(4b+2c+z)^4}{24 \cdot p} \\
& + \frac{(2b+2c-z)}{24 \cdot p} [(2b+2c-z)^2+p^2]^{3/2} - \frac{(2b+2c-z)^4}{24 \cdot p} - \left[\frac{p^3}{16} - \frac{p}{4} (4b+2c+z)^2 \right] \cdot \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2}] \\
& - \frac{5p}{16} (4b+2c+z) \sqrt{(4b+2c+z)^2+p^2} + \frac{(4b+2c+z)}{24 \cdot p} [(4b+2c+z)^2+p^2]^{3/2} - \frac{(4b+2c+z)^4}{24 \cdot p} \\
& + \frac{3}{2(b+c)^3} \left[\frac{p}{2} (2b+2c-z) \cdot \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2+p^2}] - \frac{p}{2} \sqrt{(2b+2c-z)^2+p^2} - \frac{(2b+2c-z)^3}{6 \cdot p} + \frac{[(2b+2c-z)^2+p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} \cdot (4b+2c+z) \cdot \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2}] + \frac{p}{2} \sqrt{(4b+2c+z)^2+p^2} + \frac{(4b+2c+z)^3}{6 \cdot p} - \frac{[(4b+2c+z)^2+p^2]^{3/2}}{6 \cdot p} \right] - \frac{3(2+b)}{4(b+c)^4} \left[\frac{p}{2} (2b+2c-z) \cdot \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2+p^2}] - \frac{p}{2} \sqrt{(2b+2c-z)^2+p^2} - \frac{(2b+2c-z)^3}{6 \cdot p} + \frac{[(2b+2c-z)^2+p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (4b+2c+z) \cdot \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2}] + \frac{p}{2} \sqrt{(4b+2c+z)^2+p^2} + \frac{(4b+2c+z)^3}{6 \cdot p} - \frac{[(4b+2c+z)^2+p^2]^{3/2}}{6 \cdot p} \right]
\end{aligned}$$

$$\begin{aligned}
& \frac{(4b+2c+2)^3}{6 \cdot p} - \frac{[(4b+2c+2)^2 + p^2]^{3/2}}{6 \cdot p} + \frac{p}{2}(2b+2) \cdot \ln[(2b+2) + \sqrt{(2b+2)^2 + p^2}] - \frac{p}{2} \cdot \\
& \sqrt{(2b+2)^2 + p^2} - \frac{(2b+2)^3}{6 \cdot p} + \frac{[(2b+2)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(4b+4c-2) \cdot \ln[(4b+4c-2) + \\
& \sqrt{(4b+4c-2)^2 + p^2}] + \frac{p}{2} \sqrt{(4b+4c-2)^2 + p^2} + \frac{(4b+4c-2)^3}{6 \cdot p} - \frac{[(4b+4c-2)^2 + p^2]^{3/2}}{6 \cdot p} \Big\} + \\
& \left\{ \frac{A_2(2-c)}{2(b+c)} \left[-\frac{(3b+3c-2)}{p} + \frac{\sqrt{(3b+3c-2)^2 + p^2}}{p} - \frac{(5b+3c+2)}{p} + \frac{\sqrt{(5b+3c+2)^2 + p^2}}{p} \right] + \right. \\
& \frac{A_2}{4(b+c)^3} \left[\frac{p}{2}(3b+c+2) \ln[(3b+c+2) + \sqrt{(3b+c+2)^2 + p^2}] - \frac{p}{2} \sqrt{(3b+c+2)^2 + p^2} - \frac{(3b+c+2)^3}{6 \cdot p} \right. \\
& + \frac{[(3b+c+2)^2 + p^2]^{3/2}}{6 \cdot p} + \frac{p}{2}(5b+5c-2) \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2}] - \frac{p}{2} \cdot \\
& \sqrt{(5b+5c-2)^2 + p^2} - \frac{(5b+5c-2)^3}{6 \cdot p} + \frac{[(5b+5c-2)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(3b+3c-2) \cdot \ln[\\
& (3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2}] + \frac{p}{2} \sqrt{(3b+3c-2)^2 + p^2} + \frac{(3b+3c-2)^3}{6 \cdot p} - \\
& \frac{[(3b+3c-2)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2}(5b+3c+2) \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] + \frac{p}{2} \cdot \\
& \sqrt{(5b+3c+2)^2 + p^2} + \frac{(5b+3c+2)^3}{6 \cdot p} - \frac{[(5b+3c+2)^2 + p^2]^{3/2}}{6 \cdot p} \Big] + \frac{A_2}{2(51c)^2} \left[-\frac{(3b+3c-2)}{2 \cdot p} \cdot \right. \\
& \sqrt{(3b+3c-2)^2 + p^2} - \frac{p}{2} \ln[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2}] + \frac{(3b+3c-2)^2}{2 \cdot p} + \frac{(5b+3c+2)}{2 \cdot p} \cdot \\
& \sqrt{(5b+3c+2)^2 + p^2} + \frac{p}{2} \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] - \frac{(5b+3c+2)^2}{2 \cdot p} \Big] - \\
& \frac{A_2(2+b)}{4(b+c)^3} \left[-\frac{(3b+3c-2)}{2 \cdot p} \sqrt{(3b+3c-2)^2 + p^2} - \frac{p}{2} \ln[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + p^2}] \right. \\
& + \frac{(3b+3c-2)^2}{2 \cdot p} + \frac{(5b+3c+2)}{2 \cdot p} \sqrt{(5b+3c+2)^2 + p^2} + \frac{p}{2} \ln[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + p^2}] \\
& - \frac{(5b+3c+2)^2}{2 \cdot p} - \frac{(3b+c+2)}{2 \cdot p} \sqrt{(3b+c+2)^2 + p^2} - \frac{p}{2} \ln[(3b+c+2) + \sqrt{(3b+c+2)^2 + p^2}] + \\
& \frac{(3b+c+2)^2}{2 \cdot p} + \frac{p}{2}(5b+5c-2) \sqrt{(5b+5c-2)^2 + p^2} + \frac{p}{2} \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2}] \\
& - \frac{(5b+5c-2)^2}{2 \cdot p} \Big] \Big\} + \left\{ \frac{B_2(2-c)}{2(b+c)} \left[\frac{\sqrt{(3b+3c-2)^2 + p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2 + p^2]^{1/2}} + \right. \right. \\
& \frac{\sqrt{(5b+3c+2)^2 + p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2 + p^2]^{1/2}} \Big] + \frac{B_2}{4(b+c)^2} \left[-\frac{(3b+c+2)}{2 \cdot p} \sqrt{(3b+c+2)^2 + p^2} - \frac{p}{2} \cdot \ln[\right. \\
& (3b+c+2) + \sqrt{(3b+c+2)^2 + p^2}] + \frac{(3b+c+2)^2}{2 \cdot p} - \frac{(5b+5c-2)}{2 \cdot p} \sqrt{(5b+5c-2)^2 + p^2} - \frac{p}{2} \cdot \\
& \ln[(5b+5c-2) + \sqrt{(5b+5c-2)^2 + p^2}] + \frac{(5b+5c-2)^2}{2 \cdot p} + \frac{(3b+3c-2)}{2 \cdot p} \sqrt{(3b+3c-2)^2 + p^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{f}{2} \ln \left[(3b+3c-2) + \sqrt{(3b+3c-2)^2 + f^2} \right] - \frac{(3b+3c-2)^2}{2 \cdot f} + \frac{(5b+3c+2)}{2 \cdot f} \sqrt{(5b+3c+2)^2 + f^2} \\
& + \frac{f}{2} \ln \left[(5b+3c+2) + \sqrt{(5b+3c+2)^2 + f^2} \right] - \frac{(5b+3c+2)^2}{2 \cdot f} + \frac{B_2}{2(b+c)} \left[-\frac{(3b+3c-2)}{f} \right. \\
& + \frac{\sqrt{(3b+3c-2)^2 + f^2}}{f} + \frac{(5b+3c+2)}{f} - \frac{\sqrt{(5b+3c+2)^2 + f^2}}{f} \left. \right] - \frac{B_2(2+b)}{4(b+c)^2} \left[-\frac{(3b+3c-2)}{f} + \right. \\
& \frac{\sqrt{(3b+3c-2)^2 + f^2}}{f} + \frac{(5b+3c+2)}{f} - \frac{\sqrt{(5b+3c+2)^2 + f^2}}{f} - \frac{(3b+c+2)}{f} + \frac{\sqrt{(3b+c+2)^2 + f^2}}{f} \\
& + \left. \frac{(5b+5c-2)}{f} - \frac{\sqrt{(5b+5c-2)^2 + f^2}}{f} \right] \left. \right\} + \left\{ \frac{C_2(2-c)}{2} \left[\frac{f}{[(3b+3c-2)^2 + f^2]^{3/2}} + \right. \right. \\
& \frac{f}{[(5b+3c+2)^2 + f^2]^{3/2}} \left. \right] + \frac{C_2}{4(b+c)} \left[-\frac{(3b+c+2)}{f} + \frac{\sqrt{(3b+c+2)^2 + f^2}}{f} - \frac{(5b+5c-2)}{f} + \right. \\
& \frac{\sqrt{(5b+5c-2)^2 + f^2}}{f} + \frac{(3b+3c-2)}{f} - \frac{\sqrt{(3b+3c-2)^2 + f^2}}{f} + \frac{(5b+3c+2)}{f} - \frac{\sqrt{(5b+3c+2)^2 + f^2}}{f} \left. \right] \\
& + \frac{C_2}{2} \left[\frac{\sqrt{(3b+3c-2)^2 + f^2} - (3b+3c-2)}{f \cdot [(3b+3c-2)^2 + f^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2 + f^2} - (5b+3c+2)}{f \cdot [(5b+3c+2)^2 + f^2]^{1/2}} \right. \\
& - \frac{C_2(2+b)}{4(b+c)} \left. \right] \\
& \left[\frac{\sqrt{(3b+3c-2)^2 + f^2} - (3b+3c-2)}{f \cdot [(3b+3c-2)^2 + f^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2 + f^2} - (5b+3c+2)}{f \cdot [(5b+3c+2)^2 + f^2]^{1/2}} + \right. \\
& \left. \frac{\sqrt{(3b+c+2)^2 + f^2} - (3b+c+2)}{f \cdot [(3b+c+2)^2 + f^2]^{1/2}} - \frac{\sqrt{(5b+5c-2)^2 + f^2} - (5b+5c-2)}{f \cdot [(5b+5c-2)^2 + f^2]^{1/2}} \right] \left. \right\}, \quad (A-18)
\end{aligned}$$

$$\begin{aligned}
RG_3(1, 2, 1) &= \left\{ 2(b+c)(2-c) \left[\frac{f}{[(2b+2c-2)^2 + f^2]^{3/2}} + \frac{f}{[(4b+2c+2)^2 + f^2]^{3/2}} \right] + \left[\frac{(2b+2)}{f} \right. \right. \\
& + \frac{\sqrt{(2b+2)^2 + f^2}}{f} - \frac{(4b+4c-2)}{f} + \frac{\sqrt{(4b+4c-2)^2 + f^2}}{f} + \frac{(2b+2c-2)}{f} - \frac{\sqrt{(2b+2c-2)^2 + f^2}}{f} \\
& + \left. \frac{(4b+2c+2)}{f} - \frac{\sqrt{(4b+2c+2)^2 + f^2}}{f} \right] + 2(b+c) \left[\frac{\sqrt{(2b+2c-2)^2 + f^2} - (2b+2c-2)}{f \cdot [(2b+2c-2)^2 + f^2]^{1/2}} - \right. \\
& \frac{\sqrt{(4b+2c+2)^2 + f^2} - (4b+2c+2)}{f \cdot [(4b+2c+2)^2 + f^2]^{1/2}} \left. \right] - (2+b) \left[\frac{\sqrt{(2b+2c-2)^2 + f^2} - (2b+2c-2)}{f \cdot [(2b+2c-2)^2 + f^2]^{1/2}} - \right. \\
& \frac{\sqrt{(4b+2c+2)^2 + f^2} - (4b+2c+2)}{f \cdot [(4b+2c+2)^2 + f^2]^{1/2}} + \frac{\sqrt{(2b+2)^2 + f^2} - (2b+2)}{f \cdot [(2b+2)^2 + f^2]^{1/2}} - \\
& \left. \frac{\sqrt{(4b+4c-2)^2 + f^2} - (4b+4c-2)}{f \cdot [(4b+4c-2)^2 + f^2]^{1/2}} \right] \left. \right\} + \left\{ \frac{3(2-c)}{2(b+c)^3} \left[\frac{f}{2} (2b+2c-2) \cdot \ln \left[(2b+2c-2) + \right. \right. \right. \\
& \left. \left. \sqrt{(2b+2c-2)^2 + f^2} \right] - \frac{f}{2} \sqrt{(2b+2c-2)^2 + f^2} - \frac{(2b+2c-2)^3}{6 \cdot f} + \frac{[(2b+2c+2)^2 + f^2]^{3/2}}{6 \cdot f} \right. \\
& \left. \left. \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{f}{2} (4b+2c+2) \cdot \frac{1}{\sqrt{4b+2c+2}} \left[(4b+2c+2) + \sqrt{(4b+2c+2)^2 + f^2} \right] - \frac{f}{2} \sqrt{(4b+2c+2)^2 + f^2} - \frac{(4b+2c+2)^3}{6 \cdot f} \\
& + \frac{[(4b+2c+2)^2 + f^2]^{\frac{3}{2}}}{6 \cdot f} + \frac{3}{4(b+c)^4} \left[\left[\frac{f}{12} (2b+2) - \frac{f^3}{16} (2b+2) \right] \cdot \frac{1}{\sqrt{2b+2}} + \right. \\
& \left. \sqrt{(2b+2)^2 + f^2} + \left[\frac{f^3}{16} - \frac{f}{12} (2b+2)^2 \right] \sqrt{(2b+2)^2 + f^2} - \frac{7 \cdot f}{18 \cdot 8} [(2b+2)^2 + f^2]^{\frac{3}{2}} + \right. \\
& \left. \frac{[(2b+2)^2 + f^2]^{\frac{5}{2}}}{5 \times 24 \cdot f} - \left[\frac{f}{12} (4b+4c-2) - \frac{f^3}{16} (4b+4c-2) \right] \cdot \frac{1}{\sqrt{4b+4c-2}} + \right. \\
& \left. \sqrt{(4b+4c-2)^2 + f^2} + \left[\frac{f^3}{16} - \frac{f}{12} (4b+4c-2)^2 \right] \sqrt{(4b+4c-2)^2 + f^2} - \frac{7 \cdot f}{18 \cdot 8} [(4b+4c-2)^2 + f^2]^{\frac{3}{2}} + \right. \\
& \left. + \frac{[(4b+4c-2)^2 + f^2]^{\frac{5}{2}}}{5 \times 24 \cdot f} - \left[\frac{f}{12} (2b+2c-2) - \frac{f^3}{16} (2b+2c-2) \right] \cdot \frac{1}{\sqrt{2b+2c-2}} + \right. \\
& \left. \sqrt{(2b+2c-2)^2 + f^2} - \left[\frac{f^3}{16} - \frac{f}{12} (2b+2c-2)^2 \right] \sqrt{(2b+2c-2)^2 + f^2} + \frac{7 \cdot f}{18 \cdot 8} [(2b+2c-2)^2 + f^2]^{\frac{3}{2}} + \right. \\
& \left. - \frac{[(2b+2c-2)^2 + f^2]^{\frac{5}{2}}}{5 \times 24 \cdot f} + \left[\frac{f}{12} (4b+2c+2) - \frac{f^3}{16} (4b+2c+2) \right] \cdot \frac{1}{\sqrt{4b+2c+2}} + \right. \\
& \left. \sqrt{(4b+2c+2)^2 + f^2} - \left[\frac{f^3}{16} - \frac{f}{12} (4b+2c+2)^2 \right] \sqrt{(4b+2c+2)^2 + f^2} + \frac{7 \cdot f}{18 \cdot 8} [(4b+2c+2)^2 + f^2]^{\frac{3}{2}} + \right. \\
& \left. - \frac{[(4b+2c+2)^2 + f^2]^{\frac{5}{2}}}{5 \times 24 \cdot f} + \frac{3}{2(b+c)^4} \left[\left[\frac{f^3}{16} - \frac{f}{4} (2b+2c-2)^2 \right] \cdot \frac{1}{\sqrt{2b+2c-2}} + \right. \\
& \left. \sqrt{(2b+2c-2)^2 + f^2} + \frac{5 \cdot f}{16} (2b+2c-2) \sqrt{(2b+2c-2)^2 + f^2} - \frac{(2b+2c-2)}{24 \cdot f} [(2b+2c-2)^2 + f^2]^{\frac{3}{2}} + \right. \\
& \left. + \frac{(2b+2c-2)^4}{24 \cdot f} - \left[\frac{f^3}{16} - \frac{f}{4} (4b+2c+2)^2 \right] \cdot \frac{1}{\sqrt{4b+2c+2}} + \sqrt{(4b+2c+2)^2 + f^2} - \frac{5 \cdot f}{16} \cdot \right. \\
& \left. (4b+2c+2) \cdot \sqrt{(4b+2c+2)^2 + f^2} + \frac{(4b+2c+2)}{24 \cdot f} [(4b+2c+2)^2 + f^2]^{\frac{3}{2}} - \frac{(4b+2c+2)^4}{24 \cdot f} \right] - \\
& \frac{3(2+b)}{4(b+c)^4} \left[\left[\frac{f^3}{16} - \frac{f}{4} (2b+2c-2)^2 \right] \cdot \frac{1}{\sqrt{2b+2c-2}} + \sqrt{(2b+2c-2)^2 + f^2} + \frac{5 \cdot f}{16} \cdot \right. \\
& \left. (2b+2c-2) \cdot \sqrt{(2b+2c-2)^2 + f^2} - \frac{(2b+2c-2)}{24 \cdot f} [(2b+2c-2)^2 + f^2]^{\frac{3}{2}} + \frac{(2b+2c-2)^4}{24 \cdot f} - \right. \\
& \left. \left[\frac{f^3}{16} - \frac{f}{4} (4b+2c+2)^2 \right] \cdot \frac{1}{\sqrt{4b+2c+2}} + \sqrt{(4b+2c+2)^2 + f^2} - \frac{5 \cdot f}{16} (4b+2c+2) \cdot \right. \\
& \left. \sqrt{(4b+2c+2)^2 + f^2} + \frac{(4b+2c+2)}{24 \cdot f} [(4b+2c+2)^2 + f^2]^{\frac{3}{2}} - \frac{(4b+2c+2)^4}{24 \cdot f} + \left[\frac{f^3}{16} - \right. \right. \\
& \left. \left. \frac{f}{4} (2b+2)^2 \right] \cdot \frac{1}{\sqrt{2b+2}} + \sqrt{(2b+2)^2 + f^2} + \frac{5 \cdot f}{16} (2b+2) \sqrt{(2b+2)^2 + f^2} - \frac{(2b+2)}{24 \cdot f} \cdot \right. \\
& \left. [(2b+2)^2 + f^2]^{\frac{3}{2}} + \frac{(2b+2)^4}{24 \cdot f} - \left[\frac{f^3}{16} - \frac{f}{4} (4b+4c-2)^2 \right] \cdot \frac{1}{\sqrt{4b+4c-2}} + \sqrt{(4b+4c-2)^2 + f^2} \right] \\
& - \frac{5 \cdot f}{16} (4b+4c-2) \sqrt{(4b+4c-2)^2 + f^2} + \frac{(4b+4c-2)}{24 \cdot f} [(4b+4c-2)^2 + f^2]^{\frac{3}{2}} - \frac{(4b+4c-2)^4}{24 \cdot f} \Bigg\} +
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{A_2(z-c)}{2(b+c)^2} \left[-\frac{(3b+3c-z)}{2 \cdot p} \sqrt{(3b+3c-z)^2 + p^2} - \frac{p}{2} \ln \left[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2} \right] \right. \right. \\
& + \frac{(3b+3c-z)^2}{2 \cdot p} - \frac{(5b+3c+z)}{2 \cdot p} \sqrt{(5b+3c+z)^2 + p^2} - \frac{p}{2} \ln \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2} \right] \\
& + \left. \frac{(5b+3c+z)^2}{2 \cdot p} \right] + \frac{A_2}{4(b+c)^3} \left[\left(\frac{p^3}{16} - \frac{p}{4} (3b+c+z)^2 \right) \cdot \ln \left[(3b+c+z) + \sqrt{(3b+c+z)^2 + p^2} \right] + \right. \\
& \frac{5 \cdot p}{16} (3b+c+z) \sqrt{(3b+c+z)^2 + p^2} - \frac{(3b+c+z)}{24 \cdot p} \left[(3b+c+z)^2 + p^2 \right]^{3/2} + \frac{(3b+c+z)^4}{24 \cdot p} + \left. \left(\frac{p^3}{16} \right. \right. \\
& - \frac{p}{4} (5b+5c-z)^2 \left. \right) \cdot \ln \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2} \right] + \frac{5 \cdot p}{16} (5b+5c-z) \sqrt{(5b+5c-z)^2 + p^2} \\
& - \frac{(5b+5c-z)}{24 \cdot p} \left[(5b+5c-z)^2 + p^2 \right]^{3/2} + \frac{(5b+5c-z)^4}{24 \cdot p} - \left. \left(\frac{p^3}{16} - \frac{p}{4} (3b+3c-z)^2 \right) \cdot \ln \left[\right. \right. \\
& (3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2} \left. \right] - \frac{5 \cdot p}{16} (3b+3c-z) \sqrt{(3b+3c-z)^2 + p^2} + \frac{(3b+3c-z)}{24 \cdot p} \\
& \left. \left[(3b+3c-z)^2 + p^2 \right]^{3/2} - \frac{(3b+3c-z)^4}{24 \cdot p} - \left(\frac{p^3}{16} - \frac{p}{4} (5b+3c+z)^2 \right) \cdot \ln \left[(5b+3c+z) + \right. \right. \\
& \left. \left. \sqrt{(5b+3c+z)^2 + p^2} \right] - \frac{5 \cdot p}{16} (5b+3c+z) \sqrt{(5b+3c+z)^2 + p^2} + \frac{(5b+3c+z)}{24 \cdot p} \right. \\
& \left. \left[(5b+3c+z)^2 + p^2 \right]^{3/2} - \frac{(5b+3c+z)^4}{24 \cdot p} \right] + \frac{A_2}{2(b+c)^2} \left[\frac{p}{2} (3b+3c-z) \cdot \ln \left[(3b+3c-z) + \right. \right. \\
& \left. \left. \sqrt{(3b+3c-z)^2 + p^2} \right] - \frac{p}{2} \sqrt{(3b+3c-z)^2 + p^2} - \frac{(3b+3c-z)^3}{6 \cdot p} + \frac{[(3b+3c-z)^2 + p^2]^{3/2}}{6 \cdot p} - \right. \\
& \frac{p}{2} (5b+3c+z) \cdot \ln \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2} \right] + \frac{p}{2} \sqrt{(5b+3c+z)^2 + p^2} + \\
& \frac{(5b+3c+z)^3}{6 \cdot p} - \frac{[(5b+3c+z)^2 + p^2]^{3/2}}{6 \cdot p} \left. \right] - \frac{A_2(z+b)}{4(b+c)^3} \left[\frac{p}{2} (3b+3c-z) \cdot \ln \left[(3b+3c-z) + \right. \right. \\
& \left. \left. \sqrt{(3b+3c-z)^2 + p^2} \right] - \frac{p}{2} \sqrt{(3b+3c-z)^2 + p^2} - \frac{(3b+3c-z)^3}{6 \cdot p} + \frac{[(3b+3c-z)^2 + p^2]^{3/2}}{6 \cdot p} - \right. \\
& \frac{p}{2} (5b+3c+z) \cdot \ln \left[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2} \right] + \frac{p}{2} \sqrt{(5b+3c+z)^2 + p^2} + \\
& \frac{(5b+3c+z)^3}{6 \cdot p} - \frac{[(5b+3c+z)^2 + p^2]^{3/2}}{6 \cdot p} + \frac{p}{2} (3b+c+z) \cdot \ln \left[(3b+c+z) + \sqrt{(3b+c+z)^2 + p^2} \right] \\
& - \frac{p}{2} \sqrt{(3b+c+z)^2 + p^2} - \frac{(3b+c+z)^3}{6 \cdot p} + \frac{[(3b+c+z)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (5b+5c-z) \cdot \ln \left[\right. \\
& (5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2} \left. \right] + \frac{p}{2} \sqrt{(5b+5c-z)^2 + p^2} + \frac{(5b+5c-z)^3}{6 \cdot p} - \\
& \left. \frac{[(5b+5c-z)^2 + p^2]^{3/2}}{6 \cdot p} \right] + \left\{ \frac{B_2(z-c)}{2(b+c)} \left[-\frac{(3b+3c-z)}{p} + \frac{\sqrt{(3b+3c-z)^2 + p^2}}{p} - \frac{(5b+3c+z)}{p} \right. \right. \\
& + \left. \frac{\sqrt{(5b+3c+z)^2 + p^2}}{p} \right] + \frac{B_2}{4(b+c)^2} \left[\frac{p}{2} (3b+c+z) \cdot \ln \left[(3b+c+z) + \sqrt{(3b+c+z)^2 + p^2} \right] \right. \\
& - \frac{p}{2} \sqrt{(3b+c+z)^2 + p^2} - \frac{(3b+c+z)^3}{6 \cdot p} + \frac{[(3b+c+z)^2 + p^2]^{3/2}}{6 \cdot p} + \frac{p}{2} (5b+5c-z) \cdot \left. \right.
\end{aligned}$$

$$\begin{aligned}
& \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] - \frac{p}{2} \sqrt{(5b+5c-z)^2 + p^2} - \frac{(5b+5c-z)^3}{6 \cdot p} + \\
& \frac{[(5b+5c-z)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (3b+3c-z) \cdot \ln[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] + \frac{p}{2} \cdot \\
& \sqrt{(3b+3c-z)^2 + p^2} + \frac{(3b+3c-z)^3}{6 \cdot p} - \frac{[(3b+3c-z)^2 + p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (5b+3c+z) \ln[\\
& (5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] + \frac{p}{2} \sqrt{(5b+3c+z)^2 + p^2} + \frac{(5b+3c+z)^3}{6 \cdot p} - \\
& \frac{[(5b+3c+z)^2 + p^2]^{3/2}}{6 \cdot p} \Big] + \frac{B_2}{2(b+c)} \left[- \frac{(3b+3c-z)}{2 \cdot p} \sqrt{(3b+3c-z)^2 + p^2} - \frac{p}{2} \ln[(3b+3c-z) \right. \\
& \left. + \sqrt{(3b+3c-z)^2 + p^2}] + \frac{(3b+3c-z)^2}{2 \cdot p} + \frac{(5b+3c+z)}{2 \cdot p} \sqrt{(5b+3c+z)^2 + p^2} + \frac{p}{2} \ln[\right. \\
& \left. (5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] - \frac{(5b+3c+z)^2}{2 \cdot p} \right] - \frac{B_2(z+b)}{4(b+c)^2} \left[- \frac{(3b+3c-z)}{2 \cdot p} \right. \\
& \left. \sqrt{(3b+3c-z)^2 + p^2} - \frac{p}{2} \ln[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] + \frac{(3b+3c-z)^2}{2 \cdot p} + \right. \\
& \left. \frac{(5b+3c+z)}{2 \cdot p} \sqrt{(5b+3c+z)^2 + p^2} + \frac{p}{2} \ln[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] - \frac{(5b+3c+z)^2}{2 \cdot p} \right. \\
& \left. - \frac{(3b+c+z)}{2 \cdot p} \sqrt{(3b+c+z)^2 + p^2} - \frac{p}{2} \ln[(3b+c+z) + \sqrt{(3b+c+z)^2 + p^2}] + \frac{(3b+c+z)^2}{2 \cdot p} \right. \\
& \left. + \frac{(5b+5c-z)}{2 \cdot p} \sqrt{(5b+5c-z)^2 + p^2} + \frac{p}{2} \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] - \right. \\
& \left. \frac{(5b+5c-z)^2}{2 \cdot p} \right] \Big\} + \left\{ \frac{C_2(z-c)}{2} \left[\frac{\sqrt{(3b+3c-z)^2 + p^2} - (3b+3c-z)}{p \cdot [(3b+3c-z)^2 + p^2]^{1/2}} + \frac{\sqrt{(5b+3c+z)^2 + p^2} - (5b+3c+z)}{p \cdot [(5b+3c+z)^2 + p^2]^{1/2}} \right] \right. \\
& + \frac{C_2}{4(b+c)} \left[- \frac{(3b+c+z)}{2 \cdot p} \sqrt{(3b+c+z)^2 + p^2} - \frac{p}{2} \ln[(3b+c+z) + \sqrt{(3b+c+z)^2 + p^2}] + \frac{(3b+c+z)^2}{2 \cdot p} \right. \\
& \left. - \frac{(5b+5c-z)}{2 \cdot p} \sqrt{(5b+5c-z)^2 + p^2} - \frac{p}{2} \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + p^2}] + \frac{(5b+5c-z)^2}{2 \cdot p} \right. \\
& \left. + \frac{(3b+3c-z)}{2 \cdot p} \sqrt{(3b+3c-z)^2 + p^2} + \frac{p}{2} \ln[(3b+3c-z) + \sqrt{(3b+3c-z)^2 + p^2}] - \frac{(3b+3c-z)^2}{2 \cdot p} + \right. \\
& \left. \frac{(5b+3c+z)}{2 \cdot p} \sqrt{(5b+3c+z)^2 + p^2} + \frac{p}{2} \ln[(5b+3c+z) + \sqrt{(5b+3c+z)^2 + p^2}] - \frac{(5b+3c+z)^2}{2 \cdot p} \right] \\
& + \frac{C_2}{2} \left[- \frac{(3b+3c-z)}{p} + \frac{1}{p} \sqrt{(3b+3c-z)^2 + p^2} + \frac{(5b+3c+z)}{p} - \frac{\sqrt{(5b+3c+z)^2 + p^2}}{p} \right] - \\
& \frac{C_2(z+b)}{4(b+c)} \left[- \frac{(3b+3c-z)}{p} + \frac{\sqrt{(3b+3c-z)^2 + p^2}}{p} + \frac{(5b+3c+z)}{p} - \frac{\sqrt{(5b+3c+z)^2 + p^2}}{p} \right. \\
& \left. - \frac{(3b+c+z)}{p} + \frac{\sqrt{(3b+c+z)^2 + p^2}}{p} + \frac{(5b+5c-z)}{p} - \frac{\sqrt{(5b+5c-z)^2 + p^2}}{p} \right] \Big\} \quad (A-19)
\end{aligned}$$

$$\begin{aligned}
R_{G_2}(1,2,1) = & \left\{ 2(b+c)(z-c) \left[\frac{3(2b+2c-z) \cdot p}{[(2b+2c-z)^2+p^2]^{5/2}} - \frac{3(4b+2c+z) \cdot p}{[(4b+2c+z)^2+p^2]^{5/2}} \right] + \right. \\
& (z+b) \left[\frac{p}{[(2b+z)^2+p^2]^{3/2}} + \frac{p}{[(4b+4c-z)^2+p^2]^{3/2}} - \frac{p}{[(2b+2c-z)^2+p^2]^{3/2}} - \right. \\
& \left. \left. \frac{p}{[(4b+2c+z)^2+p^2]^{3/2}} \right] \right\} + \left\{ \frac{3(z-c)}{2(b+c)^3} \left[-\frac{(2b+2c-z)}{2 \cdot p} \sqrt{(2b+2c-z)^2+p^2} - \frac{p}{2} \ln \left[\right. \right. \right. \\
& (2b+2c-z) + \sqrt{(2b+2c-z)^2+p^2} \left. \left. \right] + \frac{(2b+2c-z)^2}{2 \cdot p} + \frac{(4b+2c+z)}{2 \cdot p} \sqrt{(4b+2c+z)^2+p^2} + \right. \\
& \frac{p}{2} \ln \left[(4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2} \right] - \frac{(4b+2c+z)^2}{2 \cdot p} \left. \right] + \frac{3(z+b)}{4(b+c)^4} \left[\frac{p}{2} (2b+z) \right. \\
& \ln \left[(2b+z) + \sqrt{(2b+z)^2+p^2} \right] - \frac{p}{2} \sqrt{(2b+z)^2+p^2} - \frac{(2b+z)^3}{6 \cdot p} + \frac{[(2b+z)^2+p^2]^{3/2}}{6 \cdot p} + \frac{p}{2} \cdot \\
& (4b+4c-z) \ln \left[(4b+4c-z) + \sqrt{(4b+4c-z)^2+p^2} \right] - \frac{p}{2} \sqrt{(4b+4c-z)^2+p^2} - \frac{(4b+4c-z)^3}{6 \cdot p} \\
& + \frac{[(4b+4c-z)^2+p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (2b+2c-z) \ln \left[(2b+2c-z) + \sqrt{(2b+2c-z)^2+p^2} \right] + \frac{p}{2} \cdot \\
& \sqrt{(2b+2c-z)^2+p^2} + \frac{(2b+2c-z)^3}{6 \cdot p} - \frac{[(2b+2c-z)^2+p^2]^{3/2}}{6 \cdot p} - \frac{p}{2} (4b+2c+z) \ln \left[\right. \\
& (4b+2c+z) + \sqrt{(4b+2c+z)^2+p^2} \left. \right] + \frac{p}{2} \sqrt{(4b+2c+z)^2+p^2} + \frac{(4b+2c+z)^3}{6 \cdot p} - \\
& \frac{[(4b+2c+z)^2+p^2]^{3/2}}{6 \cdot p} \left. \right] \right\} + \left\{ \frac{A_2(z-c)}{2(b+c)^2} \left[-\frac{(3b+3c-z)}{p} + \frac{\sqrt{(3b+3c-z)^2+p^2}}{p} + \right. \right. \\
& \frac{(5b+3c+z)}{p} - \frac{\sqrt{(5b+3c+z)^2+p^2}}{p} \left. \right] + \frac{A_2(z+b)}{4(b+c)^3} \left[\frac{(3b+c+z)}{2 \cdot p} \sqrt{(3b+c+z)^2+p^2} - \right. \\
& \frac{p}{2} \ln \left[(3b+c+z) + \sqrt{(3b+c+z)^2+p^2} \right] + \frac{(3b+c+z)^2}{2 \cdot p} - \frac{(5b+5c-z)}{2 \cdot p} \sqrt{(5b+5c-z)^2+p^2} \\
& - \frac{p}{2} \ln \left[(5b+5c-z) + \sqrt{(5b+5c-z)^2+p^2} \right] + \frac{(5b+5c-z)^2}{2 \cdot p} + \frac{(3b+3c-z)}{2 \cdot p} \sqrt{(3b+3c-z)^2+p^2} \\
& + \frac{p}{2} \ln \left[(3b+3c-z) + \sqrt{(3b+3c-z)^2+p^2} \right] - \frac{(3b+3c-z)^2}{2 \cdot p} + \frac{(5b+5c+z)}{2 \cdot p} \sqrt{(5b+5c+z)^2+p^2} \\
& + \frac{p}{2} \ln \left[(5b+5c+z) + \sqrt{(5b+5c+z)^2+p^2} \right] - \frac{(5b+5c+z)^2}{2 \cdot p} \left. \right] \right\} + \left\{ \frac{B_2(z-c)}{2(b+c)} \cdot \left[\right. \right. \\
& \frac{\sqrt{(3b+3c-z)^2+p^2} - (3b+3c-z)}{p \cdot [(3b+3c-z)^2+p^2]^{1/2}} - \frac{\sqrt{(5b+3c+z)^2+p^2} - (5b+3c+z)}{p \cdot [(5b+3c+z)^2+p^2]^{1/2}} \left. \right] + \frac{B_2(z+b)}{4(b+c)^2} \left[- \right. \\
& \frac{(3b+c+z)}{p} + \frac{\sqrt{(3b+c+z)^2+p^2}}{p} - \frac{(5b+5c-z)}{p} + \frac{\sqrt{(5b+5c-z)^2+p^2}}{p} + \frac{(3b+3c-z)}{p} - \\
& \left. \left. \frac{\sqrt{(3b+3c-z)^2+p^2}}{p} + \frac{(5b+5c+z)}{p} - \frac{\sqrt{(5b+5c+z)^2+p^2}}{p} \right] \right\} + \left\{ \frac{C_2(z-c)}{2} \cdot \right.
\end{aligned}$$

$$\left[\frac{p}{[(3b+3c-2)^2+p^2]^{3/2}} - \frac{p}{[(5b+3c+2)^2+p^2]^{3/2}} \right] + \frac{C_2(2+b)}{4(b+c)} \left[\frac{\sqrt{(3b+c+2)^2+p^2} - (3b+c+2)}{p \cdot [(3b+c+2)^2+p^2]^{1/2}} + \frac{\sqrt{(5b+5c-2)^2+p^2} - (5b+5c-2)}{p \cdot [(5b+5c-2)^2+p^2]^{1/2}} - \frac{\sqrt{(3b+3c-2)^2+p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2+p^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2+p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2+p^2]^{1/2}} \right] \} (A-20)$$

$$\begin{aligned} R_{G_2}(1, 2, 2) = & \left\{ 2(b+c)(2-c) \left[\frac{p \cdot [12(2b+2c-2)^2-3p^2]}{[(2b+2c-2)^2+p^2]^{7/2}} - \frac{p \cdot [12(4b+2c+2)^2-3p^2]}{[(4b+2c+2)^2+p^2]^{7/2}} \right] + (2+b) \left[\frac{3(2b+2c-2) \cdot p}{[(2b+2c-2)^2+p^2]^{5/2}} + \frac{3 \cdot (4b+4c-2) \cdot p}{[(4b+4c-2)^2+p^2]^{5/2}} - \frac{3 \cdot (2b+2c-2) \cdot p}{[(2b+2c-2)^2+p^2]^{5/2}} - \frac{3 \cdot (4b+2c+2) \cdot p}{[(4b+2c+2)^2+p^2]^{5/2}} \right] \right\} + \left\{ \frac{3(2-c)}{2(b+c)^3} \left[-\frac{(2b+2c-2)}{p} + \frac{\sqrt{(2b+2c-2)^2+p^2}}{p} + \frac{(4b+2c+2)}{p} - \frac{\sqrt{(4b+2c+2)^2+p^2}}{p} \right] + \frac{3(2+b)}{4(b+c)^4} \left[-\frac{(2b+2c-2)}{2 \cdot p} \right. \right. \\ & \left. \left. \sqrt{(2b+2c-2)^2+p^2} - \frac{p}{2} \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2}] + \frac{(2b+2c-2)^2}{2 \cdot p} - \frac{(4b+4c-2)}{2 \cdot p} \sqrt{(4b+4c-2)^2+p^2} \right. \right. \\ & \left. \left. - \frac{p}{2} \ln[(4b+4c-2) + \sqrt{(4b+4c-2)^2+p^2}] + \frac{(4b+4c-2)^2}{2 \cdot p} + \frac{(2b+2c-2)}{2 \cdot p} \sqrt{(2b+2c-2)^2+p^2} \right. \right. \\ & \left. \left. + \frac{p}{2} \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2}] - \frac{(2b+2c-2)^2}{2 \cdot p} + \frac{(4b+2c+2)}{2 \cdot p} \sqrt{(4b+2c+2)^2+p^2} \right. \right. \\ & \left. \left. + \frac{p}{2} \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2}] - \frac{(4b+2c+2)^2}{2 \cdot p} \right] \right\} + \left\{ \frac{A_2(2-c)}{2(b+c)^2} \left[\frac{\sqrt{(3b+3c-2)^2+p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2+p^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2+p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2+p^2]^{1/2}} \right] + \frac{A_2(2+b)}{4(b+c)^3} \left[-\frac{(3b+c+2)}{p} + \frac{\sqrt{(3b+c+2)^2+p^2}}{p} - \frac{(5b+5c-2)}{p} + \frac{\sqrt{(5b+5c-2)^2+p^2}}{p} + \frac{(3b+3c-2)}{p} - \frac{\sqrt{(3b+3c-2)^2+p^2}}{p} + \frac{(5b+3c+2)}{p} - \frac{\sqrt{(5b+3c+2)^2+p^2}}{p} \right] \right\} + \left\{ \frac{B_2(2-c)}{2(b+c)} \left[\frac{p}{[(3b+3c-2)^2+p^2]^{3/2}} - \frac{p}{[(5b+3c+2)^2+p^2]^{3/2}} \right] + \frac{B_2(2+b)}{4(b+c)^2} \left[\frac{\sqrt{(3b+c+2)^2+p^2} - (3b+c+2)}{p \cdot [(3b+c+2)^2+p^2]^{1/2}} + \frac{\sqrt{(5b+5c-2)^2+p^2} - (5b+5c-2)}{p \cdot [(5b+5c-2)^2+p^2]^{1/2}} - \right. \right. \end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{\sqrt{(3b+3c-2)^2+p^2} - (3b+3c-2)}{p \cdot [(3b+3c-2)^2+p^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2+p^2} - (5b+3c+2)}{p \cdot [(5b+3c+2)^2+p^2]^{1/2}} \right\} + \left\{ \right. \\
& \frac{c_2(2-c)}{2} \left[\frac{3 \cdot (3b+3c-2) \cdot p}{[(3b+3c-2)^2+p^2]^{5/2}} - \frac{3 \cdot (5b+3c+2) \cdot p}{[(5b+3c+2)^2+p^2]^{5/2}} \right] + \frac{c_2(2+b)}{4(b+c)} \cdot \\
& \left[\frac{p}{[(3b+3c-2)^2+p^2]^{3/2}} + \frac{p}{[(5b+3c+2)^2+p^2]^{3/2}} - \frac{p}{[(3b+3c-2)^2+p^2]^{3/2}} - \right. \\
& \left. \left. \frac{p}{[(5b+3c+2)^2+p^2]^{3/2}} \right] \right\}, (A-21)
\end{aligned}$$

$$\begin{aligned}
RG_4(1,2,2) = & \left\{ 2(b+c)(2-c) \cdot \left[\frac{f \cdot [12(b+2c-2)^2-3p^2]}{[(2b+2c-2)^2+p^2]^{7/2}} + \right. \right. \\
& \frac{p \cdot [12(4b+2c+2)^2-3p^2]}{[(4b+2c+2)^2+p^2]^{7/2}} \left. \right] + \left[\frac{f}{[(2b+2c-2)^2+p^2]^{3/2}} + \frac{f}{[(4b+4c-2)^2+p^2]^{3/2}} - \right. \\
& \frac{f}{[(2b+2c-2)^2+p^2]^{3/2}} - \frac{f}{[(4b+2c+2)^2+p^2]^{3/2}} \left. \right] - 2(b+c) \cdot \left[\frac{3 \cdot (2b+2c-2) \cdot f}{[(2b+2c-2)^2+p^2]^{5/2}} \right. \\
& - \frac{3 \cdot (4b+2c+2) \cdot f}{[(4b+2c+2)^2+p^2]^{5/2}} \left. \right] + (2+b) \cdot \left[\frac{3(2b+2c-2) \cdot p}{[(2b+2c-2)^2+p^2]^{5/2}} - \frac{3(4b+4c-2) \cdot p}{[(4b+4c-2)^2+p^2]^{5/2}} + \right. \\
& \frac{3(2b+2c-2) \cdot f}{[(2b+2c-2)^2+p^2]^{5/2}} - \frac{3(4b+2c+2) \cdot f}{[(4b+2c+2)^2+p^2]^{5/2}} \left. \right] \left\} + \left\{ \frac{3(2-c)}{2(b+c)^3} \left[- \frac{(2b+2c-2)}{p} \right. \right. \\
& + \frac{\sqrt{(2b+2c-2)^2+p^2}}{p} - \frac{(4b+2c+2)}{p} + \frac{\sqrt{(4b+2c+2)^2+p^2}}{p} \left. \right] + \frac{3}{4(b+c)} \left[\frac{f}{2} (2b+2c) \cdot \right. \\
& \ln[(2b+2c-2) + \sqrt{(2b+2c-2)^2+p^2}] - \frac{f}{2} \sqrt{(2b+2c-2)^2+p^2} - \frac{(2b+2c)^3}{6 \cdot p} + \frac{[(2b+2c-2)^2+p^2]^{3/2}}{6 \cdot p} \\
& + \frac{f}{2} (4b+4c-2) \cdot \ln[(4b+4c-2) + \sqrt{(4b+4c-2)^2+p^2}] - \frac{f}{2} \sqrt{(4b+4c-2)^2+p^2} - \\
& \frac{(4b+4c-2)^3}{6 \cdot p} + \frac{[(4b+4c-2)^2+p^2]^{3/2}}{6 \cdot p} - \frac{f}{2} (2b+2c-2) \cdot \ln[(2b+2c-2) + \\
& \sqrt{(2b+2c-2)^2+p^2}] + \frac{f}{2} \sqrt{(2b+2c-2)^2+p^2} + \frac{(2b+2c-2)^3}{6 \cdot p} - \frac{[(2b+2c-2)^2+p^2]^{3/2}}{6 \cdot p} \\
& - \frac{f}{2} (4b+2c+2) \cdot \ln[(4b+2c+2) + \sqrt{(4b+2c+2)^2+p^2}] + \frac{f}{2} \sqrt{(4b+2c+2)^2+p^2} \\
& + \frac{(4b+2c+2)^3}{6 \cdot p} - \frac{[(4b+2c+2)^2+p^2]^{3/2}}{6 \cdot p} \left. \right] - \frac{3}{2(b+c)^3} \left[- \frac{(2b+2c-2)}{p} \sqrt{(2b+2c-2)^2+p^2} \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{f}{2} \ln[(2b+2c-z) + \sqrt{(2b+2c-z)^2 + f^2}] + \frac{(2b+2c-z)^2}{2 \cdot f} + \frac{(4b+2c+z)}{2 \cdot f} \cdot \\
& \sqrt{(4b+2c+z)^2 + f^2} + \frac{f}{2} \ln[(4b+2c+z) + \sqrt{(4b+2c+z)^2 + f^2}] - \frac{(4b+2c+z)^2}{2 \cdot f} \Big] + \\
& \frac{3(z+b)}{4(b+c)^4} \Big[-\frac{(2b+z)}{2 \cdot f} \sqrt{(2b+z)^2 + f^2} - \frac{f}{2} \ln[(2b+z) + \sqrt{(2b+z)^2 + f^2}] + \frac{(2b+z)^2}{2 \cdot f} + \\
& \frac{(4b+4c-z)}{2 \cdot f} \sqrt{(4b+4c-z)^2 + f^2} + \frac{f}{2} \ln[(4b+4c-z) + \sqrt{(4b+4c-z)^2 + f^2}] - \\
& \frac{(4b+4c-z)^2}{2 \cdot f} - \frac{(2b+2c-z)}{2 \cdot f} \sqrt{(2b+2c-z)^2 + f^2} - \frac{f}{2} \ln[(2b+2c-z) + \\
& \sqrt{(2b+2c-z)^2 + f^2}] + \frac{(2b+2c-z)^2}{2 \cdot f} + \frac{(4b+2c+z)}{2 \cdot f} \sqrt{(4b+2c+z)^2 + f^2} - \\
& \frac{(4b+4c-z)^2}{2 \cdot f} + \frac{f}{2} \ln[(4b+4c-z) + \sqrt{(4b+4c-z)^2 + f^2}] \Big] \Big\} + \Big\{ \frac{A_2(z-c)}{2(b+c)^2} \cdot \\
& \Big[\frac{1}{f} \Big(1 - \frac{3b+3c-z}{[(3b+3c-z)^2 + f^2]^{1/2}} \Big) + \frac{\sqrt{(5b+3c+z)^2 + f^2} - (5b+3c+z)}{f} \Big] + \frac{A_2}{4(b+c)^3} \\
& \Big[-\frac{(3b+c+z)}{2 \cdot f} \sqrt{(3b+c+z)^2 + f^2} - \frac{f}{2} \ln[(3b+c+z) + \sqrt{(3b+c+z)^2 + f^2}] + \frac{(3b+c+z)^2}{2 \cdot f} \\
& - \frac{(5b+5c-z)}{2 \cdot f} \sqrt{(5b+5c-z)^2 + f^2} - \frac{f}{2} \ln[(5b+5c-z) + \sqrt{(5b+5c-z)^2 + f^2}] + \\
& \frac{(5b+5c-z)^2}{2 \cdot f} + \frac{(3b+3c-z)}{2 \cdot f} \sqrt{(3b+3c-z)^2 + f^2} + \frac{f}{2} \ln[(3b+3c-z) + \\
& \sqrt{(3b+3c-z)^2 + f^2}] - \frac{(3b+3c-z)^2}{2 \cdot f} + \frac{(5b+3c+z)}{2 \cdot f} \sqrt{(5b+3c+z)^2 + f^2} + \frac{f}{2} \ln[\\
& (5b+3c+z) + \sqrt{(5b+3c+z)^2 + f^2}] - \frac{(5b+3c+z)^2}{2 \cdot f} \Big] - \frac{A_2}{2(b+c)^2} \Big[-\frac{(3b+3c-z)}{f} \\
& + \frac{\sqrt{(3b+3c-z)^2 + f^2}}{f} + \frac{(5b+3c+z)}{f} - \frac{\sqrt{(5b+3c+z)^2 + f^2}}{f} \Big] + \frac{A_2(z+b)}{4(b+c)^3} \Big[-\frac{(3b+c+z)}{f} \\
& + \frac{\sqrt{(3b+c+z)^2 + f^2}}{f} + \frac{(5b+5c-z)}{f} - \frac{\sqrt{(5b+5c-z)^2 + f^2}}{f} - \frac{(3b+3c-z)}{f} + \frac{\sqrt{(3b+3c-z)^2 + f^2}}{f} \\
& + \frac{(5b+3c+z)}{f} - \frac{\sqrt{(5b+3c+z)^2 + f^2}}{f} \Big] \Big\} + \Big\{ \frac{B_2(z-c)}{2(b+c)} \Big[\frac{f}{[(3b+3c-z)^2 + f^2]^{3/2}} + \frac{f}{[(5b+3c+z)^2 + f^2]^{3/2}} \Big] \\
& + \frac{B_2}{4(b+c)^2} \Big[-\frac{(3b+c+z)}{f} + \frac{\sqrt{(3b+c+z)^2 + f^2}}{f} - \frac{(5b+5c-z)}{f} + \frac{\sqrt{(5b+5c-z)^2 + f^2}}{f} + \frac{(3b+3c-z)}{f} \\
& - \frac{\sqrt{(3b+3c-z)^2 + f^2}}{f} + \frac{(5b+3c+z)}{f} - \frac{\sqrt{(5b+3c+z)^2 + f^2}}{f} \Big] - \frac{B_2}{2(b+c)} \Big[
\end{aligned}$$

$$\begin{aligned}
& \frac{\sqrt{(3b+3c-2)^2+f^2}-(3b+3c-2)}{f \cdot [(3b+3c-2)^2+f^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2+f^2}-(5b+3c+2)}{f \cdot [(5b+3c+2)^2+f^2]^{1/2}} \Big] + \frac{B_2(2+b)}{4(b+c)} \Big] \\
& \frac{\sqrt{(3b+2+2)^2+f^2}-(3b+c+2)}{f \cdot [(3b+c+2)^2+f^2]^{1/2}} - \frac{\sqrt{(5b+5c-2)^2+f^2}-(5b+5c-2)}{f \cdot [(5b+5c-2)^2+f^2]^{1/2}} + \\
& \frac{\sqrt{(3b+3c-2)^2+f^2}-(3b+3c-2)}{f \cdot [(3b+3c-2)^2+f^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2+f^2}-(5b+3c+2)}{f \cdot [(5b+3c+2)^2+f^2]^{1/2}} \Big] \Big\} + \Big\{ \\
& \frac{C_2(2-c)}{2} \Big[\frac{3 \cdot (3b+3c-2) \cdot f}{[(3b+3c-2)^2+f^2]^{5/2}} + \frac{3 \cdot (5b+3c+2) \cdot f}{[(5b+3c+2)^2+f^2]^{5/2}} \Big] + \frac{C_2}{4(b+c)} \cdot \\
& \Big[\frac{\sqrt{(3b+c+2)^2+f^2}-(3b+c+2)}{f \cdot [(3b+c+2)^2+f^2]^{1/2}} + \frac{\sqrt{(5b+5c-2)^2+f^2}-(5b+5c-2)}{f \cdot [(5b+5c-2)^2+f^2]^{1/2}} - \\
& \frac{\sqrt{(3b+3c-2)^2+f^2}-(3b+3c-2)}{f \cdot [(3b+3c-2)^2+f^2]^{1/2}} - \frac{\sqrt{(5b+3c+2)^2+f^2}-(5b+3c+2)}{f \cdot [(5b+3c+2)^2+f^2]^{1/2}} \Big] - \frac{C_2}{2} \Big[\\
& \frac{f}{[(3b+3c-2)^2+f^2]^{3/2}} - \frac{f}{[(5b+3c+2)^2+f^2]^{3/2}} \Big] + \frac{C_2(2+b)}{4(b+c)} \cdot \Big[\frac{f}{[(3b+c+2)^2+f^2]^{3/2}} \\
& - \frac{f}{[(5b+5c-2)^2+f^2]^{3/2}} + \frac{f}{[(3b+3c-2)^2+f^2]^{3/2}} - \frac{f}{[(5b+3c+2)^2+f^2]^{3/2}} \Big] \Big\} \quad (A-22)
\end{aligned}$$

REFERENCES

- Blake, J.R. 1971 A note on the image system for a stokeslet in a no-slip boundary. Proc. Camb. Phil. Soc. 70, 303-310.
- Dagan, Z., Yan, Z. & Shan, H. 1988 The axisymmetric rise of a spherical bubble at the exit of an orifice in the presence of a stagnant cap of insoluble surfactants. J. Fluid Mech. 190, 299-319.
- Gradshteyn, I.S. & Ryzhik, I.M. 1980 Tables of integrals, series and products. 4th Edition, Scripta Technica, Inc.
- Miyazaki, T. & Hasimoto, H. 1984 The motion of a small sphere in fluid near a circular pore in a plane wall. J. Fluid Mech. 145, 201-221.
- Hsu, R. & Ganatos, P. 1989 The motion of a rigid body in viscous fluid bounded by a plane wall. J. Fluid Mech. 207, 29-72.
- Liron, N. & Mochon, S. 1976 Stokes flow for a stokeslet between two parallel flat plates. J. Engr. Math. 10, 287-303.
- Liron, N. & Shahar, R. 1978 Stokes flow due to a stokeslet in a pipe. J. Engr. Math. 86, 727-744.
- Rallison, J.M., & Acrivos, A. 1978 A numerical study of the deformation and burst of a viscous drop in an extensional flow. J. Fluid Mech. 89, 191-200.
- Wu, W.Y., Weinbaum, S. & Acrivos, A. 1992 The motion of a sphere in a shear flow with suction past a plane wall with a hole and its application to particle screening. J. Fluid Mech. 243, 489-518
- Youngren, G.K. & Acrivos, A. 1975 Stokes flow past a particle of arbitrary shape: a numerical method of solution. J. Fluid Mech. 69, 377-403.2

Chapter 3 The motion of an arbitrarily shaped rigid body between two infinite parallel walls

3.1 Introduction

The creeping motion of a body between two infinite parallel walls is problem of long-standing interest with numerous biological and engineering applications. The theory may be used to model diffusion of macromolecules across the intercellular space between adjacent cells, Ganatos et al (1980). It can also be used to determine the trajectory of a foreign particle in a lubricating bearing or in studies involving sedimentation. Although the theory has numerous practical applications, it has never been adequately developed except for a spherical particle. Only several limiting cases have been obtained for a non-spherical particle, Wakiya (1958), Hsu & Ganatos (1989).

As outlined in Weinbaum, Ganatos & Yan (1990), several methods can be used to study the low Reynolds number flows involving interactions between a particle and boundaries. The theoretical methods suitable for the present research are the method of reflections, boundary collocation method, the finite element method and the boundary integral method.

The method of reflections has been used extensively to study hydrodynamic interactions between particles and boundaries in low Reynolds number flow. This iterative method, by satisfying the no-slip conditions on the surfaces of the particle and the confining boundaries alternatively, provides good results using one or two reflected fields only if the particle is far removed from the boundaries which confine the flow. Using this

technique Wakiya (1958) studied the problem of the motion of a spheroid between two parallel walls with both the velocity and the semi-major axis of the spheroid oriented parallel to the walls.

The boundary collocation method, which has been used to treat a wide variety of bounded Stokes flows and unbounded multiparticle Stokes flows, works well when the basis functions for the series expansions of the flow field are known and the no-slip boundary conditions can be satisfied on the confining boundaries of infinite extent analytically. The difficulty of this method is that for a non-spherical particle, the orientation angles of the particle enter as parameters in the inversion integrals which cancel the disturbance produced by the particle and felt on the confining walls. The result is that these integrals must be performed numerically which is exceedingly time consuming since these integrals appear as a kernel in the Fourier integral representation of the velocity field. Thus the method has only been applied for a spherical particle in a channel, Ganatos et al (1980 a,b).

The finite element method, which uses a spatial discretization of the flow field, can be used to treat a particle whose shape does not conform to a natural orthogonal coordinate system. However, it is computationally inefficient to use this method for Stokes flow problems where the fluid domain is not periodic and is of infinite extent as in the present application since a large number of nodal points would be needed to represent the three-dimensional flow domain.

The boundary integral method, which is used in the present research, is based on Ladyzhenskaya's theory (1963) of single and double layer potentials and was first applied

to the treatment of Stokes flow problems by Youngren & Acrivos (1975). It is a powerful numerical technique for treating three-dimensional Stokes flows and has a greater versatility than any of the other methods discussed previously. The flow disturbance produced by the particle is formulated as a system of linear integral equations with a distribution of stokeslets over its surface. By discretizing the surface of the particle and assuming the stokeslet strength constant over each element, the original integral equations are reduced to a system of linear algebraic equations for the unknown stokeslet densities. Recent applications of this method include the work of Lewellen (1982), Yan et al (1987) and Wu et al (1992). Although, in general, the expressions for the disturbance flows contain both single and double layer potentials, the contribution of the double layer potential vanishes identically for rigid particle and boundary motions. An important feature of this method is that the surface stress distributions, drags and torques acting on the particle can be determined without actually solving for the velocity field.

In the present chapter, we will use the fundamental solution developed in the previous chapter with the boundary integral method to obtain the first strong interaction solutions for the hydrodynamic interaction of a non-spherical particle in a channel.

3.2 Formulation for a motion of an arbitrary body between two planar walls

Consider an arbitrarily shaped particle moving slowly in a fluid between two parallel walls in an arbitrary direction. The disturbance flow field produced by the quasi-steady motion of the particle is governed by the equations

$$\mu \nabla^2 \mathbf{V} = \nabla p, \quad \nabla \cdot \mathbf{V} = 0 \quad (3.1 \text{ a,b})$$

with the boundary conditions, $\mathbf{V}=\mathbf{V}_p$ on the surface of the particle and $\mathbf{V}=\mathbf{V}_b$ on the planes, where $\mathbf{V}_p$ and $\mathbf{V}_b$ are velocities of the particle and the walls respectively.

In general the disturbance velocity $\mathbf{V}$ and pressure p can be represented by

$$\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2, \quad p = p_1 - p_2 \quad (3.2 \text{ a,b})$$

where $\mathbf{V}_1$ and p_1 are the actual fluid velocity and pressure field and $\mathbf{V}_2$ and p_2 represent the velocity and pressure of the flow in the absence of the particle. The undisturbed flow field can be any Stokes flow between two parallel walls.

Owing to the linearity of the Stokes equations (3.1 a,b) and boundary conditions, the general motion of a particle between two parallel walls can be represented by the superposition of some basic motions. The basic flow geometries considered here include a particle at an arbitrary orientation translating perpendicular or parallel to the walls, rotating about the axis parallel or perpendicular to the walls, rigidly held in Couette flow with the upper wall moving, or in Poiseuille flow between two walls. The latter two Stokes flows are defined by the following equations respectively

$$\mathbf{V}_s(\mathbf{x}_3) = S\mathbf{x}_3 \quad (3.3)$$

$$\mathbf{V}_p(\mathbf{x}_3) = 4V_c \mathbf{x}_3(H - \mathbf{x}_3)/H^2 \quad (3.4)$$

where S represents the constant velocity gradient of Couette flow, V_c is the centerline velocity of Poiseuille flow and H is wall-to-wall spacing as shown in Figure 2. All the dimensionless resistance coefficients are functions of particle-to-wall spacing H_p/c , aspect ratio b/c , wall-to-wall spacing H/c and orientation angles α, γ where c is a characteristic length of the particle. The matrix of the resistance coefficients is defined by the following

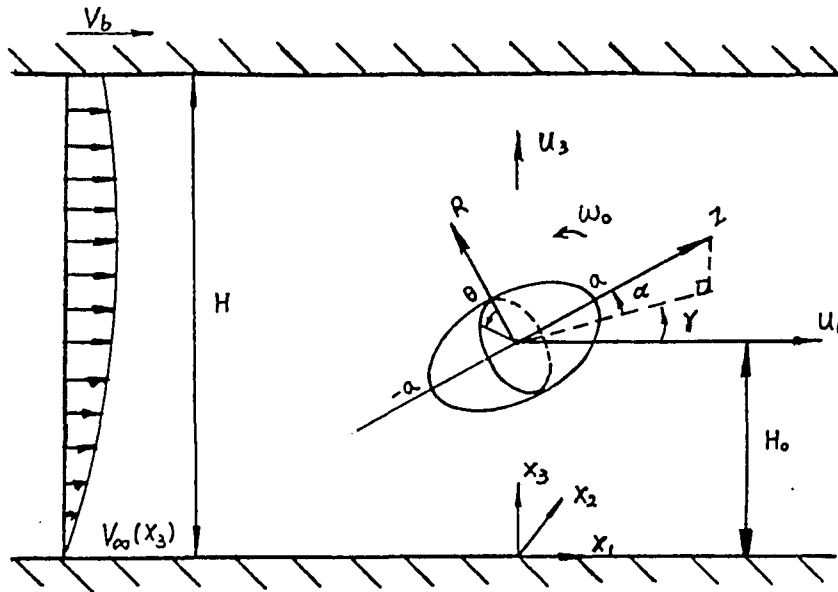


Figure 2. Geometry of a body of revolution between two parallel walls. The body position parameter s is defined as the ratio H_0/H .

equation

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix} = 6\pi\mu c \begin{pmatrix} F_1^{t_1} & F_1^{t_2} & F_1^{t_3} & F_1^{r_1} & F_1^{r_2} & F_1^{r_3} & H_0 F_1^s & F_1^p \\ F_2^{t_1} & F_2^{t_2} & F_2^{t_3} & F_2^{r_1} & F_2^{r_2} & F_2^{r_3} & H_0 F_2^s & F_2^p \\ F_3^{t_1} & F_3^{t_2} & F_3^{t_3} & F_3^{r_1} & F_3^{r_2} & F_3^{r_3} & H_0 F_3^s & F_3^p \\ \frac{4}{3}bT_1^{t_1} & \frac{4}{3}bT_1^{t_2} & \frac{4}{3}bT_1^{t_3} & \frac{4}{3}bT_1^{r_1} & \frac{4}{3}bT_1^{r_2} & \frac{4}{3}bT_1^{r_3} & \frac{2}{3}b^2T_1^s & \frac{4}{3}bT_1^p \\ \frac{4}{3}bT_2^{t_1} & \frac{4}{3}bT_2^{t_2} & \frac{4}{3}bT_2^{t_3} & \frac{4}{3}bT_2^{r_1} & \frac{4}{3}bT_2^{r_2} & \frac{4}{3}bT_2^{r_3} & \frac{2}{3}b^2T_2^s & \frac{4}{3}bT_2^p \\ \frac{4}{3}bT_3^{t_1} & \frac{4}{3}bT_3^{t_2} & \frac{4}{3}bT_3^{t_3} & \frac{4}{3}bT_3^{r_1} & \frac{4}{3}bT_3^{r_2} & \frac{4}{3}bT_3^{r_3} & \frac{2}{3}b^2T_3^s & \frac{4}{3}bT_3^p \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ b\omega_1 \\ b\omega_2 \\ b\omega_3 \\ s \\ v_c \end{pmatrix} \quad (3.5)$$

where F_1 , F_2 and F_3 are force components in the direction of x_1 -, x_2 - and x_3 -axes respectively, and T_1 , T_2 , T_3 are torque acting on the particle about the x_1 -, x_2 - and x_3 -axes respectively. Then the boundary integral technique applied here yields

$$v_i(\mathbf{X}) = \iint_{S_p} U_i^k(\mathbf{X}, \mathbf{Y}) f_k(\mathbf{Y}) dS_Y \quad (3.6)$$

where $U_i^k(\mathbf{X}, \mathbf{Y})$ is the fundamental solution, i.e. the velocity at point $\mathbf{X}$ in the i -th direction due to the stokeslet at $\mathbf{Y}$ pointing in k -th direction, and $f_k(\mathbf{Y})$ is the density function or local stress distribution. Since the fundamental solution identically satisfies the no-slip conditions on the walls the surface integration above needs only to be performed along the surface of the particle S_p .

The solution (3.6) satisfies the governing equation (3.1 a,b) and the no-slip boundary conditions on the confining walls. Applying the no-slip boundary condition on the surface of the particle results in a linear system of integral equations to be solved for the

unknown density functions $f_k(\mathbf{Y})$. The discretization on the surface of the particle and assumption of the constant density functions f_k on each element lead to the following equations

$$\mathbf{V}_i(\mathbf{X}^m) = \sum_{j=1}^N \mathbf{A}_{ik}^0(\mathbf{X}^m) f_k(\mathbf{X}^0) \quad (3.7 \text{ a})$$

$$\mathbf{A}_{ik}^0(\mathbf{X}^m) = \int \int_{\Delta_j} \mathbf{U}_i^k(\mathbf{X}^m, \mathbf{Y}) d\mathbf{S}_y \quad (3.7 \text{ b})$$

$$\mathbf{V}(\mathbf{X}^m) = \mathbf{V}_0(\mathbf{X}^m) + \boldsymbol{\omega} \times \mathbf{r}(\mathbf{X}^m) - \mathbf{V}_2(\mathbf{X}^m) \quad (3.8)$$

Here the entire surface of the particle is discretized into N elements centered at $\mathbf{X}^m$ and the unknown density functions f_k within each element are approximated by the constants equal to the values f_k at the center of the elements. The application of no-slip conditions on the surface of the particle at $\mathbf{X}^m$ generates $3N$ equations for the $3N$ unknown f_k . With the density function or the local stress functions f_k known, the total force $\mathbf{F}_i$ and torque $\mathbf{T}_i$ exerted by the fluid on the particle are

$$\mathbf{F}_i = \sum_{j=1}^N f_i(\mathbf{X}^0) \int \int_{\Delta_j} d\mathbf{S}_y \quad (3.9)$$

$$\mathbf{T}_i = \sum_{j=1}^N f_k(\mathbf{X}^0) \int \int_{\Delta_j} \epsilon_{ijk} y_l d\mathbf{S}_y \quad (3.10)$$

The integration performed over each surface element as indicated by (3.7 b) contains a singularity at the center of the element where the point force is applied. As done by Youngren & Acrivos (1975), the singularity is isolated into a square element of size ϵ and the integration performed analytically within this region and numerically over the

remaining area of the surface element. Wu et al (1992), who treated the problem of a spherical particle adjacent to a planar wall containing a circular hole, found that the computed force and torque coefficients are quite sensitive to the value of ϵ used when the particle is near the wall. These authors used a different value of ϵ for each particle-to-wall spacing, which gave the best agreement with the exact solutions available for the motion of a sphere adjacent to a planar boundary and referred to these as "optimum" values of ϵ . However for a non-spherical particle between two parallel walls, there are many parameters which affect the optimum values of ϵ , such as, particle-to-wall spacing H_0 , wall-to-wall spacing H and particle shape and orientation. Therefore it is almost impossible to find a regular pattern to obtain the optimum values of ϵ . Hereafter a unique value of ϵ , which is 10% of the largest length of the singular element, is applied to all singular elements in the calculations. Even though the accuracy in some cases may be not as good as one with a variable ϵ , the overall accuracy is still reliable as seen by comparison with existing results in the literature.

As an important special application to particles whose shape is a body of revolution, the surface of the particle is described using a cylindrical coordinate system (R, θ, Z) as shown in Figure 2. The Z -axis lies along the symmetry axis of the particle which is oriented at arbitrary cone angle α and azimuthal angle γ . The radius of the particle surface, R , is represented by a single-valued function $R_s(Z)$. For a spheroidal particle,

$$R_s(z) = b \left(1 - \left(\frac{z}{a} \right)^2 \right)^{1/2} \quad (3.11)$$

where $a < b$ for an oblate spheroid, $a > b$ for a prolate spheroid and $a=b$ for a sphere. The

characteristic length c in (3.5) is taken as the semi-major axis for a spheroid or the radius for a sphere. The transformation between the coordinate systems (R, θ, Z) and (x_1, x_2, x_3) shown in Figure 2 is given by

$$x_1 = Z \cos \alpha \cos \gamma - R \cos \theta \sin \alpha \cos \gamma + R \sin \theta \sin \gamma \quad (3.12 \text{ a})$$

$$x_2 = Z \cos \alpha \sin \gamma - R \cos \theta \sin \alpha \sin \gamma - R \sin \theta \cos \gamma \quad (3.12 \text{ b})$$

$$x_3 = Z \sin \alpha + R \cos \theta \cos \alpha + H_0 \quad (3.12 \text{ c})$$

where H_0 is the particle-to-wall spacing. The infinitesimal surface area dS_y in the solution of the boundary integral equation (3.6) reduces to the following form in the cylindrical coordinate system (R, θ, Z) ,

$$dS_y = \left[1 + \left(\frac{dR_s(Z)}{dZ} \right)^2 \right]^{1/2} R_s(Z) d\theta dZ \quad (3.13)$$

The surface of the particle is discretized by choosing $N_1 \times N_2$ elements in the z and θ directions respectively. A simple 4×4 composite Gaussian quadrature formula is applied to evaluate the two-dimensional integrals (3.7 b). As discussed earlier the singular elements are handled using the procedure presented by Youngren & Acrivos (1975).

If the particle is oriented with the azimuthal angle $\gamma=0^\circ$ the flow is planar symmetric about the x_1 - x_3 plane provided no external torque is applied about the x_1 - or x_3 -axes. Equations (3.9, 3.10) reduce to the following form in cylindrical coordinates

$$F_i = \sum_{j=1}^N f_i(\mathbf{X}^0) \iint_{A_j} dS_y, \quad i=1,3 \quad (3.14)$$

$$T_2 = \sum_{j=1}^N \left\{ \iint_{\Delta_j} (f_1(X^{(j)}) \sin \alpha - f_3(X^{(j)}) \cos \alpha) Z dS_j \right. \\ \left. + \iint_{\Delta_j} (f_1(X^{(j)}) \cos \alpha + f_3(X^{(j)}) \sin \alpha) R_s(Z) \cos \theta dS_j \right\} \quad (3.15)$$

Owing to the symmetry of the flow geometry, only an half of the surface of the particle needs to be used for calculating the unknown density functions. The density functions for a pair of symmetric points about the x_1 - x_3 plane, $f_k(x_1, x_2, x_3)$ and $f_k(x_1, -x_2, x_3)$, have the following relations

$$f_1(x_1, x_2, x_3) = f_1(x_1, -x_2, x_3) \quad (3.16 \text{ a})$$

$$f_2(x_1, x_2, x_3) = -f_2(x_1, -x_2, x_3) \quad (3.16 \text{ b})$$

$$f_3(x_1, x_2, x_3) = f_3(x_1, -x_2, x_3) \quad (3.16 \text{ c})$$

where f_i represents the density function pointing in the direction of the i -th axis. Therefore for planar symmetric flows the total number of elements (N) which the surface is subdivided into refers to the number of elements on half the surface of the particle. In the general cases in which the angle γ is not equal to zero, the total number of elements (N) refers to the number of elements on the whole surface of the particle.

In the following sections numerical solutions for a body of revolution are presented with radius of the body surface, R , given by (3.11). Tests show that a discretization of 10×10 on half the surface of the particle for the planar symmetric cases and 10×20 for the non-planar symmetric cases is sufficient for producing solutions which are converged to the number of significant figures reported.

3.3 Comparison with existing solutions for a particle near one wall or between two parallel walls

In this section the boundary integral formulas derived previously are applied to the motion of a sphere between two parallel walls and a spheroid near one planar wall or between two parallel walls. The solutions obtained are compared with existing results in the literature to test the accuracy of the solution technique which will be used for the new solutions to be presented in the next section.

3.3.1 Solutions for a sphere between two parallel walls

For the motion of a sphere translating perpendicular to the walls, which is an axisymmetric flow geometry, the force acting on it is given by

$$F_3 = 6\pi\mu a U_3 F_3^* \quad (3.17)$$

where a is the radius of the sphere, U_3 is the velocity of the sphere in the direction of x_3 -axis.

Table 3 shows a comparison between boundary integral solutions and collocation results obtained by Ganatos et al (1980 a) for the drag coefficient of a sphere moving perpendicular to the walls at various positions s and sphere-to-wall spacings H_0 . The results are in good agreement with the collocation solutions. In the worst case of $H_0=1.1$ and $s=0.5$ where the size of the sphere is relatively large compared to the size of the channel the maximum relative error is 4.2%. For $H_0 \geq 1.5$ the maximum relative error is less than 0.4%. Thus the approximate fundamental solution applied here has produced satisfactory results with the boundary integral method.

Table 3 Resistance coefficient F_3^b for a sphere moving perpendicular to the walls at various positions s .

(a) Present results. (b) Ganatos et al (1980 a,b).

H_s		$s=0.1$	$s=0.2$	$s=0.25$	$s=0.4$	$s=0.5$
1.1	a	-10.97	-11.08	-11.17	-12.12	-20.25
	b	-11.46	-11.50	-11.56	-12.48	-21.03
1.5	a	-3.213	-3.234	-3.264	-3.625	-4.796
	b	-3.206	-3.223	-3.253	-3.619	-4.780
2.0	a	-2.128	-2.138	-2.154	-2.338	-2.793
	b	-2.126	-2.135	-2.151	-2.335	-2.789
5.0	a	-1.286	-1.288	-1.291	-1.326	-1.398
	b	-1.285	-1.287	-1.290	-1.325	-1.397

Table 4 Resistance coefficients for a sphere rigidly held in Couette flow with the upper wall moving or in Poiseuille flow at position $s=0.25$.

(a) Present results. (b) Ganatos et al (1980a,b).

H_s		1.1	1.25	1.5	2.0	4.0
F_1^a	a	1.916	1.769	1.607	1.425	1.190
	b	1.900	1.758	1.599	1.420	1.188
T_2^a	a	1.038	1.033	1.028	1.023	1.013
	b	1.039	1.034	1.028	1.023	1.013
F_1^p	a	1.361	1.270	1.167	1.048	0.8858
	b	1.347	1.261	1.161	1.044	0.8858
T_2^p	a	0.2319	0.2050	0.1713	0.1286	0.06368
	b	0.2333	0.2058	0.1718	0.1287	0.06368

For the motion of a sphere translating parallel to the walls, which is a planar symmetric flow, the matrix of resistance coefficients (3.5) is reduced to the following

$$\begin{pmatrix} F_1 \\ T_2 \end{pmatrix} = 6\pi\mu a \begin{pmatrix} F_1^h & F_1^r & H_0 F_1^s & F_1^p \\ \frac{4}{3}aT_2^h & \frac{4}{3}aT_2^r & \frac{2}{3}a^2T_2^s & \frac{4}{3}aT_2^p \end{pmatrix} \begin{pmatrix} U_1 \\ a\omega \\ S \\ v_c \end{pmatrix} \quad (3.18)$$

Table 4 shows a comparison of drag and torque coefficients between the present results and the strong interaction solutions of Ganatos et al (1980 b) for a sphere rigidly held in Couette flow with the upper wall moving or in Poiseuille flow at the position of $s=0.25$. These sets of results match the collocation solutions very well. The maximum relative error, 1.04%, occurs for the coefficient F_1^p when $H_0=1.1$. At this spacing, relative errors for other coefficients, F_1^s , T_2^s , T_2^p , are 0.8%, 0.1% and 0.6% respectively.

3.3.2 Solutions for a spheroid near one wall

In this section the present results are compared with the strong interaction solutions of Hsu & Ganatos (1989) which were also obtained by the boundary integral method but with the integration in the azimuthal direction performed analytically. The matrix of resistance coefficients (3.5) is reduced to the following form for this flow geometry

$$\begin{pmatrix} F_1 \\ F_3 \\ T_2 \end{pmatrix} = 6\pi\mu c \begin{pmatrix} F_1^{t_1} & F_1^{t_3} & F_1^r & H_o F_1^s \\ F_3^{t_1} & F_3^{t_3} & F_3^r & H_o F_3^s \\ \frac{4}{3}bT_2^{t_1} & \frac{4}{3}bT_2^{t_3} & \frac{4}{3}bT_2^r & \frac{2}{3}b^2T_2^s \end{pmatrix} \begin{pmatrix} U_1 \\ U_3 \\ b\omega \\ S \end{pmatrix} \quad (3.19)$$

All coefficients presented for the spheroid are expressed as the ratio of actual one over the one in unbounded flow at the same orientation (e.g. $F_1^{t_1}/F_{1,\infty}^{t_1}$ $T_2^{t_1}/T_{2,\infty}^{t_1}$) except the torque coefficients due to the translation of the spheroid parallel or perpendicular to the walls, in which the torque is zero in unbounded flow. The formulas for the coefficients in an unbounded flow are presented in Appendix A. Table 5 shows a comparison of the drag and torque coefficients for an oblate spheroid having aspect ratio $\epsilon=0.5$ near a planar wall at various particle-to-wall spacings, H_o . The relative error for all coefficients falls within 0.5% except the torque coefficient $T_2^{t_1}$, whose error is 2.2% when particle-to-wall spacing $H_o=0.8$.

3.3.3 Solutions for a spheroid between two parallel walls with both the semi-major axis and velocity parallel to the walls

In this section the present results are compared with the asymptotic solutions of Wakiya (1958) obtained using the method of reflections.

Tables 6, 7 show a comparison of drag and torque coefficients for an oblate or prolate spheroid with an aspect ratio $\epsilon=0.5$ at the position $s=0.25$ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$.

TABLE 5 Comparison of values of resistance coefficients for an oblate spheroid near one wall with aspect ratio $\epsilon=0.5$ at various particle-to-wall spacings H_0 .

(a) Present results. (b) Hsu et al (1989)

H_0		0.8	1.1	1.5	2.0
$F_1^{h_1}$	a	1.934	1.573	1.383	1.271
	b	1.923	1.568	1.380	1.271
$F_3^{h_3}$	a	7.298	3.604	2.390	1.860
	b	7.262	3.589	2.385	1.857
$T_2^{h_1}$	a	0.1238	0.05953	0.03256	0.01891
	b	0.1211	0.05916	0.03251	0.01891
F_1^s	a	1.675	1.483	1.349	1.257
	b	1.670	1.480	1.346	1.257
T_2^s	a	1.377	1.368	1.337	1.291
	b	1.369	1.361	1.331	1.290
T_2^r	a	1.659	1.246	1.102	1.046
	b	1.645	1.245	1.102	1.046

TABLE 6 Comparison of values of resistance coefficients for a prolate spheroid between two parallel walls with a aspect ratio $\epsilon=0.5$ at position $s=0.25$.

(a) Present results. (b) Wakiya (1958).

H_0		1.00	1.25	2.50	5.00	10.00
F_1^h	a	1.56	1.42	1.18	1.08	1.04
	b	1.53	1.40	1.18	1.08	1.04
T_2^h	a	10.15	6.83	1.92	0.49	0.12
	b	*	*	2.16	0.54	0.12
F_1^s	a	1.50	1.39	1.18	1.08	1.04
	b	1.45	1.37	1.18	1.08	1.04
T_2^s	a	29.44	23.31	10.82	5.00	2.38
	b	*	*	11.24	5.04	2.39
F_1^p	a	1.48	1.37	1.17	1.08	1.04
	b	1.44	1.36	1.17	1.08	1.04
T_2^p	a	22.52	17.63	7.84	3.50	1.63
	b	*	*	8.22	3.53	1.63

* data are not available for those positions.

TABLE 7 Comparison of values of resistance coefficients for an oblate spheroid between two parallel walls with a aspect ratio $\epsilon=0.5$ at position $s=0.25$.

(a) Present results. (b) Wakiya (1958).

H_e		2.00	2.50	5.00	10.00	20.00
$F_1^{t_1}$	a	1.42	1.31	1.13	1.06	1.03
	b	1.41	1.31	1.13	1.06	1.03
$T_2^{t_1}$	a	1.78	1.02	0.22	0.05	0.01
	b	*	*	0.21	0.05	0.01
F_1^s	a	1.39	1.29	1.13	1.06	1.03
	b	1.38	1.29	1.13	1.06	1.03
T_2^s	a	15.91	12.88	6.62	3.36	1.69
	b	*	*	6.22	3.37	1.70
F_1^p	a	1.36	1.27	1.13	1.06	1.03
	b	1.35	1.27	1.13	1.06	1.03
T_2^p	a	10.08	8.26	4.34	2.22	1.15
	b	*	*	4.43	2.23	1.13

* data are not available for those positions.

At the larger particle-to-wall spacings the present results are in perfect agreement with Wakiya's solutions. As expected, a gradual decay in the accuracy of the weak interaction theory is observed as the particle-to-wall spacing H_0 becomes smaller. The torque coefficients lose accuracy much faster than the drag coefficients for both a prolate and an oblate spheroid. At a spacing of $H_0=1.0$, the percent error of the coefficients T_2^h , T_2^s , T_2^p , F_1^h , F_1^s , F_1^p are 12.5%, 3.9% , 4.8%, 1.9%, 3.3% and 2.7% respectively for a prolate spheroid. For an oblate spheroid the error are 4.5%, 6.0%, 2.0%, 0.7%, 0.7% and 0.7% respectively at a spacing of $H_0=2.0$.

3.4 Solutions for a spheroid at an arbitrary position with an orientation of arbitrary α and $\gamma=0^\circ$ between the two parallel walls

In this section the boundary integral formulas derived previously are applied to a spheroid at arbitrary α and $\gamma=0^\circ$ between two parallel walls. The flow fields considered here are those induced by rigid body motions of the spheroid in a quiescent fluid bounded by the two walls, and the flow field where a spheroid is rigidly held in place in Couette or Poiseuille flow.

Owing to the planar symmetry of the flow geometry the matrix of the resistance coefficients (3.5) is reduced to

$$\begin{pmatrix} F_1 \\ F_3 \\ T_2 \end{pmatrix} = 6\pi\mu c \begin{pmatrix} F_1^h & F_1^b & F_1^r & H_o F_1^s & F_1^p \\ F_3^h & F_3^b & F_3^r & H_o F_3^s & F_3^p \\ \frac{4}{3}bT_2^h & \frac{4}{3}bT_2^b & \frac{4}{3}bT_2^r & \frac{2}{3}b^2T_2^s & \frac{4}{3}bT_2^p \end{pmatrix} \begin{pmatrix} U_1 \\ U_3 \\ b\omega \\ S \\ V_c \end{pmatrix} \quad (3.20)$$

All the dimensionless resistance coefficients are functions of particle-to-wall spacing H_o/c , aspect ratio ϵ (defined as the ratio of the shorter axis to semi-major axis for both a prolate and an oblate spheroid), wall-to-wall spacing H/c and orientation angle α . Here c is the semi-major axis of the spheroid, i.e. $c=a$ for a prolate spheroid and $c=b$ for an oblate spheroid. By applying the reciprocity theorems, Brenner (1964), the following relations of the resistance coefficients are obtained .

$$F_1^b = F_3^h, \quad F_1^r = \frac{4}{3}T_2^h, \quad F_3^r = \frac{4}{3}T_2^b \quad (3.21 \text{ a,b,c})$$

These relations serve as a further test of the consistency of the numerical results. Since the right-hand side of the integral equation (3.7 a) only depends on the geometry of the spheroid and not on the boundary conditions satisfied on it, the density functions f_k can be obtained in a single computer run for all basic flows mentioned previously. A further reduction in the number of computer run can be made by computing the density functions f_k at the positions of $s > 0.5$ using the results at the positions of $s < 0.5$ with the following relations

$$F_1^h(s) = F_1^h(1-s), \quad F_3^h(s) = F_3^h(1-s), \quad T_2^h(s) = -T_2^h(1-s) \quad (3.22 \text{ a,b,c})$$

$$T_2^r(s) = T_2^r(1-s) \quad (3.23)$$

$$F_1^s(s) = [F_1^h(1-s) - (1-s)F_1^s(1-s)]/s \quad (3.24 \text{ a})$$

$$F_3^s(s) = [F_3^h(1-s) - (1-s)F_3^s(1-s)]/s \quad (3.24 \text{ b})$$

$$T_2^s(s) = [-2(H/c)T_2^h(1-s) + T_2^s(1-s)] \quad (3.24 \text{ c})$$

$$F_1^h(s) = F_1^h(1-s), \quad F_3^h(s) = F_3^h(1-s), \quad T_2^h(s) = -T_2^h(1-s) \quad (3.25 \text{ a,b,c})$$

$$F_1^p(s) = F_1^p(1-s), \quad F_3^p(s) = F_3^p(1-s), \quad T_2^p(s) = -T_2^p(1-s) \quad (3.26 \text{ a,b,c})$$

where the range of validity of s is $0.5 < s \leq 1.0$ in equations (3.22-3.26).

Figures 3-13 present values of the force and torque acting on an oblate or a prolate spheroid having an aspect ratio $\epsilon=0.5$ or 0.1 with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$.

Figures 3 (a,b,c,d) show the force component parallel to the walls, due to translation of the spheroid parallel to the walls. At fixed particle-to-wall spacings, $H_o=\text{constant}$ (dashed lines), the drag force increases with increasing s as the second wall is brought closer to the particle until both walls are equidistant from the particle. The effect of the second wall for an oblate spheroid is generally greater than that for a prolate spheroid in this orientation due to the smaller fluid gap between the particle and each of the walls. At fixed wall-to-wall spacings, $H=\text{constant}$ (solid lines), the spheroid experiences minimum drag when it is located midway between the two walls and increasingly experiences larger drag as it approaches either of the two walls.

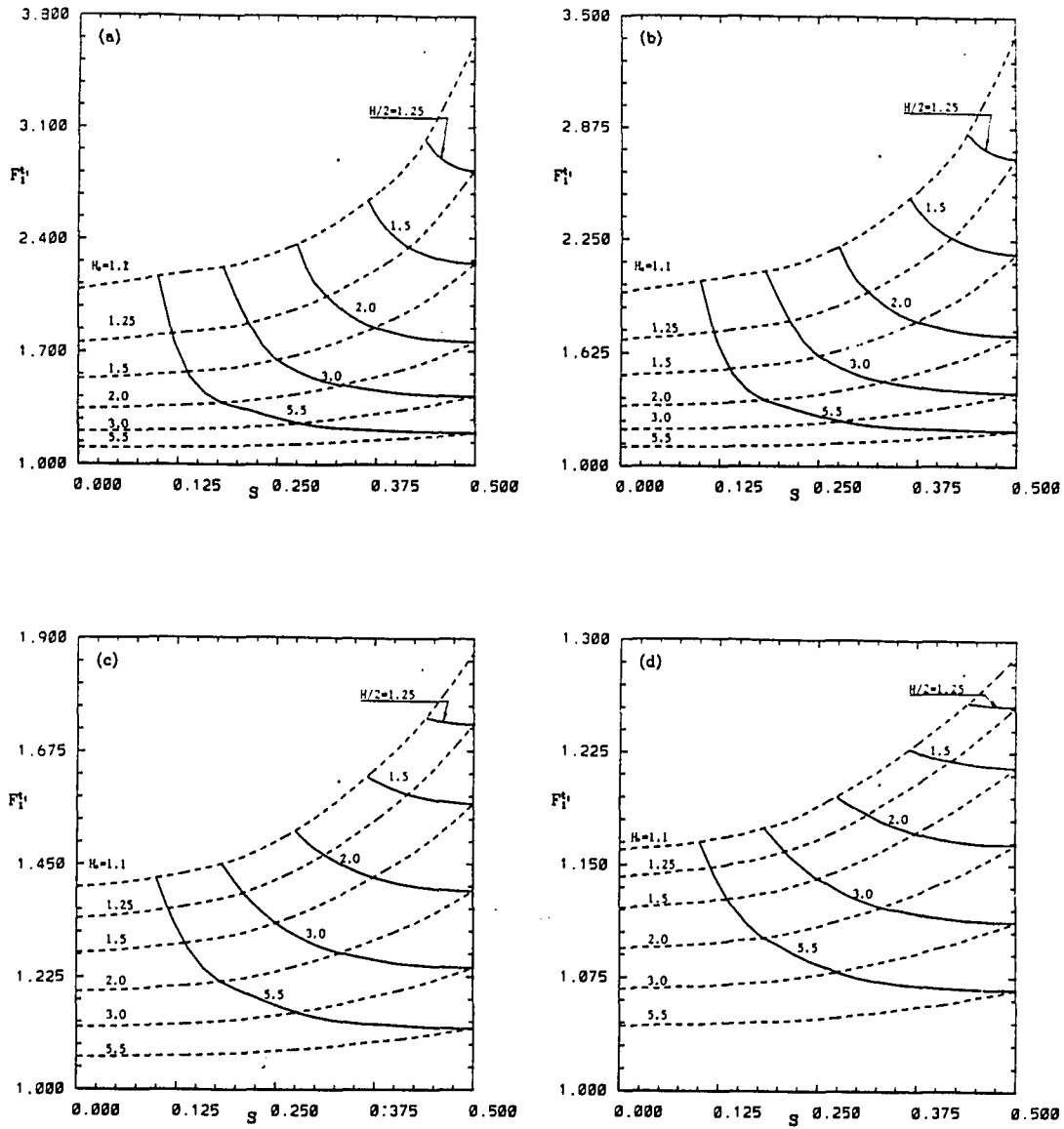


Figure 3. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

Figures 4 (a,b,c,d) present values of the force component perpendicular to the walls, due to translation of the spheroid parallel to the walls. These coefficients exhibit the same qualitative behavior as the coefficients for the parallel force component.

Figures 5 (a,b,c,d) show the torque acting on an oblate or prolate spheroid, due to translation parallel to the walls. As expected from symmetry, all torques vanish when the spheroid is located in the midway between the walls. For an oblate spheroid at small particle-to-wall spacings, the maximum torque occurs at the positions of s near 0.087 or 0.913 in a clockwise direction as viewed in Figure 2. At fixed wall-to-wall spacings, the torques increase dramatically and become infinite as the spheroid approaches either of the two walls. For a prolate spheroid having an aspect ratio $\epsilon=0.5$ at fixed H_0 , the maximum torque occurs at the positions of s near 0.25 or 0.75 in a counterclockwise direction. For an aspect ratio $\epsilon=0.1$, the maximum torque occurs when the spheroid is near either of the two walls.

Figures 6 (a,b,c,d) show the torque acting on an oblate or prolate spheroid, rotating about an axis parallel to the walls. Qualitatively, the torque coefficient exhibits a behavior which is similar to the force coefficients for translation parallel to the walls. When the spheroid is at positions $s < 0.3$ or $s > 0.7$, the second wall has almost no effect on the torque acting on the spheroid. For fixed particle-to-nearer wall spacings the torque increases as the second wall is brought closer to the particle. For fixed wall-to-wall spacings the particle experiences minimum torque midway between the two walls.

Figures 7 (a,b,c,d) present the force component parallel to the walls, acting on the spheroid rigidly held in Couette flow with the upper wall translating parallel to itself. At

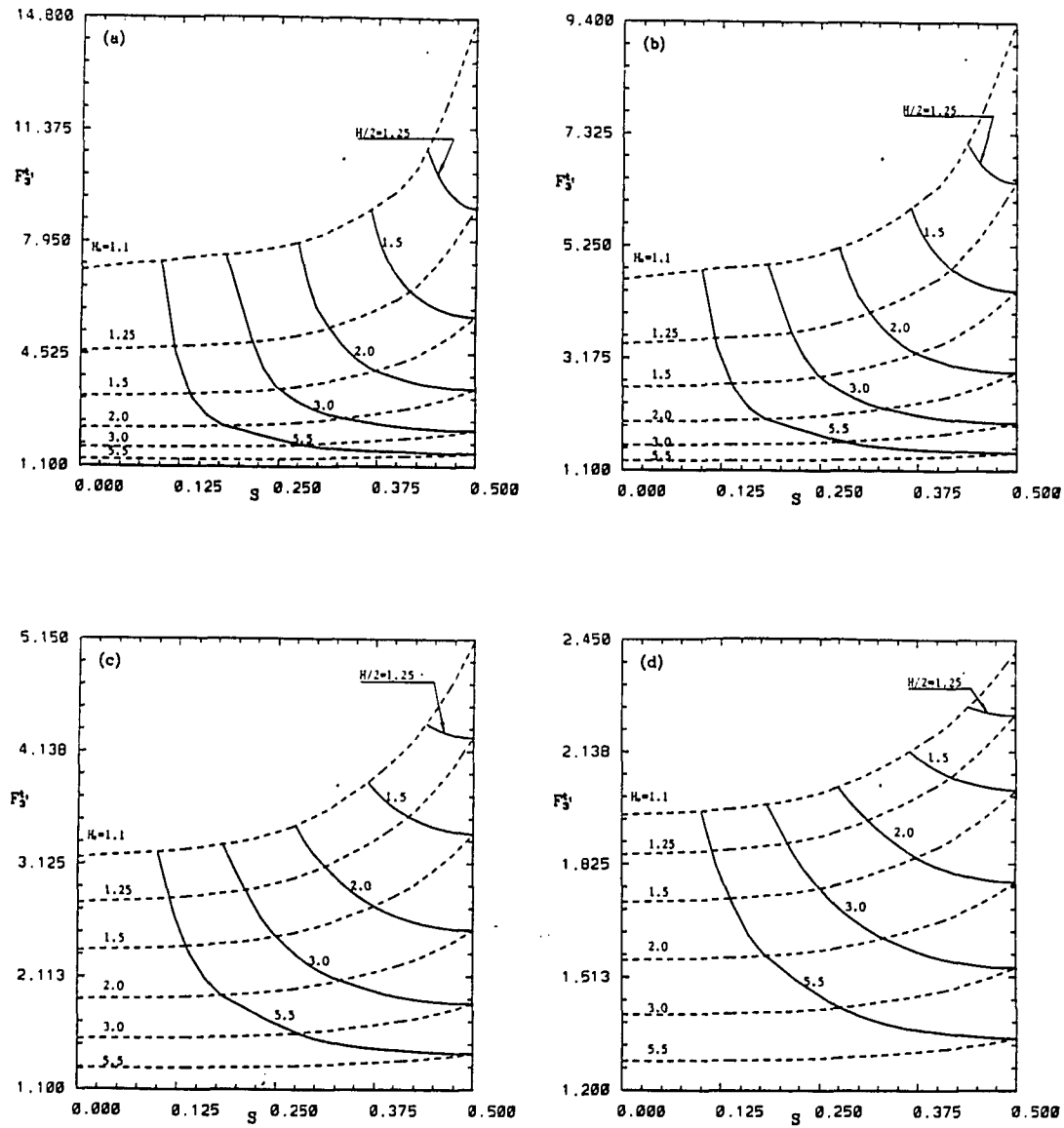


Figure 4. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

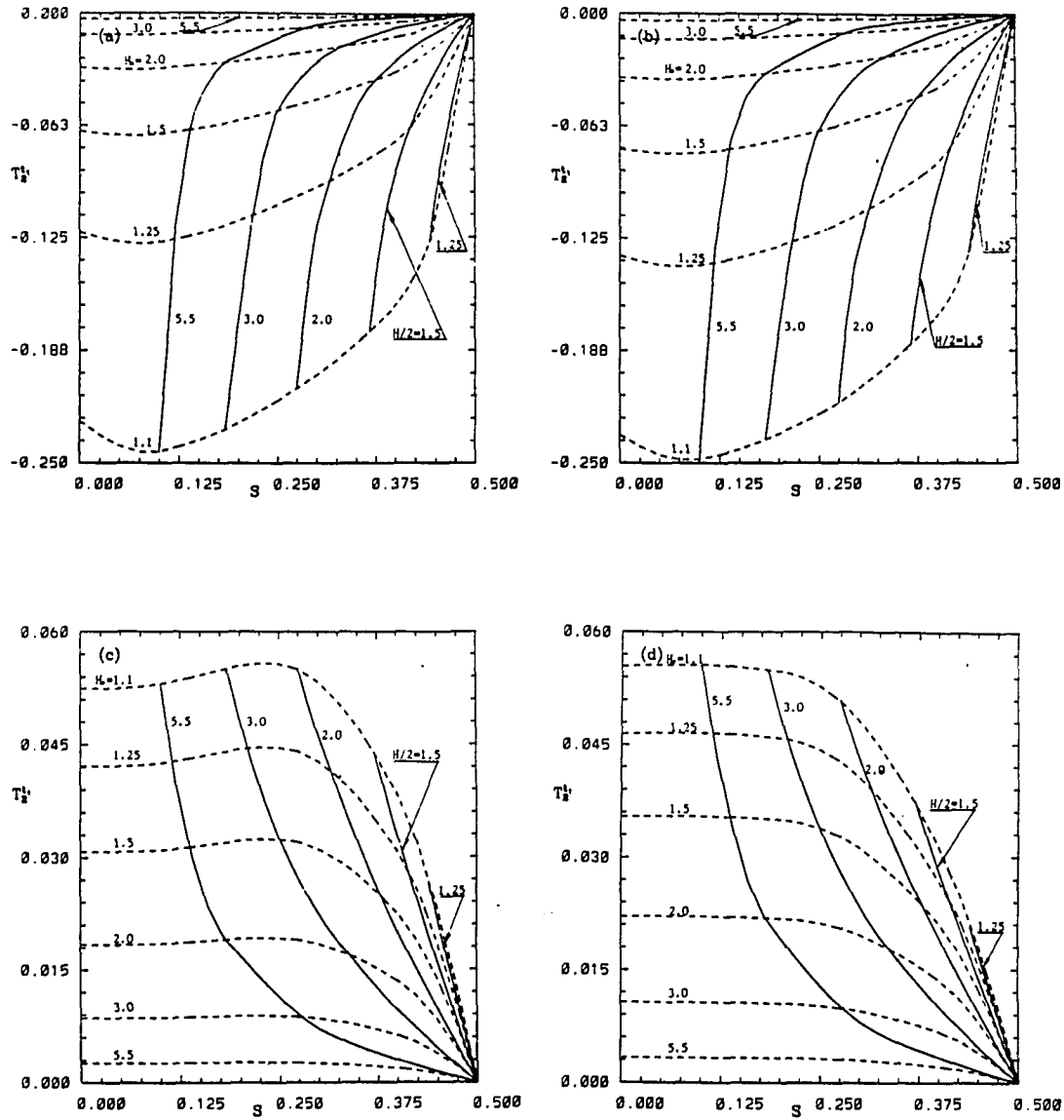


Figure 5. The torque acting on an oblate or a prolate spheroid having an aspect ratio e with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

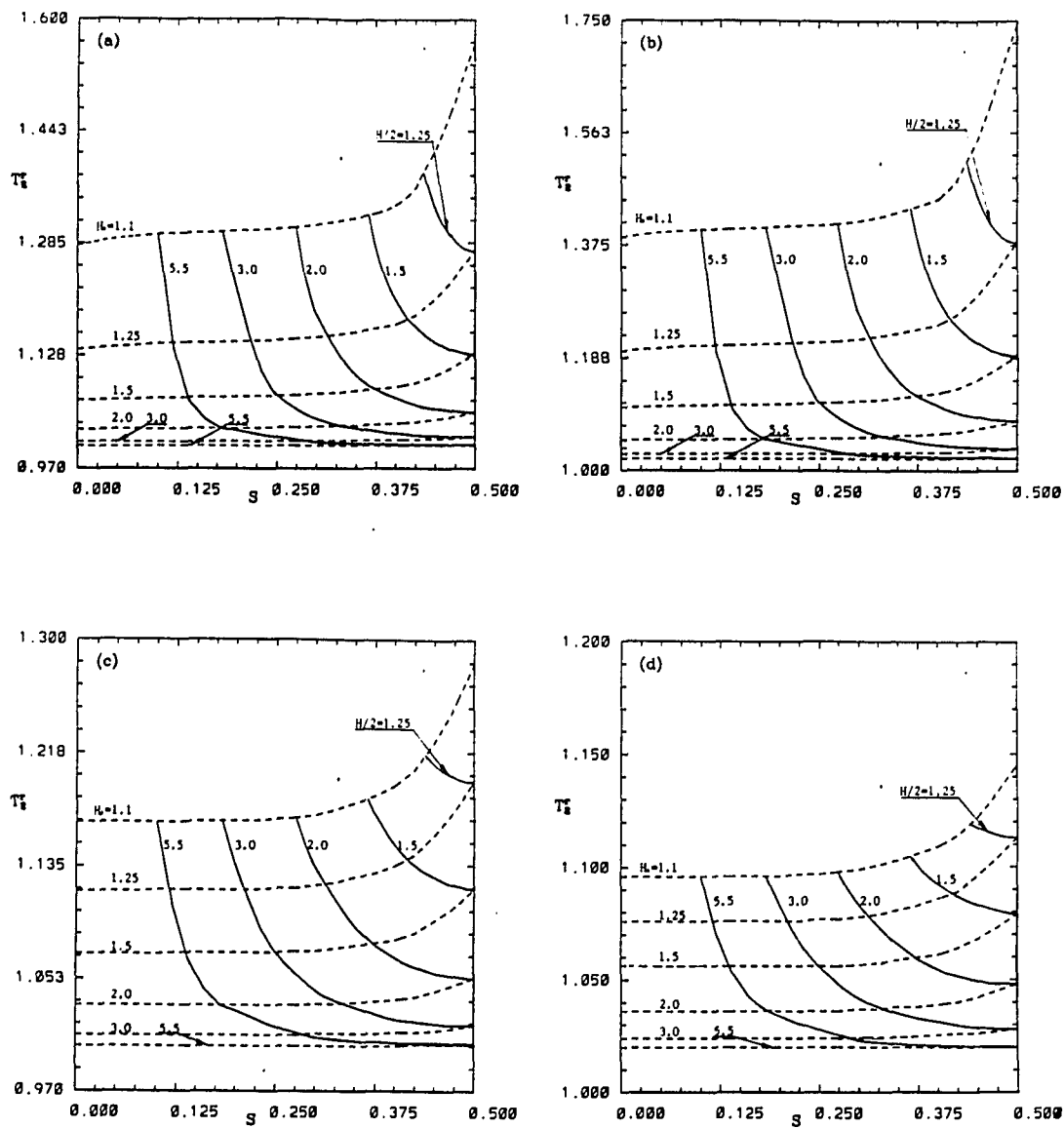


Figure 6. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rotating about an axis parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

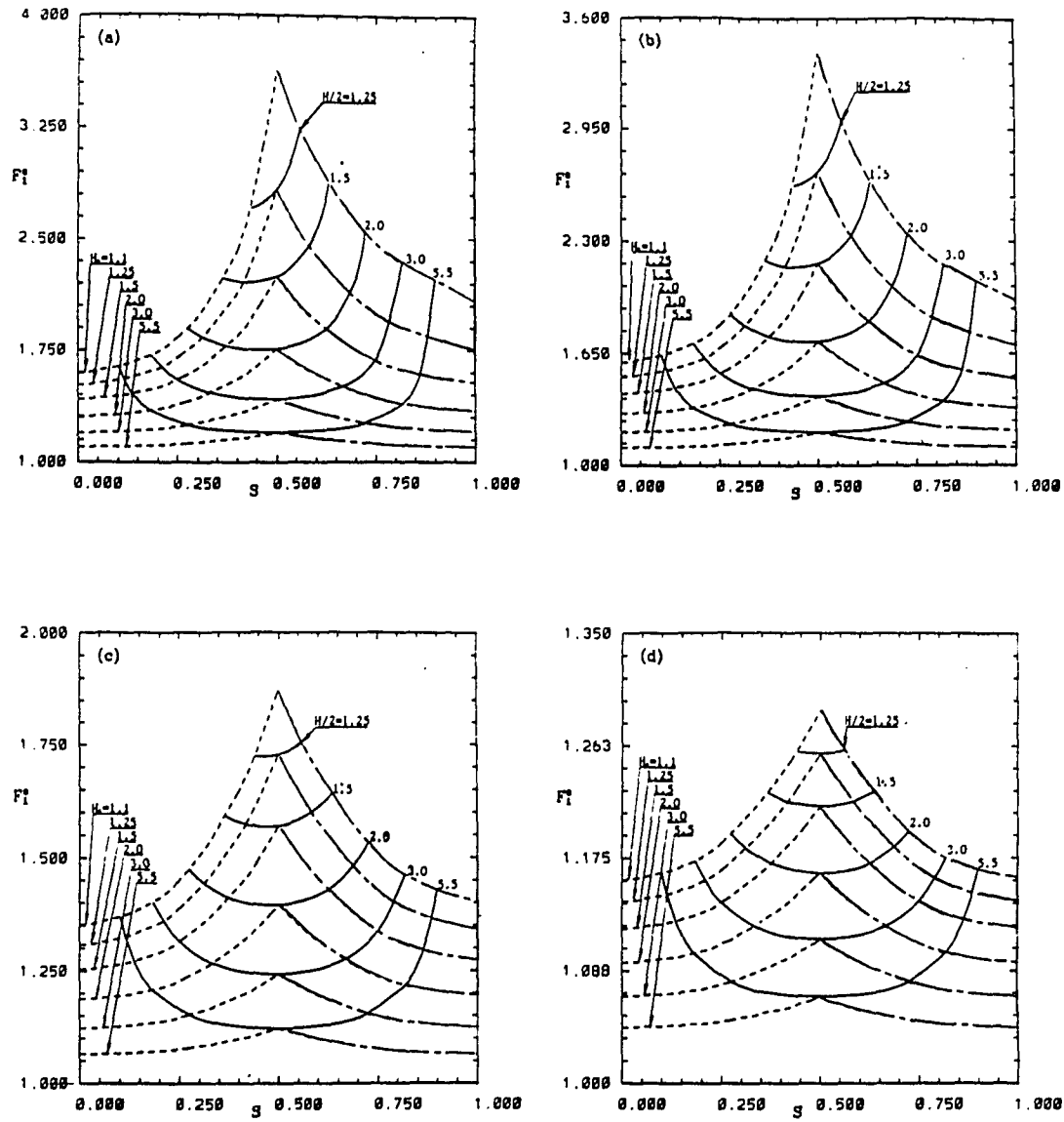


Figure 7. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$; - · - ·, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

fixed particle-to-stationary wall spacings, the drag increases as the moving wall is brought closer to the particle. At fixed wall-to-wall spacings, the minimum drag occurs at the positions of $s < 0.5$ for the smaller wall-to-stationary wall spacings and at $s=0.5$ for the larger wall-to-stationary wall spacings.

Figures 8 (a,b,c,d) present the force component perpendicular to the walls, acting on a spheroid rigidly held in Couette flow with the upper wall translating parallel to itself. These coefficients exhibit the same qualitative behavior as the parallel component coefficients in Figure 7.

Figures 9 (a,b,c,d) show the torque acting on an oblate or prolate spheroid, rigidly held in Couette flow with the upper wall translating parallel to itself. The present research reveals a new interesting phenomenon which does not exist in the case of a sphere between two parallel walls or a spheroid near one wall. For an oblate spheroid oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, with the larger wall-to-stationary wall spacings the torque increases as the spheroid moves away from the lower stationary wall, finally sharply approaching infinity as the spheroid approaches the upper moving wall. This behavior of an oblate spheroid is similar to that observed for a sphere rigidly held in Couette flow. However for a prolate spheroid at this orientation the behavior is totally different. The torque on the spheroid having an aspect ratio $\epsilon=0.1$ appears to decrease, then increase, then decrease again, change its sign, and finally sharply approach minus infinity as the spheroid approaches the upper moving wall. The torque on the spheroid having $\epsilon=0.5$ exhibits a similar behavior to that for $\epsilon=0.1$, but is not as exaggerated as in the latter case. This behavior of the torque is caused by the flow field as the spheroid approaches the moving

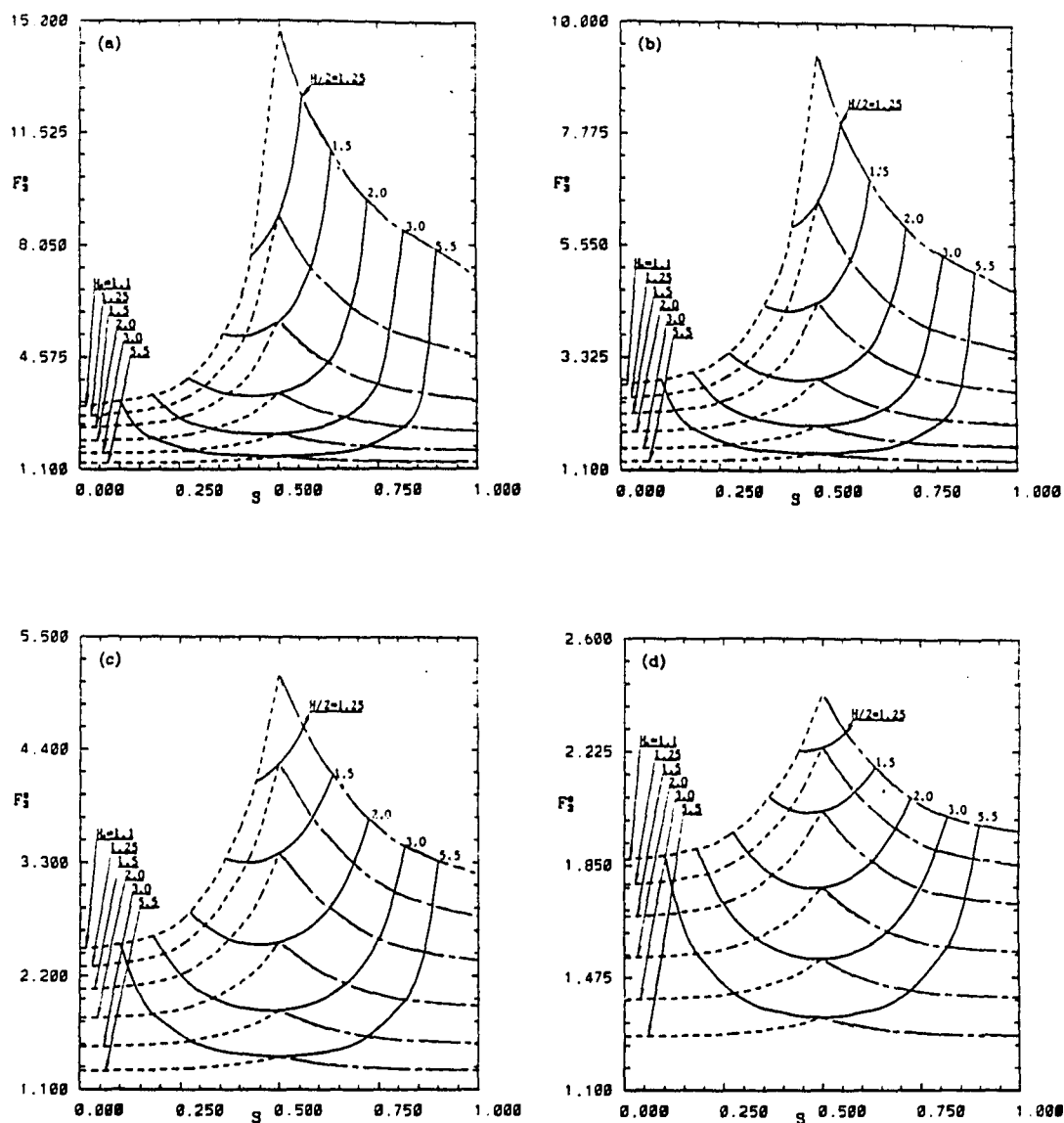


Figure 8. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$, -·-·-, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

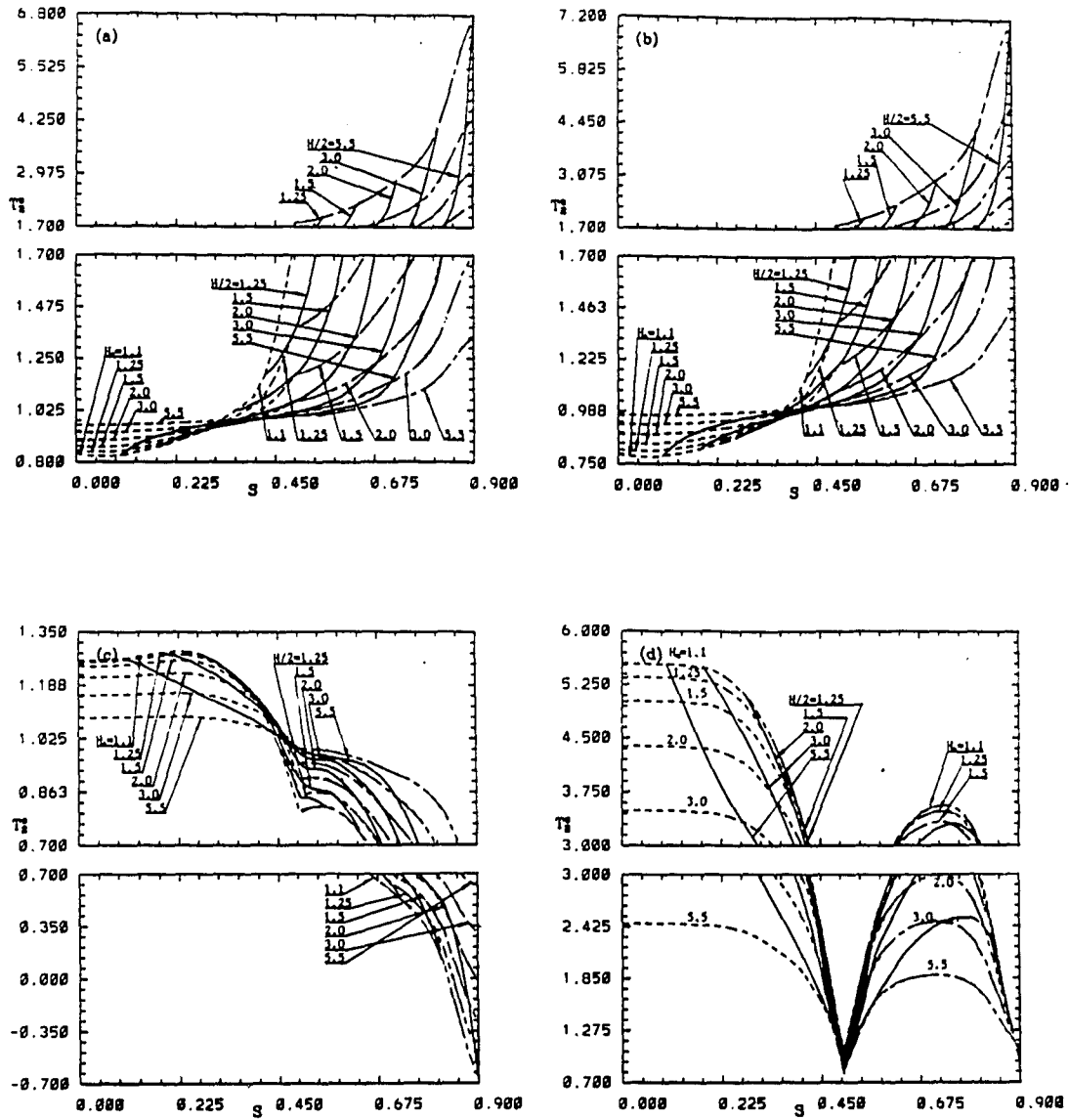


Figure 9. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2 = \text{constant}$; ----, $H_0 = \text{constant}$; — · —, $H - H_0 = \text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

wall. As the spheroid gets closer to the moving wall the fluid in the gap between the particle and the moving wall gets smaller and the pressure on the side of the spheroid which is toward to the oncoming flow and faces the moving wall increases and eventually makes the spheroid rotate in the opposite direction. Figures 10 (a,b,c,d) present the force component perpendicular to the walls, due to translation of the spheroid perpendicular to the walls. The qualitative behavior of this coefficient is similar to the one due to the translation parallel to the walls. It should be noted that the force component parallel to the walls is zero at this orientation due to symmetry of the flow.

Figures 11 (a,b,c,d) show the force component parallel to the walls, acting on a spheroid rigidly held in Poiseuille flow between two walls. The maximum force occurs when the spheroid is located midway between the two walls. As the wall-to-wall spacing increases the force acting on the particle decreases and reaches the parabolic curve for the infinite wall-to-wall spacing, in which the presence of the spheroid can be neglected.

Figures 12 (a,b,c,d) show the force component perpendicular to the walls, acting on a spheroid rigidly held in Poiseuille flow between two walls. The perpendicular component exhibits a similar qualitative behavior as the parallel component. Figures 13 (a,b,c,d) show the torque acting on a spheroid, rigidly held in Poiseuille flow between two walls. At fixed wall-to-wall spacings the torque is zero when the spheroid is located midway between two walls at this orientation due to symmetry of the flow and increases almost linearly as the spheroid approaches one of the walls. For fixed particle-to-wall spacings the torque reaches a maximum at a position s slightly smaller than 0.25 for an oblate spheroid, and at a position s slightly larger than 0.25 for a prolate spheroid.

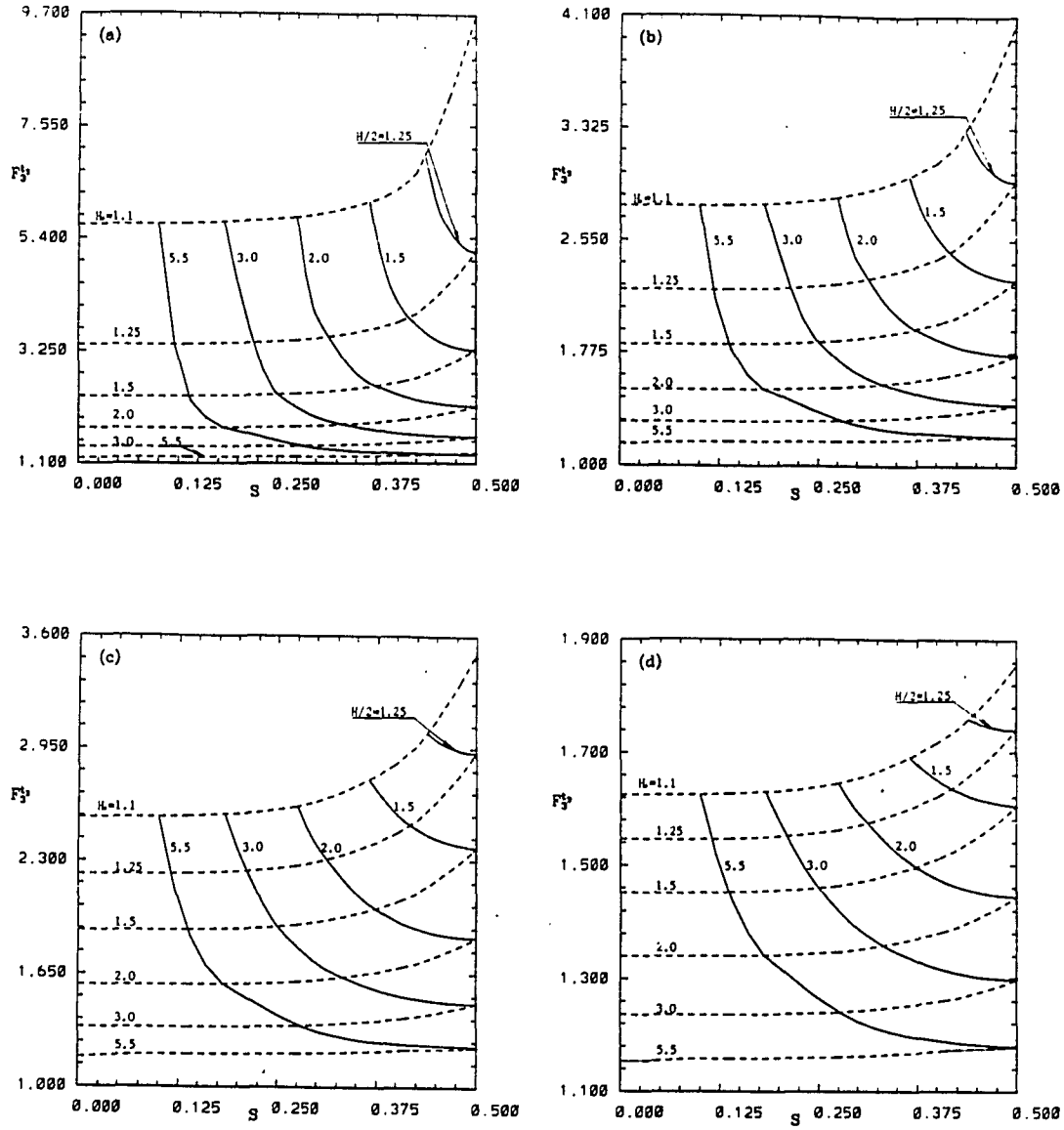


Figure 10. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, due to translation perpendicular to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

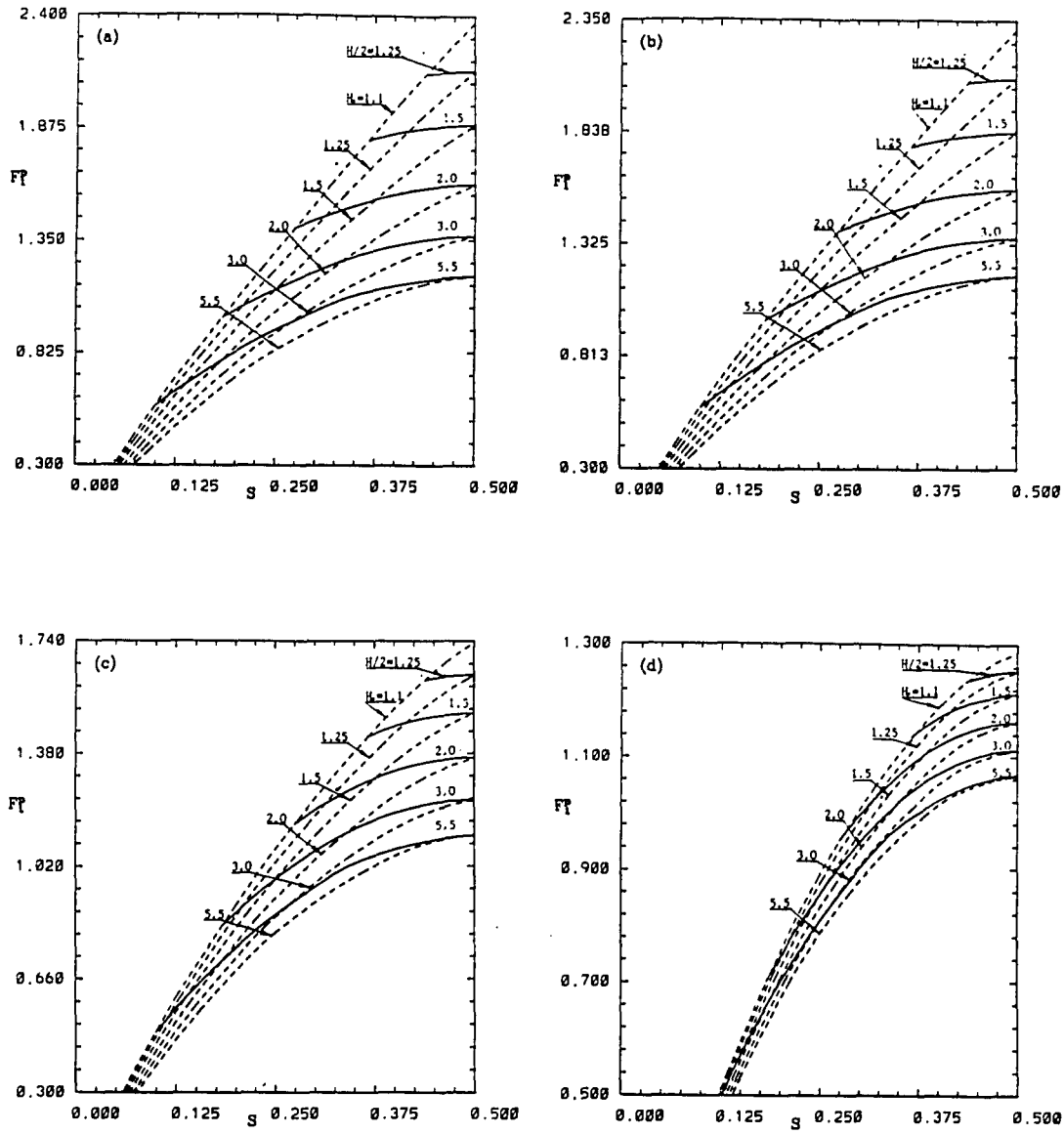


Figure 11. The force component parallel to the walls acting on an oblate or a prolate spheroid an having aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

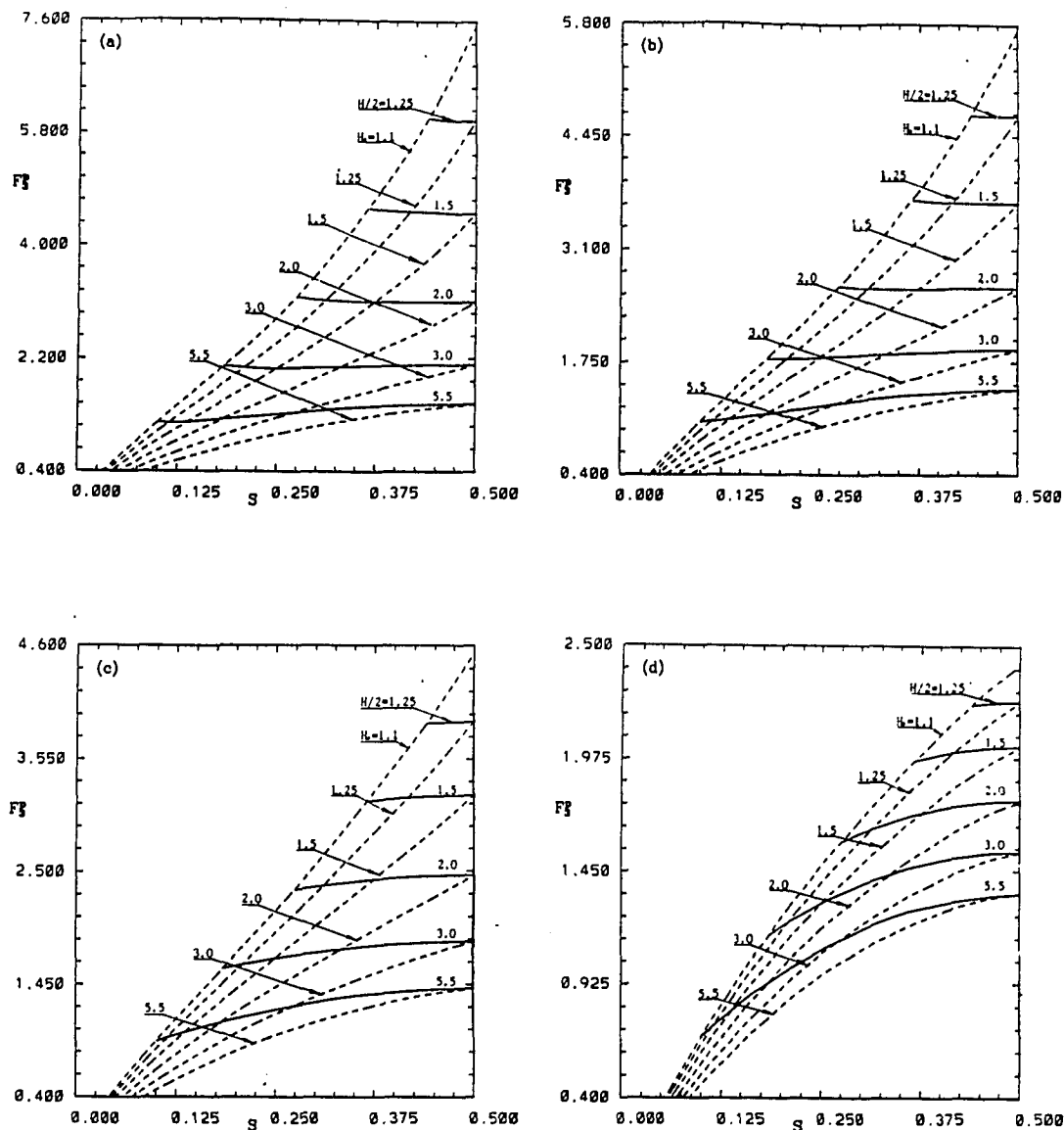


Figure 12. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

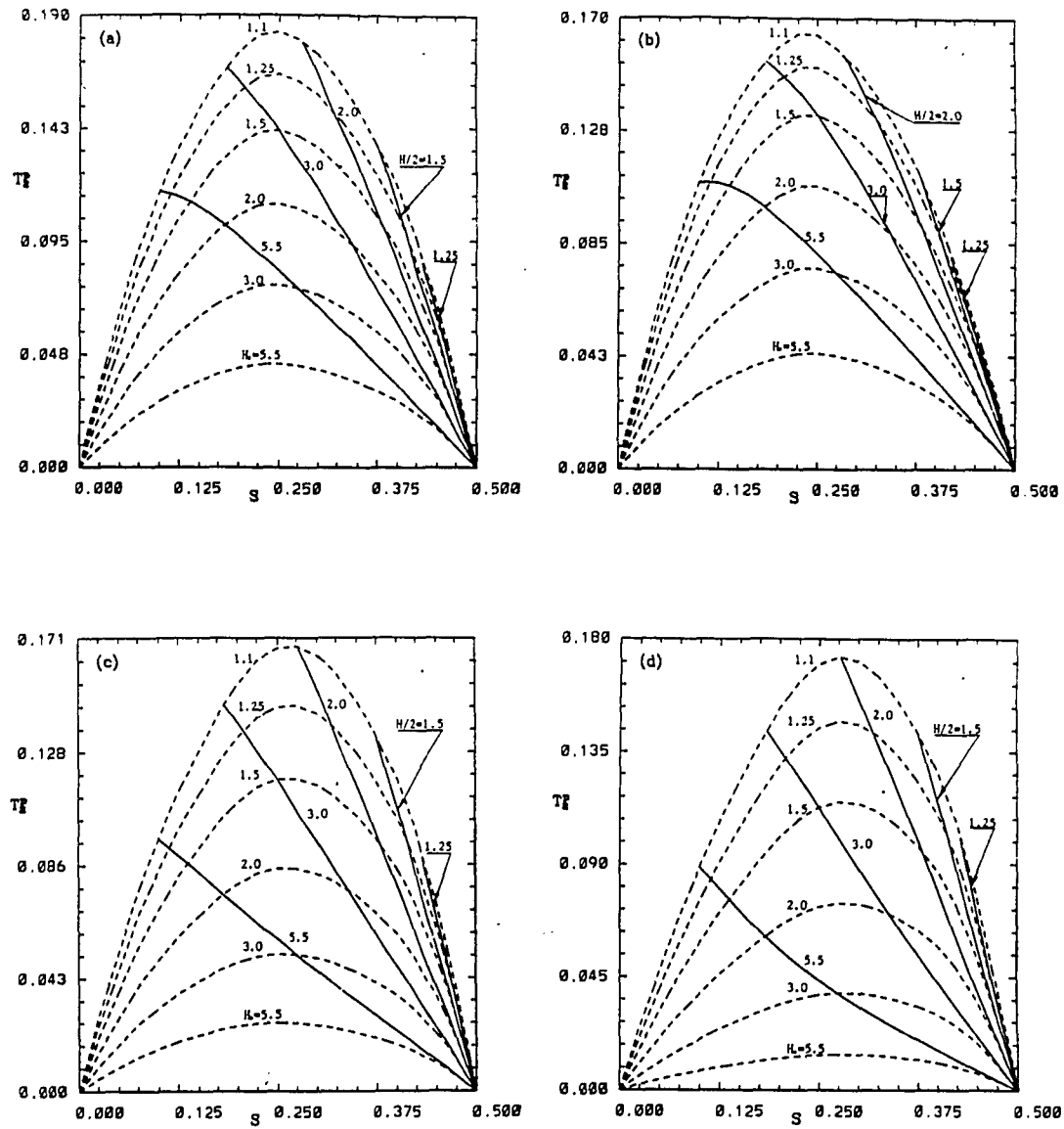


Figure 13. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=0^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

Figures 14-25 present same drag and torque coefficients as the ones presented in Figures 3-13 but with the symmetry axis of the particle oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, and include the torque coefficient due to the translation of the spheroid perpendicular to the walls which is non-zero for the present orientation. The behavior of the corresponding coefficients is similar. It is worth noting the following observations of some of the torque coefficients. First of all, for a prolate spheroid translating parallel to the walls, the torque acting on it has a different sign at the orientation of $\alpha=0^\circ$, $\gamma=0^\circ$ and $\alpha=45^\circ$, $\gamma=0^\circ$. This finding shows that for a fixed particle and wall geometry there is a critical orientation angle α ranging from 0 to 45 degree, at which the torque acting on the particle vanishes. At this critical orientation the force which pushes the spheroid to roll against the near wall is balanced by the pressure force which is built up at the tip of the spheroid and pushes it away from the near wall. Secondly the torque acting on a prolate spheroid having an aspect ratio $\epsilon=0.5$ or 0.1 with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$ does not change its sign as it approaches the moving wall in Couette flow as it does at the orientation of $\alpha=0^\circ$ and $\gamma=0^\circ$ because the pressure which is built up at the tip of the spheroid is not large enough to force it to rotate in the opposite direction. Therefore there exists a critical value of α near 0° for a prolate spheroid, at which the torque on the spheroid does not change sign as it approaches the upper moving wall.

Figures 26-36 also present the same sets of the drag and torque coefficients as the ones presented in Figures 3-13 but with the symmetry axis of the particle oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$. The torque acting on an oblate spheroid oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, translating parallel to the walls, changes its sign at the new orientation of $\alpha=90^\circ$ and

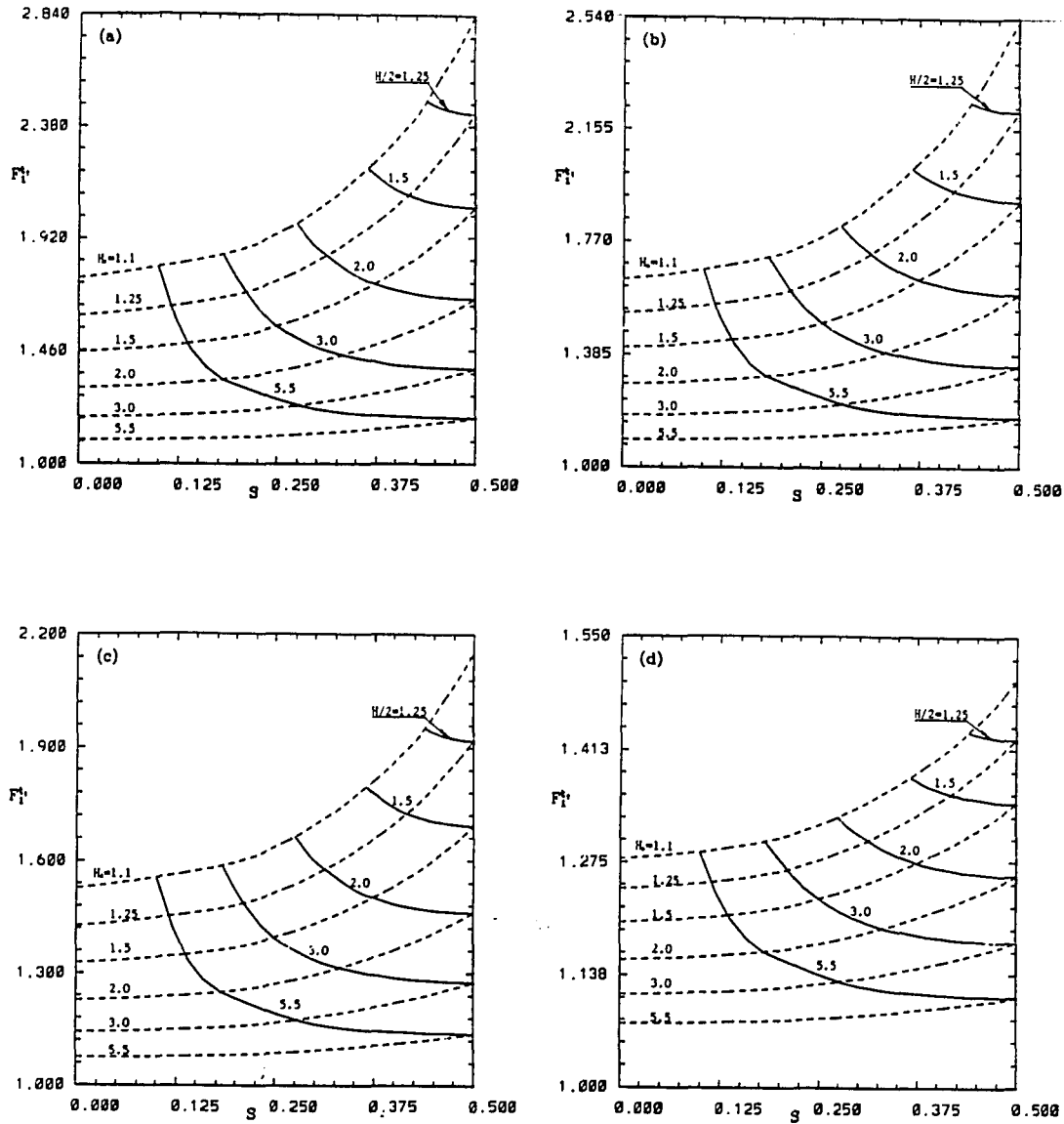


Figure 14. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

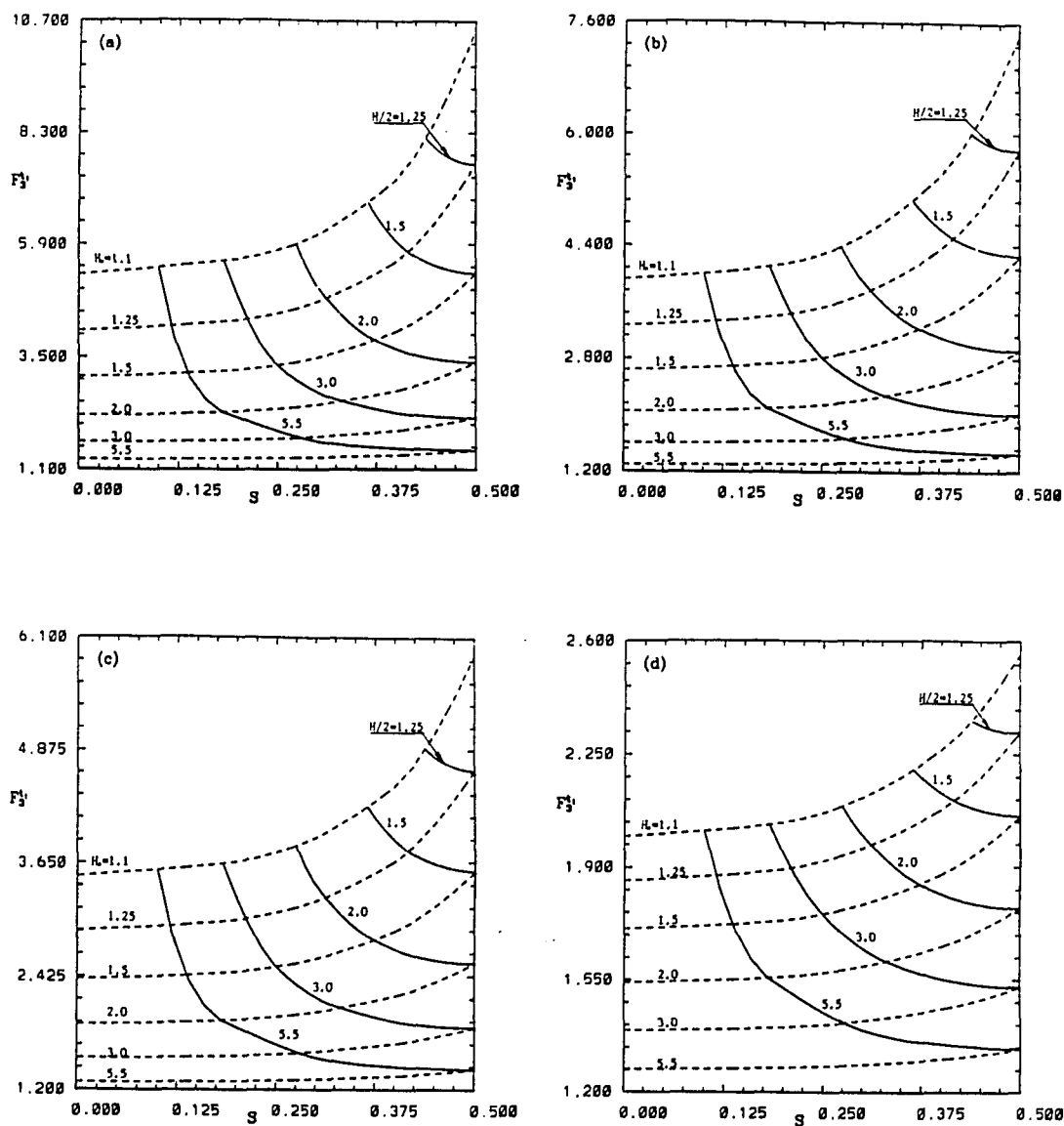


Figure 15. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

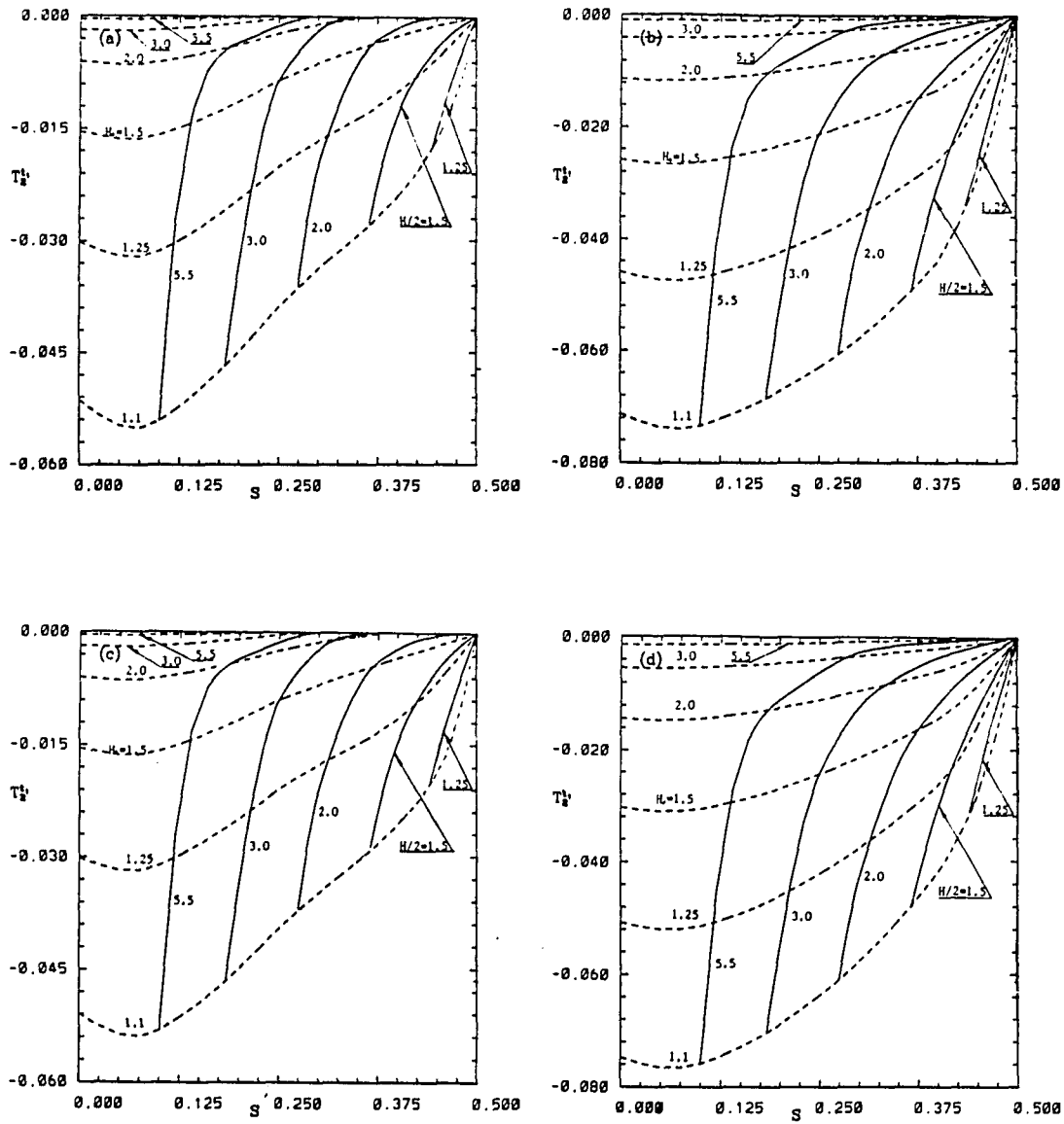


Figure 16. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

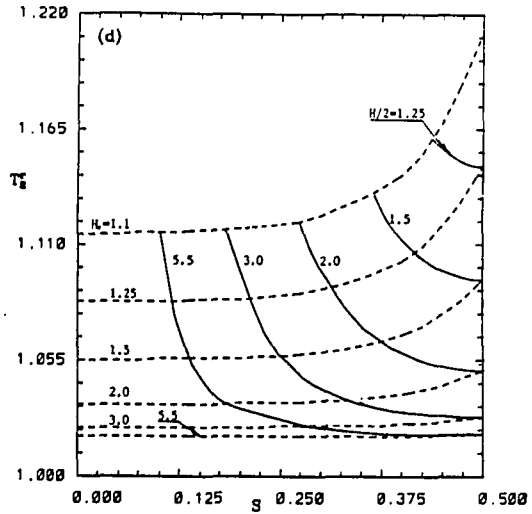
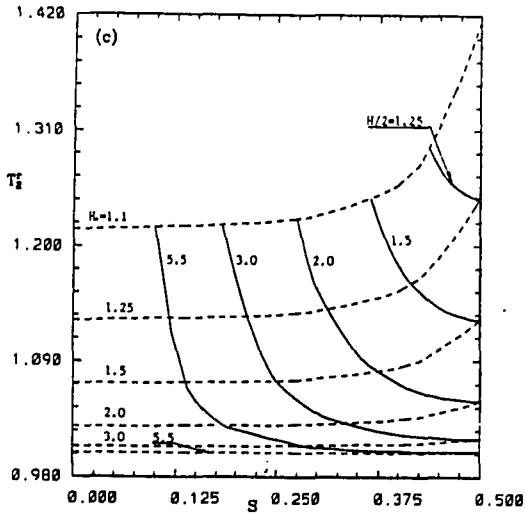
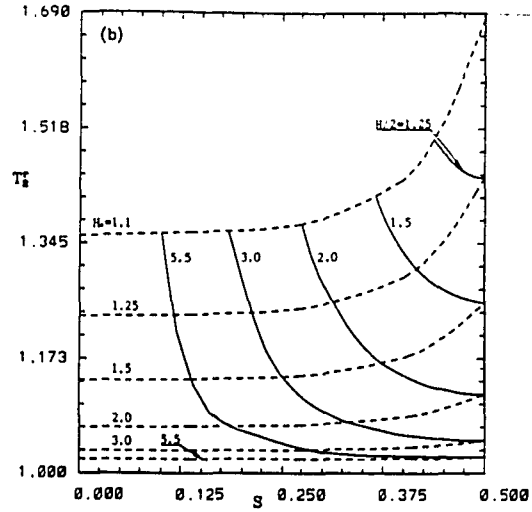
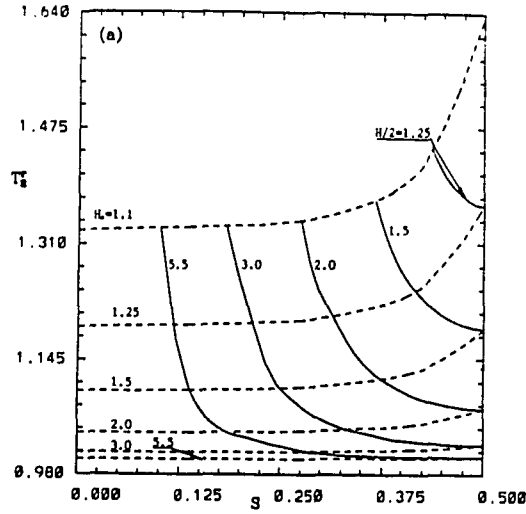


Figure 17. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rotating about an axis parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

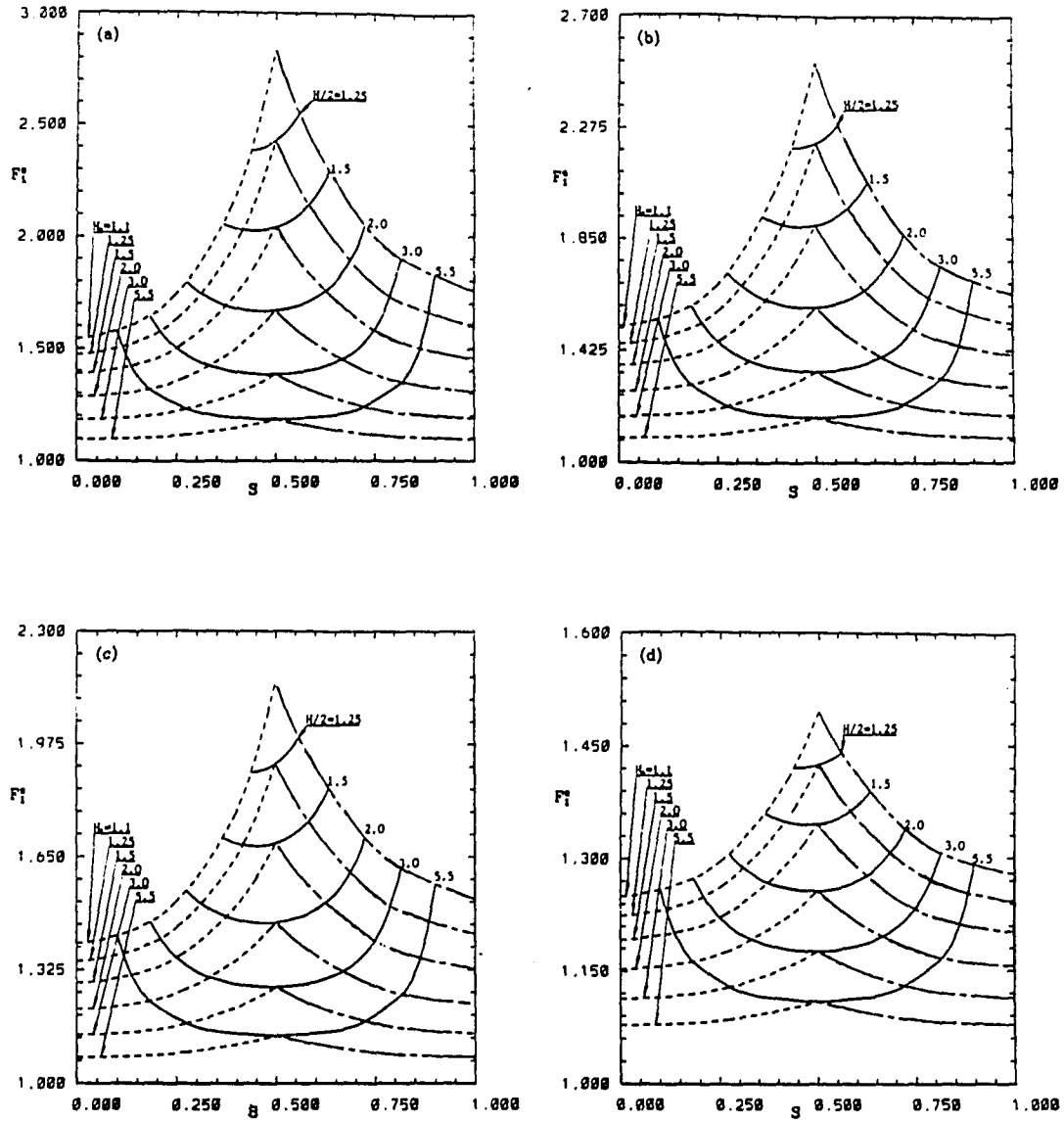


Figure 18. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$, - · - ·, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

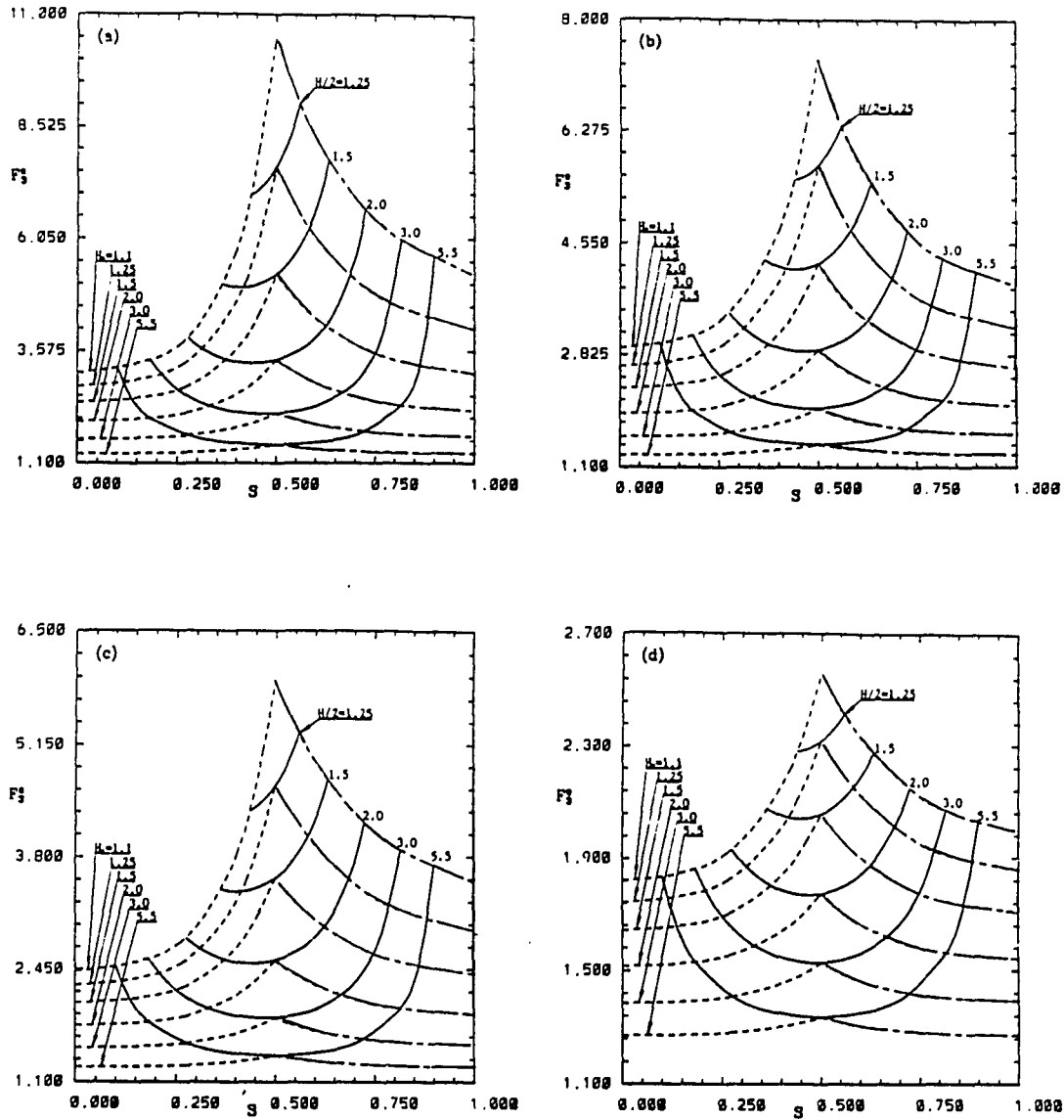


Figure 19. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$, — · —, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

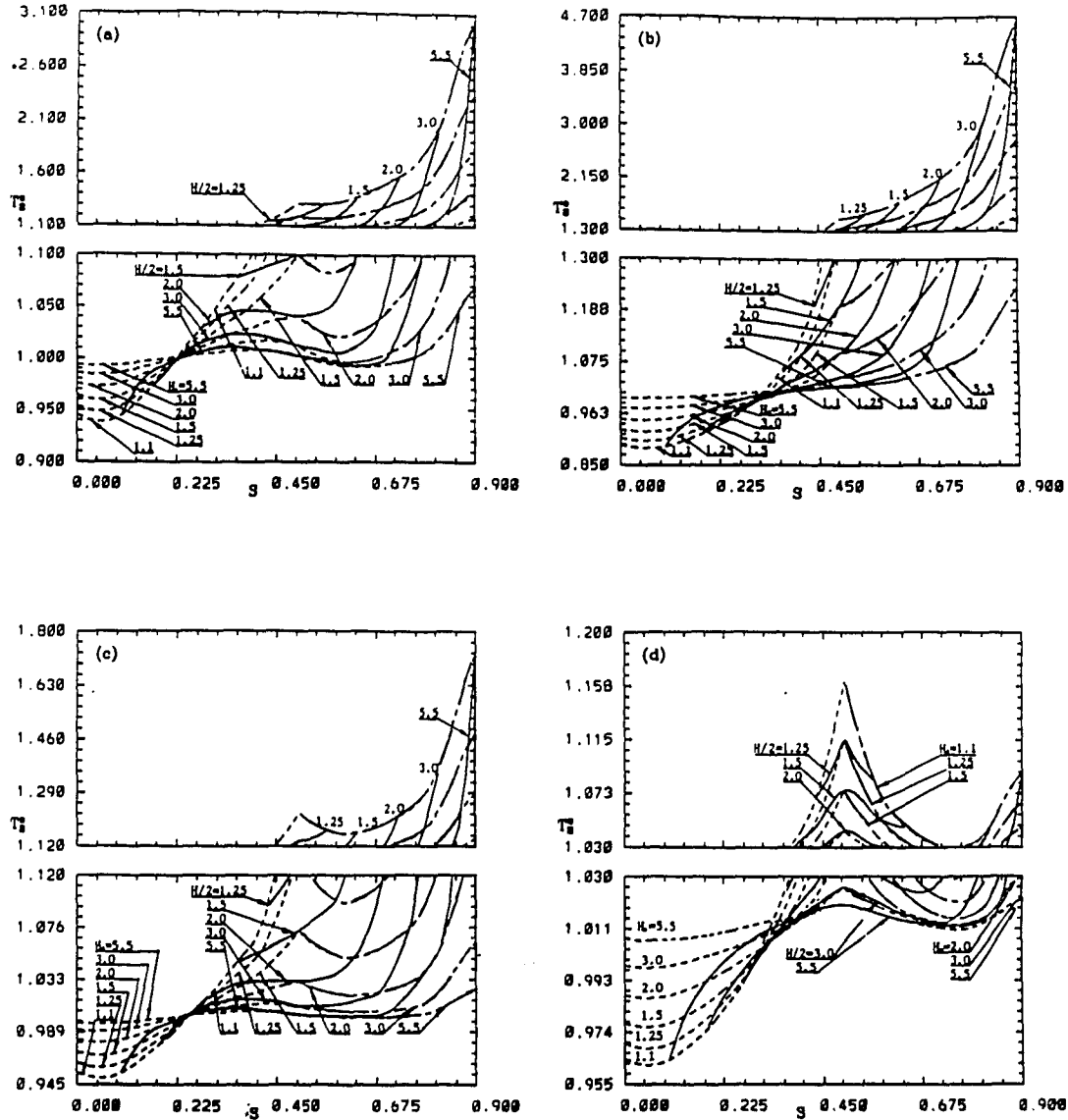


Figure 20. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2 = \text{constant}$; ----, $H_0 = \text{constant}$, ---, $H - H_0 = \text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

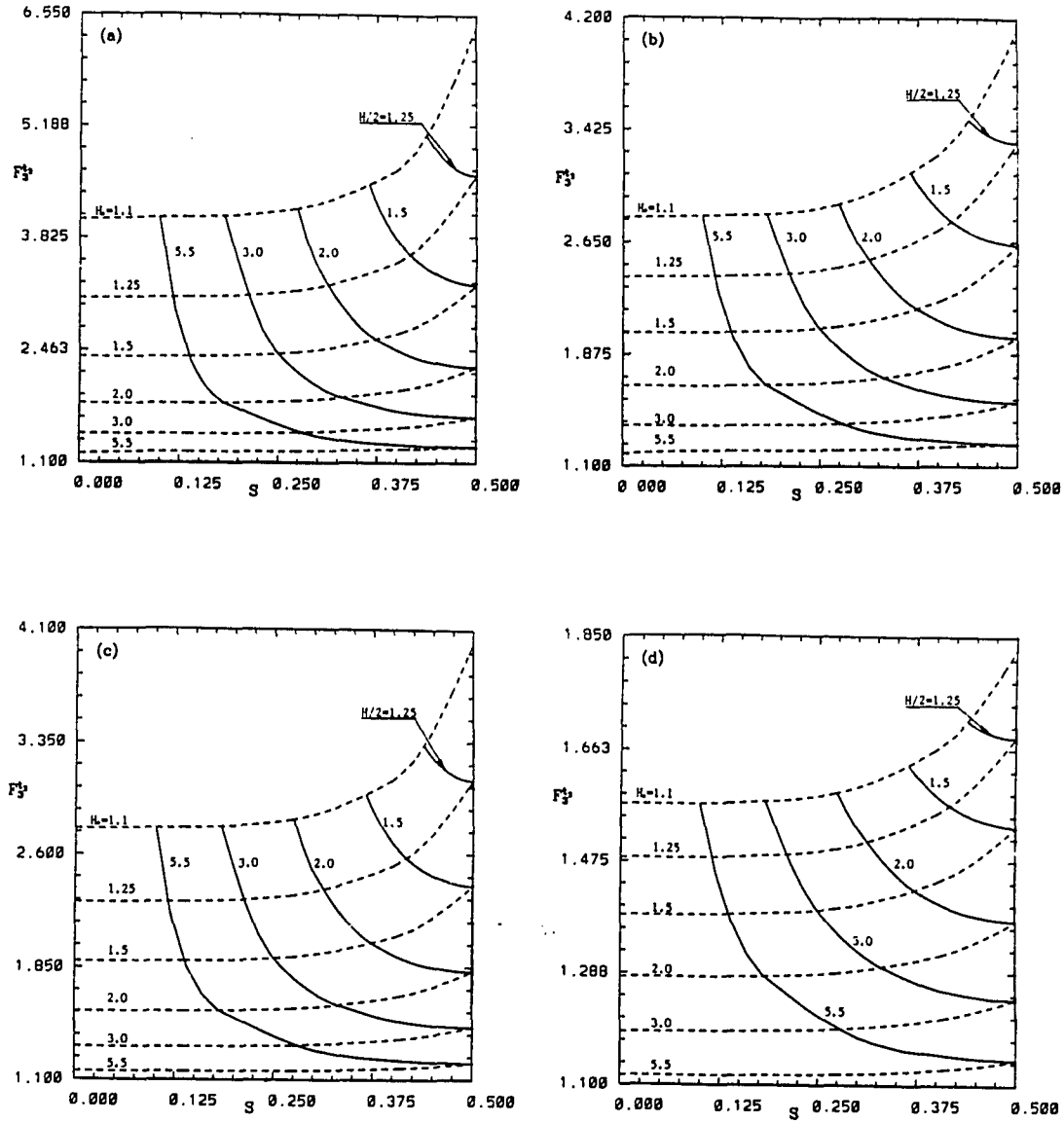


Figure 21. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation perpendicular to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

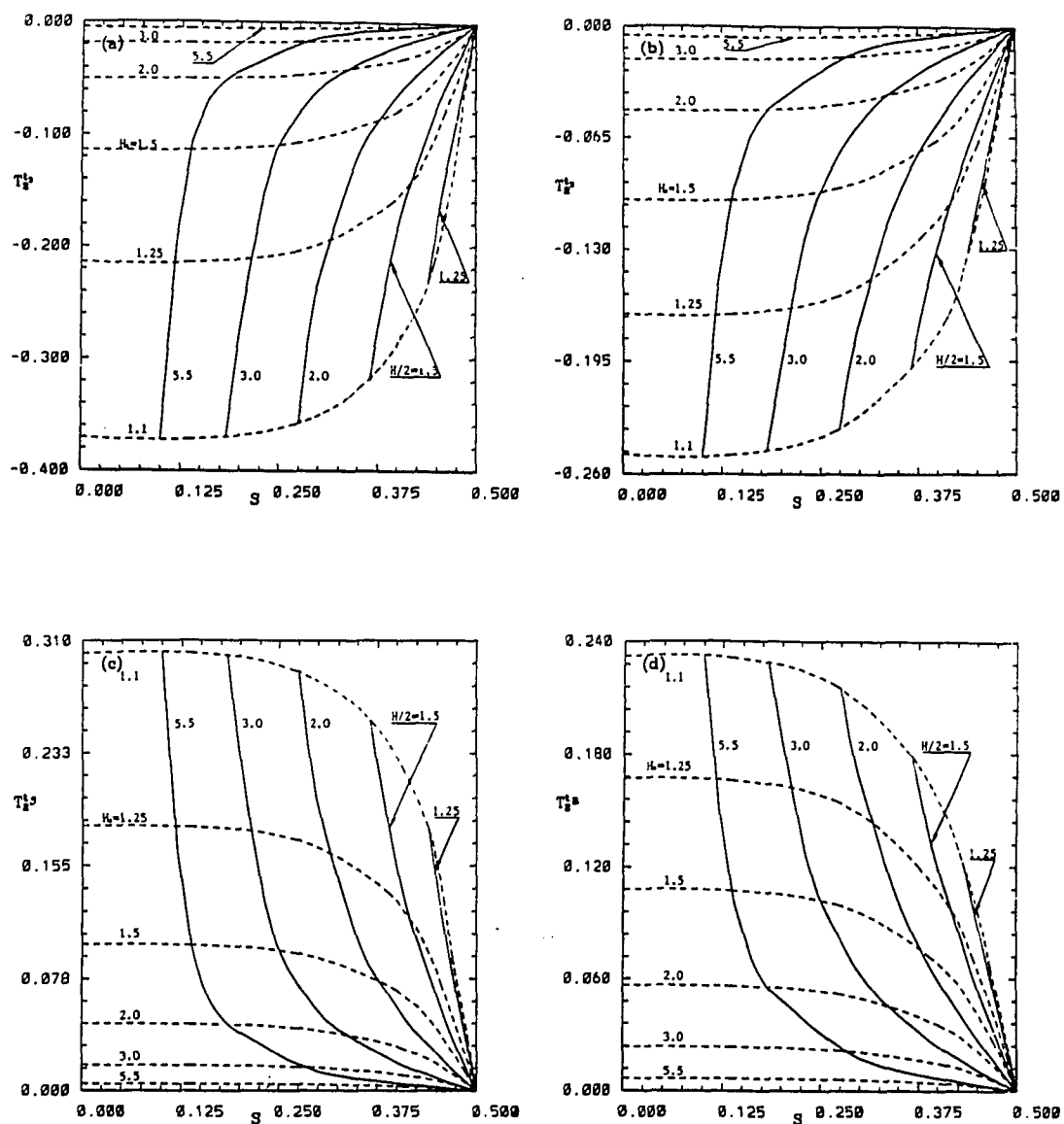


Figure 22. The torque acting on an oblate or a prolate spheroid an having aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, due to translation perpendicular to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

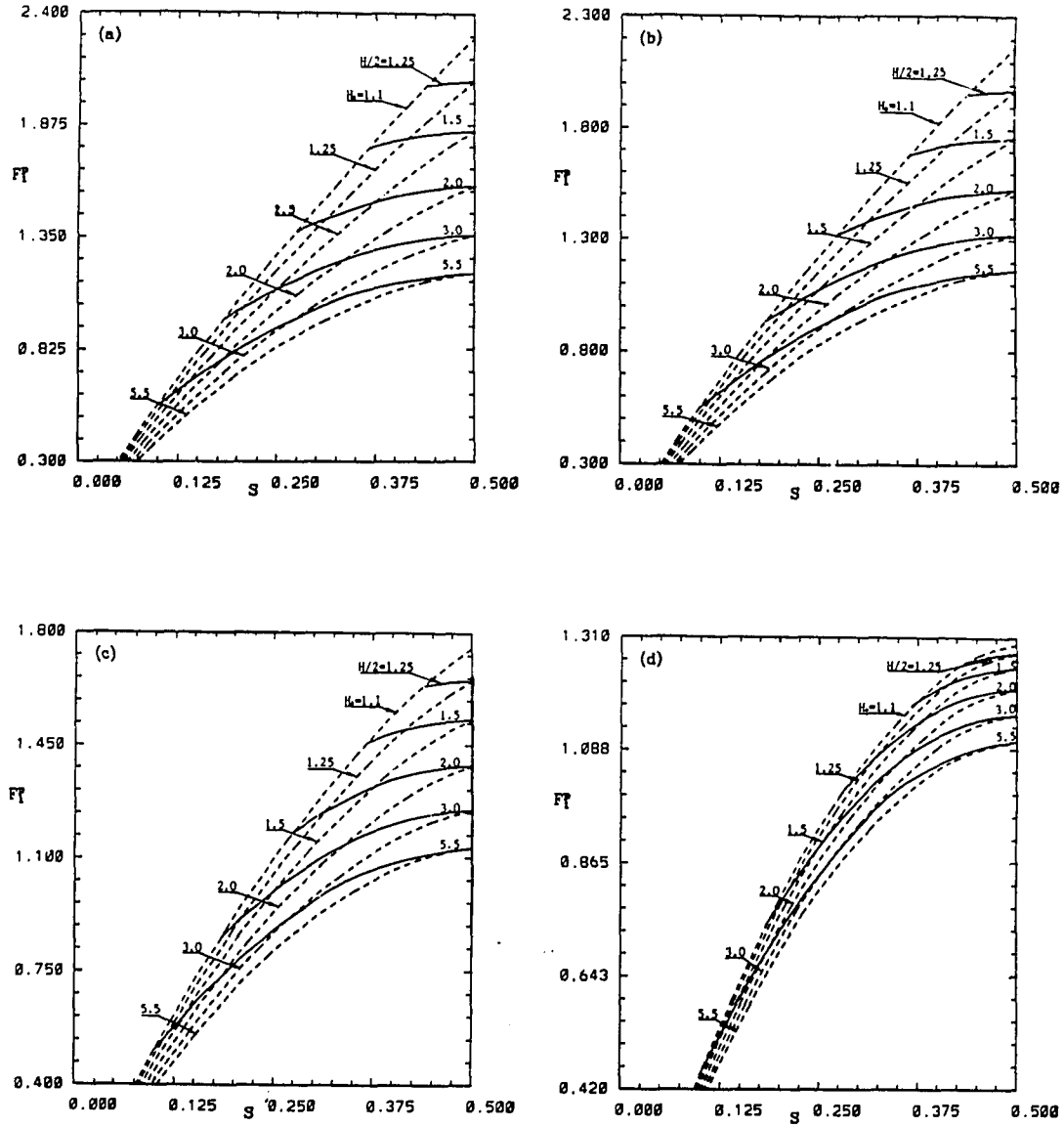


Figure 23. The force component parallel to the walls acting on an oblate or a prolate spheroid having aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

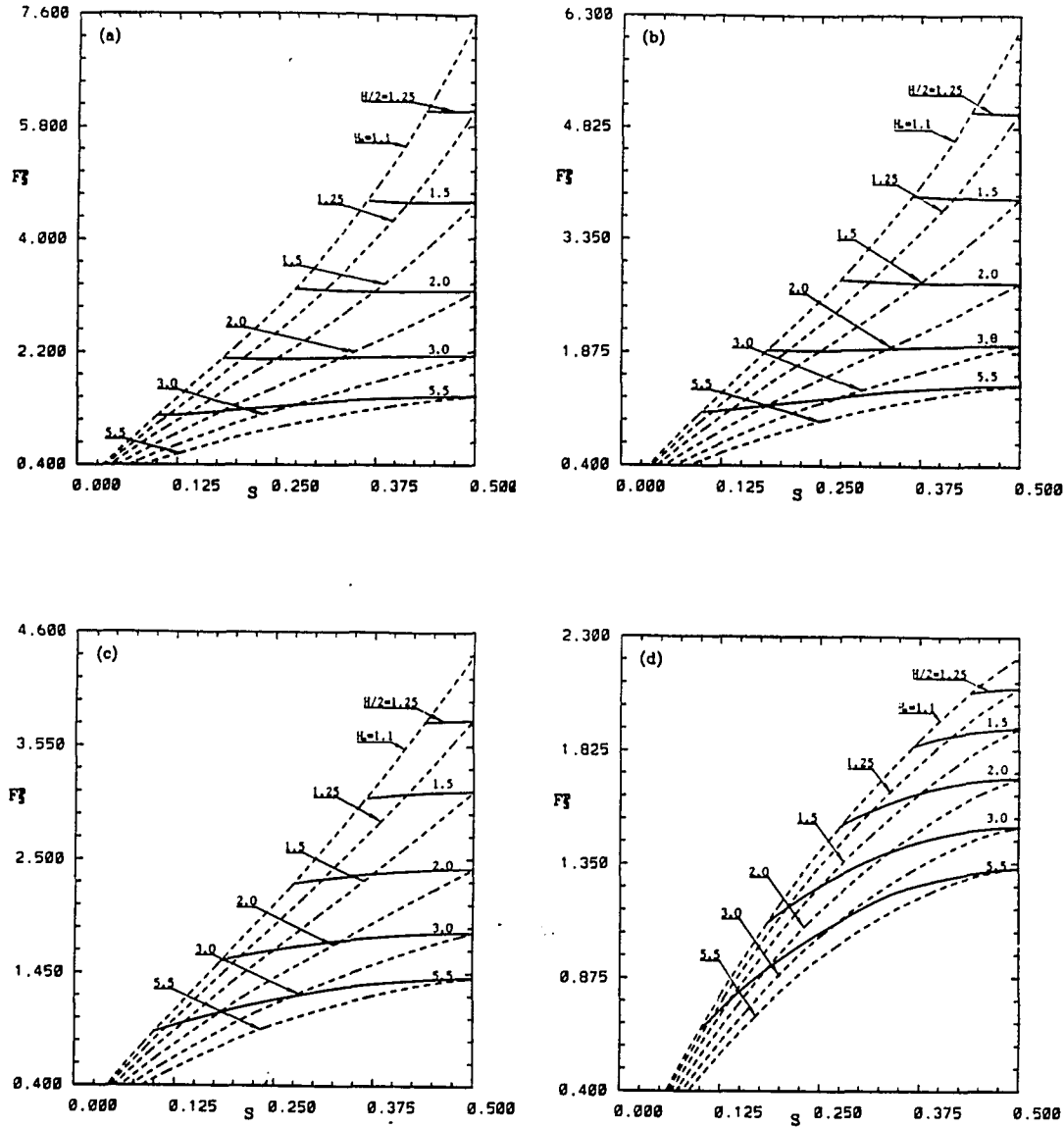


Figure 24. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

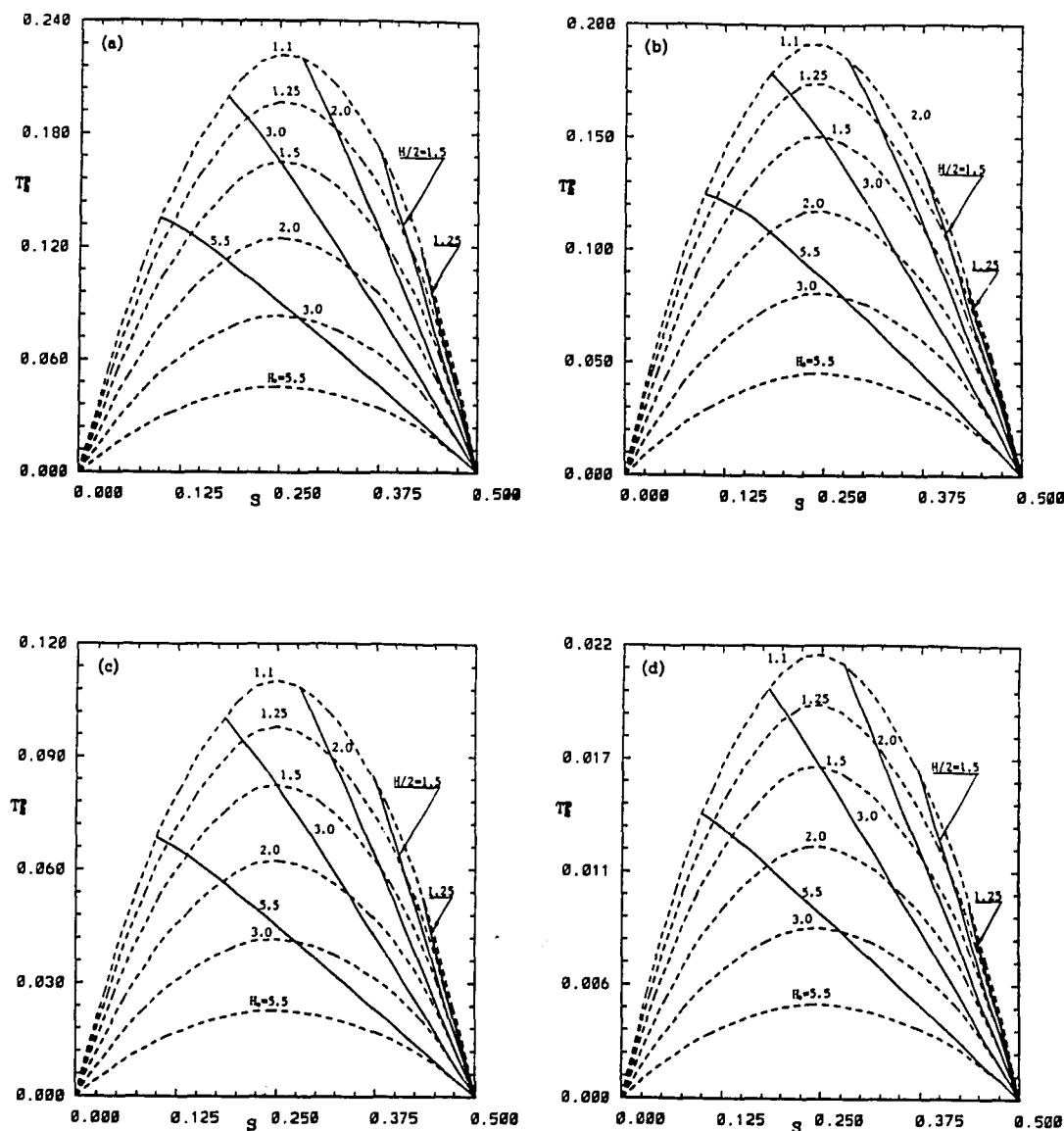


Figure 25. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=45^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

$\gamma=0^\circ$, indicating that there is a critical orientation angle α between 45 to 90 degree at which the torque acting on the particle vanishes. It should be noted that the behavior of a prolate spheroid oriented at angle α is qualitatively similar to that of an oblate spheroid oriented at angle $(90^\circ-\alpha)$ since the semi-major axes are similarly oriented. Then as expected, the phenomenon of torque reversal found in Couette flow discussed previously is also observed for an oblate spheroid in this orientation.

As a supplementary discussion of this planar symmetric flow geometry, Figures 41 and 42 show the drag coefficients $F_1^i, F_1^p, F_3^i, F_3^p$ and the torque coefficients $T_2^r, T_2^i, T_2^s, T_2^b$ as a function of the orientation angle α for aspect ratios $\varepsilon=0.5$ and 0.1 at the position $s=0.25$, with wall-to-wall spacings $H=4.4$ and 12.0. The coefficients F_1^i, F_1^p, F_3^i decrease for an oblate spheroid, increase for a prolate spheroid as the angle α increases because the drag force experienced by the spheroid decreases as its semi-major axis approaches the flow direction. But the coefficient F_3^p exhibits the opposite behavior. The torque coefficient T_2^r also decreases for an oblate spheroid, increases for a prolate spheroid as the angle α increases.

Figure 42 (b) for T_2^i clearly shows the critical angle α which was mentioned in the previous discussions at which the torque changes sign. Furthermore Figure 42 (b) for T_2^i also indicates that for a given aspect ratio there is a certain range of angle α at which the

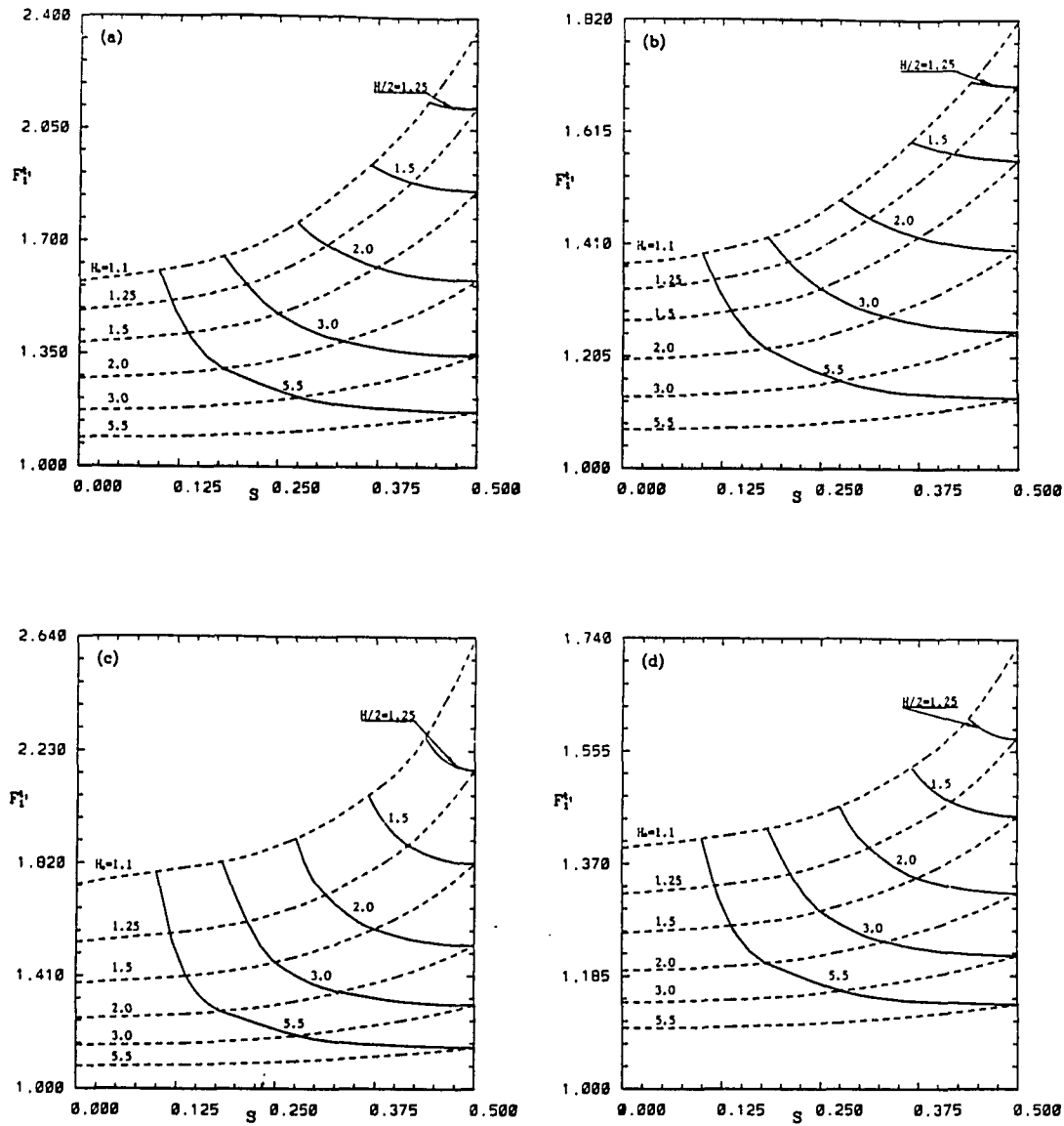


Figure 26. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

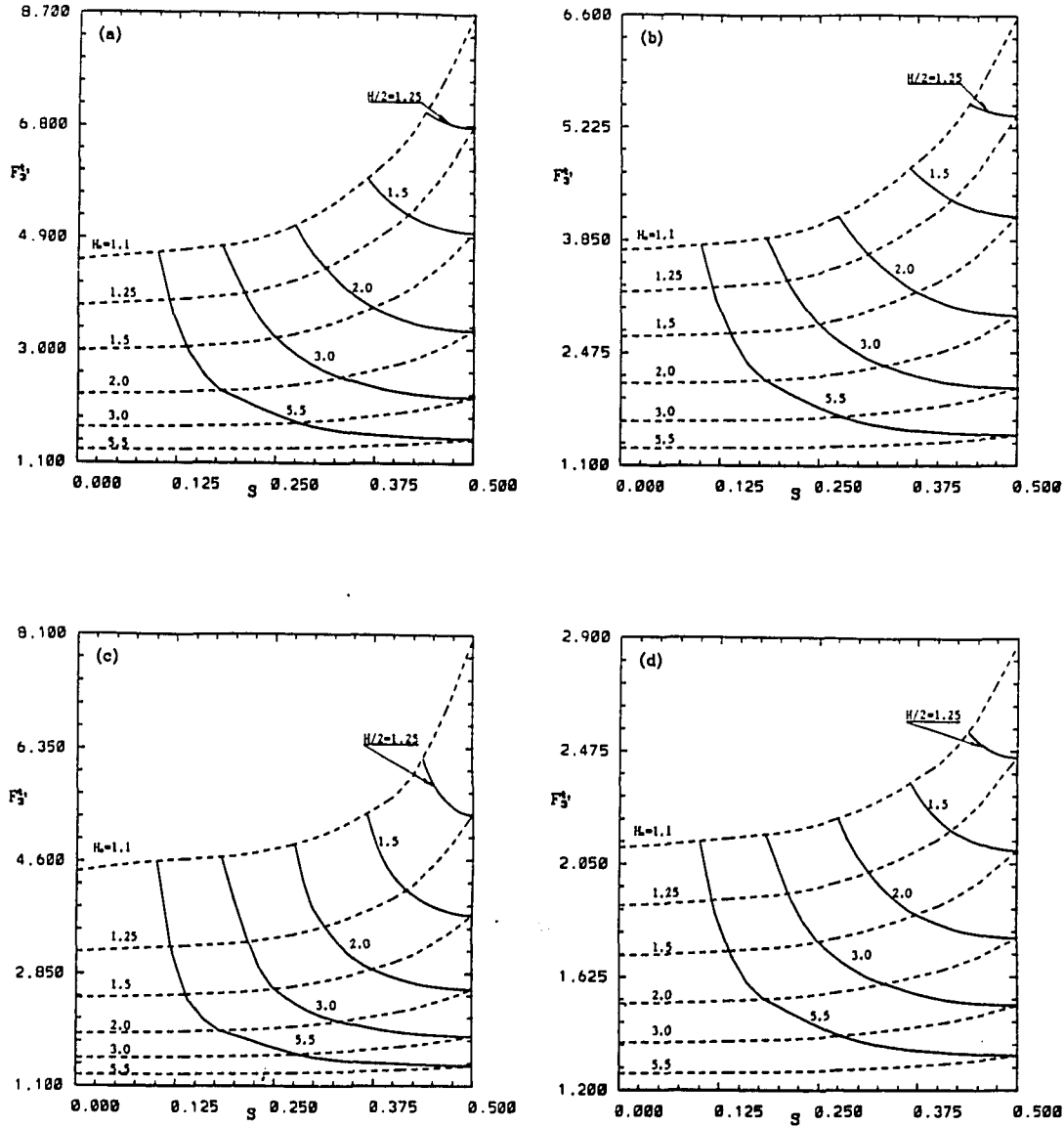


Figure 27. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

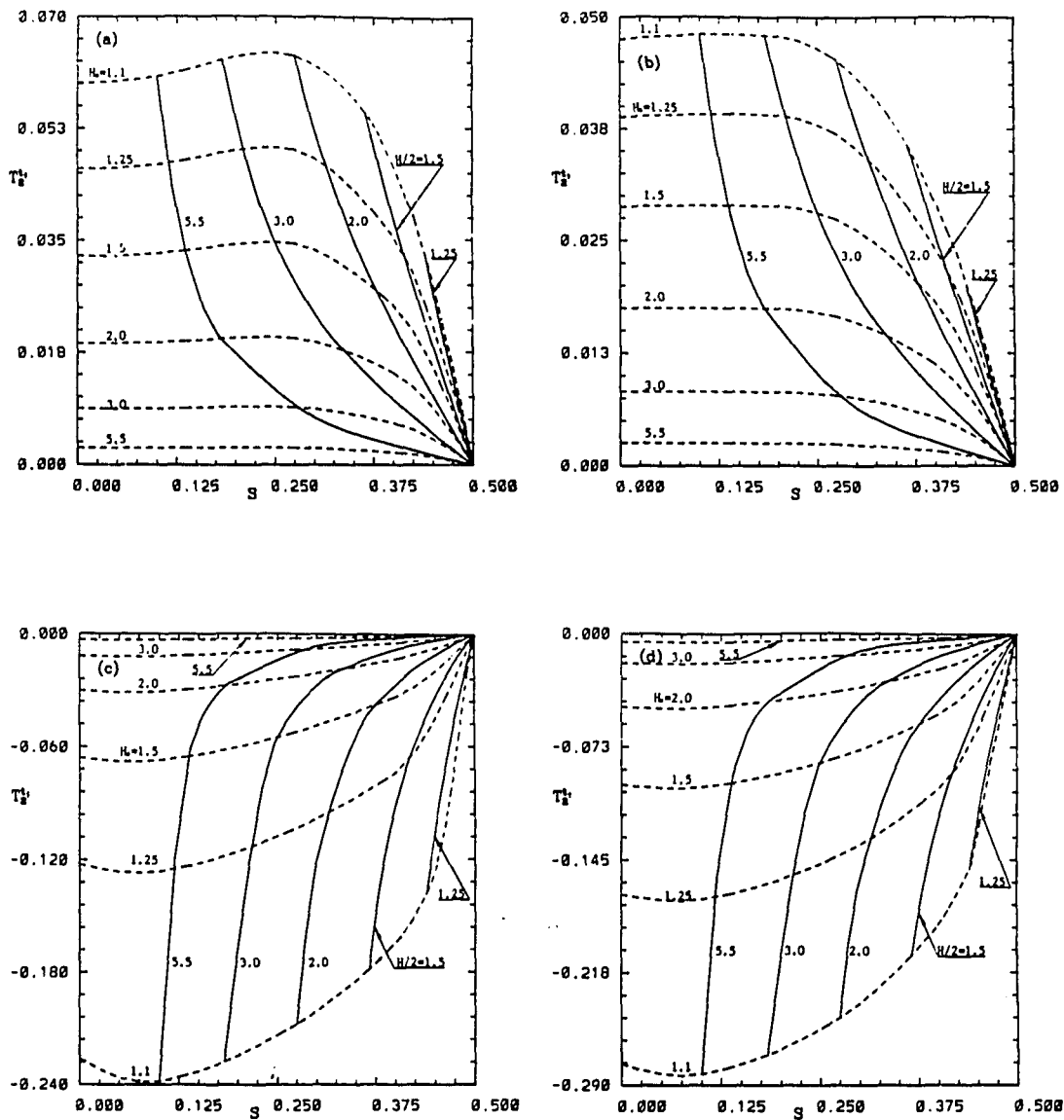


Figure 28. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

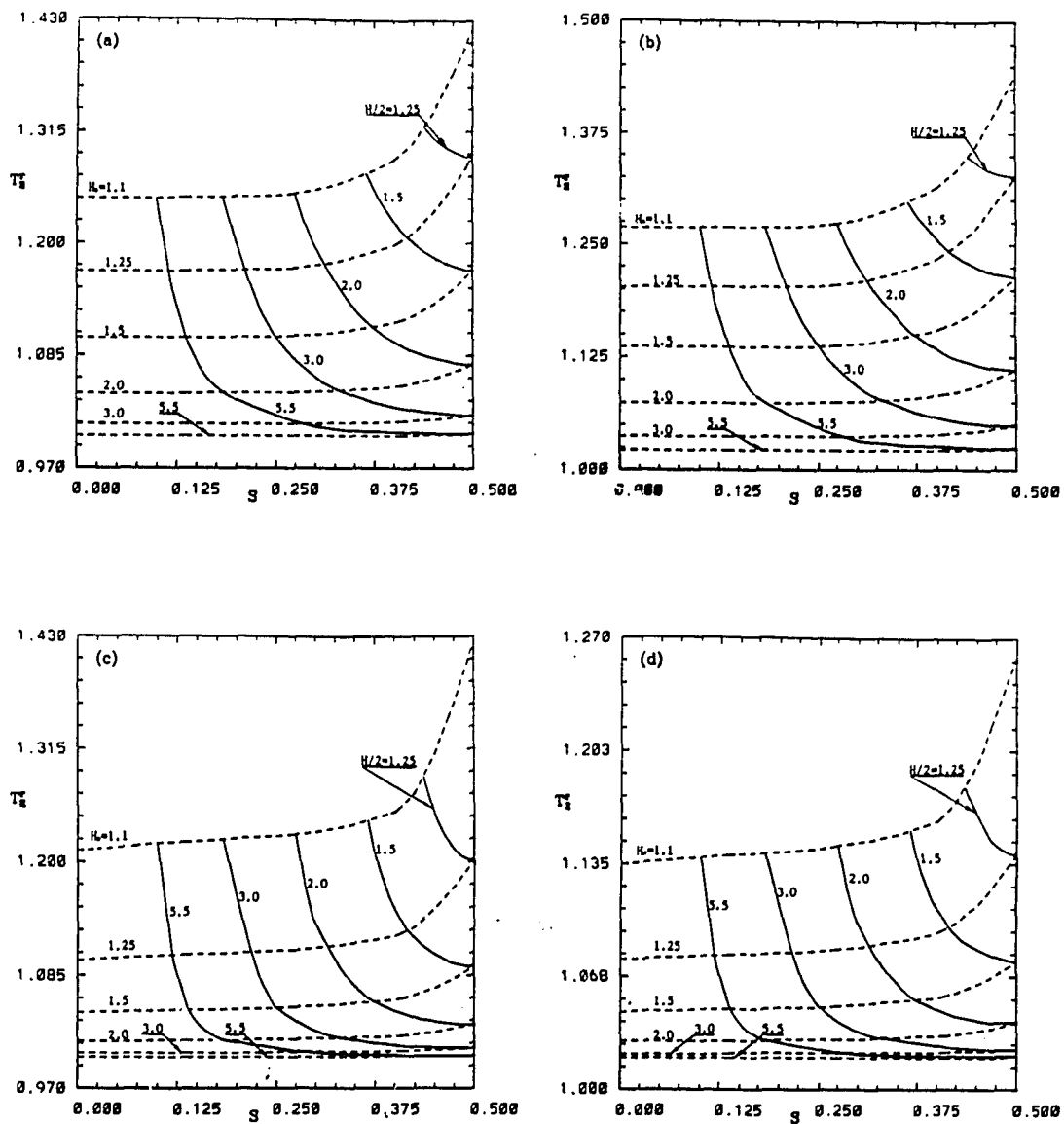


Figure 29. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rotating about an axis parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

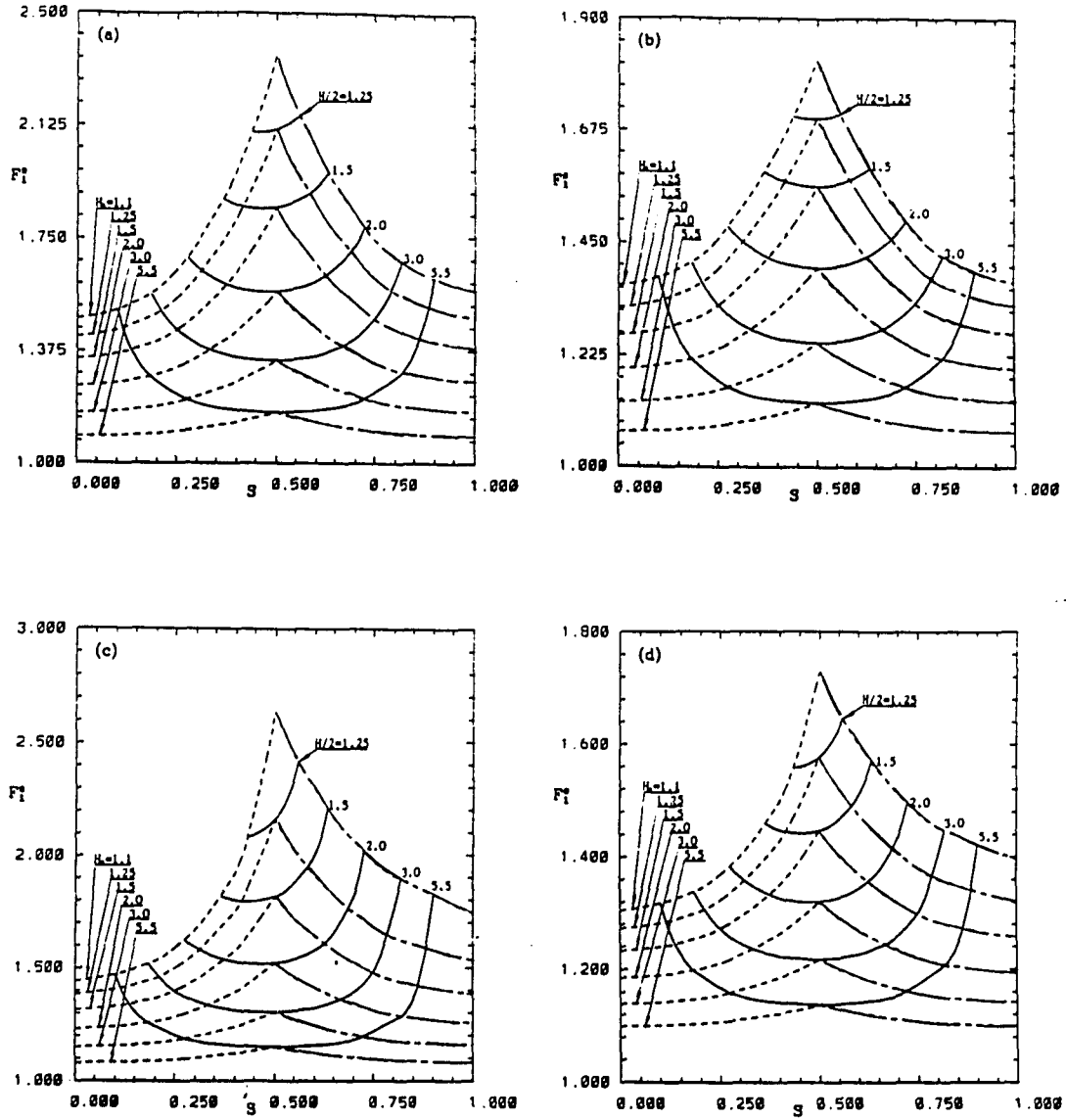


Figure 30. The force component parallel to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$, - · - ·, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

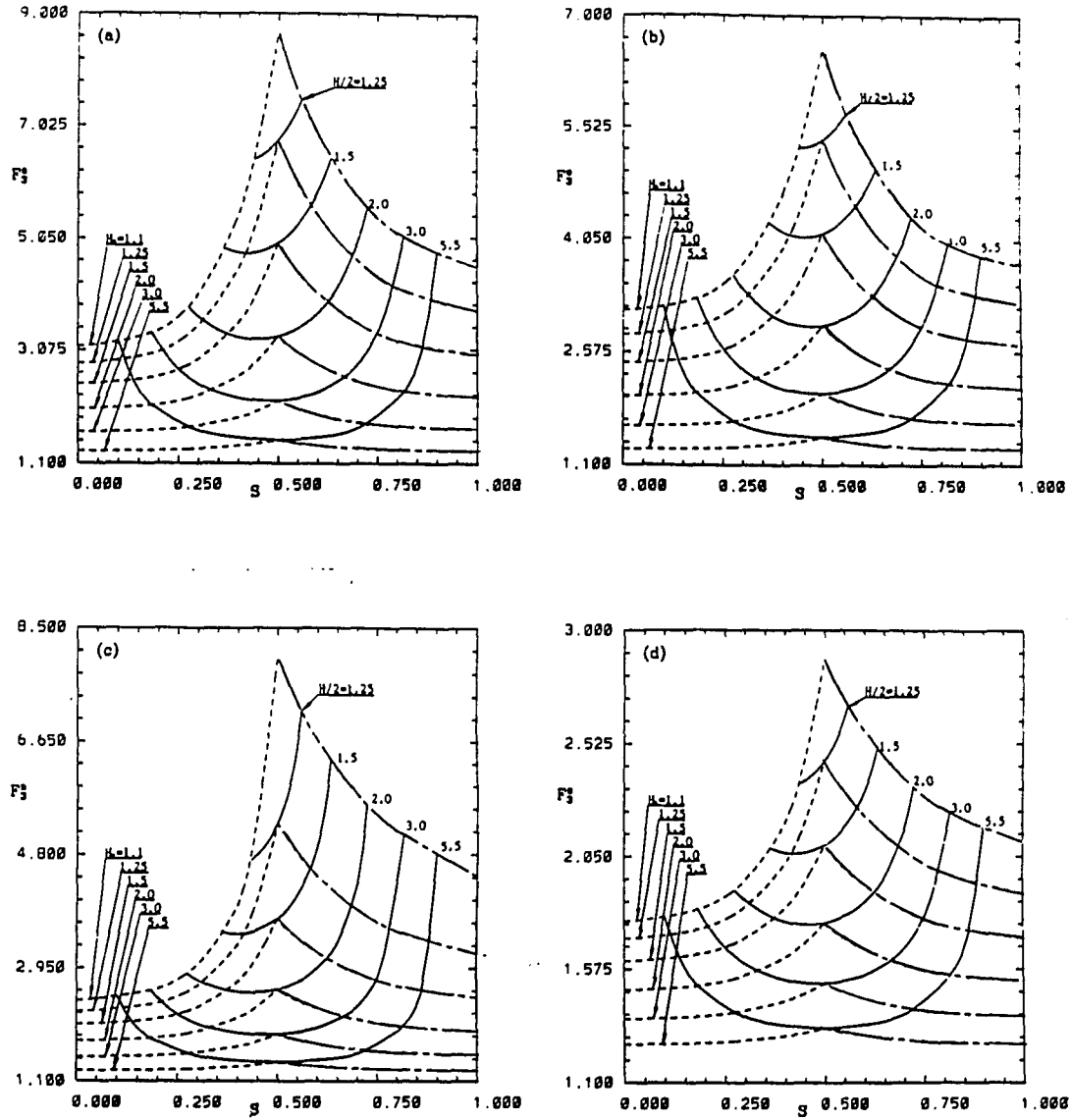


Figure 31. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$, -.-, $H-H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

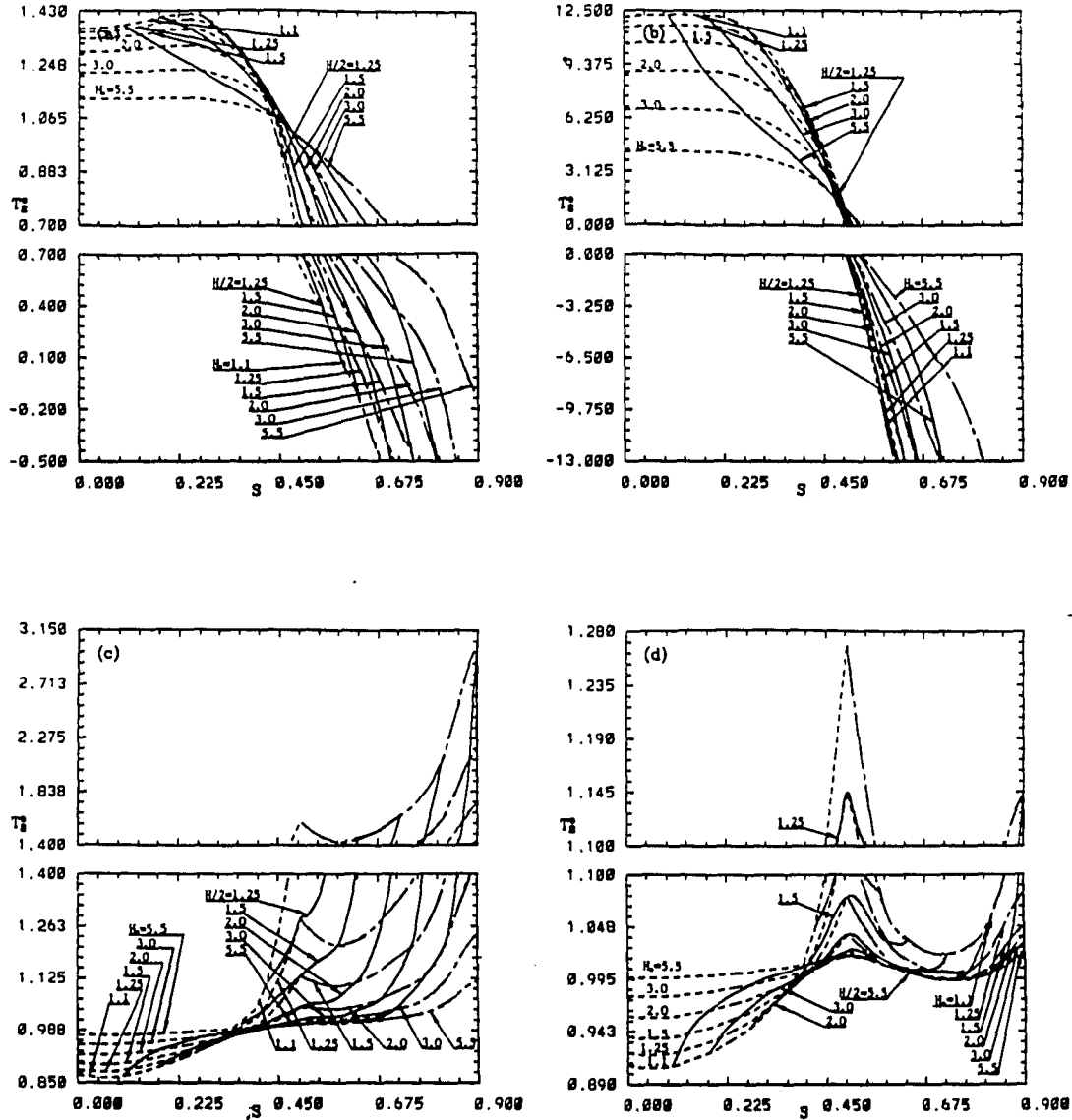


Figure 32. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in a Couette flow with the upper wall moving. —, $H/2 = \text{constant}$; ----, $H_0 = \text{constant}$, - · - ·, $H - H_0 = \text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

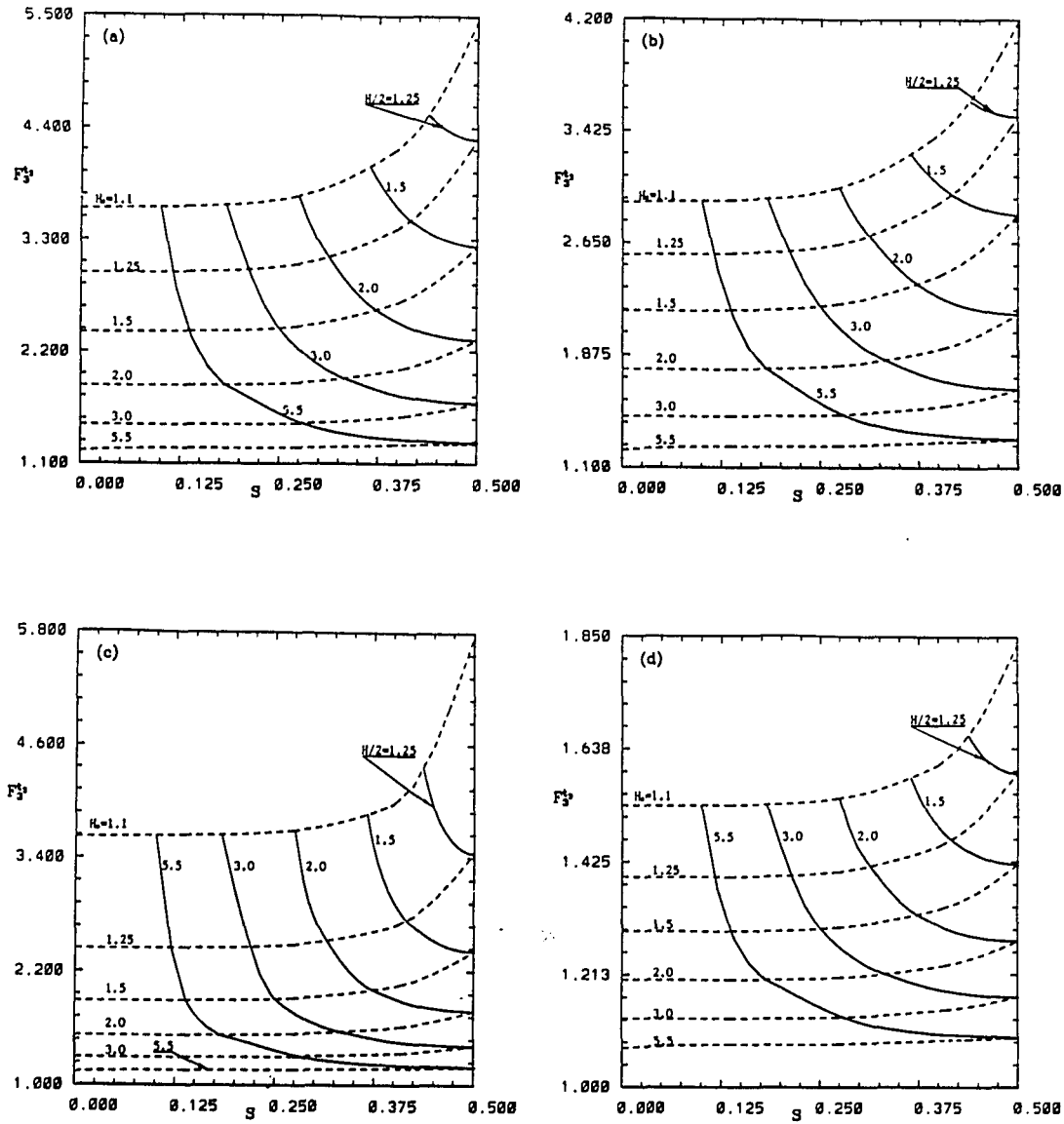


Figure 33. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, due to translation perpendicular to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

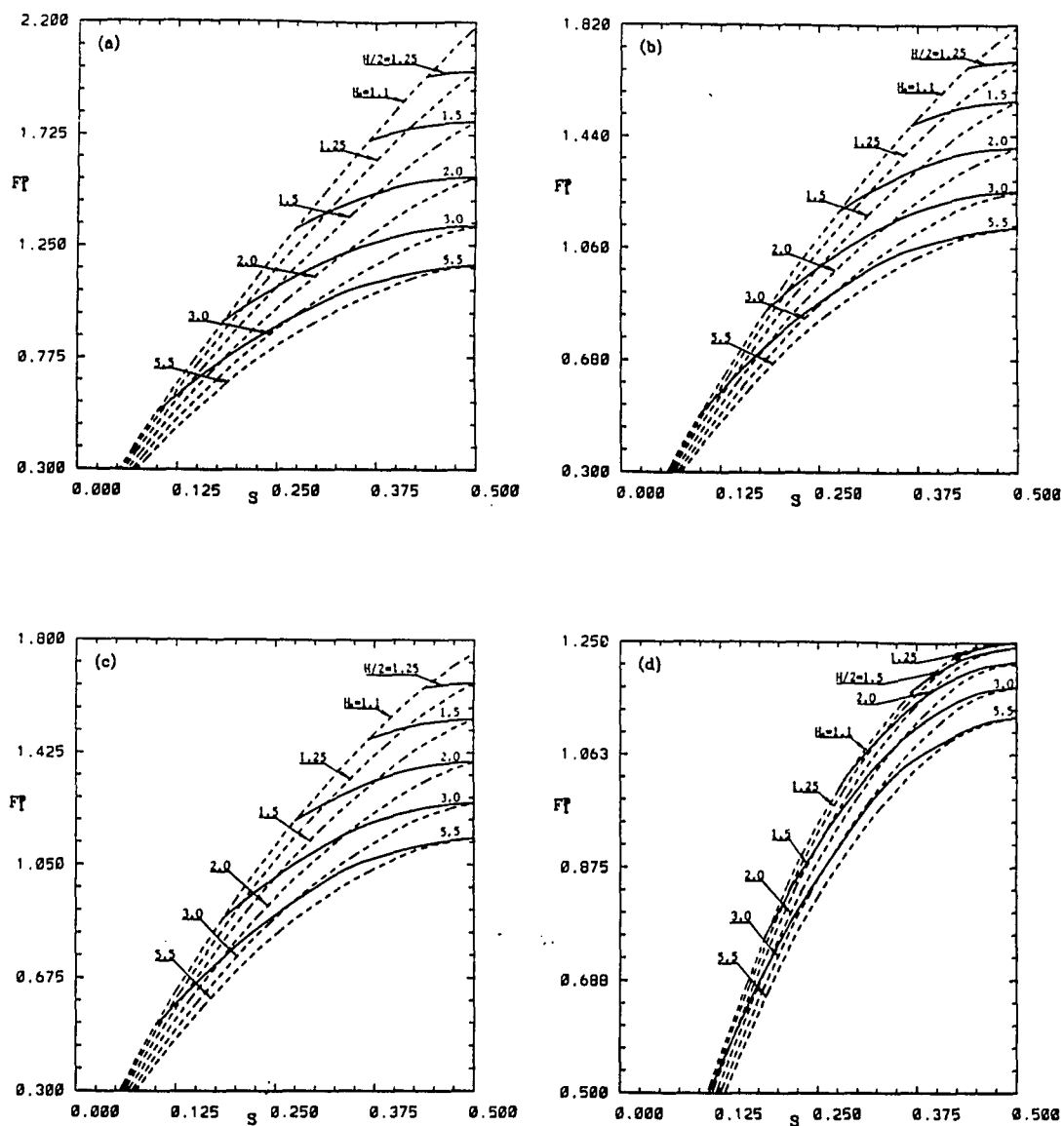


Figure 34. The force component parallel to the walls acting on an oblate or a prolate spheroid having aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

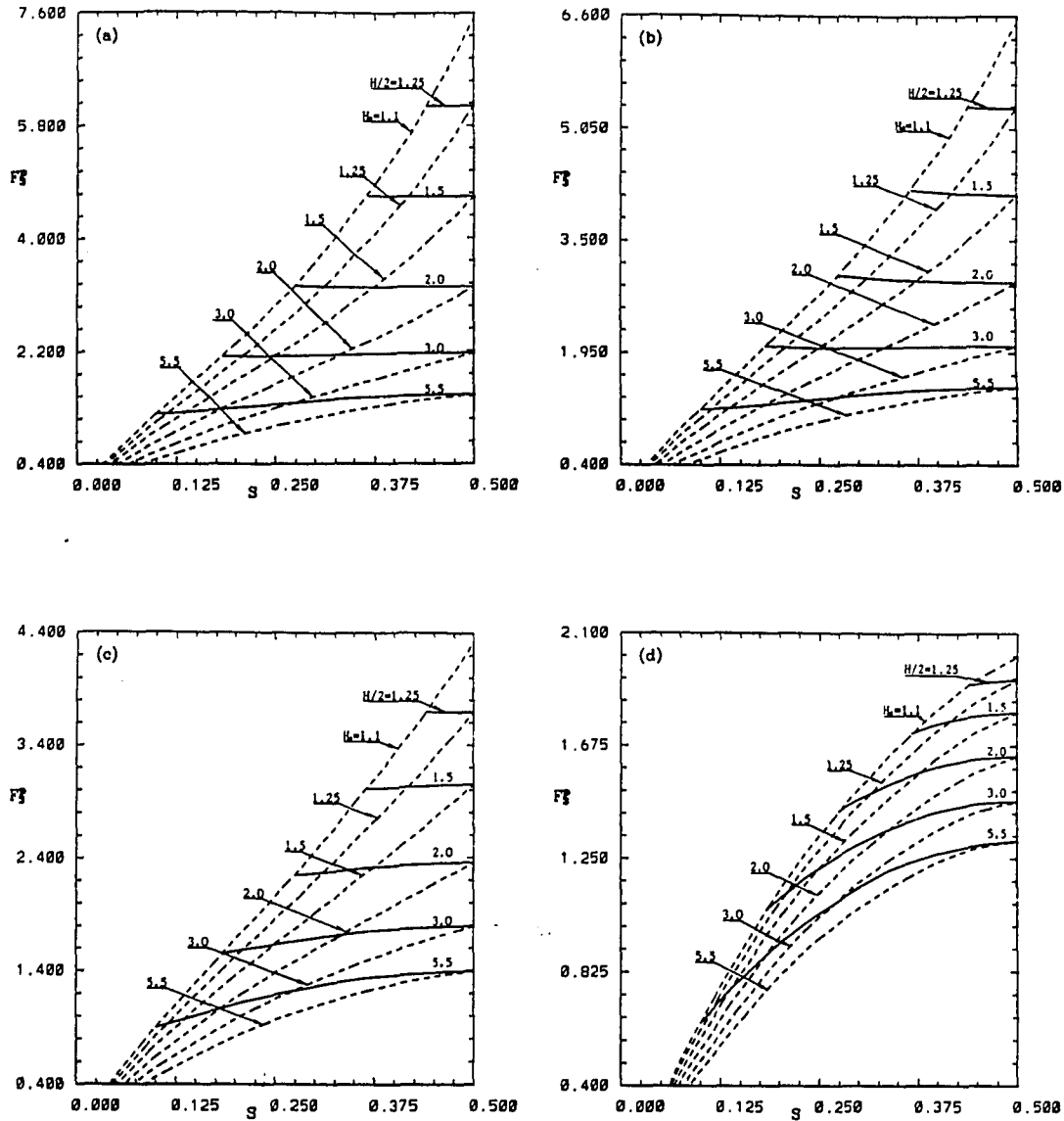


Figure 35. The force component perpendicular to the walls acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. _____, $H/2=\text{constant}$; -----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

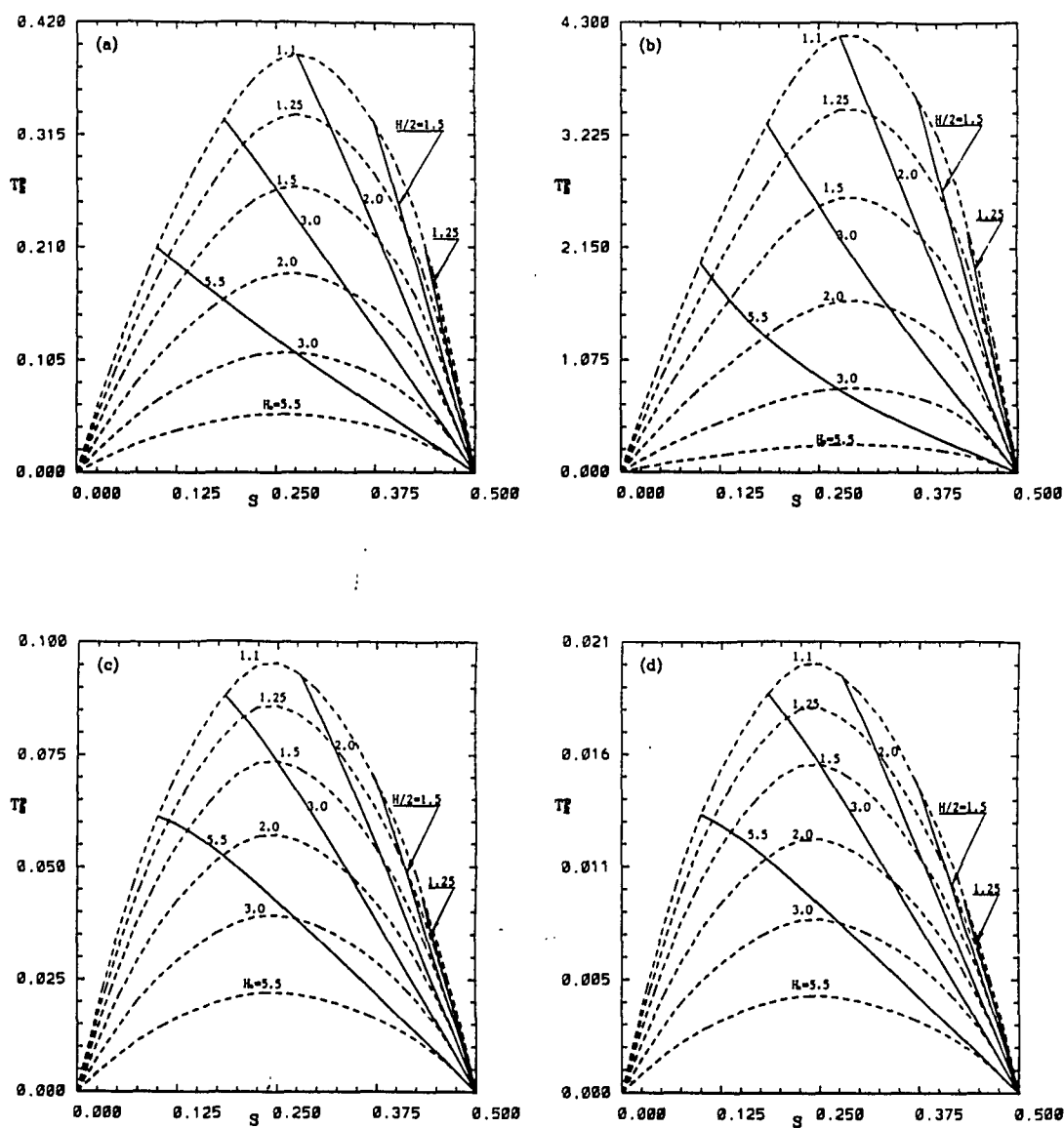


Figure 36. The torque acting on an oblate or a prolate spheroid having an aspect ratio ϵ with its symmetry axis oriented at $\alpha=90^\circ$ and $\gamma=0^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$. (a) oblate, $\epsilon=0.5$, (b) oblate, $\epsilon=0.1$, (c) prolate, $\epsilon=0.5$, (d) prolate, $\epsilon=0.1$

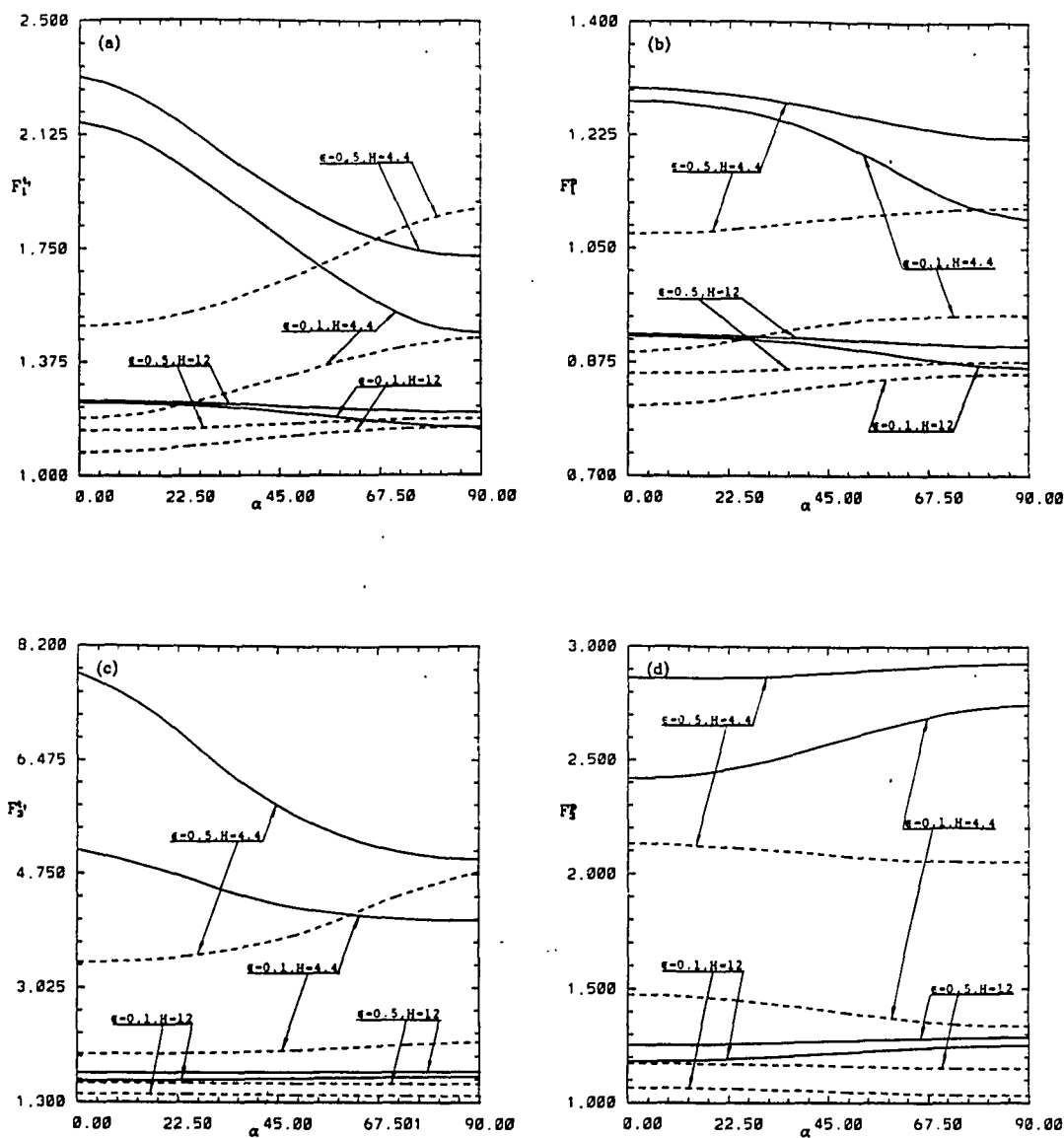


Figure 41. The resistance coefficients as a function of α (when $\gamma=0^\circ$) for an oblate or a prolate spheroid having an aspect ratio $\epsilon=0.1$ or 0.5 with the position of $s=0.25$, $H=4.4$ or $H=12.0$. (a) F_1^h , (b) F_1^p , (c) F_3^h , (d) F_3^p . —, oblate; ----, prolate.

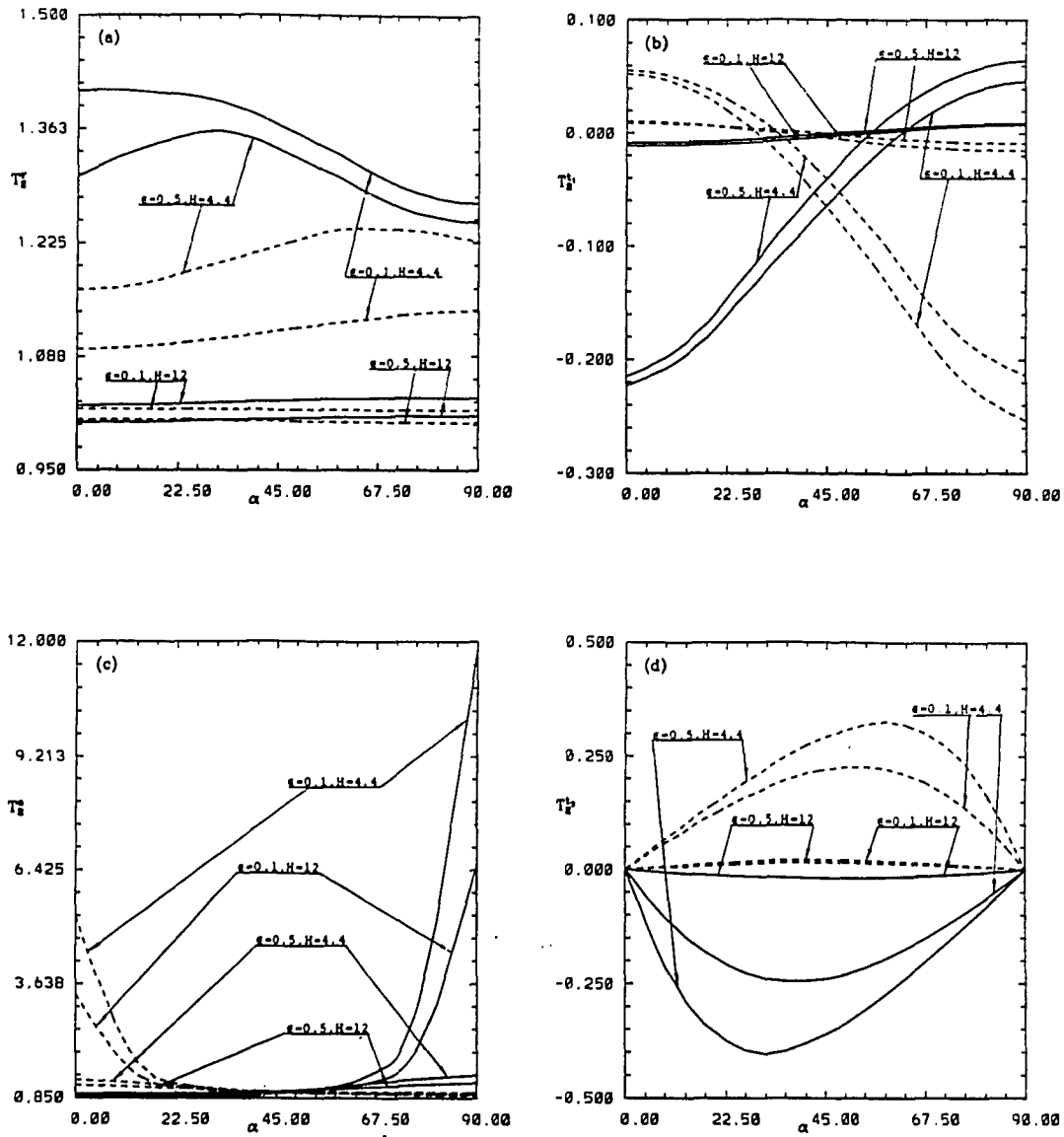


Figure 42. The resistance coefficients as a function of α (when $\gamma=0^\circ$) for an oblate or a prolate spheroid having an aspect ratio $\epsilon=0.1$ or 0.5 with the position of $s=0.25$, $H=4.4$ or $H=12.0$. (a) T_2^r , (b) T_2^{t1} , (c) T_2^s , (d) T_2^{t3} . —, oblate; ----, prolate.

torque acting on the spheroid will change its sign before reaching the midplane of the channel. At these angles there exists a second position other than midway between two walls, where the torque vanishes. This phenomenon has also been observed for a sphere, Ganatos et al (1980 b) and is due to formation of a separated region of closed streamlines between the particle and the wall which is further removed from it. At other orientations, if it exists, it is not strong enough to reverse the torque on the particle.

Figure 42 (c) shows that the torque T_2^* changes sharply near the orientation of $\alpha=90^\circ$ for an oblate spheroid and $\alpha=0^\circ$ for a prolate spheroid, which is associated with the phenomenon of torque reversal found in Couette flow discussed previously. Figure 42 (d) shows the maximum torque T_2^b acting on the spheroid occurs at an angle α smaller than 45 degrees for an oblate spheroid and larger than 45 degrees for a prolate spheroid. The torque vanishes at the angles of 0 and 90 degrees due to symmetry of the flow.

Figures 43 and 44 present the coefficients F_1^b , F_1^* , F_3^b , F_1^p and F_3^b , T_2^* , F_3^p , T_2^p as a function of the aspect ratio ϵ for an oblate or a prolate spheroid oriented at $\alpha=18^\circ$ or 45° at the position $s=0.25$, with $H=4.4$ and 12.0 . They clearly show the effect of the body shape of the spheroid on the drag and torque acting on it. All coefficients reach their maximum values for a sphere ($\epsilon=1$) except the coefficient T_2^* for the wall-to-wall spacing $H=12.0$ which first decreases first and then remains almost constant until $\epsilon=1$.

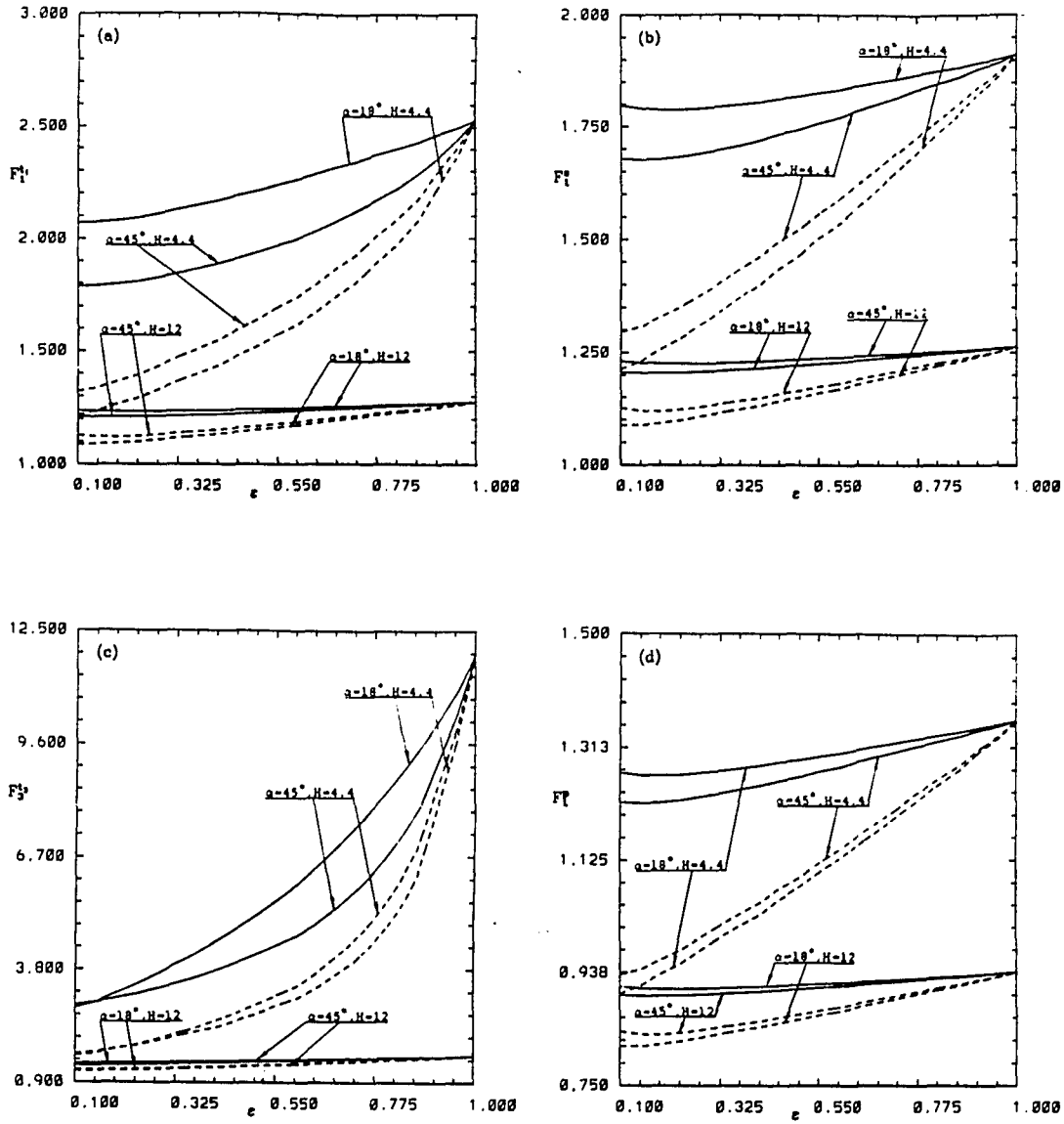


Figure 43. The resistance coefficients as a function of the aspect ratio ϵ for an oblate or a prolate spheroid with the position of $s=0.25$, $H=4.4$ or $H=12.0$, and oriented at $\alpha=18^\circ$ or 45° and $\gamma=0^\circ$. (a) F_1^i , (b) F_1^s , (c) F_3^s , (d) F_1^p . —, oblate; ----, prolate.

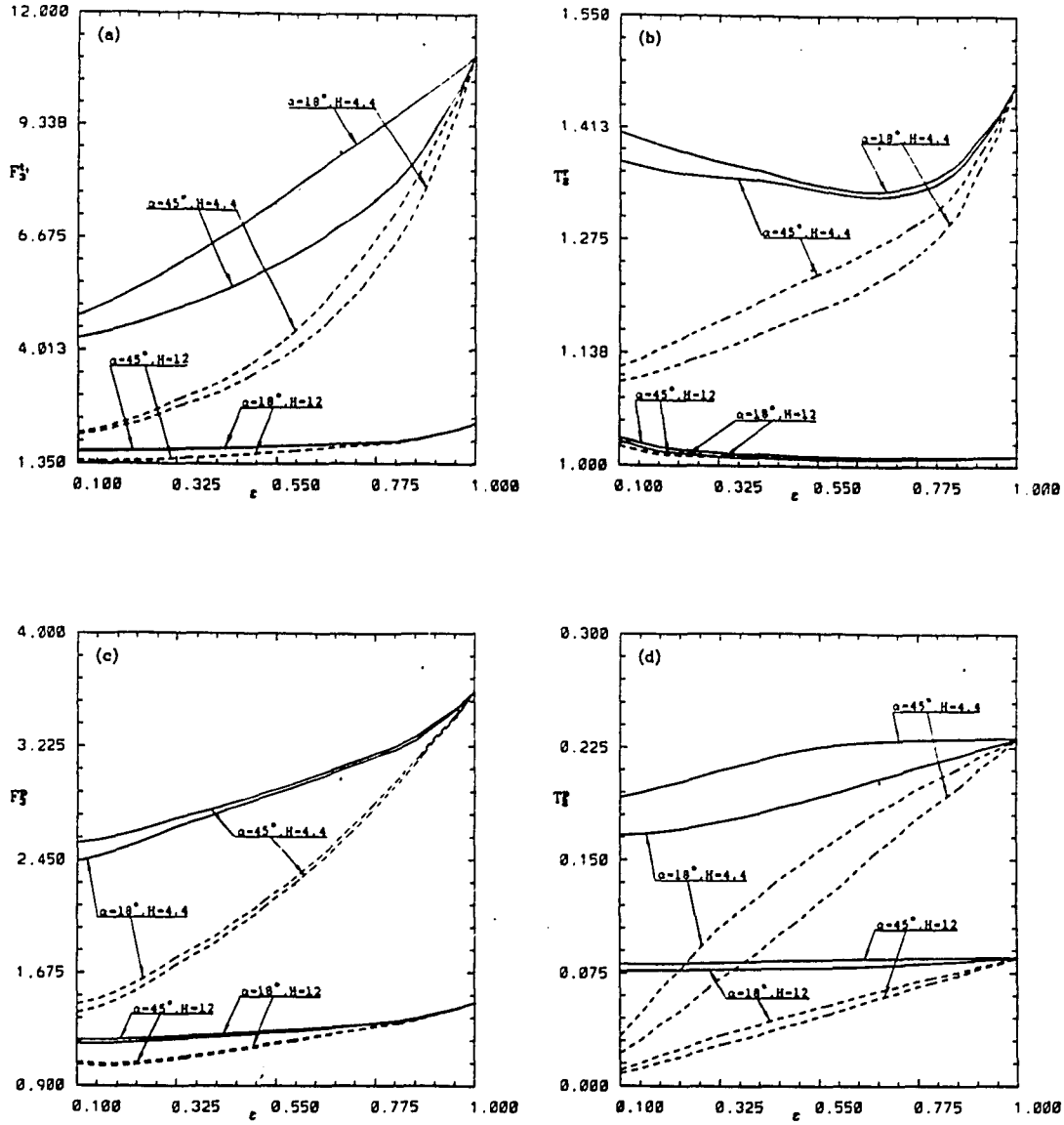


Figure 44. The resistance coefficients as a function of the aspect ratio ϵ for an oblate or a prolate spheroid with the position of $s=0.25$, $H=4.4$ or $H=12.0$, and oriented at $\alpha=18^\circ$ or 45° and $\gamma=0^\circ$. (a) F_3^I , (b) T_2^I , (c) F_3^P , (d) T_2^P . —, oblate; ----, prolate.

3.5 Solutions for a spheroid at an arbitrary position with an orientation of arbitrary α and γ between two parallel walls

In this general flow geometry, the full matrix of the resistance coefficients (3.5) is required. However, for brevity only the force component parallel to the walls acting on the spheroid, rigidly held in Poiseuille flow and due to translation of the spheroid parallel to the walls will be discussed. These two coefficients are the most important ones for the biological application to be presented in the next chapter.

Figures 37-40 present the force acting on the spheroid in above mentioned flow field at the orientations of $\alpha=0^\circ$, $\gamma=45^\circ$; $\alpha=45^\circ$, $\gamma=45^\circ$; $\alpha=0^\circ$, $\gamma=90^\circ$ and $\alpha=45^\circ$, $\gamma=90^\circ$. The results show that all forces vary with the orientation angles but exhibit a similar behavior as the corresponding ones presented for planar symmetric flow.

3.6 Concluding remarks

In the present research the boundary integral method has successfully been applied to the study of a spheroid between two parallel walls using the fundamental solution developed in the previous chapter. It reveals a new phenomenon of torque reversal for a spheroid rigidly held in Couette flow as it approaches the upper moving wall and also shows that a second zero-torque position exists for a spheroid at a certain angle α with a given aspect ratio ϵ translating between the two walls.

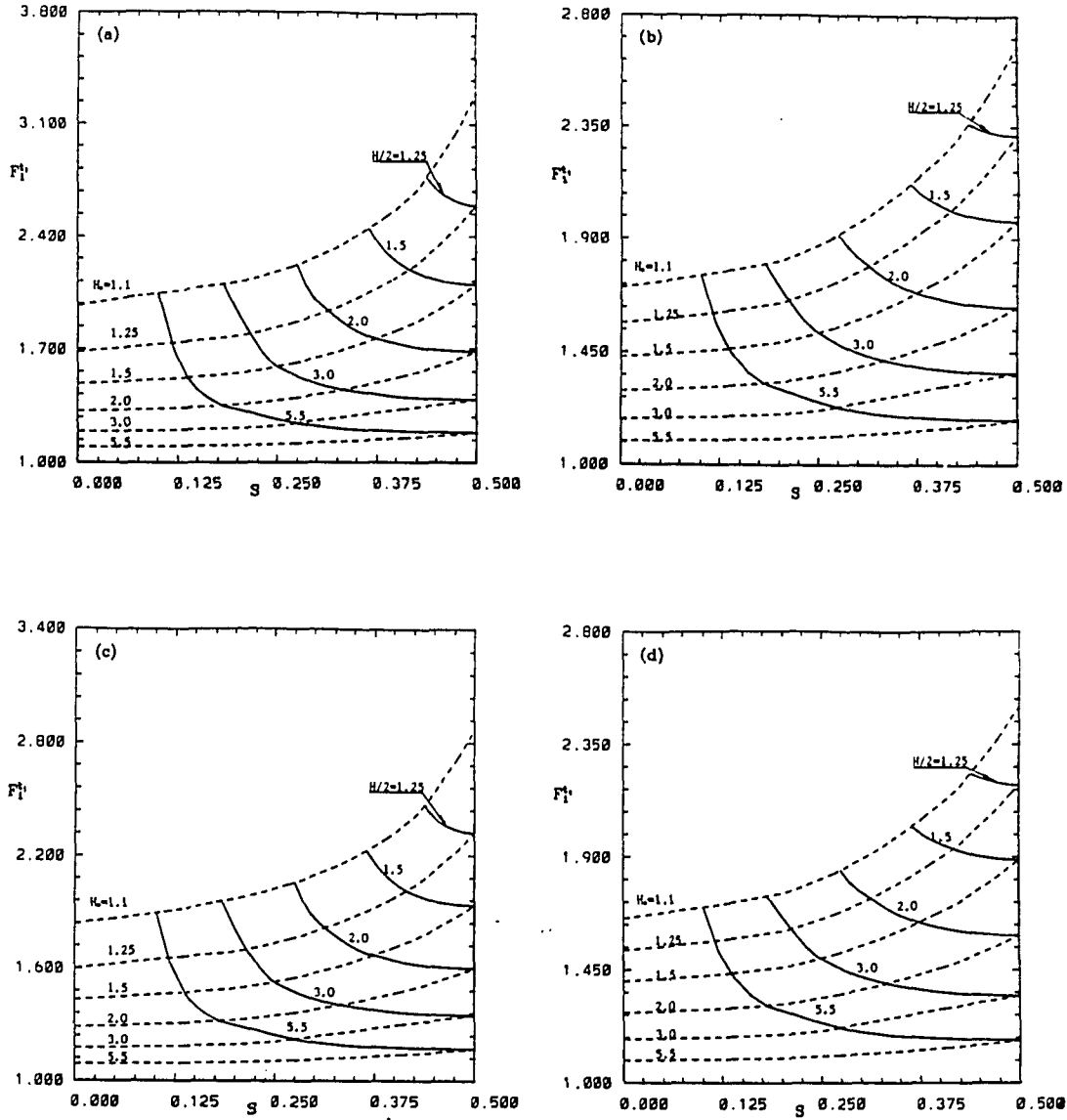


Figure 37. The force component parallel to the walls acting on an oblate spheroid having an aspect ratio $\epsilon=0.5$ with its symmetry axis oriented at (a) $\alpha=0^\circ$ and $\gamma=45^\circ$, (b) $\alpha=45^\circ$ and $\gamma=45^\circ$, (c) $\alpha=0^\circ$ and $\gamma=90^\circ$, (d) $\alpha=45^\circ$ and $\gamma=90^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$.

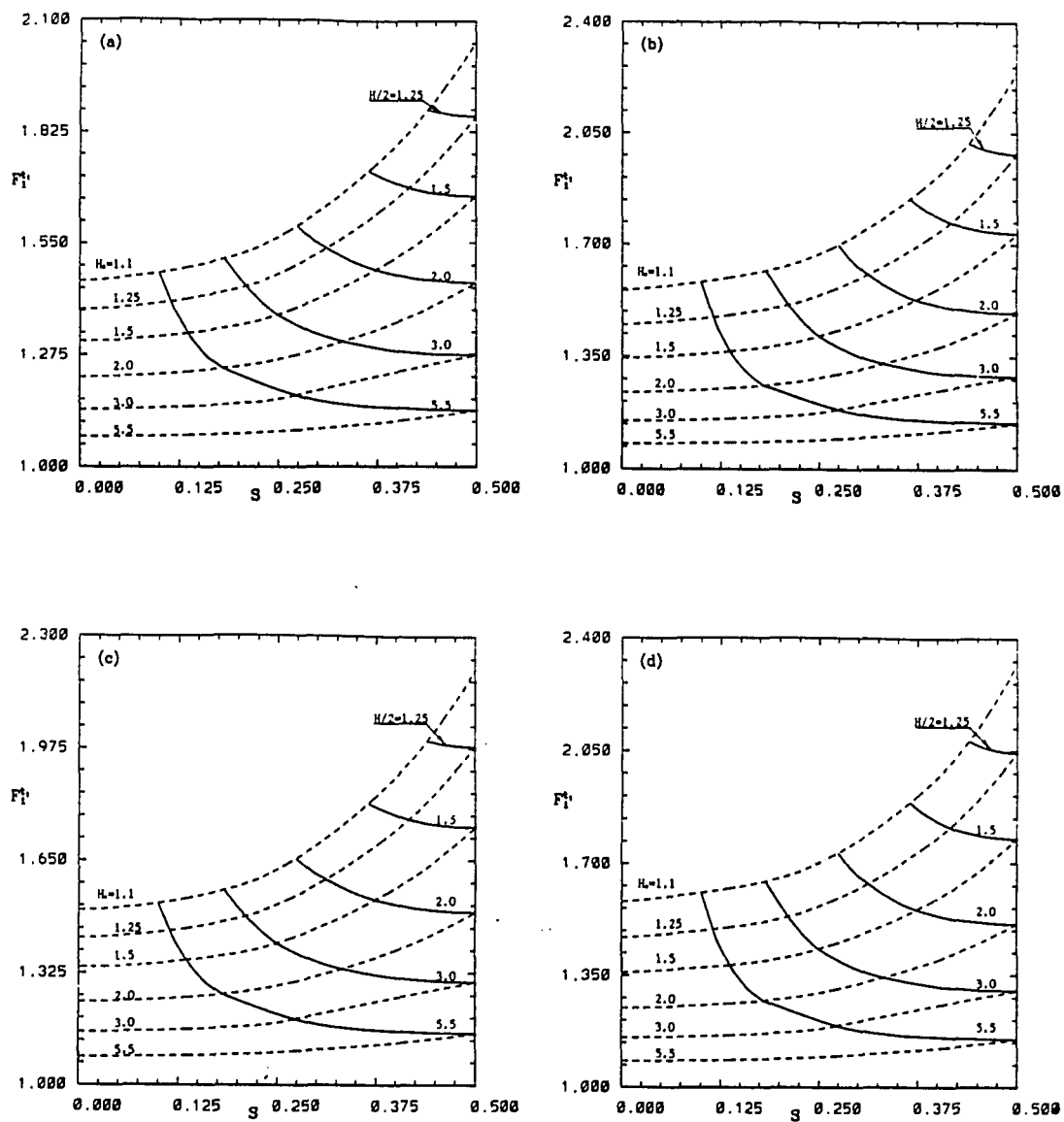


Figure 38. The force component parallel to the walls acting on a prolate spheroid having an aspect ratio $\epsilon=0.5$ with its symmetry axis oriented at (a) $\alpha=0^\circ$ and $\gamma=45^\circ$, (b) $\alpha=45^\circ$ and $\gamma=45^\circ$, (c) $\alpha=0^\circ$ and $\gamma=90^\circ$, (d) $\alpha=45^\circ$ and $\gamma=90^\circ$, due to translation parallel to the walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$.

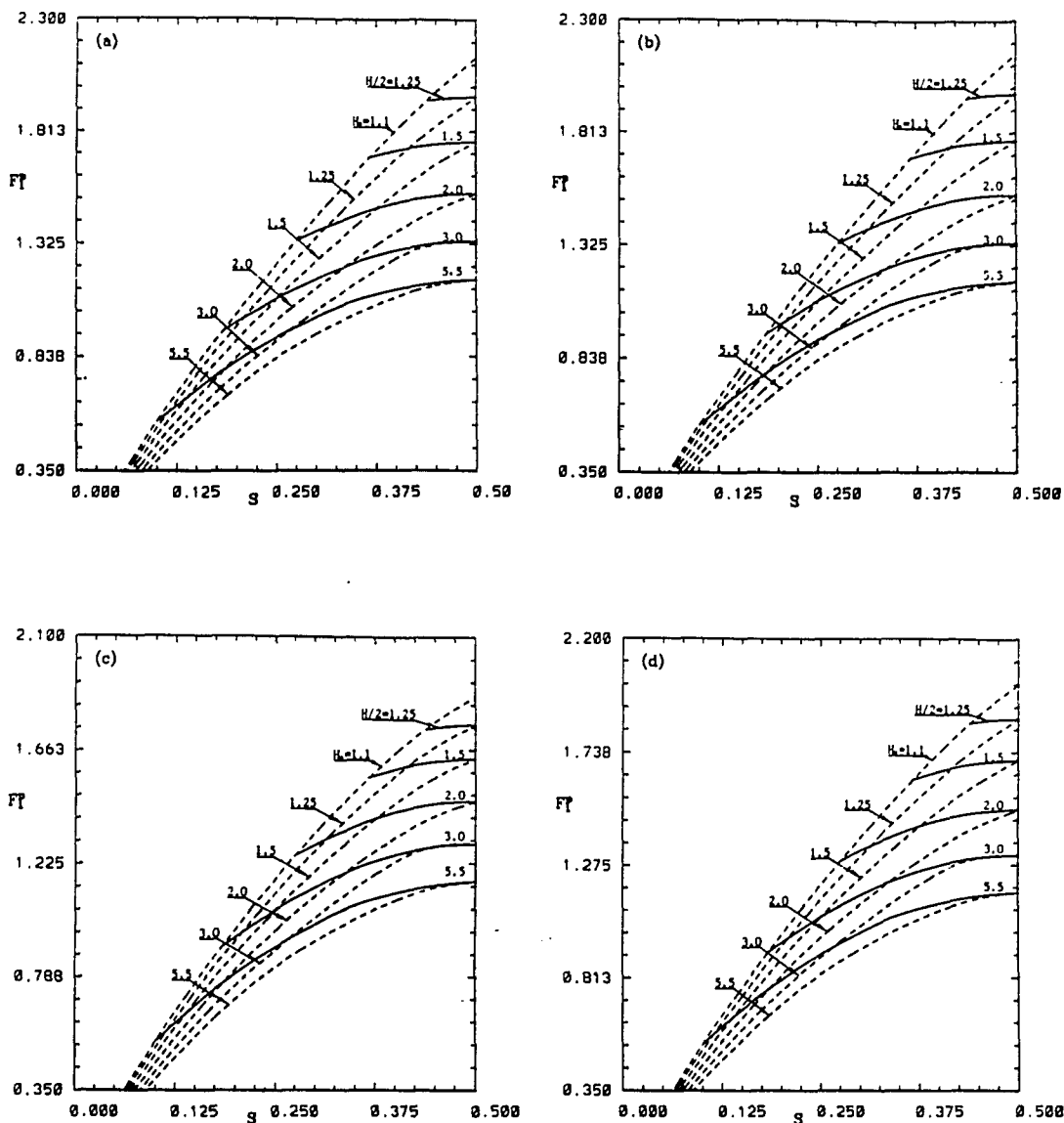


Figure 39. The force component parallel to the walls acting on an oblate spheroid having an aspect ratio $\varepsilon=0.5$ with its symmetry axis oriented at (a) $\alpha=0^\circ$ and $\gamma=45^\circ$, (b) $\alpha=45^\circ$ and $\gamma=45^\circ$, (c) $\alpha=0^\circ$ and $\gamma=90^\circ$, (d) $\alpha=45^\circ$ and $\gamma=90^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$.

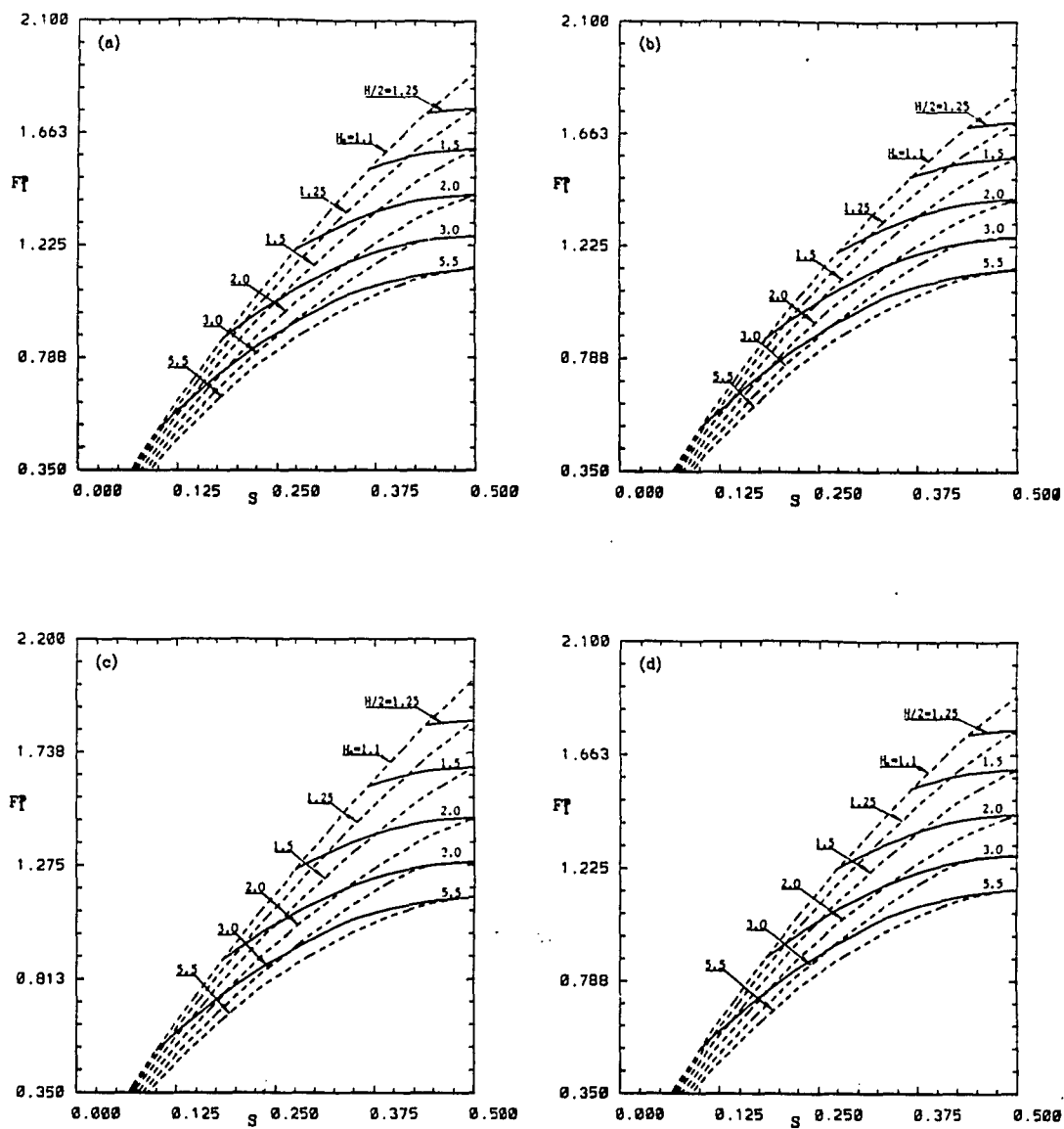


Figure 40. The force component parallel to the walls acting on a prolate spheroid having an aspect ratio $\epsilon=0.5$ with its symmetry axis oriented at (a) $\alpha=0^\circ$ and $\gamma=45^\circ$, (b) $\alpha=45^\circ$ and $\gamma=45^\circ$, (c) $\alpha=0^\circ$ and $\gamma=90^\circ$, (d) $\alpha=45^\circ$ and $\gamma=90^\circ$, rigidly held in a Poiseuille flow between two walls. —, $H/2=\text{constant}$; ----, $H_0=\text{constant}$.

Appendix B

The general solution for translation and/or rotation of an ellipsoid in an unbounded fluid or rigidly held in a shear flow was obtained by Jeffery (1922). Hsu & Ganatos (1989) have presented a summary of expressions for the forces and torques experienced by a spheroid in unbounded uniform or shear flow having planar symmetry i.e. for the case $\gamma=0^\circ$. In a general flow geometry the following formulas should be applied.

$$F_{1,\infty}^{t_1} = F_{1,\infty}^{t_1^*} \cos^2 \gamma \cos^2 \alpha + F_{1,\infty}^{t_1^{**}} (\cos^2 \gamma \sin^2 \alpha + \sin^2 \gamma) \quad (B.1)$$

$$F_{3,\infty}^{t_1} = -F_{1,\infty}^{t_1^*} \cos \gamma \sin \alpha \cos \alpha + F_{1,\infty}^{t_1^{**}} \cos \gamma \sin \alpha \cos \alpha \quad (B.2)$$

$$F_{1,\infty}^{t_3} = -F_{3,\infty}^{t_3^*} \cos \alpha \sin \alpha \cos \gamma + F_{3,\infty}^{t_3^{**}} \cos \alpha \sin \alpha \cos \gamma \quad (B.3)$$

$$F_{3,\infty}^{t_3} = F_{3,\infty}^{t_3^*} \sin^2 \alpha + F_{3,\infty}^{t_3^{**}} \cos^2 \alpha \quad (B.4)$$

$$T_{2,\infty}^s = T_{2,\infty}^{s^*} \cos^2 \gamma \cos^2 \alpha + T_{2,\infty}^{s^{**}} (\cos^2 \gamma \sin^2 \alpha + \sin^2 \gamma) \quad (B.5)$$

where the asterisked coefficients are listed in Appendix C of Hsu & Ganatos (1989).

REFERENCES

- Brenner, H. 1964** The Stokes resistance of an arbitrary particle -II. An extension. Chem. Engng. Sci. **19**, 599-629.
- Ganatos, P., Weinbaum, S. & Pfeffer, R. 1980 a,b** A strong interaction theory for the creeping motion of a sphere between plane parallel boundaries. J. Fluid Mech. **99**, 739-783.
- Ganatos, P., Weinbaum, S., Fischbarg, J. & Liebovitch, L. 1980** A hydrodynamic theory for determining the membrane coefficients for the passage of spherical molecules through an intercellular cleft. Adv. in Bioengr. 193-196.
- Hsu, R. & Ganatos, P. 1989** The motion of a rigid body in viscous fluid bounded by a plane wall. J. Fluid Mech. **207**, 29-72.
- Jeffery, G.B. 1922** The motion of ellipsoidal particle immersed in a viscous fluid. Proc.R.Soc. Lond A **102**, 161-179
- Ladyzhenskaya, O.A. 1963** The Mathematical Theory of Viscous Incompressible Flow. New York: Gordon & Breach.
- Lewellen, P.C. 1982** Hydrodynamic analysis of microporous mass transport. Ph.D dissertation, Univ. of Wisconsin-Madison.
- Wakiya, S. 1958** Res. Rep. Fac. Eng. Niigata Univ. (Japan) **7**, 1.
- Weinbaum, S., Ganatos, P. & Yan, Z. 1990** Numerical multipole and boundary integral equation techniques in Stokes flow. Annual Rev.in Fluid Mech. **22**, 275-316.
- Wu, W.Y., Weinbaum, S. & Acrivos, A. 1992** The motion of a sphere in a shear flow with suction past a plane wall with a hole and its application to particle screening. J. Fluid Mech. **243**, 489-518
- Yan, Z.Y., Weinbaum, S. & Ganatos, P. 1987** The three-dimensional hydrodynamic interaction of a fine sphere with a circular orifice at low Reynolds number. J. Fluid Mech. **174**, 39-68.
- Youngren, G.K. & Acrivos, A. 1975** Stokes flow past a particle of arbitrary shape: a numerical method of solution. J. Fluid Mech. **69**, 377-403.2

Chapter 4 Osmotic flow in an intercellular cleft

4.1 Introduction

Osmotic phenomena have a long-standing interest in the field of biological engineering. Osmosis occurs across the plasmalemma membranes of all living cells and through the intercellular clefts of the epithelial and endothelial cell layers lining all the internal body organs such as the blood capillaries and gall bladder. In the study of osmotic flow, a large number of studies have been conducted either from the standpoint of irreversible thermodynamics or using microhydrodynamic theories. Bean (1972) and Levitt (1975) have shown how both the reflection coefficient and the diffusive permeability can be related using continuum hydrodynamic theory to low Reynolds number hydrodynamic parameters which describe the arbitrary motion of a particle through a biological channel. A large amount of work has been done in the study of osmotic phenomenon without the use of strong particle-wall interaction theory due in part to the unavailability of such theories for certain flow geometries. With approximated hydrodynamic interaction solutions Anderson & Malone (1974) studied the mechanism of osmotic flow in porous membranes and developed mathematical expressions for the reflection coefficient as a function of solute dimension and shape in long cylindrical pores of circular cross section. At the same time Anderson & Quinn (1974) studied the conditions necessary for a simplified, one-dimensional treatment of the diffusion/convection process and outlined the effect of shape of the solute on the diffusion/convection process. In a general approach Anderson (1981) studied the

configuration effect on the reflection coefficient for rigid solute in capillary pores.

A more rigorous theory which includes the strong particle-wall interactions has been developed by Ganatos et al (1980), where the reflection coefficient and diffusive permeability were theoretically predicted for a spherical solute in a biological cleft. More recently Yan et al (1986) considered both the interior and entrance/exit region of a pore. The study with the strong particle-wall interactions for a non-spherical particle between two walls has never been treated in detail and is the focus of the present research. The hydrodynamic resistance of a spheroid with the strong particle-wall interactions will be utilized in the present research to predict the reflection coefficient and diffusive permeability of the osmotic flow in an cleft, which includes the effect of the shape and the orientation of the solute particle.

4.2 Osmotic phenomena in an intercellular cleft

Our model will consider an intercellular cleft of length L and width H ($H \ll L$), which is filled with a dilute solution of identical solute molecules whose shape is approximated by spheroids, e.g. albumin, globulin and fibrinogen. The solution is assumed to be sufficiently dilute that solute-solute interaction may be neglected and only the interaction between each particle and the cleft walls needs to be considered.

4.2.1 Formulation of the osmotic flow in the cleft

The total hydrodynamic force and torque acting on a spheroidal particle in a Poiseuille flow field between two walls is represented by (3.5) with $S=0$. The coefficients contained

in (3.5) account for effects of the walls on the particle and are functions of the particle aspect ratio ϵ , the orientation angles of the particle α and γ , the relative position of the particle, s and the channel width, H . The thermodynamic force per particle for an ideal solution at constant temperature is

$$\mathbf{F} = \frac{1}{N_A} \left(\frac{RT}{C} \nabla C + \bar{V}_s \nabla p \right) \quad (4.1)$$

where N_A is Avogadro's number, R is the universal gas constant and $\bar{V}_s$ is the molar volume of the particle. Equating the thermodynamic force to the hydrodynamic force and the net torque to zero yields the full force and torque balance equations for the particle which can be used to numerically compute the equilibrium concentration distribution of the solute particles in the channel. However, in most biological applications, the intercellular clefts are long and narrow and the concentration distribution in the channel may be assumed to be one-dimensional. Furthermore, the end effects may be neglected, the drift velocities in the directions other than the direction of the bulk flow are small and rotation of the particle has a negligible effect on the hydrodynamic force in the direction of the bulk flow. With these simplifications, the force balance equation between hydrodynamic and thermodynamic forces becomes

$$6\pi\mu c \left(U_1 F_1^h + V_c F_1^p \right) = \frac{1}{N_A} \left(\frac{RT}{C} \frac{dC}{dx_1} + \bar{V}_s \frac{dp}{dx_1} \right) \quad (4.2)$$

This expression may be solved for the solute particle velocity to give

$$U_1 = \frac{F}{6\pi\mu c N_A} \left(\frac{RT}{C} \frac{dC}{dx_1} + \bar{V}_s \frac{dp}{dx_1} \right) + V_c G \quad (4.3)$$

where

$$F(\epsilon, \alpha, \gamma, H, s) = 1/F_1^h, \quad G(\epsilon, \alpha, \gamma, H, s) = -F_1^p/F_1^h \quad (4.4 \text{ a,b})$$

For a given aspect ratio ϵ , particle orientation α, γ , relative position s and the channel width H , the functions $F(\epsilon, \alpha, \gamma, H, s)$ and $G(\epsilon, \alpha, \gamma, H, s)$ may be computed from the values of the drag and torque coefficients obtained in the hydrodynamic analysis. As pointed out by Brenner & Condiff (1972, 1974), Brenner & Gaydos (1977) and Anderson & Quinn (1974) the wall effect on the solute may be averaged over all orientations of the particle allowed within the channel because the solute orientation tends to be randomized for a large number of particles. In the present model we assume that all orientations of a typical solute particle are equally probable. This assumption is valid where the ratio of the major axis of the particle to the channel width is small. The solute flux is found by integrating along a cross-section of the channel

$$J_s = \frac{1}{4\pi} \int_{(\gamma)} \int_{(\alpha)} \int_{\bar{s}}^{1/2} C(x_1) U_1(x_1, \alpha, \gamma, H, s) \sin \alpha \, ds \, d\alpha \, d\gamma \quad (4.5)$$

where $\bar{s}$ is the smallest allowed position of the particle center without intersecting the walls of the channel, and (α) , (β) are all possible orientations for the particle at a certain position without intersecting the walls. The constant solute flux cross channel section allows integration along the length of the cleft L and yields the following equation

$$J_s = \frac{1}{4\pi L} \int_0^L \int_{(\gamma)} \int_{(\alpha)} \int_{\bar{s}}^{1/2} C(x_1) U_1(x_1, \alpha, \gamma, H, s) \sin \alpha \, ds \, d\alpha \, d\gamma \, dx_1 \quad (4.6)$$

Introducing (4.3) into (4.6) gives

$$J_s = \omega (RT \Delta C + \bar{V}_s \overline{c \Delta p}) + (1 - \sigma) \bar{V}_w J_w \bar{C} \quad (4.7)$$

where

$$\bar{C} = \frac{1}{L} \int_0^L C(x_1) dx_1, \quad \Delta C = \frac{1}{L} \int_0^L \frac{dC(x_1)}{dx_1} dx_1, \quad \overline{C \Delta p} = \int_0^L C(x_1) \frac{dp}{dx_1} dx_1 \quad (4.8 \text{ a,b,c})$$

and the diffusive permeability ω and reflection coefficient σ are given by

$$\omega = \frac{L_p}{\bar{V}_s} \frac{2}{3} s^{*2} e^2 (1 - s^*) \bar{F}(e, H), \quad \sigma = 1 - \frac{3}{2} (1 - s^*) \bar{G}(e, H) \quad (4.9 \text{ a,b})$$

where

$$s^* = 2\bar{s}, \quad L_p = \frac{H^2}{12\mu L}, \quad \bar{V}_s = N_A \cdot \frac{4}{3} \pi c r^2 \quad (4.10 \text{ a,b,c})$$

In the above equations c and r represent the semi-major and minor axis of a spheroid respectively, that is $c=a$, $r=b$ for a prolate spheroid and $c=b$, $r=a$ for an oblate spheroid.

The functions $\bar{F}(e, H)$ and $\bar{G}(e, H)$ can be calculated with the known functions

$F(e, \alpha, \gamma, H, s)$ and $G(e, \alpha, \gamma, H, s)$ from the hydrodynamic analysis as follows

$$\bar{F} = \frac{2}{4\pi(1-s^*)} \int \int \int_{(\gamma)(\alpha)}^{1/2} F(e, \alpha, \gamma, H, s) \sin \alpha \, ds \, d\alpha \, d\gamma \quad (4.11 \text{ a})$$

$$\bar{G} = \frac{2}{4\pi(1-s^*)} \int \int \int_{(\gamma)(\alpha)}^{1/2} G(e, \alpha, \gamma, H, s) \sin \alpha \, ds \, d\alpha \, d\gamma \quad (4.11 \text{ b})$$

This furnishes expressions for the prediction of the osmotic reflection coefficient and the diffusive permeability.

4.2.2 Numerical values of the reflection coefficient and the diffusive permeability

To illustrate the procedure of predicting ω and σ , the resistance coefficients for a prolate and an oblate spheroid having aspect ratios $\epsilon=0.1$ and 0.5 are calculated at various orientation angles α and γ for five different values of channel width. Figures 37-40 in previous chapter shows a portion of these results. Since the velocity component U_1 is an even function of α and γ , and has a periodicity of 180° the integrations about α and γ in (4.5) are carried out from 0° to 90° . Thus (4.9, 4.10) for the functions $\bar{F}(\epsilon, H)$ and $\bar{G}(\epsilon, H)$ are reduced to

$$\bar{F} = \frac{4}{\pi(1-s^*)} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_{s^*}^{\frac{1}{2}} F(\epsilon, \alpha, \gamma, H, s) \sin \alpha \, ds \, d\alpha \, d\gamma \quad (4.12)$$

$$\bar{G} = \frac{4}{\pi(1-s^*)} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_{s^*}^{\frac{1}{2}} G(\epsilon, \alpha, \gamma, H, s) \sin \alpha \, ds \, d\alpha \, d\gamma \quad (4.13)$$

Since the $\bar{s}$ is a complicated function of the orientation angles of the spheroid it is difficult to evaluate the above integrals exactly. For simplicity we assume that $\bar{s}=s$ which is the smallest position with the given wall-to-wall spacing H presented in the hydrodynamic analysis. Therefore s^* is defined as $2a/H$ for a prolate spheroid and $2b/H$ for an oblate spheroid. Furthermore, the results of the resistance coefficients in the range $s^* \leq s \leq s_0$ which corresponds to a particle-to-wall spacing $H_0=1.1$ for a given wall-to-wall spacing H , are approximated by extrapolation of the results in the range $s \geq s_0$ due to limitations of the hydrodynamic analysis. The numerical results for that range of s are

also very difficult to obtain with any other known methods. In order to evaluate the integrals above analytically with the numerical data of the resistance coefficients, the following curve fitting form is applied to the functions $F(\epsilon, \alpha, \gamma, H, s)$ and $G(\epsilon, \alpha, \gamma, H, s)$

$$\begin{aligned}
 f(\alpha, \gamma, s) = & A_1 + A_2 s + A_3 s^2 + A_4 \cos \alpha + A_5 s \cos \alpha + A_6 s^2 \cos \alpha + A_7 \cos \gamma + A_8 s \cos \gamma \\
 & + A_9 s^2 \cos \gamma + A_{10} \cos \alpha \cos \gamma + A_{11} s \cos \alpha \cos \gamma + A_{12} s^2 \cos \alpha \cos \gamma \\
 & + A_{13} \cos 2\alpha \cos \gamma + A_{14} s \cos 2\alpha \cos \gamma + A_{15} s^2 \cos 2\alpha \cos \gamma \\
 & + A_{16} \cos \alpha \cos 2\gamma + A_{17} s \cos \alpha \cos 2\gamma + A_{18} s^2 \cos \alpha \cos 2\gamma + A_{19} \cos 2\alpha \\
 & + A_{20} s \cos 2\alpha + A_{21} s^2 \cos 2\alpha + A_{22} \cos 2\gamma + A_{23} s \cos 2\gamma + A_{24} s^2 \cos 2\gamma \\
 & + A_{25} \cos 2\alpha \cos 2\gamma + A_{26} s \cos 2\alpha \cos 2\gamma + A_{27} s^2 \cos 2\alpha \cos 2\gamma
 \end{aligned} \tag{4.13}$$

where the A_i are unknown curve fitting constants. As mentioned above, the curve fitting form will be used to cover the entire range of s in the integrations.

Using the least-squares method, the total 27 unknown constants are obtained with the given data of the resistance coefficients. Tables 8 (a,b) list three non-zero constants for resistance coefficients F_1^h , F_1^p of a sphere at the five different channel widths. Tables 9 (a,b,c,d) list all 27 curve-fitting constants for resistance coefficients F_1^h , F_1^p of a spheroid with $\epsilon=0.1$, while Tables 10 (a,b,c,d) list the constants of a spheroid with $\epsilon=0.5$. At the given data points the maximum relative error is less than 0.05%. Introducing the curve-fitted functions $F(\epsilon, \alpha, \gamma, H, s)$ and $G(\epsilon, \alpha, \gamma, H, s)$ into (4.12, 4.13) and evaluating the integrals analytically yields the values of the functions $\bar{F}(\epsilon, H)$ and $\bar{G}(\epsilon, H)$ at the different channel widths, which can be used to predict the values of the diffusive permeability ω and the reflection coefficient σ . Table 11 shows the comparison of values of ω and σ for a spherical solute particle in a channel. Compared to the values obtained

Table 8 a,b Constants of the curve fitting for resistance coefficients F_1^h , F_1^p of a sphere
in equation (4.13) .

(a) for the coefficient $1/F_1^h$

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A_1	-0.1448E+01	-0.8827E+00	-0.3575E+00	-0.1094E+00	0.2017E+00
A_2	0.7171E+01	0.5443E+01	0.3890E+01	0.3717E+01	0.3210E+01
A_3	-0.7229E+01	-0.5676E+01	-0.4196E+01	-0.4316E+01	-0.4014E+01

(b) for the coefficient F_1^p/F_1^h

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A_1	-0.3554E+01	-0.2041E+01	-0.9242E+00	-0.4616E+00	-0.1542E+00
A_2	0.1732E+02	0.1190E+02	0.7690E+01	0.6080E+01	0.4799E+01
A_3	-0.1744E+02	-0.1230E+02	-0.8046E+01	-0.6490E+01	-0.5049E+01

Table 9 a,b,c,d Constants of the curve fitting for resistance coefficients F_1^h , F_1^p of a

spheroid with $\varepsilon=0.1$ in equation (4.13) .

(a) The curve fitting constants for the resistance coefficient $1/F_1^{t_1}$ of a prolate spheroid.

(b) The curve fitting constants for the resistance coefficient $1/F_1^{t_1}$ of an oblate spheroid.

(c) The curve fitting constants for the resistance coefficient $F_1^p/F_1^{t_1}$ of a prolate spheroid.

(d) The curve fitting constants for the resistance coefficient $F_1^p/F_1^{t_1}$ of an oblate spheroid.

(a) for the coefficient $1/R_1^4$ of a prolate spheroid

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A ₁	0.1137E+00	0.3982E+00	0.6333E+00	0.7037E+00	0.7752E+00
A ₂	0.2749E+01	0.1783E+01	0.9198E+00	0.8046E+00	0.6538E+01
A ₃	-0.2780E+01	-0.1886E+01	-0.9841E+00	-0.9183E+00	-0.7986E+00
A ₄	0.4620E+00	0.1574E+00	-0.4849E-01	-0.6132E-01	-0.5696E-01
A ₅	-0.2254E+01	-0.1037E+01	-0.2847E-01	0.1427E+00	0.2613E+00
A ₆	0.2300E+01	0.1150E+01	0.4856E-01	-0.1574E+00	-0.3319E+00
A ₇	-0.1285E+01	-0.8624E+00	-0.5359E+00	-0.3701E+00	-0.2195E+00
A ₈	0.4283E+01	0.2909E+01	0.1911E+01	0.1556E+01	0.1160E+01
A ₉	-0.4300E+01	-0.3017E+01	-0.2111E+01	-0.1883E+01	-0.1525E+01
A ₁₀	0.1605E+01	0.1026E+01	0.5714E+00	0.3540E+00	0.1765E+00
A ₁₁	-0.5735E+01	-0.3802E+01	-0.2299E+01	-0.1697E+01	-0.1098E+01
A ₁₂	0.5753E+01	0.3954E+01	0.2571E+01	0.2108E+01	0.1503E+01
A ₁₃	-0.9172E+00	-0.6727E+00	-0.4621E+00	-0.3477E+00	-0.2290E+00
A ₁₄	0.2760E+01	0.2036E+01	0.1466E+01	0.1317E+01	0.1093E+01
A ₁₅	-0.2767E+01	-0.2104E+01	-0.1595E+01	-0.1557E+01	-0.1395E+01
A ₁₆	-0.1171E+01	-0.7484E+00	-0.4066E+00	-0.2580E+00	-0.1382E+00
A ₁₇	0.4214E+01	0.2777E+01	0.1587E+01	0.1159E+01	0.7448E+00
A ₁₈	-0.4230E+01	-0.2894E+01	-0.1761E+01	-0.1418E+01	-0.9963E+00
A ₁₉	0.3353E+00	0.3309E+00	0.2960E+00	0.2412E+00	0.1726E+00
A ₂₀	-0.6635E+00	-0.7691E+00	-0.8274E+00	-0.8209E+00	-0.7333E+00
A ₂₁	0.6528E+00	0.7643E+00	0.8763E+00	0.9464E+00	0.9212E+00
A ₂₂	0.9030E+00	0.6078E+00	0.3696E+00	0.2597E+00	0.1616E+00
A ₂₃	-0.3006E+01	-0.2036E+01	-0.1263E+01	-0.1014E+01	-0.7453E+00
A ₂₄	0.3020E+01	0.2117E+01	0.1389E+01	0.1217E+01	0.9684E+00
A ₂₅	0.6265E+00	0.4529E+00	0.3049E+00	0.2312E+00	0.1575E+00
A ₂₆	-0.1894E+01	-0.1361E+01	-0.9273E+00	-0.8162E+00	-0.6684E+00
A ₂₇	0.1900E+01	0.1410E+01	0.1006E+01	0.9596E+00	0.8467E+00

(b) for the coefficient $1/F_1^h$ of an oblate spheroid

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A ₁	0.2475E+00	0.3731E+00	0.4183E+00	0.4775E+00	0.5570E+00
A ₂	0.1061E+01	0.8163E+00	0.1013E+01	0.1251E+01	0.1494E+01
A ₃	-0.1027E+01	-0.7653E+00	-0.9879E+00	-0.1314E+01	-0.1742E+01
A ₄	-0.2077E+00	-0.2363E+00	-0.1195E+00	-0.7753E-01	-0.8643E-02
A ₅	0.1083E+01	0.1213E+01	0.7260E+00	0.5787E+00	0.2176E+00
A ₆	-0.1142E+01	-0.1348E+01	-0.8803E+00	-0.7948E+00	-0.3842E+00
A ₇	-0.3322E+01	-0.1969E+01	-0.8244E+00	-0.4802E+00	-0.1666E+00
A ₈	0.1358E+02	0.8635E+01	0.4002E+01	0.2751E+01	0.1263E+01
A ₉	-0.1380E+02	-0.9282E+01	-0.4611E+01	-0.3547E+01	-0.1875E+01
A ₁₀	0.4581E+01	0.2636E+01	0.1007E+01	0.5389E+00	0.1273E+00
A ₁₁	-0.1919E+02	-0.1200E+02	-0.5287E+01	-0.3459E+01	-0.1355E+01
A ₁₂	0.1952E+02	0.1293E+02	0.6148E+01	0.4542E+01	0.2132E+01
A ₁₃	-0.2137E+01	-0.1318E+01	-0.6068E+00	-0.3800E+00	-0.1657E+00
A ₁₄	0.8491E+01	0.5541E+01	0.2725E+01	0.1968E+01	0.1036E+01
A ₁₅	-0.8625E+01	-0.5939E+01	-0.3112E+01	-0.2492E+01	-0.1473E+01
A ₁₆	-0.3611E+01	-0.2029E+01	-0.8233E+00	-0.4705E+00	-0.1797E+00
A ₁₇	0.1457E+02	0.8753E+01	0.3877E+01	0.2575E+01	0.1224E+01
A ₁₈	-0.1478E+02	-0.9378E+01	-0.4432E+01	-0.3272E+01	-0.1760E+01
A ₁₉	-0.3588E+00	-0.1832E+00	-0.9597E-01	-0.6278E-01	-0.4296E-01
A ₂₀	0.1313E+01	0.6665E+00	0.3399E+00	0.2351E+00	0.1794E+00
A ₂₁	-0.1305E+01	-0.6721E+00	-0.3430E+00	-0.2501E+00	-0.2074E+00
A ₂₂	0.2444E+01	0.1357E+01	0.5330E+00	0.2923E+00	0.9838E-01
A ₂₃	-0.9947E+01	-0.5942E+01	-0.2595E+01	-0.1684E+01	-0.7530E+00
A ₂₄	0.1009E+02	0.6365E+01	0.2971E+01	0.2152E+01	0.1103E+01
A ₂₅	0.1522E+01	0.8593E+00	0.3552E+00	0.2025E+00	0.7899E-01
A ₂₆	-0.6111E+01	-0.3683E+01	-0.1659E+01	-0.1102E+01	-0.5369E+00
A ₂₇	0.6195E+01	0.3936E+01	0.1888E+01	0.1391E+01	0.7641E+00

(c) for the coefficient F_1^P/F_1^h of a prolate spheroid

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A ₁	0.4532E+00	0.6293E+00	0.7247E+00	0.5943E+00	0.3785E+00
A ₂	0.8421E+01	0.6502E+01	0.4389E+01	0.3251E+01	0.2705E+01
A ₃	-0.8420E+01	-0.6608E+01	-0.4433E+01	-0.3239E+01	-0.2645E+01
A ₄	-0.1136E+01	-0.1269E+01	-0.1276E+01	-0.9980E+00	-0.6127E+00
A ₅	-0.5823E+01	-0.3153E+01	-0.2173E+00	0.1408E+01	0.2124E+01
A ₆	0.5821E+01	0.3285E+01	0.2511E+00	-0.1472E+01	-0.2246E+01
A ₇	-0.3563E+01	-0.2781E+01	-0.2116E+01	-0.1532E+01	-0.8868E+00
A ₈	-0.2734E-02	0.5972E-01	0.1314E+01	0.2727E+01	0.3157E+01
A ₉	-0.7784E-01	-0.1608E+00	-0.1468E+01	-0.3002E+01	-0.3414E+01
A ₁₀	0.5332E+01	0.4217E+01	0.3235E+01	0.2352E+01	0.1368E+01
A ₁₁	0.9137E+00	0.3856E+00	-0.1805E+01	-0.4126E+01	-0.4886E+01
A ₁₂	-0.8042E+00	-0.2453E+00	0.2023E+01	0.4528E+01	0.5268E+01
A ₁₃	-0.2276E+01	-0.1815E+01	-0.1400E+01	-0.1017E+01	-0.5890E+00
A ₁₄	-0.3164E+00	-0.6633E-01	0.8517E+00	0.1800E+01	0.2079E+01
A ₁₅	0.2800E+00	0.1059E-01	-0.9470E+00	-0.1982E+01	-0.2249E+01
A ₁₆	-0.3213E+01	-0.2467E+01	-0.1842E+01	-0.1323E+01	-0.7631E+00
A ₁₇	0.4912E+00	0.3136E+00	0.1208E+01	0.2382E+01	0.2739E+01
A ₁₈	-0.5551E+00	-0.4199E+00	-0.1352E+01	-0.2622E+01	-0.2957E+01
A ₁₉	0.9398E+00	0.8293E+00	0.6921E+00	0.5021E+00	0.2882E+00
A ₂₀	0.1090E+01	0.4632E+00	-0.3496E+00	-0.8577E+00	-0.1002E+01
A ₂₁	-0.1085E+01	-0.4806E+00	0.3761E+00	0.9376E+00	0.1087E+01
A ₂₂	0.2120E+01	0.1613E+01	0.1194E+01	0.8542E+00	0.4905E+00
A ₂₃	-0.6190E+00	-0.3727E+00	-0.8551E+00	-0.1561E+01	-0.1756E+01
A ₂₄	0.6643E+00	0.4470E+00	0.9548E+00	0.1724E+01	0.1902E+01
A ₂₅	0.1346E+01	0.1038E+01	0.7821E+00	0.5625E+00	0.3237E+00
A ₂₆	-0.2077E+00	-0.1403E+00	-0.5339E+00	-0.1016E+01	-0.1150E+01
A ₂₇	0.2311E+00	0.1808E+00	0.5938E+00	0.1120E+01	0.1244E+01

(d) for the coefficient F_1^p/F_1^h of an oblate spheroid

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A ₁	0.1282E+01	0.8178E+00	0.4447E+00	0.3178E+00	0.1923E+00
A ₂	0.2507E+01	0.3646E+01	0.4423E+01	0.4123E+01	0.3919E+01
A ₃	-0.2434E+01	-0.3443E+01	-0.4304E+01	-0.4025E+01	-0.3864E+01
A ₄	-0.2666E+01	-0.1739E+01	-0.9599E+00	-0.6394E+00	-0.3539E+00
A ₅	0.4175E+01	0.1874E+01	0.1309E+00	0.3056E+00	0.3431E+00
A ₆	-0.4292E+01	-0.2233E+01	-0.3775E+00	-0.5460E+00	-0.4926E+00
A ₇	-0.8009E+01	-0.4500E+01	-0.2039E+01	-0.1186E+01	-0.5645E+00
A ₈	0.2328E+02	0.1164E+02	0.3517E+01	0.2053E+01	0.9701E+00
A ₉	-0.2366E+02	-0.1272E+02	-0.4273E+01	-0.2773E+01	-0.1357E+01
A ₁₀	0.1161E+02	0.6548E+01	0.2999E+01	0.1766E+01	0.8589E+00
A ₁₁	-0.3250E+02	-0.1597E+02	-0.4532E+01	-0.2684E+01	-0.1329E+01
A ₁₂	0.3305E+02	0.1753E+02	0.5614E+01	0.3697E+01	0.1872E+01
A ₁₃	-0.5011E+01	-0.2829E+01	-0.1315E+01	-0.7713E+00	-0.3698E+00
A ₁₄	0.1438E+02	0.7155E+01	0.2233E+01	0.1330E+01	0.6456E+00
A ₁₅	-0.1461E+02	-0.7808E+01	-0.2706E+01	-0.1784E+01	-0.8965E+00
A ₁₆	-0.6947E+01	-0.3830E+01	-0.1733E+01	-0.1007E+01	-0.4851E+00
A ₁₇	0.2002E+02	0.9644E+01	0.2791E+01	0.1605E+01	0.7737E+00
A ₁₈	-0.2034E+02	-0.1054E+02	-0.3411E+01	-0.2182E+01	-0.1083E+01
A ₁₉	0.3270E+00	0.2520E+00	0.1875E+00	0.1619E+00	0.1135E+00
A ₂₀	0.9300E+00	0.8740E+00	0.6893E+00	0.2924E+00	0.1295E-01
A ₂₁	-0.9044E+00	-0.8168E+00	-0.6649E+00	-0.2817E+00	-0.2750E-02
A ₂₂	0.4629E+01	0.2530E+01	0.1126E+01	0.6481E+00	0.3084E+00
A ₂₃	-0.1365E+02	-0.6596E+01	-0.1929E+01	-0.1091E+01	-0.5109E+00
A ₂₄	0.1387E+02	0.7196E+01	0.2344E+01	0.1475E+01	0.7148E+00
A ₂₅	0.2748E+01	0.1510E+01	0.6927E+00	0.4053E+00	0.1971E+00
A ₂₆	-0.7841E+01	-0.3716E+01	-0.1076E+01	-0.6260E+00	-0.3119E+00
A ₂₇	0.7968E+01	0.4060E+01	0.1317E+01	0.8509E+00	0.4345E+00

Table 10 a,b,c,d Constants of the curve fitting for resistance coefficients $F_1^{t_1}$, F_1^p of a spheroid with $\varepsilon=0.5$ in equation (4.13) .

(a) The curve fitting constants for the resistance coefficient $1/F_1^{t_1}$ of a prolate spheroid.

(b) The curve fitting constants for the resistance coefficient $1/F_1^{t_1}$ of an oblate spheroid.

(c) The curve fitting constants for the resistance coefficient $F_1^p/F_1^{t_1}$ of a prolate spheroid.

(d) The curve fitting constants for the resistance coefficient $F_1^p/F_1^{t_1}$ of an oblate spheroid.

(a) for the coefficient $1/F_1^4$ of a prolate spheroid

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A ₁	-0.91979	-0.32314	0.090461	0.26706	0.46883
A ₂	5.7716	3.7848	2.5190	2.3757	2.0783
A ₃	-5.8416	-3.9581	-2.7091	-2.7371	-2.5773
A ₄	0.99696	0.40960	0.12578	0.066184	0.02698
A ₅	-4.0676	-1.7852	-0.58278	-0.33562	-0.14207
A ₆	4.1537	1.9305	0.67019	0.41866	0.19056
A ₇	-1.7845	-1.0963	-0.46399	-0.26504	-0.10894
A ₈	7.0563	4.6366	2.1232	1.4095	0.71034
A ₉	-7.1118	-4.9339	-2.3988	-1.7697	-0.99882
A ₁₀	2.7069	1.6629	0.69856	0.39645	0.16030
A ₁₁	-10.735	-7.0613	-3.2190	-2.1265	-1.0616
A ₁₂	10.817	7.5142	3.6387	2.6727	1.4966
A ₁₃	-1.0005	-0.61755	-0.26653	-0.15399	-0.06522
A ₁₄	3.9297	2.5897	1.2035	0.80548	0.41409
A ₁₅	-3.9295	-2.7543	-1.3591	-1.0093	-0.57958
A ₁₆	-1.9677	-1.1965	-0.50589	-0.28963	-0.12239
A ₁₇	7.7626	5.0355	2.2890	1.5116	0.76460
A ₁₈	-7.8208	-5.3493	-2.5791	-1.8899	-1.0639
A ₁₉	0.19447	0.22413	0.16789	0.12128	0.07079
A ₂₀	-0.59233	-0.78561	-0.64047	-0.53431	-0.36548
A ₂₁	0.57517	0.79844	0.68944	0.63544	0.48109
A ₂₂	1.3036	0.80180	0.35201	0.20979	0.09687
A ₂₃	-5.0653	-3.3029	-1.5306	-1.0396	-0.55684
A ₂₄	5.1040	3.5087	1.7219	1.2932	0.76486
A ₂₅	0.74386	0.46210	0.21047	0.12955	0.06428
A ₂₆	-2.8420	-1.8600	-0.87987	-0.61156	-0.34556
A ₂₇	2.8631	1.9745	0.98773	0.75618	0.46874

(b) for the coefficient $1/F_1^h$ of an oblate spheroid

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A ₁	-0.42820	-0.13192	0.11003	0.24205	0.41548
A ₂	3.4511	2.6372	2.1272	2.2340	2.1806
A ₃	-3.4639	-2.7041	-2.2388	-1.5277	-2.6631
A ₄	0.08707	-0.03414	-0.05401	-0.04314	-0.01794
A ₅	-0.30434	0.17748	0.27470	0.24649	0.13790
A ₆	0.29811	-0.20367	-0.32315	-0.31059	-0.20215
A ₇	-3.3196	-1.8738	-0.77916	-0.45863	-0.18923
A ₈	13.379	8.0619	3.6196	2.4378	1.2118
A ₉	-13.579	-8.6533	-4.1298	-3.0594	-1.7021
A ₁₀	5.0573	2.8380	1.1691	0.68289	0.27774
A ₁₁	-20.438	-12.256	-5.4666	-3.6558	-1.7968
A ₁₂	20.744	13.157	6.2422	4.5933	2.5329
A ₁₃	-1.8602	-10.601	-0.44936	-0.26707	-0.11274
A ₁₄	7.4673	4.5333	2.0669	1.4033	0.70954
A ₁₅	-7.5803	-4.8646	-2.3564	-1.7574	-0.99252
A ₁₆	-3.9483	-2.1322	-0.84761	-0.48700	-0.20323
A ₁₇	15.845	9.1357	3.9058	2.5574	1.2685
A ₁₈	-16.075	-9.7970	-4.4506	-3.2039	-1.7706
A ₁₉	-0.84169	-0.49620	-0.24539	-0.16105	-0.08660
A ₂₀	3.2444	1.9869	1.0121	0.74811	0.45882
A ₂₁	-3.2726	-2.0890	-1.1118	-0.90298	-0.60548
A ₂₂	2.5915	1.3874	0.53522	0.29714	0.11429
A ₂₃	-10.481	-6.0132	-2.5302	-1.6185	-0.76183
A ₂₄	10.632	6.4485	2.8867	2.0368	1.0769
A ₂₅	1.4838	0.78662	0.29826	0.16131	0.05917
A ₂₆	-6.0122	-3.4225	-1.4258	-0.89425	-0.40922
A ₂₇	6.0989	3.6688	1.6270	1.1267	0.58176

(c) for the coefficient F_1^p/F_1^h of a prolate spheroid

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A ₁	-1.8695	-0.99402	-0.39655	-0.19146	-0.06029
A ₂	10.978	7.8325	5.5337	4.8346	4.2938
A ₃	-11.057	-8.0712	-5.6968	-5.0046	-4.3835
A ₄	1.3857	0.59201	0.13792	0.051009	0.00851
A ₅	-5.6119	-2.5976	-0.65282	-0.27416	-0.04323
A ₆	5.6993	2.8283	0.75685	0.34560	0.05660
A ₇	-2.9046	-1.4767	-0.53102	-0.22869	-0.06313
A ₈	11.611	6.2446	2.4072	1.1839	0.39421
A ₉	-11.742	-6.6236	-2.7120	-1.4789	-0.55011
A ₁₀	4.4332	2.2501	0.80667	0.34740	0.09602
A ₁₁	-17.726	-9.5148	-3.6558	-1.7970	-0.60010
A ₁₂	17.928	10.092	4.1176	2.2445	0.83674
A ₁₃	-1.6264	-0.82329	-0.29575	-1.2851	-0.03559
A ₁₄	6.5032	3.4799	1.3385	0.66564	0.22284
A ₁₅	-6.5779	-3.6897	-1.5061	-0.83267	-0.31123
A ₁₆	-3.0413	-1.5777	-0.55637	-0.23851	-0.06421
A ₁₇	12.132	6.6720	2.5174	1.2348	0.39630
A ₁₈	-12.247	-7.0771	-2.8306	-1.5419	-0.55027
A ₁₉	0.31460	0.25987	0.16390	0.085866	0.03027
A ₂₀	0.91005	-0.81380	-0.57179	-0.35167	-0.15178
A ₂₁	0.90197	0.81680	0.61541	0.42123	0.20263
A ₂₂	2.0000	1.0417	0.37026	0.15950	0.04320
A ₂₃	-7.9759	-4.4043	-1.6747	-0.82507	-0.26583
A ₂₄	8.0507	4.6713	1.8828	1.0299	0.36981
A ₂₅	1.1205	0.58197	0.20712	0.089946	0.02456
A ₂₆	-4.4684	-2.4587	-0.93463	-0.46475	-0.15104
A ₂₇	4.5109	2.6065	1.0489	0.57995	0.20948

(d) for the coefficient F_1^p/F_1^h of an oblate spheroid

	H/2=1.25	H/2=1.5	H/2=2.0	H/2=3.0	H/2=5.5
A ₁	-0.59520	-0.39998	-0.25425	-0.14823	-0.05523
A ₂	5.8265	5.2191	4.8321	4.5866	4.2525
A ₃	-5.8536	-5.2285	-4.8558	-4.6773	-4.3161
A ₄	-0.89831	-0.53304	-0.19529	-0.07860	-0.02323
A ₅	3.5850	2.2861	0.91469	0.40515	0.14832
A ₆	-3.5863	-2.4427	-1.0628	-0.50557	-0.21103
A ₇	-7.1386	-3.4475	-1.1209	-0.46159	-0.12056
A ₈	28.698	14.774	5.1628	2.3980	0.74226
A ₉	-29.049	-15.815	-5.8928	-2.9963	-1.0349
A ₁₀	10.856	5.2437	1.7007	0.69830	0.18209
A ₁₁	-43.642	-22.475	-7.8356	-3.6286	-1.1220
A ₁₂	44.177	24.062	8.9461	4.5351	1.5651
A ₁₃	-4.0132	1.9472	-0.63714	-0.26211	-0.06872
A ₁₄	16.133	8.3437	2.9347	1.3603	0.42225
A ₁₅	-16.331	-8.9304	-3.3498	-1.6984	-0.58803
A ₁₆	-7.3590	-3.4672	-1.1015	-0.45009	-0.11605
A ₁₇	29.601	14.866	5.0705	2.3398	0.71283
A ₁₈	-29.989	-15.922	-5.7813	-2.9230	-0.99299
A ₁₉	-1.1065	-0.58534	-0.24967	-0.12346	-0.03862
A ₂₀	4.1101	2.2183	0.95641	0.54486	0.19330
A ₂₁	-4.1654	-2.3336	-1.0426	-0.65834	-0.25401
A ₂₂	4.7925	2.2445	0.70567	0.28613	0.07279
A ₂₃	-19.278	-9.6264	-3.2518	-1.4905	-0.44912
A ₂₄	19.532	10.314	3.7115	1.8650	0.62731
A ₂₅	2.6473	1.2343	0.38517	0.15430	0.03894
A ₂₆	-10.648	-5.2950	-1.7771	-0.80435	-0.24087
A ₂₇	10.788	5.6740	2.0304	1.0067	0.33696

Table 11 Comparison of values of the diffusive permeability ω and reflection coefficient σ for a spherical solute particle at the different channel widths.

s^*	σ			$\omega \bar{V}_s / L_p$		
	ϵ_s	ϵ_{s1}	error %	ϵ_s	ϵ_{s1}	error %
0.182	*	0.0762		*	0.0134	
0.333	0.2150	0.2162	0.56%	0.0302	0.0302	0.00%
0.500	0.4130	0.4103	0.65%	0.0407	0.0408	0.24%
0.667	0.6280	0.6222	0.92%	0.0377	0.0382	1.32%
0.800	0.7970	0.7920	0.63%	0.0259	0.0263	1.54%

where ϵ_s are solutions of Ganatos et al (1980) and ϵ_{s1} are present solutions. The asterisk indicates the data is not available in this position.

by Ganatos et al 1980, the present results with the general approach for a spheroidal solute particle have a maximum relative error of 1.54% at the worst position. Tables 12 (a,b) show these computed values of ω and σ for a spheroidal solute particle in a channel. From the obtained results we have the following observations

(1) For a given aspect ratio $\varepsilon=0.1$ or 0.5 the exact shape of the spheroid, (oblate or prolate), does not have a major effect on the computed diffusive permeability and reflection coefficient. That is because the semi-major axis of the spheroid is used to define the relative particle size to the channel width. It has been shown that the force and torque coefficients of an oblate and a prolate spheroid having the same ratio of semi-minor to semi-major axes exhibit the same qualitative behavior.

(2) Both the diffusive permeability and the reflection coefficient increase as the aspect ratio of the spheroid increases, if the aspect ratio is defined as the ratio of semi-minor to semi-major axes of the spheroid.

4.3 Concluding remarks

In the present research a strong hydrodynamic interaction theory which includes the effect of the shape and orientation of the solute particle has been applied to the study of osmotic flow in an intercellular cleft. The conclusions drawn on the diffusive permeability and reflection coefficient are based on theoretical prediction and need to be verified experimentally. If necessary the torque acting on the spheroid can be included in the derivation and a modified lubrication theory can be developed for the region which the boundary integral method cannot reach as part of a future research effort.

Table 12 a,b The theoretical prediction of the diffusive permeability ω and reflection coefficient σ for a spheroidal solute particle at the different channel widths

(a) for a spheroid with $\epsilon=0.1$

	σ		$\omega \bar{V}_s / L_p$	
s^*	$\epsilon_o=0.1$	$\epsilon_p=0.1$	$\epsilon_o=0.1$	$\epsilon_p=0.1$
0.182	0.0545	0.0583	0.00015	0.00015
0.333	0.1667	0.1776	0.00037	0.00040
0.500	0.3399	0.3589	0.00055	0.00062
0.667	0.5480	0.5708	0.00058	0.00069
0.800	0.7286	0.74820	0.00045	0.00056

(b) spheroid with $\epsilon=0.5$

	σ		$\omega \bar{V}_s / L_p$	
s^*	$\epsilon_o=0.5$	$\epsilon_p=0.5$	$\epsilon_o=0.5$	$\epsilon_p=0.5$
0.182	0.0599	0.0634	0.0036	0.0037
0.333	0.1783	0.1886	0.0085	0.0089
0.500	0.3558	0.3732	0.0123	0.0132
0.667	0.5644	0.5848	0.0124	0.0135
0.800	0.7421	0.7600	0.0093	0.0102

REFERENCES

Anderson, J.L. 1981 Configurational effect on the reflection coefficient for rigid solutes in capillary pores. J. Theor. Biology **90**, 405-426.

Anderson, J.L. & Malone, D.M. 1974 Mechanism of osmotic flow in porous membranes. Biophysical J. **14**, 957-982.

Anderson, J.L. & Quinn, J.A. 1974 Restricted transport in small pores. Biophysical J. **14**, 130-150.

Bean, C.P. 1972 In membranes - A series of advances. G. Eisenman, Editor. Marcel-Dekker, Inc. New York, Vol. 1

Brenner, H. & Condiff, D.W. 1972, 1974 Transport mechanics in system of orientable particles, III, IV. J. Colloid Interface Sci. **41**, 228-274, **47**, 199-264.

Brenner, H. & Gaydos, L.J. 1977 The constrained Brownian movement of spherical particles in cylindrical pores of comparable radius. J. Colloid Interface Sci. **58**, 312-356.

Ganatos, P., Weinbaum, S., Fischbarg, J. & Liebovitch, L. 1980 A hydrodynamic theory for determining the membrane coefficients for the passage of spherical molecules through an intercellular cleft. Adv. in Bioengr. 193-196.

Levitt, D. 1975 General continuum analysis of transport through pores. Biophysical J. **15**, 533-551.

Yan, Z.Y., Weinbaum, S. & Pfeffer, R. 1986 On the fine structure of osmosis at the entrance and exit of biological-membrane pores. J. Fluid Mech. **162**, 415-438.