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MULTIPHOTON IONIZATION PROCESSES IN STRONG LASER FIELDS

City University of New York PH.D. 1982

MULTIPHOTON IONIZATION PROCESSES IN STRONG LASER FIELDS

by

PREDRAG KRSTIC

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

MULTIPHOTON IONIZATION PROCESSES IN STRONG LASER FIELDS

by

PREDRAG KRSTIC

Adviser: Professor Marvin H. Mittleman

Multiphoton ionization of hydrogen in ultrastrong laser fields is studied. The previous calculations of this process yield differing result for the transition rate. We show the relations between them and difficulties with each of them. One difficulty is that the finite spatial and time extent of the laser field has been omitted. It is also found that a laser field, which is sufficiemtly intense to be labeled ultrastrong, makes the electron move relativistically so that it becomes necessary to use Volkov states to describe the electron in the laser field. The transition rate is obtained, using a CO laser as an example, and it is found that the transition rate rises as the laser intensity rises. This is a consequence of the use of relativistic kinematics and is not true non-relativistically.

We also discuss the multiple peaks observed in the energy spectrum of electrons resulting from multiphoton ionization of atoms by lasers. When the laser intensity is large enough for the ponderomotive force to result in appreciable broading of the peaks we

show the shape of the broadened peaks contains useful information. We show that the multiphoton ionization probability as a function of laser intensity can be obtained but that the free-free cross sections, which are in principle also obtainable, are probably not obtainable in practice.

Finally, we describe the theory of the absorption of more than minimum number of photons needed to ionize an atom by an intense laser. The basic approximation used is that the atom is adiabatically deformed by the laser and an impulsive interaction then results in multiphoton absorption. In our first calculation we allow only one resonant excited state to be included in the adiabatic deformation. In our second we also allow the lowest energy continuum to be included. The two results are then compared.

A C K N O W L E D G E M E N T S

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L I S T O F I L L U S T R A T I O N S

INTRODUCTION

In the last twenty years, owing to the development of high intensity lasers, there is increased interest, both theoretical and experimental, in multiphoton-induced atomic transitions. These are electronic transitions which involve the net absorption, emission or scattering of more than one photon. In particular, multiphoton ionization is a process of light-induced ionization of an atom if the energy of the photons is smaller than the ionization potential of the atom. Multiphoton processes are important in applications such as isotope separation, laser-induced fusion, gas breakdown, laser-induced chemical reactions, etc. They also provide a new tool for high resolution exploaration of excited atomic states. Multiphoton transitions generally require large radiation intensities for their observation. This can be seen on the basis of a simple perturbation theory argument. If the two-photon absorption is a two-step process, through a resonant intermediate state, then the arrival of the second photon must occur in the life-time τ of the intermediate state. This is typically $\tau \sim 10^{-8}$ sec, and the linear dimension of the atom is $a_0 \sim 10^{-8}$ cm. Then the necessary photon flux is of the order $\frac{1}{a_0^2 \tau} \sim 10^{24}$ cm^{-2} sec^{-1}. If the two-photon absorption is via the virtual intermediate state, with typical life-time of 10^{-15} sec then the required photon flux is considerable higher, of the order 10^{31} cm^{-2} sec^{-1}. From these arguments it is clear that the necessary photon flux for N-photon transition increases with increase of N. In other words, multiphoton processes require electromagnetic fields highly occupied with photons. But, in the limit of such high photon occupation numbers, the electromagnetic field can be described classically

$$A(\vec{r},t)=\frac{1}{2}\sum_{\vec{k},\lambda}\left(\vec{a}_{\vec{k},\lambda}\exp i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t-\phi_{\vec{k}\lambda})+c.c.\right) \quad (1)$$

$|\vec{a}_{\vec{k}\lambda}|$ is the amplitude of the vector potential in the mode $\vec{k}$ and in the direction of unit vector of polarization λ . The connection with the quantum-mechanical description of the radiation field is the relation ($\hbar = c = 1$)

$$|\vec{a}_{\vec{k}\lambda}|=\left(\frac{8\pi N_{\vec{k}\lambda}}{\omega_{\vec{k}}V}\right)^{1/2} \quad (2)$$

$N_{\vec{k},\lambda}$ is the occupation number of photons with the polarization λ in the mode with frequency $\omega_{\vec{k}}$. $\phi_{\vec{k},\lambda}$ is the phase factor which, for a single mode laser, can be absorbed into the choice of the time origin.

The theotetical, nonrelativistic treatments of an atom in the electromagnetic field start from the Schrodinger equation

$$\left(i\frac{\partial}{\partial t}-H\right)\Psi=0 \quad (3)$$

In the interaction representation

$$H=\frac{\left(\vec{P}_N-ze\vec{A}(\vec{r}_N,t)\right)^2}{2M}+\sum_{i=1}^{Z}\frac{\left(\vec{P}_{ei}+e\vec{A}(\vec{r}_{ei},t)\right)^2}{2m}+V(\vec{r}_{e1},\vec{r}_{e2},\cdots,\vec{r}_{eZ},\vec{r}_N) \quad (4)$$

$\vec{P}_N$ and $\vec{P}_{ei}$ are momentum operators of the nucleus and the electrons, respectively; Z is the number of electrons in the atom and, $\vec{r}_N$ and $\vec{r}_{ei}$

are the radius-vectors of position of the nucleus and the electrons, respectively. If we introduce the transformation to the center-of-mass coordinates $\vec{r}_i$ and $\vec{\rho}$

$$\vec{\rho} = \frac{M\vec{r}_N + \sum_{i=1} m\vec{r}_{ei}}{M_A}$$

$$\vec{r}_i = \vec{r}_{ei} - \vec{r}_N \qquad (5)$$

$$M_A = M + Zm$$

and neglect the terms which are $\frac{m}{M}$ times smaller than the leading terms, we get

$$H_{CM} = \frac{P_\rho^2}{2M_A} + \frac{1}{2\mu}\sum_{i=1}^{Z}\left(\vec{P}_i + e\vec{A}(\vec{\rho}+\vec{r}_i,t)\right)^2 + V(\vec{r},\vec{\rho}) \qquad (6)$$

μ is the reduced mass of the electron, $\mu = \frac{mM}{m+M}$. It is consistent with the approximations made above to replace the reduced mass by the mass of the electron m, and the mass of the atom M_A by the mass of the nucleus M. In the new coordinates, the vector-potential $\vec{A}$ is a function of the coordinates through the factor $\exp\pm i\vec{k}\cdot(\vec{\rho}+\vec{r}_i)$. If $\vec{r}_i$ is limited to the size of the atom in relevant matrix element, as often is the case, then in the range of optical frequences

$$\langle \vec{k}\cdot\vec{r}_i \rangle \ll 1 \qquad (7)$$

We set the dipole approximation by expanding in powers of $\vec{k}\cdot\vec{r}_i$ and

keeping only the leading term $\exp(\pm i\vec{k}\cdot\vec{\rho})$. The center-of mass coordinate will couple to the laser through the remaining exponential factor, so that the motion of the atom as a whole can be affected by the laser. The effect of this residual coupling to the motion of the atom is often small because of the large atomic mass. If $\vec{\rho}$ is a slowly varing function of time, its effect is a slow drift of the mode phases (Doppler shift). It can be removed by the redefinition of the phases of the modes. Neglecting that coupling, the kinetic energy of the center-of-mass can be dropped from H_{CM} and $\vec{\rho}$ can be fixed at the origin, $\vec{\rho}=0$. In that case, in the center-of-mass coordinates and in dipole approximation, the Hamiltonian in the Schrodinger equation (3) can be written as

$$H_I = \frac{1}{2m}\sum_{i=1}^{Z}\left(\vec{P}_i + e\vec{A}(t)\right)^2 + V(\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_Z) \qquad (8)$$

Any physical observable is invariant to the arbitrary transformation of the phase of the wave function (gauge invariance). The wave function can be transformed by the unitary transformation

$$\Psi_I = e^{-i\phi_j}\Psi_j \qquad (9)$$

The new Ψ_j satisfies the transformed Schrodinger equation

$$\left(i\frac{\partial}{\partial t} - H_j\right)\Psi_j = 0 \qquad (10)$$

with

$$H_j = e^{i\Phi_j} H_1 e^{-i\Phi_j} - \dot{\Phi}_j \tag{11}$$

We will refer to H_1 as the Hamiltonian in the first gauge. The choice of Φ_j which removes A^2 from H_1 is

$$\Phi_2 = \frac{Ze^2}{2m} \int^t dt' A^2(t') \tag{12}$$

This leads to

$$H_2 = \sum_{i=1}^{Z} \frac{P_i^2}{2m} + \frac{e}{m} \sum_{i=1}^{Z} \vec{A}(t)\cdot\vec{P}_i + V \tag{13}$$

H_2 is usually refered as the Hamiltonian in $\vec{p}\cdot\vec{A}$ gauge. The coupling of the atom with the electromagnetic field is described by the term

$$H_2' = \frac{e}{m} \sum_{i=1}^{Z} \vec{A}(t)\cdot\vec{P}_i \tag{14}$$

If $V = 0$, then the solution of eq. (10) describes the motion of a free electron in the radiation field. The corresponding wave function in $\vec{p}\cdot\vec{A}$ gauge is

$$\chi_{\vec{q}} = \exp i\left(\vec{q}\cdot\vec{r} - \frac{q^2}{2m}t - \frac{e}{m}\int^t \vec{q}\cdot\vec{A}(t')dt'\right) \tag{15}$$

The choice

$$\phi_3 = e \sum_{i=1}^{Z} \vec{r}_i \cdot \vec{A}(t) \tag{16}$$

leads to

$$H_3 = \sum_{i=1}^{Z} \frac{p_i^2}{2m} + e \sum_{i=1}^{Z} \vec{r}_i \cdot \vec{E}(t) + V \tag{17}$$

where $\vec{E} = -\dot{\vec{A}}$ is the classical electric field. The term that represents interaction of an atom and the electromagnetic field is here

$$H_3' = e \sum_{i=1}^{Z} \vec{r}_i \cdot \vec{E}(t) \tag{18}$$

H_3 is known as the Hamiltonian in $\vec{r} \cdot \vec{E}$ gauge. The wave function of a "free" electron in the radiation field in this gauge is

$$\chi_{\vec{q}} = \exp i\left((\vec{q} + e\vec{A}(t)) \cdot \vec{r} - \frac{1}{2m} \int^{t} (\vec{q} + e\vec{A}(t'))^2 dt' \right) \tag{19}$$

The third choice ϕ_4, which leads to the so called Kramers gauge, is

$$\phi_4 = \frac{e}{m} \sum_{i=1}^{Z} \int^{t} \vec{p}_i \cdot \vec{A}(t')\, dt' + \frac{Ze^2}{2m} \int^{t} A^2(t')\, dt' \tag{20}$$

The coordinate of a classical electron moving in the electromagnetic field with no other interactions is

$$\vec{\alpha}(t) = \frac{e}{m} \int^{t} dt' \vec{A}(t') \tag{21}$$

Then H_4 , induced by ϕ_4, can be written in the form

$$H_4 = \sum_{i=1}^{Z} \frac{p_i^2}{2m} + V(\vec{r} + \vec{\alpha}(t)) \tag{22}$$

and so, the transformation generated by ϕ_4 is a space translation to an accelerated frame of reference.

If any problem of the calculation of a physical observable, defined out of the laser, is solved exactly in any of the mentioned gauges, the results will be exactly the same. Only if approximations are made, the results can be gauge dependent.

In a typical experiment of atomic transitions induced by a laser the atom is initially in some atomic state outside the laser field. The laser-atom interaction is then switched on, after some time switched off, and the atomic state is observed again. The switching process can be accomplished either by turning the laser on and off, or by moving the atom through the laser beam. In each case, the switching process will almost always take place slowly on the atomic time scale. That is, it is realistic to suppose that the atom will adjust adiabatically to the laser field, as it rises from zero to its maximum and then goes back to zero.

The Hamiltonian H_j can be written in the form

$$H = H_0 + H' \tag{23}$$

where H_0 is the "bare" atomic Hamiltonian in the absence of the laser field, and H' represents the laser-atom interaction. The index of the gauge was dropped in (23). The "bare" atomic wave functions are eigen-states of H_0, which correspond to eigenvalues W_n

$$(W_n - H_0)u_n = 0 \;,\; u_n(\vec{r},t) = u_n(\vec{r})e^{-iW_n t} \tag{24}$$

The amplitude for the transition into a final state u_f is

$$\tau_{fi} = (S-1)_{fi} = -i\langle u_f^{(-)}, H'\Psi_i^{(+)}\rangle \tag{25}$$

Here $\Psi_i^{(+)}(\vec{r},t)$ is the total wave function of H, which satisfies

$$\left(i\frac{\partial}{\partial t} - H\right)\Psi_i^{(+)} = 0 \tag{26}$$

with the initial condition

$$\Psi_i^{(+)}(-T) = u_0(\vec{r},-T) \tag{27}$$

T is some large time such that for $t \leq -T$ the laser-atom interaction was off, and u_0 is the initial state of the atom. $u_f^{(-)}$ is the eigen-state of H_0, with ingoing boundary conditions. Equivalently, we can write

$$\tau_{fi} = (S-1)_{fi} = -i\langle \Psi_f^{(-)}, H'u_i\rangle \tag{28}$$

where $\Psi_f^{(-)}$ is the total wave function of H, which satisfies eq. (26), such that

$$\Psi_f^{(-)}(T) = U_f(\vec{r}, T) \tag{29}$$

and for $t \geq T$ the laser-atom interaction is off. The brackets in (25) and (28) denote both space and time integration. The time integration extends at least from $-T$ to T. This was the basic framework in which the most of the theoretical attacks on the multiphoton processes have been done.

The atomic unit for the electric field is

$$E_0 = \frac{e}{2a_0^2} = 25.7 \times 10^8 \frac{V}{cm} \tag{30}$$

From that we can construct the atomic unit for the intensity of the laser field

$$I_0 = \frac{c E_0^2}{4\pi} \approx 1.75 \times 10^{16} \frac{W}{cm^2} \tag{31}$$

If we estimate the laser-atom interaction in the dipole approximation as

$$e E \langle \vec{r} \rangle \approx e E a_0 = R_y \left(\frac{I}{I_0}\right)^{1/2} \tag{32}$$

where I is the laser intensity, it is obvious that for $I \ll I_0$ the interaction energies of the laser with the atom are much smaller than the separation of the typical atomic energy levels.That allows for the po-

sibility of a straightforward use of the non-resonant perturbation theory in the laser-atom interaction H'. (It is assumed that no resonance between the laser frequency and atomic transitions will occur). We can expand the full wave function $\Psi_i^{(+)}$ in (25) in terms of the full Green's function of the problem

$$\Psi_i^{(+)} = u_0 + G^{(+)} H' u_0 \tag{33}$$

where

$$\left(i\frac{\partial}{\partial t} - H\right) G^{(+)} = 1 \tag{34}$$

and $G^{(+)}$ is defined to vanish at initial time. Furthermore, $G^{(+)}$ can be expanded in terms of the unperturbed Green's function $G_0^{(+)}$ as

$$G^{(+)} = G_0^{(+)}\left(1 + \sum_{n=1}^{\infty} \left(H' G_0^{(+)}\right)^n\right) \tag{35}$$

where

$$\left(i\frac{\partial}{\partial t} - H_0\right) G_0^{(+)} = 1 \tag{36}$$

$G_0^{(+)}$ can be written in terms of the unperturbed (or "bare") eigenstates u_n of the atom, as

$$G_0(\vec{r},t;\vec{r}',t') = -i\,\theta(t-t') \sum u_n \rangle e^{-(\eta + i W_n)(t-t')} \langle u_n \tag{37}$$

As is well known, a single photon transition can be described, in the lowest order of the perturbation theory, in terms of the matrix element

$$\langle u_f | H' | u_0 \rangle \tag{38}$$

i.e. in terms of the zeroth order term in the expansion (35). The description of the N-photon process is obtained, in the lowest, nonvanishing order, from the N-th order term of the perurbation expansion, and is described by the matrix element

$$\langle u_f | H' G_0^{(+)} (H' G_0^{(+)})^{N-1} u_0 \rangle \tag{39}$$

The transition rate and the cross section, which are proportional to the square of the transition amplitude, will be obviously proportional to the N-th power of the laser intensity. So, N photon ionization rate can be written in the form [1]

$$W_{fi}^{(N)} = \sigma_N I^N \tag{40}$$

If I is the photon flux in cm^{-2} sec^{-1} , then σ_N is the total generalized cross section (in cm^{2N} sec^{N-1})

$$\sigma_N = \frac{(2\pi\alpha_F)^N}{4\pi^2} m p \omega^{N-1} \int d\Omega_p | K_{f0}^{(N)} |^2 \tag{41}$$

p is the momentum of the outgoing photo-electron, $\alpha_F = (137)^{-1}$, and integration is over all angles of propagation of the photo-electron; $K_{f0}^{(N)}$ is N-th order matrix element

$$K_{fo}^{(N)} = \sum_{i_{N-1}} \sum_{i_{N-2}} \cdots \sum_{i_2} \sum_{i_1} \langle f | \hat{\varepsilon}\cdot\vec{r} | i_{N-1} \rangle \langle i_{N-1} | \hat{\varepsilon}\cdot\vec{r} | i_{N-2} \rangle \cdots \langle i_2 | \hat{\varepsilon}\cdot\vec{r} | i_1 \rangle \times$$

$$\times \langle i_1 | \hat{\varepsilon}\cdot\vec{r} | 0 \rangle \times \left[\left(E_0 - E_{i_{N-1}} + (N-1)\omega \right) \cdots \left(E_0 - E_{i_1} + \omega \right) \right]^{-1} \tag{42}$$

For the low laser intensities and off resonance, and for the relatively low photon multiplicities, the perturbation theory up to the first nonvanishing order is in the satisfactory agreement with the carefully done experiments on multiphoton ionization. But even then, the calculation includes N-1 summations (for the photon multiplicity N), and the various methods have been developed for is [2]

Certain aspects of the multiphoton, off-resonant ionization such as angular distribution of the photo-electrons, can be discussed directly from (41). So, for N-photon ionization of the S state, the differential cross section with a linearly polarized light behaves as [3]

$$\frac{d\sigma_N^{(L)}}{d\Omega} \sim \sum_{n=0}^{N} M_L^{(N)} \cos^{2n+j}\theta \tag{43}$$

θ is the angle between thelaser electric field and the momentum of the photo-electron; $j = 0$ for N even, and $j = 2$ for N odd. For the circularly polarized light

$$\frac{d\sigma_N^{(C)}}{d\Omega} \sim \sin^{2N}\theta \tag{44}$$

Then, for N odd, in the case of the linearly polarized light, and for every N, in the case of the circular polarization, $\frac{d\sigma_N}{d\Omega}=0$ if $\theta=\frac{\pi}{2}$. That says that electrons tend to be emitted in the direction of the quantization axis. Much simpler angular dependence with the circularly polarized light is the consequence of the dipole selection rules. So, for the circular polarization and an initial S state, for an absorption, we have $\Delta M=1$, and so $\Delta L=1$. Then, the only open channel, allowed by the dipole selection rules is $L=M=N$, for N photon ionization. In the case of the linear polarization, in every single absorption of a photon, the angular momentum of the electron is raised or lowered by 1. The final electron state is then composed of all $L\leq M$. The same reasoning leads us to expect that linearly polarized light will be much more effective in multiphoton ionization than the circularly polarized one. With the linearly polarized light, photo-electrons will have the statistical distribution of angular momentum in the final state. Photo-electrons, produced by the circularly polarized light, will have very high angular momentum ($L=N$). But the high angular momentum components of the final wave function have the small amplitudes in the vicinity of the atom, which reduces the matrix element of the transition and yields smaller transition rates [6].

De Witt [4] compared perturbation theory results for linear and circular polarization of light, in the case of hydrogen. He found that for the small photon multiplicities (up to $N=22$), ionization by the circularly polarized light is greatly exceeded by the one with the linearly polarized light, Fig. 1. This conclusion is not in agreement with some experimental results for low photon multiplicities. For exam-

ple, Fox at al [68] obtained that 3 photon ionization of cesium with circularly polarized light is about twice as effective as with linearly polarized light.

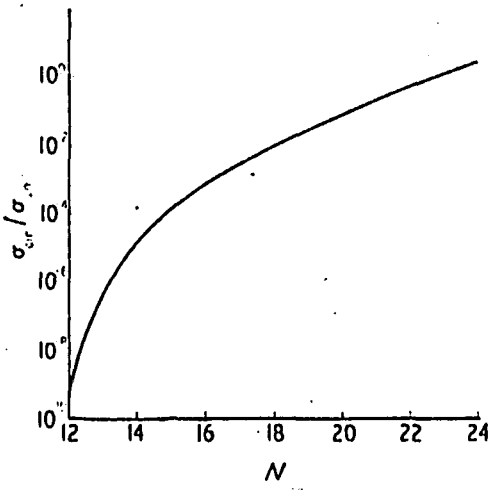

Fig. 1

Fig. 2 gives the measured dependences of the 13 photon ionization rates of krypton, on the laser intensities in the range of $\sim 10^{13}$ W/cm^2 and off resonance [5]. The ionization rates obey the law $\sim I^{13}$, as is expected from the perturbation theory for this case, and that was not affected by the light polarization value. But, the absolute values of the ionization rate were very different, showing the linear polarization of the light as nearly as 70 times more effective than the circular one.

One can pursue the first nonvanishing order perturbation theory further, to account for the distortion of the atomic states by the laser. That distortion causes changes in the wave functions, shifts in energy eigen-states and induced width of the states, since the laser can induce the transitions between the states. In the language of QED, the shifts of the states arise from the fact that an atomic state absorbs and reemits a large number of photons when it is exposed to the

radiation field, and before the transition is made into or out the state. Such emissions and absorptions are virtual (i.e. the photons remain absorbed for a very short time of the order of 10^{-15} sec) and the atom remains in the same state, while their net effect is to modify the energy of the state. The observed transitions then take place between these "dressed" states.

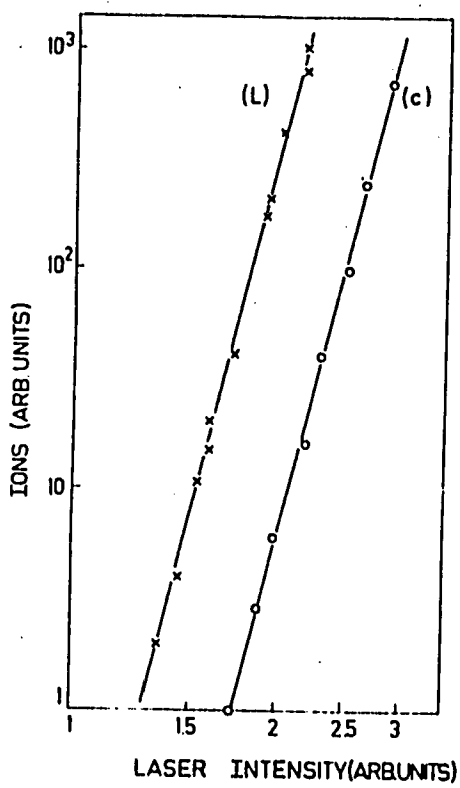

Fig. 2

The calculation of the "dressing" of the states is complex in practice, even for simple atoms. For example, one can calculate the ac Stark shifts [7], which are in the lowest order proportional to the mean value of the square of the laser electric field, by constructing the dynamic polarizabilities of the states. As the wave functions are known exactly only for hydrogen, even for helium such calculations must use some approximation of atomic structure [8]. That is the main reason that hydrogen is often the candidate for theoretical analysis. It

can furnish the clear information on the multiphoton processes and provide a correct starting point for the study of more complicated atoms. Hydrogen is not always a convenient candidate for the experimental work.

Resonant multiphoton processes exibit large cross sections and so a lot of theoretical and experimental work have been done in them. These studies showed that many results characteristic of off resonance perurbation theory are simply invalid. Furthermore, for sufficiently high laser intensities, the states that are otherwise nonresonant may become resonant, due to the energy shift of the dressed states. The theoretical treatment of the resonant multiphoton ionization usually uses the fact that the resonant state (or states) is a dominant one among all intermediate states. So, one can partition the Hilbert space of atomic states into the states of interest (initial state, resonant states, the final state), while the influence of the rest of the states is treated in the ordinary perturbation theory. For example, for the resonant two-photon ionization, through a resonante intermediate state, one can define the projection operator

$$P = |u_0\rangle\langle u_0| + |u_1\rangle\langle u_1| + \int d^3\vec{q}\, |u_{\vec{q}}^{(+)}\rangle\langle u_{\vec{q}}^{(+)}| \qquad (45)$$

which projects onto the Hilbert space of the states of interest. The complement of P is Q = 1-P. Then the Schrodinger equation of the problem can be written as a pair of the coupled equations [9],[16]

$$P\left(i\frac{\partial}{\partial t} - H\right)P\Psi = PHQ\Psi \qquad (46a)$$

$$Q\left(i\frac{\partial}{\partial t}-H\right)Q\Psi=QHP\Psi \tag{46b}$$

where has been used $Pi\frac{\partial}{\partial t}Q=0$. If one define a causal Green's function with outgoing boundary conditions by

$$Q\left(i\frac{\partial}{\partial t}-H\right)QG_Q^{(+)}=Q \tag{47}$$

then it can be used to obtain a formal solution of (46b)

$$Q\Psi=QG_Q^{(+)}QHP\Psi \tag{48}$$

If this is substituted back into (46a), the equation for $P\Psi$ is obtained

$$P\left(i\frac{\partial}{\partial t}-H-HQG_Q^{(+)}QH\right)P\Psi=0 \tag{49}$$

That equation contains the effective Hamiltonian

$$H_{eff}=P\left(H+HQG_Q^{(+)}QH\right)P \tag{50}$$

but, as the resonant states are excluded from $G_Q^{(+)}$ by the Q operation, we can safely use a perurbation method described earlier, in powers of the laser field. Particularly, to the zero'th order in atom-laser inter-

action, one can use the Green's function (37), where the initial, resonant and final states are excluded from the sum. This method will be discussed in more detail in Part III, in the context of the resonant ionization into multiple continua.

Many peculiarities were pointed out in connection with resonant multiphoton ionization. We can expect to get a sharp increase in the multiphoton ionization cross section when one of the factors in the denominator of the matrix element (42) tends to zero. Then the shift and the width of the corresponding resonant state, which are laser intensity dependent, become very important. As a consequence, it could be expected that a large deviation from the law I^N for the N photon ionization rate would occur. The result would also depend on the detuning. Further, the angular distribution of the photoelectrons can be changed by some relaxation mechanisms of the resonant state [10]. Studing these angular distributions can give the information about the particular relaxation mechanisms.

It was shown [11], in an example of the two-photon ionization of the alkali atoms via an intermediate resonant state, that the frequently used f-splitting scheme could be inadequate if the laser bandwidth is sufficiently wide to excite more than one hyperfine level of the resonant state.That could result in interference in ionization from each of these levels.

The calculations are mostly done in the dipole approximation, for the reason that the multipole expansion is very rapidly converging one. But, it was shown [12] on the example of two-photon ionization of litium, that the dipole approximation is not allways valid in a resonant multiphoton ionization. It happens that at some frequency of the

laser the dipole transition is not resonant to any intermediate state, and at the same time, the quadrupole transition into some excited state is resonant. So, in the limited frequency range, the quadrupole channel will give the dominant contribution to the ionization rate.

There are not many experiments on multiphoton ionization processe, the reasons certainly being the difficulties with which such experiments are done, as well as the uncertainty of the results. The most serious problems are the photon correlation effects, the change of the interaction volume with laser intensity, instrumental saturation and gas breakdown.

Early experiments on multiphoton ionization showed that the picture of the multiphoton process was not as simple as given by the perturbation theory, even if there were no obvious resonances and the laser intensities were relativelly low. There is another class of experiments, for example by Evans at al [13] , who measured the probability of the three photon ionization of cesium, using a ruby laser, in the range of photon fluxes $4\times10^{27} \div 3\times10^{29}\ cm^{-2}sec^{-1}$. The laser frequency was near the resonance with one (9D) bound state energy. The ionization probability as a function of the photon flux F is shown at Fig. 3. For the photon fluxes less than $3\times10^{28}\ cm^{-2}sec^{-1}$ the slope of the graph (in log-log scale) is ≈ 3, which is expected from the perturbation theory. The variation of the probability over the small range ($5\times10^{28} \div 7\times10^{28}\ cm^{-2}sec^{-1}$) is such that it decreases as F increases. After that, through a certain region ($\lesssim 10^{29}\ cm^{-2}sec^{-1}$), the probability increases as F^3 , and then the slope decreases, being constant over a small region. Then the slope slowly increases. Up to the photon fluxes of $3\times10^{28}\ cm^{-2}sec^{-1}$ the energy of the 9D level stays far enough from resonance, so

that the perturbation theory is applicable, giving the correct result. With the increase of the laser intensity, the energy of the "dressed" 9D state is shifted toward the resonance. That means that the relevant factor ($E_{9D} - E_o - n\omega$) in the denominator of the matrix element (42) becomes small, which increases the transition rate, but also requires the more exact treatment of the term. Including the shift and the width of the level it leads to the form ($E_{9D} - E_o - n\omega$)$^2 + \Gamma_{9D}^2$ of the mentioned denominator factor in the transition rate (E_{9D} is the shifted level).

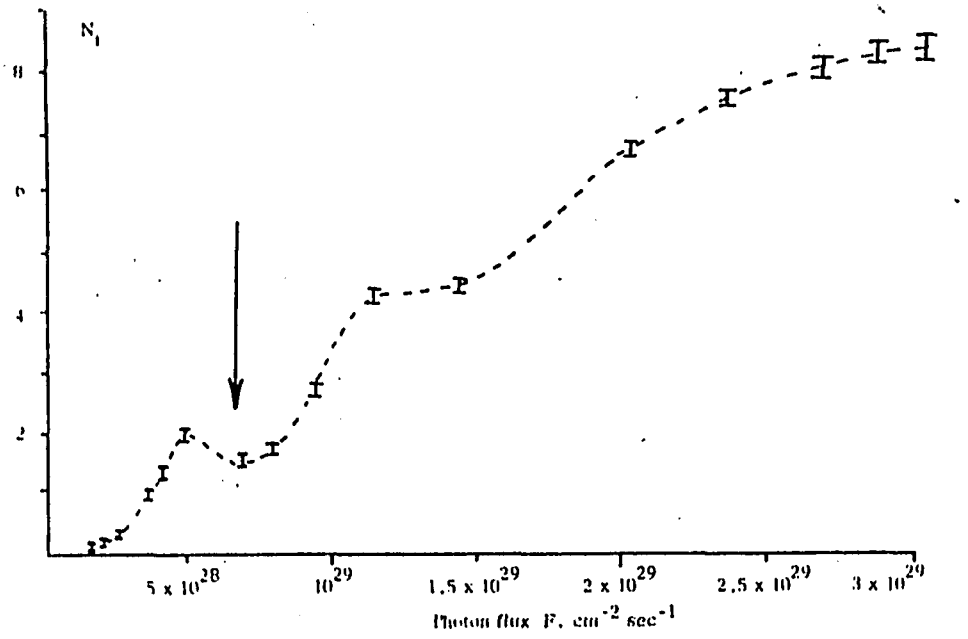

Fig. 3

After passing the minimum, that factor increases with the laser intensity (or photon flux F). That can compensate the increase with I^N (N = 3) in the ionization rate (40), causing a small dip at Fig. 3. The fact that the counted number of ions remains constant over the region of F can be explained by the complex distribution of the light intensity in the focus. For example, Chin [14] measured the transverse spatial distribution of the photon flux at the focus of the ruby

laser, for the photon fluxes about 10^{29} $cm^{-2}sec^{-1}$. He found the existence of multiple foci, i.e. more than one region where the light intensity has a local maximum. As the laser intensity increases in Evans's experiment, the ionization saturation first occurs at the primary focus, when the ionization rate reaches the value $\approx \frac{1}{\tau}$, τ is the duration of the laser pulse. After that, the number of ions produced per pulse remains constant, until the intensity is not large enough for ionization to occur at the secondary focus, or even in the region between the two. That explains the increase at curve at Fig. 3, after the flat region.

Mainfray at al [15] did the experiment on the resonant multiphoton ionization of cesium, by a tunable neodimium-glass laser, about the intensity of 10^{8} W/cm^2. The laser bandwidth was very narrow, 0.4 Å. The necessary photon multiplicity to cause ionization was 4, and it went via the three photon excitation of 6f resonant level (λ_{6f}= 10589 Å). Fig. 4 shows the number of the counted cesium ions as a function of the laser intensity Γ, for two different laser wavelengths, corresponding to the detuning from the resonance by 0.4 Å. In Fig. 5, η_{exp} as a function of the laser wavelength is shown in the vicinity of the resonance, and in Fig. 6 the ionization rate as a function of wavelength is given. It is interesting to note that the shift of the resonant level is less than the laser bandwidth when the laser frequency is just on the resonance (Fig. 6). But the shape of the curve indicates that the Stark shift of 6f level is important out of resonance. The dip at 10579 Å could be a consequence of an interference between the nearly resonant intermediate state and the non-resonant backround.

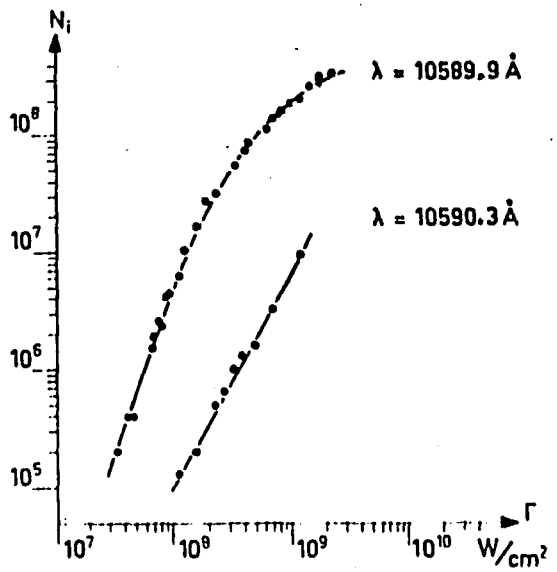

Fig. 4

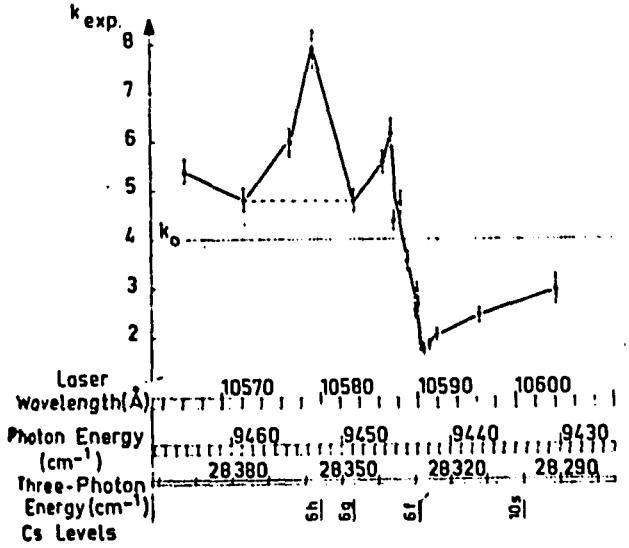

Fig. 5

In another experiment by the same group [5] , multiphoton ionization of the noble gases (xenon, krypton, argon) was studied by the very short pulses laser (30 psec), at intensities about 10^{13} W/cm², and with a tunable frequency about 10640 Å. In Fig. 7 the results for 10644 Å are shown. The slopes of the lines are 11 for Xe, 13 for Kr and 14 for Ar. These exactly correspond to the values, predicted by

the perturbation theory.

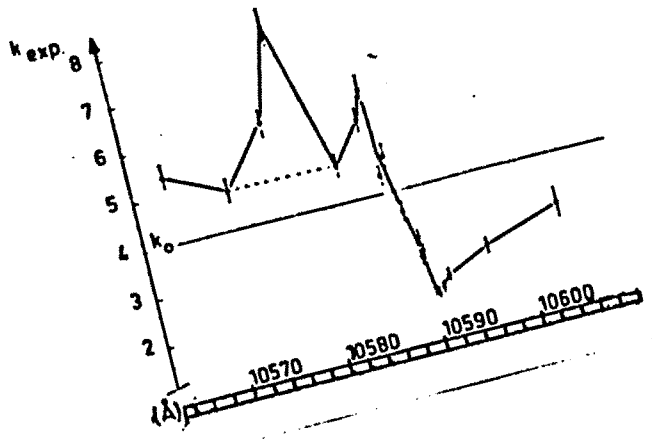

Fig. 6

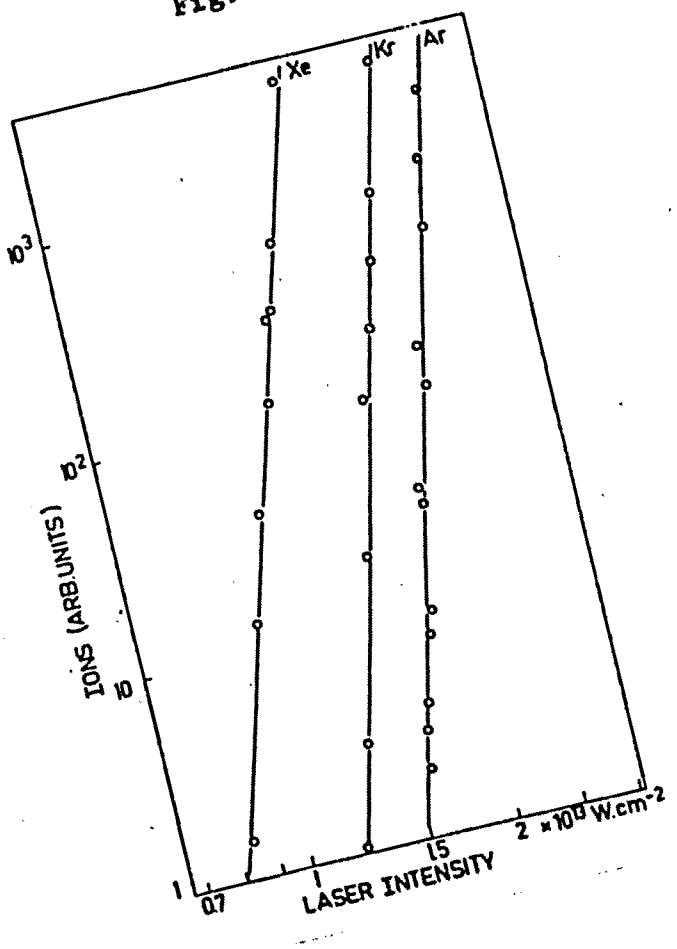

Fig. 7

When the laser wavelength was tuned over a range of 80 Å, no resonances were measured and there were no changes in the slope of the curves, nor in the number of the counted ions, although there were several levels allowable for the resonant multiphoton excitation. The explanantion of the disapearance of the resonances in this experiment, and the consequent agreement with the perturbation theory, even at relatively high laser intensities, could be in the short duration of the laser pulse. The resonances were observed under the same experimental conditions, but with a much longer pulse duration (10 nsec). When the laser pulse is too short, the resonant channel (which takes a much longer time to occur than nonresonant) doesn't have enough time to take place, and multiphoton ionization is governed by the nonresonant process.

The purpose of this Introduction was to show qualitativelly some features of the multiphoton ionization in the range of laser intensities where the perturbation theory in atom-laser interaction and its corrections or extensions are practically applicable. A short review of the work that was done on multiphoton ionization in ultrastrong laser fields ($I \gtrsim I_0$, where I_0 is the atomic unit for light intensity, defined by (31)), is given in Part I. We shall show that the nonrelativistic treatment of the multiphoton ionization in such a strong light fild is unacceptable. The relativistic approach to this problem, shown on the example of hydrogen atom, will be given in Sections 1.2, 1.3 and 1.4.

In Part II it will be discussed the possibility of getting the multiphoton ionization probability as a function of the laser intensity from the measured energy spectrum of photo-electrons.

Part III will treat the multiphoton ionization into multiple continua, the effect which was experimentaly observed recently.*

* As a basis and the main sources of references for this Introduction we used references [16] and [17].

P A R T I

MULTIPHOTON IONIZATION OF HYDROGEN IN ULTRASTRONG LASER FIELDS

§ 1.1 I n t r o d u c t i o n. In recent years, due to the development of ultrahigh intensity lasers ($> 10^{15} W/cm^2$), interest has arisen in the theoretical and the experimental studies of behaviour of atoms in such fields. Multiphoton ionization of atoms with very high photon multiplicity has become possible. The theoretical treatment of such high photon multiplicities is difficult and uncertain, even at low laser intensities when the perturbation theory in laser-atom interaction is still applicable. The reason for this is the nesessity of performing a large number of summations, and also the fact that at low laser frequences the possibility of a great number of resonant and quasiresonant states arises and this can complicate the situation very much.

The physics which one usually has in mind when considers the process of multiphoton ionization in a high intensity laser field is the following: An atom is in its ground state in the absence of the laser field. Then, in the rest frame of the atom, the laser electric field amplitude is adiabatically increased from zero to an ultrastrong plateau value and held constant for the time T. It is then adiabatically decreased to zero and the probability of ionization is measured. Since the initial and the final measurements on the atom are made in the absence of the laser, there is no difficulty in defining the atomic state. The experiment is repeated with the same plateau value of the electric field, but varing T, and the variation of the curve of ionization probability versus T yields the ionization rate per unit

time for ultrastrong fields. The experiment is then repeated for different plateau values in order to obtain the ionization rate as a function of field strength.

In order to reach the high plateau value of the laser intensity, it must be increased adiababtically from zero to this value. A similar decrease occurs when the atom emerges from the laser. This implies that the atom is in the changing field for a finite length of time. It can be ionized during that time, and if the ionization probability is essentialy unity during that interval, then the measurement which we are describing would be very difficult to perform.

When an atom is illuminated with a sufficiently intense laser field, the motion of electron is determined by that field more than by the interaction with the nucleus. As a result, atom could be totally destroyed with the states so mixed and shifted, that the perturbative picture of transitions between the dressed stationary states under the influence of the laser looses any meaning. It is more likely that the transitions occur, between the free electron states in the laser, "dressed" by the potential of the electron in the field of the nucleus. Then the transitions occur in the process of the scattering of such states on the nucleus potential. A lot of work have been done on the behaviour of atoms in ultrastrong laser field, based on this picture.

Our work will be also based on the same picture, i.e. taking in the first approximation the nuclear potential as negligible, and then treating it as a perturbation on the otherwise free electron states in a laser. The justification for that can be roughly taken as based on the supposition that the electric field of the laser is larger than

some average value of the electric field that electron experiences due to the nucleus*. Quantitavely, the stated condition can be written as

$$E \gg E_0 \tag{1.1.1}$$

where E is the laser electric field and E_0 is the atomic unit for electric field, defined in the Introduction, eq. (30) . If we define the dimensionless quantity

$$X = \frac{eA}{2mc^2} \tag{1.1.2}$$

where A is vector potential of the laser field, then in the case of the CO_2 laser (photon energy $\hbar\omega \approx 0.12$ eV), which we will use as a particular example, the condition (1.1.1) has the form

$$X \gg 1 \tag{1.1.3}$$

For $E \sim E_0$, the peak oscillatory velocity of a free electron in the laser field is

$$\frac{v}{c} = \frac{eE}{m\omega c} = 2X \tag{1.1.4}$$

That expression, together with the condition (1.1.3) says that the electron in such a field should be treated relativistically. It was pointed out by many authors before [6], [12], [19], [20] . On the

* In this Part we have in mind a particular example of hydrogen atom.

other side, we have already seen in the Introduction that dipole approximation is not always valid even in low laser field limit [12]. But, if we consider electron in a laser field which is so strong that in first approximation the atomic potential could be neglected, then the Lorentz force on the electron is

$$m\vec{a} = -e\left[\vec{E}(\vec{r},t) + \frac{\vec{v}}{c}\times\vec{B}(\vec{r},t)\right] \qquad (1.1.5)$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields of the laser. If the field is not too strong, the magnetic force term can be neglected. Then the electron acceleration, velocity and the displacement remain paralel to $\vec{E}$, and so $\vec{k}\cdot\vec{r}\approx 0$, leading to the dipole approximation. But for laser intensities sufficiently large, the electron experiences a net force along $\vec{k}$, which contradicts the dipole approximation. In QED language, this net magnetic force along $\vec{k}$ can be described as "Compton scattering", but this is not an ordinary Compton scattering, of a photon by a free electron, as (usually) a large number of low energy photons is included in the process. In our calculation of multiphoton ionization probability of hydrogen we will include all multipoles of the laser field and treat the electron states relativistically.

All the work that have been done so far on the multiphoton ionization in ultrastrong laser fields have been based on the nonrelativistic, dipole approximation and the hydrogen atom and CO_2 laser have been used, as the particular examples. We will now briefly describe the previous work.

The Space Translational Approximation (S T A) starts from the

fourth (Kramers) gauge, which was described in the Introduction. In the case of the hydrogen atom the interaction potential ((22)) can be written as

$$V(\vec{r}+\vec{\alpha}(t)) = -\frac{e^2}{|\vec{r}+\vec{\alpha}|} = -\frac{e^2}{2\pi^2}\int \frac{e^{i\vec{q}\cdot(\vec{r}+\vec{\alpha})}}{q^2} d^3q \qquad (1.1.6)$$

where $\vec{\alpha}(t)$ was defined in (21) and represents the displacement of the classical electron in the electromagnetic field. This describes the shift to the accelerated frame of reference in which the electron "sees" nucleus as oscillating. Expanding $\exp(i \vec{q}\cdot\vec{\alpha}(t))$ in the Fourier series, one gets for a linearly polarized light

$$V(\vec{r}+\vec{\alpha}) = -\frac{e^2}{2\pi^2}\sum \int J_n(\vec{\alpha}_0\cdot\vec{q}) \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2} J_0(\vec{\alpha}_0\cdot\vec{q}) e^{in\omega t} d^3q \qquad (1.1.7)$$

where $\vec{\alpha}_0 = \frac{e\vec{a}}{mc\omega}$. Averaging this in time, over a period of the light oscillation, one gets [21] the time independent part of (1.1.7)

$$V_0(\vec{r}) = -\frac{e^2}{2\pi^2}\int \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2} J_0(\vec{\alpha}_0\cdot\vec{q}) d^3q = -\frac{e^2}{2\pi}\int \frac{d\varphi}{|\vec{r}+\vec{\alpha}_0 \sin\varphi|} \qquad (1.1.8)$$

For the case of the circularly polarized light, this gets particularly simple form in terms of the complete elliptic integral of the first kind K [22]

$$V_0(\vec{r}) = -\frac{2}{\pi}\left(r^2+\alpha_0^2+2\alpha_0 r_\perp\right)^{-1/2} K\left(\frac{4\alpha_0 r_\perp}{(r^2+\alpha_0^2+2\alpha_0 r_\perp)^{1/2}}\right) \qquad (1.1.9)$$

$$\hat{k}\cdot\vec{r}_\perp = \vec{A}\cdot\hat{n} = 0$$

which can be interpreted as the potential due to nucleus, with the nuclear charge spred uniformly over a ring of radius α_0 . S T A consists in describing the "dressing" of the atom by the laser field as caused by the effective potential $V_0(\vec{r})$. In that way, one part of the strong laser field is incorporated into atomic Hamiltonian, while the remaining, oscillatory part is treated as perturbation. If one wants to find ionization probability of the atom in the laser field, in this formulation, he will find eigenvalues and eigenstates of the Hamiltonian

$$H_o = \frac{p^2}{2m} + V_0(\vec{r}) \qquad (1.1.10)$$

numerically, and then he will compute the ionization probability due to the time-dependent perturbation

$$H' = -\frac{e^2}{2\pi^2} \sum_{n\neq 0} \int \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2} J_n(\vec{\alpha}_0 \cdot \vec{q})\, e^{in\omega t} d^3q \qquad (1.1.11)$$

Lima at al [23] calculated the deformation of the ground state of hydrogen, due to the potential $V_0(\vec{r})$, for the circularly polarized, strong CO_2 laser. He introduced the strength parametar of the lasser field $\lambda = \frac{\lambda}{a_0} \times$. His result was that with increasing λ the radial distribution of the electron was shifted further from the nucleus. For example, for $\lambda = 100$ it was peaked at about 100 a_0 . As a consequence, the binding potential of the electron to the nucleus is decreased, and for larger values of λ it decreases as λ^{-1}.

There is concern about the validity of this method. The effecti-

ve potential V_0 , used in S T A , contains the first order contribution to the dressing of the atomic states by the laser, and is obtained by the time averaging. Gersten and Mittleman [22] showed that if one includes higher order contributions into S T A method, the series converges very slowly (as $\frac{1}{\omega}$). That strongly limits the application of the method, making it inapplicable for the case of the CO_2 laser. Therefore, the picture of expanding the atom in a strong laser field could be wrong.

Faisal [24] calculated the ionization probability for hydrogen in an ultrastrong laser field using the exact expression for the S matrix

$$S_{fi} = -i \langle U_f^{(-)}, H_1' \Psi_i^{(+)} \rangle \qquad (1.1.12)$$

which was defined in the Introduction, eq. (25). He did his calculation in the first gauge, with the Hamiltonian H_1' given in eq. (8). $\Psi_i^{(+)}$ is the exact solution of the Schrodinger equation with the Hamiltonian H_1, which evolved from the ground state of hydrogen at $t=-\infty$. Faisal made an unitary transformation of the Hamiltonian, transforming it in the Kramers gauge, and then approximated the new wave function by the unperturbed ground state U_0 . So, he made the zeroth order S T A, neglecting $\vec{\alpha}(t)$ in the effective potential (1.1.6). In the first gauge this approximation for $\Psi_i^{(+)}$ can be written as

$$\Psi_i^{(+)} = e^{i\Phi_4} U_0 \qquad (1.1.13)$$

The S matrix (1.1.12) is now

$$S_{f0} \simeq -i \langle u_f^{(-)}, H_1' e^{i\Phi_4} u_0 \rangle \tag{1.1.14}$$

Faisal approximated the unperturbed final state $u_f^{(-)}$ by the free electron plane wave (i.e. he neglected the Coulomb interaction in the final state) which leads to

$$S_{\vec{q}0} \approx -i \langle \chi_{\vec{q}}^{(-)}, H_1' u_0 \rangle \tag{1.1.15}$$

where $\chi_{\vec{q}}^{(-)}$ is the wave function of free electron in laser, which corres ponds to the Hamiltonian H_1 ($V \simeq 0$) in (8)

$$\chi_{\vec{q}}^{(-)} = \exp i \left(\vec{q} \cdot \vec{r} - \frac{1}{2m} \int^t (\vec{q} + e\vec{A})^2 dt' \right) \tag{1.1.16}$$

But (1.1.15) is just the first term of the perturbative expansion of the exact S matrix

$$S_{\vec{q}0} = -i \langle \Psi_{\vec{q}}^{(-)}, H_1' u_0 \rangle \tag{1.1.17}$$

in the electron-nucleus interaction V. $\Psi_{\vec{q}}^{(-)}$ was defined in Introduction, by (29). Using the Lippman-Schwinger equation

$$\Psi_{\vec{q}}^{(-)} = \chi_{\vec{q}}^{(-)} + G^{(-)} V \chi_{\vec{q}}^{(-)} \tag{1.1.18}$$

and expanding the full Green's function $G^{(-)}$ in the electron-nucleus interaction

$$G^{(-)} = G_0^{(-)} + G_0^{(-)} V G_0^{(-)} + \ldots \tag{1.1.19}$$

one gets (1.1.15), if neglect other than the first term. Time and space integrations in (1.1.15) can be easily performed to give the T matrix

$$T_\ell(\vec{q}) = (\ell\omega - X^2 m)\,\tilde{U}_0(\vec{q})\,F_\ell(\vec{q}) \qquad (1.1.20)$$

and the energy conserving condition

$$E_q = W_0 + \ell\omega - X^2 m \qquad (1.1.21)$$

$\tilde{U}_0(\vec{q})$ is the Fourier transform of the initial state with eigen-energy W_0 , $\vec{q}$ is the final momentum, X is the dimensionless quantity defined in eq. (1.1.2) , and F_ℓ is the "generalized Bessel function", that corresponds to the photon multiplicity ℓ [18]

$$F_\ell(\vec{q}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\, \exp i\left(\frac{2X}{\omega}\vec{q}\cdot\hat{a}\sin\varphi + \frac{mX^2}{2\omega}\sin 2\varphi + \ell\varphi\right) \qquad (1.1.22)$$

The condition $q^2 \geqslant 0$, together with (1.1.21) yields

$$\ell \geqslant \frac{X^2 m - W_0}{\omega} = \frac{\tilde{W}_0}{\omega} \qquad (1.1.23)$$

where $\tilde{W}_0$ is the "effective" ionization potential.

If one pursues this further, to calculate the ionization rate from the Faisal T matrix (1.1.20), the calculation becomes difficult. The reason is that F_ℓ has the saddle points in the complex plane. The

saddle points in the integrand of (1.1.22) are at

$$\cos\varphi=\frac{1}{2mx}\left(-\vec{q}\cdot\hat{a}\pm\left((\vec{q}\cdot\hat{a})^2+2m(x^2m-l\omega)\right)^{1/2}\right) \quad (1.1.24)$$

Appliing the energy conserving condition (1.1.21) one finds

$$(\vec{q}\cdot\hat{a})^2+2m(x^2m-l\omega)=-q^2(1-(\hat{q}\cdot\hat{a})^2)+2mw_0<0 \quad (1.1.25)$$

and so all four saddle points are always in the complex plane. Using the standard method [25] we found

$$F_l\approx\frac{1}{\sqrt{x\pi}}\left(\frac{\omega}{m}\right)^{1/2}\frac{1}{Y_0^{1/4}}e^{-\frac{1}{x}\frac{m}{8\omega}Y_0^{3/2}}\cos\left(3\frac{\vec{q}\cdot\hat{a}}{\omega}x^2\left(1-\left(\frac{\vec{q}\cdot\hat{a}}{m}\right)^2\right)-l\left(\frac{\vec{q}\cdot\hat{a}}{m}\right)+l\left(\frac{\pi}{2}\right)\right)$$

$$Y_0=\left(\frac{q}{m}\right)^2\left(1-(\hat{q}\cdot\hat{a})^2\right)+2\frac{R_y}{m} \quad (1.1.26)$$

which yields

$$|T_l|^2=\frac{16}{x}\left(\frac{R_y}{\omega}\right)\left(\frac{R_y}{m}\right)\alpha_F\frac{1}{m}\left(l-\frac{x^2m}{\omega}\right)^{-2}\frac{1}{Y_0^{1/2}}e^{-\frac{1}{x}\frac{m}{4\omega}Y_0^{3/2}} \quad (1.1.27)$$

For the laser fields for which $x\sim 1$, and for CO_2 laser, the exponent in (1.1.27) is of order 1. For very strong fields, $x\gg 1$, and we can approximate the exponential factor by 1. The ionization rate for photon multiplicity l is

$$W_\ell = 2\pi \int \frac{d^3q}{(2\pi)^3} \delta(W_q - \ell\omega - W_0) |T_\ell(\vec{q})|^2 \qquad (1.1.28)$$

and after q integration we finaly get

$$W_\ell = \frac{16}{\pi\chi} \frac{R_y^2}{\omega} \alpha_F \left(\ell - \frac{\chi^2 m}{\omega}\right)^{-2} \arcsin \frac{\left(\ell - \frac{\tilde{W}_a}{\omega}\right)^{1/2}}{\left(\ell - \frac{\chi^2 m}{\omega}\right)^{1/2}} \qquad (1.1.29)$$

Summation over all ℓ yields the total ionization rate

$$W = \frac{4}{\chi} \frac{R_y}{\hbar} \alpha_F \qquad (1.1.30)$$

For the circularly polarized light, Faisal obtained the T matrix for the ionization and the energy conserving condition in the form

$$T_\ell(\vec{q}) = (\mp i)^\ell (\ell\omega - \chi^2 m) \tilde{u}_0(\vec{q}) J_\ell\left(2\chi \frac{q_\perp}{\omega}\right) e^{\pm i\ell\theta} \qquad (1.1.31a)$$

$$E_q = W_0 + \ell\omega - 2\chi^2 m \qquad (1.1.31b)$$

where $\vec{q}_\perp \cdot \hat{k} = 0$, $\vec{q}_\perp \cdot \hat{a} = \cos\theta$. The effective ionization potential was then

$$\tilde{W}_0 = 2\chi^2 m - W_0 \qquad (1.1.32)$$

The first applicable calculation of multiphoton ionization in an ultrastrong laser field was due to Keldish [26], for the hydrogen atom and a linearly polarized laser. He started from the exact S matrix (1.1.17) and expanded $\Psi_{\vec{q}}^{(-)}$ in the electron-nucleus interaction,

using (1.1.18) and (1.1.19). This leads to the approximate S matrix, given by eq, (1.1.15). So, the Keldysh's and the Faisal's calculations were similar, the only difference was in the gauge used. Keldysh used $\vec{r}\cdot\vec{E}$ gauge, with $\chi_g^{(-)}$ given by (19) and $H_3' = e\vec{r}\cdot\vec{E}$. The time integration in the S matrix gave the energy conserving condition (1.1.20) which is the same as the Faisal's one. Keldysh used the functions, similar to those given by (1.1.22), which had the principal contibution from the saddle points in complex plane. On condition $\frac{g^2}{2m} \ll R_Y$ he finaly got the result for the total ionization rate in the form

$$\omega = \frac{\sqrt{3\pi}}{2^{5/4}} \frac{R_Y}{\hbar} \left(\frac{E}{E_0}\right)^{1/2} e^{-\frac{2}{3}\frac{E_0}{E}} = \frac{\sqrt{3\pi}}{2^{3/4}} \frac{R_Y}{\hbar} \left(\frac{\hbar\omega}{R_Y} \frac{1}{\alpha_F}\right)^{1/2} \chi^{1/2} e^{-\frac{1}{\chi}\frac{mc^2}{6\hbar\omega}d_F^3} \quad (1.1.33)$$

valid in the limit $\gamma \ll 1$, where γ is a dimensionless parametar, defined as

$$\gamma = \frac{\hbar\omega}{R_Y} \frac{E_0}{2E} = \frac{\alpha_F}{4\chi} \quad (1.1.34)$$

For $\chi \sim 1$ and CO_2 laser, the exponent in (1.1.33) is of order 1,and in the ultrastrong laser limit ($\chi \gg 1$) it can be approximated by 0. But preexponential factor now increases with the laser electric field, the behaviour that is oposite to the one in the Faisal's result.

Pert [27] followed the same steps as Keld sh and Faisal did, except that he worked on in $\vec{p}\cdot\vec{A}$ gauge, with the free electron wave function in laser given by (15) and the interaction H_2' given by (14). He obtained a T matrix for the absorption of ℓ photons in the form

$$T_\ell(\vec{Q}) = \ell\omega\, \tilde{u}_0(\vec{Q})\, J_\ell(\vec{\alpha}_0 \cdot \vec{Q}) \tag{1.1.35}$$

with the energy conserving condition

$$E_Q = \ell\omega + W_0 \tag{1.1.36}$$

Using (1.1.28), that yields the ionization rate

$$\omega_\ell = 16\, R_Y \frac{E_0}{E} \frac{1}{\ell^2} \int_0^{\lambda_\ell} dx\, J_\ell^2(x) \tag{1.1.37}$$

where

$$\lambda_\ell = 2\frac{E}{E_0}\left(\frac{R_Y}{\hbar\omega}\right)^2\left(\frac{\ell\hbar\omega}{R_Y} - 1\right)^{1/2} = 2x\frac{mc^2}{\hbar\omega}\alpha_F\left(\frac{\ell\hbar\omega}{R_Y} - 1\right)^{1/2} \tag{1.1.38}$$

The integral in (1.1.37) diverges logaritmicaly as $\lambda_\ell \to \infty$, leading to the result

$$\omega_\ell \sim \frac{R_Y}{\hbar}\frac{1}{\ell^2}\frac{E_0}{E}\ln\frac{E}{E_0} \tag{1.1.39}$$

Summing over ℓ, the total ionization rate is

$$\omega \sim \frac{R_Y}{\hbar}\frac{\hbar\omega}{R_Y}\frac{E_0}{E}\ln\frac{E}{E_0} \overset{CO_2}{\sim} \frac{R_Y}{\hbar}\frac{\alpha_F}{2}\frac{1}{x}\ln x \tag{1.1.40}$$

which is different from both the Keldysh and the Faisal result.

Gersten and Mittleman [6] started from the exact expression for the S matrix

$$S_{\vec{g}0} = -i\langle u_{\vec{g}}^{(-)}, H' u_0\rangle - i\langle u_{\vec{g}}^{(-)}, H' G^{(+)} H' u_0\rangle \qquad (1.1.41)$$

where $G^{(+)}$ is the full causal Green's function of the problem

$$\left(i\frac{\partial}{\partial t} - H\right) G^{(+)} = 1 \qquad (1.1.42)$$

and $u_{0,\vec{g}}$ are the solutions of the unperturbed Hamiltonian

$$\left(i\frac{\partial}{\partial t} - T - V\right) u_{0,\vec{g}} = 0 \qquad (1.1.43)$$

The first term in (1.1.41) is a one photon transition, which can be discarded on the ground of energy conservation. The Green's function in the second term was then expanded in the electron-nucleus interaction, using (1.1.19). An additional approximation used was the replacement of the $u_{\vec{g}}^{(-)}$, which is a continuum function of an electron in the field of the nucleus, by a plane wave. The calculation was done in $\vec{p}\cdot\vec{A}$ gauge.

$G_0^{(+)}$ in the $\vec{p}\cdot\vec{A}$ gauge can be written as

$$G_0^{(+)}(\vec{r},t;\vec{r}',t') = -i\,\theta(t-t')\lim_{\eta\to 0}\int\frac{d^3K}{(2\pi)^3}\, e^{i\vec{K}(\vec{r}-\vec{r}')-i(E_K-i\eta)(t-t')} \times$$

$$\times \sum_{n,n'} J_n(\alpha_0\cdot\vec{K})\, J_{n'}(\alpha_0\cdot\vec{K})\, e^{-i\omega(nt-n't')} \qquad (1.1.44)$$

The result for the T matrix was

$$T_\ell(\vec{q}) = \ell\omega\, \tilde{U}_0(\vec{q})\, J_\ell(\vec{\alpha}_0\cdot\vec{q})\, J_0(\vec{\alpha}_0\cdot\vec{q}) \qquad (1.1.45)$$

with the energy conserving condition $E_q = \ell\omega + W_0$. That yields the transition rate for ionization of photon multiplicity ℓ in the form

$$\omega_\ell = 16\, R_Y \frac{E_0}{E} \frac{1}{\ell^2} \int_0^{\lambda_\ell} dx\, J_\ell^2(x)\, J_0^2(x) \qquad (1.1.46)$$

where λ_ℓ was defined in (1.1.38). When the parameter $2x\frac{mc^2}{\hbar\omega}\alpha_F$ is much larger than $\ell_{min} \approx \frac{R_Y}{\hbar\omega}$ (the ultrastrong laser limit), the upper limit in the integral can be replaced by infinity. The use of the approximative formula

$$\int_0^\infty dx\, J_\ell^2(x)\, J_0^2(x) \approx \frac{1}{2\pi\ell}\,, \quad \ell > 0 \qquad (1.1.47)$$

leads to

$$\omega_\ell \approx \frac{8}{\pi} R_Y \frac{E_0}{E} \frac{1}{\ell^3} \qquad (1.1.48)$$

That yields the total ionization rate as

$$\omega = \sum_{\ell=\ell_{min}} \omega_\ell = \frac{4}{\pi} \frac{R_Y}{\hbar} \left(\frac{\hbar\omega}{R_Y}\right)^2 \frac{E_0}{E} = \frac{2}{\pi} \frac{R_Y}{\hbar} \frac{\hbar\omega}{R_Y} \alpha_F \frac{1}{X} \sim \frac{2}{\pi} \frac{R_Y}{\hbar} \alpha_F^2 \frac{1}{X} \qquad (1.1.49)$$

$$\omega = \sum_{\ell=\ell_{min}} \omega_\ell = \frac{4}{\pi} \frac{K_Y}{\hbar} \left(\frac{\hbar\omega}{R_Y}\right)^{-} \frac{E_0}{E} = \frac{4}{\pi} \frac{K_Y}{\hbar} \frac{\hbar\omega}{R_Y} \alpha_F \frac{1}{X} \sim \frac{4}{\pi} \frac{K_Y}{\hbar} \alpha_F^{*} \frac{1}{X} \qquad (1.1.49)$$

In both the Pert and the Gersten and Mittleman calculations the "effective" ionization potential was $\tilde{w}_0 = Ry$.

If compare the T matrices for photon multiplicity ℓ , obtained by Gersten and Milttleman (1.1.45) and the one obtained by Pert (1.1.35), we see that the difference between two is in additional factor $J_0(\vec{\alpha}_0 \cdot \vec{q})$ in former calculation. On the other side, if the first term $\langle u_{\vec{q}}^{(-)}, H' u_0 \rangle$ in eq. (1.1.41) of Gersten-Mittleman calculation is retained, approximation of the final state as a plane wave $\lambda_{\vec{q}}$ allows (1.1.41) to be rewritten as .

$$S_{\vec{q},0} \approx -i \langle \lambda_{\vec{q}}^{(-)}, (1 + H' G_0^{(+)}) H' u_0 \rangle \qquad (1.1.50)$$

If the laser is of finite spatial extent, we could use the indentity

$$\chi_{\vec{q}} = \lambda_{\vec{q}} + G_0^{(+)} H' \lambda_{\vec{q}} \qquad (1.1.51)$$

which would make the forms (1.1.41) and $\langle \chi_{\vec{q}}, H' u_0 \rangle$ identical. But, if the laser is infinite in its spatial extent, from the definition of $G_0^{(+)}$ there arises an additional factor $J_0(\vec{\alpha}_0 \cdot \vec{q})$. That is, the relation (1.1.51) must be changed to the form [16]

$$J_0(\vec{\alpha}_0 \cdot \vec{q}) \chi_{\vec{q}} = \lambda_{\vec{q}} + G_0^{(+)} H' \lambda_{\vec{q}} \qquad (1.1.52)$$

which explaines the difference in factor J_0 for the two calculations of the multiphoton ionization rate.

We can now discuss the range of validity of the above mentioned four derivations of the ionization rate of hydrogen in strong laser

fields. As we have already seen, the calculations of Faisal, Keldysh, Pert and Gersten-Mittleman can be understood as the first term in the perturbative expans-ion in the electron-nucleus interaction V of the exact S matrix

$$S_{\vec{q},0} = -i \langle \Psi_{\vec{q}}^{(-)}, H' u_0 \rangle \tag{1.1.53}$$

Expanding this, by the use of eq. (1.1.18) and (1.1.19) one gets

$$S_{\vec{q},0} = -i \left(\langle \chi_{\vec{q}}^{(-)}, H' u_0 \rangle + \langle \chi_{\vec{q}}^{(-)}, V G_0 H' u_0 \rangle + \langle \chi_{\vec{q}}^{(-)}, V G_0 V G_0 H' u_0 \rangle \right)$$

(1.1.54)

(finite laser spatial extent is supposed). Strictly speaking, the way to investigate the convergence of the perturbation serie is to find the next term of the expanssion (1.1.54) , and reestablish the condition

$$\left| \langle \chi_{\vec{q}}^{(-)}, V G_0 H' u_0 \rangle \right| << \left| \langle \chi_{\vec{q}}^{(-)}, H' u_0 \rangle \right| \tag{1.1.55}$$

Still, we can use the usual perturbation theory argument that the condition of the convergence of the procedure is that the dominant interaction is much larger than the perturbation, which gives

$$\left| \frac{\hbar e \vec{a}}{mc} \cdot \nabla_r \right| >> V(r) \tag{1.1.56}$$

Considering that the Coulomb interaction is important in the vicinity of the atom of range a_0 yields $ea\sqrt{\frac{R_Y}{mc^2}} \gg R_Y$

This is equivalent to

$$X \gg \frac{\alpha E}{4} \tag{1.1.57}$$

or (Keldysh)

$$\gamma = \frac{\omega\sqrt{2mR_y}}{eE} \ll 1 \tag{1.1.58}$$

or (Gersten-Mittleman)

$$\frac{\bar{E}}{\bar{\omega}} \gg 1 \ , \ \bar{E} = \frac{E}{E_0} \ , \ \bar{\omega} = \frac{\omega}{R_y/\hbar} \tag{1.1.59}$$

The three of the above mentioned calculations lead to the conclusion that the ionization probability decreases as the laser electric field increases (in ultrastrong laser limit), while Keldysh came to the opposite conclusion. Reiss [28] discussed the possibility that the ionization rate for the strong laser fields can have more succesive maxima and minima. He used the " momentum translation approximation " (M T A) to find the approximation of the full wave function of the electron in the nuclear potential – laser interaction. It consists of the following: One starts from the Hamiltonian in the first gauge given by (8) , and makes an unitary transformation $exp{-i(e\vec{A}\cdot\vec{r})}$ to the $\vec{r}\cdot\vec{E}$ gauge. The new Hamiltonian is (17)

$$H_3 = \frac{p^2}{2m} + e\vec{r}\cdot\vec{E} + V(r) \tag{1.1.60}$$

The M T A neglects the electron-laser interaction in (1.1.60), approximating the solution of the Schrodinger equation

$$\left(i\frac{\partial}{\partial t}-H_3\right)\Psi_i = 0 \qquad (1.1.61)$$

by the $u_0(r)$, where u_0 is the eigen function of the unperturbed Hamiltonian $H_0 = T+V$. In that way, the approximate wave function of H_1 is

$$\Psi_i = e^{-ie\vec{A}\cdot\vec{r}}\Psi_i \approx e^{-ie\vec{A}\cdot\vec{r}}u_0 \qquad (1.1.62)$$

The justification of the approximation is the following: The unitary transformation $\exp -i(e\vec{A}\cdot\vec{r})$ transforms, in the case of small laser frequences, the laser-atom interaction $\frac{e}{m}\vec{A}\cdot\vec{p}+\frac{e^2A^2}{2m}$ to the much smaller laser-atom interaction $e\vec{r}\cdot\vec{E}$, which can be treated as a perturbation and neglected in the first approximation. The interaction of the electron with a laser is then described by the large exponent in (1.1.62). The conditions of validity of this approximation could be expressed as

$$\left|\frac{e}{mc}\vec{A}\cdot\vec{p}_r\right| >> e\vec{E}\cdot\vec{r} \qquad (1.1.63a)$$

$$|e\vec{E}\cdot\vec{r}| << V(r) \qquad (1.1.63b)$$

For hydrogen atom, the first condition yields $\frac{a_0}{\lambda} << 1$, while the second gives $X << \alpha_F\frac{R_y}{\hbar\omega}$ .For a CO_2 laser, the condition $\frac{a_0}{\lambda} << 1$ is already fulfiled, but the second one gives $X << 1$. So, in that case, the method is applicable for the lasers of intermediate intensities, but certain-

ly not for ultrastrong intensities. In fact, the second condition, $(\chi \ll \alpha_F \frac{R\chi}{\hbar\omega})$ is equivalent to the condition $E \ll E_0$, where E_0 is the atomic unit for the electric field, and that clearly restricts the range of the applicability of M T A.

To find the transition probability by this method, one can start from the exact S matrix

$$S_{\vec{q},0} = -i \left\langle u_{\vec{q}}^{(-)}, H_1' e^{-ie\vec{A}\cdot\vec{r}} \psi_i^{(+)} \right\rangle \qquad (1.1.64)$$

where $\psi_i^{(+)}$ is the full wave function of the Hamiltonian H_3 , defined in (1.1.60) and which evolves from the initial state u_0 at $t=-\infty$.

$u_{\vec{q}}^{(-)}$ is the final state that satisfies the wave equation, unperturbed by the laser. If we expand $\psi_i^{(+)}$ in perturbative serie in laser-electron interaction

$$\psi_i^{(+)} = u_0 + G^{(+)} H_3' u_0 = u_0 + G_A^{(+)} H_3' u_0 + \ldots \qquad (1.1.65)$$

where $G_A^{(+)}$ is unperturbed atomic Green's function, and if we keep only the first term of that expanssion, we get M T A. Using the usual perurbation theory arguments, this series converges under the conditions, stated above. We will not further pursue the Reiss calculation,but we will rather present his discussion on some features of the high intensity multiphoton transitions. Reiss showed that the S matrix (1.1.64) is propotional to the matrix, $(\psi_i \approx u_0)$,

$$M_{\vec{q},0} = \left\langle u_{\vec{q}}^{(-)}, e^{-ie\vec{A}\cdot\vec{r}} u_0 \right\rangle \qquad (1.1.66)$$

For a weak laser we can expand the exponential in $\vec{A}\cdot\vec{r}$ to get

$$M_{\vec{q},0} = \langle u_{\vec{q}}^{(-)}, u_0 \rangle - i \langle u_{\vec{q}}^{(-)}, e\vec{A}\cdot\vec{r}\, u_0 \rangle + \dots \qquad (1.1.67)$$

The right hand side of this expression can never exhibit maxima as a function of $|\vec{a}|$, which is an essential feature of the perturbation theory result for the multiphoton transitions, as was presented in Introduction. But the left hand side can. Starting from zero field, when $M_{\vec{q}0}=0$, it can lead to the extremum for $e\vec{A}\cdot\vec{r}=\frac{\pi}{2}$ (in some average sense). A minimum can be expected for $e\vec{A}\cdot\vec{r}=\pi$, and so on. Due to the fast oscillatory character of the exponential factor, we can eventually expect zero for $|\vec{a}|\to\infty$. In short, the term (1.1.66) can exhibit oscillatory properties as a function of the laser intensity.

An additional aspect of multiphoton ionization is the possibility, that in some range of low laser frequances and high intensities, multiphoton ionization can be considered as a tunneling process. The idea is due to Keldysh [26] , and later was made clearer by Perelomov at al [19], [30]. The physics behind the idea is the following: If the atom is in a laser field, the total instantenous potential energy of the electron is the sum of the interaction energies with the nucleus and with the laser (hydrogen is an example)

$$V = -\frac{e^2}{r} - eEz \qquad (1.1.68)$$

This potential energy has a maximum $-2e\sqrt{eE}$ for $z=\sqrt{\frac{e}{E}}$ (in $\hat{z}=\hat{E}$ direction) and the barrier has an approximate width of $\frac{Ry}{eE}$. If the chan-

ge of the laser field, during the time of the passage of the electron through the barrier can be neglected, then the ionization can be considered as a tunneling through a barrier, determined by the instanteneous value of the laser electric field. By increasing E_0 the width of the barrier decreases. As long as the time of flight of the electron through the barrier is much less than the period of oscillation of $\vec{E}$, the tunneling remains independent on frequency, depending only on barrier width, which depends on E_0. In this picture the electron doesn't see separate photons, but rather a slowly oscillating potential barier, and that can't be described as a multiphoton process. The picture depends on the supposition that electron states are essentialy unperturbed inside the barrier, which yields the condition $E \ll E_0$. On the other through a barrier of width $\frac{R_Y}{eE}$ is $\tau = \frac{\sqrt{2mR_Y}}{eE}$ ron of momentum $\sqrt{2mR_Y}$ through a barrier of width $\frac{R_Y}{eE}$ is $\tau = \frac{\sqrt{2mR_Y}}{eE}$, and then the condition that the tunneling is the dominant process in a ionization event can be written as $\gamma = \frac{\tau}{T} \ll 1$ or, equivalently $X \gg \frac{\alpha_F}{2}$, which is just the condition for the convergence of the perturbative series in V (the electron-nucleus interaction) of the exact S matrix (1.1.17). The idea of tunneling is not applicable for the CO_2 laser, because the conditions $X \ll 1$ ($E \ll E_0$) and $X \gg \frac{\alpha_F}{2}$ contradict each other. It is rather applicable in the range of the microwave frequences.

The formula for the ionization rate of hydrogen by the static electric field through tunneling process was given by Landau [32]

$$\omega_{stat}(E) = 8 \frac{R_Y}{\hbar} \frac{E_0}{E} e^{-\frac{2}{3}\frac{E_0}{E}} \qquad (1.1.69)$$

The Keldysh result (1.1.33) contains the same exponential factor, but a different preexponential factor. Perelomov [30] corrected that factor by the similar quasiclassical calculations as had been done by Landau, in DC electric field case. He matched the wave function inside the barrier, where he neglected the laser field, with the wave function out of the barrier, where he neglected electron-nucleus interaction. The resulting expression for the ionization probability was

$$W \sim \frac{1}{E} e^{-\frac{E_0}{E}} \qquad (1.1.70)$$

which resembles DC field result.

An experiment of Bayfield and Koch [34] was done on microwave ionization of highly excited states of hydrogen (principal quantum number $63 \ll n \ll 69$). The photon multiplicity, necessary for ionization was about 200. At Fig. 1.1 are results for ionization probability as a function of the parameter γ , defined earlier (eq. (1.1.34)). The flatness on the three curves at higher radiation intensities is due to instrumental saturation. The result resembles the behaviour that could be due the tunneling ionization. Although the radiation field intensities were in the range of applicability of the perturbation theory, the slope of the curves (in log-log scale) was about 1.5-2 , very different than 200, which could be explained by the tunneling exponential dependence, (1.1.70). An important, independent result of this experiment was that more than 90% of the effect of the applied radiation field was ionization, rather than bound-bound transitions.

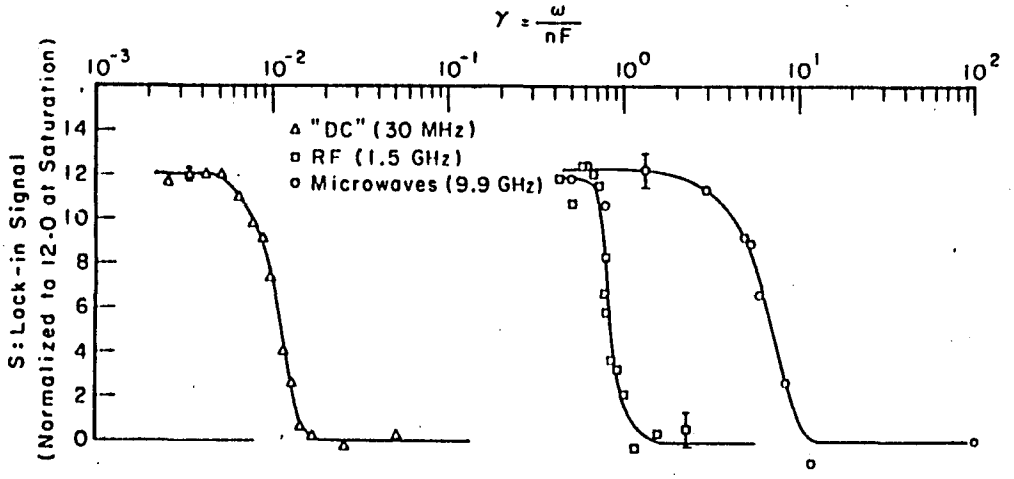

Fig. 1.1

The most disturbing fact about the calculations, mentioned in this Section is the gauge dependence of the results. As we have already discussed in the Introduction, the initial and final states of the atom are defined out of the laser. This was the assumption which was taken implicitely in the definition of the S matrix in all these treatments, but not treated explicitely in the calculations. We want to show now what consequences arise when the laser is explicitely treated as finite in space and so inhomogeneous, even in the nonrelativistic, dipole approximation. In practice, the vector potential of the electromagnetic field of the laser is a function of both $\vec{r}$ and t. Here we will restrict ourself to the linearly polarized laser "beam", that is to the case in which the vector potential of the laser is given as

$$\vec{A} = \vec{a}(\vec{r}_\perp)\cos\varphi \qquad (1.1.71)$$

where $\varphi = \omega t - \vec{k}\cdot\vec{r}$, $\vec{A}\cdot\vec{k} = 0$ and $\vec{r}_\perp$ is the part of $\vec{r}$, perpendicular to $\vec{k}$. The cases of the time inhomogeneous amplitude of the vector potential as well as a circularly polarized laser we will leave for the Sections

2,3 and 4. We are supposing that $a(\vec{r}_\perp)$ is slowly varing on the scale of the laser wavelength, the electron wavelength and the atom wavelength.

Let's briefly discuss the evolution of a free electron going into and out of the laser defined by (1.1.71). Its motion is governed by the Hamiltonian

$$H_1 = \frac{P_r^2}{2m} + \frac{e}{mc}\,\vec{A}\cdot\vec{P}_r + \frac{e^2A^2}{2mc^2} \qquad (1.1.72)$$

It is well known both experimentaly and theoreticaly [35],[36] for the spatialy inhomogeneous laser, that the time averaged squared field term (kinetic energy term $\frac{e^2A^2}{2mc^2}$ in (1.1.72)) acts in the nonrelativistic approximation as a ponderomotive potential, affecting the energy of the electron. This ponderomotive potential is

$$U_p^{n.r.}(\vec{r}) = \frac{e^2a^2(\vec{r})}{4mc^2} = \chi^2(\vec{r})\,mc^2 \qquad (1.1.73)$$

Its effect is to continuosly decrease the kinetic energy of the time averaged motion of the electron as it enters the laser beam, and to increase it, as the electron leaves the beam. Because of the slowly varing character of the laser field (1.1.71) with $\vec{r}_\perp$, time averaged electronic motion is essentialy classical, and the wave function can be found in the usual eikonal approximation [37] . That yields the solution of the Schrodinger equation

$$\left(i\hbar\frac{\partial}{\partial t} - H_1\right)\chi_{\vec{q}} = 0 \qquad (1.1.74)$$

in the form

$$\chi_{\vec{q}} = \exp\frac{i}{\hbar}\left(\int^{\vec{r}} \vec{f}\cdot d\vec{r}' - \frac{2\chi c}{\omega}\,\vec{f}\cdot\hat{a}\sin\omega t - \frac{mc^2}{2\omega}\chi^2\sin 2\omega t - \frac{q^2}{2m}t\right) \tag{1.1.75}$$

where $\vec{q}$ is the momentum of the electron outside the beam, and $\vec{f}$ is the field dependent momentum of the electron, which satisfies

$$q^2 = f^2 + 2\chi^2 m^2 c^2 \tag{1.1.76}$$

The integral in the exponent of (1.1.75) is over the classical trajectoria of the electron. The electron, born (by ionization) inside the laser with some time averaged momentum $\vec{f}$ will leave the laser, having the momentum $\vec{q}$, defined by (1.1.76).

It can be simply proved [38] (by expanding the wave function in the Fourier serie in ωt) that if the Hamiltonian, describing the atom in the classical, single mode electromagnetic field, is a periodic function of time, then the energy transfer to the particles is limited to integer multiples of $\hbar\omega$. Thus we have an appearent contradiction in which, on the one hand we know that only an integer number of photons is transfered, and on the other we know that the electron emerges with a continuous distribution of energy. It can be easily resolved. The key is that proton looses some energy. This may be seen in the following way. The hydrogen atom couples to the ponderomotive potential essentialy through the electron, since the coupling of the

proton to the radiation field is $\frac{m}{M}$ smaller. As the atom enters the laser beam, it is slowed up by something like $U_p^{n.r.}(\vec{R})$, until the ionization occurs at some point $\vec{R}_0$. From that point the proton is essentially decoupled from the field, and so it leaves the laser, having lost the energy $U_p^{n.r.}(\vec{R})$. The electron absorbs an integer number of $\hbar\omega$ and is expeled from the laser by the ponderomotive potential (1.1.73) so that its energy upon leaving the laser is

$$E_2 = W_0 + \ell\omega + U_p^{n.r.}(\vec{R}_0) \qquad (1.1.77)$$

where W_0 is its original energy in the hydrogen atom. Detailed analysis of this situation will be given in Section 1.3. Let's start from the Hamiltonian in the center of mass system, which was derived in the Introduction, eq. (6)

$$H_{CM} = \frac{P_R^2}{2M} + \frac{P_r^2}{2m} + \frac{e}{mc}\vec{A}\cdot\vec{P}_r + \frac{e^2A^2}{2mc^2} + V(r) \qquad (1.1.78)$$

We treat the oscillating terms as a perturbation, which can cause transitions between the states of the atom and take them out of the Hamiltonian, the rest is

$$H_{CM}^{(0)} = \frac{P_R^2}{2M} + X^2mc^2 + \frac{P_r^2}{2m} + V(r) \qquad (1.1.79)$$

We are supposing that the atomic wavelength is short on the scale of the characteristic change of the laser field amplitude, and then, the Schrodinger equation

$$\left(i\hbar\frac{\partial}{\partial t} - H_{CM}^{(0)}\right)\Psi = 0 \qquad (1.1.79a)$$

can be solved in eikonal approximation for the atom's motion, with the ansatz

$$\Psi = U_n(r)\, e^{-i\frac{W_n}{\hbar}t}\, \chi(\vec{R},t) \qquad (1.1.80)$$

$U_n(r)$ is the unperturbed eigen-function of the electron, corresponding to the eigenvalue W_n of the Hamiltonian

$$H_r = \frac{p_r^2}{2m} + V(r) \qquad (1.1.81)$$

and $\chi(R,t)$ satisfies

$$\left(i\hbar\frac{\partial}{\partial t} - \frac{P_R^2}{2M} - \chi^2 mc^2\right)\chi = 0 \qquad (1.1.82)$$

Here the term $\chi^2 mc^2$ (ponderomotive potential) represents the effective potential in which the center of mass of the atom moves. For the case of the laser beam $a = a(\vec{R}_\perp)$ and the solution of eq. (1.1.82) in eikonal approximation is

$$\chi = \exp\frac{i}{\hbar}\left(P_{\parallel} R_{\parallel} + \int^{\vec{R}_\perp} d\vec{R}'_\perp\, P_\perp(\vec{R}'_\perp) - \frac{P_i^2}{2M}t\right) \qquad (1.1.83)$$

with the initial conditions $\vec{P} = \vec{P}_i$, $t = -\infty$ and

$$P_\perp^2 = P_{\perp i}^2 - 2Mm\chi^2 c^2 \qquad (1.1.84a)$$

$$P_{\parallel} = P_{\parallel i} \qquad (1.1.84b)$$

$\vec{P}_i$ is the momentum of the atom out of the beam. The line integral in (1.1.83) is along the classical trajectory of the atom. The wave function $\mathcal{X}$ describes the evolution of the atom, entering the spatialy inhomogeneous laser field, due to the increasing oscillatory kinetic energy of the electron. The conclusion that the atom couples to the laser in a manner similar to the free electron, is consistent with the supposition that, in the first approximation, electronic motion for an atom in the strong laser field is similar to the motion of the free, unbound electron in the same field. Paricularly, an atom entering the laser beam can be reflected back if

$$\frac{P_i^2}{2M} < U_p = \chi^2 m c^2 \qquad (1.1.85)$$

The semiclassical motion of the center of mass in (1.1.80) can be carried further to a completely classical description.This is justified under the conditions of the slowness of the change of the laser spatial distribution, stated above.

To go in the classical description of the center of mass motion we aga-in start from the equation (1.1.78) and simply drop the kinetic energy operator $\frac{P_R^2}{2M}$ and treat $\vec{R}$ as a prescribed function of t, which is obtained from the classical Hamiltonian

$$H_{cl}^{(0)} = \frac{P^2}{2M} + \chi^2 m c^2 \qquad (1.1.86)$$

Using this prescription in the Schrodinger equation

$$\left(i\hbar\frac{\partial}{\partial t} - mc^2\chi^2(R(t)) - V(r)\right)\Psi_0 = 0 \qquad (1.1.87)$$

we get

$$\Psi_0 = u_0(r)\exp -\frac{i}{\hbar}\int_{-\infty}^{t}\left(mc^2\chi^2(R(t')) + W_0\right)dt' \qquad (1.1.88)$$

To find the multiphoton ionization rate, we start from the exact S matrix

$$S_{\vec{q},0} = -\frac{i}{\hbar}\left\langle \Psi_{\vec{q}}^{(-)}, H'\Psi_0 \right\rangle \qquad (1.1.89)$$

where $\Psi_{\vec{q}}^{(-)}$ is the full wave function of the Hamiltonian

$$H_1 = \frac{P_r^2}{2m} + \frac{e}{mc}\vec{A}\cdot\vec{P}_r + \frac{e^2A^2}{2mc^2} + V(r)$$

with ingoing boundary conditions, and H' is defined by

$$\left(i\hbar\frac{\partial}{\partial t} - H_1\right)\Psi_0 = -H'\Psi_0 \qquad (1.1.90)$$

which yields

$$H' = \frac{e}{mc}\vec{a}\cdot\vec{p}_r\cos\varphi + \frac{e^2a^2}{4mc^2}\cos 2\varphi \qquad (1.1.91)$$

We approximate $\psi_{\vec{q}}^{(-)}$ by $\chi_{\vec{q}}^{(-)}$ given by eq. (1.1.75), which yields the approximate S matrix

$$S_{\vec{q},0} \approx -i \langle \chi_{\vec{q}}^{(-)}, H' \psi_0 \rangle \tag{1.1.92}$$

Performing the time and the space integration, one gets

$$S_{\vec{q},0} = -2\pi i \sum_{\ell=-\infty}^{\infty} \delta(W_{\vec{q}} - \ell\omega - x^2 m - W_0)\, \ell\omega\, \tilde{U}_0(\vec{f})\, F_\ell(\vec{f}) \tag{1.1.93}$$

which gives for the T matrix for absorption of ℓ photons

$$T_\ell(\vec{q}) = \ell\omega\, \tilde{U}_0(\vec{f})\, F_\ell(\vec{f}) \tag{1.1.94}$$

The energy conserving condition

$$W_{\vec{q}} = \ell\omega + x^2 m + W_0 \tag{1.1.95}$$

can be also written in the form

$$\frac{\vec{f}^2}{2m} = \ell\omega + W_0 \tag{1.1.96}$$

where $\vec{f}$ is the momentum of the electron at the place of the laser field where the ionization occured. From the condition $f^2 \geq 0$ we get

$$\ell > -W_0/\omega \qquad (1.1.97)$$

and so W_0 is now the effective ionization potential. The squared T matrix can be obtained in the form

$$|T_\ell|^2 = \frac{16\pi}{\ell^2}\left(\frac{R_y}{\omega}\right)^2 \alpha_F \frac{1}{m} |F_\ell|^2 \qquad (1.1.98)$$

The presence of ℓ^2 term in the denominator of the expression above says that the dominant contribution to the ionization comes for relatively small ℓ., of order $\ell\omega \sim R_y$. That conclusion is not affected by the ℓ dependence of the "generalized Bessel functions" F_ℓ , defined by (1.1.22). In fact, as will be discussed in details in Section 1.4, these functions have the stationary phase points in a very broad range of ℓ, allowed by the energy conserving condition (1.1.96), and in ultrastrong laser field limit. Apliing the standard stationary phase technique, that yields the approximate expression

$$\overline{|F_\ell(\vec{f})|^2} \approx \frac{1}{\pi\chi^2}\frac{\omega}{m} \qquad (1.1.99)$$

which is independent on ℓ. The averaging in (1.1.99) is over the final momentum $\vec{q}$ of the electron. Using (1.1.98) and (1.1.99) we obtained the ionization rate for the photon multiplicity ℓ as

$$\omega_\ell = \frac{16}{\pi\chi^2}\frac{R_y}{m\omega}\alpha_F\sqrt{2m\omega}\,\frac{\left(\ell - \frac{R_y}{\omega}\right)^{1/2}}{\ell^2} \qquad (1.1.100)$$

and the total ionization rate (sum over ℓ) as

$$\omega = \frac{8}{x^2} \frac{Ry}{\hbar} \alpha_F^2 \qquad (1.1.101)$$

In that way, the explicit treatment of laser as a spatialy finite yields the multiphoton ionization rate different than any of the previously mentioned in this Section. In derivation of that result we clearly defined the initial and the final state of the atom out of the laser. As a consequence, multiphoton ionization has to be considered as a two step process. The atom is first ionized, absorbing ℓ photons and then, the electron is continuosly accelerated out of the laser beam. The energy conserving condition that we obtained, (1.1.95), describes that process in a "proper" way, i.e.

$$\frac{f^2}{2m} = \ell\omega + W_0 \qquad (1.1.102a)$$

$$q^2 = f^2 + 2m^2x^2 \qquad (1.1.102b)$$

where $\vec{f}$ is the momentum of the free electron inside the laser, and that momentum appears to interact with the laser field in (1.1.92).

On the other side, the description of multiphoton ionization in an ultrastrong laser as a one step process in the calculations of Faisal and Keldysh leads to a confusing energy conservation condition

$$W_q = \ell\omega + W_0 - 2mx^2 \qquad (1.1.103)$$

with $\ell > \frac{2mx^2}{\omega}$. But, incorrect phusics contained in that condition has a large influence to the calculated observable - ionization probability. As a consequence of the large minimum ℓ, the order of the

"generalized Bessel functions" is much higher than $\frac{Ry}{\omega}$. That causes that these functions have saddle points in complex plane, rather than stationary phase points, which certainly influences their behaviour and changes the result.

In the calculations of Pert and Gersten-Mittleman the energy conserving condition was

$$W_q = \ell\omega + W_0 \qquad (1.1.104)$$

which represent the energy conservation for just the first step of the process - the absorption of ℓ photons by the atom. That means that the electron momentum $\vec{q}$ is the momentum of the electron inside the laser, and so the recognition of the final state was done inside the laser. The same is true for the calculations of Faisal and Keldysh. As the identification of any of the initial and final state inside the laser is not a gauge invariant procedure, that explains the differences between the results.

Still, the majority of the mentioned results for the multiphoton ionization probability (all except the Keldysh one) have one common property : They decrease as the ultrastrong laser field increases. This rather suprising result could be understood in the following way [16] : The ionization of the atom in an ultrastrong laser is essentialy scattering of the otherwise free electron in the laser field on the nuclear potential. In the situation in which the electronic motion is dominated by the laser, its state is essentialy given by the free electron wave function in the laser, i.e. it is a plane wave with the laser oscillation superimposed. It is well known that a free

electron can not either gain or lose energy in a plane electromagnetic wave [39], and so energy transfer can only occur due to the presence of the third body. In this case it is the electron-nucleus collision, which acts as a perturbation of the free electron in the laser field. The oscillation amplitude of the plane wave function increases for large fields, and one can speak of an oscillating velocity, whose amplitude is $V_0=\frac{eE}{m\omega}=2\chi c$. It is also known [40] that electron-nucleus collisions become less effective as the relative velocity goes up. So, the net effect is a decrease in the ionization rate as the laser electric field E rises.

The principal goal of this Part is to find multiphoton ionization rate for hydrogen in an ultrastrong, spatialy and/or time inhomogeneous, linearly or circularly polarized, monochromatic laser wave, taking the relativistic effects into account. That goal will be achieved in Section 1.4. In Sections 1.2 and 1.3 will be considered the motion of an electron (initial state) and the atom (final state) in the laser field, respectively.

§ 1.2 The motion of an electron in a laser of the finite spatial and time extent.

Let us suppose that the laser is a one mode traveling electromagnetic wave, defined by the three-dimensional vector potential

$$\vec{A} = a(\vec{r},t)\left(\hat{e}_1 \cos\varphi + \varepsilon\hat{e}_2 \sin\varphi\right) \qquad (1.2.1)$$

where $\varphi=\omega t-\vec{k}\cdot\vec{r}$ is the real phase, $|\vec{k}|=\omega$, $\hat{e}_1$ and $\hat{e}_2$ are the orthonormal vectors of polarization and the dimensionless parameter $\varepsilon=0$

for a linearly polarized wave and $\epsilon = 1$ for a circularly polarized wave. The real amplitude a and also $\hat{e}_1, \hat{e}_2, \vec{\kappa}$ are slowly varying functions of position and time on the scale of the laser wavelength. They satisfy the gauge condition

$$\nabla \cdot \vec{A} = 0 \ , \ \vec{\kappa} \cdot \hat{e}_1 = \vec{\kappa} \cdot \hat{e}_2 = 0 \qquad (1.2.2)$$

We will confine ourselves mostly on the two special cases of the space and time inhomogenity of the laser - laser "beam" and laser "pulse". By the laser beam we mean the situation when the laser wave is finite in the direction perpendicular to the direction of the wave propagation $\vec{k}$, but infinite along $\vec{k}$. This can be described by the vector potential of the form (1.2.1), whose amplitude is a function of $\vec{r}_\perp$ only, i.e. $a = a(\vec{r}_\perp)$ where $\vec{r}_\perp \cdot \vec{k} = 0$. By the laser pulse we mean the traveling wave train, which is of infinite spatial extent in the direction perpendicular to $\vec{k}$, and which is described by the vector potential of the form (1.2.1), where $a = a(\varphi)$. $a(\varphi)$ is here amplitude envelope of the wave train, with $a(\pm\infty) = 0$ and we suppose

$$\frac{d}{d\varphi}\left(\ln a(\varphi)\right) \ll 1 \qquad (1.2.3)$$

Volkov [29] has obtained the exact wave function of a Dirac electron in an infinite, traveling, plane, linearly polarized, one mode electromagnetic wave. He started from the vector potential of the form (1.2.1), with $\epsilon = 0$ and $\vec{a}$ and $\vec{k}$ constant with $\vec{a} \cdot \vec{k} = 0$. Solving the Dirac equation, he got the wave function of the electron in the

form

$$\Psi_{\vec{q}} = C_q \left(A_q + B_q \cos\varphi \right) \underset{\sim}{z}_0 \exp i \left(\vec{q}\cdot\vec{r} - E_q t - a_q \sin\varphi - \frac{b}{2} \sin 2\varphi - b\varphi \right) \tag{1.2.4}$$

where $E_q = +\left(q^2 + m^2 \right)^{1/2}$ and

$$A_{\vec{q}} = E_q + \beta m + \vec{\alpha}\cdot\vec{q} \tag{1.2.5}$$

$$B_{\vec{q}} = \frac{mx}{\nu} \left(\vec{q}\cdot\hat{a} + i \left(E_q + \beta m - \vec{q}\cdot\hat{\kappa} \right) \vec{\Sigma}\cdot(\hat{\kappa}\times\hat{a}) + i\vec{\Sigma}\cdot\hat{\kappa}\,\vec{q}\cdot(\hat{\kappa}\times\hat{a}) + \right.$$

$$\left. \frac{i}{3} \vec{\alpha}\cdot\vec{\Sigma}\,\vec{q}\cdot(\hat{\kappa}\times\hat{a}) - \vec{\alpha}\cdot\hat{a} \left(\vec{q}\cdot\hat{\kappa} - \beta m - E_q \right) + \vec{q}\cdot\hat{a}\,\vec{\alpha}\cdot\hat{\kappa} \right) \tag{1.2.6}$$

$$a_q = \frac{2mx}{\omega\nu}\,\vec{q}\cdot\hat{a} \qquad b = \frac{m^2x^2}{\omega\nu} \tag{1.2.7}$$

$$\nu = E_q - \vec{q}\cdot\hat{\kappa} \qquad x = \frac{e|\vec{a}|}{2m} \tag{1.2.8}$$

$\underset{\sim}{z}_0$ is the unit bispinor and C_q is the normalization constant. If the normalization constant is obtained by choosing a finite volume of linear dimension D as the working volume, then the condition

$$\left(\Psi_q, \Psi_q \right) = 1 \tag{1.2.9}$$

yields the leading term in powers of large D as

$$C_q^{-2} = 2 \left(E_q + m \right) \left(E_q + \frac{m^2x^2}{\nu} \right) \tag{1.2.10}$$

The field dependence of the normalization constant is a consequence of the infinite laser extent in all spatial directions.

We want to generalize Volkov's result to the one in the laser of the finite spatial extent and/or the finite time duration. We start with the vector potential (1.2.1) which satisfies the condition (1.2.2). The Dirac equation is

$$\left(i\frac{\partial}{\partial t} - (\vec{\alpha}\cdot\vec{\pi} + \beta m)\right)\Psi = 0 \qquad (1.2.11)$$

where $\vec{\pi} = \vec{p}_r + e\vec{A}$ .Following Volkov, we set

$$\Psi = \left(i\frac{\partial}{\partial t} + \vec{\alpha}\cdot\vec{\pi} + \beta m\right)Z \qquad (1.2.12)$$

The equation for Z, after matrix multiplication is

$$\left(\frac{\partial^2}{\partial t^2} - \nabla_r^2 - 2ie\cos\varphi\,\vec{a}\cdot\nabla_r + e^2a^2\cos^2\varphi + m^2 + e\vec{\Sigma}\cdot\vec{H} + ie\vec{\alpha}\cdot\vec{E}\right)Z = 0$$

$$\left(\frac{\partial^2}{\partial t^2}\right. \qquad (1.2.13)$$

(1.2.13)

where

$$H = \nabla\times\vec{A} \approx \vec{k}\times\vec{a}\sin\varphi\ ,\quad E = -\frac{\partial\vec{A}}{\partial t} \approx \omega\vec{a}\sin\varphi \qquad (1.2.14)$$

In the last step was used supposition that $a(\vec{r},t)$ is a slowly varying function on the spatial scale λ and the time scale $\frac{1}{\omega}$. If $a(\vec{r},t)$ is also slowly varying on the scale of the electron wavelength, then the time averaged motion of the electron is essentialy the classical one.That justifies the use of the eikonal method to solve the

equation (1.2.13). In the first order eikonal approximation, the time averaged motion of the electron is described by the common factor $\exp(iS_0(r,t))$, such that the second and the higher derivatives of $S_0(r,t)$ can be neglected. Making the ansatz

$$z = C_q \exp i \left(S_0(\vec{r},t) - S(\vec{r},t,\varphi) \right) \qquad (1.2.15)$$

where S contains fast oscillating motion, dependent on φ , and with the assumption

$$\left[S, \frac{\partial S}{\partial \varphi} \right] = 0 \qquad (1.2.16)$$

we get from (1.2.13)

$$(\nabla S_0)^2 - \left(\frac{\partial S_0}{\partial t}\right)^2 + 2\omega \left(\frac{\partial S_0}{\partial t} + \hat{k}\cdot\nabla S_0\right)\frac{\partial S}{\partial \varphi} + 2e\cos\varphi\, \vec{a}\cdot\nabla S_0 +$$

$$\frac{e^2a^2}{2} + \frac{e^2a^2}{2}\cos 2\varphi + m^2 + e\omega a g \sin\varphi = 0 \qquad (1.2.17)$$

where

$$g = \vec{\Sigma}\cdot(\hat{k}\times\hat{a}) + i\vec{\alpha}\cdot\hat{a} \qquad (1.2.18)$$

with the property

$$g^2 = 0 \qquad (1.2.19)$$

In the first approximation, based on the slowly varying nature of $a(r,t)$, we can separate (1.2.17) into the two equations, one describing the time averaged motion, and the other describing the fast oscillatory motion of the electron. That yields

$$\left(\frac{\partial S_0}{\partial t}\right)^2 - \left(\nabla S_0\right)^2 = \frac{e^2a^2}{2} + m^2 \qquad (1.2.20)$$

$$2\omega\left(\frac{\partial S_0}{\partial t} + \hat{k}\cdot\nabla S_0\right)\frac{\partial S}{\partial \varphi} + 2e\cos\varphi\,\vec{a}\cdot\nabla S_0 + \frac{e^2a^2}{2}\cos 2\varphi + \qquad (1.2.21)$$
$$+ e\omega a g \sin\varphi = 0$$

The fact that g is the only matrix occuring in S, and that it is independent on φ means that (1.2.16) is satisfied.

Equation (1.2.20) has the form of the relativistic Hamilton-Jacoby equation of a free particle with the rest mass $\left(\frac{e^2a^2}{2} + m^2\right)^{1/2}$. Adopting the metric in which $u^\mu u_\mu = 1$, where u^μ is a contravariant component of the velocity four vector, we can write S_0 as the action integral [36], [33]

$$S_0 = -\int d\lambda \left(\frac{e^2a^2}{2} + m^2\right)^{1/2}\left(\frac{\partial \xi^\mu}{\partial \lambda}\frac{\partial \xi_\mu}{\partial \lambda}\right)^{1/2} + \text{const.} \qquad (1.2.22)$$

where λ is an arbitrary parameter (not necessarily the proper time τ) and $\xi^\mu(\lambda)$ is the four-vector coordinete of the particle. The constraint

$$\frac{d\xi^\mu}{d\lambda}\frac{d\xi_\mu}{d\lambda} = 1 \qquad (1.2.23)$$

is imposed. Integration in (1.2.22) is along the ray path in space-time, for some prescribed boundary conditions. The Euler-Lagrange

equations, which define the family of the physical ray paths, can be obtained by the variation of the action integral (1.2.22) which, together with (1.2.23) gives [33]

$$\frac{d}{ds}\left(m^*(\xi)\frac{\partial \xi_\mu}{\partial s}\right)=\frac{\partial m^*(\xi)}{\partial \xi_\mu} \qquad (1.2.24)$$

where

$$m^*=\left(\frac{e^2a^2(\xi)}{2}+m^2\right)^{1/2}=m\left(1+2\chi^2\right)^{1/2} \qquad (1.2.25)$$

ξ is the position four vector and dimensionless parameter χ was defined in (1.1.2). The equation (1.2.24) has been already derived by Kible [36] on the basis of the pure classical considerations.

Alternatively, following Weinberg's approach [41],[42] (which was given in a different and nonrelativistic context), we can set up the problem of the time averaged part of the solution of eq. (1.2.14) as

$$D(\xi,-i\nabla)\Psi_0=0 \qquad (1.2.26)$$

where ∇ is the four gradient of the position, and

$$D=-\Box+m^2\left(1+2\chi^2(\xi)\right) \qquad (1.2.27)$$

$\Box$ is the four Laplacian $\partial^\mu\partial_\mu$. In the first eikonal approximation, Ψ_0 is supposed in the form

$$\Psi_0 = \exp i(S_0(\xi)) \tag{1.2.28}$$

where S_0 is real, and

$$D(\xi, -i\nabla)\Psi_0 \approx D(\xi, K)\Psi_0 = 0 \tag{1.2.29}$$

with

$$K(\xi) = \nabla S_0(\xi) \tag{1.2.30}$$

Then it follows

$$D(\xi, K) = 0 \tag{1.2.31}$$

and, using (1.2.27)

$$K^2 = m^{*2} \tag{1.2.32}$$

From (1.2.23), ray paths are given as functions of a parameter λ

$$\frac{\partial X_\mu}{\partial \lambda} = \frac{\partial D}{\partial K_\mu} \quad , \quad \frac{\partial K_\mu}{\partial \lambda} = -\frac{\partial D}{\partial X_\mu} \tag{1.2.33}$$

and finaly, the eikonal $S_0(\xi)$ is defined by the path integral (1.2.22) along the world line, defined by (1.2.33). If the four momentum of the electron, before entering the laser was $p_\mu^{(i)}$, or after leaving the

laser is $p_\mu^{(f)}$, then to define the ingoing wave $\psi_0^{(+)} = \exp i S_0^{(+)}$ we should choose the constant of integration in (1.2.22) so that

$$S_0^{(+)}(\xi) = p^{(i)}{}_\mu \xi_\mu \tag{1.2.34}$$

before entering the laser. To define outgoing wave $\psi_0^{(-)} = \exp i S_0^{(-)}$ we have to choose the constant, so that

$$S_0^{(-)}(\xi) = p^{(f)}{}_\mu \xi_\mu \tag{1.2.35}$$

In the spirit of the eikonal approximation, we can interpret ∇S_0 as the classical four momentum of the electron, whose time development is governed by the Hamiltonian

$$H(\vec{p}, \xi) = \left(p^2 + m^{*2}\right)^{1/2} \tag{1.2.36}$$

From this form of the Hamiltonian, as well as from equations (1.2.20), (1.2.22), (1.2.24) and (1.2.33) with (1.2.32) it can be concluded that electron moves in a laser field as a relativistic free particle with a variable rest mass m^*. This can be understood with the simple example of the laser beam. An electron, subjected to the oscillating electromagnetic field $\vec{A} = \vec{a} \cos\varphi$, oscillates along the axis defined by $\hat{a}$, with the velocity

$$v = \frac{2 \times mc^2}{W_{rel}} \cos\varphi \tag{1.2.37}$$

W_{rel} is the total relativistic energy of the electron, in the "inertial"

system in which the electron is at rest outside the laser. On the other hand, $\vec{v} = \frac{\vec{p}}{W_{rel}}$ so that the square of the electron oscillatory momentum averaged over a period of laser oscillation T is

$$\overline{p^2} = \frac{e^2 a^2}{2} = 2\chi^2 m \qquad (1.2.38)$$

$\vec{p}$ is the three-momentum of the electron, in the same inertial system in which W_{rel} was defined. In that way, the relativistic energy of the electron in the laser, averaged over T, in the reference system in which the electron is at rest outside the laser is

$$\overline{W_{rel}} = \left(\overline{p^2} + m^2\right)^{1/2} = m\left(1+2\chi^2\right)^{1/2} \qquad (1.2.39)$$

This can be interpreted as the rest effective mass in that reference system. Any motion of the electron, which changes slowly on the time scale T, is influenced by the oscillatory motion through its average effect: The change of the effective rest mass of the electron.

Nonrelativistically, if $\chi \ll m$, the variable rest mass appears as an effective potential energy function

$$U_p = m^* - m = m\left(\sqrt{2\chi^2+1} - 1\right) \approx m\chi^2 \qquad (1.2.40)$$

This is known as the ponderomotiva potential.

Now we will study the motion of electron in the two special cases of the laser wave - in the laser "beam" and in the laser "pulse". which were defined at the begining of this Section. The Hamiltonian (1.2.36) is conserved in the case of the laser beam and we can separe-

te the time in (1.2.20). We can also separate the $\vec{r}_{\parallel}$ dependence, as $\vec{a}$ is only a function of $\vec{r}_{\perp}$. Making the ansatz

$$S_0 = F_b(\vec{r}_{\perp}) + p_{\parallel} r_{\parallel} - Et \tag{1.2.41}$$

where $p_{\parallel}$ and E are constants, we get from (1.2.20)

$$(\nabla_{\vec{r}_{\perp}} F_b)^2 + \frac{e^2 a^2}{2} + m^2 + p_{\parallel}^2 - E^2 = 0 \tag{1.2.42}$$

Then, it follows

$$F_b(\vec{r}_{\perp}) = \int^{\vec{r}_{\perp}} \vec{f}_{\perp} \cdot d\vec{r}_{\perp}' \tag{1.2.43}$$

where $f_{\perp}^2 = E^2 - p_{\parallel}^2 - m^2 - \frac{e^2 a^2}{2}$ .Matching this solution with the boundary conditions (1.2.34) or (1.2.35) in $t = \mp\infty$, we get

$$S_0 = \int^{\vec{r}} \vec{f} \cdot d\vec{r} - Et \tag{1.2.44}$$

where $E = (q^2 + m^2)^{1/2}$ and

$$f_{\perp}^2 = q_{\perp}^2 - \frac{e^2 a^2(\vec{r}_{\perp})}{2}, \quad f_{\parallel} = q_{\parallel} \tag{1.2.45}$$

D, defined by (1.2.29) is here

$$D(\vec{r}_{\perp}, \vec{f}) = f^2 - q^2 + \frac{e^2 a^2(\vec{r}_{\perp})}{2} \tag{1.2.46}$$

To construct $\vec{f}$, we introduce a family of ray paths through "phase space" [42]

$$\vec{r} = \vec{r}(s), \vec{f} = \vec{f}(s) \tag{1.2.47}$$

which is , for any initial conditions ($\vec{r}(s_0), \vec{f}(s_0)$) defined by

$$\frac{\partial \vec{r}}{\partial s} = \frac{\partial D}{\partial \vec{f}} \quad , \frac{\partial \vec{f}}{\partial s} = -\frac{\partial D}{\partial \vec{r}} \tag{1.2.48}$$

Along any ray path D is constant, i.e.

$$\frac{\partial D}{\partial s} = 0 \tag{1.2.49}$$

If we choose ($\vec{r}_0$,$\vec{q}$) at $s = s_0$, which satisfy the condition

$$D(\vec{r}_0, \vec{q}) = 0 \tag{1.2.50}$$

then $\vec{r}$ and $\vec{f}$ are determined for all s and still

$$D(\vec{r}, \vec{f}) = 0 \tag{1.2.51}$$

The problem is essentialy a two-dimensional, and consists of the construction of $\vec{f}_\perp$, as $\vec{f}_{||} = \vec{q}_{||}$ is defined for all s.

Let's consider an "initial" surface Γ in the coordinate space, with unit normal $\hat{n}(\vec{r})$, and suppose that for each $\vec{r}_0$ on Γ the conditions

$$D(\vec{r}_0, \vec{q}) = 0$$
$$\hat{q} = \hat{n}(\vec{r}_0) \qquad (1.2.52)$$

determine an unique $\vec{q}(\vec{r}_0)$. The second of equations (1.2.52) ensures that $\nabla_{\vec{r}} F_b$ is normal to Γ on Γ, so that S is constant over Γ. For any given $\vec{r}$, there is an unique path leading from Γ to $\vec{r}$ and defined by

$$\vec{r}(s_0) \in \Gamma \;,\; \vec{f}(s_0) = \vec{q}(\vec{r}(s_0)) \qquad (1.2.53)$$

Then $\vec{f}(\vec{r})$ is given as

$$\vec{f}(\vec{r}) = f(s) \qquad (1.2.54)$$

where $\vec{r}(s) = \vec{r}$, and integral in (1.2.44) can be written as

$$\int_{s_0}^{\hat{s}} \vec{f}(s) \cdot \frac{\partial \vec{r}}{\partial s} ds \qquad (1.2.55)$$

$\vec{f}(\vec{r})$ is always normal to the surface of constant $F_b(\vec{r})$, passing through $\vec{r}$. In the direction of $\vec{k}$ the wave fronts are plane surfaces, perpendicular to $\vec{k}$ and $\vec{q}_{\parallel}$. An electron, entering the laser beam, will be deflected in the plane perpendicular to $\vec{k}$, due to the variation of $\vec{\alpha}(\vec{r}_{\perp})$. If $|\vec{q}|$ is not large enough, so that f becomes negative, the electron will be reflected back, leaving the beam with the same kinetic energy as it entered the beam.

Now, consider the case of the laser "pulse". Writting again the relation (1.2.20), we have

$$\left(\frac{\partial S_0}{\partial t}\right)^2 - \left(\nabla S_0\right)^2 = \frac{e^2 a^2(\varphi)}{2} + m^2 \qquad (1.2.56)$$

In order to match the boundary conditions (1.2.34) or (1.2.35), we make the ansatz

$$S_0 = \vec{q}\cdot\vec{r} - E_q t + F_p^{\pm}(\varphi) \qquad (1.2.57)$$

with $E_q = (q^2 + m^2)^{1/2}$. Then, from (1.2.56) we get

$$-2\omega\left(E_q - \vec{q}\cdot\hat{k}\right)\frac{\partial F}{\partial \varphi} = \frac{e^2 a^2(\varphi)}{2} \qquad (1.2.58)$$

and so

$$F_p = -\frac{e^2}{4\omega\left(E_q - \vec{q}\cdot\hat{k}\right)} \int^{\varphi} a^2(\varphi) \qquad (1.2.59)$$

As far as the spatial dependence is considered, this is essentially a one dimensional problem, and S_0 is uniquely determined by (1.2.59). The time dependent "ponderomotive potential" acts in this case only in the direction of the wave propagation. In fact, being overtaken by the laser pulse, the electron is subjected to the increasing repulsive potential, so that its total energy E^* inside the laser pulse is increasing as long as $a(\varphi)$ is increasing, i.e.

$$E^* = -\frac{\partial S_0}{\partial t} = E_q + \frac{e^2 a^2(\varphi)}{4\left(E_q - \vec{q}\cdot\hat{k}\right)} = E_q + V^*(\varphi) \qquad (1.2.60)$$

under the action of the "force"

$$\mathcal{F}^* = -\frac{\partial V^*}{\partial r_{\parallel}} = \frac{e^2}{4(E_{\vec{q}} - \vec{q}\cdot\hat{k})} \frac{\partial}{\partial \varphi} (a^2(\varphi))\, \vec{k} \qquad (1.2.61)$$

the electron gains the momentum, in the direction of the wave propagation

$$p_{\parallel}^* = \frac{\partial S_0}{\partial r_{\parallel}} = q_{\parallel} + \frac{e^2 a^2(\varphi)}{4(E_{\vec{q}} - \vec{q}\cdot\hat{k})} \qquad (1.2.62)$$

After $a(\varphi)$ passes its maximum, and starts to decrease, the force changes its sign, and when the electron is left by the pulse (at $t=+\infty$) it has the same momentum as it had before being overtaken by the pulse. This is in full agreement with the adiabatic hypothesis, as it was in the case of the laser beam.

The process of acceleration of free electrons by long, ultrastrong laser pulses was discussed recently by Lebouf [43] , in the context of its application for the heating of ultradense plasmas. The electron is accelerated to ultrarelativistic energies by the action of the ponderomotive potential, due to the spatial inhomogenity of the laser wave. It will be expelled from the focus, thereby depleting that region. Although the interaction of ions with the laser field is much weaker (by the factor $\frac{m}{M}$) than the one with electrons, ions will be pulled away from the focus region, gaining high energies, because of very strong electrostatic, charge separation forces.

It is useful to discuss whether the laser pulse or the laser beam is the better description of the real physical situation of the interaction of the laser wave with an electron. That obviously depends

on certain parameters such as the duration of the pulse τ , the width L of the beam in the focal region, and the electron velocity v . The electron has a greater probability of entering the beam (from the side) if $|\vec{v}_\perp|\tau \gg L$. If τ is of order 10^{-8} sec, that gives $v_\perp \gg 10^8 L$. For electrons of kinetic energy of the order of 1 eV, that requires $L \ll$ 5 mm, which is mostly satisfied. But the electron will be reflected from such beam if the laser intensity is larger than~$10^{13} \frac{W}{cm^2}$ (in optical region). So, at a given laser intensity, even if $v_\perp \tau \gg L$ is fulfilled, the electrons with an energy, less than the critical one, will be overtaken by the laser pulse, being not able to penetrate the laser beam. But, if the laser pulse is long, these electrons will leave the laser from the side, as though it were the beam. So, the effect of a long laser pulse on the cloud of free, noninteracting electrons is to change the energy distribution of the cloud, depleting the region $E < mX^2$. The effect is better if the pulses are longer.

Now we shall take care of the oscillatory part S of Z (equations (1.2.15), (1.2.17), (1.2.21)). From (1.2.44) and (1.2.57) it follows, that for both the laser beam and the laser pulse

$$\frac{\partial S_0}{\partial t} + \hat{\kappa} \cdot \nabla S_0 = \mathcal{Q}_{11} - E_{\mathcal{Q}} \qquad (1.2.63)$$

and $\vec{a} \cdot \nabla S_0 = \vec{a} \cdot \vec{f}_\perp$ for the beam, while $\vec{a} \cdot \nabla S_0 = \vec{a} \cdot \vec{\mathcal{Q}}_\perp$ for the pulse. So, from (1.2.21) we have

$$S = \frac{e\vec{a} \cdot \vec{f}^{(i)}}{\omega(E_{\mathcal{Q}} - \mathcal{Q}_{11})} \sin\varphi + \frac{e^2 a^2}{4\omega(E_{\mathcal{Q}} - \mathcal{Q}_{11})} \sin 2\varphi - \frac{ea}{2(E_{\mathcal{Q}} - \mathcal{Q}_{11})} g \sin\varphi \qquad (1.2.64)$$

where

$$\vec{f}_\perp^{(1)} = \vec{f}_\perp^{p} = \vec{q}_\perp$$
$$\vec{f}_\perp^{(2)} = \vec{f}_\perp^{b} = \vec{f}_\perp \qquad (1.2.65)$$

With the known S_0 , S we can determine Z in (1.2.15), which can be substituted into (1.2.12), to obtain Ψ . So, for the case of the laser pulse we have

$$\Psi_q = C_q \left(i\frac{\partial}{\partial t} - i\vec{\alpha}\cdot\nabla + e\vec{\alpha}\cdot\vec{a}\cos\varphi + \beta m \right) \exp i \left(\vec{q}\cdot\vec{r} - E_q t - \right.$$
$$\left. - S^{(p)}(\varphi) - \frac{m^2}{\omega\nu}\int^{\varphi} \chi^2(\varphi')d\varphi' \right) \underset{\sim}{z}_0 \qquad (1.2.66)$$

For the laser beam we get

$$\Psi_q = C_q \left(i\frac{\partial}{\partial t} - i\vec{\alpha}\cdot\nabla + e\vec{\alpha}\cdot\vec{a}\cos\varphi + \beta m \right) \exp i \left(\int^{\vec{r}} \vec{f}\cdot d\vec{r} - \right.$$
$$\left. - E_q t - S^{(b)}(\varphi) \right) \underset{\sim}{z}_0 \qquad (1.2.67)$$

$S^{(b)}(\varphi)$ is also a slowly varying function of $\vec{r}_\perp$, but we neglected the derivatives in $\vec{r}_\perp$ of $a(r_\perp)$. Application of the derivatives , matrix multiplication and some rearangements yield for the case of the laser pulse essentialy the Volkov solution

$$\Psi_q^{(P)} = C_q \left(A_q + B_q \cos\varphi\right) \underset{\sim}{z}_o \, e^{i\left(\vec{q}\cdot\vec{r} - E_q t - a_q \sin\varphi - \frac{b}{2}\sin 2\varphi - \int^{\varphi} b(\varphi')d\varphi'\right)} \tag{1.2.68}$$

where A_q, B_q, a_q, b, ν, χ are given by (1.2.5), (1.2.6), (1.2.7) and (1.2.8). For the laser beam we obtained

$$\Psi_q^{(b)} = C_q \left(A_f - \frac{m^2\chi^2}{\nu}(1+\vec{\alpha}\cdot\hat{K}) + B_f \cos\varphi\right) \underset{\sim}{z}_o \times$$
$$\times \exp i\left(\int^{\vec{r}_1} \vec{f}_\perp \cdot d\vec{r}_1' + f_{11} r_{11} - E_q t - a_f \sin\varphi - \frac{b}{2}\sin 2\varphi\right) \tag{1.2.69}$$

where $f_\perp^2 = q_\perp^2 + 2\chi^2 m$, $f_{11} = q_{11}$, and A_f, B_f, a_f have the same forms as in the case of the pulse, except that for every occurring of $\vec{q}$ it should be replaced by the electron momentum inside the laser $\vec{f}$. In $\Psi_q^{(P)}$, it should be understood that $\vec{a} = a(\varphi)$ and in $\Psi_q^{(b)}$, $\vec{a} = \vec{a}(\vec{r}_1)$.

The whole discussion, given above in this Section, is also valid in the case of the circularly polarized laser wave, which is described by the vector potential of the form (1.2.1), with $\varepsilon = 1$. But, as here $\vec{A}^2 = a^2$, we can expect the disapearance of the term $\sin 2\varphi$ in the phase of the wave functions (1.2.68), (1.2.69). This is a consequence of the fact that the amplitude of the vector potential is constant during one oscillation period, for the circularly polarized laser. That causes the length of the vector of electron oscillatory momentum to be constant in period T. Then, the time average value of the momentum is equal to its amplitude and so the electron effective rest mass in the laser is now $m^* = m\,(1+4\chi^2)^{1/2}$.

Repeating exactly the same steps as we did for the linearly polarized laser, we have in the case of the laser pulse

$$\psi_q^{(p)\,cir} = C_q \left(A_q + B_q^{(1)} \cos\varphi + B_q^{(2)} \sin\varphi \right) \underset{\sim}{\xi}_0 \times$$
$$\times \exp i \left(\vec{q}\cdot\vec{r} - E_q t - a_q^c \sin(\varphi-\theta) - 2\int^{\varphi} b(\varphi')\,d\varphi' \right) \qquad (1.2.70)$$

where $B_q^{(i)}$ is B_q in (1.2.6), with $\hat{a}$ replaced by $\hat{e}_i$, i=1,2. A_q is given by (1.2.5) and b by (1.2.7), $a_q^c = \frac{2mx}{\omega\nu} q_\perp$ and $\hat{q}_\perp \cdot \hat{e}_1 = \cos\theta$.

In the case of the laser beam we get

$$\psi_q^{(b)\,cir} = C_q \left(A_f - \frac{2m^2x^2}{\nu}(1+\vec{\alpha}\cdot\hat{\kappa}) + B_f^{(1)} \cos\varphi + B_f^{(2)} \sin\varphi \right) \underset{\sim}{\xi}_0 \times$$
$$\times \exp i \left(\int^{\vec{r}_\perp} \vec{f}_\perp \cdot d\vec{r}_\perp' + f_\parallel r_\parallel - E_q t - a_f^c \sin(\varphi-\theta) \right) \qquad (1.2.71)$$

where A_f, $B_f^{(i)}$, a_f^c and θ are as in the case of the laser pulse, except that $\vec{q}$ is replaced by $\vec{f}$, and

$$f_\perp^2 = q_\perp^2 - 4x^2 m$$
$$f_\parallel = q_\parallel \qquad (1.2.72)$$

Finaly, we must also obtain the normalization constants C_q for the wave functions, given by (1.2.68), (1.2.69), (1.2.70) and (1.2.71). We do this by working in a finite volume of linear dimension D. Then, if the cross section of the laser beam has a dimension L, the condition

$$(\psi_q, \psi_q) = 1 \qquad (1.2.73)$$

yields

$$C_q^{-2} = 2E_q (E_q + m) D^3 \left(1 + O\left(\frac{L^2}{D^2}\right)\right) \qquad (1.2.74)$$

We assume $L/D \ll 1$

If the duration of the laser pulse is τ , then the condition (1.2.73) yields

$$C_q^{-2} = 2E_q(E_q+m)D^3\left(1+O\left(\frac{c\tau}{D}\right)\right) \qquad (1.2.75)$$

and we assume $c\tau/D \ll 1$.

& 1.3 The motion of an atom in an ultra-strong laser of the finite spatial and time extent. As was pointed out in & 1.1, an atom cannot pick up energy monotonicaly in a harmonicaly oscillating electromagnetic field. However, if the atomic parameters - charge, dipole moment, polarizability etc. could change with time or space, continuous acceleration can be produced. This has been the subject of many theoretical [51],[52],[53],[54],[55] and experimental [56],[57],[58],[59],[60],[61] work.

The absorption and isotropic reradiation by a spontaneous emission of the resonance radiation results in an average driving force on the atom, and the so called radiation pressure in the direction of the incident light. Suppose we have an atom in the ground state, and the frequency of the laser pulse is close to the frequency of the transition between the ground and an upper state. Then, absorption of a photon leaves atom in the upper state, and the momentum of the photon, $\hbar\vec{k}$, is transfered to the atom. If the intensity of the laser is low, a photon will be emitted from the upper state due to fluorescence, in random directions and in usual dipole pattern. Then the net scattering

force, which describes the linear momentum per sec, gained by the atom, is proportional [56]

$$\vec{F} \sim \frac{\hbar \vec{K}}{\tau} \tag{1.3.1}$$

where τ is the natural lifetime of the excited state. In the process of acceleration the atom tends to decouple itself from the laser by the Doppler shift of the laser frequency, which causes the detuning of the laser from the resonance, and the atom is returned to the ground state [55]. Various methods were proposed for recoupling the atom [55] as well as increasing the efficiency of the process [58].

If the atom is in a spatialy inhomogeneous laser wave, it can exibit ponderomotive forces on its induced dipole moment. It is easy to see it from the action of the Lorentz force on the atomic dipole moment [52]

$$\vec{F} = (\vec{p} \cdot \nabla)\vec{E} + \frac{1}{c}\frac{d\vec{p}}{dt} \times \vec{B} \tag{1.3.2}$$

where $\vec{p} = \alpha \vec{E}$, and α is the polarizability of the atom in a particular state. To simplify, let's suppose that α is independent of the laser intensity. Then, by the use of the Maxwell equation $\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$ and the identity $(\vec{E} \cdot \nabla)\vec{E} = \nabla(\frac{1}{2}E^2) - \vec{E} \times (\nabla \times \vec{E})$ one can get

$$\vec{F} = \alpha \left(\nabla \left(\frac{1}{2} E^2 \right) + \frac{1}{c}\frac{\partial}{\partial t}(\vec{E} \times \vec{B}) \right) \tag{1.3.3}$$

which yields the time averaged force as

$$\overline{\vec{F}} = \alpha \left(\nabla \left(\frac{1}{4} E_0^2 \right) + \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{E_0^2}{2} \right) \hat{\kappa} \right) \qquad (1.3.4)$$

E_0 is the amplitude of the laser electric field, and the averaging is over a period of oscillation of the electromagnetic field. In the case of a traveling laser pulse, infinite in the direction perpendicular to the direction of its propagation, it follows from (1.3.4)that the force on the atom acts in the direction of the wave propagation at the front edge of the pulse, and in oposite direction at the back edge of the pulse [52], (Fig. 1.1).

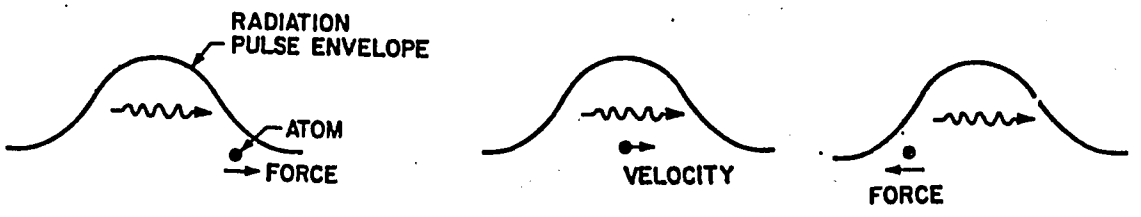

Fig. 1.1 [52]

A remarkable application of these forces is in laser separation of isotopes. One can selectively accelerate, trap or separate the atoms or molecules because of their large cross sections at specific resonances. Selective excitation can be done by virtue of isotope shift of the resonance levels.

In an ultrastrong laser wave it could be that the concept of the light pressure and the ponderomotive forces in the sense described above, looses any meaning. In the ultrastrong laser field the mixing and shifting of the states, as well as their width, are so large that the concept of the isolated atomic states is not acceptable any more.

For the same reasons, one shouldn't expect the resonance behaviour of the atom excitation and ionization at certain frequences of the laser. In such situation, the motion of the electron will be governed more by the laser than by the binding potential of the nucleus, and it is consistent with such picture to suppose that the motion of the atom in an ultrastrong inhomogeneous laser field is governed dominantly by the processes, that govern the motion of a free electron in such field. As we have seen in & 1.2, such process is the acceleration of the electron in a laser field due to the electron's effective mass. The electron effective rest mass is in this case, in the first approximation, the time or space changable atomic parameter which causes continuous energy transfer from the electromagnetic field to the atom.

In & 1.1 we have already discussed the motion of the hydrogen atom in the ultrastrong laser "beam", due to the interaction with the "ponderomotive" potential (1.1.73), in nonrelativistic, dipole approximation. Since the coupling of the electromagnetic field to the nucleus is $\frac{m}{M}$ smaller than to the electron (M is the mass of the nucleus), the hydrogen atom couples to the ponderomotive potential essentialy through the electron. That coupling is described as a residual coupling in the center of mass system, as the field which interacts with the electron is dependent on the center of mass coordinate. Although nor the dipole neither the nonrelativistic approximation are acceptable in the case we discuss, such analysis could help to understand in a simple way the basic features of the atomic motion.The motion of the hydrogen atom in the ultrastrong laser pulse will be treated relativistically later in this Section. If we make a nonrela-

tivistic approximation ($x \ll 1$) of the relativistic result evaluated for that case, we get that the atom, being overtaken by the laser pulse, gains momentum

$$\Delta \vec{p} = m \chi^2(\phi) \hat{K} \qquad (1.3.5)$$

and, at the same time, it has the potential energy

$$V^* = m \chi^2(\phi) \qquad (1.3.6)$$

As it is being left by the pulse, an atom looses its momentum (the force changes the direction) and the overall change of the atomic momentum, when the atom is free of laser again is zero. Here we supposed that the atom was in its ground state upon leaving the laser, as it had been before entering the laser. But, as we will see in & 1.4, the ionization rate for the hydrogen atom increases with the laser intensity even in the ultrastrong laser limit, and the chanses that the atom will survive in the laser become very small. In that way, any applicable discussion about the possibility of the acceleration of atoms by an ultrastrong laser could be irrelevant.

As we saw in & 1.2 for the case of a free electron, ponderomotive potential $U_p = \frac{e^2 a^2}{4m}$ is a nonrelativistic approximation of the more general change in effective rest mass of the electron, which arisses when the electron motion is treated relativistically. We now want to find the relativistic generalization of the solution (1.1.80) for the center of mass motion. We describe the atom in the laser field in terms of the center of mass and relative coordinates by

$$\left(i\frac{\partial}{\partial t} - T_R - (\vec{\alpha}\cdot\vec{\pi} + \beta m + V(r))\right)\Psi = 0 \qquad (1.3.7)$$

where $\vec{R}$ is the center of mass coordinate and $\vec{r}$ is the position of the electron relative to the proton. $\vec{\alpha}$ and β are the usual Dirac matrices of the electron, and

$$\vec{\pi} = \vec{p}_r + e\vec{A} \qquad (1.3.8)$$

First we deal with a linearly polarized laser, with the vector potential in the form

$$\vec{A} = \vec{a}(\vec{R},t)\cos\varphi\ ,\ \varphi = \omega t - \vec{K}(\vec{R}+\vec{r}) \qquad (1.3.9)$$

We suppose that the amplitude $\vec{a}(R,t)$ is a slowly varying function of time on the scale of the period of the laser oscillation $T = \frac{2\pi}{\omega}$. It is also a slowly varying function of space on the scale of the laser wavelength and the atom wavelength. We have in mind a particular example of CO_2 laser, whose frequency of oscillation is much smaller than the atomic characteristic frequency (by the factor of about 100). The potential V(r) is the pure Coulomb potential, as we deal with hydrogen atom. We take

$$T_R = \frac{p_R^2}{2(M+m)} \qquad (1.3.10)$$

so that internal motion of the atom is described relativistically, but the center of mass is described nonrelativistically. This is permissi-

ble, since M≫m. In fact, in tha case when the motion of the atom is nonrelativistic, which we suppose, the transformation from the laboratory frame to the rest frame of the nucleus is Galilean, so that the atom has the same time as it is in the laboratory system. The relativistic electron is connected to the laboratory frame and to the rest frame of the atom by the Lorentz transformation. The motion of the atom is described by (1.3.7) as a plane wave, although deformed, because of the coupling with the electronic motion. Our supposition of slowness of the change of the amplitude of the laser field allows us to suppose that the motion of the atom is only slightly changed in the characterisic time of the electron motion and the laser oscillations. So, we can introduce a local inertial system, connected with the atom, and then describe the Galilean transformation to the laboratory system by the slowly varing phase of the plane wave (eikonal approximation).

Similarly as in & 1.2, equation (1.2.13), we set

$$\Psi = \left(i\frac{\partial}{\partial t} - T_R - V + \vec{\alpha}\cdot\vec{\pi} + \beta m \right) Z \qquad (1.3.11)$$

Substituting this in eq.(1.3.7), after matrix multiplication we get the equation for Z

$$\left((E-V)^2 - m^2 - \vec{\pi}^2 - e\vec{\Sigma}\cdot\vec{H} - ie\vec{\alpha}\cdot\vec{E} \right) Z = 0 \qquad (1.3.12)$$

where we used the short hand

$$E = i\frac{\partial}{\partial t} - T_R = i\frac{\partial}{\partial t} + \frac{\nabla_R^2}{2M} \qquad (1.3.13)$$

and

$$\vec{H} = \nabla \times \vec{A} \quad , \quad \vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{e}\nabla_r V$$

In derivation of (1.3.12) we neglected the terms of the order $\frac{1}{\tau}$, where τ is characteristic time of the change of the laser field amplitude $\vec{a}(R,t)$. The commutator $[\vec{\pi}, T_R]$ is also neglected on the basis of the smallness of the atom wavelength compared to the laser wavelength and to the characteristic length L of the spatial change of $\vec{a}$. The terms, that oscillate in time can be treated as a perturbation, which causes the absorption and emission of an integer number of photons, but doesn't significantly influence the continuous acceleration of the system. These will be droped from (1.3.12). The dropped part of "Hamiltonian" in (1.3.12) is

$$\mathcal{H}' = 2e\vec{A}\cdot\vec{p}_r + \frac{1}{2}e^2a^2\cos 2\varphi + e\omega a g \sin\varphi \qquad (1.3.14)$$

where g is a matrix, defined by (1.2.18). In that way we are left with equation

$$\left[(E-V)^2 - m^2 - p_r^2 - \frac{1}{2}e^2a^2(\vec{R},t) + \vec{\alpha}\cdot(\vec{p}_r V)\right] Z_0 = 0 \qquad (1.3.15)$$

Since $\vec{\alpha}$ mixes large and small components of the electron, its expectation value is of the order $\frac{v}{c}$, where $v \sim \frac{\hbar}{md}$, d is a linear dimension of the atom. Then

$$\frac{c\vec{\alpha}\cdot(\vec{p}_r V)}{2EV} \sim c\frac{v}{c}\frac{mv}{mc^2} = \left(\frac{v}{c}\right)^2 \sim \alpha_F^2 \qquad (1.3.16)$$

where $\alpha_F = (137)^{-1}$ is the fine structure cinstant. Also. it follows

$$\frac{V^2}{2EV} \sim \frac{V}{mc^2} \sim \frac{e^2}{d\,mc^2} \approx \frac{e^2}{\hbar c}\frac{v}{c} \sim \alpha_F^2 \qquad (1.3.17)$$

So, we can neglect V^2 and $\vec{\alpha}(\vec{p}_r V)$ in comparison to $2E\,V$ which, applied in (1.3.15), gives the equation

$$\left(E^2 - 2EV - m^2 - p_r^2 - 2m^2 X^2(\vec{R},t)\right) Z_0 = 0 \qquad (1.3.18)$$

where X was defined in (1.1.2).

Assume now that the vector potential amplitude is only a function of $\vec{\rho}$, where $\vec{\rho}$ is that part of $\vec{R}$ which is perpendicular to $\vec{k}$ ($\vec{k}$ defines the propagation direction of the laser wave). This is just the case of the laser "beam", defined at the begining of & 1.2. To solve equation (1.3.18) we use that $X(\vec{\rho})$ is a slowly varying function and set, similarly as in the theory of atomic-atomic collisions [40]

$$Z_0 = \sum_n \chi_n(\vec{R})\, U_n(\vec{R},r)\, e^{-i\frac{p_i^2}{2M}t - i W_{n0} t} \qquad (1.3.19)$$

where $U_n(R,r)$, a slowly varying function of $\vec{R}$, is an eigenfunction of the "distorted" Schrodinger equation which corresponds to the eigenvalue E_n' , i.e.

$$\left(E_n' - V(r) - \frac{p_r^2}{2m^*}\right) U_n(\vec{R},r) = 0 \qquad (1.3.20)$$

The electron rest mass m is here replaced by the effective rest mass

$$m^* = \frac{P_i^2}{2M} + \frac{1}{2M}\frac{\nabla_R^2 \chi(\vec{R})}{\chi(\vec{R})} + m + E'_{n0} \qquad (1.3.21)$$

and $W_{n0} = m + E'_{n0}$. E'_{n0} is an eigenvalue of equation (1.3.20) with $m^* = m$, which evolves in E'_n in the presence of the laser.

The ansatz (1.3.19) is based on the large mass difference between the electron and the nucleus. This difference implies much slower motion of nucleus with respect to the electronic motion, and we can separate Hamiltonian in (1.3.18) in nuclear and electronic part, as in (1.3.19). The slowness of the change of $\vec{a}(\vec{p})$ implies the weak coupling of the nuclear and electronic motion, and $\vec{R}$ in $U_n(\vec{R},r)$ can be considered as a slowly varying parameter, rather than a dynamical variable. As U_n form a complete orthonormal set, using (1.3.19) in (1.3.18) and multipliing from the left by U_n^* , we get equation for center of mass wave functions χ

$$\left(\varepsilon_n^2 - 2E'_n\varepsilon_n - m^2(1+2\chi^2)\right)\chi_n(\vec{R}) = 0 \qquad (1.3.22)$$

where we neglected E'_n compared to m . The terms which represent a kinematic coupling of nuclear and electronic motion were also neglected. In (1.3.22), $i\frac{\partial}{\partial t}$ in E was replaced by $\frac{P_i^2}{2M} + m + E'_{n0} = \varepsilon_n$, and so $E \to \varepsilon_n - T_R = W_{n0}$. In the first eikonal approximation, we make the ansatz [41]

$$\chi_n(\vec{R}) = \exp i\left(\int^{\vec{R}} d\vec{R}' \cdot \vec{P}_n(\vec{R}')\right) \qquad (1.3.23)$$

where derivatives of $\vec{P}_n(\vec{R})$ can be neglected. Then (1.3.22) gives

$$\left(\varepsilon_n - \frac{P_n^2}{2M}\right)^2 - 2E_n'\left(\varepsilon_n - \frac{P_n^2}{2M}\right) - m^2(1+2\chi^2) = 0 \qquad (1.3.24)$$

For the hydrogen-like atoms energy eigenvalues are

$$E_{n0}' = -\frac{m\mathcal{Z}^2e^4}{2n^2} \qquad (1.3.25)$$

and so

$$E_n' = E_{n0}'\,\frac{\varepsilon_n - \frac{P_n^2}{2M}}{m} \qquad (1.3.26)$$

Substituting (1.3.26) into (1.3.24), it follows

$$\varepsilon_n - \frac{P_n^2}{2M} = \frac{m\sqrt{1+2\chi^2}}{\left(1-\frac{2E_{n0}'}{m}\right)^{1/2}} \approx (m+E_{n0}')\sqrt{1+2\chi^2} \qquad (1.3.27)$$

and finaly

$$|\vec{P}_{n\perp}| = \left(P_{i\perp}^2 - 2M(m+E_{n0}')\left(\sqrt{1+2\chi^2}-1\right)\right)^{1/2}$$
$$|\vec{P}_{n\parallel}| = |\vec{P}_{i\parallel}| \qquad (1.3.28)$$

where $|\vec{P}_i|$ is the classical momentum of the atom at $t=-\infty$. The center of mass momentum $\vec{P}_n(\vec{R})$ is the classical momentum, whose evolution is governed by the Hamiltonian

$$H_0(\vec{P}_n, \vec{R}) = \frac{P_n^2}{2M} + U_{Pn}(\vec{R}) \qquad (1.3.29)$$

where

$$U_{pn}(\vec{R}) = (m + E'_{n0})\left(\sqrt{1+2\chi^2} - 1\right) \qquad (1.3.30)$$

U_{pn} is the relativistic generalization of the ponderomotive potential (1.1.73), and it becomes $m\chi^2$ in the weak field limit. The largest error, associated with this solution of equation (1.3.15) is of the order of $\left(P_n(\vec{R})L\right)^{-1}$, and is sufficiently small to be negligible for macroscopic beams. The line integral in (1.3.23) is taken along the path of the classical motion. To find such paths, we can use the procedure of Weinberg [41], [42], as we did in & 1.2 for the free electron motion. We have defined D by equation (1.2.26) and it is here

$$D(\vec{R}, \vec{P}_n) = \frac{P_n^2}{2M} + U_{pn}(\vec{R}) - \frac{P_i^2}{2M} \qquad (1.3.31)$$

To construct $\vec{P}_n$ and find the path, we have to solve the equations of motion

$$\frac{\partial \vec{R}}{\partial s} = \frac{\partial D}{\partial \vec{P}_n} \;, \quad \frac{\partial \vec{P}_n}{\partial s} = -\frac{\partial D}{\partial \vec{R}} \qquad (1.3.32)$$

where s is an arbitrary parameter (which could be a measure of time along the path). For any given initial conditions

$$\vec{R}_0 = \vec{R}(s_0) \;, \quad \vec{P}_i = \vec{P}_n(s_0) \qquad (1.3.33)$$

these equations uniquely define $\vec{P}_n$ and the path of the atom. Note

that in the case of the laser beam, we deal with, $\vec{P}_{n\parallel} = \vec{P}_{i\parallel}$ and the problem is a two dimensional one.

So, neglecting the kinematic coupling of the nuclear and electronic motion, we defined the stationary states of the center of mass motion $\chi_n(\vec{R})$ in the (electronic) potential energy $U_{pn}(\vec{R})$. In addition, the atom that enèters the laser adiabatically compresses in its size, in every state. Inelastic transitions in the potential U_{pn} are adiabatically improbable. In particular, for the hydrogen ground state, we have

$$W_0 = m - Ry$$

$$U_0 = e^{-\eta r}, \quad \eta = \frac{1}{a_0}\left(1 - \frac{1}{2}\alpha_F^2\right)\sqrt{1+2\chi^2} \qquad (1.3.34)$$

$$U_{p0}(\vec{R}) = (m - Ry)\left(\sqrt{1+2\chi^2} - 1\right)$$

In the case of the laser "pulse", of the kind as was defined at the begining of & 1.2, amplitude of the vector potential is a slowly varying function of ϕ , where $\phi = \omega t - \vec{k}\cdot\vec{R}$. In that way, $\chi = \chi(\phi)$ and we can solve equation (1.3.18), making the ansatz

$$Z_0 = \sum_n U_n(r,\phi)\, e^{i\int^{\phi} S_n(\phi)d\phi' + i\vec{P}_i\cdot\vec{R} - i\frac{P_i^2}{2M} - iW_{n0}t} \qquad (1.3.35)$$

where $W_{n0} = m + E'_{n0}$. U_n is the eigenfunction, which corresponds to the eigenvalue E'_n of the Schrodinger equation

$$\left(E'_n - V(r) - \frac{p_r^2}{2m^*}\right) U_n(r,\phi) = 0 \qquad (1.3.36)$$

where

$$m^* = m_n^* = W_{no} - \omega S(\phi) - \frac{\omega^2 S_n^2(\phi)}{2M} + \frac{P_{i\parallel}\omega}{M} S_n(\phi) = \mathcal{E}_n \qquad (1.3.37)$$

$\mathcal{E}_m$ obeys the equation

$$\mathcal{E}_n^2 - 2E_n'\mathcal{E}_n - m^2(1+2X^2(\phi)) = 0 \qquad (1.3.38)$$

E_n' is a function of $S_n(\phi)$ and, from (1.3.25) it follows

$$E_n' = E_{no}' \frac{\mathcal{E}_n}{m} \qquad (1.3.39)$$

On the other hand, (1.3.38) gives

$$\mathcal{E}_n = (m + E_{no}')\sqrt{1+2X^2(\phi)} \qquad (1.3.40)$$

The last two terms on the right hand side of (1.3.37) are much smaller than $\omega S_n(\phi)$ as they have the large atomic mass in the denominator, compared to ω^2 and $P_i\omega$ in numerator. Neglecting these two terms, we get S_n of the order of $\frac{mX}{\omega}$. With such S_n we can easily see that

$$\frac{\omega^2 S_n^2}{2M} : \omega S \sim \frac{m}{M} \ll 1, \quad \frac{P_{i\parallel}\omega S}{M} : \omega S \sim \frac{v_{\parallel}}{c} \ll 1 \qquad (1.3.41)$$

which justifies the approximations we made. So,

$$W_{no} - \omega S_n(\phi) = (m + E_{no}')\sqrt{1+2X^2} \qquad (1.3.42)$$

i.e.

$$S_n(\phi) = -\frac{1}{\omega}(m + E'_{n0})(\sqrt{1+2\chi^2} - 1) \qquad (1.3.43)$$

The solution above doesn't depend on the mass of the atom. Particularly, if $M \to \infty$, then $\vec{R}$ tends to a constant, and so $\phi \to \omega t$. The motion of the atom is described by the "time" dependent Hamiltonian

$$H = \frac{P^2}{2M} + (m + E'_{n0})(\sqrt{1+2\chi^2(\phi)} - 1) \qquad (1.3.44)$$

where the role of the time is taken by ϕ. The atom, being overtaken by the laser pulse, moves in the increasing repulsive potential $(m+E'_{n0})(\sqrt{1+2\chi^2} - 1)$, gaining a small momentum in the direction of the wave propagation

$$\Delta\vec{p} = (m + E'_{n0})(\sqrt{1+2\chi^2} - 1)\hat{\kappa} \qquad (1.3.45)$$

The corresponding change in kinetic energy of the atom will be small, of the order $\frac{m}{M}$, and can be neglected, as we neglected the other terms of that order, whenever they appeared in the calculation. Like in the case of the laser beam, nonelastic transitions between the different electronic states are adiabatically improbable.

In particular, for hydrogen in the ground state we have the same relations, given by (1.3.34) for both the case of the laser pulse and the laser beam.

Until now we found the phase of the Dirac wave function contained in Z_0, eq.(1.3.11). We can now use (1.3.11) to obtain Ψ

which satisfies the wave equation (1.3.7). In that way we get the wave function for the atom in the laser "beam"

$$\Psi_n = C_n \left(m\sqrt{1+2\chi^2} + \beta m + \vec{\alpha}\cdot\vec{\Pi} \right) u_n \underset{\sim}{\xi}_0 \times$$

$$\times \exp i \left(\int^{\vec{R}} d\vec{R}' \cdot \vec{p}_n(\vec{R}') - \frac{p_i^2}{2M} - W_{on} t \right) \qquad (1.3.46)$$

The wave function for the atom in the laser "pulse" is

$$\Psi_n = C_n \left(m\sqrt{1+2\chi^2} + \beta m + \vec{\alpha}\cdot\vec{\Pi} \right) u_n \underset{\sim}{\xi}_0 \times$$

$$\times \exp i \left(\int^{\phi} S_n(\phi')\, d\phi' + \vec{p}_i \cdot \vec{R} - \frac{p_i^2}{2M} t - W_{on} t \right) \qquad (1.3.47)$$

Here, $\underset{\sim}{\xi}_0$ is a unit spinor, $W_{on} = m + E'_{on}$ and C_n is normalization constant. u_n, $\vec{p}_n$, and S_n were defined by (1.3.20), (1.3.28) or (1.3.36) and (1.3.43), respectively, for the boundary conditions that electron was in a state which corresponded to the eigenvalue E'_{no} while the center of mass of the atom was moving with the momentum $\vec{p}_i$ before entering the laser. In derivation of (1.3.46) and (1.3.47) from (1.3.11) we have neglected E'_{no} and V(r) in comparison with m.

The semiclassical motion of the center of mass in (1.3.7) may be carried further to a completely classical description. This is justified when the amplitude of the laser field is slowly varying in the sense discussed below (1.3.9). For example, in the case of the laser beam, the classical description would require a wave packet with an initial extent ΔX_0, which is small in comparison to spatial extent of the beam. If the experiment lasts a time τ, then the packet will

have grown to a size $\Delta X(\tau)$, given by [44],[55]

$$(\Delta X(\tau))^2 = (\Delta X_0)^2 + \left(\frac{\hbar}{2M}\frac{\tau}{\Delta X_0}\right)^2 \qquad (1.3.48)$$

ΔX can be optimized with respect to ΔX_0 and then one obtains

$$\Delta X_0 = \left(\frac{\hbar\tau}{2M}\right)^{1/2} \qquad (1.3.49)$$

from which

$$(\Delta X(\tau))^2 = \frac{\hbar}{M}\tau \qquad (1.3.50)$$

We can ask how long the experiment can last, such that the wave packet remains classical, i.e. that $\Delta X(\tau) \ll L$. For example, for $\tau = 10^{-5}$ sec, which is much longer than necessary either in the case of the real laser beam or laser pulse, we get $\Delta X(\tau) \sim 10^{-3}$ mm. L can be anticipated to be of the order of 1 mm $>>$ 10^{-3} mm. The finite size of the packet also introduces momentum uncertainties of the order $\Delta p = \frac{\hbar}{\Delta X_0}$, which gives a deflection of the atom by a small angle $\Delta\theta = \frac{\Delta p}{p} \sim 10^{-8}$ rad. This is negligible, so that the atom indeed behaves as a classical particle.

To go to the classical description of the center of mass motion, we again start from the equation (1.3.7), drop the kinetic energy operator T_R, and treat $\vec{R}(t)$ as a prescribed function of t, which is obtained from the classical Hamiltonian

$$H = \frac{p^2}{2M} + (m + E'_{n_0})\left(\sqrt{1+2\chi^2} - 1\right) \qquad (1.3.51)$$

where $X=X(R(t),t)$ is a slowly varying function of time. Repeating the same procedure which we have used below eq.(1.3.7) to obtain eq.(1.3.18), we get

$$\left(\frac{\partial^2}{\partial t^2} + 2iV\frac{\partial}{\partial t} + m^2 + p_r^2 + 2m^2X^2(t)\right) Z_0 = 0 \qquad (1.3.52)$$

The solution of this equation can be expanded in the eigenfunctions of the Schrodinger equation

$$\left(E_n' - \frac{p_r^2}{2m^*} - V(r)\right) u_n(r,t) = 0 \qquad (1.3.53)$$

where $m^* = m(1+2X^2)^{1/2}$. In that way

$$Z_{on} = u_n \exp i\left(\int_{-\infty}^{t} (m^*(t') + E_n'(t'))dt'\right. \qquad (1.3.54)$$

Again, due to the slowness of change of $X(t)$, the transitions between the states of the different n are adiabatically improbable. The transitions can occur only by the action of the oscillating perturbation $\mathcal{H}'$, which is given by (1.3.14). The atom, being initially in the ground state, will stay in that state, only adiabatically distorting the state. The distortion cames from the change in the effective mass of the electron, while the atom moves into the relativistic ponderomotive potential of the laser. The size of the atom is schrinking. (The idea that the electron in atom changes its rest mass in a way similar to the above one, was already mentioned by Cohen-Tannoudji [31] in context of dressing of an atom by high frequency laser).

We can now use (1.3.11) (without T_R) to obtain the Dirac spinor of the state n. It follows

$$\Psi_n = C_n(m^*+m)\left(\frac{1}{\frac{\vec{\sigma}\vec{\pi}}{m^*+m}}\right) \underset{\sim}{u}_o u_n \exp\left(-i\int_{-\infty}^{t}(m^*+E_n')dt'\right. \qquad (1.3.55)$$

This wave function is normalized to the electron, and the normalization constant will be laser field dependent due to the fact that the bound electron is inside the laser. The normalizing condition

$$(\Psi_n, \Psi_n) = 1 \tag{1.3.56}$$

applied to the "distorted" hydrogen ground state, yields

$$C_0 = \frac{1}{\sqrt{2}\,m}\left(1+\chi^2+(1+2\chi^2)^{1/2}+\chi^2(1+\cos 2\phi)\right)^{-1/2} \tag{1.3.57}$$

So, for that state we can finally write

$$\Psi_0 = \frac{1+(1+2\chi^2)^{1/2}}{\sqrt{2}\left(1+2\chi^2+(1+2\chi^2)^{1/2}+\chi^2\cos 2\phi\right)^{1/2}}\left(\frac{1}{\frac{\vec{\epsilon}\cdot\vec{\pi}}{m(1+(1+2\chi^2)^{1/2})}}\right) \underset{\sim}{U_0}\, U_0(\vec{r},t) \tag{1.3.58}$$

where

$$U_0(r,t) = \frac{\eta^{3/2}}{\sqrt{\pi}} \exp\left(-i\int_{-\infty}^{t}(m-R_y)(1+2\chi^2)^{1/2}dt'\right) e^{-\eta r} \tag{1.3.59}$$

$$\eta = \frac{1}{a_0}\left(1-\frac{\alpha E^2}{2}\right)(1+2\chi^2(t))^{1/2} \tag{1.3.60}$$

$\phi = \omega t - \vec{K}\cdot\vec{R}$, and φ in π is $\varphi = \omega t - \vec{K}\cdot\vec{r} - \vec{K}\cdot\vec{R}$. Note that for $\chi \to 0$ this solution goes to the first relativistic approximation (up to the order α_F^2) of the "bare" ground state of hydrogen. Our description of the atom is to treat the center of mass motion nonrelativi-

stically, while the relativistic effects come from the oscillations of the electron in the strong laser field. At the same time, the motion of the electron due to the interaction with the nucleus only, stays essentialy nonrelativistic.

All derivations and the discussions through this Section, can be applied also in the case of the circularly polarized laser, defined by the vector potential (1.2.1), $\varepsilon = 1$. The results are clear if one recognises that an equation analogous to equation (1.3.15) is now

$$\left((E-V)^2 - m^2 - p_r^2 - e^2 a^2(\vec{R},t) + \vec{\alpha}\cdot(\vec{p}_r V)\right) z_0 = 0 \qquad (1.3.61)$$

From this it follows that $|\vec{p}_{n\perp}|$ in (1.3.28) is here

$$|\vec{p}_{n\perp}| = \left(p_{i\perp}^2 - 2M(m + E'_{n0})\left(\sqrt{1+4\chi^2} - 1\right)\right)^{1/2} \qquad (1.3.62)$$

while $S_n(\phi)$ in (1.3.43) is

$$S_n(\phi) = -\frac{1}{\omega}(m + E'_{n0})\left(\sqrt{1+4\chi^2} - 1\right) \qquad (1.3.63)$$

That implies that Ψ_n for the laser beam and the laser pulse have exactly the same forms as in (1.3.46) and (1.3.47), respectively, except that $|\vec{p}_{n\perp}|$ and $S_n(\phi)$ are now defined by (1.3.62) and (1.3.63), respectively. The normalized ground state of hydrogen, in the classical description of the center of mass motion, in the circularly polarized laser, is

$$\Psi_0 = \frac{1+(1+4x^2)^{1/2}}{\sqrt{2}\,(1+4x^2+(1+4x^2)^{1/2})} \begin{pmatrix} 1 \\ \frac{\vec{c}\cdot\vec{\pi}}{m(1+(1+4x^2)^{1/2})} \end{pmatrix} \underset{\sim}{u_0}\, u_0 \qquad (1.3.64a)$$

where

$$u_0 = \frac{\eta^{3/2}}{\sqrt{\pi}} \exp\left(-i\int_{-\infty}^{t}(m-Ry)(1+4x^2)^{1/2}dt'\right) e^{-\eta r} \qquad (1.3.64b)$$

and

$$\eta = \frac{1}{a_0}\left(1-\frac{\alpha_E^2}{2}\right)(1+4x^2)^{1/2} \qquad (1.3.64c)$$

& 1.4 Calculation of the ionization rate.

1.4.1 Linearly polarized laser. Having obtained the "distorted wave" description of the initial state, we may now write the exact S matrix for the ionization of the hydrogen, initialy in the ground state as [45]

$$S_{\vec{q};0} = -i\left\langle \Psi^{(-)}_{\vec{q}}, H_i \chi_i \right\rangle \qquad (1.4.1)$$

where the perturbation operator H_i is defined by

$$\left(i\frac{\partial}{\partial t} - H\right)\chi_i = -H_i\chi_i \qquad (1.4.2)$$

H is the full Hamiltonian in (1.3.7) with $T_R=0$ as we accepted the classical description of center of mass motion of the atom as the simpler one. $\Psi_q^{(-)}$ is the exact wave function, with incoming wave boundary conditions, which satisfies

$$\left(i\frac{\partial}{\partial t}-H\right)\Psi_q^{(-)}=0 \tag{1.4.3}$$

and $\mathcal{X}_i$ was defined by (1.3.50). Substituting $\mathcal{X}_i$, as expressed in the form (1.3.11), into (1.4.2), with the use of eq. (1.3.52), we get

$$H_i\mathcal{X}_i=\mathcal{H}'\underset{\sim}{Z}_0 \tag{1.4.4}$$

where $\mathcal{H}'$ was defined in (1.3.14). $\underset{\sim}{Z}_0=Z_0\underset{\sim}{\xi}_0$, where Z_0 is given by (1.3.54) and $\underset{\sim}{\xi}_0$ is a constant bispinor. It is convenient to write down operator $\mathcal{H}'$ again

$$\mathcal{H}'=2e\,\vec{A}\cdot\vec{p}_r+\frac{1}{2}e^2a^2\cos 2\varphi+e\omega a g\sin\varphi$$

For the CO_2 laser, $\hbar\omega\approx 0.1eV\ll R_y$, and on that basis the third term on the right hand side can be neglected in comparison to the first. Their quotient is approximately

$$\frac{\hbar e\omega a g}{e a c\,\hbar/a_0}\sim\frac{\hbar\omega}{R_Y}\alpha_F\sim(\alpha_F)^2 \tag{1.4.5}$$

It can be shown that the first and the second term of $\mathcal{H}'$ are of the same order of magnitude. In fact, we shall see later that the principal contribution to the T matrix comes from $\varphi\approx\frac{\pi}{4}$, such that $\cos 2\varphi\sim\frac{\alpha_F}{X}$.

The comparison of the two gives

$$\sim \frac{e a \hbar c \, 1/a_0}{e^2 a^2 \alpha_F / \chi} \sim \frac{R_Y}{mc^2} \frac{\chi}{\alpha_F^2} \frac{1}{\chi} \sim \frac{1}{\chi} \qquad (1.4.6)$$

Using (1.3.58) we can finaly write

$$H_i \chi_i \approx \frac{1}{\sqrt{2}} \left(1+\chi^2+(1+2\chi^2)^{1/2}+\chi^2(1+\cos 2\phi)\right)^{-1/2} \times$$

$$\times \begin{pmatrix} 4\chi \cos\varphi \, \hat{a}\cdot\vec{p}_r + 2m\chi^2 \cos 2\varphi \\ 0 \end{pmatrix} \underset{\sim}{U}_0 \, U \qquad (1.4.7)$$

where $\underset{\sim}{U}_0$ is the unit spinor, and U was defined by (1.3.59),(1.3.60).

The approximation which we shall use to calculate the S matrix (1.4.1) is the same one that have been used by Faisal, Keldysh, Pert and essentialy also by Gersten-Mittleman. This is to expand the exact final state in perturbation series in V(r) and keep only the first term, in which the electron-nucleus interaction is absent. As was discussed in &1.1, the condition for convergence of such a series could be written as $\chi \gg \alpha_F$. The difference between our and the previous calculations that we treat the electron relativistically, and the laser wave is of a finite spatial and/or time extent, allowing the initial state to be distorted due to the nonoscillating terms of the interaction Hamiltonian. The proton motion is described classicaly. Its coupling to the laser field is neglected in the final state, since it is $\frac{m}{M}$ smaller than it is through the indirect coupling of the nucleus and the laser in the initial state. The free proton, after the

atom is ionized, moves unaccelerated motion. The electron in the final state is described by the modified Volkov state, the modification resulting from the slowly varying nature of the laser amplitude. The details of this state were discussed in § 1.2. It is in the case of the laser "pulse" given by (1.2.68), and by (1.2.69) in the case of the laser "beam".

From (1.4.7) it is imediately clear that only the "large" components of the final state survive in the S matrix. The electron coordinate in the final state must be transformed into the same coordinate as the one in the initial state, i.e. $\vec{r}_e = \vec{r} + \vec{R}$. The S matrix, for the case of the l a s e r b e a m takes the form

$$S_{\vec{q},0} = -\frac{i}{\sqrt{2}} \int dt\, d\vec{r}\; C_q \left(A^{+}_{fi_b} + B^{+}_{fi_b} \cos\varphi \right) e^{-i\int^{\vec{R}+\vec{r}} d\vec{r}' \cdot \vec{f}(\vec{r}') + iE_q t} \times$$

$$\times e^{i\left(a_f \sin\varphi + \frac{b}{2}\sin 2\varphi\right)} \frac{4X\cos\varphi\, \hat{a}\cdot\vec{p}_r + 2mX^2\cos 2\varphi}{\left(1+X^2+\sqrt{1+2X^2}+X^2(1+\cos 2\phi)\right)^{1/2}} \times$$

$$\times u_0(r,\eta)\, e^{-iw_0 t - i\int_{-\infty}^{t} dt' U_p(\vec{R}(t'))} \qquad (1.4.8)$$

where

$$a_f = \frac{2mX}{\omega\nu}\, \hat{a}\cdot\vec{f}(\vec{R}+\vec{r}) \approx \frac{2mX}{\omega\nu}\, \hat{a}\cdot\vec{f}(\vec{R}) \qquad (1.4.9a)$$

$$b = \frac{m^2X^2}{\omega\nu} \quad , \quad \nu = E_q - \hat{k}\cdot\vec{f}(\vec{R}) \qquad (1.4.9b)$$

$$U_p = (m - R_y)\left(\sqrt{1+2X^2} - 1\right) \quad , \quad w_0 = m - R_y \qquad (1.4.9c)$$

A_{fib} , B_{fib} are the matrix elements of the large components of the Dirac matrices of the modified Volkov state (1.2.69), between initial and final spin states,

$$A_{fib} = \left(E_q + m - \frac{m^2 x^2}{\nu}\right)\delta_{fi} \tag{1.4.10a}$$

$$B_{fib} = \frac{mx}{\nu}\left(\vec{f}\cdot\hat{a} + i\vec{\epsilon}\cdot\hat{\kappa}\vec{f}\cdot(\hat{a}\times\hat{\kappa}) + i\vec{\epsilon}\cdot(\hat{\kappa}\times\hat{a})\left(\vec{f}\cdot\hat{\kappa} - E_q - m\right)\right)_{fi} \tag{1.4.10b}$$

It is convenient to express ϕ as $\phi = \varphi + \vec{\kappa}\cdot\vec{r}$, and then to use φ as the integration variable, instead of t. It is also useful to define the functions $F_\ell^{(n)}$ by

$$\frac{e^{i a_f \sin\varphi + i\frac{b}{2}\sin 2\varphi}\cos^n\varphi}{\left(1 + x^2 + \sqrt{1+2x^2} + x^2(1+\cos 2\phi)\right)^{1/2}} = \sum_{\ell=-\infty}^{\infty} \frac{e^{-i\ell\varphi} F_\ell^{(n)}}{(1+2x^2)^{1/4}\left(1+\sqrt{1+2x^2}\right)^{1/2}}$$

(1.4.11)

which allows us to write

$$\frac{e^{i a_f \sin\varphi + i\frac{b}{2}\sin 2\varphi}\cos^n\varphi\cos 2\varphi}{1 + x^2 + \sqrt{1+2x^2} + x^2(1+\cos 2\phi)^{1/2}} = \sum_{\ell=-\infty}^{\infty} \frac{e^{-i\ell\varphi}\left(F_{\ell-2}^{(n)} + F_{\ell+2}^{(n)}\right)}{(1+2x^2)^{1/4}\left(1+\sqrt{1+2x^2}\right)^{1/2}}$$

(1.4.12)

We can exploit the slowly varying nature of $\vec{f}(\hat{r})$ and U_p to expand the exponential factors

$$\int_{\vec{R}}^{\vec{R}+\vec{r}} d\vec{r}'\cdot\vec{f}(\vec{r}') \approx \int_{\vec{R}} d\vec{r}'\cdot\vec{f}(\vec{r}') + \vec{f}(\vec{R})\cdot\vec{r} + \ldots \tag{1.4.13a}$$

$$\int_{-\infty}^{t} dt' U_p(R(t')) \approx \int_{-\infty}^{\varphi/\omega + \hat{k}\cdot\vec{R}} dt' U_p(\vec{R}(t')) + U_p(\vec{R})\,\hat{k}\cdot\vec{r} + \ldots \quad (1.4.13b)$$

Then (1.4.8) can be rewritten as

$$S_{\vec{q},0} \approx -\frac{i}{\sqrt{2}} \sum_{\ell} \int \frac{d\varphi}{\omega} \int d^3r\, C_q \exp i \left(- \int_{\vec{R}} d\vec{r}' \cdot \vec{f}(\vec{r}') - \vec{f}(\vec{R})\cdot\vec{r} + \right.$$

$$\left. + \frac{E_q - W_0}{\omega} (\varphi + \vec{k}\cdot\vec{r} + \vec{k}\cdot\vec{R}) \right) \times \left\{ A^{+}_{fib} \left(4\times \hat{a}\cdot\vec{p}_r\, F_\ell^{(1)} + m\times^2 \left(F_{\ell-2}^{(0)} + F_{\ell+2}^{(0)} \right) \right) + \right.$$

$$\left. + B^{+}_{fib} \left(4\times \hat{a}\cdot\vec{p}_r\, F_\ell^{(2)} + m\times^2 \left(F_{\ell-2}^{(1)} + F_{\ell+2}^{(1)} \right) \right) \right\} U_0(r,\eta) \times$$

$$\times \exp\left(-i \left(\int_{-\infty}^{\varphi/\omega + \hat{k}\cdot\vec{R}} dt' U_p(\vec{R}(t')) + \hat{k}\cdot\vec{r}\, U_p(\vec{R}) \right) \right) \quad (1.4.14)$$

The functions $F_\ell^{(n)}$ are slowly varying functions of $\vec{k}\cdot\vec{r}$, and this dependence may be neglected, since $K/\eta \ll 1$. In effect, this replaces ϕ by φ in (1.4.11) and (1.4.12). Then the r integration can be performed as a Fourier transform of the initial state

$$\int d^3r\, e^{-i\vec{f}\cdot\vec{r} + i\frac{E_q - W_0}{\omega}\vec{k}\cdot\vec{r} - i\hat{k}\cdot\vec{r}\, U_p(\vec{R})} \qquad u_0(r,\eta) = \tilde{u}_0\left(\vec{f} - \hat{k}(E_q - W_0 - U_p), \eta\right) =$$

$$= 8\sqrt{\pi}\,\eta^{5/2} \left(\left(\vec{f} - \hat{k}(E_q - W_0 - U_p)\right)^2 + \eta^2 \right)^{-2} \quad (1.4.15)$$

The momentum operator $\vec{p}_r$ at each occurance in (1.4.14) can be replaced by $\vec{f}(\vec{R})$, since $\hat{a}\cdot\hat{k} = 0$. The remaining φ integration gives

$$\int_{-\infty}^{\infty} \frac{d\varphi}{\omega} e^{i\frac{\varphi}{\omega}(E_q - W_0 - \ell\omega) - i\int_{-\infty}^{\varphi/\omega + \hat{k}\cdot\vec{R}} dt' U_p(\vec{R}(t'))} \approx 2\pi\,\delta\left(E_q - W_0 - \ell\omega - U_p(\vec{R}_0)\right) \quad (1.4.16)$$

where we have dropped an irrelevant phase factor. In obtaining (1.4.16) we have again exploited the slowly varying nature of $\vec{a}(\vec{R})$ or $U_p(\vec{R})$, and have labeled as $\vec{R}_0$ the point in the laser beam in which the ionization takes place. The observed cross section will contain an ensemble average of this parameter, in effect allowing for the ionization to occur at any point within the laser beam. This energy δ-function, taken with $U_p(\vec{R}_0)$ as given by (1.3.34) is the relativistic generalization of the energy conserving condition (1.1.77).

In the case of the l a s e r p u l s e , changing again the electron coordinate in the final state as $\vec{r}_e \sim \vec{r} + \vec{R}$, S matrix takes the form

$$S_{\vec{q},0} = -\frac{i}{\sqrt{2}} \int dt\, d^3r\, C_q \left(A^+_{fip} + B^+_{fip}\cos\varphi\right) e^{-i\vec{q}\cdot(\vec{r}+\vec{R}) + iE_q t + i\int_{-\infty}^{\varphi} b(\varphi')d\varphi'} \times$$

$$\times\, e^{i\left(a_q \sin\varphi + \frac{b}{2}\sin 2\varphi\right)} \frac{\left(4x\cos\varphi\, \hat{a}\cdot\vec{p}_r + 2mx^2\cos 2\varphi\right)}{\left(1 + x^2 + \sqrt{1+2x^2} + x^2(1+\cos 2\phi)\right)^{1/2}} \times$$

$$\times\, u_0(r,\eta)\, e^{-iw_0 t - i\int_{-\infty}^{t} dt' U_p(\phi(t'))} \qquad (1.4.17)$$

where

$$a_q = \frac{2mx}{\omega\nu}\,\hat{a}\cdot\vec{q}\,, \quad b = \frac{m^2x^2}{\omega\nu}\,, \quad \nu = E_q - \vec{q}\cdot\hat{u} \qquad (1.4.18)$$

and A_{fip}, B_{fip} are the matrix elements of the large components of the Dirac matrices of the Volkov state (1.2.69), between the initial and the final spin states. In that way

$$A_{fip} = (E_q + m)\,\delta_{fi} \qquad (1.4.19)$$

and B_{fip} is the same as the one given in (1.4.10b), with $\vec{\xi}$ replaced by $\vec{q}$. In the initial state $\chi=\chi(\phi)$ while in the final state $\chi=\chi(\varphi)$. We express $\varphi=\phi+\vec{\kappa}\cdot\vec{r}$ and use now ϕ as integration variable, instead of t in (1.4.17). We can further write

$$\int_{-\infty}^{t} dt' U_p(\phi(t')) = \int_{-\infty}^{\phi/\omega+\vec{\kappa}\cdot\vec{R}} dt' U_p(\phi(t')) \tag{1.4.20}$$

and exploit the slowly varying nature of $a(\varphi)$ to expand

$$\int_{-\infty}^{\varphi} b(\varphi')d\varphi' = \int_{-\infty}^{\phi-\vec{\kappa}\cdot r} b(\varphi')d\varphi' \approx \int_{-\infty}^{\phi} b(\varphi')d\varphi' - b(\phi)\vec{\kappa}\cdot\vec{r}+\ldots \tag{1.4.21}$$

Then the r integration in (1.4.17) can be performed similarly as in (1.4.15), while the remaining ϕ integral yields the energy conserving condition

$$E_q = W_0 + l\omega + U_p(\phi_0) + b(\phi_0)\omega \tag{1.4.22}$$

Here we labeled by ϕ_0 the phase of the laser pulse, when ionization occurs. The observed cross section will contain an ensemble average of this parameter.

In & 1.2 we concluded that the electron, which was born by the ionization and then left by the laser pulse, would gain momentum -b $\vec{k}$. It is convenient to replace in the S matrix

$$q_{\parallel} = f_{\parallel} - b\omega \tag{1.4.23}$$

and to recognize $\vec{f}$ as the electron momentum inside the laser pulse, where $\vec{f}_\perp = \vec{q}_\perp$. The S matrices, in both the case of the laser beam and the laser pulse can be rewritten as

$$S^{(b)}_{q,0} = -2\pi i \sum_\ell \delta\left(E_q - W_0 - \ell\omega - U_p(\vec{R}_0)\right) T^{(b)}_{\vec{q},0}(\ell) \qquad (1.4.24)$$

$$S^{(p)}_{q0} = -2\pi i \sum_\ell \delta\left(E_q - W_0 - \ell\omega - U_p(\phi_0) + b(\phi_0)\omega\right) T^{(p)}_{q,0}(\ell) \qquad (1.4.25)$$

respectively. Accounting (1.4.23), it can be shown by strightforward calculation that the T matrices in (1.4.24) and (1.4.25) could be written in the same form, which is

$$T_{\vec{q},0}(\ell) = \frac{1}{\sqrt{2}} C_q \Big\{ A^+_{fi} \left(4x\,\hat{a}\cdot\vec{f}\, F^{(1)}_\ell + mx^2 \left(F^{(0)}_{\ell-2} + F^{(0)}_{\ell+2} \right) \right) +$$

$$+ B^+_{fi} \left(4x\,\hat{a}\cdot\vec{f}\, F^{(2)}_\ell + mx^2 \left(F^{(1)}_{\ell-2} + F^{(1)}_{\ell+2} \right) \right) \Big\} \frac{\tilde{u}_0\left(\vec{f} - \ell\vec{u}, \eta\right)}{(1+2x^2)^{1/4}\left(1+\sqrt{1+2x^2}\right)^{1/2}} \qquad (1.4.26)$$

where

$$A_{fi} = \left(W_0 + \ell\omega + U_p + m - \frac{m^2x^2}{\nu}\right)\delta_{fi} \qquad (1.4.27a)$$

$$B_{fi} = \frac{mx}{\nu}\left(\vec{f}\cdot\hat{a} - i2\cdot\hat{k}\,\vec{f}\cdot(\hat{a}\times\hat{k}) - i\vec{2}\cdot(\hat{u}\times\hat{a})\left(\vec{f}\cdot\hat{u} - W_0 - \ell\omega - U_p - m\right)\right)_{fi} \qquad (1.4.27b)$$

$$\nu = W_0 + \ell\omega + U_p - \vec{f}\cdot\hat{u} \qquad (1.4.27c)$$

In the case of the laser beam, $\vec{f} = \vec{f}(\vec{R}_0)$ is the momentum of the electron resulting from ℓ photon ionization at $\vec{R}_0$ inside the laser. Then the electron moves as a classical particle, governed by the Hamiltonian

$$H(\vec{f}, \vec{r}_e) = \left(p^2 + m^2 (1 + 2\chi^2(\vec{r}_e)) \right)^{1/2} \qquad (1.4.28)$$

and $\vec{f}$ becomes $\vec{q}$ as the electron emerges from the laser. Energy conservation yields

$$E_q = \sqrt{q^2 + m^2} = \left(f^2 + m^2 (1 + 2\chi^2(\vec{R}_0)) \right)^{1/2} \qquad (1.4.29)$$

where

$$q_\perp^2 = f_\perp^2 + 2m^2 \chi^2(\vec{R}_0) \qquad (1.4.30)$$

and the supposition that the laser is uniform in the $\vec{k}$ direction yields $f_{||} = q_{||}$. On the other hand, in the case of the laser pulse, $\vec{f}(\phi_0)$ is the electron momentum in laser which results after ℓ photon ionization at the pulse phase ϕ_0, and then the electron evolves as a classical particle, governed by the Hamiltonian (1.4.28), with $\chi = \chi(\varphi_e)$, and $\vec{f}$ becomes $\vec{q}$ as the electron emerges from the laser pulse. The assumption that the pulse is uniform in the direction perpendicular to $\vec{k}$ yields $\vec{f}_\perp = \vec{q}_\perp$ and in addition we have the relation (1.4.23) between the k components of $\vec{f}$ and $\vec{q}$.

The functions, defined by (1.4.11) can be written as

$$F_\ell^{(n)} = \int_0^{2\pi} \frac{d\varphi}{2\pi} \cos^n\varphi\, e^{i\Lambda(\varphi)} \frac{\left(1+2\chi^2+\sqrt{1+2\chi^2}\right)^{1/2}}{\left(1+\chi^2+\sqrt{1+2\chi^2}+\chi^2(1+\cos 2\varphi)\right)^{1/2}} \tag{1.4.31a}$$

where

$$\Lambda(\varphi) = \ell\varphi + a_f \sin\varphi + \frac{b}{2}\sin 2\varphi \tag{1.4.31b}$$

As we shall see later, for the laser fields so strong that $\chi \sim 1$, and $\ell_{min} \sim \frac{R_y}{\omega} \gg 1$, the parameters a_f , b are also very large and are such that

$$\frac{a_f}{b} = \frac{2\hat{a}\cdot\vec{f}}{m\chi} \approx \frac{\alpha_F}{\chi} \ll 1$$
$$\frac{\ell}{b} \approx \frac{R_y}{m\chi^2} \approx \left(\frac{\alpha_F}{\chi}\right)^2 \ll\ll 1 \tag{1.4.32}$$

where we used the fact that $\vec{f}$ is a nonrelativistic momentum. This is sugested by the form of $\tilde{U}_o(\vec{f}-\ell\vec{k},\eta)$, which maximizes sharply for $f_\perp$ and $\vec{f}\cdot\vec{k} - \ell k$ of the order of $\sqrt{2mR_y}$.

The above discussion allows a stationary phase evaluation of the integral in (1.4.31a). The stationary phase condition $\frac{\partial\Lambda}{\partial\varphi}=0$ yields four points, for which $\cos\varphi = s_\pm$, where

$$s_\pm = -\frac{a_f}{4b} \pm \left(\frac{1}{2} - \frac{\ell}{2b} + \left(\frac{a_f}{4b}\right)^2\right)^{1/2} =$$
$$= -\frac{\hat{a}\cdot\vec{f}}{2m\chi} \pm \left(\left(\frac{\hat{a}\cdot\vec{f}}{2m\chi}\right)^2 + \frac{1}{2} - \frac{\ell\omega\nu}{2m^2\chi^2}\right)^{1/2} \tag{1.4.33}$$

From (1.4.32) it follows that the phase stationary points are approximately at $\varphi \approx \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$, so that $\cos 2\varphi \approx 0$. Thus the denominator in the integrand of (1.4.31a) becomes a constant, and

$$F_\ell^{(n)} \approx \int_0^{2\pi} \frac{d\varphi}{2\pi} \cos^n\varphi \, e^{i\Lambda(\varphi)} \tag{1.4.34}$$

This function, for $n=0$, gives a generalized Bessel function which has been previously encountered in a similar context [18], and recently discussed extensively [46]. We shall not need the results of that discussion since the stationary phase evaluation can be used in our case.

Using (1.4.32), the first term on the right hand side of (1.4.33) is of the order of α_F, the first term in the root is of the order α_F^2 while the third is of the order α_F. Expanding the root, we have

$$S_\pm = \pm \frac{1}{\sqrt{2}} \left(1 - \frac{\ell\omega\nu}{2m^2\chi^2} \mp \frac{\hat{a}\cdot\vec{f}}{\sqrt{2}m\chi} + \left(\frac{\hat{a}\cdot\vec{f}}{2m\chi}\right)^2 + \cdots \right) \tag{1.4.35}$$

so that the corresponding stationary phase points are $\varphi_\pm, -\varphi_\pm$

$$\begin{aligned} \varphi_+ &= \frac{\pi}{4} + \frac{\ell\omega\nu}{2m^2\chi^2} + \frac{\hat{a}\cdot\vec{f}}{\sqrt{2}m\chi} - \left(\frac{\hat{a}\cdot\vec{f}}{2m\chi}\right)^2 + \cdots \\ \varphi_- &= \frac{3\pi}{4} - \left(\frac{\ell\omega\nu}{2m^2\chi^2} - \frac{\hat{a}\cdot\vec{f}}{\sqrt{2}m\chi} - \left(\frac{\hat{a}\cdot\vec{f}}{2m\chi}\right)^2 + \cdots \right) \end{aligned} \tag{1.4.36}$$

Then,

$$F_\ell^{(0)} = \sum_{\mu=1}^{4} \frac{e^{i\Lambda_\mu + i\frac{\pi}{4}\,\mathrm{sgn}\,\Lambda''_\mu}}{\sqrt{2\pi\,|\Lambda''_\mu|}} \tag{1.4.37}$$

where $\Lambda_\mu, \Lambda''_\mu$ are the values of the Λ and $\frac{d^2\Lambda}{d\varphi^2}$ at the phase stationary points. Since Λ and Λ'' are odd functions in φ, the summation over μ gives

$$F_\ell^{(0)} \approx \left(\frac{2}{\pi|\Lambda''_+|}\right)^{1/2} \cos\left(\Lambda_+ + \frac{\pi}{4}\,\mathrm{sgn}\,\Lambda''_+\right) + \left(\frac{2}{\pi|\Lambda''_-|}\right)^{1/2} \cos\left(\Lambda_- + \frac{\pi}{4}\,\mathrm{sgn}\,\Lambda''_-\right) \tag{1.4.38}$$

It can be shown that

$$\Lambda''_\pm = -\sqrt{2}\,\frac{m^2x^2}{\omega\nu}\left(\pm\sqrt{2} + \frac{\hat{a}\cdot\vec{f}}{mx} + \cdots\right) \approx \mp\,\frac{2m^2x^2}{\omega\nu} \tag{1.4.39}$$

Then

$$F_\ell^{(0)} \approx \left(\frac{\omega\nu}{\pi m^2x^2}\right)^{1/2}\left(\cos\left(\Lambda_+ - \frac{\pi}{4}\right) + \cos\left(\Lambda_- + \frac{\pi}{4}\right)\right) \tag{1.4.40}$$

$T_{\vec{q},0}(\ell)$, given by (1.4.26), can be simplified by the use of the recurence relations

$$4x\,\hat{a}\cdot\vec{f}\,F_\ell^{(1)} + mx^2\left(F_{\ell+2}^{(0)} + F_{\ell-2}^{(0)}\right) = \frac{2\ell\omega\nu}{m}\,F_\ell^{(0)} \tag{1.4.41a}$$

$$4x\,\hat{a}\cdot\vec{f}\,F_\ell^{(2)} + mx^2\left(F_{\ell+2}^{(1)} + F_{\ell-2}^{(1)}\right) = \frac{2\ell\omega\nu}{m}\,F_\ell^{(1)} + \frac{\omega\nu}{m}\left(F_{\ell+1}^{(0)} - F_{\ell-1}^{(0)}\right) \tag{1.4.41b}$$

The result is

$$T_{\vec{q},0}(\ell)=\frac{C_{\vec{q}}}{\sqrt{2}\,(1+2x^2)^{1/4}\left(1+\sqrt{1+2x^2}\right)^{1/2}}\left\{A^{+}_{fi}\,\frac{2\ell\omega\nu}{m}\,F^{(0)}_{\ell}+\right.$$

$$\left.+B^{+}_{fi}\left(\frac{2\ell\omega\nu}{m}\,F^{(1)}_{\ell}+\frac{\omega\nu}{m}\left(F^{(0)}_{\ell+1}-F^{(0)}_{\ell-1}\right)\right)\right\}\tilde{U}_0(\vec{f}-\ell\vec{u},\eta) \qquad (1.4.42)$$

Since

$$\Lambda_{\pm}=\ell\varphi_{\pm}+a_f\sin\varphi_{\pm}+\frac{b}{2}\sin 2\varphi_{\pm} \qquad (1.4.43)$$

and

$$\frac{\partial\Lambda_{\pm}}{\partial\ell}=\varphi_{\pm}+\frac{\partial\varphi_{\pm}}{\partial\ell}\left(\ell+a_f\cos\varphi_{\pm}+b\cos 2\varphi_{\pm}\right)=\varphi_{\pm} \qquad (1.4.44)$$

we can write

$$\begin{aligned}\Lambda_{+}(\ell\pm1)&\approx\Lambda_{+}(\ell)\pm\varphi_{+}\\ \Lambda_{-}(\ell\pm1)&\approx\Lambda_{-}(\ell)\pm\varphi_{-}\end{aligned} \qquad (1.4.45)$$

Then it follows

$$F^{(1)}_{\ell}=\frac{1}{2}\left(F^{(0)}_{\ell+1}+F^{(0)}_{\ell-1}\right)=\frac{1}{2}\left(\frac{\omega\nu}{\pi m^2x^2}\right)^{1/2}\left(\cos\left(\Lambda_{+}(\ell+1)-\frac{\pi}{4}\right)+\right.$$

$$\left.+\cos\left(\Lambda_{+}(\ell-1)-\frac{\pi}{4}\right)+\cos\left(\Lambda_{-}(\ell+1)+\frac{\pi}{4}\right)+\cos\left(\Lambda_{-}(\ell-1)+\frac{\pi}{4}\right)\right)\approx$$

$$\approx\frac{1}{\sqrt{2}}\left(\frac{\omega\nu}{\pi m^2x^2}\right)^{1/2}\left(\cos\left(\Lambda_{+}(\ell)-\frac{\pi}{4}\right)-\cos\left(\Lambda_{-}(\ell)+\frac{\pi}{4}\right)\right) \qquad (1.4.46)$$

where we used $\varphi_{\pm}(\ell) \approx \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$. On the other hand

$$F^{(0)}_{\ell+1} - F^{(0)}_{\ell-1} = \left(\frac{\omega\nu}{\pi m^2 \chi^2}\right)^{1/2} \Big(\cos\left(\Lambda_+(\ell) + \varphi_+(\ell) - \tfrac{\pi}{4}\right) -$$

$$- \cos\left(\Lambda_+(\ell) - \varphi_+(\ell) - \tfrac{\pi}{4}\right) + \cos\left(\Lambda_-(\ell) - \varphi_-(\ell) + \tfrac{\pi}{4}\right) -$$

$$- \cos\left(\Lambda_-(\ell) - \varphi_-(\ell) + \tfrac{\pi}{4}\right) \approx$$

$$\approx -\sqrt{2}\left(\frac{\omega\nu}{\pi m^2 \chi^2}\right)^{1/2} \left(\sin\left(\Lambda_+(\ell) - \tfrac{\pi}{4}\right) + \sin\left(\Lambda_-(\ell) + \tfrac{\pi}{4}\right)\right) \qquad (1.4.47)$$

Therefore, it follows

$$2\ell F^{(1)}_{\ell} + F^{(0)}_{\ell+1} - F^{(0)}_{\ell-1} \approx \sqrt{2}\left(\frac{\omega\nu}{\pi m^2 \chi^2}\right)^{1/2} \Big(\ell \Big(\cos\left(\Lambda_+(\ell) - \tfrac{\pi}{4}\right) -$$

$$- \cos\left(\Lambda_-(\ell) + \tfrac{\pi}{4}\right)\Big) - \left(\sin\left(\Lambda_+(\ell) - \tfrac{\pi}{4}\right) + \sin\left(\Lambda_-(\ell) + \tfrac{\pi}{4}\right)\right)\Big) =$$

$$= -2\sqrt{2}\left(\frac{\omega\nu}{\pi m^2 \chi^2}\right)^{1/2} \sin\left(\frac{\Lambda_+(\ell) + \Lambda_-(\ell)}{2}\right) \sqrt{\ell^2+1}\, \sin\left(\frac{\Lambda_+(\ell) - \Lambda_-(\ell)}{2} - \frac{\pi}{4} + \operatorname{tg}^{-1}\frac{1}{\ell}\right)$$

(1.4.48)

In this expression we can neglect 1 in comparison to ℓ , as $\ell \gtrsim 100$ or, in another words, we can neglect $F^{(0)}_{\ell+1} - F^{(0)}_{\ell-1}$ in comparison to $\ell F^{(1)}_{\ell}$ in the expression (1.4.42).

We can now write the T matrix, given by (1.4.42) in the form

$$T_{\vec{q},0}(\ell) = T^{(0)}_{\vec{q},0}(\ell) + i\,\vec{\sigma}_{fi}\cdot(\hat{u}\times\hat{a})\, T^{(1)}_{\vec{q},0}(\ell) + i\,\vec{\sigma}_{fi}\cdot\hat{u}\; T^{(2)}_{\vec{q},0}(\ell) \qquad (1.4.49)$$

where

$$T^{(0)}_{\vec{q},0}(\ell) = 2C_q' \frac{\ell\omega\nu}{m}\left((E_p + m - \frac{m^2\chi^2}{\nu}) F_\ell^{(0)} + \frac{m\chi}{\nu} \vec{f}\cdot\hat{a}\, F_\ell^{(1)} \right) \tag{1.4.50a}$$

$$T^{(1)}_{\vec{q},0}(\ell) = 2C_q' \ell\omega\chi \left(\vec{f}\cdot\hat{\kappa} - E_p - m \right) F_\ell^{(1)} \tag{1.4.50b}$$

$$T^{(2)}_{\vec{q},0}(\ell) = 2C_q' \ell\omega\chi\, \vec{f}\cdot(\hat{a}\times\hat{\kappa})\, F_\ell^{(1)} \tag{1.4.50c}$$

and

$$C_q' = \frac{C_q\, \tilde{U}_0(\vec{f} - \ell\vec{\kappa}, \eta)}{\sqrt{2}\,(1+2\chi^2)^{1/4}\left(1+\sqrt{1+2\chi^2}\right)^{1/2}}\,, \quad E_p = W_0 + \ell\omega + U_p \tag{1.4.51}$$

Now, we could obtain the spin flip and the non spin flip T matrices, but these will depend upon our choice of the axis of quantization. For example, if we choose

$$\hat{\kappa} = \hat{z}\,, \quad \hat{\kappa}\times\hat{a} = \hat{y}\,, \quad \hat{a} = \hat{x} \tag{1.4.52}$$

then, using $(\sigma_z)_{fi'} = \pm\delta_{fi'}$ where "+" corresponds to the spin up, as well as $(\sigma_{x,y})_{fi'} = 0$, $i = f$, and $(\sigma_y)_{fi'} = \mp i$ for $i \neq f$, where "−" corresponds to the initial spin up, we get

$$T_{NSF} = T^{(0)}_{\vec{q},0}(\ell) \pm i\, T^{(2)}_{q,0}(\ell) \tag{1.4.53}$$

$$T_{SF} = \pm\, T^{(1)}_{\vec{q},0}(\ell) \tag{1.4.54}$$

In the last two formulas the upper signs correspond to the initial spin up, lower to the initial spin down. Since these results are axis

dependent, we will not pursue this further, but instead we will find the total squared T matrix. From (1.4.49) it is clear that the spin and non spin flip amplitudes add incoherently, and so the result of averaging over the initial spins, and summing over the final spins yields

$$\overline{|T_{\vec{q},0}(\ell)|^2} = 2\left(|T^{(0)}_{\vec{q},0}(\ell)|^2 + |T^{(1)}_{\vec{q},0}(\ell)|^2 + |T^{(2)}_{\vec{q},0}(\ell)|^2\right) \qquad (1.4.55)$$

which is independent of the spin quantization axis. Explicitely, it can be written as

$$\overline{|T_{\vec{q},0}(\ell)|^2} = \frac{4C_q^2 |U_0(\vec{f}-\ell\vec{\kappa},\eta)|^2 \ell^2\omega^2}{(1+2x^2)^{1/2}(1+\sqrt{1+2x^2})}\Big\{ x^2 (F_\ell^{(1)})^2 \big(f^2+(E_p+m)^2 -$$

$$-2\hat{\kappa}\cdot\vec{f}(E_p+m)\big)\Big) + \frac{\nu^2}{m^2}\left(E_p+m-\frac{m^2x^2}{\nu}\right)^2 (F_\ell^{(0)})^2 +$$

$$+2\frac{\nu}{m}x\,\hat{a}\cdot\vec{f}\,F_\ell^{(0)}F_\ell^{(1)}\left(E_p+m-\frac{m^2x^2}{\nu}\right)\Big\} \qquad (1.4.56)$$

The transition rate for ℓ photon ionization is then

$$\omega_\ell = 2\pi\int \frac{d^3q}{(2\pi)^3}\,\delta(E_q - E_p')\,\overline{|T_{\vec{q},0}(\ell)|^2} \qquad (1.4.57)$$

where $E_p' = E_p$ for the case of the laser beam , and $E_p' = E_p - \frac{m^2x^2}{\nu}$ for the case of the laser pulse. The functions $F_\ell^{(n)}$, obtained in (1.4.40) and (1.4.46) depend upon $\Lambda_\pm(\ell)$, which were given by (1.4.43) and which are rapidly varying functions of $\vec{q}$. To show that it is true it is enough to show that $\Lambda_\pm(\ell)$ are rapidly varying functions of $\vec{f}$. There exists, at least in principle, a one to one correspondence between

$\vec{f}$ and $\vec{q}$. It can be easily derived that

$$\nabla_{\vec{f}} \Lambda_{\pm} \approx \sqrt{2}\,\frac{mx}{\omega\nu}\left(\hat{a} + \left(\frac{3\hat{a}\cdot\vec{f}}{4} \pm \frac{mx}{2}\left(\left(\frac{\hat{a}\cdot\vec{f}}{2mx}\right)^2 + \frac{1}{2} - \frac{\ell\omega\nu}{2m^2x^2}\right)^{1/2}\right)\frac{\hat{K}}{\nu}\right) \tag{1.4.58a}$$

and so

$$\nabla_{\vec{f}}\left(\Lambda_+ + \Lambda_-\right) \approx 2\sqrt{2}\,\frac{mx}{\omega\nu}\left(\hat{a} + \left(\frac{3\hat{a}\cdot\vec{f}}{4}\right)\frac{\hat{K}}{\nu}\right) \tag{1.4.58b}$$

$$\nabla_{\vec{f}}\left(\Lambda_+ - \Lambda_-\right) \approx \sqrt{2}\,\frac{m^2x^2}{\omega\nu^2}\left(\left(\frac{\hat{a}\cdot\vec{f}}{2mx}\right)^2 + \frac{1}{2} - \frac{\ell\omega\nu}{2m^2x^2}\right)^{1/2}\hat{K} \tag{1.4.58c}$$

where we used $\sin\varphi_{\pm} \approx \frac{1}{\sqrt{2}}$. Since $\frac{m}{\omega} \sim 10^6$, $\Lambda_{\pm}$ and $\Lambda_+ \pm \Lambda_-$ are very rapidly varying functions of $\vec{f}$, with no stationary points. This implies that the forms $\exp(\pm i\Lambda_{\pm})$, $\exp(\pm i(\Lambda_+ \pm \Lambda_-))$ are rapidly varying functions of $\vec{f}$ which do not have saddle points. That says that the oscillating terms in

$$\left(F_{\ell}^{(0)}\right)^2 \approx \frac{\omega\nu}{\pi m^2x^2}\left(1 + \frac{\sin 2\Lambda_+ - \sin 2\Lambda_-}{2} + \cos(\Lambda_+ + \Lambda_-) + \sin(\Lambda_+ - \Lambda_-)\right) \tag{1.4.59}$$

averages to zero , performing the integration in (1.4.57), and we can write

$$\overline{\left(F_{\ell}^{(0)}\right)^2} \approx \frac{\omega\nu}{\pi m^2x^2} \tag{1.4.60}$$

In that sense we get

$$\overline{(F_\ell^{(1)})^2} \simeq \frac{\omega\nu}{2\pi m^2 x^2} \tag{1.4.60b}$$

and

$$\overline{F_\ell^{(0)} F_\ell^{(1)}} = 0 \tag{1.4.60c}$$

From (1.4.57), with the use of (1.4.56),(1.4.60), it follows

$$\omega_\ell = 2\pi \int \frac{d^3q}{(2\pi)^3} \delta(E_q - E_p') \frac{c_q^2 |\tilde{u}_0(\vec{f}-\ell\vec{k},\eta)|^2}{(1+2x^2)^{1/2}(1+\sqrt{1+2x^2})} \times$$

$$\frac{2\ell^2\omega^3\nu}{\pi m^2}\left\{ f^2 + (E_p+m)^2 - 2\hat{k}\cdot\vec{f}(E_p+m) + \right.$$

$$\left. + \frac{2\nu^2}{m^2x^2}\left(E_p + m - \frac{m^2x^2}{\nu}\right)^2 \right\} \tag{1.4.61}$$

If we set $\vec{f}=0$ in (1.4.23) and (1.4.30) and use the energy conserving conditions, contained in the δ function in (1.4.61), we get the condition, satisfied with the minimal ℓ in both the cases of the laser pulse and the laser beam

$$\ell_{min}\omega + (m-R_Y)\sqrt{1+2x^2} = \pm m\sqrt{1+2x^2} \tag{1.4.62}$$

The positive solution of this equation is

$$\ell_{min}\omega = R_Y\sqrt{1+2x^2} \tag{1.4.63}$$

so that we may interpret $R_y\sqrt{1+2\chi^2}$ as the binding energy of the atom, in the field of laser, which is by factor $\sqrt{1+2\chi^2}$ larger than the corresponding nonrelativistic result, obtained in & 1.1. This increase in binding energy is consistent with the conclusion that the atom shrinks in the laser, while the electron acquires the effective mass $m^*=m\sqrt{1+2\chi^2}$.

The integral in (1.4.61) may be conveniently done in cylindrical coordinates, with $\vec{k}$ as the cylindrical axis. There is no remaining dependence upon the asimutal angle of $\vec{q}$, so that integral can be done imediately, giving a factor 2π. In the case of the laser beam, the energy δ function can be used to perform $q_\perp$ integral with the relation (1.4.30) used to relate $f_\perp$ to $q_\perp$. So

$$\omega_\ell = (2\pi)^2 \int \frac{dq_\perp q_\perp dq_\parallel}{(2\pi)^3} \delta(E_q - E_p) \overline{\left| T_{\vec{q},0}(\ell)\right|^2} =$$

$$= \frac{E_p}{2\pi} \int dq_\parallel dq_\perp \delta(q_\perp - (E_p^2 - q_\parallel^2 - m^2)^{1/2}) \left| T_{\vec{q},0}(\ell)\right|^2 =$$

$$= \frac{E_p}{2\pi} \frac{c_q^2 4\ell^2\omega^2(E_p+m)}{(1+2\chi^2)^{1/2}(1+\sqrt{1+2\chi^2})\pi m^2} \int dq_\parallel \nu \left| U_0(\vec{f} - \ell\vec{u}, \eta)\right|^2 \left(\frac{\nu^2(E_p+m)}{m^2\chi^2} - \nu\right) \tag{1.4.64}$$

Using the identity

$$(\vec{f} - \ell\vec{u})^2 + \eta^2 = 2\ell\omega\nu \tag{1.4.65}$$

which follows from the energy conserving condition, we have

$$\tilde{U}_0(\vec{f}-\ell\vec{u},\eta)=\frac{2\sqrt{\pi}\,\eta^{5/2}}{(\ell\omega\nu)^2} \tag{1.4.66}$$

It yields

$$\omega_\ell=\frac{4}{\pi\ell^2\omega a_0^5 m^4}\frac{(1+2\chi^2)^2}{\chi^2(1+\sqrt{1+2\chi^2})}\int\frac{d\mathcal{Q}_\parallel}{(E_p-\mathcal{Q}_\parallel)^2}\left((E_p-\mathcal{Q}_\parallel)(E_p+m)-m^2\chi^2\right) \tag{1.4.67}$$

There is a maximum value of $|\mathcal{Q}_\parallel|$, implied by the energy conserving condition $E_{\mathcal{Q}}=E_p$, and by the condition $f_\perp^2\geq 0$. From these we get

$$\mathcal{Q}_\parallel^2\leq Q^2=E_p^2-m^2(1+2\chi^2) \tag{1.4.68}$$

and so

$$\int_{-Q}^{Q}\frac{d\mathcal{Q}_\parallel}{(E_p-\mathcal{Q}_\parallel)^2}\left((E_p+m)(E_p-\mathcal{Q}_\parallel)-m^2\chi^2\right)=$$

$$=(E_p+m)\ln\frac{E_p+Q}{E_p-Q}+\frac{m^2\chi^2}{E_p+Q}-\frac{m^2\chi^2}{E_p-Q} \tag{1.4.69}$$

From (1.4.68), assuming $\ell\omega$ nonrelativistic, it follows $Q\ll E_p$, and we can expand

$$\ln\frac{E_p+Q}{E_p-Q}\approx 2\frac{Q}{E_p}$$

$$\frac{m^2\chi^2}{E_p+Q}-\frac{m^2\chi^2}{E_p-Q}\approx-\frac{2m^2\chi^2 Q}{E_p^2} \tag{1.4.70}$$

Finaly, the ionization rate of hydrogen in the laser beam, for the pho-

ton multiplicity ℓ is

$$\omega_\ell = \frac{g}{\pi}\frac{m\alpha E^5}{\ell^2\omega}\sqrt{2m}\,\frac{(1+2X^2)^{9/4}}{X^2(1+\sqrt{1+2X^2}}\,(\ell\omega-\ell_{min}\omega)^{1/2}\left(1+\frac{m}{E_p}-\frac{m^2X^2}{E_p}\right) \tag{1.4.71}$$

ω_ℓ, as a function of ℓ, increases from zero at $\ell=\ell_{min}$, and having maximum at $\ell\approx\frac{4}{3}\ell_{min}$, decreases with ℓ. For large ℓ ($\ell\gg\ell_{min}$), it behaves as $\ell^{-3/2}$. The expression (1.4.71) grossly overestimates the real contribution for very large $\ell\sim\frac{m}{\omega}$, since in that region the stationary phase points move into the complex plane, and the resulting $F_\ell^{(n)}$ are exponentialy decreasing functions of ℓ. Still, even in (1.4.71) the contribution of so large ℓ to W is negligible, and we can neglect ℓ dependence of E_p in (1.4.71) and simply take $E_p\approx m\sqrt{1+2X^2}$.

The total ionization rate is obtained by summing over all ℓ

$$\sum_{\ell=\ell_{min}}^{\infty}\omega_\ell \sim \int_{\ell_{min}}^{\infty} d\ell\,\frac{(\ell-\ell_{min})^{1/2}}{\ell^2} = \frac{\pi}{2\sqrt{\ell_{min}}} \tag{1.4.72}$$

That yields, after some rearangements

$$\omega_b \approx g\frac{R_Y}{\hbar}\alpha_F^2\,\frac{(1+2X^2)(1+\sqrt{1+2X^2})}{X^2} \tag{1.4.73}$$

The subscript b in ω_b stands for the laser beam.

In the case of the laser pulse, we again use the energy δ function to perform $q_\perp$ integral in (1.4.61). The relation (1.4.23) is used to relate $f_{||}$ and $q_{||}$. In this case it is convenient to transform $q_{||}$ integration into ν integration, where $\nu = E_q - q_{||}$. Then, using

$$\frac{dq_{\parallel}}{d\nu} = -\frac{\nu^2+m^2+q_{\perp}^2}{2\nu^2}, \quad E_q = \frac{\nu^2+m^2+q_{\perp}^2}{2\nu} \tag{1.4.74}$$

in (1.4.61), we get

$$\omega_\ell = (2\pi)^2 \int \frac{q_{\perp} dq_{\perp} d\nu}{(2\pi)^3} \frac{\nu^2+m^2+q_{\perp}^2}{2\nu^2} \delta(E_q - E_p') \overline{|T_{q,0}(\ell)|^2} =$$

$$= (2\pi)^2 \int \frac{q_{\perp} dq_{\perp} d\nu}{(2\pi)^3} \frac{\nu^2+m^2+q_{\perp}^2}{2\nu^2} \frac{\nu}{q_{\perp}} \delta\left(q_{\perp} - (2E_p\nu - m^2(1+2\chi^2) - \nu^2)^{1/2}\right) \times$$

$$\times |T_{q,0}(\ell)|^2 = \frac{4(1+2\chi^2)^2}{\pi \ell^2 \omega a_0^5 m^4} \frac{(E_p+m)}{\chi^2(1+\sqrt{1+2\chi^2}} \int \frac{d\nu}{\nu} \tag{1.4.74}$$

In the last step in (1.4.74) we applied relation (1.4.65), which can be proved to be valid in the case of the laser pulse, too. Limits of integration for ν can be found, from the energy conserving condition in (1,4,74), requiring $q_{\perp}^2 \geq 0$. That gives

$$E_p - Q \leq \nu \leq E_p + Q \tag{1.4.75}$$

where Q was defined by (1.4.68). Integration over ν in (1.4.74) is contained in

$$\int_{E_p-Q}^{E_p+Q} \frac{d\nu}{\nu} = \ln \frac{E_p+Q}{E_p-Q} \approx 2\frac{Q}{E_p} \tag{1.4.76}$$

and so, finaly

$$\omega_\ell = \frac{8}{\pi} \frac{m\alpha_F^5}{\ell^2\omega} \sqrt{2m} \frac{(1+2\chi^2)^{9/4}}{\chi^2(1+\sqrt{1+2\chi^2})} \left(\ell\omega - \ell_{min}\omega\right)^{1/2} \left(1+\frac{m}{E_p}\right) \tag{1.4.77}$$

Summing over ℓ , we get the total ionization rate for the case of the laser pulse

$$\omega_p = 16 \frac{Ry}{\hbar} \alpha_F^2 \frac{(1+2\chi^2)\sqrt{1+2\chi^2}}{\chi^2} \qquad (1.4.78)$$

So, for both the case of the linearly polarized laser beam and the laser pulse, tho total ionization rate shows the same behaviour in the ultrastrong laser field limit ($\chi \gg 1$) : It increases with χ as χ, i.e. it is proportional to the laser electric field amplitude. This is a consequence of the relativistic dynamic of the electron. The nonrelativistic results on multiphoton ionization in an ultrastrong laser field, presented in & 1.1, show the decrease of the ionization rate with the increase of the laser electric field. Our nonrelativistic considerations in & 1.1 which, trating the laser of the finite spatial extent (laser beam), yields the ionization rate (1.1.101)

$$\omega_b^{n.r.} = \frac{8}{\chi^2} \frac{Ry}{\hbar} \alpha_F^2$$

which is just the nonrelativistic limit ($\chi \ll 1$) of the relativistic result for the laser beam (1.4.73).

The results (1.4.73) and (1.4.78) could be understood on the basis of the same physical arguments we used in & 1.1 to explain the nonrelativistic results. There we discussed that the amplitude $v_0 = 2\chi c$ of the oscillation velocity of the electron in the laser field increases with the increase of the field, causing the scattering of the electron on the nucleus to become less effective. The effect of relativity is to dump the oscillation amplitude and the amplitude of the oscillation

velocity. It is easy to show that the relativistic amplitude of the velocity of oscillation is $v_0 = \frac{2\chi c}{\sqrt{1+4\chi^2}}$. For $\chi \gg 1$, v_0 increases with χ very slowly and tends to a constant, $v \to c$ as $\chi \to \infty$. So, there is no decrease in ionization probability, based on the increase of the relative velocity of the electron and the nucleus. In addition, we have shown that the atom compresses in the strong laser field. In effect, the electron spends more time in the vicinity of the nucleus and that increases the cross section for the scatering, thus increasing the ionization rate.

As we have already discussed, the condition for the convergence of the perturbation series in V(r), which was used in the final state in both our calculation in this Section and in nonrelativistic calculations in & 1.1 is $\chi \gg \alpha_F$. On the other hand, the nonrelativistic approximation is correct if $\chi \ll 1$. These two conditions contradict each other, and strictly speaking, there is no a range of the laser field intensity, for the CO_2 laser, in which nonrelativistic calculations in & 1.1 could be applied. Still, the ionization rate (1.4.73) (or (1.4.78), as a function of χ has a broad minimum for χ of the order of 1, and in that region one can expect that the ionization rate is a flat or even a decreasing function of the laser electric field.

It is interesting to note that $\frac{\omega_p}{\omega_b} \to 2$ as $\chi \to \infty$.

1.4.2 C i r c u l a r l y p o l a r i z e d l a s e r. Many of the details of the calculation of the ionization rate with a circularly polarized ultrastrong laser are similar to the calculation with a linearly polarized laser. We will not repeat such details here, but rather present the main points of the derivation.

Again, we start from the exact S matrix (1.4.1) and neglect the electron-nucleus interaction in the final state, so that the final state is approximated by the wave function in laser, which evolves into a plane wave at $t=+\infty$. It is given by (1.2.71) for the circularly polarized laser pulse, and by (1.2.72) for the circularly polarized laser beam. For the initial state, we use the "dressed" initial state, which evolves by the action of the "ponderomotive" potential, from the ground state of hydrogen at $t=-\infty$, and is given by (1.3.64). Following the same steps as in the derivation of (1.4.7), we get

$$H_i \chi_i \approx \frac{1}{\sqrt{2}} \left(1+4x^2+\sqrt{1+4x^2}\right)^{1/2} \begin{pmatrix} 4x\,(\cos\varphi\,\hat{e}_1+\sin\varphi\,\hat{e}_2)\cdot\vec{p}_r \\ 0 \end{pmatrix} \underset{\sim}{u} \tag{1.4.79}$$

where

$$\underset{\sim}{u} = e^{-iW_0 t - i\int_{-\infty}^{t} U_p(t')dt'} u_0(r,\eta)\,\underset{\sim}{u}_0$$

$$U_p(x(t)) = (m-R_Y)\left(\sqrt{1+4x^2}-1\right)$$

$$u_0 = e^{-\eta r}\,,\quad \eta = \frac{1}{a_0}\left(1-\frac{\alpha_F^2}{2}\right)\sqrt{1+4x^2}$$

In the case of the l a s e r b e a m , using (1.2.72) for the final state, it follows

$$S_{\vec{q},0} = -\frac{i}{\sqrt{2}} \int dt\,d^3r\, C_q \left(A^{+}_{fib} + B^{(1)+}_{fib}\cos\varphi + B^{(2)+}_{fib}\sin\varphi\right) e^{-i\int^{\vec{R}+\vec{r}} d\vec{r}'\cdot\vec{f}(\vec{r}')+iE_q t} \times$$

$$\times\, e^{i a_f \sin(\varphi-\theta)} \frac{4x\,(\cos\varphi\,\hat{e}_1+\sin\varphi\,\hat{e}_2)\cdot\vec{p}_r}{\left(1+4x^2+(1+4x^2)^{1/2}\right)^{1/2}}\, u_0(r,\eta)\, e^{-iW_0 t - i\int_{-\infty}^{t} dt' U_p} \tag{1.4.80}$$

where

$$a_f = \frac{2mx}{\omega\nu} f_\perp \,, \quad \nu = E_q - \hat{k}\cdot\vec{f}(\vec{R}), \quad f_\perp^2 = q_\perp^2 - 4m^2x^2, \quad f_{11} = q_{11}$$
$$\vec{f}_\perp \cdot \hat{e}_1 = \cos\theta \tag{1.4.81}$$

A_{fib}, $B^{(j)}_{fib}$ are the matrix elements of the "large" components of the Dirac matrices of the final state, between initial and final spin states. i.e.

$$A_{fib} = \left(E_q + m - \frac{2m^2x^2}{\nu}\right)\delta_{fi} \tag{1.4.82a}$$

$$B^{(j)}_{fib} = \frac{mx}{\nu}\left(\vec{f}\cdot\hat{e}_j - i\vec{\sigma}\cdot\hat{k}\,\vec{f}\cdot(\hat{e}_j\times\hat{k}) - i\vec{\sigma}\cdot(\hat{k}\times\hat{e}_j)(\vec{f}\cdot\hat{k} - E_q - m)\right)_{fi} \tag{1.4.82b}$$
$$j = 1, 2$$

Expanding

$$e^{i a_f \sin(\varphi-\theta)\cos\varphi} = \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} G_\ell^{(1)} \tag{1.4.83a}$$

$$e^{i a_f \sin(\varphi-\theta)\sin\varphi} = \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} G_\ell^{(2)} \tag{1.4.83b}$$

we get

$$S\vec{q}_{10} = -\frac{i}{\sqrt{2}}\int dt\, d^3r \frac{C_q}{(1+4x^2+(1+4x^2)^{1/2})^{1/2}} \Big\{ A^+_{fib} 4x \left(G_\ell^{(1)}\hat{e}_1 + G_\ell^{(2)}\hat{e}_2\right)\cdot\vec{p}_r +$$
$$+ B^{(1)+}_{fi}\left(4x\left(\frac{G^{(1)}_{\ell+1}+G^{(1)}_{\ell-1}}{2}\hat{e}_1 + \frac{G^{(2)}_{\ell+1}+G^{(2)}_{\ell-1}}{2}\hat{e}_2\right)\cdot\vec{p}_r\right) +$$
$$+ B^{(2)+}_{fi}\left(4x\left(\frac{G^{(1)}_{\ell+1}-G^{(1)}_{\ell-1}}{2i}\hat{e}_1 + \frac{G^{(2)}_{\ell+1}-G^{(2)}_{\ell-1}}{2i}\hat{e}_2\right)\cdot\vec{p}_r\right)\Big\} u_0(r)\,\gamma$$
$$\times \exp i\left(E_q t - \int d\vec{r}\cdot\vec{f}(\vec{r}) - \ell\varphi - w_0 t - \int_{-\infty}^{t} dt'\, U_p(R(t'))\right) \tag{1.4.84}$$

We can perform r and t integration in the same way we did for the linearly polarized laser, which yields

$$S^{(b)}_{\vec{q},0} = -2\pi i \sum_{\ell} \delta\left(E_q - W_0 - \ell\omega - U_p(\vec{R}_0)\right) T^{(b)}_{\vec{q},0}(\ell) \qquad (1.4.85)$$

where

$$T^{(b)}_{q,0}(\ell) = \frac{1}{\sqrt{2}} \frac{C_q\, \tilde{u}_0(\vec{f}-\ell\vec{\kappa}, \eta)}{(1+4\chi^2+(1+4\chi^2)^{1/2})^{1/2}} 2\chi \Big\{ 2\, A^{+}_{fib} \left(G^{(1)}_{\ell}\hat{e}_1 + G^{(2)}_{\ell}\hat{e}_2\right)\cdot\vec{f} +$$

$$+ B^{(1)+}_{fib}\left(\left(G^{(1)}_{\ell+1} + G^{(1)}_{\ell-1}\right)\hat{e}_1\cdot\vec{f} + \left(G^{(2)}_{\ell+2} + G^{(2)}_{\ell-1}\right)\hat{e}_2\cdot\vec{f}\right) -$$

$$- B^{(2)+}_{fib}\left(\left(G^{(1)}_{\ell+1} - G^{(1)}_{\ell-1}\right)\hat{e}_1\cdot\vec{f} + \left(G^{(2)}_{\ell+1} - G^{(2)}_{\ell-1}\right)\hat{e}_2\cdot\vec{f}\right)\Big\} \qquad (1.4.85)$$

This can be simplified, using the recurence relation

$$\hat{e}_1\cdot\vec{f}\, G^{(1)}_{\ell} + \hat{e}_2\cdot\vec{f}\, G^{(2)}_{\ell} = (-1)^{\ell+1} e^{i\ell\theta} \frac{\nu}{2m\chi} \ell\omega\, J_{\ell}(a_f) \qquad (1.4.86)$$

where $J_{\ell}(a_f)$ is the Bessel function of the first kind

$$J_{\ell}(a_f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i a_f \sin\varphi - i\ell\varphi}\, d\varphi \qquad (1.4.87)$$

Introducing the short hand

$$C'_q(\ell) = \frac{(-1)^{\ell} e^{i\ell\theta} C_q\, \tilde{u}_0(\vec{f}-\ell\vec{\kappa}, \eta)\, \ell\omega\nu}{\sqrt{2}\, m\, (1+4\chi^2+(1+4\chi^2)^{1/2})^{1/2}} \qquad (1.4.88)$$

(1.4.85) can be written as

$$T^{(b)}_{\vec{q},0}(\ell) = C'_{\vec{q}}(\ell)\left\{-2A^{+}_{fi;b}J_{\ell}(a_f) + B^{(1)+}_{fi;b}\left(e^{i\theta}\left(1+\frac{1}{\ell}\right)J_{\ell-1} + \right.\right.$$

$$\left.\left. + e^{-i\theta}\left(1-\frac{1}{\ell}\right)J_{\ell-1}(a_f)\right) - i\,B^{(2)+}_{fi;b}\left(e^{i\theta}\left(1+\frac{1}{\ell}\right)J_{\ell+1} - e^{-i\theta}\left(1-\frac{1}{\ell}\right)J_{\ell-1}(a_f)\right)\right\} \tag{1.4.89}$$

$\frac{1}{\ell}$ could be neglected, comparing to 1, as ℓ is very large (of the order of 100 or larger). If we compare the T matrix in (1.4.89) with the one in (1.4.42), we see that the role of the "generalized Bessel functions" $F_{\ell}^{(n)}$ in the case with a linearly polarized laser is here played by the "ordinary" Bessel functions. The reason for this is the absence of the term $\sim \sin 2\varphi$ in the phase of the Volkov state with a circularly polarized laser: As the vector potential "circulates", its intensity stayes constant during the period of oscillation. The same is true for the vector of the oscillating momentum of the electron in the laser field.

Using $A_{fi;b}$ and $B_{fi;b}$ explicitely, after some rearangements, (1.4.89) can be rewritten in the form

$$T^{(b)}_{\vec{q},0}(\ell) = T^{(0)}_{\vec{q},0}(\ell)\,\delta_{fi} + \vec{c}_{fi}\cdot\hat{k}\,T^{(3)}_{\vec{q},0}(\ell) + \vec{c}_{fi}\cdot\hat{z}_1\,T^{(1)}_{\vec{q},0}(\ell) + \vec{c}_{fi}\cdot\hat{z}_2\,T^{(2)}_{\vec{q},0}(\ell) \tag{1.4.90}$$

where $\hat{z}_1, \hat{z}_2$ are the compex unit vectors

$$\hat{z}_1 = \frac{\hat{e}_1 + i\hat{e}_2}{\sqrt{2}}\ , \quad \hat{z}_2 = \frac{\hat{e}_1 - i\hat{e}_2}{\sqrt{2}} \tag{1.4.91}$$

which obey the properties

$$\hat{z}_i \cdot \hat{z}_j^* = \delta_{ij} \;,\; \hat{z}_j \cdot \hat{\kappa} = \hat{z}_j^* \cdot \hat{\kappa} = 0 \qquad i,j = 1,2 \tag{1.4.92}$$

and

$$\begin{aligned}
T^{(0)}_{\vec{q},0}(\ell) &= C'_q(\ell)\left(-2\left(E_q + m - \frac{2m^2\chi^2}{\nu}\right) J_\ell(a_f) + \frac{m\chi}{\nu} f_\perp \left(J_{\ell+1}(a_f) + J_{\ell-1}(a_f)\right)\right) \\
T^{(3)}_{\vec{q},0}(\ell) &= C'_q(\ell)\frac{m\chi}{\nu} f_\perp \left(J_{\ell+1}(a_f) - J_{\ell-1}(a_f)\right) \\
T^{(1)}_{\vec{q},0}(\ell) &= C'_q(\ell)\sqrt{2}\, e^{-i\theta}\frac{m\chi}{\nu}\left(\vec{f}\cdot\hat{\kappa} - E_q - m\right) J_{\ell-1}(a_f) \\
T^{(2)}_{\vec{q},0}(\ell) &= -C'_q(\ell)\sqrt{2}\, e^{i\theta}\frac{m\chi}{\nu}\left(\vec{f}\cdot\hat{\kappa} - E_q - m\right) J_{\ell+1}(a_f)
\end{aligned} \tag{1.4.93}$$

Like in the case of the linearly polarized laser, the spin flip and non spin flip T matrices will depend on the choice of the axis of quantization. If choose that axis in $\vec{k}$ direction, we get

$$T_{NSF} = T^{(0)}_{\vec{q},0}(\ell) \pm T^{(3)}_{\vec{q},0}(\ell) \tag{1.4.94a}$$

$$T_{SF} = \begin{cases} \sqrt{2}\, T^{(1)}_{\vec{q},0}(\ell) & \text{, initial spin up} \\ \sqrt{2}\, T^{(2)}_{\vec{q},0}(\ell) & \text{, initial spin down} \end{cases} \tag{1.4.94b}$$

Spin flip and non spin flip amplitudes add incoherently, and the result, averaging over initial spin and summing over the final spin directions, yields

$$\overline{\left|T_{\vec{q},0}(\ell)\right|^2} = 2\left(\left|T^{(0)}_{\vec{q},0}(\ell)\right|^2 + \left|T^{(1)}_{\vec{q},0}(\ell)\right|^2 + \left|T^{(2)}_{\vec{q},0}(\ell)\right|^2 + \left|T^{(3)}_{\vec{q},0}(\ell)\right|^2\right) \tag{1.4.95}$$

which is independent of the spin quantization axis. This can be written as

$$\overline{|T_{\vec{q},0}(\ell)|^2} = 8\,|C^{\prime}_{q}(\ell)|^2 \left\{ (E_q+m)\left(\frac{m^2x^2}{\nu}\left(J^2_{\ell+1}+J^2_{\ell-1}-4J^2_{\ell}\right)+(E_q+m)J^2_{\ell}\right)+\right.$$

$$\left. +2\frac{m^4x^4}{\nu}\left(2J^2_{\ell}-J^2_{\ell+1}-J^2_{\ell-1}\right)-\ell\omega\left(E_q+m-\frac{2m^2x^2}{\nu}\right)J^2_{\ell}\right\} \qquad (1.4.96)$$

where has been used the relation

$$J_{\ell+1}(a_f)+J_{\ell-1}(a_f)=\frac{2\ell}{a_f}\,J_{\ell} \qquad (1.4.97)$$

The ionization rate for ℓ photon ionization with the circularly polarized laser can be found from

$$\omega_{\ell}=2\pi\int\frac{d^3q}{(2\pi)^3}\,\delta\left(E_q-\ell\omega-W_0-U_p(\vec{R}_0)\right)\overline{|T_{\vec{q},0}(\ell)|^2} \qquad (1.4.98)$$

We will perform q integration in cylindrical coordinates, with $\vec{k}$ axis as the cylindrical axis. Performing the $q_{\perp}$ and angular integration, it yields

$$\omega_{\ell}=\frac{4m^3\alpha_F^5}{(E_p+m)\omega^2\ell^2}\frac{(1+4x^2)^2}{\left(1+\sqrt{1+4x^2}\right)}\int_{-Q}^{Q}\frac{dq_{\parallel}}{(E_p-q_{\parallel})}\left\{(E_p+m)\left(\frac{m^2x^2}{E_p-q_{\parallel}}\times\right.\right.$$

$$\times\left(J^2_{\ell+1}(s)+J^2_{\ell-1}(s)-4J^2_{\ell}(s)\right)+(E_p+m)J^2_{\ell}(s)\Big)+\frac{2m^4x^4}{(E_p-q_{\parallel})^2}\times$$

$$\left.\times\left(2J^2_{\ell}(s)-J^2_{\ell+1}(s)-J^2_{\ell-1}(s)\right)-\ell\omega\left(E_p+m-\frac{2m^2x^2}{E_p-m}\right)J^2_{\ell}(s)\right\} \qquad (1.4.99)$$

where Q is determined from the condition $f_\perp^2 \geq 0$

$$Q = \left(\ell\omega - R_y\sqrt{1+4X^2}\right)^{1/2}\left(2m\sqrt{1+4X^2} + \left(\ell\omega - R_y\sqrt{1+4X^2}\right)\right)^{1/2} \quad (1.4.100)$$

and

$$S = \frac{2mX}{\omega(E_p - q_{11})}\left(Q^2 - q_{11}^2\right)^{1/2} \quad (1.4.101)$$

The condition $f^2 = 0$ yields the effective ionization potential $R_y\sqrt{1+4X^2}$ so that

$$\ell\omega \geq R_y\sqrt{1+4X^2} \quad (1.4.102)$$

in order that ionization occurs.

The argument S of the Bessel functions in (1.4.99) is very large, even if $\ell\omega$ is just above ionization threshold. For example, if $Q^2 - q_{11}^2 \sim \ell m^*\omega$, it follows that $s \sim \left(\frac{m^*}{\omega}\right)^{1/3} \gg 1$ and that gives us the right to use the asymptotic form of $J_\ell(S)$ $(S \to \infty)$

$$J_\ell(S) \approx \left(\frac{2}{\pi S}\right)^{1/2}\cos\left(S - \ell\frac{\pi}{2} - \frac{\pi}{4}\right) \quad (1.4.103)$$

The argument of the cosine in this expression is a very rapidly oscillating function of $\vec{q}, \ell$, and averaging over the rapidly oscillating terms in (1.4.98), we can use

$$\overline{J_\ell^2(S)} \approx \overline{J_{\ell+1}^2(S)} \approx \frac{1}{\pi S} \quad (1.4.104)$$

Substituting this in (1.4.99), we have

$$\omega_\ell \approx \frac{2m^2\alpha_F^5}{\pi\omega\ell^2}\frac{(E_p+m-\ell\omega)}{(E_p+m)X}\frac{(1+4X^2)^2}{1+\sqrt{1+4X^2}}\int_{-Q}^{Q}\frac{dq_{11}\left((E_p+m)(E_p-q_{11})-2m^2X^2\right)}{(E_p-q_{11})^2\left(Q^2-q_{11}^2\right)^{1/2}} \tag{1.4.104}$$

Because of ℓ^2 factor in the denominator, ω_ℓ is a fast decreasing function of ℓ , $\ell > \ell_{min}$, and neglecting the contribution of ℓ such that $\ell \sim \frac{m}{\omega}$ we can neglect $\ell\omega$ and q_{11} in comparison to m. That yields

$$\omega_\ell \approx \frac{2m^2\alpha_F^5}{\pi\omega\ell^2 X}\frac{(1+4X^2)^2}{(1+\sqrt{1+4X^2}}\left(1+\frac{m}{E_p}-\frac{2m^2X^2}{E_p^2}\right)\int_{-Q}^{Q}\frac{dq_{11}}{(Q^2-q_{11}^2)^{1/2}} =$$

$$= \frac{2m^2\alpha_F^5}{\omega\ell^2 X}\frac{(1+4X^2)^2}{1+\sqrt{1+4X^2}}\left(1+\frac{m}{E_p}-\frac{2m^2X^2}{E_p^2}\right) =$$

$$= \frac{m^2\alpha_F^5}{\omega\ell^2}\frac{(1+4X^2)}{X}\left(1+\sqrt{1+4X^2}\right) \tag{1.4.105}$$

The total ionization rate for ℓ photon ionization of hydrogen with the circularly polarized laser beam is

$$\omega_b^{(cir)} = 4\frac{Ry}{\hbar}\alpha_F\frac{\sqrt{1+4X^2}}{X}\left(1+\sqrt{1+4X^2}\right) \tag{1.4.106}$$

In the similar way, the total ionization rate with the circularly polarized laser pulse is

$$\omega_p^{(cir)} = 8\frac{Ry}{\hbar}\alpha_F\frac{1+4X^2}{X} \tag{1.4.107}$$

Like with the linearly polarized laser, the ionization rate inc-

reases linearly with the amplitude of the laser electric field, if $\chi \gg 1$. Here is also $\frac{\omega_p}{\omega_b} \to 2$ if $\chi \to \infty$. But, while for a linearly polarized laser, ω_ℓ increases from zero to the maximum, as ℓ increases from ℓ_{min} to $\frac{4}{3}\ell_{min}$, and then decreases as $\ell^{-3/2}$, for a circularly polarized laser ω_ℓ decreases as ℓ^{-2} as ℓ increases from ℓ_{min} . Another difference is in order of magnitude. The comparison of the total ionization rates with a linearly and a circularly polarized laser yields

$$\frac{\omega_b^{lin}}{\omega_b^{cir}} = \frac{\omega_p^{lin}}{\omega_p^{cir}} = \sqrt{2}\,\alpha_F << 1 \quad , \chi \to \infty \tag{1.4.108}$$

This is rather supprising..As was discussed in the Introduction, in the low intensity laser limit, linearly polarized laser is much more effective than a circularly polarized one, and that can be understood on the basis of simple physical arguments.

P A R T II

ENERGY SPECTRUM OF ELECTRONS FROM MULTIPHOTON IONIZATION

& 2.1 I n t r o d u c t i o n. The subject of multiphoton ionization by intense lasers has been of experimental interest for some time, but it is only recently that experiments have been done which yield information on the energy spectrum of the electrons. Compton at al [47] have seen two peaks in the spectrum, which they identify as five photon resonant ionization of Xe (Fig. 2.1). The peaks were interpreted as the probabilities of leaving the residual Xe^+ in either the $^2P_{1/2}$ or $^2P_{3/2}$ state, since the energy spacing between the peaks (1.6 eV) agreed with the energy separation of these two states. Somewhat earlier, Agostini at al [35] measured the electron distributions in the nonresonant six and eleven photon ionization of Xe, with a much more intense laser and also saw two peaks in the former case (Fig. 2.2). These were identified as six and seven photon ionization respectively, since the peaks were separated by the energy of a single photon (2.34 eV, ionization potential was 12.27 eV). No evidence of the multiplicity due to more than one possible final state of Xe^+ was found. But since the first experiment was a resonant ionization and the second nonresonant, this is not necessarily a contradiction. Finally, Kruit at al [48] measured the electron distribution in the resonant five photon ionization of Xe. They saw the doublet structure that corresponds to the fine structure of Xe^+ observed by Compton. But they also saw this structure repeated at electron energies larger by $\hbar\omega$ and $2\hbar\omega$ than the energy of the lowest doublet, the phenomena observed in Agostini expe-

riment. These were identified as (5+1) and (5+2) photon ionization, (Fig. 2.3).

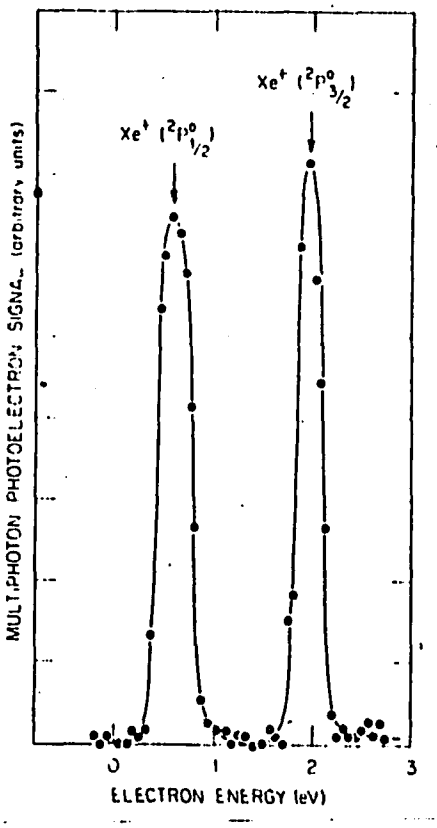

Fig. 2.1

Compton's experiment was performed witha peak laser intensity of $10^9 \frac{W}{cm^2}$, which was too small for the forces, due to the ponderomotive potential (1.1.73) in a spatialy inhomogeneous laser, to affect the measured energy distributions of the electrons. The same is true for Kruit's experiment ($10^{11} \frac{W}{cm^2}$). But the Agostini experiment was performed at intensities of order $10^{13} \frac{W}{cm^2}$, so that forces were observed which shifted and broadened the peaks in the electron distribution. So, the peak that corresponds to the eleven photon ionization is shifted in energy by about 3.5 eV, which is approximately the maximum energy that the electron can gain by the ponderomotive forces (~4 eV).

If the spatial distribution of the laser intensity at the focus is known, than the shape of the broadened low energy peak can be used to obtain information on the multiphoton ionization probability as a function of laser intensity. This is discussed in & 2.2.

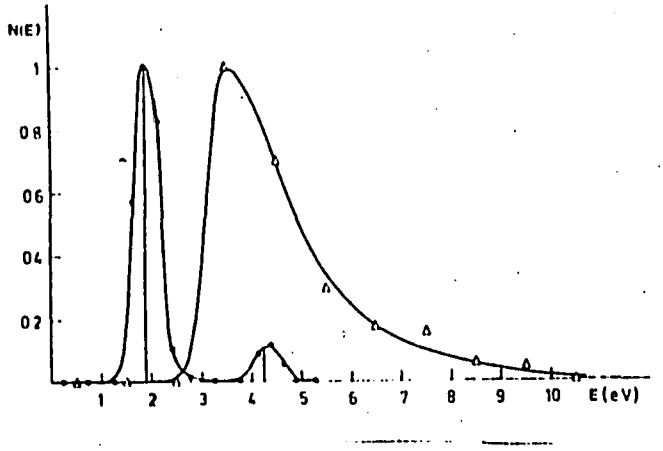

Fig. 2.2 Energy spectra of the emited electrons for two photon energies: triangles, $\hbar\omega$=1.17 eV, I=4×10^{13} W/cm^2, E_{max}=4 eV; circles, $\hbar\omega$= 2.34 eV, I=8×10^{12} W/cm^2, E_{max}=0.2 eV (E_{max} is the maximum energy gained in the ponderomotive potential).

The upper peak, due to absorption of an extra photon, will also be broadened, and this in principle can be used to extract information on the one photon inverse bremsstrahlung cross section. Gontier at al [49] have discussed this second peak and pointed out that it arises from the two separate (non interfering) effects. The first, which they called "above threshold ionization" (ATI) results from an absorption of the last photon, while the electron is in the field of its parent ion. The second, "inverse bremsstrahlung process" (IBP) is a two step process and involves a collection of atoms. It arises from the electron collision with another atom (or ion) as it travels out of the laser from its point of origin. The processes are at least in principle distinguishable by different pressure dependences (linear

in the case of ATI, but quadratic in the case of IBP), so that we shall assume that they can be separately measured. The first of these ATI, is the more interesting one. This is a one step process: the electron undergoes transition from a discrete initial state of atom to the final continuum state via virtual intermediate states. The process can happen in many different and interfering ways. But the dominant path is the almost on shell one (Part III of the Thesis). That means that the transitions which the most closely conserve energy in the intermediate states tend to dominate the total transition probability. Such transitions are available here, since the absorption of the last photon can take place as a free-free transition in the field of the parent ion. Looking at the ionization this way, a simple analysis below shows that the second peak in the Agostini experiment is unlikely to contain much of this first process (ATI).

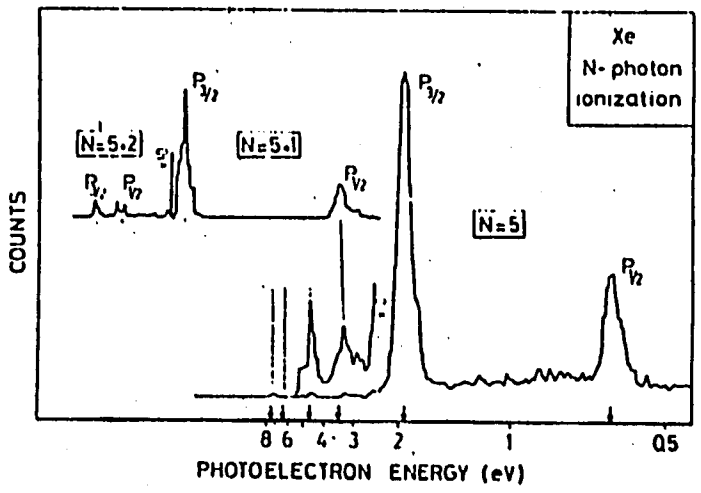

Fig. 2.3

The mean time T between photon arrivals at the atom being ionized is given by

$$T = \frac{\hbar \omega}{I \mathcal{E}} \qquad (2.1.1)$$

where $\hbar\omega$ is the photon energy, I is the laser intensity and σ some cross section for absorption of the photon to form the second peak by ATI. If v is the velocity of the electron in the lower peak, then v T will be the distance that the electron will move before the absorption takes place. If the distance is much larger than ρ , the range of the potential around the ion in which the absorption takes place, the absorption is an unlikely event. We can crudely evaluate this ratio for the conditions of the Agostini experiment. Using (2.1.1) one gets

$$\frac{v I}{\rho} = \frac{v \hbar\omega}{I \sigma \rho} \approx 29 a_0^3 \frac{1}{\sigma \rho} \qquad (2.1.2)$$

With reasonable σ and ρ ($\alpha_F a_0^2$ and a_0 respectively) it yields result greater than unity, indicating that the electron has most likely left the influence of its parent ion before ATI occurs. However, the ratio of ATI and IBP will depend also upon the pressure of the atoms in the laser beam.

The second process depends on the IBP cross section, and in & 2.3 we show how the shape of the upper peak can in principle be used to extract information concerning the free-free cross section as a function of electron energy and laser intensity. However, the details appear to be too complex to be useful.

Our analysis in & 2.2 and & 2.3 includes only the transport of the electrons out of the laser beam due to the ponderomotive forces. At certain conditions, the transport of electrons due to the other forces, such as diffusion or electrostatic forces between electrons and ions can become important.

& 2.2 A n a l y s i s o f t h e l o w e r p e a k. We assume that the electrons are collected at a small rectangular hole in the containment vessal of dimension a , in a direction perpendicular to the laser beam, and length b , along the beam. The hole is a distance d from the focus of the beam whose radius is R, as in Fig. 2.4. We assume that $d \gg a \gg R$, and that b is sufficiently small, so that variation of the laser intensity along the beam direction can be neglected. (This restriction will be removed below). Fig. 2.5 shows a plane perpendicular to the laser beam. At a point (r_0, ϕ_0, z_0) an electron is ejected in the azimutal direction Ψ_0 with a z-component of the velocity v_z. There is a range of the angle Ψ_0, called $\Delta\Psi_0$, for which the electron will reach the collector. It can be obtained by integration of the classical equations of motion of an electron, moving in a potential V(r), which is taken to be ponderomotive potential (1.1.73)

$$V(r) = \frac{e^2 E_L^2(r)}{4 m \omega^2} = K\, I(r) \qquad (2.2.1)$$

where $\vec{E}_L$ is the electric field of the laser, and the laser intensity I(r) is assumed to be azimutally symetric. We assume that the laser pulse duration is long enough, so that the electron is most likely to leave the beam, before it is left by the laser pulse.

The trajectory equation of the electron in the potential V(r) is

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{L\, dr}{\sqrt{2}\, r^2 \left(E_{K0} + V(r_0) - E_z - V(r) - \frac{L^2}{2r^2} \right)^{1/2}} \qquad (2.2.2)$$

where

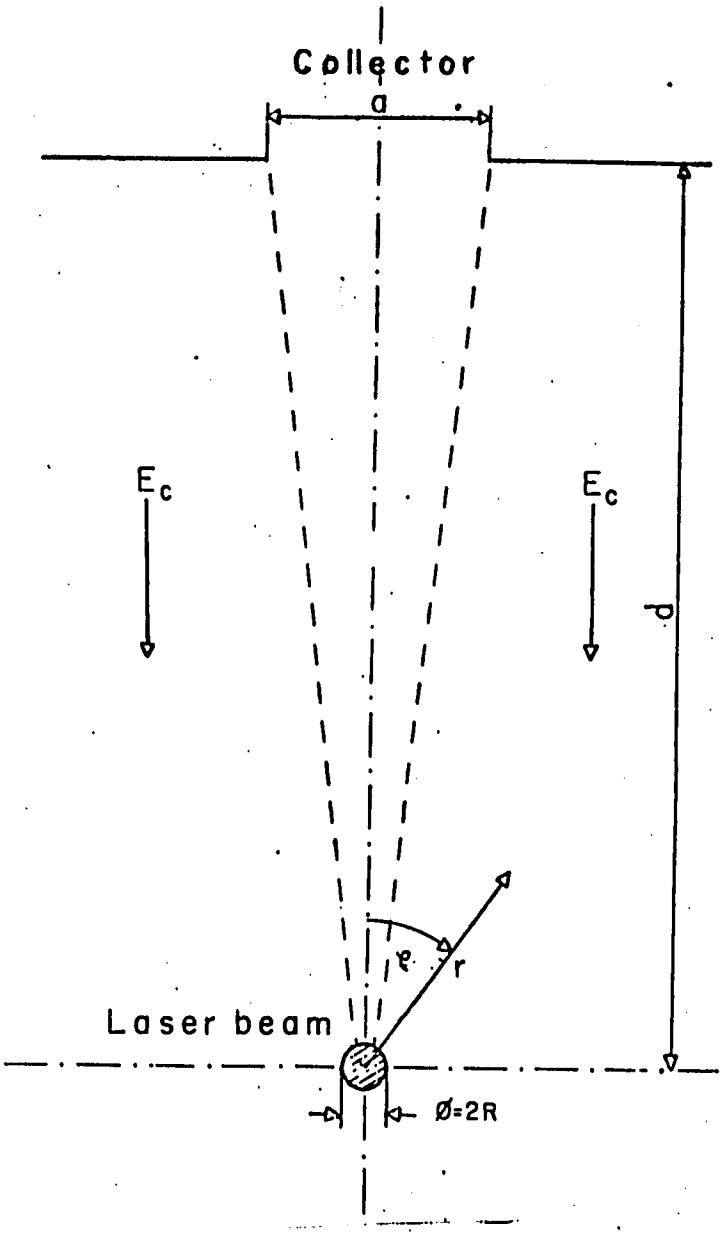

Fig. 2.4

$$L = m v_\perp r_0 \sin(\Psi_0 - \phi_0) \tag{2.2.3}$$

is the electron angular momentum about the laser beam axis,

$$E_{K0} = \nu \hbar \omega - U_I \tag{2.2.4}$$

is the initial kinetic energy of the electron, and

$$E_z = \frac{1}{2} m v_z^2 \tag{2.2.5}$$

In (2.2.4) ν is the number of photons absorbed during the ionization and U_I is the binding energy of the electron with account being taken of the residual state of the ion. $v_\perp$ and v_z are related by

$$E_{K0} = \frac{1}{2} m (v_{\perp 0}^2 + v_z^2) = \frac{1}{2} m v_H^2 \tag{2.2.6}$$

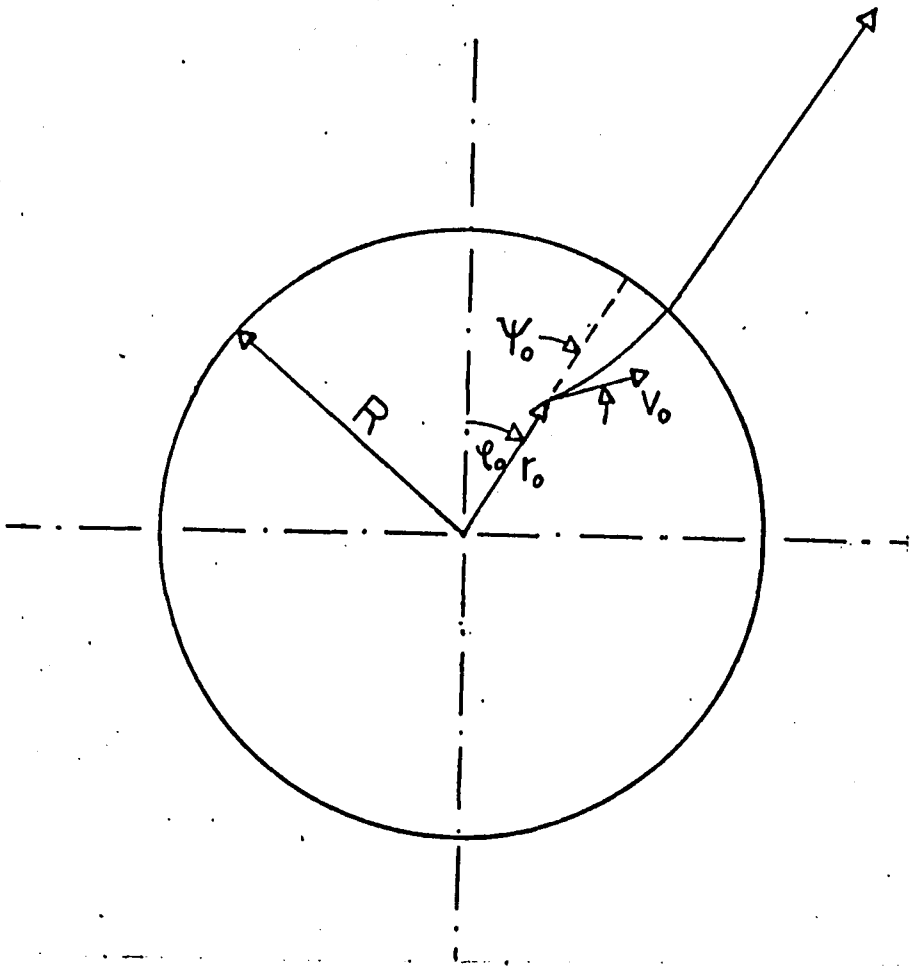

Fig. 2.5

An electron, which is collected, will pass through the point

$r=\infty, \phi=0$ (Fig. 2.4), with an allowable range in ϕ given by

$$\Delta\phi \approx \frac{a}{d} << 1 \tag{2.2.7}$$

This will map into allowable range in Ψ_0 , which will be collected. This range can be obtained from (2.2.4) as

$$\Delta\Psi_0 = \Delta\phi \left(\frac{\partial f}{\partial \Psi_0}\right)^{-1} \tag{2.2.8}$$

where

$$f(r_0,\phi_0,\Psi_0,v_z) = \int_{r_0}^{\infty} \frac{dr\, L}{\sqrt{2}\, r^2 \left(E_{\kappa 0}+V(r_0)-E_z-V(r)-\frac{L^2}{2r^2}\right)^{1/2}} = -\phi_0 \tag{2.2.9}$$

The right hand side of (2.2.8) is to be considered as a function of (r_0,ϕ_0,v_z), with Ψ_0 eliminated by solving (2.2.9) as $\Psi_0=\Psi_0(r_0,\phi_0,v_z)$. Electrons, born at a point z_0, with z component of velocity v_z, may or may not reach the opening in the chamber, in order to be counted. But, since we are approximating the laser by a beam which is uniform in z direction, the entire system (beam plus chamber aperture) can be considered to be periodic in z, with period b . Then for every electron (z_0,v_z) which is not collected, there is one at (z_0-b, v_z) which is. So the aperture in effect will collect all electrons originating at z_0 , for all v_z, and no other electrons.

Let $P(I,\Psi_0)$ be the probability per unit time per atom of ionization at the point (r_0,ϕ_0,z_0), with electron velocity specified by (Ψ_0,v_z) (Fig. 2.5). Then the total rate of collection of electrons, with energy between E and E+dE is

$$n b r_0 dr_0 \int_{-v_M}^{v_M} dv_z \int_0^{2\pi} d\phi_0 \, P(I,\Psi_0)\, \Delta\Psi_0(r_0,\phi_0,\Psi_0,v_z) \qquad (2.2.10)$$

where Ψ_0 is a function of ϕ_0 (eq.(2.2.9)), and v_M is given by (2.2.6). This can be identified with the detection rate per unit energy, integrated over all possible v_z (collection efficiency is supposed to be 100%). The energy of the detected electron can be related to its point of origin by

$$E = E_{K0} + K\, I(r_0) \qquad (2.2.11)$$

and if $I(r_0)$ is a monotonically decreasing function of r_0, then it can be inverted to give $r_0(I)$, and

$$dr_0 = \frac{dE}{K} \left(\frac{dI}{dr_0}(r_0,I) \right)^{-1} \qquad (2.2.12)$$

Then (2.2.10) and (2.2.12) can be assembled to give

$$\frac{1}{2v_M} \int_{-v_M}^{v_M} dv_z \, \frac{1}{2\pi} \int_0^{2\pi} d\phi_0 \, P(I,\Psi_0)\, \Delta\Psi_0(r_0,\phi_0,\Psi_0,v_z) =$$

$$= \frac{K}{2\pi n b} \, \frac{N(E}{r_0(I)} \, \frac{dI}{dr_0}(r_0,I) \qquad (2.2.13)$$

where N(E) is the experimental counting rate per unit energy range, and Ψ_0 is considered to be a function of $(r_0(I),\phi,v)$ from (2.2.9). Equation (2.2.13) is an integral equation for P, the quantity of theoretical interest, in terms of N(E), the measured spectrum. Its inversion would require extensive numerical analysis which, we show below, can be

avoided. Before doing so, we discuss it qualitatively.

If the multiphoton ionization rate is a rapidly rising function of I, as seems likely in the Agostini experiment, then the most electrons will be created at the region of the greatest intensity. This region will produce electrons of the greatest energy, since the (repulsive) ponderomotive potential is the greatest there. Consequently, we expect the broadened (lower) peak of the distribution function to have its maximum near the high energy end. This is not the case in the Agostini experiment (Fig. 2.2), so there must be other broadening mechanism at work there.

The complexities, encountered above, can be avoided by using a small collecting electric field E_c (Fig. 2.4), which is large enough to collect all electrons formed in the multiphoton ionization process. However, it should be too small to distort the distribution of the photo electrons. This is expressed by

$$e E_c R << V(0) = K I(0) \qquad (2.2.14)$$

where I(0) is the peak laser intensity. These conditions are easily satisfied. We can also allow for a variation of the laser intensity along the direction of the laser beam. This complicates the dynamics of the electron and the geometry of the collection, but this complication can be avoided by making b , the size of the aperture, large enough. This is easily seen in the following way: The laser intensity will decrease as the distance from the focal plane increases. The ionization probability, a rapidly rising function of intensity, will therefore be smaller, the greater the distance from the focal plane. If b is large enough, then essentialy all the ionized electrons will

be collected, since few will be produced far from the focal plane.

We can now obtain a relation between the ionization probability and the counting rate, in a manner analogous to the one used in obtaining (2.2.13). There are several differences: First, the use of the collecting field means that electrons with any Ψ_0 will be collected. Second, the laser intensity is now a function of (r_0, z_0). The integral over (r_0, z_0) can be converted to an integral over (I, z_0) with a Jacobian $\frac{\partial I(r_0, z_0)}{\partial r_0}$. We again assume that I is a monotonic function of r_0, for all z_0. Since we suppose azimutal symmetry of the laser intensity, integration over ϕ will give simply 2π. The total rate of collection of electrons with energy between E, E+dE is

$$n\, 2\pi \frac{dE}{K} \int_{-\infty}^{\infty} dz_0 \frac{r_0(I, z_0)}{\left|\frac{\partial I}{\partial r_0}(I, z_0)\right|} \theta(I(0, z_0) - I) \int_0^{2\pi} d\Psi_0 P(I, \Psi_0) = N(E(I))\, dE \qquad (2.2.15)$$

where $I = \frac{E - E_{K0}}{K}$ is a constant for the given E, so that integration in (2.2.15) represents essentialy a surface integration over an equiintensity cylindrical surface, defined by

$$I(r_0, z_0) = \frac{E - E_{K0}}{K} \qquad (2.2.16)$$

The intensity change is in the direction of the gradient, perpendicular to the surface. Equation (2.2.15) relates the total ionization rate (integrated over all directions of the electron) for a given laser intensity, to the experimental counting rate, which is the purpose of this Section. The collecting field has eliminated the geometric complications of (2.2.13).

& 2.3 Analysis of the upper peak. We shall assume that the different pressure dependences of the ATI and the IBP processes have been used to isolate each, and we shall discuss only the inverse bremsstrahlung process contribution to the upper peak.

The spectrum of the upper peak is obtained in the following way: An atom is ionized at a point (r_0, ϕ_0, z_0), where the laser intensity is assumed to be $I(r_0, z_0)$. The electron moves under the influence of the ponderomotive potential and the collecting field E_c to some point in the laser field (r_1, ϕ_1, z_1), where it collides with another atom and absorbs a laser photon. It then escapes from the laser and is collected with the energy

$$E = E_{k0} + \hbar\omega + K I(r_0) \qquad (2.3.1)$$

We have assumed that another collision is unlikely, and this is born out by the experimental absence of the third peak in the Agostini experiment. An analysis is similar to that given in the preceding Section.

Suppose that some number ΔN_0 of electrons, created in a volume element $\Delta S_0 \Delta \ell_0$ (ΔS_0 is the surface element of the equiintensity surface, $d\ell_0$ is the element of the normal to the surface) with an angle Ψ_0 , and the component of the velocity v_z. The motion of the electron in such electronic beam is defined by the equation of orbit (2.2.2). The fraction of electrons in the beam, which absorb an additional photon along the path is

$$\Delta_{+1} \approx \Delta N_0 \, n \int_\sigma d\ell \, \sigma_T(\vec{P}_i, \vec{E}_L) \qquad (2.3.2)$$

Here $\sigma_T(\vec{p}_i, \vec{E}_L)$ is the total cross section for one photon inverse bremsstrahlung, $\vec{E}_L$ is the laser electric field and $\vec{p}_i$ is the momentum of the electron, just before the collision at (r_1, ϕ_1, z_1). $\vec{p}_i$ is a function of the point at which the electron is initially produced, its velocity at that point, and the point of the collision. The magnitude of $\vec{p}_i$ is given by

$$\frac{p_i^2}{2m} = E_{K0} + K\left(I(r_0, z_0) - I(r_1, z_1)\right) \qquad (2.3.3)$$

but its direction must be obtained from a solution of the equations of motion. Finally, the integral over dl follows the path of the electron, determined from these equations.

To obtain the total number of electrons, which are subjected to the inverse bremsstrahlung, and are ejected from the laser beam with the energy in an interval E,E+dE, we have to integrate (2.3.2) over the equiintensity surface I(r,z) and so, over all possible electron trajectories. Using (2.2.13) and (2.2.15) we get

$$N_1(E) = \frac{n^2}{K} \int_{-\infty}^{\infty} \frac{dz_0 v_0(I)}{\left|\frac{\partial I}{\partial r_0}\right|} \theta\left(I(0, z_0) - I\right) \int_{r_0}^{\infty} dl \int_0^{2\pi} d\phi_0 \times$$

$$\times \int_0^{2\pi} d\psi_0 P(I, \psi_0) \frac{1}{2v_M} \int_{-v_M}^{v_M} dv_z \sigma_T(\vec{p}_i, \vec{E}_L) \qquad (2.3.4)$$

The Heavyside function arises from the fact that there is a maximum value of I, for each z_0, which is $I(0, z_0)$. This is an extremely complicated function, which can be in principle used to obtain σ_T, once P has been obtained by the methods of § 2.2. Furthermore, the analysis of

the lower peak could be done with the correction to the upper peak. So, the measured energy distribution of electrons in the lower peak is

$$N_M(E) = N_0(E) - N_1(E+\hbar\omega) \tag{2.3.5}$$

where $N_0(E)$ is given in & 2.2. If the upper peak distribution at $E+\hbar\omega$ is much smaller than the lower peak one at E, this correction can be neglected.

We can illustrate the method by a simple example. We neglect any z-dependence of the laser intensity, as we did in the first part of & 2.2, and we assume that E_{K0} in (2.2.) is very small. We also note that (2.2.14) allows us to neglect the effect of the collecting field on the electron while it is inside the laser beam. The electron will then be expelled radially from its point of creation. Moreover, $\vec{p}_i$ will then depend upon the conditions of the ionization event only through $I(r_0)$, which is held constant in (2.3.4). The integral over dl will then be an integral over the radial line, starting from r_0. Then, integrals over v_z, ψ_0, z_0, ϕ_0 can be done analytically. The integral over v_z is simpy 2 v_M , and integral over the orbit is decoupled from the rest of the expression. Then, (2.3.4) and (2.2.15) give

$$\frac{N_1(E)}{n\,N(E-\hbar\omega)} = \int_0^{2\pi} \frac{d\phi_0}{2\pi} \int_{r_0(I)}^{\infty} dr_i\, \sigma_T\left(\vec{p}_i(r_i); E_L(I)\right) \tag{2.3.6}$$

where E is defined by (2.3.1) and is in that way related to I. $\vec{p}_i$ is directed radially, with the magnitude and the direction given by

$$\begin{aligned} p_i(r_i) &= \left(2mK(I-I(r_i))\right)^{1/2} \\ \vec{p}_i \cdot \vec{E}_L &= \cos\phi_0 \end{aligned} \tag{2.3.7}$$

If the dependence of γ_T upon the laser intensity is known and simple, then it is possible to invert this equation to get γ_T analytically, but in general the method, described in this Section seems to be too complicated to be useful for extracting γ_T from the shape of the second peak.

P A R T I I I

MULTIPHOTON IONIZATION INTO MULTIPLE ENERGY CONTINUA

& 3.1 I n t r o d u c t i o n . Recent experiments on the energy distribution of electrons produced in laser induced multiphoton ionization processes have shown structure in these distributions. This was discussed in Part II of the Thesis and is due at least three separate phenomena. In the first (Compton at al,[47]) the residual ion can be left in more than one bound state and the energy of the ionized electrons will reflect this (Fig. 2.1). We shall not be concerned with this process here. In the second the ionized electron can collide with another atom while still in the laser beam and absorb additional photons during the collision (inverse bremsstrahlung) thereby changing its energy. Some experimental aspects of this process were presented in & 2.3 and we will not be concerned with it here. Finally, during the ionization process the electron can have absorbed any number of photons and so can leave the ionizing event with a multiplicity of energies (Agostini at al [35] ,Fig. 2.2;P. Kruit at al [48] ,Fig. 2.3). In addition each of these energies can be broadened by the ponderomotive potential which acts to expel the electron from its place of birth inside the laser field. This broadening has some interesting theoretical and experimental ramifications which have been discussed in Part I and Part II of the Thesis and will not be pursued further here.

This Part will be concerned with a theory of the absorption of more than the minimum number of photons to ionize. The process has been

called "above threshold ionization" (ATI)[49]. The interesting aspect of the process is the mechanism for the absorption of these extra photons. It is the result of the combination of two interactions, that of the electron with the electromagnetic field and of the electron with its residual ion. Each of the interactions alone cannot accomplish the absorptions. It is currently impossible to treat the interactions together in anything like an exact way so one must resort to some expansion in one of them. If the laser interaction is treated as a perturbation then the multiple photon absorptions would require multiple interaction. This implies the need for some higher order in a perturbation series which is usually difficult to obtain. If the electron-ion interaction is treated as a perturbation then multiple photon absorptions can be accomplished at a single scattering but the multiple scatterings by this interaction must be neglected. The latter description is the simplest and so we shall adopt it but we shall present two calculations of this type whose comparison will allow some assesment of the importance of the multiple scattering effect. We shall assume the fluorescence by the atom plays no role in this process.

As an example, our calculation will describe an ionization which passes through a single bound state resonance. The number of photons which connect the ground state to the resonant bound state need not be specified for our calculation. It only determines the matrix element which connects these states. However, for the sake of simplicity we assume a one photon transition. We also assume that the absorption of only a single photon by the resonantly excited state will cause ionization. Our first calculation will treat this last absorption to the continuum as a weak perturbation. The unperturbed state will therefore be

the two state rotating wave approximation to the atomic wavefunction. This will be used to find the ionization rate by perturbation theory in & 3.2. In & 3.3 we extend this calculation by allowing for a strong single photon absorption from the excited bound state but we then treat all additional absorptions as perturbations. In both calculations we resort to an approximation in which only one electron is active and all others are treated as spectators. & 3.4 contains a numerical comparison of the two calculations and a discussion of them.

& 3.2 L o w e s t o r d e r c a l c u l a t i o n . The "zero order" atomic state will be described by the two state rotating-wave approximation with the allowance for the slow variation with time of the laser intensity. We find it convenient to write the two orthonormal states ($\hbar = c = 1$) as ([16], Chapter II)

$$\Phi_{\pm} = (2ch\mu)^{-\frac{1}{2}} \left(e^{\pm\mu/2 + i\omega t/2} u_0 \pm e^{\mp\mu/2 - i\theta - i\omega t/2} u_1 \right) \times$$

$$\times \exp -i \left((W_0 + W_1)t/2 \pm \int^t dt' \mathcal{E}(t') \right) \qquad (3.2.1)$$

where u_0 and u_1 are the ground and resonantly excited states, respectively, and W_0 and W_1 are their respective energies. The laser-atom coupling in $\vec{p}\cdot\vec{A}$ gauge is

$$\Lambda(t/T) = \left(u_0 \left| \frac{e}{m\omega} \vec{p}\cdot\vec{E}(t/T) \right| u_1\right) = |\Lambda(t/T)| e^{i\theta} \qquad (3.2.2)$$

where $E(t/T)$ is the amplitude of the laser electric field and T is the time scale of the slow time variation. The parameters μ and $\mathcal{E}$ are defined in terms of this matrix element and so are also slowly

varying functions of time. The detuning parameter, μ , is given by varying functions of time. The detuning parameter, μ , is given by

$$\sinh\mu = \frac{\Delta\omega}{|\Lambda|} \tag{3.2.3}$$

where

$$\Delta\omega = \omega - W_{10} \tag{3.2.4}$$

and the Rabi frequency is

$$\mathcal{E} = \left(\Delta\omega^2 + |\Lambda|^2\right)^{1/2} \tag{3.2.5}$$

The laser intensity ($\sim E^2$) is assumed to vanish in the remote future and past where the states $\phi_\pm$ take on simple forms. For $\Delta\omega > 0$ and $|t| \to \infty$

$$\phi_+ \sim u_0 \,, \quad \phi_- \sim u_1 \tag{3.2.6}$$

and for $\Delta\omega < 0, |t| \to \infty$

$$\phi_+ \sim u_1 \,, \quad \phi_- \sim u_0 \tag{3.2.7}$$

Since we are assuming that the entire process happens too quickly for fluorescence to occur, then for $\Delta\omega > 0$ ($\Delta\omega < 0$) the initial state is ϕ_+ (ϕ_-) and will remain so as the laser switches on adiabatically.

The exact S matrix for the ionization process is

$$S_{\vec{q},0} = -i \left\langle \chi_{\vec{q}}, V \Psi_i^{(+)} \right\rangle \tag{3.2.8}$$

where $\Psi_i^{(+)}$ is the exact wave function which we shall approximate by ϕ_+ or ϕ_- depending on the sign of $\Delta\omega$. The ionized electron wave function is taken to be the one given by (15)

$$\chi_{\vec{q}} = \exp i(\vec{q}\cdot\vec{r} - \vec{\alpha}_0\cdot\vec{q}\sin\omega t - E_q t) =$$

$$= \sum_{n=-\infty}^{\infty} J_{-n}(\vec{\alpha}_0\cdot\vec{q}) \exp i(\vec{q}\cdot\vec{r} - (E_q - n\omega)t) \quad (3.2.9)$$

where α_0 is a measure of the coupling of the active electron to the laser field

$$\vec{\alpha}_0(t/T) = \frac{e}{m\omega^2}\vec{E}(t/T) \quad (3.2.10)$$

In obtaining $\chi_{\vec{q}}$ and $\phi_{\pm}$ were neglected corrections of order T^{-1}. Finally V is the interaction potential of the active electron with the residual ion, in the approximation in which the remaining electrons are frozen into their initial states. This form of the S matrix allows for an adiabatic deformation of the atom by the laser and then a single impulsive interaction in which the electron is ejected with the absorption of n photons.

Substitution of (3.2.1) and (3.2.9) into the S matrix, with the performance of the time integration results in

$$S_{\vec{q},0} = -2\pi i \sum_n \delta(E_q - W_0 - n\omega + \tfrac{1}{2}(\Delta\omega - \mathcal{E}))\, T_{\vec{q},0}(\ell) \quad (3.2.11)$$

where the result for the T matrix for n photon ionization is

$$T_{\vec{q},0}(n) = (2ch\mu)^{-1/2}\Big(J_{-n}(\vec{\alpha}_0\cdot\vec{q})\, e^{|\mu|/2}(\lambda_{\vec{q}}, V u_0) + \quad (3.2.12)$$

$$+ J_{-n+1}(\vec{\alpha}_0\cdot\vec{q})\, e^{-|\mu|/2 - i\theta}(\lambda_{\vec{q}}, V u_1)\Big)$$

where $\lambda_{\vec{q}} = e^{i\vec{q}\cdot\vec{r}}$. The time integral which leads to this result was performed in the limit of large T, with the use of (3.3.13). The parameters μ and α_0 in (3.2.12) are evaluated at $t=0$, which means the value of the laser amplitude that occurs over almost all of the time. (This is discussed in more detail in & 3.3). The absolute value signs on μ in (3.2.12) allow for either sign of $\Delta\omega$. Then, the transition rate for n photon ionization can be obtained from (3.2.12) as

$$\omega_n = \int \frac{d^3q}{(2\pi)^2}\, \delta(E_q - n\omega - W_0)\, |T_{\vec{q},0}(\ell)|^2 \qquad (3.2.13)$$

where $E_q = q^2/2m$ and we have neglected the small energy shift of the bound state by the laser, which occurs in the delta function of (3.2.11). We shall defer further discussion of this result until the last Section.

& 3.3 I m p r o v e d a p p r o x i m a t i o n f o r t h e i n i t i a l w a v e f u n c t i o n . In this Section the exact wave function is approximated by one which contains u_0, u_1, and $u_{\vec{q}}^{(+)}$, the lowest energy continuum of ionization. It will also be treated in the rotating wave approximation. It is assumed to have the form

$$\psi_i^{(+)} = \alpha(t)\, u_0 e^{-iW_0 t} + \beta(t)\, u_1 e^{-iW_1 t} + \int \frac{d^3k}{(2\pi)^3}\, \gamma_{\vec{k}}(t)\, u_{\vec{k}}^{(+)}\, e^{-iE_k t} \qquad (3.3.1)$$

This is essentially the phenomenological model of Beers and Armstrong [50] who gave a set of equations which determine the coefficients α, β, and $\gamma_{\vec{k}}$. They can be derived more rigorously ([16], Chapter VI) with essentially the same result. The wave function (3.3.1) is a pro-

jection of the full wave function of the atom in laser on to the states U_0, U_1, and $U_K^{(+)}$, i.e. $\Psi_i^{(+)} = P\Psi^{(+)}$, where P is the projection operator, defined by (45) in Introduction of the Thesis. The three states are excluded from the Green function $G_Q^{(+)}$, which was defined by equation (47), and we can expand $G_Q^{(+)}$ in a perturbation series in powers of the laser field. Then, substitution of $P\Psi^{(+)}$ into equation (49) and successive projections from the left by U_0^*, U_1^*, U_K^* yield the set of equations for the amplitudes α, β, γ_K, which is in the rotating wave approximation

$$i\dot{\alpha} = \frac{1}{2}\Lambda e^{i\Delta\omega t}\beta(t) + \frac{1}{4}\int \frac{d^3K}{(2\pi)^3} H_{0K} e^{i(\Delta\omega_K + \Delta\omega)t}\gamma_{\vec{K}}(t) \quad (3.3.2a)$$

$$i\dot{\beta} = \frac{1}{2}\Lambda^* e^{-i\Delta\omega t}\alpha(t) + \frac{1}{2}\int \frac{d^3K}{(2\pi)^3} \Lambda_{\vec{K}} e^{i\Delta\omega_K t}\gamma_{\vec{K}}(t) \quad (3.3.2b)$$

$$i\dot{\gamma}_{\vec{K}} = \frac{1}{2}\Lambda_{\vec{K}}^* e^{-i\Delta\omega_K t}\beta(t) + \frac{1}{4} H_{0\vec{K}}^* e^{-i(\Delta\omega_K + \Delta\omega)t}\alpha(t) \quad (3.3.2c)$$

where $\Delta\omega_K = \omega + W_1 - E_K$ and

$$\Lambda_{\vec{K}} = \left(U_1 \left| \frac{e}{m\omega}\vec{E}(t/T)\cdot\vec{P} \right| U_K^{(+)}\right) \quad (3.3.3)$$

$$H_{0\vec{K}} = \sum_m{}' \frac{\left(U_0 \left| \frac{e}{m\omega}\vec{E}(t/T)\cdot\vec{P}\right| U_n\right)\left(U_n \left| \frac{e}{m\omega}\vec{E}(t/T)\cdot\vec{P}\right| U_K^{(+)}\right)}{W_{on} + \omega} \quad (3.3.4)$$

$\Lambda_{\vec{K}}$ is the coupling of U_1 to the continuum and $H_{0\vec{K}}$ is the two photon direct coupling of U_0 to the continuum, since the prime on the sum in (3.3.4) is meant to exclude U_1 from the sum. Since U_1 is resonantly coupled to the ground state and to the continuum, the terms $H_{0\vec{K}}$ are one order in $\frac{|\Lambda|}{\omega}$ smaller than the Λ and $\Lambda_{\vec{K}}$ terms, which is the same

order as terms dropped in the rotating wave approximation, and they can be dropped. The set of equations can be solved, [16] but we will not pursue it here since in our case it is more useful to make an expansion, which is completely equivalent to the one in (3.3.1), by using the states $\Phi_{\pm}$, (3.2.1) , instead of U_0 and U_1 . This takes the form

$$\Psi_i^{(+)} = A_+\Phi_+ + A_-\Phi_- + \int \frac{d^3K}{(2\pi)^3} \gamma_{\vec{K}}(t) U_{\vec{K}}^{(+)} e^{-iE_K t} \qquad (3.3.5)$$

where the initial condition for $\Delta\omega>0$ is at $t=-\infty$, $A_+=1, A_-=\gamma_{\vec{K}}=0$, and for $\Delta\omega<0$, $A_-=1, A_+=\gamma_{\vec{K}}=0$. We shall choose the first case and then give the results for both at the end of our calculation. The states $\Phi_{\pm}$ individually diagonalize the Hamiltonian in the two state rotating wave approximation. Therefore, in the case in which $A_-(-\infty)=0$, its growth depends upon its coupling to A_+ via the continuum. This is a small effect which we shall neglect so that the equations that result from (3.3.2) and (3.3.5) are

$$i\dot{A}_+ = \frac{1}{2}\int \frac{d^3K}{(2\pi)^3} \xi_{\vec{K}+}(t)\gamma_{\vec{K}} \exp i\left(\Delta\omega_K t - \frac{1}{2}\int_{-\infty}^{t} dt'(\varepsilon(t')-\Delta\omega)\right) \qquad (3.3.6)$$

$$i\dot{\gamma}_{\vec{K}} = \frac{1}{2}\xi^*_{\vec{K}+}(t) \exp -i\left(\Delta\omega_K - \frac{1}{2}\int_{-\infty}^{t} dt'(\varepsilon(t')-\Delta\omega)\right) A_+ \qquad (3.3.7)$$

where

$$\xi_{\vec{K}+}(t) = e^{-\mu/2 - i\theta}(2ch\mu)^{-1/2}\Lambda_{\vec{K}}(t) \qquad (3.3.8)$$

We can use (3.3.7) to obtain $\gamma_{\vec{K}}(t)$ and substitute it back into (3.3.6). This results in an integral equation for A_+. It may be solved by the

assumption (Weisskopf and Wigner,[62])

$$A_+(t) = e^{-\frac{1}{2}\int_{-\infty}^{t} dt' \Gamma(t')} \tag{3.3.9}$$

The further assumption that $\Gamma \ll \mathcal{E}$ leads to the explicit result

$$\Gamma(t) = \frac{\pi e^{-\mathcal{A}}}{4ch\mu} \int \frac{d^3K}{(2\pi)^3} |\Lambda_{\vec{K}}|^2 \delta(\Delta\omega_K) \tag{3.3.10}$$

where we neglected $\mathcal{E}$ compared to ω and have also neglected a small imaginary part of Γ, which can be interpreted as a radiative correction to the bound eigenvalues. The interpretation of this result is clear: The inclusion of the continuum in (3.3.5) has led to an exponential decay of the bound states, but since the pumping rate between U_0 and U_1 in $\phi_\pm$ is much larger then the ionization rate (from U_1), the states U_0 and U_1 maintain their relative amplitude and phase and decay at the same rate.

We can now turn to the calculation of the ionization rate by substituting

$$\Psi_+ = \phi_+ e^{-\frac{1}{2}\int_{-\infty}^{t} dt' \Gamma(t')} + \int \frac{d^3K}{(2\pi)^3} \gamma_{\vec{K}}(t) U_{\vec{K}}^{(+)} e^{-iE_K t} \tag{3.3.11}$$

for $\Psi_i^{(+)}$ in the S matrix, (3.2.8). We use (3.3.7) to eliminate $\gamma_{\vec{K}}$ and arrive at the form

$$S_{\vec{q},0} = -i \int dt \sum_n J_{-n}(\vec{\alpha}_0 \cdot \vec{q}) e^{i(E_q - n\omega)t} \left\{ \left((\lambda_{\vec{q}}, V u_0) e^{\mu/2 + i\omega t/2} + \right.\right.$$

$$\left.\left. + (\lambda_{\vec{q}}, V u_1) e^{-\mu/2 - i\theta - i\omega t/2} \right) e^{-\frac{i}{2}\left((W_0 + W_1 + \Delta\omega)t + \int_{-\infty}^{t} dt' (\varepsilon(t') - \Delta\omega) \right)} \times \right.$$

$$\times e^{-\frac{1}{2}\int_{-\infty}^{t} dt' \Gamma(t')} (2ch\mu)^{-1/2} - \frac{i}{2} \int \frac{d^3K}{(2\pi)^3} (\lambda_{\vec{q}}, V u_{\vec{K}}^{(t)}) \int_{-\infty}^{t} dt' \zeta_{\vec{K}t}(t') \times$$

$$\left. \times e^{-i(\Delta\omega_K t' - \frac{1}{2}\int_{-\infty}^{t'} dt''(\varepsilon(t'') - \Delta\omega) - \frac{1}{2}\int_{-\infty}^{t'} dt'' \Gamma(t''))} \right\} \qquad (3.3.12)$$

The time integrations may now be done in the limit of large T by the use of the generic relations

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} dt\, e^{iEt} f_1(t/T) = 2\pi \delta(E) f(0) \qquad (3.3.13)$$

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} dt\, e^{iE_1 t} f_1(t/T) \int_{-\infty}^{t} dt' e^{iE_2 t'} f_2(t'/T) = 2\pi^2 \delta(E_1)\delta(E_2) f_1(0) f_2(0) \qquad (3.3.14)$$

This results in

$$S_{\vec{q},0} = -2\pi i \sum_{n=2}^{\infty} \delta(E_q - n\omega - W_0)\, T_{\vec{q},0}^{(n)} \qquad (3.3.15)$$

where we have again, as in & 3.2, neglected small shifts of W_0 of the order of ε . Then $T_{\vec{q},0}^{(n)}$ can be interpreted as the T matrix element for the n photon ionization. It is given by

$$T_{\vec{q},0}(n) = \Big\{ J_{-n}(x)(\lambda_{\vec{q}}, V u_0) e^{\mu/2} + (\lambda_{\vec{q}}, V u_1) J_{-n+1}(x) e^{-\mu/2 - i\theta} -$$

$$- i\frac{\pi}{2} J_{-n+2}(x) \int \frac{d^3k}{(2\pi)^3} (\lambda_{\vec{q}}, V u^{(+)}_{\vec{k}}) \delta(E_k - 2\omega - W_0) \Lambda^*_{\vec{k}}(0) e^{-\mu/2 - i\theta} \Big\} \times$$

$$\times e^{-\frac{1}{2}\int_{-\infty}^{0} dt' \Gamma(t')} (2ch\mu)^{-1/2} \qquad (3.3.16)$$

where μ means $\mu(t=0)$ and

$$x = \vec{\alpha}_0(0) \cdot \vec{q} \qquad (3.3.17)$$

The ionization rate into the nth continuum is then obtained from (3.2.13).

The n=2 case, which is the lowest energy continuum, is already contained in (3.3.11) and so must be dealt with carefully. For n=2 (3.3.16) is

$$T_{\vec{q},0}(2) = \Big\{ J_2(x)(\lambda_{\vec{q}}, V u_0) e^{\mu/2} - e^{-\mu/2 - i\theta} \Big(J_1(x)(\lambda_{\vec{q}}, V u_1) +$$

$$+ i\frac{\pi}{2} J_0(x) \int \frac{d^3k}{(2\pi)^3} (\lambda_{\vec{q}}, V u^{(+)}_{\vec{k}}) \delta(E_k - 2\omega - W_0) \Big(u^{(+)}_{\vec{k}}, \frac{e}{m\omega} \vec{E}(0) \cdot \vec{p}\, u_1 \Big) \Big) \Big\} \times$$

$$\cdot e^{-1/2 \int_{-\infty}^{0} dt' \Gamma(t')} (2ch\mu)^{-1/2} \qquad (3.3.18)$$

We use the identity

$$(\lambda_{\vec{q}}, V u_1) = -\frac{(n-1)}{x} \Big(\lambda_{\vec{q}}, \frac{e}{m\omega} \vec{E}(0) \cdot \vec{p}\, u_1 \Big) \qquad (3.3.19)$$

(which is obtained from the Schrodinger equation for u_1 , and the

delta function in (3.2.13)) to revrite (3.3.18) as

$$T_{\vec{q},0}^{(2)} = \Big\{ J_2(x)\left(\lambda_{\vec{q}}, V u_0\right) e^{i\mu/2} + e^{-i\mu/2 - i\theta}\Big(\frac{J_1(x)}{x}\left(\lambda_{\vec{q}}, \frac{e}{m\omega}\vec{E}(0)\cdot\vec{p}\,u_1\right) -$$

$$- i\frac{\pi}{2} J_0(x) \int \frac{d^3k}{(2\pi)^3}\left(\lambda_{\vec{q}}, V u^{(+)}_{\vec{k}}\right)\delta(E_q - E_k)\left(u^{(+)}_{\vec{k}}, \frac{e}{m\omega}\vec{E}(0)\cdot\vec{p}\,u_1\right)\Big\} \times$$

$$\times e^{-\frac{1}{2}\int_{-\infty}^{0} dt' \Gamma(t')} (2\mathrm{ch}\mu)^{-\frac{1}{2}} \qquad (3.3.20)$$

The rotating wave approximation is one in which only on shell or near on shell terms are retained in intermediate states. This is equivalent to the replacement

$$\left(E_q + i\eta - E_k\right)^{-1} \longleftrightarrow -i\pi\,\delta(E_q - E_k) \qquad (3.3.21)$$

in the intermediate state of the last term of (3.3.20). This replacement results in

$$T_{\vec{q},0}^{(2)} = \Big\{ J_2(x)\left(\lambda_{\vec{q}}, V u_0\right) e^{i\mu/2} + e^{-i\mu/2 - i\theta}\Big(\frac{J_1(x)}{x}\left(\lambda_{\vec{q}}, \frac{e}{m\omega}\vec{E}(0)\cdot\vec{p}\,u_1\right) +$$

$$+ \frac{1}{2} J_0(x)\left(\lambda_{\vec{q}}, V G^{(+)}(E_q)\frac{e}{m\omega}\vec{E}(0)\cdot\vec{p}\,u_1\right) - \frac{1}{2} J_0(x)\left(\lambda_{\vec{q}}, V u_0\right)\Big\} \times$$

$$\times e^{-\frac{1}{2}\int_{-\infty}^{0} dt' \Gamma(t')} (2\mathrm{ch}\mu)^{-1/2} \qquad (3.3.22)$$

where $G^{(+)}(E_q)$ is the Green's function of the electron in the absence of the laser, in the restricted set of states contained in (3.3.1). The last term results from the fact that u_0 is absent as an intermediate state in the last term of (3.3.20). We may now use the solution of the

Lippman-Schwinger equation in the form

$$u_{\vec{q}}^{(+)} = \lambda_{\vec{q}} + G^{(+)}(E_2) V \lambda_{\vec{q}} \tag{3.3.23}$$

to eliminate $G^{(+)}$ from (3.3.22). The result is

$$T_{\vec{q},0}(2) = \left\{ \frac{1}{2} e^{-\mu/2 - i\theta} J_0(x) \left(u_{\vec{q}}^{(-)}, \frac{e}{m\omega} \vec{E}(0)\cdot\vec{p}\, u_1 \right) + \right.$$
$$\left. + e^{\mu/2} (\lambda_{\vec{q}}, V u_0) \left(J_2(x) - \frac{\epsilon - \Delta\omega}{2\omega} J_0(x) \right) - \frac{x}{2} J_2(x) e^{-\mu/2 - i\theta} (\lambda_{\vec{q}}, V u_1) \right\} \times$$
$$\times e^{-\frac{1}{2}\int_{-\infty}^{0} dt' \Gamma(t')} (2ch\mu)^{-1/2} \tag{3.3.24}$$

where we have used (3.2.3) and (3.2.5) and a Bessel function recurrence relation, (1.4.97). The first term (for x=0) is just the first Born approximation for ionization from the state U_1 , weighted by the amplitude for finding this state in ϕ_+ . The remaining terms are higher order corrections in the laser interaction.

Returning to (3.3.16) it can be compared to the result in & 3.2, (3.2.12)

$$T_{\vec{q},0}^{(3)}(n) = e^{-\frac{1}{2}\int_{-\infty}^{0} dt' \Gamma(t')} \left(T_{\vec{q},0}^{(2)}(n) - \right.$$
$$\left. - i\frac{\pi}{2} J_{-n+2}(x) \int \frac{d^3k}{(2\pi)^3} (\lambda_{\vec{q}}, V u_{\vec{k}}^{(t)}) \delta(E_k - 2\omega - W_0) \Lambda_{\vec{k}}^{*}(0) e^{-\mu/2 - i\theta} (2ch\mu)^{-1/2} \right) \tag{3.3.25}$$

where the superscripts on T are included to indicate the Section of this Part in which they were obtained.

& 3.4 Discussion and numerical results. The difference between the two methods given above can be seen by comparison of (3.2.1) and (3.3.11) which are our two approximations to the exact wave function. The inclusion of the lowest continuum in (3.3.11) is clearly the first of a sequence of calculations, each of which would include an additional continuum. The sequence converges to the exact wave function in the rotating wave approximation.

The inclusion of the first continuum in (3.3.11) results in the exponential decay with time of the bound states. This is reflected the first term of (3.3.25) in which the direct ionization from the bound state, $T^{(2)}$, is modified by the exponential decay of that state in the time interval from $t=-\infty$ to $t=0$ at which time the impulsive scattering takes place. The reduction of this contribution is compensated by the contribution from the second term of (3.3.25) which represents scattering from the lowest ($n=2$) continuum into all continua, including itself. This term contains the factor $(\lambda_2, V\psi_k^{(+)})$ which is the exact off-shell (for $n\neq 2$) T matrix for the electron scattering by the ion in the absence of the laser. This is an example of the fact that the appearance of the laser in an atomic process makes possible the observation of scattering parameters which are otherwise not observable.

We can illustrate this process with the Cs atom. A laser, whose wavelength is about 459 nm will be resonant with the $6S_{1/2}$ - $7P_{1/2}$ transition and the single (double, triple) absorption in the $7P_{1/2}$ state will result in a continuum electron of about 1.5 eV (4.2 eV, 6.9 eV). We have calculated the ionization rates for absorption of n photons, for

$n = 2,3,4,5$ by using (3.2.13) and our two results for $T_{\vec{q},0}(n)$, (3.2.12) and (3.3.24), (3.3.25)

$$W_m = \int d\Omega_q \, dE_q \, \delta(E_q - n\omega - W_0) \left| T_{\vec{q},0}(n) \right|^2 \qquad (3.4.1)$$

which is the same as (3.2.13), the only difference being in the normalization of the free electron wave function. As a matter of convenience, we will in this Section use the plane waves, normalized as

$$(\lambda_{\vec{q}}, \lambda_{\vec{q}'}) = \pi \, \delta(E_q - E_{q'}) \qquad (3.4.2)$$

We shall briefly dicuss the structure of the Cs atom. The closed shell of Cs acts on the valence electron as a spherically symmetric distribution of charges, causing the outer electron to move in a Coulomb like potential which can be written, neglecting the exchange effects, in the form

$$V(r) = -\frac{Z(r)e^2}{r} \qquad (3.4.3)$$

$Z(r)$ is the effective charge that the electron sees. Its value at nucleus is the atomic number and it decreases to 1 far enough from the nucleus. As a consequence, the term values of Cs are hydrogen like

$$W_n = -\frac{Ry}{(n+\mu(l))^2} \qquad (3.4.4)$$

where n is the usual principal quantum number, and μ is the "quantum defect", caused by the screening of the nucleus potential by the closed shell electrons. μ is a function of angular quantum number l, but almost independent on n. It is common to use the "effective quantum

number" $\nu = \mu(l) + n$ instead of μ . This is the basic terminology of the "Quantum Defect Method" [63], [64] which is extensivelly used with alkali atoms.

The term diagram of Cs is shown at Fig. 4.1. The configuration of the ground state is $6S_{1/2}$, with an ionization potential of 3.87 eV, i.e. with the effective quantum number $\nu_0 = 1.875$. The 7P state contains two fine structure states, $P_{1/2}$ and $P_{3/2}$, which differ in energy by 0.02 eV. We will suppose that the laser bandwidth is narrow enough that these two states can be distinguished, and then take into account only one of them, for example $7P_{1/2}$. This state has ionization potential of 1.16 eV, i.e. the effective quantum number $\nu_1 = 3.420$..

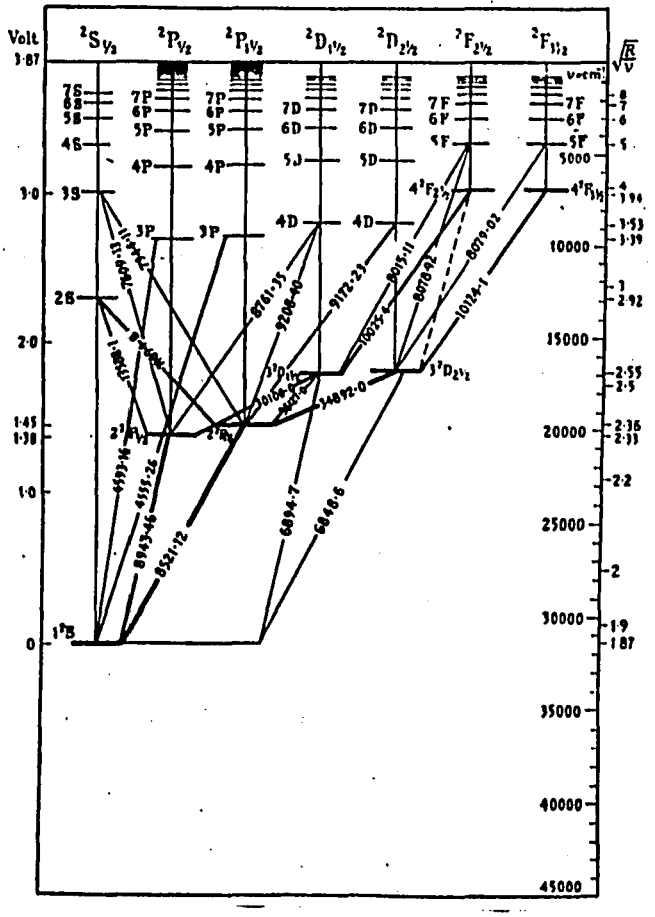

Fig. 3.1 [65]

In the Quantum Defect Theory it was shown [64] that the radial wave functions of the bound states can be taken (in atomic units) as

$$R_{n\ell}(r) \underset{r\ \text{large}}{=} z^{1/2} K(\nu,\ell)\, W_{\nu,\ell+1/2}\left(\frac{2zr}{\nu}\right) \qquad (3.4.5)$$

where W is the Whittaker function, $z = 1$ in our case, and K is the normalizing factor

$$K(\nu,\ell) = \left[\zeta(\nu)\,\nu^2\,\Gamma(\nu+\ell+1)\,\Gamma(\nu-\ell)\right]^{-1/2} \qquad (3.4.6)$$

with

$$\zeta(\nu) = 1 + \frac{2}{\nu^3}\frac{\partial\mu(W)}{\partial W} \qquad (3.4.7)$$

From the term diagram at Fig. 4.1 it can be found that μ is, as a function of energy W, almost a stright line, with the slope $\frac{\partial\mu}{\partial W} \approx 0.03$ eV for the S series, and $\frac{\partial\mu}{\partial W} \approx 0.29$ eV for the P series. Then, from (3.4.7) it follows that $\zeta(\nu_0) \approx \zeta(\nu_1) \approx 1$. The asympthotic form of the Whittaker function is

$$W_{\nu,\ell+1/2} = \left(\frac{2r}{\nu}\right)^{\nu} e^{-r/\nu} \sum_{t=0}^{t_0} b_t(\nu,\ell)\, r^{-t} + O(r^{-t_0-1}) \qquad (3.4.8)$$

where

$$b_0 = 1\,, \quad b_t = \frac{\nu}{2t}\left[\ell(\ell+1) - (\nu-t)(\nu-t+1)\right] b_{t-1} \qquad (3.4.9)$$

The criterion for the termination of the series in t was stated by Bates and Damgard [63], as

$$\nu+\ell-1 \leq t_0 \leq \nu+\ell \qquad (3.4.10)$$

That criterion led to a satisfactory agreement with experimental values of bound-bound and bound-free matrix elements of $\vec{r}$ and is based on the compromis between the necessity to keep in the series as many terms as possible and the necessity to avoid the terms whose principal contribution comes from near the nuclus ($r < a_0$).

We need the radial wave functions (3.4.5) of the bound states $6S_{1/2}$ and $7P_{1/2}$ of Cs to calculate the bound-free matrix elements of V

$$(\lambda_{\vec{q}}, V u_j) = -(n-j)\,\omega\,(\lambda_{\vec{q}}, u_j)\ , \quad j=0,1 \qquad (3.4.11)$$

Expanding the free electron wave functions $\lambda_{\vec{q}}$ in partial waves, we can write

$$(\lambda_q, V u_j) = -(n-j)\, i^{\ell_j} \frac{\omega}{Ry} \left(\frac{Ry}{2\pi}\right)^{1/2} (2\ell_j+1)^{1/2} \zeta_q^{-1/2} P_{\ell_j}(\hat{q}\cdot\hat{E}) R_j \qquad (3.4.12)$$

where

$$\zeta_q = \frac{e^2}{\hbar v_q} \qquad (3.4.13)$$

and

$$R_j = \int_0^\infty j_{\ell_j}(qr)\, r\, R_{\nu_j \ell_j}\, dr \qquad (3.4.14)$$

P_{ℓ_j} and j_{ℓ_j} are the Legendre polinimials and the spherical Bessel functions, respectively. Only the ℓ_j component of the free electron survi-

ves in (3.4.12). The radial integrals in (3.4.14) were calculated with the use of (3.4.5), (3.4.8) and (3.4.10). That could be done analytically, but we rather broke up the integral into two, performing the integration separately in the regions $r\epsilon(0,1)$ and $r\epsilon(1,\infty)$. In that way we were able to control the validity of the criterion (3.4.10) in this case. We found that the contribution of the region (0,1) does not exceed 5% of the R_j in the worst case (n = 5), with a similar conclusion for the region $(1,\infty)$ if $t > t_0$. The values of R_j, $j = 0,1$ for $n = 2,3,4,5$ are presented in Table 3.1.

	2	3	4	5
R_0	5.198	0.399	-0.343	-0.440
R_1	-4.429	-0.886	-0.052	0.124

Table 3.1

Bound-bound matrix element Λ, defined by (3.2.2), was obtained from the Cs oscillator strengths of Stone [66]. One can write

$$|\Lambda|^2 = e^2 E^2 \frac{3\hbar}{2m\omega} f \qquad (3.4.15)$$

and using $f = 0.00284$ for $6S_{1/2} \rightarrow 7P_{1/2}$ transition, we found

$$|\Lambda| = 2.131 \times 10^{-8} \sqrt{I}\ eV \qquad (3.4.16)$$

where I is the laser intensity in W/cm^2. The sign of Λ was decided using the method of Seaton [67] for calculation of the radial bound-bound integrals. That leads to

$$\Lambda = |\Lambda| e^{-i\frac{\pi}{2}} \qquad (3.4.17)$$

The bound-free matrix element (3.3.3)

$$\left(u_{\vec{k}}, \frac{e}{m\omega}\vec{E}(0)\cdot\vec{p}\,u_1\right) = i\left(\frac{R\gamma}{4\pi}\right)^{1/2}\left(\frac{I}{I_0}\right)^{1/2}\left\{\sqrt{\frac{(\ell+1)(2\ell+3)}{2\ell+1}}\, g_{\ell\to\ell+1} P_{\ell+1}(\hat{k}\cdot\hat{E}) + \right.$$

$$\left. + \sqrt{\frac{\ell(2\ell-1)}{2\ell+1}}\, g_{\ell\to\ell-1} P_{\ell-1}(\hat{k}\cdot\hat{E})\right\} \qquad (3.4.17)$$

was found by the use of the relevant radial integrals $g_{\ell_i\to\ell_j}$, calculated by Seaton [67]. We get $(\ell=1)$

$$g_{1\to0} = -3.205 \quad , \quad g_{1\to2} = 6.906 \qquad (3.4.18)$$

The decay parameter Γ , defined by (3.3.10), can be calculated using (3.4.17) and (3.4.18), which yields

$$\Gamma = 5.19\, I \ \text{sec}^{-1} \qquad (3.4.19)$$

where I is again in W/cm^2.

To calculate the free-free matrix element

$$\left(\lambda_{\vec{q}(n)}, V\, u^{(+)}_{\vec{k}}\right) , \quad n>2 , \quad k = q(2) \qquad (3.4.20)$$

we approximate the lowest continuum wave function $u^{(+)}_{\vec{k}}$ by the Coulomb wave function

$$u^{(+)}_{\vec{k}} = \frac{\sqrt{mk}}{2\pi}\, e^{\frac{\pi}{2}\xi_k}\, \Gamma(1-i\xi_k)\, F(i\xi_k, 1, i(kr-\vec{k}\cdot\vec{r}))\, e^{i\vec{k}\cdot\vec{r}} \qquad (3.4.21)$$

Then

$$\left(\lambda_{\vec{q}}, V\, u^{(+)}_{\vec{k}}\right) = -\frac{1}{(2\pi)^2}\,\frac{1}{(\xi_q \xi_k)^{1/2}}\, J_{\vec{q},\vec{k}} \qquad (3.4.22)$$

where J is (expressed in atomic units)

$$J_{\vec{q},\vec{K}} = \int e^{i(\vec{K}-\vec{q})\vec{r}} F(i\zeta_K, 1, i(Kr - \vec{K}\cdot\vec{r})) \frac{d^3r}{r} = 4\pi \frac{(q^2-K^2)^{-i\zeta_K}}{((\vec{q}-\vec{K})^2)^{1-i\zeta_K}} \quad (3.4.23)$$

The right hand side can be expanded in Legendre polinomials $P_\ell(\hat{q}\cdot\hat{K})$ as

$$J_{\vec{q},\vec{K}} = \sum_{\ell=0}^{\infty} A_\ell P_\ell(\hat{q}\cdot\hat{K}) \quad (3.4.24)$$

where

$$A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} J_{\vec{q},\vec{u}}(x) P_\ell(x)\, dx \quad (3.4.25)$$

Using the addition theorem of spherical harmonics, we further have

$$J_{\vec{q},\vec{u}} = 2\pi \sum_{\ell=0}^{\infty} B_\ell \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\hat{q}\cdot\hat{E}) Y_{\ell m}(\hat{u}\cdot\hat{E}) \quad (3.4.26)$$

As the matrix element (3.4.20) is coupled in $T^{(n)}_{\vec{q},0}, n>2$ to the matrix element (3.4.17), the only components that will survive in (3.4.26) are S and D, with $m=0$. Keeping only these two terms, we finally get

$$(\lambda_{\vec{q}}, V u_{\vec{u}}) = -\frac{1}{4\pi(\zeta_K \zeta_q)^{1/2}} \left(B_0 + 5B_2 P_2(\hat{q}\cdot\hat{E}) P_2(\hat{u}\cdot\hat{E})\right) \quad (3.4.27)$$

where

$$B_0 = \frac{2}{qK\zeta_K} \sin\zeta_K \ln\frac{q+K}{q-K} \quad (3.4.28)$$

$$B_2 = \frac{1}{qK(1+i\zeta_K)(2+i\zeta_K)} \left(\frac{3(q^2+K^2)^2 - 4K^2q^2(1+\zeta_K)^2}{2q^2K^2\zeta_K} \sin\zeta_K \ln\frac{q+K}{q-K} - \right.$$

$$\left. - \frac{3q^2+K^2}{qK} \cos\zeta_K \ln\frac{q+K}{q-K} \right) \quad (3.4.29)$$

The values of B_0, B_2 for n = 3,4,5 are presented in Table 3.2.

	B_0	Re B_2	Im B_2
3	-3.0331	-5.2595	-6.8234
4	0.2789	-1.3839	-1.7954
5	1.3902	0.3851	0.4996

Table 3.2

The gama function $\Gamma(1-i\zeta_K)$, which appears in the normalization of the Coulomb wave function, was calculated using Stirling's formula

$$\Gamma(1+z) \approx \sqrt{2\pi z}\, e^{-z} z^{z} \qquad (3.4.30)$$

with $\zeta_K = 2.9807$. That yields

$$Re\,\Gamma(1-i\zeta_K) \approx e^{-\frac{\pi}{2}\zeta_K} \times 1.5036, \quad Im\,\Gamma(1-i\zeta_K) = -e^{-\frac{\pi}{2}\zeta_K} \times 2.6989 \qquad (3.4.31)$$

After integration over the energy delta function, ionization rate (3.4.1) can be written in the form

$$W(2) = \frac{Ry}{\hbar} \frac{e^{-\int_{-\infty}^{0} dt' \Gamma(t')}}{2ch\mu} \int_{-1}^{1} dt \left(\frac{e^{-\mu/2}}{2\sqrt{6}} \sqrt{\frac{I}{I_0}} \left(g_{1\to 0} + \sqrt{10}\, g_{1\to 2} P_2(t) \right) J_0(at) + \right.$$

$$\left. + e^{\mu/2} \frac{n\hbar\omega}{Ry} \frac{1}{\sqrt{\zeta_K}} R_0 J_2(at) + e^{-\mu/2} \frac{at}{2} \frac{(n-1)\hbar\omega}{Ry} \frac{\sqrt{3}}{\sqrt{\zeta_K}} R_1 P_1(t) J_2(at) \right)^2 \qquad (3.4.32)$$

$$W(n>2)=\frac{R_y}{\hbar}\frac{e^{-\int_{-\infty}^{0}dt'\Gamma(t')}}{2ch\mu}\int_{-1}^{1}dt\,\Big|e^{\mu/2}\frac{n\hbar\omega}{R_y}\frac{1}{\sqrt{\gamma_g}}R_0J_n(at)+$$

$$+e^{-\mu/2}\frac{(n-1)\hbar\omega}{R_y}\frac{\sqrt{3}}{\sqrt{\gamma_g}}R_1P_1(t)J_{n-1}(at)-$$

$$-\frac{i}{4\sqrt{6}}e^{-\mu/2}\sqrt{\frac{I}{I_0}}\frac{1}{\sqrt{\gamma_g\gamma_K}}e^{\frac{\pi}{2}\gamma_K}\Gamma(1-i\gamma_K)\Big(B_0g_{1\to0}+\sqrt{10}\,B_2g_{1\to2}P_2(t)\Big)\times$$

$$\times J_{n-2}(at)\Big|^2 \qquad (3.4.33)$$

where

$$a=2\left(\frac{R_y}{\hbar\omega}\right)^2\frac{1}{\gamma_g}\sqrt{\frac{I}{I_0}}\,,\quad g=g(n) \qquad (3.4.34)$$

and the remaining integration is essentially integration over the direction of the photoelectron. We are interested in the range of laser intensities up to 10^{15} W/cm^2, where the parameter a, (3.4.34), can exceed 1. In that case the series expanssion of the Bessel functions is not a satisfactory one. The integrals in (3.4.32) and (3.4.33) were done numerically, on the computer P D P 10, Digital, up to the four significant digits.

The ionization rates (3.4.32) and (3.4.33) can be written in the form

$$W(n)=W_0(n)\,e^{-\int_{-\infty}^{0}dt'\Gamma(t')} \qquad (3.4.35)$$

At Fig. 3.2 is presented $W(2)$ as a function of the laser intensity, in the range 10^9 W/cm^2 - 10^{15} W/cm^2. Although the W_0 values are very large, the ionization rate (3.4.35) is very small. This is due to the fact that even for very short exposures of the atom to the laser, the

decay parameter $\int \Gamma dt$ will be very large (see (3.4.19)). The smalness of the transition rates (3.4.35) arises from the fact that the ionization probability will be essentially unity so that the rate of ionization will be small for a small additional time. For this reason, and since it is difficult to measure absolute ionization rates we only present the ratios of W(n)/W(2) from the results of & 3.3, calculated from (3.4.32) and (3.4.33). This ratio is independent of the exponential decay factor. Fig. 3.3 shows this ratio for laser intensities ranging from 10^9 to 10^{15} W/cm^2. In the final calculations of W's, μ was taken to be zero. As expected, the higher continua are only weakly populated even at the higher intensities. The slopes at the lower intensities are roughly linear (on the log-log plot) and roughly correspond to the perturbation theory result $W(n)/W(2) \sim I^{n-2}$. The structure of the curves results from the interference of the last two terms of (3.3.16). Our results are smaller than those obtained experimentally by Kruit [48] for Xe by a factor of 10 or so for $n=3$ and 100 for $n=4$. Although the experiments and theory describe different targets, it is still suprising that the results differ by such a large factor. One possible explanation is that the intensity distribution in a focused laser beam differs significantly from the average value. Then the fact that the ionization rates are rapidly rising functions of the intensity could account for the higher experimental results. At higher intensities, $I \sim 10^{13}$ W/cm^2, we see a departure from the general linear behaviour which is attributable to the structure of the Bessel functions and an interference among the three terms.

We have also calculated ionization rates from the T matrix, (3.2.12) in & 3.2. Their relation to those obtained from & 3.3 contains

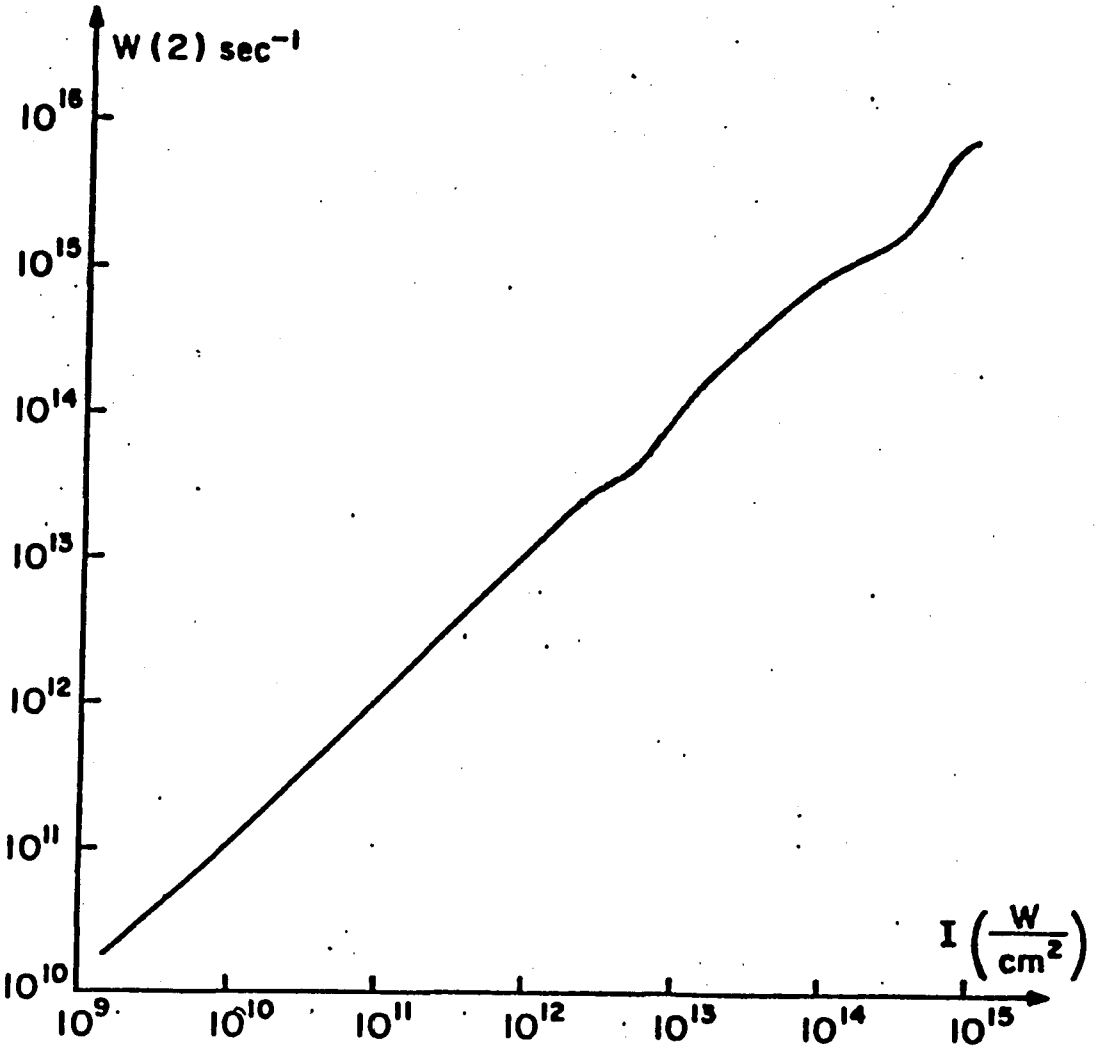

Fig. 3.2

a factor $e^{-\int r dt}$ which depends upon the additional parameter of the exposure time. However, this will enter as an overall multiplicative factor in the ratio $\frac{W^{(3)}(n)}{W^{(2)}(n)}$. Our results for this ratio as a function of n and I show that no such factor can bring these ratios close to unity for the entire range of these parameters. This shows that the additional process included in $T^{(3)}$ significantly changes the results from those of $T^{(2)}$. We have pointed out above that the two calculations can be considered to be the start of a sequence of calculations which

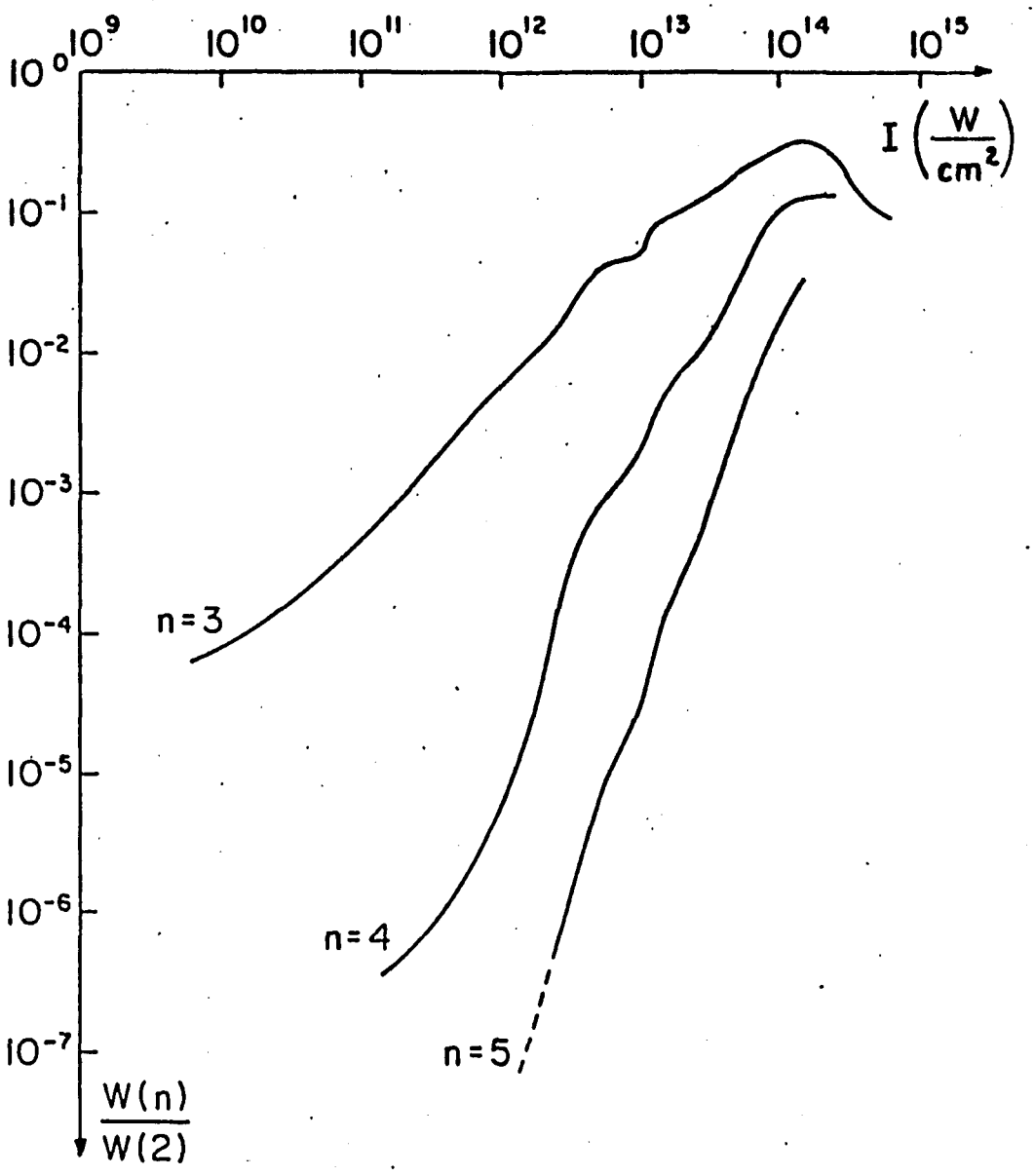

Fig. 3.3

converge to the exact result. The fact that the results are so different could indicate that the sequence is far from convergence. The next calculation in the sequence would start from a wave function such as (3.3.1), but it would be necessary to include a second continuum (n = 3) in it. The possible lack of convergence could also account for the discrepancy when we compared with the experiment.

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