Non-Abelian Hydrodynamics and Anomalies in Quark Gluon Plasma

by

HAILONG LI

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2006

UMI Number: 3232027

UMI

UMI Microform 3232027

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

This manuscript has been read and accepted for the
Graduate Faculty in Physics in satisfaction of the
dissertation requirement for the degree of Doctor of Philosophy.

V. P. Nair

Date

Chair of Examining Committee

S. Catto

Date

Executive Officer

A. P. Polychronakos

P. Orland

D. Karabali

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Non-Abelian Hydrodynamics and Anomalies in Quark Gluon Plasma

by

Hailong Li

Advisor: Prof. V. P. Nair

Elementary properties of quantum chromodynamics and some important experimental findings in relativistic heavy ion collisions are briefly reviewed. A model for non-Abelian hydrodynamics is constructed. This model is then applied to a quark gluon plasma with an extension to incorporate various anomalies of the system.

Acknowledgments

My greatest gratitude goes to my advisor, Prof. V. P. Nair, for his solid guidance and tremendous support. Without him, this research work would not have been possible. I deeply thank Prof. Sultan Catto, for his constant encouragement and support. I thank my friend Rashmi Ray, for many inspiring discussions. I truly appreciate the help provided by all the people in the Physics Program of The City University of New York. Last but not least, I thank my parents and my wife, for their patience, support, and love.

Contents

1 Quantum Chromodynamics and Quark Matter 1

1.1 QCD and color confinement.. 1

1.2 Recent Experimental Findings.. 5

2 Non-Abelian Hydrodynamics 8

2.1 Introduction.. 8

2.2 Canonical Formulation of Ordinary Hydrodynamics..........................10

2.3 A Model for Non-Abelian Hydrodynamics....................................... 19

2.3.1 Non-Abelian Current...19

2.3.2 The Action for non-Abelian Hydrodynamics............................23

2.3.3 Generalization to Multi-Component Fluid...............................31

2.4 Discussion.. 37

3 Anomalies in QGP as a non-Abelian Fluid 40

3.1 Introduction.. 40

3.2 Incorporation of Anomalies... 44

3.3 Conclusions... 53

A Charge Density Algebra 55

B Wess Zumino Term and Anomaly for Flavor Dynamics 58

Biliography 63

Chapter 1

Quantum Chromodynamics and Quark Matter

The study of quark matter is important for investigating fundamental problems of quantum chromodynamics as well as for understanding the evolution of the early Universe. In section 1.1 we briefly review the elementary properties of quantum chromodynamics. We note some important experimental findings of relativistic heavy ion collisions in section 1.2.

1.1 QCD and Color Confinement

The fundamental theory of strong interactions between quarks and gluons is quantum chromodynamics (QCD). The QCD Lagrangian density is given by

$$\mathfrak{L} = \sum_{\alpha=1}^{6}\sum_{i,j=1}^{3} \overline{\psi}^{i}_{\alpha}\left(i\gamma^{\mu}D^{ij}_{\mu} - m_{\alpha}\delta^{ij}\right)\psi^{j}_{\alpha} - \frac{1}{4}F^{\mu\nu}_{a}F^{a}_{\mu\nu}, \tag{1.1}$$

where ψ is the spinor of quark fields, $\overline{\psi} \equiv \psi^{\dagger}\gamma_0$ is the Dirac conjugate of ψ, and m is the quark mass matrix. The covariant derivative is given by

$$\begin{aligned} D^{ij}_{\mu} &= (\partial_{\mu} + A_{\mu})^{ij} \\ &= \partial_{\mu}\delta^{ij} - ig_{s}A^{a}_{\mu}T^{ij}_{a}, \end{aligned} \tag{1.2}$$

where $A_\mu^a, a = 1, 2, \ldots, 8$ are the gluon fields. T_a are the generators of the $SU(3)$ color symmetry in the fundamental representation, they are Hermitian, traceless 3×3-matrices. g_s is the strong interaction coupling constant. $F_a^{\mu\nu}$ is the gluon field strength tensor

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g_s f_{abc} A_b^\mu A_c^\nu, \tag{1.3}$$

where f_{abc} are the structure constants of $SU(3)$.

In QCD, quarks and gluons are the analogs of electrons and photons in QED. Quarks interact with each other by exchanging gluons just as electrons interact with each other by exchanging photons. However, unlike photons, which are chargeless, gluons carry color charge. As a consequence, while photons cannot directly interact with each other, there are direct interactions between gluons. Thus QCD is an intrinsically non-linear theory. This ultimately leads to the remarkable property of asymptotic freedom through the mechanism of antiscreening by polarization of virtual gluons. Mathematically, the running coupling constant is given by [1]

$$\alpha(q^2) = \frac{12\pi}{(33 - 2N_f)\ln(q^2/\Lambda_{QCD}^2)}, \tag{1.4}$$

where q is the four momentum transferred in strong interactions, Λ_{QCD} is the

QCD scale parameter, $\Lambda_{QCD} \simeq 200MeV$, and N_f is the number of quark flavors. For QCD, $N_f = 6$. Therefore $\alpha(q^2)$ decreases as q increases, which means interactions with very large momentum transfer can be treated perturbatively. However, to study the structure of hadrons, their spectra and interactions, we need to work in a regime where the effective coupling as given by (1.4) is large at the hadron dimensions. Presently, we don't have the analytic techniques to solve the field equations in this regime. Specifically, color confinement, though widely believed to be true, has not been analytically proven.

Experimentally, color confinement corresponds to the fact that color charged quarks cannot be individually found. Mathematically, color confinement says that free particles must exist as color singlet. We can understand confinement qualitatively. As we try to separate two quarks, the gluon fields form a string between them. Because the gluons carry color charges, the force between the quark pair remains constant as the distance between the quarks increases. As a consequence, the energy in the string increases linearly with the distance. At some point, it is energetically favorable for a quark-antiquark pair to appear. Then the string breaks, resulting in two mesons instead of two individual quarks.

On the other hand, because of asymptotic freedom, at very large densities and/or temperatures, the interactions between quarks and gluons should become weak enough to set them free [2]. Thus it is possible for the quarks and gluons to enter such a deconfined phase in which they propagate freely over the entire volume occupied by the system. Such a system is called a quark gluon plasma (QGP). This is not in contradiction with color confinement because the deconfined quarks and gluons as a whole form a color singlet and we still cannot separate out an individual quark. The mechanism for deconfinement in dense nuclear matter is screening of color charge [3]. This is analogous to the Debye screening in ordinary plasma.

Deconfinement phase transition can happen under two scenarios. The first is for nuclear matter to be compressed to extreme extent. The nucleons then start to overlap and screening effect comes into play, setting the quarks free. This can happen in the core of neutron stars where nuclear matter is compressed by strong gravitational force.

The second scenario is for nuclear matter to be heated up. Particle number grows as the temperature increases to above particle mass, resulting in high density. Numerical studies of lattice QCD indicates that when the temperature of nuclear matter increases to above $T_c \approx 170MeV$, deconfinement transition happens and a quark gluon plasma is produced. It is believed that in the first few microseconds after the creation of the Universe in the Big Bang all matter is in the state of such a hot QGP. As the Universe expands the plasma cools down and the quarks condense into hadrons. This scenario also leads to the only experimental method to create QGP. There is a strong hope to observe QGP by colliding heavy nuclei at ultra-high energies, creating a high temperature high density "fireball". An important experimental program is being run at the Relativistic Heavy Ion Collider (RHIC) facility of Brookhaven National Laboratory, and another will begin soon at the Large Hadron Collider (LHC) of CERN. Study of QGP can not only improve our understanding of the strong interactions and hadron structures but also provide insights into the evolution of the early Universe.

1.2 Recent Experimental Findings

Since the RHIC started running in 2000, a large body of data has been collected. Although definitive evidence for the creation of QGP has not yet been found, many novel phenomena have been observed. Here we only note some of the most interesting findings. A more comprehensive review of current experimental status can be found in reference [4].

It was expected by naive theoretical considerations that the QGP should behave like a weakly coupled, almost ideal gas. However, experimental data suggests otherwise. At the energy scale of RHIC (up to several times T_c), remarkable phenomena such as jet quenching and elliptic flow strongly indicate a highly dense, strongly interacting quark gluon plasma, the so called sQGP. Jets produced from hard scattering of quarks are observed to be suppressed in central Au+Au collisions [4]. This jet quenching results from interactions with a dense medium. Jets are strongly absorbed by the fireball; the energy appears to be transferred to hydrodynamic motion. Elliptic flow, on the other hand, occurs in non-central collisions. In this case, the overlap region is anisotropic. The momenta of the hadrons that emerge from the fireball have an elliptical distribution [4]. The elliptical pattern indicates that large pressure must have built up in the collision center, resulting in pressure gradient dependent on azimuthal angle, causing momentum space anisotropy. This means that the quarks and gluons in the fireball behave collectively. Thus the QGP is like a liquid instead of a gas. Were it a gas, the hadrons would have emerged in an isotropic pattern. Furthermore, the details of the anisotropic pattern indicate that the viscosity of this liquid is almost zero. It is the most ideal fluid ever observed.

It is fair to say that at this point the study of QGP is driven by experiments; the large amount of new and often unexpected phenomena provide golden opportunities to understand strong interactions and hadron structures. Theoretical work has yet to catch up.

Chapter 2

Non-Abelian Hydrodynamics

2.1 Introduction

Currently, most of the theoretical work in the study of QGP has been based on perturbative QCD at high temperatures, assuming weak coupling [5]. While this could be a good description at high temperatures and for plasma states that are not too far from equilibrium, a hydrodynamic description may be more suitable for nondilute plasmas or for situations far from equilibrium. Indeed, as we noted in Chapter 1, experimental findings indicate that the hot QGP behaves like a strongly coupled, dense, liquid instead of a weakly coupled, dilute, gas. This implies that the effective interaction in QGP is in fact much stronger than expected on the basis of the perturbation theory.

It is well known that the hydrodynamic equations can be derived from particle dynamics by taking suitable averages of Boltzmann-type equations. Specifically for the QGP, some work along these lines was done many years ago using single particle kinetic equations [6]. These kinetic equations form a hierarchy, the so-called BBGKY hierarchy (named after Born, Bogoliubov, Green, Kirkwood, and Yvon), involving higher and higher correlated N-particle distribution functions. To be able to solve them, one needs to truncate the hierarchy, very often at just the single particle distribution function. Therefore, the feasibility of solving these equations limits the kinetic approach to dilute systems near equilibrium, where the truncation can be justified. However, hydrodynamic equations can also be derived from very general principles, showing that they have a much wider regime of validity, and, indeed in practice, we apply them over such a wider range. This is the universality of hydrodynamics. In the context of QGP, which is a non-Abelian system, in analogy with the ordinary hydrodynamics, we aim to derive an a priori non-Abelian hydrodynamics, which incorporates the non-Abelian degrees of freedom, coupling to a non-Abelian gauge field, etc. This theory may be valid for strongly coupled, dense, systems.

In section 2.2 we review the canonical formulation of ordinary hydrodynamics. We give in section 2.3 the non-Abelian generalization [7], and discuss the results in section 2.4.

2.2 Canonical Formulation of Ordinary Hydrodynamics

For simplicity we consider an isentropic fluid, that is, entropy is constant and does not appear in our theory. Our discussion here largely follows the review paper by Jackiw, Nair, Pi and Polychronakos [8]. The conceptual basis of the hydrodynamics is very simple: it is just a set of local conservation laws for the energy-momentum tensor $T^{\mu\nu}$ and for the current j^{μ}

$$\partial_{\mu} T^{\mu\nu} = 0, \tag{2.1}$$

$$\partial_{\mu} j^{\mu} = 0. \tag{2.2}$$

In the nonrelativistic case, this is equivalent to the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.3}$$

which ensures matter conservation, that is, time independence, of $N = \int d^3 r \rho$, and the Euler equations

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{f}, \tag{2.4}$$

which is the expression of a nonrelativistic force law. Here $\mathbf{f}$ is the force density. We ignore dissipation and take the force density to be given by pressure P: $\mathbf{f} = -\frac{1}{\rho}\nabla P$. For isentropic motion P is a function only of ρ, so $\mathbf{f}$ can also be written as $-\nabla V'(\rho)$:

$$\mathbf{f} = -\frac{1}{\rho}\nabla P = -\nabla V'(\rho), \tag{2.5}$$

with the prime denoting the derivative with respect to the argument. $V'(\rho)$ is the enthalpy, $\rho V'(\rho) - V(\rho) = P(\rho)$.

Dynamics of any particular system is most economically presented when a canonical/action formulation is available. Also, a canonical formulation is important for quantization. We note that the equations (2.3, 2.4, 2.5) can be obtained by Poisson bracketing with the Hamiltonian

$$H = \int d^3r\Big(\frac{1}{2}\rho\mathbf{v}^2 + V(\rho)\Big) = \int d^3r\varepsilon, \tag{2.6}$$

$$\frac{\partial\rho}{\partial t} = \{H,\rho\}, \tag{2.7}$$

$$\frac{\partial\mathbf{v}}{\partial t} = \{H,\mathbf{v}\}, \tag{2.8}$$

provided the non-vanishing brackets of the fundamental $(\rho,\mathbf{v})$ variables are taken to be [9]

$$\{v^i(\mathbf{r}),\rho(\mathbf{r}')\} = \partial_i\delta(\mathbf{r}-\mathbf{r}'), \tag{2.9a}$$

$$\{v^i(\mathbf{r}),v^j(\mathbf{r}')\} = -\frac{w_{ij}(\mathbf{r})}{\rho(\mathbf{r})}\delta(\mathbf{r}-\mathbf{r}'), \tag{2.9b}$$

where $w_{ij} \equiv \partial_i v^j - \partial_j v^i$ is the vorticity. Fields in the brackets are at equal time, hence the time argument is suppressed. Poisson brackets for a more general, magnetohydrodynamic system have been explicitly constructed, from which (2.9a, b) result as special formulae. It can be verified that the Jacobi identity is satisfied by the above brackets.

We then seek a Lagrangian whose canonical variables lead to the Poisson brackets (2.9a, b) and to the Hamiltonian (2.6). In more mathematical terms, we look for a canonical 1-form and a symplectic 2-form that lead to the algebra (2.9a, b) [10]. Without losing generality, we consider a Lagrangian that is first order in time. Denoting all variables by the generic symbol ξ^i, the most general first-order Lagrangian is

$$L = a_i(\xi)\ \dot{\xi}^i - H(\xi). \tag{2.10}$$

Although we are interested in fields defined on space-time, for illustration purpose it suffices to consider variables $\xi^i(t)$ that are functions only of time. The Euler-Lagrange equation derived from (2.10) is

$$f_{ij}(\xi)\ \dot{\xi}^j = \frac{\partial H(\xi)}{\partial \xi^i}, \tag{2.11}$$

where

$$f_{ij}(\xi) = \frac{\partial a_j(\xi)}{\partial \xi^i} - \frac{\partial a_i(\xi)}{\partial \xi^j}. \tag{2.12}$$

The first term in (2.10) gives the canonical 1-form: $a_i(\xi)\, \dot{\xi}^i\, dt = a_i(\xi) d\xi^i$, while f_{ij} gives the symplectic 2-form: $da_i(\xi) d\xi^i = \frac{1}{2} f_{ij}(\xi) d\xi^i d\xi^j$.

If the matrix f_{ij} has the inverse f^{ij}, then from (2.11) we have

$$\dot{\xi}^i = f^{ij}(\xi) \frac{\partial H(\xi)}{\partial \xi^j}. \tag{2.13}$$

If one wants this equation of motion to result from a Poisson bracket with the Hamiltonian

$$\begin{aligned} \dot{\xi}^i &= \{H(\xi), \xi^i\} \\ &= \{\xi^j, \xi^i\} \frac{\partial H(\xi)}{\partial \xi^j}, \end{aligned} \tag{2.14}$$

one needs to postulate the fundamental bracket as

$$\{\xi^i, \xi^j\} = -f^{ij}(\xi). \tag{2.15}$$

The Poisson bracket between functions of ξ is then defined by

$$\{F_1(\xi), F_2(\xi)\} = -\frac{\partial F_1(\xi)}{\partial \xi^i} f^{ij} \frac{\partial F_2(\xi)}{\partial \xi^j}. \tag{2.16}$$

It can be verified that the Jacobi identity holds for (2.15) by virtue of (2.12).

Our formulation of the hydrodynamics actually concerns the inverse of the above discussion. From (2.9a, b) we know the form of f^{ij} and that the Jacobi identity is satisfied. We then wish to determine the inverse f_{ij}, and also a_i from (2.12). Since we know the Hamiltonian from (2.6), the expression of the Lagrangian (2.10) can be obtained straightforwardly.

However, an obstacle may arise: If there exists a quantity $C(\xi)$ whose Poisson brackets with all the ξ^i vanish, then

$$0 = \{\xi^i, C(\xi)\} = -f^{ij}\frac{\partial}{\partial\xi^j}C(\xi). \tag{2.17}$$

That is, f^{ij} has zero modes $\frac{\partial}{\partial\xi^j}C(\xi)$, and the inverse to f^{ij}, namely the symplectic 2-form f_{ij}, does not exist. Totally commuting quantities like $C(\xi)$ are called "Casimirs". As we will see below, the algebra (2.9a, b) indeed has Casimirs.

Consider the quantity

$$C(\mathbf{v}) \equiv \int d^3r\epsilon^{ijk}v^i\partial_j v^k = \int d^3r\mathbf{v}\cdot\mathbf{w}, \tag{2.18}$$

it Poisson commutes with both ρ and $\mathbf{v}$. Thus the symplectic 2-form does not exist, or in other words, f^{ij} has no inverse. To overcome this obstacle we need to utilize the so called Clebsch parametrization [11], using an idea of Lin [12].

Consider the vector field $\mathbf{v}$. Any three-dimensional vector, which involves three functions, can be presented as

$$\mathbf{v} = \nabla\theta + \alpha\nabla\beta, \tag{2.19}$$

with three suitably chosen scalar functions θ, α, and β. This is the Clebsch parametrization. The vorticity then becomes

$$\mathbf{w} = \nabla\alpha \times \nabla\beta \tag{2.20}$$

and the Lagrangian is taken as

$$L = -\int d^3r\rho\left(\dot{\theta} + \alpha\,\dot{\beta}\right) - H_{\mathbf{v}=\nabla\theta+\alpha\nabla\beta} \tag{2.21}$$

with $\mathbf{v}$ in H expressed as in (2.19). Thus there are two canonical pairs: (ρ, θ) and $(\rho\alpha, \beta)$, that is,

$$\{\theta(\mathbf{r}), \rho(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}')$$

$$\{\beta(\mathbf{r}), \rho\alpha(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}'). \tag{2.22}$$

The phase space is 4-dimensional, corresponding to the four observables ρ and $\mathbf{v}$. A straightforward calculation shows that the Poisson brackets (2.9a, b) are reproduced using (2.22), with $\mathbf{v}$ constructed by (2.19).

Now we observe that in the Clebsch parametrization $C(\mathbf{v})$ is given by

$$C(\mathbf{v}) = \int d^3r\epsilon^{ijk}\partial_i\theta\partial_j\alpha\partial_k\beta \tag{2.23}$$

which is just a surface integral

$$C(\mathbf{v}) = \int d\mathbf{S} \cdot (\theta \mathbf{w}). \tag{2.24}$$

In this form, $C(\mathbf{v})$ has no bulk contribution, and presents no obstacle to constructing a symplectic 2-form and a canonical 1-form in terms of ρ, θ, α, and β, which are defined in the bulk, that is, for all finite $\mathbf{r}$.

Note that the Lagrangian in (2.21) can also be written as

$$\begin{aligned} L &= -\int d^3r\Big(\rho\big(\dot{\theta} + \alpha\,\dot{\beta}\big) + \rho\mathbf{v} \cdot (\nabla\theta + \alpha\nabla\beta)\Big) \\ &\quad + \int d^3r\Big(\frac{1}{2}\rho\mathbf{v}^2 - V(\rho)\Big) \\ &= -\int d^3r j^\mu(\partial_\mu\theta + \alpha\partial_\mu\beta) + \int d^3r\Big(\frac{1}{2}\rho\mathbf{v}^2 - V(\rho)\Big) \end{aligned} \tag{2.25}$$

We have used covariant notation for the canonical 1-form, with $j^\mu = (\rho, \rho\mathbf{v})$ and $\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)$, in order to generalize the formulation to relativistic case. The form of (2.25) suggests a relativistic Lagrangian density

$$\mathfrak{L} = -j^\mu a_\mu - f\Big(\sqrt{j^\mu j_\mu}\,\Big), \tag{2.26}$$

where

$$a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta. \tag{2.27}$$

The function f depends on the Lorentz invariant $n^2 = j^\mu j_\mu = \rho^2 - \mathbf{j}^2$. Thus we have the Eckart decomposition [13]

$$j_\mu = nu_\mu,\ u^\mu u_\mu = 1. \tag{2.28}$$

f encodes the specific dynamics (equation of state).

The energy-momentum tensor for $\mathfrak{L}$ is

$$\begin{aligned} T^{\mu\nu} &= 2\frac{\partial \mathfrak{L}}{\partial g_{\mu\nu}} + g^{\mu\nu}\mathfrak{L} \\ &= g^{\mu\nu}\mathfrak{L} - \frac{j^\mu j^\nu}{n} f'(n). \end{aligned} \tag{2.29}$$

Equations of motion follow from varying the fields. Variation of j^μ gives

$$a_\mu = -\frac{j_\mu}{n} f'(n), \tag{2.30}$$

so that (2.29) becomes

$$T^{\mu\nu} = g^{\mu\nu}(nf'(n) - f(n)) - nu^\mu u^\nu f'(n). \tag{2.31}$$

Comparing (2.31) with the conventional formula [14]

$$T^{\mu\nu} = g^{\mu\nu}P - u^\mu u^\nu (P + e) \tag{2.32}$$

identifies the proper energy density e and the pressure P (which coincides with $\mathfrak{L}$). Thus

$$e = f(n), \tag{2.33}$$

$$P = nf'(n) - f(n). \tag{2.34}$$

Variation of θ gives the current continuity equation

$$\partial_\mu j^\mu = 0. \tag{2.35}$$

Varying α and β we have

$$u^\mu \partial_\mu \alpha = u^\mu \partial_\mu \beta = 0. \tag{2.36}$$

The curl of (2.30) gives

$$\begin{aligned} &(\partial_\nu u_\mu - \partial_\mu u_\nu) f'(n) + (u_\mu \partial_\nu n - u_\nu \partial_\mu n) f''(n) \\ &= \partial_\mu u_\nu - \partial_\nu u_\mu \\ &= \partial_\mu \alpha \partial_\nu \beta - \partial_\nu \alpha \partial_\mu \beta \end{aligned} \tag{2.37}$$

Contracting this with u^μ and using (2.36) we get

$$u^\mu (\partial_\nu u_\mu - \partial_\mu u_\nu) f'(n) + (g_{\nu\mu} - u_\nu u_\mu) \partial^\mu n f''(n) = 0 \tag{2.38}$$

This is the relativistic Euler equation.

As we mentioned at the beginning of this section, the hydrodynamic equations can be represented by the conservation laws for the energy-momentum tensor and the current. Indeed, the vanishing of the divergence of T^μ_ν implies

$$\begin{aligned} 0 &= \partial_\mu T^\mu_\nu \\ &= n u^\mu (\partial_\nu u_\mu - \partial_\mu u_\nu) f'(n) + (g_{\mu\nu} - u_\mu u_\nu) n f''(n) \partial^\mu n \\ &\quad - \partial_\mu (n u^\mu) u_\nu f'(n). \end{aligned} \tag{2.39}$$

Using the continuity equation (2.35) we have

$$u^{\mu}(\partial_{\nu}u_{\mu} - \partial_{\mu}u_{\nu})f'(n) + (g_{\mu\nu} - u_{\mu}u_{\nu})f''(n)\partial^{\mu}n = 0, \tag{2.40}$$

which is exactly the Euler equation (2.38).

2.3 A Model for Non-Abelian Hydrodynamics

In this section we generalize hydrodynamics to systems with non-Abelian charges.

2.3.1 Non-Abelian Current

Before presenting a specific model, we give a general analysis of the non-Abelian current $J_a^{\mu} = (\rho_{a,}\mathbf{J}_a)$.

The conventional formula for the current of a single, non-Abelian point particle, moving in a 4-dimensional space-time $\{t, \mathbf{r}\}$, along a space-time path $X^{\mu}(\tau)$ (τ parametrizes the path) is

$$J_a^{\mu}(t, \mathbf{r}) = \int d\tau \mathfrak{Q}_a(\tau) \frac{dX^{\mu}(\tau)}{d\tau} \delta(X^0(\tau) - t)\delta(\mathbf{X}(\tau) - \mathbf{r}), \tag{2.41}$$

where $\mathfrak{Q}_a$ is the classical color charge of the particle. This current is covariantly conserved

$$(D_{\mu}J^{\mu})_a \equiv \partial_{\mu}J_a^{\mu} + f_{abc}A_{\mu}^{b}J_c^{\mu} = 0 \tag{2.42}$$

provided that Ω_a satisfies the standard classical equation of motion for non-Abelian particles - the Wong equation [15]

$$\frac{d\Omega_a(\tau)}{d\tau} + f_{abc}\frac{dX^\mu(\tau)}{d\tau}A^b_\mu(X(\tau))\Omega_c(\tau) = 0. \tag{2.43}$$

Here A^a_μ is the potential for a non-Abelian theory based on a gauge group with structure constants f_{abc}.

In the many-particle case Ω_a and X^μ are indexed by a discrete particle label n, which is summed in the definition of the current J^μ_a. In a continuum limit $n \to \mathbf{x}$ we have

$$J^\mu_a(t,\mathbf{r}) = \int d^3x d\tau \Omega_a(\tau,\mathbf{x})\frac{\partial X^\mu(\tau,\mathbf{x})}{\partial\tau}\delta(X^0(\tau,\mathbf{x}) - t)\delta(\mathbf{X}(\tau,\mathbf{x}) - \mathbf{r}), \tag{2.44}$$

$$\frac{\partial\Omega_a(\tau,\mathbf{x})}{\partial\tau} + f_{abc}\frac{\partial X^\mu(\tau,\mathbf{x})}{\partial\tau}A^b_\mu(X(\tau,\mathbf{x}))\Omega_c(\tau,\mathbf{x}) = 0. \tag{2.45}$$

Notice that, since we replace a sum over n with an integral over $\mathbf{x}$, $\Omega_a(\tau,\mathbf{x})$ is now a charge density. The parametrization may be fixed at $X^0(\tau,\mathbf{x}) = \tau$ so that the equations read

$$\rho_a(t,\mathbf{r}) = \int d^3x \Omega_a(t,\mathbf{x})\delta(\mathbf{X}(t,\mathbf{x}) - \mathbf{r})$$

$$\mathbf{J}_a(t,\mathbf{r}) = \int d^3x \Omega_a(t,\mathbf{x})\partial_t\mathbf{X}(t,\mathbf{x})\delta(\mathbf{X}(t,\mathbf{x}) - \mathbf{r}) \tag{2.46}$$

$$\partial_t \mathfrak{Q}_a(t,\mathbf{x}) + f_{abc}[A_0^b(t,\mathbf{X}(t,\mathbf{x})) + \partial_t \mathbf{X}(t,\mathbf{x})\mathbf{A}^b(t,\mathbf{X}(t,\mathbf{x}))]\mathfrak{Q}_c(t,\mathbf{x}) = 0. \tag{2.47}$$

To evaluate the integrals in (2.46) we consider the corresponding expressions for the Abelian charge density and current

$$\rho(t,\mathbf{r}) = \int d^3x \delta(\mathbf{X}(t,\mathbf{x}) - \mathbf{r})$$

$$\mathbf{j}(t,\mathbf{r}) = \mathbf{v}(t,\mathbf{r})\rho(t,\mathbf{r}) = \int d^3x\ \dot{\mathbf{X}}\ (t,\mathbf{x})\delta(\mathbf{X}(t,\mathbf{x}) - \mathbf{r}). \tag{2.48}$$

$\mathbf{x}$ describes the fluid coordinate $\mathbf{X}$ at initial time $t = 0$,

$$\mathbf{X}(t,\mathbf{x}) = \mathbf{x}, \tag{2.49}$$

thus the initial density is normalized to unity. The $\mathbf{x}$ integral evaluates $\mathbf{X}$ at an expression $\boldsymbol{\chi}(t,\mathbf{x})$ which is inverse to $\mathbf{X}(t,\mathbf{x})$

$$\mathbf{X}(t,\boldsymbol{\chi}(t,\mathbf{r})) = \mathbf{r}$$

$$\boldsymbol{\chi}(t,\mathbf{X}(t,\mathbf{x})) = \mathbf{x}. \tag{2.50}$$

There is also a Jacobian for the $\mathbf{x} \rightarrow \mathbf{X}$ transformation. Thus

$$\rho(t,\mathbf{r}) = \frac{1}{\det\left(\frac{\partial X^i}{\partial x^j}\right)|_{\mathbf{x}=\boldsymbol{\chi}}} \tag{2.51}$$

$$\mathbf{v}(t,\mathbf{r}) = \partial_t \mathbf{X}(t,\mathbf{x})|_{\mathbf{x}=\boldsymbol{\chi}} \tag{2.52}$$

In (2.46) the $\mathbf{x}$ integration also evaluates $\mathbf{X}$ at $\boldsymbol{\chi}$ and there is still the Jacobian factor $\left(\det \frac{\partial X^i}{\partial x^j}\right)^{-1}$ which is given by (2.51). Consequently we have

$$\rho_a(t,\mathbf{r}) = Q_a(t,\mathbf{r})\rho(t,\mathbf{r})$$

$$\mathbf{J}_a(t,\mathbf{r}) = Q_a(t,\mathbf{r})\rho(t,\mathbf{r})\mathbf{v}(t,\mathbf{r}), \tag{2.53}$$

or

$$J_a^\mu(t,\mathbf{r}) = Q_a(t,\mathbf{r})j^\mu(t,\mathbf{r}), \tag{2.54}$$

where

$$Q_a(t,\mathbf{r}) = \mathfrak{Q}_a(t,\mathbf{x})|_{\mathbf{x}=\boldsymbol{\chi}} \tag{2.55}$$

$$\rho(t,\mathbf{r})Q_a(t,\mathbf{r}) = \int d^3x\mathfrak{Q}_a(t,\mathbf{x})\delta(\mathbf{X}(t,\mathbf{x})-\mathbf{r}). \tag{2.56}$$

Moreover, differentiating (2.56) with respect to time and using (2.3) and (2.47) results in an equation for $\frac{\partial Q^a}{\partial t}$,

$$\partial_t Q_a(t,\mathbf{r}) + \mathbf{v}(t,\mathbf{r})\cdot\nabla Q_a(t,\mathbf{r}) = -f_{abc}[A_0^b(t,\mathbf{r}) + \mathbf{v}(t,\mathbf{r})\cdot\mathbf{A}^b(t,\mathbf{r})]Q_c(t,\mathbf{r}) \tag{2.57}$$

which can also be written as

$$j^\mu(D_\mu Q)_a = 0. \tag{2.58}$$

This is analogous to the Abelian equation (2.36). Equations (2.57) and (2.58) can be understood from the fact that the covariantly conserved current (2.42) factorizes according to (2.54) into a group variable Q_a and a conserved Abelian current j^μ. Consistency of (2.3), (2.42) and (2.54) then enforces (2.58).

We recognize that (2.57) and (2.58) are the field generalizations of the particle Wong equation (2.43). The decomposition of the non-Abelian current in (2.54) is the non-Abelian version of the Eckart decomposition (2.28). Indeed, (2.54) may be further factored as in (2.28)

$$J_a^\mu(t,\mathbf{r}) = Q_a(t,\mathbf{r})n(t,\mathbf{r})u^\mu(t,\mathbf{r}). \tag{2.59}$$

In 2.3.2, we are guided in our construction of a dynamical model for non-Abelian hydrodynamics by the above properties of the non-Abelian current, which follow from the very general arguments, based on a particle picture for the substratum of a fluid.

2.3.2 The Action for Non-Abelian Hydrodynamics

For a fluid with non-Abelian charges, we still have conservation of the energy-momentum tensor, which gives the continuity equation and Euler equation that specifies the dynamics. In addition, the non-Abelian current must be covariantly conserved as indicated by equation (2.42). We construct an action the variation of which gives the said kinematical and dynamical equations.

The model that we construct is based on a group with group elements g, and anti-Hermitian Lie algebra elements with generators T^a satisfying

$$[T^a, T^b] = f^{abc} T^c, \tag{2.60}$$

and normalized by

$$tr(T^a T^b) = -\frac{1}{2}\delta^{ab}. \tag{2.61}$$

We expect the canonical 1-form implied by the action to lead to such Poisson brackets that the charge density algebra is represented canonically, i.e., reproduces (2.60)

$$\{\rho_a(t,\mathbf{r}), \rho_b(t,\mathbf{r})\} = f_{abc}\rho_c(t,\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'). \tag{2.62}$$

It is well known that the canonical term which leads to (2.62) is the field theoretical version of the Kirillov-Kostant form, which for a particle is

$$I_{KK} = 2\int dttr\left(Kg^{-1}\ \dot{g}\right) \tag{2.63}$$

where K is a fixed element of the Lie algebra [16].

Inspired by this observation, we take the Lagrangian density to be the following generalization of the Abelian expression(2.26)

$$\mathfrak{L} = j^{\mu} 2tr(Kg^{-1}D_{\mu}g) - f(n) + \mathfrak{L}_{gauge}. \tag{2.64}$$

Here j^μ is an Abelian vector field (current) which can also have the Eckart decomposition as in (2.28)

$$j^\mu = (\rho, \rho\mathbf{v}) = nu^\mu, u^\mu u_\mu = 1. \tag{2.65}$$

The covariant derivative

$$D_\mu g = \partial_\mu g + A_\mu g \tag{2.66}$$

involves a dynamical non-Abelian gauge potential $A_\mu = A_\mu^a T_a$ whose dynamics is provided by $\mathfrak{L}_{gauge}$. K is a fixed, constant element of the Lie algebra. The first term in $\mathfrak{L}$ contains the canonical 1-form for our theory and determines the Poisson brackets, while $f(n)$ describes the fluid dynamics. The theory is invariant under gauge transformations with group element U

$$g \to U^{-1} g$$

$$A_\mu \to U^{-1}(A_\mu + \partial_\mu)U. \tag{2.67}$$

According to (2.64), the current to which A_μ^a couples is of the Eckart form (2.59)

$$J_a^\mu = Q_a j^\mu \tag{2.68}$$

with

$$Q \equiv Q_a T^a = gKg^{-1}. \tag{2.69}$$

In Appendix A we show that the canonical structure of the Lagrangian density (2.64) leads to the charge density algebra (2.62). Our analysis in Appendix A follows the method of Bak, Jackiw and Pi [17].

We now derive the equations of motion by varying the group element g and the Abelian current j^{μ} in the action ($\mathfrak{L}_{gauge}$ omitted)

$$I = \int dtd^3x[j^{\mu}2tr(Kg^{-1}D_{\mu}g) - f(n)]. \tag{2.70}$$

First, we compute the variation of g. Let

$$g^{-1}\delta g = v, \delta g = gv. \tag{2.71}$$

Then

$$\begin{aligned}\delta(g^{-1}D_{\mu}g) &= \delta[g^{-1}(\partial_{\mu}g + A_{\mu}g)] \\ &= -vg^{-1}\partial_{\mu}g + g^{-1}\partial_{\mu}(gv) - vg^{-1}A_{\mu}g + g^{-1}A_{\mu}gv \\ &= \partial_{\mu}v + [g^{-1}\partial_{\mu}g, v] + [g^{-1}A_{\mu}g, v] \\ &= \partial_{\mu}v + [g^{-1}D_{\mu}g, v].\end{aligned} \tag{2.72}$$

Substituting (2.72) into the variation of I in (2.70), we have

$$\delta I = \int dtd^3xj^{\mu}2[tr(Kg^{-1}\partial_{\mu}v) + tr(K[g^{-1}D_{\mu}g, v])].$$

Integrating by parts, and rearranging the trace with K gives

$$\delta I = -\int dtd^3x\{\partial_{\mu}j^{\mu}2tr[Kv] + j^{\mu}2tr([g^{-1}D_{\mu}g, K]v)\}. \tag{2.73}$$

First consider arbitrary v: the vanishing of δI requires

$$\partial_\mu j^\mu K + j^\mu [g^{-1} D_\mu g, K] = 0, \tag{2.74}$$

or, after sandwiching the above between $g \ldots g^{-1}$, and noting (2.69),

$$\partial_\mu j^\mu Q + j^\mu [D_\mu g g^{-1}, Q] = 0. \tag{2.75}$$

But,

$$\begin{aligned} D_\mu Q &= D_\mu (g K g^{-1}) \\ &= D_\mu g K g^{-1} + g K D_\mu g^{-1} \\ &= D_\mu g g^{-1} g K g^{-1} - g K g^{-1} D_\mu g g^{-1} \\ &= [D_\mu g g^{-1}, g K g^{-1}], \end{aligned} \tag{2.76}$$

thus

$$\partial_\mu j^\mu Q + j^\mu D_\mu Q = D_\mu (j^\mu Q) = D_\mu J^\mu = 0. \tag{2.77}$$

This is equation (2.42), which states that the non-Abelian current J^μ is covariantly conserved. Next we consider the special variation

$$v = \theta K, \tag{2.78}$$

where θ is an arbitrary function on space-time. This immediately gives

$$\partial_\mu j^\mu = 0. \tag{2.79}$$

This is the continuity equation (2.35).

It remains to derive the Euler equation. This is accomplished by varying j^{μ}; stationary variation requires

$$2tr(QD_{\mu}gg^{-1}) = f'(n)u_{\mu} \tag{2.80}$$

which we call the non-Abelian Bernoulli equation. Taking the curl of (2.80), we have

$$\partial_{\mu}(2tr[QD_{\nu}gg^{-1}]) - \partial_{\nu}(2tr[QD_{\mu}gg^{-1}]) = \partial_{\mu}(f'(n)u_{\nu}) - \partial_{\nu}(f'(n)u_{\mu}). \tag{2.81}$$

To simplify, observe that the first term in (2.81) equals

$$\begin{aligned}
&\partial_{\mu}(2tr[QD_{\nu}gg^{-1}]) \\
&= 2tr(\partial_{\mu}QD_{\nu}gg^{-1} + Q\partial_{\mu}D_{\nu}gg^{-1} - QD_{\nu}gg^{-1}\partial_{\mu}gg^{-1}) \\
&= 2tr\begin{pmatrix} D_{\mu}QD_{\nu}gg^{-1} - A_{\mu}QD_{\nu}gg^{-1} + QA_{\mu}D_{\nu}gg^{-1} \\ +Q\partial_{\mu}D_{\nu}gg^{-1} - QD_{\nu}gg^{-1}\partial_{\mu}gg^{-1} \end{pmatrix} \\
&= 2tr(D_{\mu}QD_{\nu}gg^{-1} + QD_{\mu}D_{\nu}gg^{-1} - QD_{\nu}gg^{-1}D_{\mu}gg^{-1}).
\end{aligned} \tag{2.82}$$

This can be further simplified using (2.76)

$$\partial_{\mu}(2tr[QD_{\nu}gg^{-1}]) = 2tr(QD_{\mu}D_{\nu}gg^{-1} - QD_{\mu}gg^{-1}D_{\nu}gg^{-1}). \tag{2.83}$$

After antisymmetrization in (μ,ν), the left hand side of (2.81) reads

$$\begin{aligned}
&2tr\{Q[([D_{\mu},D_{\nu}]g)g^{-1}] - [D_{\mu}gg^{-1},D_{\nu}gg^{-1}]\} \\
&= 2tr(QF_{\mu\nu} - [Q,D_{\mu}gg^{-1}]D_{\nu}gg^{-1}).
\end{aligned} \tag{2.84}$$

When (2.76) is used again, (2.81) becomes

$$2tr[D_\mu Q D_\nu g g^{-1}] + 2tr(QF_{\mu\nu}) = \partial_\mu(f'(n)u_\nu) - \partial_\nu(f'(n)u_\mu). \tag{2.85}$$

Finally, contracting with $j^\mu = nu^\mu$ and using (2.58) produces the relativistic, non-Abelian Euler equation

$$nu^\mu \partial_\mu(u_\nu f'(n)) - n\partial_\nu f'(n) = 2tr(J^\mu F_{\mu\nu}). \tag{2.86}$$

Apart from a factor n, the left hand side of (2.86) is equal to that of (2.38). The right hand side describes the non-Abelian Lorentz force acting on the charged fluid.

The energy-momentum tensor for the matter part of the Lagrangian density (2.64) is the same as in (2.31) when (2.80) is used to eliminate $tr(Kg^{-1}D_\mu g)$

$$T^{\mu\nu} = -g^{\mu\nu}(nf'(n) - f(n)) + u^\mu u^\nu n f'(n) \tag{2.87}$$

The divergence of the energy-momentum tensor reads

$$\partial_\mu T^{\mu\nu} = \partial_\mu(nu^\mu)u^\nu f'(n) + n[u^\mu \partial_\mu(u^\nu f'(n)) - \partial^\nu f'(n)]. \tag{2.88}$$

The first term vanishes by virtue of the continuity equation (2.35) and the rest is evaluated from Euler equation (2.86), leaving

$$\partial_\mu T^{\mu\nu} = 2tr[J_\mu F^{\mu\nu}]. \tag{2.89}$$

This is canceled by the divergence of the gauge-field energy-momentum tensor

$$\partial_\mu T^{\mu\nu}_{gauge} = -2tr[J_\mu F^{\mu\nu}]. \tag{2.90}$$

So, just like in the Abelian case, energy-momentum conservation implies the non-Abelian Euler force equation (2.86). Covariant conservation of the non-Abelian current has to be enforced additionally. This is achieved by variation of the group element g, which leads to equation (2.77).

It's interesting to examine the nonrelativistic limit of (2.86). The Eckart decomposition implies

$$\begin{aligned} j^\mu &= (\rho, \rho\mathbf{v}) = nu^\mu, u^\mu u_\mu = 1, \\ u^\mu &= \frac{(1,\mathbf{v})}{\sqrt{1-\mathbf{v}^2}}, n = \rho\sqrt{1-\mathbf{v}^2}\,. \end{aligned} \tag{2.91}$$

In the nonrelativistic limit we have

$$n \approx \rho - \frac{1}{2}\rho\mathbf{v}^2, \tag{2.92}$$

$$u^\mu \approx (1,\mathbf{v}). \tag{2.93}$$

Also,

$$f = n + V(n), \tag{2.94}$$

when $V(n) = 0, f = n$, this corresponds to the free theory in the Abelian case.

Using (2.92), (2.93), (2.94), we find that the nonrelativistic limit for the spatial component of (2.86) gives the Euler equation with a non-Abelian Lorentz force

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = force + Q^a \mathbf{E}_a + \mathbf{v} \times Q^a \mathbf{B}_a \tag{2.95}$$

where *force* is the pressure caused by the potential V (and is therefore Abelian in nature), while non-Abelian force term involve the chromoelectric and chromomagnetic fields

$$\mathbf{E}^i_a = \mathbf{F}^a_{0i}, \mathbf{B}^i_a = -\frac{1}{2}\epsilon_{ijk}\mathbf{F}^a_{jk}. \tag{2.96}$$

It is seen that our non-Abelian fluid moves effectively in a single direction specified by $\mathbf{j} = \rho \mathbf{v}$. Nevertheless, it experiences a non-Abelian Lorentz force.

In 2.3.3 we generalize the action to describe a non-Abelian fluid with several independent directions of motion.

2.3.3 Generalization to Multi-Component Fluid

In the previous subsection we construct a hydrodynamic action for a non-Abelian fluid with a single flow, i.e., a single Abelian current j^μ. The non-Abelian current J^μ is proportional to j^μ by the Eckart decomposition

$$J^\mu_a = Q_a j^\mu. \tag{2.97}$$

It's possible to generalize our Lagrangian density (2.64) to give rise to several fluid components carrying various densities and moving with various velocities. This is achieved by choosing several directions in the Lie algebra, $K_{(s)}$, and coupling to different Abelian currents

$$\mathfrak{L} = \sum_{s=1}^{r} j^{\mu}_{(s)} 2tr[K_{(s)} g^{-1} D_{\mu} g] - f(n_{(1)}, n_{(2)}, \ldots, n_{(r)}) + \mathfrak{L}_{gauge}$$

$$j^{\mu}_{(s)} = (\rho_{(s)}, \rho_{(s)} \mathbf{v}_{(s)}) = n_{(s)} u^{\mu}_{(s)}$$

$$u^{\mu}_{(s)} u_{(s)\mu} = 1$$

$$n_{(s)} = \sqrt{j^{\mu}_{(s)} j_{(s)\mu}} \tag{2.98}$$

(Sums over s are indicated explicitly; the summation convention does not apply.) Evidently the current which couples to the gauge potential is now

$$J^{\mu} = \sum_{s=1}^{r} Q_{(s)} j^{\mu}_{(s)}, \text{ with } Q_{(s)} = g K_{(s)} g^{-1}. \tag{2.99}$$

Similar to equation (2.73), variation of the group element g now leads to

$$\delta I = -\int dt d^3x \sum_{s=1}^{r} \left\{ \partial_{\mu} j^{\mu}_{(s)} 2tr[K_{(s)} v] + j^{\mu}_{(s)} 2tr([g^{-1} D_{\mu} g, K_{(s)}] v) \right\}. \tag{2.100}$$

Again, for arbitrary v, the vanishing of δI implies

$$D_\mu J^\mu = 0, \text{ with } J^\mu = \sum_{s=1}^{r} Q_{(s)} j^\mu_{(s)}. \tag{2.101}$$

We also need the conservation of individual $j^\mu_{(s)}$. This is achieved by choosing special forms for v

$$v_{(s')} = \theta K_{(s')}. \tag{2.102}$$

Under the choice (2.102), equation (2.100) leads to

$$\delta I = -\int dtd^3x\{\partial_\mu j^\mu_{(s)} 2tr\left(K_{(s')}K_{(s')}\right)\theta + j^\mu_{(s')} 2tr\left(g^{-1}D_\mu g\left[K_{(s')}, K_{(s')}\right]\right)\theta$$
$$+\sum_{s\neq s'} \left(\partial_\mu j^\mu_{(s)} 2tr\left(K_{(s)}K_{(s')}\right)\theta + j^\mu_{(s)} 2tr\left(g^{-1}D_\mu g\left[K_{(s)}, K_{(s')}\right]\right)\theta\right)\}. \tag{2.103}$$

The first commutator vanishes; so does the second when $K_{(s)}$ and $K_{(s')}$ commute, i.e., when they belong to the Cartan subalgebra. Also $2tr\left(K_{(s)}K_{(s')}\right) = -K^a_{(s)}K^a_{(s')}$; for $s' = s$ this is constant, while for $s' \neq s$ it vanishes when it is arranged that distinct elements of the Cartan subalgebra are selected.

Therefore, we choose the $K_{(s)}$ to belong to the Cartan subalgebra of the Lie algebra and the total number of different channels equals the rank r of the group. Then for δI in (2.103) to vanish $j^\mu_{(s)}$ must be conserved

$$\partial_\mu j^\mu_{(s)} = \partial_t \rho_{(s)} + \nabla \cdot (\rho_{(s)} \mathbf{v}_{(s)}) = 0. \tag{2.104}$$

The Wong equation which follows from the conservation of the non-Abelian current now reads

$$\sum_{s=1}^{r} j^\mu_{(s)} D_\mu Q_{(s)} = 0. \tag{2.105}$$

Varying the individual $j^\mu_{(s)}$ produces the Bernoulli equations

$$2tr[Q_{(s)} D_\mu g g^{-1}] = f^{(s)} u_\mu, \text{ with } f^{(s)} \equiv \frac{\partial}{\partial n_{(s)}} f(n_{(1)}, n_{(2)}, \ldots, n_{(r)}). \tag{2.106}$$

Again, the curl of the above can be cast in the form

$$\partial^\mu \left(u^\nu_{(s)} f^{(s)} \right) - \partial^\nu \left(u^\mu_{(s)} f^{(s)} \right) = 2tr[D^\mu Q_{(s)} D^\nu g g^{-1}] + 2tr[Q_{(s)} F^{\mu\nu}]. \tag{2.107}$$

When contracted with $j^\mu_{(s)} = n_{(s)} u^\mu_{(s)}$, this leads to the Euler equation

$$n_{(s)} u^\mu_{(s)} \partial_\mu \left(u^\nu_{(s)} f^{(s)} \right) - n_{(s)} \partial^\nu f^{(s)} = 2tr\left[j^\mu_{(s)} D_\mu Q_{(s)} D^\nu g g^{-1} \right] + 2tr[j_{(s)\mu} Q_{(s)} F^{\mu\nu}]. \tag{2.108}$$

However, unlike in the single channel case, the right side does not simplify since $j_{(s)\mu} Q_{(s)}$ cannot be replaced by J_μ because the latter requires summing over s. Also the first term in the right hand side of (2.108) does not vanish since (2.105) requires summation over s.

The energy-momentum tensor is

$$T^{\mu\nu} = -g^{\mu\nu}\left(\sum_{s=1}^{r} n_{(s)}f^{(s)} - f\right) + \sum_{s=1}^{r} u^{\mu}_{(s)}u^{\nu}_{(s)}n_{(s)}f^{(s)}. \tag{2.109}$$

Its divergence of course reproduces (2.88)

$$\partial_{\mu}T^{\mu\nu} = \sum_{s=1}^{r} \left\{\partial_{\mu}\left(n_{(s)}u^{\mu}_{(s)}\right)u^{\nu}_{(s)}f^{(s)} + n_{(s)}\left[u^{\mu}_{(s)}\partial_{\mu}\left(u^{\nu}_{(s)}f^{(s)}\right) - \partial^{\nu}f^{(s)}\right]\right\}. \tag{2.110}$$

The first term in the brackets vanishes according to (2.104) and the remainder is evaluated from (2.108) as

$$\sum_{s=1}^{r} \left(2tr\left[j^{\mu}_{(s)}D_{\mu}Q_{(s)}D^{\nu}gg^{-1}\right] + 2tr[j_{(s)\mu}Q_{(s)}F^{\mu\nu}]\right).$$

Since now we are summing over all channels, it follows from (2.99) and (2.105) that, as before,

$$\partial_{\mu}T^{\mu\nu} = 2tr[J_{\mu}F^{\mu\nu}]. \tag{2.111}$$

A more transparent picture of what is happening is given if the dynamical potential separates

$$f(n_{(1)}, n_{(2)}, \ldots, n_{(r)}) = \sum_{s=1}^{r} f_{(s)}(n_{(s)}) \tag{2.112}$$

$$f^{(s)} = f'_{(s)} \tag{2.113}$$

Then the left hand side of (2.108) refers only to variables labeled s, while the right hand side may be rewritten with the help of (2.98) and (2.105) to give

$$n_{(s)}u^{\mu}_{(s)}\partial_{\mu}\left(u^{\nu}_{(s)}f'_{(s)}\right) - n_{(s)}\partial^{\nu}f'_{(s)}$$
$$= 2tr[J_{\mu}F^{\mu\nu}] - 2\sum_{s'\neq s}^{r} tr\left[j_{(s')\mu}\left(Q_{(s')}F^{\mu\nu} + D^{\mu}Q_{(s')}D^{\nu}gg^{-1}\right)\right] \qquad (2.114)$$

Thus in the addition to the Lorentz force, there are forces arising from the other channels $s' \neq s$. Note that for separated dynamics (2.112), the energy-momentum tensor also separates

$$T^{\mu\nu} = \sum_{s=1}^{r} T^{\mu\nu}_{(s)} = \sum_{s=1}^{r} \left\{-g^{\mu\nu}\left(n_{(s)}f'_{(s)} - f_{(s)}(n_{(s)})\right) + u^{\mu}_{(s)}u^{\nu}_{(s)}n_{(s)}f'_{(s)}\right\}, \qquad (2.115)$$

but the divergence of individual $T^{\mu\nu}_{(s)}$ does not vanish. This reflects that energy is exchanged between the different channels and with the gauge field, as is also evident from the equation of motion (2.114). It is clear that this fluid moves with r different velocities $\mathbf{v}_{(s)}$.

The single-channel Euler equation (2.86) is expressed in terms of physically relevant quantities (currents, chromomagnetic fields); the many-channel equation (2.108) involves, additionally, the gauge group element g. One may simplify that equation by going to special gauge, for example $g = I$, so that the right hand side of (2.108) reduces to

$$2tr[j_{(s)\mu}D^{\mu}Q_{(s)}D_{\nu}gg^{-1}] + 2tr\Big[j^{\mu}_{(s)}Q_{(s)}F_{\mu\nu}\Big] = 2tr\Big[K_{(s)}j^{\mu}_{(s)}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})\Big] \qquad (2.116)$$

while the Wong equation (2.105) becomes

$$\sum_{s=1}^{r} j^{\mu}_{(s)}[A_{\mu}, K_{(s)}] = 0. \qquad (2.117)$$

It is interesting that in this gauge the non-linear terms in $F^{\mu\nu}$ disappear.

2.4 Discussion

In section 2.2 we reviewed the canonical formulation of ordinary particle based fluid dynamics. We have presented two non-Abelian generalizations in section 2.3. Both versions use a fluid generalization of the Kirillov-Kostant form that naturally encodes the charge density algebra (2.62). This is needed if the charge density is to be identified as the generator of non-Abelian symmetry transformations. The first version is given in section 2.3.2 and features a single density. In section 2.3.3, we generalized the first version so that the density is specified in terms of a set of Abelian densities equal in number to the rank r of the Lie algebra. Since the charge density at a point in the fluid is an element of the Lie algebra, diagonalization shows that an invariant specification must use r

eigenvalues. Alternatively, we may use the r Casimir invariants of ρ_a at a point to characterize it. Therefore the appearance of r Abelian currents in the Lagrangian is entirely natural.

The two versions also differ in the formula for the current. Our first version, which is mathematically more concise, gives the non-Abelian Eckart decomposition, (2.68), while the second version does not allow the factorization of the current into a non-Abelian charge density and an Abelian velocity, rather it is a sum of such factorized expressions, a generalized Eckart decomposition as in (2.99). The Eckart decomposition shows that we can choose a local Lorentz frame for which a given fluid element can be brought to rest. The charge density is then related to the charge carried by this element. By contrast, in the second version, with a generalized Eckart decomposition, we see that even if we choose a frame where one Lie algebra component of the velocity is zero, the other color velocities need not be. Thus the latter applies to a situation where color separating flows can occur. Physically, it is not yet clear what kinematic regimes of a QGP, for example, would admit or require such flows.

The currents we have obtained are the non-Abelian analogues of the irrotational part of the Abelian current, even though the vorticity is nonvanishing. The other components can be easily incorporated, if needed , by generalizing the Lagrangian in (2.98) as

$$\mathfrak{L} = \sum_{s=1}^{r} j^{\mu}_{(s)} \{2tr[K_{(s)} g^{-1} D_{\mu} g] + a_{\mu(s)}\} - f(n_{(1)}, n_{(2)}, \ldots, n_{(r)}) + \mathfrak{L}_{gauge} \qquad (2.118)$$

where $a_{\mu(s)}$ is given by

$$a_{\mu(s)} = \alpha_{(s)} \partial_{\mu} \beta_{(s)}. \qquad (2.119)$$

The final fluid equations remain unchanged.

There also exists a field based realization of ordinary fluid dynamics which is provided by the Madelung "hydrodynamical" rewriting of the Schrodinger equation [18]. This presentation can be extended to yield a non-Abelian hydrodynamics [7]. In this model the fluid equations are much less appealing because they involve velocities in all group directions. We know of no compelling physical reason for preferring this field based model over the particle based one.

Chapter 3

Anomalies in QGP as a non-Abelian Fluid

3.1 Introduction

In Chapter 2 we have presented a dynamical model for a non-Abelian fluid, i.e., a fluid with internal symmetry. The hydrodynamic action we construct is quite general and may be applied to any fluid with non-Abelian charges. But this generality also means that to capture the features intrinsic to the particular fluid under study, some extensions to the action are needed. Our interest, as stated in the introduction to Chapter 2, is in the quark gluon plasma. The QGP as a real physical system has a vast array of characteristic properties, many of which are not yet known to us and a lot of these are subjects of experimental investigations. This is in contrast with a hypothetical fluid the basic properties of which can be clearly defined and completely enumerated. As a result, if we apply our model to QGP, we can only expect to capture some of the many dynamical features, and only one or a few at a time. Thus possible utilizations of our model to QGP are distinguished by different extensions to the original action (2.64). For example, to study the thermodynamics of QGP, one might extend the action to include terms that incorporate temperature, heat conduction, etc. In this Chapter we focus our attention on the various anomalies that may be present in the dynamics of QGP.

QGP is a plasma of quarks interacting with each other and the $SU(3)$ color gauge field (gluons) as well as the electroweak gauge fields; its constituents and underlying dynamics belong to a subset of the standard model. In the standard model all gauge anomalies cancel. However, in the QGP, various anomalies can arise due to the following reasons:

1. At zero temperature, the hypercharge $U(1)$ anomalies cancel exactly between quarks and leptons. However, for a hot QGP, the thermal distribution will cause the Dirac sea of some quarks to shift, altering the corresponding number distribution. As a result, the anomalies no longer cancel between leptons and quarks. Of course, anomalies have to cancel at the fundamental level. For unthermalized particles which are not described by the fluid action, their contribution to the anomalies should be calculated from loop diagrams. On the other hand, our fluid action can be understood as an effective action which results from the standard model action with some degrees of freedom integrated

out; we don't need to do loop calculations with it. So there has to be an explicit term in the fluid action that cancels the contribution from unthermalized particles.

2. Depending on the temperature, some quark flavors may be absent. Take the early universe, for example, when the temperature is higher than 178 *GeV* (the mass of the top quark), we have all 6 flavors of quarks. As the universe expands and cools down to less than 178 *GeV* but higher than 4.5 *GeV* (the mass of the bottom quark), we essentially have 5 flavors of quarks in the fluid of QGP. And so on. So there are two possibilities: to have complete generations of quarks or integer plus half generations of quarks in the QGP.

The gauge anomalies are proportional to the trace factor d^{abc}. $d^{abc} \equiv \frac{1}{4} Tr(T^a \{T^b, T^c\})$, T^a are the generators of the gauge group, they can be the weak isospin generators I^i $(i = 1,2,3)$ or the weak hypercharge $U(1)$ generator Y. In fact, as far as the anomalies are concerned, we only need the I^3 component of the weak isospin. The reason is that, the W boson, which has a heavy mass of about 80 *GeV*, consists of the b^1 and b^2 gauge fields. When the temperature is higher than the mass of the top quark, we have 3 complete generations of quarks, and the weak isospin anomalies cancel, as we will show below. When we have 5 or fewer flavors of quarks involved, the temperature range of interest is much lower than the mass of the *W* boson. So we can always safely neglect the I^1 and I^2 components.

The gauge anomalies arise from the *bbb*, *bbc*, and *ccc* triangle diagrams [19]. We will examine d^{abc} for each of these.

1. *bbb* graph

$$d^{abc} \propto Tr(I^3\{I^3,I^3\}) \propto Tr(I^3)$$

This is zero for complete generations of quarks and non-zero for integer plus half generations of quarks.

2. *bbc* graph

$$d^{abc} \propto Tr(Y\{I^3,I^3\}) \propto Tr(Y)$$

Only left-handed fermions contribute to this trace. This is non-zero for any number of quarks.

3. *ccc* graph

$$d^{abc} \propto Tr(Y^3)$$

Both left-handed and right-handed fermions can contribute to this trace. This is also non-zero for any number of quarks.

Since $Tr(Y)$ can be non-zero, we can also have mixed gravitational anomaly which is proportional to $Tr(Y)Tr(R \wedge R)$, where R is the Riemann curvature tensor. This can be important when there is a strong background gravitational field; the early universe is an example. And of course, we have the axial $U(1)$ anomaly of QCD as usual.

In summary, the anomalies that may be present are

1. Complete generations of quarks: axial $U(1)$ anomaly of QCD, hypercharge $U(1)$ anomalies.

2. Integer plus half generations of quarks: mixed gravitational anomaly, axial $U(1)$ anomaly of QCD, hypercharge $U(1)$ anomalies, and I^3 isospin anomaly.

The presence of anomalies will change the fluid equations as the currents are no longer conserved.

In the next section we present a extension that incorporates the anomalies.

3.2 Incorporation of Anomalies

The Lagrangians (2.64) and (2.98) contain the color degrees of freedom of the QGP, but to capture the flavor dynamics we need to incorporate the gauge anomalies. The axial $U(1)$ anomaly of QCD and the mixed gravitational anomaly should be accounted for as well. We present this extension for the three quarks case (up, down, and strange). We do this for two reasons: First, the temperature of the RHIC experiments is in the range of the three quarks scenario, so our results may be directly relevant. Second, with one plus half generations of quarks, the model has all possible anomalies.

1. *axial* $U(1)$ *anomaly of QCD*

As we know, the axial current has a non-zero divergence

$$\partial_\mu j^\mu = \frac{3}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr(F_{\mu\nu}F_{\alpha\beta}). \tag{3.1}$$

To produce this anomaly we just need to add to the Lagrangian two terms

$$\int d^4x j^\mu \partial_\mu \lambda + \int \frac{3}{4\pi^2}\lambda Tr(F \wedge F), \tag{3.2}$$

where $F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$.

2. *mixed gravitational anomaly*

For three quarks $Tr(Y)$ is non-zero, so the mixed gravitational anomaly will be present [20]. The Wess-Zumino term that gives this anomaly can be derived from the index density

$$\begin{aligned}\mathcal{I}_6 &= \frac{1}{384\pi^3}Tr(F_Y)Tr(R\wedge R)\\ &= \frac{1}{128\pi^3}d(A_Y Tr(R\wedge R)).\end{aligned} \tag{3.3}$$

We have

$$\begin{aligned}\mathcal{S}_{WZ} &= \int_{M_5}\frac{1}{128\pi^3}(A_Y^g - A_Y)Tr(R\wedge R)\\ &= \int_{M_5}\frac{1}{128\pi^3}d\theta Tr(R\wedge R)\\ &= \int_{M_5}\frac{1}{128\pi^3}d(\theta Tr(R\wedge R))\\ &= \int_{\mathbb{R}_4}\frac{1}{128\pi^3}\theta Tr(R\wedge R).\end{aligned} \tag{3.4}$$

The corresponding baryon number current is introduced to the action by the term

$$\int d^4x B^\mu \partial_\mu \theta.$$

The axial current couples to the gravitational field in the same way, so we need to include a term

$$\int \frac{1}{128\pi^3}\lambda Tr(R\wedge R).$$

3. *gauge anomalies*

The flavor anomalies of QCD can be accounted for by an effective action given by Witten, and Kaymakcalan et al [21]. The context of their study is QCD at low energies. In that case, the dynamics can be presented by a non-linear σ-model involving a matrix valued field $\widetilde{U}$ which is the Goldstone field for chiral symmetry breaking. Witten and Kaymakcalan et al introduce a Wess-Zumino term in the action which describes all processes which are mediated by flavor anomalies. This term, $\Gamma_{WZ}\left(\widetilde{U}, A_L, A_R\right)$, has a lengthy expression and we will record it in Appendix B. Although this result is for low energy theory, the topological nature of the anomalies shows that the anomaly structure should be the same at high energies [22]. The Goldstone field $\widetilde{U}$ transforms similarly to the parameters of gauge transformations. Based on these observations, we introduce a group valued dynamical variable U, which is an element of $U(N_f)$. N_f is the number of quark flavors, in our case $N_f = 3$. We expect $\Gamma_{WZ}(U, A_L, A_R)$ to give all the flavor anomalies. Correspondingly we introduce two non-Abelian chiral currents to the action

$$\int d^4x Tr(J_L^\mu D_\mu U U^{-1}) + \int d^4x Tr(J_R^\mu U^{-1} D_\mu U),$$

where

$$D_\mu U = \partial_\mu U - iA_{L\mu} U + iUA_{R\mu}. \tag{3.5}$$

Under chiral transformations

$$U \to U' = U_L U U_R^{-1} \tag{3.6}$$

$$A_{L\mu} \to A'_{L\mu} = U_L A_{L\mu} U_L^{-1} - i\partial_\mu U_L U_L^{-1} \tag{3.7}$$

$$A_{R\mu} \to A'_{R\mu} = U_R A_{R\mu} U_R^{-1} - i\partial_\mu U_R U_R^{-1}. \tag{3.8}$$

We can now display the full fluid action

$$\begin{aligned}
\mathcal{S} = & \int d^4x \sum_{s=1}^{2} K^\mu_{(s)} Tr(I_{(s)} D_\mu g g^{-1}) + \int d^4x B^\mu \partial_\mu \theta + \int d^4x j^\mu \partial_\mu \lambda \\
& + \int \frac{1}{128\pi^3} \theta Tr(R \wedge R) + \int \frac{1}{128\pi^3} \lambda Tr(R \wedge R) \\
& + \int \frac{3}{4\pi^2} \lambda Tr(F \wedge F) \\
& + \int d^4x Tr(J^\mu_L D_\mu U U^{-1}) + \int d^4x Tr(J^\mu_R U^{-1} D_\mu U) \\
& + \Gamma_{WZ}(U, A_L, A_R) - \int d^4x F(l, r, k_{(s)}, n, m) + \mathcal{S}_{gauge}.
\end{aligned} \tag{3.9}$$

The first term is essentially the first term in (2.98), with different notations and the group specified to $SU(3)$. $F(l,r,k_{(s)},n,m)$ is the analog of $f(n_{(s)})$, but has more components, since there are more currents now.

$$\begin{aligned}
l &= \sqrt{B^\mu B_\mu}\,, B^\mu = l u^\mu_B, \\
r &= \sqrt{j^\mu j_\mu}\,, j^\mu = r u^\mu_j, \\
k_{(s)} &= \sqrt{K^\mu_{(s)} K_{(s)\mu}}\,, K^\mu_{(s)} = k_{(s)} u^\mu_{(s)}, \\
n &= \sqrt{Tr(J^\mu_L J_{L\mu})}\,, \\
m &= \sqrt{Tr(J^\mu_R J_{R\mu})}\,.
\end{aligned} \tag{3.10}$$

For three quarks, keeping only the I^3 isospin component, we have

$$A_L = b\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} + c\begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} = \frac{c-b}{2\sqrt{6}}\lambda_0 + \frac{b}{2}\lambda_3 + \frac{b}{2\sqrt{3}}\lambda_8, \tag{3.11}$$

$$A_R = c\begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \frac{c}{2}\lambda_3 + \frac{c}{2\sqrt{3}}\lambda_8, \tag{3.12}$$

where b and c are the I^3 component of the isospin gauge field and hypercharge gauge field respectively. $\lambda_a (a = 0, 1, \ldots 8)$ are the standard $U(3)$ generators, normalized as

$$Tr(\lambda_a \lambda_b) = 2\delta_{ab}. \tag{3.13}$$

The color current has the same fluid equations as before. So here we derive the fluid equations for the new currents. Variation with respect to θ gives

$$\nabla_\mu B^\mu = \frac{1}{128\pi^3}\frac{1}{4}(R_{\mu\nu})^\alpha{}_\beta (R_{\delta\tau})^\beta{}_\alpha \tilde{\epsilon}^{\mu\nu\delta\tau}, \tag{3.14}$$

where ∇_μ is the covariant derivative, in flat space $\nabla_\mu = \partial_\mu$. $\tilde{\epsilon}^{\mu\nu\delta\tau}$ is the

Levi-Civita tensor. Variation with respect to B^μ gives

$$\partial_\mu \theta - \frac{1}{l}\frac{\partial F}{\partial l} B_\mu = 0 \tag{3.15}$$

Taking the curl of (3.15) and contracting with u_B^μ, we get the Euler equation for B^μ

$$u_B^\mu \partial_\mu \left(u_{B\nu} \frac{\partial F}{\partial l} \right) = \partial_\nu \left(\frac{\partial F}{\partial l} \right). \tag{3.16}$$

Variation with respect to λ gives

$$\nabla_\mu j^\mu = \frac{1}{128\pi^3}\frac{1}{4}(R_{\mu\nu})^\alpha{}_\beta (R_{\delta\tau})^\beta{}_\alpha \tilde{\epsilon}^{\mu\nu\delta\tau} + \frac{3}{16\pi^2}\tilde{\epsilon}^{\mu\nu\alpha\beta} Tr(F_{\mu\nu}F_{\alpha\beta}), \tag{3.17}$$

where $F_{\mu\nu}$ is the field tensor of the color gauge field. Variation with respect to j^μ gives

$$\partial_\mu \lambda - \frac{1}{r}\frac{\partial F}{\partial r} j_\mu = 0 \tag{3.18}$$

Taking the curl of (3.18) and contracting with u_j^μ, we get the Euler equation for j^μ

$$u_j^\mu \partial_\mu \left(u_{j\nu} \frac{\partial F}{\partial r} \right) = \partial_\nu \left(\frac{\partial F}{\partial r} \right). \tag{3.19}$$

Variation with respect to U gives

$$iTr\left(\lambda_a D_\mu \mathfrak{J}_R^\mu\right) + iTr\left[\lambda_a U^{-1}\left(D_\mu \mathfrak{J}_L^\mu\right) U\right] = [Anomaly], \tag{3.20}$$

where

$$\mathfrak{J}_R^\mu = U^{-1} J_L^\mu U, \mathfrak{J}_L^\mu = U J_R^\mu U^{-1}, \tag{3.21}$$

$$D_\mu \mathfrak{J}_R^\mu = \partial_\mu \mathfrak{J}_R^\mu - i\left[A_{R\mu}, \mathfrak{J}_R^\mu \right], \tag{3.22}$$

$$D_\mu \mathfrak{J}_L^\mu = \partial_\mu \mathfrak{J}_L^\mu - i\left[A_{L\mu}, \mathfrak{J}_L^\mu \right]. \tag{3.23}$$

Under chiral transformations, $\mathfrak{J}_R^\mu$ and $\mathfrak{J}_L^\mu$ transform in a way as their subindices indicate

$$\mathfrak{J}_R^\mu \to \mathfrak{J}_R^{\mu\prime} = U_R \mathfrak{J}_R^\mu U_R^{-1}, \tag{3.24}$$

$$\mathfrak{J}_L^\mu \to \mathfrak{J}_L^{\mu\prime} = U_L \mathfrak{J}_L^\mu U_L^{-1}. \tag{3.25}$$

The term [*Anomaly*] has a lengthy expression, again we will record it in Appendix B. Variation with respect to $J_L^{\mu a}$ gives

$$J_{L\mu} = \frac{n}{\frac{\partial F}{\partial n}} (D_\mu U U^{-1}). \tag{3.26}$$

Variation with respect to $J_R^{\mu a}$ gives

$$J_{R\mu} = \frac{m}{\frac{\partial F}{\partial m}} (U^{-1} D_\mu U). \tag{3.27}$$

Let

$$u_L^\mu = \frac{J_L^\mu}{n}, u_R^\mu = \frac{J_R^\mu}{n}, \tag{3.28}$$

then

$$Tr(u_L^\mu u_{L\mu}) = Tr(u_R^\mu u_{R\mu}) = 1. \tag{3.29}$$

Substituting (3.26, 3.27) into (3.29), we have

$$\left(\frac{\partial F}{\partial n}\right)^2 = \left(\frac{\partial F}{\partial m}\right)^2. \tag{3.30}$$

Taking the curl of (3.26) yields

$$\begin{aligned} &\partial_\nu\left(\frac{\partial F}{\partial n}\right)u_{L\mu} - \partial_\mu\left(\frac{\partial F}{\partial n}\right)u_{L\nu} + \left(\frac{\partial F}{\partial n}\right)(D_\nu u_{L\mu} - D_\mu u_{L\nu}) \\ &= [D_\nu, D_\mu]UU^{-1} - [D_\mu UU^{-1}, D_\nu UU^{-1}]. \end{aligned} \tag{3.31}$$

Contracting with u_L^μ and taking the trace, we have

$$\begin{aligned} &\frac{1}{2}\partial_\nu\left[\left(\frac{\partial F}{\partial n}\right)^2\right] - \frac{1}{2}\partial_\mu\left[\left(\frac{\partial F}{\partial n}\right)^2\right]Tr(u_L^\mu u_{L\nu}) - \left(\frac{\partial F}{\partial n}\right)^2 Tr(u_L^\mu D_\mu u_{L\nu}) \\ &= i\left(\frac{\partial F}{\partial n}\right)Tr(F_{L\mu\nu}u_L^\mu) - i\left(\frac{\partial F}{\partial m}\right)Tr(F_{R\mu\nu}u_R^\mu), \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} F_{L\mu\nu} &= \partial_\mu A_{L\nu} - \partial_\nu A_{L\mu} - i[A_{L\mu}, A_{L\nu}], \\ F_{R\mu\nu} &= \partial_\mu A_{R\nu} - \partial_\nu A_{R\mu} - i[A_{R\mu}, A_{R\nu}]. \end{aligned} \tag{3.33}$$

This is the non-Abelian version of the Euler equation. Applying the same procedure to equation (3.27) we get

$$\frac{1}{2}\partial_\nu\left[\left(\frac{\partial F}{\partial m}\right)^2\right]-\frac{1}{2}\partial_\mu\left[\left(\frac{\partial F}{\partial m}\right)^2\right]Tr(u_R^\mu u_{R\nu})-\left(\frac{\partial F}{\partial m}\right)^2 Tr(u_R^\mu D_\mu u_{R\nu})$$
$$= i\left(\frac{\partial F}{\partial n}\right)Tr(F_{L\mu\nu}u_L^\mu)-i\left(\frac{\partial F}{\partial m}\right)Tr(F_{R\mu\nu}u_R^\mu). \qquad (3.34)$$

This equation is not independent of (3.32). Indeed, using (3.26, 3.27, 3.28, 3.30) we can transform (3.34) into (3.32).

3.3 Conclusions

In this chapter we have extended the Lagrangian (2.64) to incorporate the various anomalies that may arise in a quark gluon plasma. We find that which anomalies will be present depends on the number of quark flavors that are abundant in the QGP. More precisely, it depends on whether complete generations or integer plus half generations of quarks are involved. In the former case axial $U(1)$ anomaly of QCD and hypercharge $U(1)$ anomalies arise; in the latter case mixed gravitational anomalies and I^3 isospin anomaly arise in additional to the said anomalies.

We have constructed an action for a QGP with up, down, and strange quarks abundant. Thus there are one and half generations of quarks. We incorporate the anomalies by introducing Wess-Zumino terms and corresponding currents to the action. The equations of motion manifest the anomalies. For the flavor dynamics we have introduced two currents, one left handed and the other right handed. The divergence of the chiral currents has a very long expression. This should not come as a surprise as we expect a vast array of flavor processes to happen in the QGP.

Appendix A

Charge Density Algebra

The portion of the Lagrangian density (2.64) that determines the Poisson bracket is

$$\mathfrak{L}_{canonical} = \rho 2tr(Kg^{-1}\partial_t g) = \rho 2tr(Q\partial_t gg^{-1}), \text{ with } Q = gKg^{-1}. \tag{A1}$$

Parametrizing the group element as $g(\varphi) = e^{T^a\varphi_a}$, we see that $\partial_t gg^{-1}$ has the form $-\partial_t\varphi_a C^a_{\ b}(\varphi)T^b$, where the non-singular matrix $C^a_{\ b}(\varphi)$ is defined by

$$C^a_{\ b}(\varphi)T^b = -\frac{\partial g(\varphi)}{\partial\varphi_a}g^{-1}(\varphi). \tag{A2}$$

Thus

$$\mathfrak{L}_{canonical} = \rho\partial_t\varphi_a C^a_{\ b}Q_b = \partial_t\varphi_a C^a_{\ b}\rho_b \tag{A3}$$

and the momentum conjugate to φ_a is

$$\Pi^a = C^a_{\ b}\rho_b. \tag{A4}$$

Then (A3) reads

$$\mathfrak{L}_{canonical} = \Pi^a\partial_t\varphi_a. \tag{A5}$$

With inverse to $C^a_{\ b}$ denoted as $c^a_{\ b}$, we can write

$$\rho_a = c^a_{\ b}\Pi^b. \tag{A6}$$

The non-Abelian charge density ρ_a is a function of $(t,\mathbf{r})$ and for (2.62) we need the bracket with another density evaluated at $(t,\mathbf{r}')$. Since the dependence of $c^a_{\ b}$ on φ involves no spatial derivatives of φ, it is clear that the brackets will be local in $(\mathbf{r}-\mathbf{r}')$; just as is the bracket between φ and Π.

$$\begin{aligned}
\{\rho_a(t,\mathbf{r}),\rho_b(t,\mathbf{r}')\} &= \left(c^a_{\ a'}\frac{\partial c^b_{\ b'}}{\partial\varphi_{a'}}\Pi^{b'} - a \leftrightarrow b\right)\delta(\mathbf{r}-\mathbf{r}') \\
&= \left(-c^a_{\ a'}c^b_{\ c'}\frac{\partial C^{c'}_{\ c''}}{\partial\varphi_{a'}}c^{c''}_{\ b'}\Pi^{b'} - a \leftrightarrow b\right)\delta(\mathbf{r}-\mathbf{r}') \\
&= \left(-c^a_{\ a'}c^b_{\ c'}\frac{\partial C^{c'}_{\ c''}}{\partial\varphi_{a'}}\rho_{c''} - a \leftrightarrow b\right)\delta(\mathbf{r}-\mathbf{r}')
\end{aligned} \tag{A7}$$

To evaluate the derivative with respect to φ, observe that from (A2) we have

$$\begin{aligned}
\frac{\partial C^{c'}_{\ c''}}{\partial\varphi_{a'}} &= \frac{\partial}{\partial\varphi_{a'}}\left(2tr\left(\frac{\partial g}{\partial\varphi_{c'}}g^{-1}T^{c''}\right)\right) \\
&= 2tr\left(\frac{\partial^2 g}{\partial\varphi_{a'}\partial\varphi_{c'}}g^{-1} - C^{c'}_{\ d'}T^{d'}C^{a'}_{\ d''}T^{d''}\right)T^{c''}.
\end{aligned} \tag{A8}$$

The first term in the parentheses is symmetric in (a',c'); when inserted in (A7) it produces a symmetric contribution in (a,b) and does not contribute when antisymmetrization in (a,b) is effected. What is left establishes (2.62).

$$
\begin{aligned}
\{\rho_a(t,\mathbf{r}),\rho_b(t,\mathbf{r}')\} &= \left(c^a{}_{a'}c^b{}_{c'}C^{c'}{}_{d'}C^{a'}{}_{d''}2tr\left(T^{d'}T^{d''}T^{c''}\right)\rho_{c''} - a \leftrightarrow b\right)\delta(\mathbf{r}-\mathbf{r}') \\
&= -\left(2tr\left(T^aT^bT^{c''}\right)\rho_{c''} - a \leftrightarrow b\right)\delta(\mathbf{r}-\mathbf{r}') \\
&= -2f^{abd}tr\left(T^dT^{c''}\right)\rho_{c''}\delta(\mathbf{r}-\mathbf{r}') \\
&= f^{abc}\rho_c(t,\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'). \qquad \text{(A9)}
\end{aligned}
$$

Appendix B

Wess Zumino Term and Anomaly for Flavor Dynamics

We display explicitly the Wess-Zumino term $\Gamma_{WZ}(U, A_L, A_R)$ and the term $[Anomaly]$ in (3.20).

$$
\begin{aligned}
&\Gamma_{WZ}(U, A_L, A_R) \\
&= C\int_{M_5} Tr(\alpha^5) + 5iC\int_{\mathbb{R}_4} Tr(A_L\alpha^3 + A_R\beta^3) \\
&\quad - 5C\int_{\mathbb{R}_4} Tr[(dA_LA_L + A_LdA_L)\alpha + (dA_RA_R + A_RdA_R)\beta] \\
&\quad + 5C\int_{\mathbb{R}_4} Tr[dA_LdUA_RU^{-1} - dA_Rd(U^{-1})A_LU] \\
&\quad + 5C\int_{\mathbb{R}_4} Tr(A_RU^{-1}A_LU\beta^2 - A_LUA_RU^{-1}\alpha^2) \\
&\quad + \frac{5C}{2}\int_{\mathbb{R}_4} Tr\left[(A_L\alpha)^2 - (A_R\beta)^2\right] + 5iC\int_{\mathbb{R}_4} Tr(A_L^3\alpha + A_R^3\beta) \\
&\quad + 5iC\int_{\mathbb{R}_4} Tr[(dA_RA_R + A_RdA_R)U^{-1}A_LU - (dA_LA_L + A_LdA_L)UA_RU^{-1}] \\
&\quad + 5iC\int_{\mathbb{R}_4} Tr(A_LUA_RU^{-1}A_L\alpha + A_RU^{-1}A_LUA_R\beta) \\
&\quad + 5C\int_{\mathbb{R}_4} Tr\left[A_R^3U^{-1}A_LU - A_L^3UA_RU^{-1} + \frac{1}{2}(UA_RU^{-1}A_L)^2\right],
\end{aligned} \tag{B1}
$$

$$
\begin{aligned}
&[Anomaly] \\
= \;& (5iC)\epsilon^{\mu\nu\sigma\tau} Tr(\lambda_a \beta_\mu \beta_\nu \beta_\sigma \beta_\tau) \\
&+ (5iC)\left(\frac{1}{12}\right)(-2if^{abc})\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{c_\nu b_\tau D^{b3} Tr(\lambda_c \beta_\sigma)\} \\
&+ (5iC)\left(\frac{1}{12\sqrt{3}}\right)(-2if^{abc})\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{c_\nu b_\tau D^{b8} Tr(\lambda_c \beta_\sigma)\} \\
&+ (-10iC)\left(\frac{\sqrt{6}}{4}\right)\delta^{a0}\epsilon^{\mu\nu\sigma\tau}\partial_\mu(b_\nu \partial_\sigma b_\tau) \\
&+ (-10iC)\left(\frac{1}{6\sqrt{6}}\right)\delta^{a0}\epsilon^{\mu\nu\sigma\tau}\partial_\mu(c_\nu \partial_\sigma c_\tau) \\
&+ (10iC)\left(\frac{1}{6\sqrt{6}}\right)\delta^{a0}\epsilon^{\mu\nu\sigma\tau}\partial_\mu\left(c_\nu \overleftrightarrow{\partial_\sigma} b_\tau\right) \\
&+ (-10iC)\left(\frac{1}{6}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\left(D^{a3} c_\nu \overleftrightarrow{\partial_\sigma} b_\tau\right) \\
&+ (-10iC)\left(\frac{1}{6\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\left(D^{a8} c_\nu \overleftrightarrow{\partial_\sigma} b_\tau\right) \\
&+ (-10iC)\left(\frac{4}{9}\right)\epsilon^{\mu\nu\sigma\tau} c_\mu \partial_\nu c_\sigma Tr(\lambda_a \beta_\tau) \\
&+ (-10iC)\left(\frac{1}{6}\right)\epsilon^{\mu\nu\sigma\tau} c_\mu \partial_\nu c_\sigma Tr(\lambda_a [\lambda_3, \beta_\tau]) \\
&+ (-10iC)\left(\frac{1}{6\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau} c_\mu \partial_\nu c_\sigma Tr(\lambda_a [\lambda_8, \beta_\tau]) \\
&+ (-10iC)\left(\frac{2}{3}\sqrt{\frac{2}{3}}\right)\delta^{a0}\epsilon^{\mu\nu\sigma\tau}\partial_\mu c_\nu \partial_\sigma c_\tau \\
&+ (-10iC)\left(\frac{1}{3}\right)\delta^{a3}\epsilon^{\mu\nu\sigma\tau}\partial_\mu c_\nu \partial_\sigma c_\tau \\
&+ (-10iC)\left(\frac{1}{3\sqrt{3}}\right)\delta^{a8}\epsilon^{\mu\nu\sigma\tau}\partial_\mu c_\nu \partial_\sigma c_\tau \\
&+ (-5C)\left(\frac{1}{2}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\tau Tr(\lambda_a \beta_\nu [\beta_\sigma, U^{-1}\lambda_3 U])\} \\
&+ (-5C)\left(\frac{1}{2\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\tau Tr(\lambda_a \beta_\nu [\beta_\sigma, U^{-1}\lambda_8 U])\} \\
&+ (-5C)\left(\frac{1}{6}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{(c_\nu - b_\nu) Tr(\lambda_a \beta_\sigma \beta_\tau)\}
\end{aligned}
$$

$$+ (5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu b_\nu Tr(\lambda_a[\lambda_8, U^{-1}\lambda_3 U\beta_\sigma\beta_\tau])$$
$$+ (5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu b_\nu Tr(\lambda_a[\lambda_8, U^{-1}\lambda_8 U\beta_\sigma\beta_\tau])$$
$$+ (5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{c_\sigma b_\tau Tr(\lambda_a[\beta_\nu, \lambda_3])\}$$
$$+ (5iC)\left(\frac{1}{12\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{c_\sigma b_\tau Tr(\lambda_a[\beta_\nu, \lambda_8])\}$$
$$+ (-5iC)\left(\frac{1}{4}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{c_\sigma b_\tau Tr(\lambda_a[\beta_\nu, \lambda_3 U^{-1}\lambda_3 U])\}$$
$$+ (-5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{c_\sigma b_\tau Tr(\lambda_a[\beta_\nu, \lambda_3 U^{-1}\lambda_8 U])\}$$
$$+ (-5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{c_\sigma b_\tau Tr(\lambda_a[\beta_\nu, \lambda_8 U^{-1}\lambda_3 U])\}$$
$$+ (-5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{c_\sigma b_\tau Tr(\lambda_a[\beta_\nu, \lambda_8 U^{-1}\lambda_8 U])\}$$
$$+ (-5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu b_\tau Tr(\lambda_a[\lambda_3, \beta_\nu\beta_\sigma])$$
$$+ (-5iC)\left(\frac{1}{12\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu b_\tau Tr(\lambda_a[\lambda_8, \beta_\nu\beta_\sigma])$$
$$+ (5iC)\left(\frac{1}{4}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu b_\tau Tr(\lambda_a[\lambda_3, \beta_\nu\beta_\sigma U^{-1}\lambda_3 U])$$
$$+ (5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu b_\tau Tr(\lambda_a[\lambda_3, \beta_\nu\beta_\sigma U^{-1}\lambda_8 U])$$
$$+ (5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu b_\tau Tr(\lambda_a[\lambda_8, \beta_\nu\beta_\sigma U^{-1}\lambda_3 U])$$
$$+ (5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu b_\tau Tr(\lambda_a[\lambda_8, \beta_\nu\beta_\sigma U^{-1}\lambda_8 U])$$

$$+ (-5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\sigma c_\tau Tr(\lambda_a\beta_\nu\lambda_3)\}$$
$$+ (-5iC)\left(\frac{1}{12\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\sigma c_\tau Tr(\lambda_a\beta_\nu\lambda_8)\}$$
$$+ (5iC)\left(\frac{1}{4}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\sigma c_\tau Tr(\lambda_a\beta_\nu U^{-1}\lambda_3 U\lambda_3)\}$$
$$+ (5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\sigma c_\tau Tr(\lambda_a\beta_\nu U^{-1}\lambda_3 U\lambda_8)\}$$
$$+ (5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\sigma c_\tau Tr(\lambda_a\beta_\nu U^{-1}\lambda_8 U\lambda_3)\}$$
$$+ (5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\sigma c_\tau Tr(\lambda_a\beta_\nu U^{-1}\lambda_8 U\lambda_8)\}$$

$$+ (5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\nu c_\sigma Tr(\lambda_a\lambda_3\beta_\tau)\}$$
$$+ (5iC)\left(\frac{1}{12\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\nu c_\sigma Tr(\lambda_a\lambda_8\beta_\tau)\}$$
$$+ (-5iC)\left(\frac{1}{4}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\nu c_\sigma Tr(\lambda_a U^{-1}\lambda_3 U\lambda_3\beta_\tau)\}$$
$$+ (-5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\nu c_\sigma Tr(\lambda_a U^{-1}\lambda_3 U\lambda_8\beta_\tau)\}$$
$$+ (-5iC)\left(\frac{1}{4\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\nu c_\sigma Tr(\lambda_a U^{-1}\lambda_8 U\lambda_3\beta_\tau)\}$$
$$+ (-5iC)\left(\frac{1}{12}\right)\epsilon^{\mu\nu\sigma\tau}\partial_\mu\{b_\nu c_\sigma Tr(\lambda_a U^{-1}\lambda_8 U\lambda_8\beta_\tau)\}$$

$$+ (-10C)\left(\frac{1}{8}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu\partial_\nu c_\sigma b_\tau Tr(\lambda_a[\lambda_3, U^{-1}\lambda_3 U])$$
$$+ (-10C)\left(\frac{1}{8\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu\partial_\nu c_\sigma b_\tau Tr(\lambda_a[\lambda_3, U^{-1}\lambda_8 U])$$
$$+ (-10C)\left(\frac{1}{8\sqrt{3}}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu\partial_\nu c_\sigma b_\tau Tr(\lambda_a[\lambda_8, U^{-1}\lambda_3 U])$$
$$+ (-10C)\left(\frac{1}{24}\right)\epsilon^{\mu\nu\sigma\tau}c_\mu\partial_\nu c_\sigma b_\tau Tr(\lambda_a[\lambda_8, U^{-1}\lambda_8 U]), \qquad \text{(B2)}$$

where

$$C = \frac{-i}{80\pi^2}, \tag{B3}$$

$$\alpha = dUU^{-1},\ \beta = U^{-1}dU,\ \alpha_\mu = \partial_\mu UU^{-1},\ \beta_\mu = U^{-1}\partial_\mu U, \tag{B4}$$

$$U^{-1}\lambda_a U = D^{ba}\lambda_b, \tag{B5}$$

$$[\lambda_a, \lambda_b] = 2if^{abc}\lambda_c. \tag{B6}$$

Bibliography

[1] D. J. Gross and F. Wilczek, *Phys. Rev. Lett.* **30**, 1343 (1973); H. D. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973).

[2] J. C. Collins, M. J. Perry, *Phys. Rev. Lett.* **34,** 1353 (1975); G. Baym, S. A. Chin, *Phys. Lett. B* **62,** 241 (1976); B. A. Freedman, L. D. McLerran, *Phys. Rev. D* **16**, 1169 (1977); G. Chapline, M. Nauenberg, *Phys. Rev. D* **16,** 450 (1977).

[3] L. Susskind, *Phys. Rev. D* **20**, 2610 (1979); H. Satz, *Nucl. Phys.* **A418**, 447c (1984).

[4] See for example, T. Ludlam and L. McLerran, *Physics Today*, **56**, 48 (2003); for more detailed explorations, see for example *Quark-Gluon Plasma. New Discoveries at RHIC: Case for the Strongly Interacting Quark-Gluon Plasma. Contributions from the RBRC Workshop held May 14-15, 2004*, *Nucl. Phys.* **A750**, 1 (2005).

[5] R. Pisarski, *Physica* **A158**, 246 (1989); *Phys. Rev. Lett.* **63**, 1129 (1989); E. Braaten and R. Pisarski, *Phys. Rev. D* **42**, 2156 (1990); for a recent update and later references, see J. Andersen, E. Braaten, E. Petitgirard and M. Strickland, hep-ph/0205085.

[6] U. Heinz, *Ann. Phys.* **161**, 48 (1985); H.-T. Elze and U. Heinz, *Phys. Rep.* **183**, 81 (1989); for a recent update and later references, see D. Litim and C. Manuel, *Phys. Rept.* **364**, 451 (2002).

[7] B. Bistrovic, R. Jackiw, H. Li, V.P. Nair and S.-Y. Pi, *Phys. Rev. D* **67**, 025013 (2003).

[8] R. Jackiw, V. P. Nair, S.-Y. Pi and A. P. Polychronakos, *J. Phys.* **A37,** R327 (2004).

[9] P. Morrison and J. Greene, *Phys. Rev. Lett.* **45**, 790 (1980), (E) **48** 569 (1982).

[10] L. Faddeev and R. Jackiw, *Phys. Rev. Lett.* **60**, 1692 (1988).

[11] A. Clebsch, *J. Reine Angew. Math.* **56**, 1 (1859).

[12] C. C. Lin, *International School of Physics E. Fermi (XXI)*, G. Careri, ed. (Academic Press, New York 1963).

[13] C. Eckart, *Phys. Rev.* **54**, 920 (1938).

[14] See, for example, L. Landau and E. Lifshitz, *Fluid Mechanics* (2nd ed. Pergamon, Oxford UK 1987), or S. Weinberg, *Gravitation and Cosmology* (Wiley, New York 1972).

[15] S. K. Wong, *Nuovo Cim.* **A65**, 689 (1970).

[16] A. Balachandran, S. Borchardt and A. Stern, *Phys. Rev. D* **17**, 3247 (1978); A. Balachandran, G. Marmo, B.-S. Skagerstam and A. Stern, *Gauge Symmetries and Fibre Bundles*, Lecture Notes in Physics 188 (Springer-Verlag, Berlin, 1982).

[17] D. Bak, R. Jackiw and S.-Y. Pi, *Phys. Rev. D* **49**, 6778, (1994) (Appendix).

[18] E. Madelung, *Z. Phys.* **40**, 332 (1926).

[19] See, for example, V. P. Nair, *Quantum Field Theory: A Modern Perspective* (Springer, New York 2005).

[20] L. Alvarez-Gaumé and E. Witten, *Nucl. Phys.* **B234**, 269 (1984).

[21] E. Witten, *Nucl. Phys.* **B223**, 422 (1983); Ö. Kaymakcalan, S. Rajeev and J. Schechter, *Phys. Rev. D* **30**, 594 (1984).

[22] See, for example, V. P. Nair, Ref. [15].