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Hydrogen atom in ultraintense laser fields: Photoionization and spectroscopy study

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HYDROGEN ATOM IN ULTRAINTENSE LASER FIELDS:PHOTOIONIZATION AND SPECTROSCOPY STUDY

by

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Abstract

HYDROGEN ATOM IN ULTRAINTENSE LASER FIELDS:PHOTOIONIZATION AND SPECTROSCOPY STUDY

by

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Adviser: Prof. Marvin H. Mittleman

The subject of investigation is the photoionization process when atomic hydrogen is subjected to ultraintense laser field. Laser field is at constant ultraintense level.

That brings outs effects that such fields have on atomic structure. In that respect our approach is different from the previous methods where laser field is adiabatically switched on, thus covering the range of intensity from zero to ultraintense.

Our focus is on the binding energy of a dressed metastable state of an H atom in linearly and circularly polarized laser field . We use the Rayleigh-Ritz variational method to determine the energy of a bound state and then calculate the width of that variational minimum to determine a photoionization rate.

We use a nonrelativistic approach and the dipole approximation whose validity we proved aposteriori. We found no bound state for circularly polarized laser light but find a very losely bound state for a linearly polarized laser light.

The binding energy of this state behaves as $I^{-1/2}(\ln I)^2$. The width of the state has the same behavior with intensity and the coefficient is roughly given.

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Introduction

The investigation of the interaction of intense electromagnetic radiation with atoms has formed one of the important branches of atomic physics in recent years. Although two photon processes like Rayleigh and Raman scattering had been understood in an early stage of Quantum Mechanics, multiphoton processes in which atomic transitions involve absorption, emission or scattering of more than one photon had been left aside.

The prime reason was the small probability for multiphoton interaction with an atom under experimental conditions with the small radiation intensity of ordinary light sources. For example, for an atom to be able to absorb two photons in a simple process of nonresonant photoionization it has to be exposed to a photon flux of about $\frac{1}{a_0^2\tau} = 10^{31}$ $cm^{-2}s^{-1}$, where a_0 is Bohr radius and τ is a lifetime of a typical intermediate state, $\tau \approx 10^{-15}s$.

Discovery and rapid development of lasers met the requirement of high occupation number for electromagnetic radiation in multiphoton processes.

Multiphoton processes represent a large set of processes where the inter

action of intense laser radiation with molecules,atoms , ions and electrons creates a variety of new phenomena such as mulyiphoton ionization (MPI),multiphoton excitation (MPE), nonlinear efects in atomic gasses as well as the observation of the straightforward influence of an intense radiation on the processes of atom - atom and electron - atom collisions[1,2]. Although a great deal of experimental and theoretical work had been done in understanding those phenomena much remains to be explored. Multiphoton phenomena could be formally treated in a rigorous quantum mechanical way or by applying semiclassical methods.

The first approach the whole system, atom and radiation field, is treated quantum mechanically. The semiclassical approach treats the atom quantum mechanically and the radiation field satisfies the classical Maxwell equations[3].

The quantum mechanical description of the transverse vector potential of the electromagnetic field is given as

$$\vec{A}(\vec{r}) = \sum_{\vec{k},\lambda} \left[\frac{2\pi\hbar c^2}{\omega_k V}\right]^{1/2} (a_{k\lambda}\hat{\varepsilon}_{k\lambda}e^{i\vec{k}\cdot\vec{r}} + a^{+}_{k\lambda}\hat{\varepsilon}_{k\lambda}e^{-i\vec{k}\vec{r}}) \tag{1.1.1}$$

where $a^{*}_{k\lambda}$, $a_{k\lambda}$ are creation, destruction operators in the $\vec{k}$, λ mode, $\hat{\varepsilon}_{n\lambda}$ the unit polarization vector of $\vec{k},\lambda$ mode and V is the quantization volume, and

$$\vec{k}\cdot\hat{\varepsilon}_{n\lambda} = 0.$$

A large number of photons in a laser mode allows a transition to a classical description of the laser field[4,5], so that

$$\vec{A}(\vec{r},t) = \sum_{K,\lambda} \frac{c}{2\omega_k} \{\vec{E}_{k\lambda} \exp[i(\vec{k}\cdot\vec{r}-\omega_k t - \phi_{k\lambda}]$$

$$+ \vec{E}^{*}_{k\lambda} \exp[-i(\vec{k}\cdot\vec{r}-\omega_k t - \phi_{k\lambda})]\} \tag{2}$$

where

$$\vec{E}_{k\lambda} = \left[\frac{8\pi h\omega_{k\lambda}N_{k\lambda}}{V}\right]^{1/2}\hat{\varepsilon}_{k\lambda} \tag{3}$$

the electric field of a laser and $N_{k\lambda} >> 1$ is the number of photons in the $\vec{k},\lambda$ mode, and $\phi_{k,\lambda}$ is Bialinicki-Birula phase operator.

The intensity of a laser field[1] can be measured in units of the Coulomb field which for the hydrogen atom in the ground state is

$$I_0 = \frac{cE_0^2}{8\pi} \approx 0.88 \times 10^{16}\ \mathrm{W/cm^2} \tag{4}$$

where

$$E_0 = \frac{e}{2a_0^2} = 2.57 \times 10^8\ \mathrm{W/cm} \tag{5}$$

is Coulomb electric field. We can estimate the laser atom interaction in the dipole approximation as

$$eE\,\langle\vec{r}\rangle \approx eEa_0 = Ry\,\left(\frac{I}{I_0}\right)^{1/2} \tag{6}$$

where I is the laser intensity and $Ry \approx 13.6$ eV.

Out of the large number of multiphoton phenomena we will focus on the process of multiphoton ionization.An atom with an ionization potential Ei can only be ionized by photons with energy less than Ei if the photon flux is strong enough[2] .To describe the process, virtual states are introduced to which the atom can be excited by an integer number of photons. For $I \ll I_0$ the laser-atom interaction energy is much smaller then the typical separation of atomic energy levels so one can use perturbation theory, the perturbation being the laser atom interaction, H'.

Calcululations of the nonresonant ionization rate where performed[7,8,9] with results for the N photon ionization rate which are written

$$W^{(N)}_{fi} = \sigma_N \, I^N \tag{7}$$

σ_N being total generalized cross section σ_N $[cm^{2N}s^{N-1}]$

$$\sigma_N = \frac{(2\pi\alpha_F)}{4\pi^2} \, m p \omega^{N-1} \int d\Omega_p \left| k_{fo}{}^{(n)} \right|^2 \tag{8}$$

$\alpha_F = (137)^{-1}$, p is the momentum of the outgoing photo-electron and $k_{fo}{}^{(N)}$ is the N-th order matrix element.

$$k_{fo}{}^{(N)} = \sum_{in-1} \sum_{in-2} \sum_{i2} \sum_{i1} \langle f | \hat{\varepsilon}\cdot\vec{r} | in-1 \rangle \langle in-1 | \hat{\varepsilon}\cdot\vec{r} | in-2 \rangle \ldots \tag{9}$$

$$\times \langle i \, | \hat{\varepsilon}\cdot\vec{r} | 0 \rangle \times [(E_0 - E_{in-1} + (N-1)\omega) \ldots (E_0 - E_{i1} + \omega)]$$

It has taken almost ten years of experimenting[10,11,12] before one was capable of measuring the order of nonlinearity in ionization rate described by eq.(7) in correct way. For low laser intensities perturbation theory is in the satisfactory agreement with experiments on multiphoton ionization[13] .

Also the angular distribution of emerging photoelectrons could be discussed using this approach as well as a comparison between different light polarizations.[8,9]

Another part of the problem is that the laser atom interaction induces a distortion of atomic states, so that the wave function and its eigenvalues depend on the intensity of the laser field.

Those distortions are calculated for low intensities yielding the dynamic Stark effect[13,14] , but for intermediate and high intensities it remains to be investigated.

Even for low intensities one can not assume that the wave function does not change so that calculations done by using perturbation theory have to be carried to a higher order. That makes calculations difficult to perform. For example one can calculate the A.C. Stark shifts which are, in the lowest order, proportional to the mean value of the square of the laser electric field, by constructing the dynamic polarizability of the states.[15] The energy shift being proportional to intensity

$$\Delta E \sim \alpha I \tag{10}$$

where α is a constant, that is frequency dependent.

The understanding of the A.C. Stark shift was important in the treatment of resonant multiphoton ionization.→[12,15]

Resonant multiphoton ionization can be coupled with a nonlinear[16,17] effects of harmonic generation as in the case of the 5-photon ionization of xenon.There a third harmonic generation is a competting process to the multiphoton ionization.

Free electrons that emerge in the process of MPI can gain kinetic energy from the electromagnetic field either in units of the photon energy or continously.

The absorption of one whole photon can only take place in the presence of a third body , needed in order to fulfil the conservation of energy and momentum law.In the process of Compton scattering the absorption of an arbitrary amount of energy can occur.

The first observation of additional photon absorption had been reported by Agostini et. al[19,20]. The target atom was xenon, ionized by 532 nm light, giving rise to the absorption of one additional photon. Subsequentely experiments done by Kruit and the group at Amsterdam FOM Institute[21,22,23] in addition to better resolution showed a new feature. It had been observed that the first peak in the energy spectrum of free electrons emerging in MPI process have been suppresed as laser intensity increases.The efect was explained by recognizing the role of the ponderomotive potential.

Ponderomotive potential, which we are to address in chapter II, is equal to the time-averaged classical wiggle kinetic energy of a classical electron under the influence of a time periodic electric field E.

In photoionization of xenon[24,25,26] by a 0.1 nsec 1064 nm Nd:YAG pulse the required intensity is approximately $I = 1\text{-}3\times10^{13}$ W/cm^2 and $U_p = \frac{e^2E^2}{4m\omega^2} = 1.1\times10^{13}\ I(\mathrm{W/cm^2})$ eV so that U_p energy actually exceeds the light energy quantum $h\nu = 1.165$ eV at intensities where xenon ionize. It was shown from the energy conservation point of view that shift of ionization potential by the U_p amount gives an explanation for the disappearance of the first peak in (ATI) experiments done by Kruit, et al.

Due to anticipated development of high and ultra high intensity laser fields in the near future we have $I \gg I_0$. Therefore there is an interest in experimental and theoretical studies of atoms in such strong fields. New technique of pulse compression combined with broad spectrum laser amplifiers (excimer or solid state) are pushing laser intensities up to fantastic values of $I > 10^{22}$ W/cm^2.[28]

This problem is more complicated than the problem of an atom in low intensity laser fields.

Experimental results of ionization rates at such high intensities are difficult to interpret due to the fact that imprecisely known temporal fluctuations in intensity of laser can affect the observed rates in a profound way.

This makes quantitative comparison between approximate theories and experiment very difficult.

In order to achieve some comparison between various approximate theories Berson[29] proposed and solved exactly a model of one electron atom in a circularly polarized monochromatic electromagnetic field in dipole approximation. The model is described by the Hamiltonian

$$H = \frac{\vec{P}+e\vec{A}(t)}{2m}^2 + V(r) \tag{11}$$

where the vector potential describing propagation in the $\hat{z}$ direction is

$$\vec{A}(t) = \frac{E}{\omega}\,(a_x \cos\omega t - a_y \sin\omega t) \tag{12}$$

and the attractive potential is

$$V(\vec{r}) = \frac{4\pi}{2m} \delta(\vec{r}) \frac{\partial}{\partial r} r \tag{13}$$

This makes the model unrealistic in that its (essentially) zero range potential fails to describe the Coulomb potential which is, we will see, relevant for ionization at ultra high intensities.

Berson used a Floquet[6] theory to obtain an implicit equation for the total ionization rate. Muller et al.[27] have also used the model to investigate the A.C. Stark shift and the spectrum of photoelectrons emerging in the process of photoionization.

The first applicable calculation of multiphoton ionization in an ultrastrong laser field had been done by Keldish[30] for the hydrogen atom in a linearly polarized laser. He started from an exact expression for the S matrix

$$S_{q_{10}} = -i \langle \psi_q{}^{(-)}, H' \phi_0 \rangle \tag{14}$$

where ϕ_0 is the initial state of the atom

$$\phi_0 = u_0(x) e^{-i\omega qt/h}$$

and where $\psi_q{}^{(-)}$ is exact time reversed wave function.

$$\lim_{t \to t\omega} \psi^{(-)}{}_q = U_q{}^{(-)} (\vec{x})\, e^{-i\omega qt/h} = \phi_q{}^{(-)} \tag{15}$$

the spatial part of the wave functions satisfy the unperturbed Schrodinger equation

$$(W_n - H_0)\, u_n = 0 \tag{16}$$

The perturbation is

$$H' = H(t) - H_0. \tag{17}$$

Keldysh expanded $\psi_q^{(-)}$ in the electron nucleus interaction

$$\psi_q^{(-)} = \chi_q^{(-)} + G^{(-)} V \chi_q^{(-)} \tag{18}$$

where

$$G^{(-)} = G_0^{(-)} + G_0^{(-)} V G_0^{(-)} + \ldots \tag{19}$$

and $G_0^{(-)}$ satisfies

$$\left(i\frac{\partial}{\partial t} - H_0\right) G_0^{(-)} = 1. \tag{20}$$

χ_q is wave function of a free electron in an electromagnetic field, the Volkov state[34].

After many approximations Keldysh obtained ionization rate as

$$W \approx \frac{Ry}{h}\, (3h)^{1/2}\, 2^{-5/4}\, (E/E_0)^{1/2} \tag{21}$$

so that ionization rate rises with laser intensity.

Reiss[31] performed a similar calculation obtaining results for the ionization rate and angular distribution of emerging electrons. The result is known as Reiss-Keldysh theory of (MPI).

The other major results obtained in calculation of multiphoton ionization are done by Pert[32] and Gersten & Mittleman.[33] Both of these calculations have one common feature and that is that each of them is S-matrix type of calculation and that both of them perform a perturbative expansion in electron nucleus interaction V of the exact S matrix

$$S_{q_{10}} = -i < \psi_q^{(-)}, H'U_0> \tag{22}$$

Expanding this using (18) and (19) one gets

$$S_{q_{10}} = -i\ (<\chi_{\vec{q}}^{(-)}, H'U_0> + <\chi_{\vec{q}}^{(-)}, VG_0H'U_0> + <\chi_q^{(-)}, VG_0VG_0H'U_0> + \ldots) \tag{23}$$

Of course this means that $H' >> V(r)$, H' being the laser atom interaction. Those two calculations come to the conclusion that the ionization rate decreases with intensity.

Pert, working in $\vec{p}\vec{A}$ gauge, arrived to the result for the ionization rate as

$$W \sim E^{-1}\ln(E/E_0) \tag{24}$$

with a T matrix for absorption of ℓ photons.

$$T_1 = \ell h\omega\ J\ell(\alpha_0\vec{q})\Omega_0(q) \tag{25}$$

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Pert, working in $\vec{p}\vec{A}$ gauge, arrived to the result for the ionization rate as

$$W \sim E^{-1} \ell n(E/E_0) \tag{24}$$

with a T matrix for absorption of ℓ photons.

$$T_l = \ell h\omega \, J\ell(\alpha_0 \vec{q}) \Omega_0(q) \tag{25}$$

where Jl is a Bessel function of ℓ-th order, and $\alpha_0 = eE(0)/m\omega^2$ is the parameter that describes the amplitude of the laser field.

Gersten, Mittleman calculations started also from supposedly exact S matrix

$$S_{q,0} = -i \langle \phi_q^{(-)}, H'\phi_0 \rangle - i \langle (\phi_q^{(-)}, H' G^{(+)} H' \phi_0 \rangle \tag{26}$$

where $G^{(+)}$ is the full Green's function for the problem and

$$\phi_q^{(-)} = \lim_{t \to \infty} \psi_q^{(-)}. \tag{27}$$

The essential approximation is the neglect of the electron-proton interaction compared to the electron laser interaction in $G^{(+)}$.
Another approximation is replacement of $\phi q^{(-)}$ by λ_q, a plane wave, which allowed analytic performance of the integrals.

The results obtained by Keldysh and Reiss on the one side and those obtained by Mittleman and Gersten and Pert are disturbingly different. The difference is due to the fact that calculations are approximate and done in different gauges.

In order to understand the problem let us state the experimental conditions. An atom is in the ground state in the absence of the laser field. In the rest frame of the atom the laser field amplitude is adiabatically increased from zero to an ultraintense plateau value and held for constant time T. It is then adiabatically decreased to zero and the probability for ionization is measured. Approximation methods used above take the initial state of the system at $t \rightarrow -\infty$ to be the ground state of the atom. However, there are inconsistancies associated with such treatment. The main one is in using ultraintense field approximation while the laser field is covering the whole range of intensities, from zero to $I \gg I_0$.

However, the approximation that laser-atom intraction is much larger then Coulomb interaction requires that we look at the problem of a Hydrogen atom in a laser field of constant (ultrastrong) intensity . Consequentialy, ground state wave function is a "dressed" state of an atom in the laser field.

Is it possible to make an experiment that measures the ionization rate for the conditions described above ?

It is possible, in principle,although it could be a difficult experiment to performe. One way would be by determining the ratio of the number of atoms that survive photoionizatin when laser pulse doubles its duration.It is easy to see that the width of the ground state is $\Gamma = \frac{1}{T}$

$\left[\ln(\frac{N_1}{N_2})\right]$, where we assumed the same switch on and switch off time. The other approach may involve creation of the region of spatialy homogenous laser field.

This work will concentrate on the photoionization rate and the "dressing " of atomic states of a hydrogen atom subjected to an ultrastrong laser that has a constant intensity level. In order to obtain ionization rate we will use a variational Rayleigh-Ritz method to find a ground state of a dressed atom. The width of that state will give us a proper intensity dependence of ionization .ou-9

CHAPTER II

Our treatment of photoinization and atomic structure of a hydrogen atom in the ultrastrong laser field is based on number of approximations which we will justify in this chapter.

2.1 The Hamiltonian for the Laser Atom Interaction

Our starting Hamiltonian with no aproximations in the interaction picture is

$$H = \frac{(\vec{P}_N - e\vec{A}(\vec{r}_{n_1}t))^2}{2M} + \frac{(\vec{P}_e + e\vec{A}(\vec{r}_{e_1}t))^2}{2m} + + V(\vec{r}e,\vec{r}_N) \tag{2.1.1}$$

$\vec{P}_N$ and $\vec{P}_e$ are momentum operators of the nucleus and the electron respectively. $\vec{r}_N$ and $\vec{r}_e$ are the position vectors of the nucleus and the electron.

If we introduce the transformation to the center of mass coordinates

$$\vec{\rho} = \frac{M\vec{r}_N + m\vec{r}_e}{M_A}$$

$$\vec{r}_1 = \vec{r}_e - \vec{r}_N \tag{2.1.2}$$

$$M_A = M + m$$

we get

$$H_{cm} = \frac{\vec{P}_{\rho}^{\,2}}{2M_A} + \frac{1}{2\mu}\left(\vec{P}_1 + e\vec{A}\,(\vec{\rho} + \vec{r}_1,\, t)\right)^2 + V(\vec{r},\vec{\rho}) \tag{2.1.3}$$

where $\mu = \frac{mM}{m+M}$ the reduced mass of the electron.

An important approximation we are going to make in the expression for Hamiltonian Eq. (1.3) is the dipole approximation for the radiation field, vector potential. The vector potential $\vec{A}$ is a function of coordinates through the factor exp $\pm$ $(i\vec{k}(\vec{\rho}+\vec{r}_1))$. We will assume that for the range of laser frequencies we are interested in the wave function of the electron would be limited in size so that approximation[1]

$$\langle \vec{k}\cdot\vec{r}_1 \rangle \ll 1 \tag{2.1.4}$$

is justified.

We are going to prove the validity of our assumption in Chapter 3. The center of mass coordinate will couple to the laser through the remaining exponential factor, so that the motion of the whole atom can be affected by the laser.

This is in general a small effect. It manifests itself through a slow drift of the mode phase (Doppler shift), that could be removed by the redefinition of the mode phase. However, it has to be properly addressed in order to understand problems of (ATI)[4,5] and deflection[3] of atoms in external fields coupled with a laser.

2.2 Transition to a Classical Description of the Laser

In the introduction to this thesis we said that purely quantum mechanical and semiclassical treatment of this problem give the same result due to the fact that the radiation field has a high photon occupation number. Results obtained by Glauber[6] and Bialinicki-Birula[7] confirm it. We will follow the later approach which is known as the phase representation.

For the radiation field Eq. (1.5) $\phi_{k\lambda}$ coordinates are introduced[1]

$$0 \leq \phi_{k\lambda} \leq 2\pi \tag{2.2.1}$$

so that for the state of $n_{k\lambda}$ photons in $\vec{k},\lambda$ mode

$$|n_{k\lambda}\rangle = \frac{1}{(2\pi)^{1/2}} e^{in_{k\lambda}\phi_{k\lambda}} \tag{2.2.2}$$

and the number operator is

$$\hat{n}_{k\lambda} = \frac{1}{i} \frac{\partial}{\partial\phi_{k\lambda}}. \tag{2.2.3}$$

This means that creation (destruction) operators are

$$a^{+}_{k\lambda} = \left(\frac{1}{i} \frac{\partial}{\partial\phi_{k\lambda}}\right)^{1/2} e^{i\phi_{k\lambda}}$$

$$a_{k\lambda} = e^{-i\phi_{k\lambda}} \left(\frac{1}{i} \frac{\partial}{\partial\phi_{k\lambda}}\right)^{1/2} \tag{2.2.4}$$

Let us assume that laser mode occupation number could be represented as

$$n_{k\lambda} = N_{k\lambda} + \nu_{k\lambda} \tag{2.2.5}$$

$N_{k\lambda}$ being an average occupation number and $\nu_{k\lambda}$ is a variation from it.

If we make the unitary transformation so that

$$|n_{k\lambda}\rangle = e^{iN_{k\lambda}\phi_{k\lambda}}|\nu_{k\lambda}\rangle \tag{2.2.6}$$

The operators also change to

$$\begin{aligned} a^{+}_{k\lambda} &= \left(N_{k\lambda} + \frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}}\right)^{1/2} e^{i\phi_{k\lambda}} \\ a_{k\lambda} &= e^{-i\phi_{k\lambda}} \left(N_{k\lambda} + \frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}}\right)^{1/2} \end{aligned} \tag{2.2.7}$$

Because $N_{k\lambda} >> \frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}}$ we can expand

$$\begin{aligned} a_{k\lambda} &= e^{-i\phi_{k\lambda}} \sqrt{N_{k\lambda}} \left(1 + \frac{1}{2N_{k\lambda}}\frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}} + \ldots\right)^{1/2} \\ a^{+}_{k\lambda} &= \sqrt{N_{k\lambda}} \left(1 + \frac{1}{2N_{k\lambda}}\frac{1}{i}\frac{\partial}{\partial\phi_{k\lambda}} + \ldots\right)^{1/2} \end{aligned} \tag{2.2.8}$$

We can now write the radiation field vector potential as

$$\vec{A} = \sum_{k\lambda} \left(\frac{2\pi\hbar c^2 N_{k\lambda}}{\omega_{k\lambda}V}\right)^{1/2} \{\hat{\varepsilon}_{k\lambda}\exp[i(\vec{k}\vec{r}-\omega_k t-\phi_{k\lambda}) + h.c.\} + \delta\vec{A} \tag{2.2.9}$$

where

$$\delta\vec{A} = \frac{1}{2} \sum_{k\lambda} \left(\frac{2\pi hc^2}{N_{k\lambda}\omega_k V}\right)^{1/2} \{\hat{\varepsilon}_{k\lambda}\exp[i(\vec{k}\vec{r}-\omega_k t-\phi_{k\lambda}]\ 1/i\ \partial/\partial\phi_{k\lambda} + h.c.\} \tag{2.2.10}$$

which is a small correction if $N_{k\lambda} \gg 1$. In that case one can write $\vec{A}(\vec{r},t)$ in the form 1.1.5. Here the phase parameter $\phi_{k\lambda}$ is still an operator. For a single mode laser the one phase parameter can be absorbed into a translation of t. For multimode lasers only one of the phase parameter could be absorbed in the way described above.

Multimode laser fields has been treated[37] by performing averaging for each mode.

In our calculations we will restrict ourself to the case of a single mode laser.

2.3 Floquet Theory of Formalism for Periodic Hamiltonian

The Schrodinger equation for the system we are discussing here is written as (h = 1)

$$(H\ (\vec{r},t) - i\frac{\partial}{\partial t})\ \psi\ (\vec{r},t) = 0 \tag{2.3.1}$$

where $H(\vec{r},t)$ is the total Hamiltonian. We can write

$$H\ (\vec{r},t) = H_0(\vec{r}) + H'(\vec{r},t) \tag{2.3.2}$$

where $H'(\vec{r},t)$ is the perturbation due to the interaction between the system and the monchromatic field. $H'(\vec{r},t)$ is a periodic interaction

$$H'(\vec{r},t+\tau) = H'(\vec{r},t) \tag{2.3.4}$$

where $\tau = 2\pi/\omega$ and ω is the radiation field angular frequency. The unperturbed Hamiltonian $H_0(\vec{r})$ has a complete orthonormal set of eigenfunctions

$$H_0(\vec{r})|\alpha(\vec{r})\rangle = E^0{}_\alpha|\alpha(\vec{r})\rangle, \ \langle\beta(\vec{r})|\alpha(\vec{r})\rangle = \delta_{\alpha\beta} \tag{2.3.5}$$

It can be shown[9,10,11] that a solution of a Schrodinger equation with a periodic Hamiltonian can be related to the solution of another Schrodinger equation with a time independent Hamiltonian represented by an infinite matrix. The solution for the wave function here is

$$\psi(\vec{r},t) = e^{-i\varepsilon t}\ \phi(\vec{r},t) \tag{2.3.6}$$

where

$$\phi(\vec{r},t+\tau) = \phi(\vec{r},t) \tag{2.3.7}$$

a function periodic in time, and ε is a parameter, unique up to the multiples of $2\pi n/\tau$, called the Floquet characteristic exponent or the quasi-energy. The term quasi energy reflects the formal analogy of the states 2.3.4 with the Bloch eigenstates in a solid with the quasimomentum $\vec{k}$. Our periodic function can be expanded in the Fourier series so that

$$\psi_\alpha(\vec{r},t) = \exp(-i\varepsilon_\alpha t) \sum_{n=-\infty}^{\infty} C_\alpha{}^{(n)}(\vec{r})\exp(-in\omega t)$$

$$C_\alpha^{(n)}(\vec{r}) = \sum_{\beta}^{0} \phi_{\alpha\beta}^{(n)} | \beta(\vec{r})> \tag{2.3.8}$$

It can be shown[11] that the only important terms in the summation over n are those that are close to average field occupation number $\dot{N}$.

That means the function ψ can be expanded into variations around N where

$$n = N + \nu \tag{2.3.9}$$

Using the Bialinicki-Birula formalism

$$\phi(\vec{r},t) = \sum_{\nu=\infty}^{\infty} \phi(\vec{r}) e^{i\nu\phi} \tag{2.3.10}$$

where time t is absorbed into Bialinicki-Birula phase ϕ. $\omega t + \phi = \phi$. So

$$\psi_a(\vec{r},t) = e^{-i\varepsilon_\alpha t} \sum_{\nu=\infty}^{\infty} \phi(r) e^{i\nu\phi} \tag{2.3.11}$$

This method was first introduced by Shirley.

The quasienergy ε is going to be treated as a complex value

$$\varepsilon = W + i\eta \tag{2.3.12}$$

where we are going to make an approximation about the value of η in our calculation of binding energy.

2.4 Ponderomotive Potential and Gauge Invariance for Linear and Circular Polarization

The Hamiltonian (2.2.1) in dipole approximation is

$$H = (\frac{(\vec{P}(\vec{r}) + e\vec{A}(t,\rho))^2}{2m} + V(\vec{r}) + \frac{P^2(\rho)}{2M} \qquad (2.4.1)$$

where for linear polarization (c = 1)

$$\vec{A}(t,\vec{\rho}) = \hat{z}\ \frac{E(\rho)}{\omega}\cos\omega t \qquad (2.4.2)$$

and for circular polarization

$$\vec{A}(t,\vec{\rho}) = \frac{E(\rho)}{\omega}\ (\hat{x}\cos\omega t - \hat{y}\sin\omega t) \qquad (2.4.3)$$

$\vec{r}$ is internal and ρ the center of mass coordinate of the hydrogen atom.

For linear polarization the time average of A^2 term is

$$U_p(\rho) = \frac{e^2E^2(\rho)}{4m\omega^2} \qquad (2.4.4.)$$

and for circular it is

$$U_p(\rho) = \frac{e^2E^2(\rho)^2}{2m\omega^2} \qquad (2.4.5)$$

The time average of the $\vec{A}$ term in 2.4.1 is the ponderomotive potential U_p. It can not be sensibly removed from H by a contact transformation of the form $e^{-iU_p t}$ since the kinetic energy operator of the center of mass would then induce a term of the form $\sim (\nabla_\rho U_p(\rho))t$ which becomes large with increasing t even though $U_p(\rho)$ is very small. $U_p(\rho)$ is a slowly varying function of the C.M. coordinate ρ. Instead $U_p(\rho)$ must be treated as a potential which modifies the center of mass motion and induces a shift in the local-ionization potential of the atom.

For circular polarization the A^2 term is time independent and it needs no further discussion. However, for linear polarization there is an additional term of the form $U_p(\rho)\cos 2\omega t$ in H. This may be removed from H by the contact transformation $\exp(iU_p(\rho)\,\frac{\sin 2\omega t}{2\omega})$, which also has an effect on the operator $P^2(\rho)/2M$ that should be examined.

$$e^{i\frac{U_p}{2\omega}\sin 2\omega t}\left(\frac{P^2(\rho)}{2M}\right)e^{-\frac{iU_p}{2\omega}\sin 2\omega t}$$
$$= \frac{P^2(\rho)}{2M} - \frac{1}{2M\omega}\nabla_\rho U_p \sin 2\omega t\, P_\rho + \frac{i}{4M\omega}\nabla^2{}_\rho U_p \sin 2\omega t$$
$$+ \frac{1}{8M\omega^2}(\nabla_\rho U_p)^2 \sin^2\omega t \qquad (2.4.6)$$

Of all those terms only the last one gives a nonoscilatory contribution as a "new ponderomotive potential." It is however negligible because of the factors M^{-1} and two derivatives with respect to ρ even at the very large intensities considered here.

The remaining new terms are time oscilatory couplings to the center of mass motion which are negligibly small here.

We see that the only essential difference between the circular and linear polarization cases lies in $\vec{p}.\vec{A}$ term of H. We shall see in Chapter III that it is an important difference.

The term that has been eliminated from the linear polarization case is the one which introduces the "modified Bessel functions" into the analysis of the multiphoton processes (Brown and Kibble (1964)). We see that their occurance is unnecessary when the field is treated in dipole approximation. In the circular polarization case only the usual Bessel functions arise.

For the purposes of the next Chapter we will neglect the center of mass motion and treat external conditions as constant. The origin may be taken at $\rho = 0$ so that $\vec{A}(\vec{\rho},t) \equiv \vec{A}(t)$.

CHAPTER III

In this chapter we will examine the possibility that atomic hydrogen exposed to the ultrastrong laser field forms a metastable bound state.

Calculations will be performed for linearly and circularly polarized laser light. Taking into account approximations discussed in Chapter II we will look for a state as a variational minimum, applying the Rayleigh-Ritz variational principle.

3.1 Feshbach Formalism and Green's Function for a Hydrogen Atom in Ultrastrong Laser Field

Our starting point is the Floquet theory for the complex eigenvalue of the atom.

The Schrodinger equation ($\hbar = 1$, $c = 1$)

$$\left(i\frac{\partial}{\partial t} - H\right)\psi = 0 \tag{3.1.1}$$

where

$$H = \frac{[\vec{p}(\vec{r}) + e\vec{A}(t)]^2}{2m} + V(\vec{r}) \tag{3.1.2}$$

is the Hamiltonian of the atom plus the single mode radiation field which is described in the phase representation (Bialinicki-Birula 1976) expanded about a state with a very large number of photons as we had described in Chapter II. The wave function

$$\psi = e^{i\epsilon t} \sum_{\nu=-\infty}^{\infty} \phi(\vec{r}) e^{i\nu\phi} \tag{3.1.3}$$

is a Floquet wave function, with complex quasienergy $\epsilon = w + i\eta$. So our Schrodinger equation has a form

$$[w + i\eta + i\omega\frac{\partial}{\partial\phi} - \frac{(\vec{p} + e\vec{A})^2}{2m} - V]\ \psi = 0 \tag{3.1.4}$$

where $0 \le \phi < 2\pi$ and $\vec{A}$ is a function of ϕ obtained by replacing $\omega t \to \phi$ in either (2.4.2.) or (2.4.3.). In order to get our dressed ground state wave function we make use of a projection operator

$$P = \frac{1}{2\pi} \int_{2\pi}^{\pi} d\phi \tag{3.1.5}$$

and its complement $Q = 1 - P$. Out of the complete set of functions projection operator P picks out one with unperturbed mode of the field, $\nu = 0$.

The Schrodinger equation (3.1.1.) may be rewritten as a pair of coupled equations

$$P\ (i\omega\frac{\partial}{\partial\phi} - H)\ P\psi = PHQ\psi \tag{3.1.6}$$

$$Q\ (i\omega\frac{\partial}{\partial\phi} - H)\ Q\psi = QHP\psi \tag{3.1.7}$$

where we have used $Pi\frac{\partial}{\partial\phi}\ Q = 0$. If we define a causal Green's function with outgoing boundary conditions as

$$Q\ (w + i\eta + i\omega\frac{\partial}{\partial\phi} - h)\ QG_Q{}^{(+)} = Q\delta(\vec{r},\vec{r}')\delta(\phi,\phi') \tag{3.1.8}$$

then it may be used to obtain a formal solution

$$Q\psi = QG_QQHP\psi \tag{3.1.9}$$

This can be put back in the equation for $P\psi$ which is now

$$P\ (w + i\eta + i\omega\frac{\partial}{\partial\phi} - h - v)\ P\psi = 0 \tag{3.1.10}$$

where

$$h = \frac{p^2}{2m} + V + U_p + \delta h,\ \delta h = \frac{e}{m}\ \vec{p}.\vec{A},\ v = P\delta hG_Q\delta hP \tag{3.1.11}$$

For linear polarization

$$\delta h = \frac{e}{m}\ k_z\ \frac{E}{\omega}\ \cos\phi + U_p\ \cos 2\phi \tag{3.1.12}$$

and

$$U_p = \frac{e^2E^2}{4m\omega^2} \tag{3.1.13}$$

We made an argument in Chapter II for the neglect $U_p\cos 2\phi$ term. In circular polarization

$$U_p = \frac{e^2E^2}{2m\omega^2},\ \delta h = \frac{e}{m}\ k\ \ \frac{E}{\omega}\cos\phi. \tag{3.1.14}$$

the only ϕ dependence is in δh. We used a fact that

$$\frac{\partial}{\partial\phi}\ P\psi = 0 \tag{3.1.15}$$

and that

$$P\delta h = 0 \tag{3.1.16}$$

in writing down equation 3.1.10. The essential approximation in our method will be to neglect V, the electron-nucleus interaction, compared to δh the electron-laser coupling in the intermediate state in (3.1.11). Then G_Q is replaced by

$$Q\ (w + i\eta + i\omega\frac{\partial}{\partial\phi} - \frac{p^2}{2m} - U_p - \delta h)\ QG^{(+)} = Q \tag{3.1.17}$$

This can be transformed to momentum space where this becomes a first order equation (in ϕ) for $G^{(+)}(k)$.

It can be rewritten as

$$(\varepsilon + i\eta + i\omega\frac{\partial}{\partial\phi} - \varepsilon_k - \frac{e}{m\omega}\ Ek_1\cos\phi)\ G_k{}^{(+)}(\phi,\phi') =$$

$$\delta(\phi-\phi') - \frac{1}{2\pi} - \frac{eE}{m\omega} k_1 \bar{G}_k^{(+)} (\phi') \qquad (3.1.18)$$

and $k_1 = k_z$ for the linearly polarized laser, or $k_1 = k = \sqrt{k_x^2+k_y^2}$ for the circularly polarized laser.

This first-order-separable-integro-diferential equation is tediously solvable.

The procedure for solving its is given in Appendix I. The result is

$$G_k^{(+)}(\phi,\phi') = -\frac{1}{2\omega} \gamma'_k(\phi)\gamma'_k(\phi')\theta(\phi-\phi') -$$

$$- \frac{1}{2\omega\Delta_k(\eta)} \gamma'^*_k(\phi)\gamma'_k(\phi') \{-\gamma_k(\phi)\gamma^*_k(\phi')(\gamma'^*_k-\gamma'_k) -$$

$$(3.1.19)$$

$$- (Z_k^*\gamma'^*_k - Z_k\gamma'_k) + \frac{1}{\pi} \gamma_k(\gamma'^*_k\gamma^*_k(\phi') - \gamma'_k\gamma_k(\phi))\}$$

$$= g_k^{(1)} (\phi,\phi') + \frac{1}{\Delta_k(\eta)} g_k^{(2)}(\phi,\phi') \qquad (3.1.19)$$

Here θ is a step function $\theta(x) = \frac{x}{|x|}$. The remainding factors are

$$\gamma'_k(\phi) = \exp i(\nu_k\phi + \chi_k\sin\phi)$$

$$\gamma'_k = \gamma'_k(\pi) = \exp i\nu_k\pi, \quad \gamma_k(\phi) = \int_{-\pi}^{\phi} d\phi' \, \gamma'_k(\phi')$$

$$Z_k = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \gamma'_k(\phi)\gamma_k(\phi) \qquad (3.1.20)$$

and

$$\nu_k = \frac{\varepsilon_k - \varepsilon}{\omega}, \; \chi_k = \alpha k_1 \tag{3.1.21}$$

and

$$\alpha = \frac{eE}{m\omega^2} \tag{3.1.22}$$

is the parameter which will gauge the strength of the field.

In addition

$$\Delta_k(\eta) = (I_k(\eta)e^{\pi\eta/\omega}\gamma'_k + I^*_k(-\eta)\gamma'^*_k e^{-\pi\eta/\omega}) \tag{3.1.23}$$

where

$$I_k(\eta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi e \; (-\frac{\eta}{\omega}\phi)\gamma'_k(\phi) \int_{-\pi}^{\phi} ds e \; (\frac{\eta}{\omega}.s) \; \gamma'_k(s) \tag{3.1.24}$$

We have anticipated the $\eta \to 0+$ limit everywhere with the exception of the denominator Δ_k where the limit will produce a principle value and delta function in the usual way.[43] We use this result to construct v(k) (3.1.11) which is given in detail in Appendix II.

$$v = P\delta hG_Q\delta hP \tag{3.1.25}$$

which means, that v(k) could be written as

$$v(k) = \frac{1}{(2\pi)^2} \text{Re} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\phi d\phi' \omega a k_1 \cos\phi \; G_k^{(+)}(\phi,\phi') \cos\phi' \omega a k_1 \qquad (3.1.26)$$

Using the result $\frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \cos\phi G_k^{(+)} = \bar{G}_k^{(+)}(\phi')$ and after some algebra shown in Appendix II we can write

$$v(k) = (\varepsilon - = \varepsilon_k) \left(1 - \frac{1}{\nu_k P(x_k, \nu_k)}\right) \qquad (3.1.27)$$

where

$$P(x,\nu) = (\sin \pi\nu)^{-1} \text{Re}\Delta_k = \sum_{n=-\infty}^{\infty} \frac{J_n^2(x)}{n + \nu} \qquad (3.1.28)$$

The sum on the right hand side could be evaluated[41]

$$\sum_{n=-\infty}^{\infty} \frac{J_n^2(x)}{n + \nu} = \frac{\pi}{\sin\pi\nu} J_\nu(x) J_{-\nu}(x) \qquad (3.1.29)$$

Here both sides of the equation have the same poles and residues as a function of ν and no other singularities and the difference of the two expressions vanishes faster than ν^{-1} for large ν.

Then in this approximation equation (3.1.10) can be rewritten as ($P\psi = \phi$)

$$(\varepsilon - \varepsilon_k)F(x_k,\nu_k)\phi(\vec{k}) = \int d^3k'\tilde{V}(\vec{k}-\vec{k})\phi(\vec{k}') \qquad (3.1.30)$$

where $\tilde{V}$ is the Fourier transform of the Coulomb potential

$$\tilde{V}(\vec{k}-\vec{k}') = \int \frac{d^3}{(2\pi)^3}\left(-\frac{e^2}{|r-r'|}\right) e^{i\vec{k}\vec{r}} = -\frac{e^2}{2\pi^2|\vec{k}-\vec{k}'|^2} \qquad (3.1.31)$$

and

$$F(x,\nu) = \frac{\sin\pi\nu}{\pi\nu}\,(J\nu(x)J_{-\nu}(x))^{-1} \qquad (3.1.32)$$

Now we can scale out the large parameter α that occurs in $x_k = \alpha k_i$ by

$$\vec{\ell} = \alpha\vec{k} \qquad (3.1.33)$$

The equation then becomes

$$(\varepsilon' - \varepsilon_e)F\left(\ell_i,\ \varepsilon - \frac{\varepsilon_\ell}{\alpha^2\omega}\right)\phi(\vec{\ell}) =$$
$$= \alpha \int d^3\ell'\tilde{V}(\vec{\ell}-\vec{\ell}')\phi(\vec{\ell}')$$

where $\varepsilon' = \alpha^2\varepsilon$ and we have used the explicit form of V. For sufficiently large α, F can be replaced by $F(\ell_i,0)$. We return to the $\vec{k}$ variables, and use the substitution

$$\phi(\vec{k}) = AJ_0(x_k)U(\vec{k}) \qquad (3.1.35)$$

to obtain

$$(\varepsilon - \varepsilon_n)U(\vec{K}) = \int d^3k' J_0(\alpha k_\perp)\tilde{V}(\vec{K}-\vec{K}')J_0(\alpha k'_\perp)U(\vec{K}') \qquad (3.1.36)$$

Here A is a normalization constant which plays no role in this section. The effect of the laser is to destroy the translational invariance of the effective potential in momentum space and therefore to make it non-local in configuration space. The rapidly oscillating Bessel functions weaken the potential and so the existence of bound state is questionable.

3.2 Rayleigh-Ritz Variational Method for Determining Bound State Energy

The Rayleigh-Ritz variational method that we are going to use here is somewhat simplified in the sense that we are interested in obtaining only one eigenvalue.[40] In other words we are interested in the very existance of the bound state and that is the reason we use just one eigen function instead of a set of n linearly independent functions that could give us upper bounds of n lowest eigenvalues.

For the problem with which we are concerned, Eq. (3.1.36), the bound state should arise from a (negatively) large value of the modified potential energy while the expectation value of the kinetic energy, ε_K, remains small.

The rapid oscillation in the potential therefore must be compensated for by a similar oscillation of the wave function. Therefore we make the substitution

$$U(\vec{K}) = J_0(x_K)U_1(\vec{K}) \qquad (3.2.1)$$

and then multiply (3.1.36) by (3.2.1) and integrate over the k volume

$$\varepsilon = (T + V)/N \tag{3.2.2}$$

where

$$\begin{aligned} N &= \int d^3k\,|U_1(\vec{k})|^2 J_0^2(x_k) \\ T &= \int d^3k\,|U_1(\vec{k})|^2 J^2{}_0(x_k)\varepsilon_k \\ V &= \int d^3k d^3k' U_1^*(\vec{k}) J^2{}_0(x_k)\tilde{V}(\vec{k}-\vec{k}') J^2{}_0(x'_k) U_1(\vec{k}') \end{aligned} \tag{3.2.3}$$

The choice of the trial $U_1(\vec{k})$ function is such that its Fourier transformation to configuration space resembles the form of hydrogen like wave function, and reflects the symmetry of the laser radiation.

3.3 Linear Polarization case

For the case of linear polarization we choose

$$U_1(\vec{k}) = \frac{\xi^6}{(k^2{}_z+\xi^2)(k\ ^2+\xi^2\beta^2)^2} \tag{3.3.1}$$

and treat ξ and β as optimization parameters. Detailed evaluation of integrals (3.2.3) are given in Appendix III. The integrals are performed for large α with the result that is given as an expansion in $(\ell n\lambda)^{-1}$ terms. We kept terms of the order $(\ell n\lambda)^0$ and smaller.

The results are

$$N = \frac{2\lambda^2}{3\beta^6\alpha\beta}\left(\ell n\lambda+\pi\chi_1 - \frac{1}{2} + 0\left(\frac{1}{\lambda^2}\right)\right) \tag{3.3.2}$$

where

$$\lambda = \xi\alpha,\ \chi = \int_0^1 dxJ^2{}_0(x) + \int_1^\infty dx(J^2{}_0(x) - \frac{1}{\pi x}) \approx 0.846 \qquad (3.3.3)$$

and

$$T = \frac{\lambda^4}{6\beta^6\alpha^5 m} (\beta^2 \ell n\lambda + \beta^2 (\pi\chi - \frac{1}{2}) + 1 + 0 (\frac{1}{\lambda^2})) \qquad (3.3.4)$$

The result for V is much more complicated and difficult to obtain. It is first rewritten as

$$V = - \frac{e^2\lambda^4}{\pi^2\alpha^4} \int_0^\infty \int_0^\infty \frac{d\ell_z d\ell_z'}{(\ell_z{}^2+1)(\ell'_z{}^2+1)} J^2{}_0(\lambda\ell_z J^2{}_0(\lambda\ell'_z)$$

$$[g((\ell+\ell')^2 + g((\ell-\ell')^2)] \qquad (3.3.5)$$

where

$$g(x^2) = \frac{2\pi^2}{\beta^4} \int_0^\infty \frac{ydy}{(y+1)^2} \int_0^1 \frac{ds}{(sy+1)^2} [x^4+2x^2\beta^2 y(1+s) + y^2\beta^4(1-s)^2] \quad (3.3.6)$$

which is the integral of the Coulomb potential over the components of the momenta perpendicular to the $\hat{z}$ axis. It is clear that a replacement of

$J^2{}_0(\lambda\ell_z) \rightarrow \frac{1}{\pi}\ (\pi\ell_z)^{-1}$ (its asymptotic form) results in logarithmic singularities in the integrals which coupled with

$$\lim_{x\to 0} g(x^2) \rightarrow \frac{2\pi^2}{\beta^6}\left(\frac{1}{3}\ \ell n\ \frac{\beta}{x} + \frac{1}{36}\right) \qquad (3.3.7)$$

yields the dominant behavior of

$$V \sim -\frac{\lambda^2}{\alpha^4\beta^6}\left[(\ell n\lambda)^3 + 0\ (\ell n\lambda)^2\right] \qquad (3.3.8)$$

The more careful approach (Appendix III) is taken by writing

$$J^2{}_0(\lambda\ell) = \frac{\varepsilon(\lambda\ell-1)}{\pi\lambda\ell} + \left(J^2{}_0(\lambda\ell) - \frac{\varepsilon(\lambda\ell-1)}{\pi\lambda\ell}\right) \qquad (3.3.9)$$

where $\varepsilon(x)$ is the step function $\varepsilon(x) = 1/2\ (1 + x/{}_{|x|})$. The first term yields the dominant term and the second its corrections. Replacing $J^2{}_0(\lambda\ell)$ by Eq. (3.3.9) and placing it in the integral for V gives

$$V = V_{11} + 2V_{12} + V_{22} \qquad (3.3.10)$$

The part of V which results from the use of the first term of (   ) in place of both $J^2{}_0$ factors is

$$V_{11} = \frac{e^2\lambda^2}{\pi^4\alpha^4}\int_{\frac{1}{\lambda}}^{\infty}\int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell\, d\ell'}{\ell\ell'(\ell^2+1)(\ell^2+1)}\left[g((\ell+\ell')^2) + g((\ell-\ell')^2)\right] \qquad (3.3.11)$$

If we use (3.3.7) to replace the last factor in V_{11} by

$$g(\ell,\ell') \to \frac{2\pi^2}{\beta^2}\left(-\frac{1}{3}\,\ell n\left(\ell^2 - \frac{\ell'^2\pi}{\beta^2} + \frac{1}{18}\right) + q(\ell,\ell')\right) \tag{3.3.12}$$

and thereby defining $q(\ell,\ell')$. It has the property that it vanishes for ℓ or ℓ' equal zero. It therefore contributes a small corrections to V_{11} when $\lambda = e\alpha\xi$ is large. The result

$$V_{11} = -\frac{2e^2\lambda^2}{\pi^2\alpha^4\beta^6}\left[\frac{2}{9}\,(\ell n\lambda)^3 + \frac{1}{3}\,(\ell n\lambda)^2\,(2\ell n\beta + \frac{1}{6}) + 0\,(\ell n\lambda)\right] \tag{3.3.13}$$

The second term V_{12} gets the dominant contribution from the small (ℓ,ℓ') contribution in g's Eq. (A.3.33). Integration of (A.3.40) yields

$$V_{12} = -\frac{2e^2\lambda^2}{\pi^2\alpha^4\beta^6}\left(\frac{\pi}{6}\,(\ell n\lambda)^2 + \frac{\pi}{3}\,\ell n\lambda\,\ell n\beta\right) \tag{3.3.14}$$

where the second term arises because we allow for the possibility that β is also large.

We proved (Appendix III) that the contribution to V from the V_{22} term is negligable with respect to V_{11} and V_{12} because its contribution is of the order of terms already dropped in (3.3.13) and (3.3.14). Then we obtain.

$$V = -\frac{2e^2\lambda^2}{\pi^2\alpha^4\beta^6}\,[\frac{2}{9}\,(\ell n\lambda)^3 + (\ell n\lambda)^2\,(\frac{2}{3}\,\ell n\beta + \frac{1}{18} + \frac{\pi}{3}) + \\ + \frac{2\pi}{3}\,\ell n\lambda\,\ell n\beta + 0\,(\ell n\lambda) \tag{3.3.15}$$

Having evaluated integrals N, T and V, we can write down the expression for

ε Eq. (3.2.1). Keeping only the leading terms in $(\ell n\lambda)^{-1}$ expansion the result is

$$\varepsilon = \frac{1}{4m\alpha^2} (\lambda^2\beta^2 - \frac{8}{3\pi^2} \frac{\alpha}{a_0} ((\ell n\lambda)^2 + 3\ell n\lambda\ell n\beta + a_1\ell n\lambda + a_2\ell n\beta)) \quad (3.3.16)$$

where a_0 is the Bohr radius, and $a_1 = 3/4 + \pi\,(\frac{3}{2} - \chi)$, $a_2 = \frac{3}{2} + 3\pi\,(1 - \chi)$ and χ is given in (3.3.3)

At this point we will make use of the variational principle and find an energy minimum as a function of parameters λ and β. Straightforward evaluation of $\frac{\partial E}{\partial\lambda} = 0$ gives

$$\lambda^2\beta^2 = \frac{4}{3\pi^{-2}} \frac{\alpha}{a_0} (2\ell n\lambda + 3\ell n\beta + a_1) \quad (3.3.17)$$

and variation with respect to β leads to another equation

$$\lambda^2\beta^2 = \frac{4}{3\pi^2} \frac{\alpha}{a_0} (2\ell n\lambda + 3\ell n\beta + a_1) \quad (3.3.18)$$

These can be combined to give

$$\lambda = \beta^3\exp(a_1 - a_2) \quad (3.3.19)$$

and

$$\frac{(\lambda^1)^{8/3}}{\ell n\lambda^1} = \frac{4}{\pi^2} \frac{\alpha}{a_0} \exp[\frac{2}{3}\, a_1 + \frac{2}{9}\, a_2],\ \lambda^1 = \lambda\exp[a_2\, \frac{2}{3}] \quad (3.3.20$$

which can be inverted numerically to give $\lambda(\alpha)$.

We can do an approximate analytic inversion of Eq. (3.3.20) which is

$$\lambda = \left(\frac{3}{2\pi^2} \frac{\alpha}{a_0} \ell n \left(\frac{4}{\pi^2} e^{\frac{2}{3}a_1 + \frac{2}{9}a_2} \frac{\alpha}{a_0}\right)\right)^{3/8} e^{1/4(a_1-a_2)} \tag{3.3.21}$$

which for $\frac{\alpha}{a_0} = 2016$ yields $\lambda = 19.0$ whereas the numerical inversion of Eq. (3.3.20) yields $\lambda = 20.0$, 5% error.

Evaluation of λ and β as a function of α gives $\varepsilon = \varepsilon(\alpha)$ in this form

$$\begin{aligned} \varepsilon &= -\frac{8}{3\pi^2} \mathrm{Ry} \frac{a_0}{\alpha} \left(\left(\ln\lambda\, e^{-3/8+\frac{a_2}{3}}\right)^2 + 0\,(\ln\lambda)^0\right) \\ &\approx -\frac{3}{8\pi^2} \mathrm{Ry} \frac{a_0}{\alpha} \left(\ln \left(\frac{3}{2\pi_2} e^{-1+\frac{8}{9}a_2} \frac{\alpha}{a_0} \ln \left(\frac{4}{\pi^2} e^{2/3 a_1+\frac{2}{9}a_2}\frac{\alpha}{a_0}\right)\right)\right)^2 \\ &= -\frac{3}{8\pi^2} \mathrm{Ry} \frac{a_0}{\alpha} \left(\ln \left(0.77 \frac{\alpha}{a_0} \ln \left(5.07 \frac{\alpha}{a_0}\right)\right)\right)^2 . \end{aligned} \tag{3.3.22}$$

In obtaining this result we find that the kinetic energy term is negligible compared to the potential energy.

It is well known[27,36] that the start of the continuum is shifted upward by the ponderomotive potential. Then ε is the binding energy of the atom in the presence of the field. We see that it is a slowly decreasing function of the intensity in this asymptotic domain.

The result for $\lambda = \lambda(\alpha)$ Eq. (3.3.20) and the fact that $\beta = \beta(\lambda)$ shows that roughly

$$\xi \sim \alpha^{-5/8} \text{ and } \beta\xi = \eta \sim \alpha^{-1/2} \tag{3.3.23}$$

The momentum space wave function $\phi(\vec{k})$ that describes the electron when the hydrogen atom is subjected to the ultrastrong laser field (large α) peaks for small $k \sim \beta\xi \sim \alpha^{-1/2}$ and still smaller $k_z \sim \alpha^{-1}$. $k_z \sim \alpha^{-1}$ is due to Bessel function in the form of $\phi(\vec{k})$. (Eq. 3.1.35)

The configuration space wave function therefore extends to large r, but larger in z direction.

At this point we can judge the validity of our nonrelativistic approach and that of the dipole approximation. The fact that only low momenta are important in the wave function shows that the nonrelativistic treatment is justified.

The interference between the influence of the ultrastrong laser on the motion of the electron and that of the Coulomb potential creates this interesting effect of a losely bound electron that performs nonrelativistic motion.

The accuracy of the dipole approximation can be judged from the size of the wavefunction in the direction of laser propagation compared to the laser wavelength.

We had

$$\phi(k) = AJ^2{}_0(\sigma k_z)/(k_K{}^2+\xi^2)(k\ {}^2+\xi^2\beta^2)^2 \qquad (3.3.24)$$

so the extent of the wavefunction in configuration space in the r direction is of the order $(\xi\beta)^{-1}$. For the neodimium glass laser at $I \sim 10^{20}$ W/cm^2, $\lambda \approx 1.6\ \mu$

$$\frac{(\xi\beta)^{-1}}{\lambda_{laser}} = \frac{1}{2}\left(\frac{\alpha}{a_0}\right)^{1/2} \alpha F \frac{h\omega}{Ry}\left(\frac{3}{2\pi^2}\,\ell n\left(5.07\,\frac{\alpha}{a_0}\right)\right)^{-1/2} \qquad (3.3.25)$$

where $\alpha_F = (137)^{-1}$ is the fine structure constant. The dipole approximation is still well justified, since $(\xi\beta)^{-1}/\lambda$laser = (0.043)

Is there any restriction on the frequency ω of the laser? We did our calculation with the approximation of large $\alpha = \frac{eE}{mw^2}$ and our fundamental approximation was made on the intermediate state Green's function.

We have neglected V, the electron proton interaction compared to the electron field interaction. This is roughly

$$\frac{e^2}{\bar{r}} << \omega\alpha\bar{k} \tag{3.3.26}$$

where $\bar{r}$ and $\bar{k}$ are some average values of the distance and momenta, respectively, of the electron in the intermediate state.

If we use the uncertainty principle

$$\bar{r}.\bar{k} \sim 1 \tag{3.3.27}$$

then this result is

$$\omega >> e^2/\alpha \approx 2Ry \frac{a_0}{\alpha} \tag{3.3.28}$$

which is a very weak restriction on the frequency for the large values of α considered here.

3.4 Binding Energy - Circularly Polarized Laser Field

The analysis of a binding energy for a hydrogen atom in circularly polarized light starts with the Hamiltonian

$$H = \frac{[\vec{p}(\vec{r}) + e\vec{A}(t)]^2}{2m} + V(\vec{r}) \tag{3.4.1}$$

where

$$\vec{A}(t) = \frac{E}{\omega}\,(\hat{x}\cos\omega t - \hat{y}\sin\omega t) \tag{3.4.2}$$

Variational calculation of

$$\varepsilon = \frac{T}{N} + \frac{V}{N} \tag{3.4.3}$$

follows the same approach as given in Appendix III.

We have to calculate integrals T, V and N.

The choice of wave function is such that it has the same type of symmetry

$$U_1(k) = \frac{\xi^4}{(k^2+\xi^2)(k_z^2+\xi^2\beta^2)} \tag{3.4.4}$$

the details of this calculation are given in Appendix IV. The first integral

$$N = \int d^3k \; J^2_0\,(\alpha k\,)\left|U_1\,(\frac{k}{\xi},\beta)\right|^2 \tag{3.4.5}$$

transforms after scaling $\frac{\vec{k}}{\xi} = \vec{\ell}$

$$N = 2\pi\xi^3 \int_0^\infty \frac{d\ell\;\ell\;J^2_0(\lambda\ell\,)}{(\ell^{\,2}+1)^2} \int_{-\infty}^\infty \frac{d\ell_Z}{(\ell_Z^{\,2}+\beta^2)^2} \tag{3.4.6}$$

The result of this integration is

$$N = 2\pi\xi^3 \frac{\pi}{2\beta^3} \left(-\frac{\lambda}{2}\frac{\partial}{\partial\lambda}\right) K_0(\lambda)I_0(\lambda) \tag{3.4.7}$$

where K_0 and I_0 are modified Bessel functions[42].

Using the asymptotic[4] expansion of K_0 and I_0 for λ large

$$N = \frac{\pi^2\xi^3}{2\beta^3}\left(\frac{1}{2\lambda} + \frac{3}{16\lambda^3} + \ldots\right) \tag{3.4.8}$$

Similar calculations for the T integral

$$T = \frac{1}{2m}\int d^3k k^2 J^2_0(\alpha k\,)\left|U_1(\frac{k}{s},\beta)\right|^2 \tag{3.4.9}$$

give the result

$$T = \frac{\xi^5\pi^2}{2m\beta^3}\left[1 + (\beta^2-1)\left(-\frac{\lambda}{2}\frac{\partial}{\partial\lambda}\right)\right] K_0(\lambda)I_0(\lambda) \tag{3.4.10}$$

Finally, asymptotic expansion in $K_0(\lambda)\ I_0(\lambda)$ in (3.4.1) yields

$$T = \frac{\xi^5\pi^2}{2m\beta^3}\left[\frac{1}{2\lambda}\left(\frac{\beta^2+1}{2}\right) + \frac{1}{16\lambda^3}\left(\frac{3\beta^2-1}{2}\right) + 0\left(\frac{1}{\lambda^5}\right)\right] \tag{3.4.11}$$

Evaluation of the integral V is the hardest part in the evaluation of ε.

$$V = -\frac{e^2}{2\pi^2}\int d^3k d^3k' U_1\left(\frac{k}{\xi},\beta\right) J^2{}_0(\alpha k\ )\frac{1}{|\vec{k}-\vec{k}'|^2} J^2{}_0(\alpha k\ ) U_1\left(\frac{k'}{\xi},\beta\right) \tag{3.4.12}$$

The usual scaling $\vec{k} = \vec{\ell}\xi$, $\vec{k}' = \vec{\ell}'\xi$ and $\xi\alpha = \lambda$ is applied so that

$$V = \frac{e^2}{2\pi^2}\frac{\xi^4}{\lambda^2}\int_0^\infty\int_0^\infty d\ell\, d\ell'\left(f_0(\lambda\ell\ ) + \frac{1}{\pi}\right)g(\ell\ ,\ \ell\ ')\left(f_0(\lambda\ell\ ) + \frac{1}{\pi}\right) \tag{3.4.13}$$

where we defined

$$f_0(\lambda\ell\ ) = \lambda\ell\ J^2{}_0(\lambda\ell\ ) - \frac{1}{\pi} \tag{3.4.14}$$

and

$$g(\ell\ ,\ell\ ') = \int_0^\infty \int_0^\infty d\ell_z d\ell_z' \int_0^{2\pi}\int_0^{2\pi} d\phi d\phi' U(\ell,\beta)\ \frac{1}{|\dot{\ell}-\ell|^2}\ U(\ell',\beta) \quad (3.4.15)$$

Evaluation of V breaks into the evaluation of three integrals

$$V_{11} = \frac{1}{\pi^2}\int_0^\infty \int_0^\infty d\ell\ d\ell\ 'g(\ell\ ,\ell\ ')$$

$$V_{12} = \frac{2}{\pi}\int_0^\infty \int_0^\infty d\ell\ d\ell\ g((\ell\ ,\ell\ ')f_0(\lambda\ell\ )$$

$$V_{22} = \int_0^\infty \int_0^\infty d\ell\ d\ell\ f_0(\lambda\ell\ )g(\ell\ ,\ell\ ')f_0(\lambda\ell\ ') \quad (3.4.16)$$

Integral V_{11} is independent of λ.

The evaluation of V_{12} requires the evaluation of

$$\Upsilon(\ell\ ) = \int_0^\infty d\ell\ 'g(\ell\ ,\ell\ ') \quad (3.4.17)$$

the most contribution around $\ell\ \to 0$

$$\Upsilon(\ell\ ) = -\frac{2\pi}{\beta^3}\ \ln\ \ell \quad (3.4.18)$$

We integrate V_{12} by parts

$$V_{12} = \frac{2}{\pi} \int_0^\infty d\ell \; f_0(\lambda\ell\,)\gamma(\ell\,) \tag{3.4.19}$$

with $v = \gamma$ and $du = d\ell \; f_0(\lambda\ell\,)$ so that

$$V_2 = - \frac{2}{\pi} \int_0^\infty d\ell \; f_1(\lambda\ell\,)\gamma'(\ell\,) \tag{3.4.21}$$

or

$$V_2 = - \frac{2}{\pi} \int_0^\infty \bar{f}_1(\lambda\ell\,)(\ell\;\gamma'(\ell\,)) \tag{3.4.21}$$

and

$$\bar{f}_1(\lambda\ell\,) = \frac{1}{\lambda\ell} \; f_1(\lambda\ell\,)$$

$$f_1(\lambda\ell\,) = \int_0^{\lambda\ell} f_0(t)dt \tag{3.4.22}$$

Integrating it by parts once more one gets

$$V_{12} = \frac{1}{\lambda} \; \frac{4\pi k_{21}}{\beta^3} \tag{3.4.23}$$

where

$$k_{21} = -\frac{1}{\pi} - \int_0^1 dt\,t\ln t J^2_0(t) - \int_1^\infty dt\ln t\left[tJ^2_0(t) - \frac{1}{\pi}(1+\sin 2t)\right] -$$

$$-\frac{1}{2\pi}\int_1^\infty \frac{dt}{t}\cos 2t \tag{3.4.24}$$

k_{21} is evaluated numerically and $k_{21} = 0.43715$.

The last term, V_{22} is unchanged by $\ell \to \ell\,'$ so that

$$V_{22} = 2\int_0^\infty d\ell \int_0^\ell d\ell\,' f_0(\lambda\ell)g(\ell,\ell')f_0(\lambda\ell\,') \tag{3.4.25}$$

Let

$$f_0(y) = \bar{f}_0(y) + 1/\pi \sin 2y \tag{3.4.26}$$

where

$$\bar{f}_0(y) = yJ^2_0(y) - \frac{1}{\pi}(1 + \sin 2y) \tag{3.4.27}$$

that breaks the integral V_{22} break into three integrals, $V_{22}(1)$, $V_{22}(2)$, $V_{22}(3)$.

$$V_{22}{}^{(1)} = \frac{4\pi^3}{\lambda\beta^3} \int_0^1 dxK(x) \int_0^\infty ds\ \bar{f}_0(s)\bar{f}_0(sx) \tag{3.4.28}$$

where $s = \lambda\ell$ and $K(x)$ is the eliptic integral of the first kind.

$$V_{22} = \frac{4\pi^3}{\lambda\beta^3} K_{22}{}^{(1)}$$
$$K_{22}{}^{(1)} \tag{3.4.29}$$

$k_{22}{}^{(1)}$ being a constant independentof λ which we evaluated numerically.

Similarly the second term

$$V_{22} = \frac{4\pi^2}{\lambda\beta^2} k_{22}{}^{(2)} \tag{3.4.30}$$

where

$$k_{22}{}^{(2)} = \int_0^1 dxK(x) \int_0^\infty ds \left[\bar{f}_0(s) \sin 2sx + \bar{f}_0(sx) \sin 2s\right] \tag{3.4.31}$$

is a convergent integral. The last term $V_{22}{}^{(3)}$ is evaluated and

$$V_{22} = \frac{1}{4\beta^3\lambda} (3\ln 2 + \ln 2\lambda) \tag{3.4.32}$$

All integrations are done in the limit of large λ where we kept terms up to the order of $O(\frac{1}{\lambda^3})$. Then combining those results we get that

$$V = -\frac{e^2}{2\pi^2}\frac{\xi^4}{\lambda^2}\left[v_1 + \frac{4\pi^2 k_{21}}{\lambda\beta^3} + \frac{4\pi^3}{\lambda\beta^3}k_{22}(1) + \frac{4\pi^2}{\lambda\beta^3}k_{22}(2) + \right.$$

$$\left. + \frac{\pi}{4\beta^3\lambda}(3\ln 2 + \ln 2\lambda)\right] \tag{3.4.33}$$

We are able to investigate the variational minimum of the total energy of a hydrogen atom in ultrastrong circularly polarized laser field.

Performing a variational principle outlined in Appendix IV on the total energy

$$\varepsilon = \frac{\lambda^2}{2m\alpha^2}\left[\frac{\beta^2+1}{2} - \frac{1}{4\lambda^2}\right] - \frac{2e^2\beta^3}{\pi^4\alpha}\left[v_1(\beta) + \frac{A'}{\lambda\beta_3} + \frac{\pi}{\gamma\lambda\beta^3}\ln\lambda\right] \tag{3.4.34}$$

where A' is defined by (A.4.80).

so that $\frac{\partial E}{\partial\lambda}$ leads to

$$\lambda^3 = -\frac{2m\alpha e^2}{\pi^4(\beta^2+1)}A' - \frac{e^2\pi}{8}(\ln\lambda - 1)$$

Since $A' > 0$ and $\lambda < 0$, that makes $\varepsilon > 0$ and we found no bound state for the case of circularly polarized laser field.

CHAPTER IV

4.1 Photoionization Rate of a Hydrogen Atom in Ultraintense Laser Field

The calculation of the width of the ground state obtained as, a variational minimum in Chapter III will give us the information about the photoionization rate since that is the only open channel for the hydrogen atom in the ultraintense laser field.

Going back to the formalism outlined in Chapter III for the problem of binding energy we use Eq. (3.10) to proceed

$$P[W + i\eta + i\omega\frac{\partial}{\partial\phi} - h - v]\, P\psi = 0 \tag{4.1}$$

Again $P\psi = \phi$ and the operator

$$P\, i\frac{\partial}{\partial\phi}\, Q = 0,\ PQ = 0,\ P^2 = P$$

The width of the state then in this formalism is the imaginary part of the expectation value for the operator $v = p\delta h G_Q \delta h P$.

We now turn to calculation of the width of the state. It is obtainable as the expectation value of the non-hermitian part of the operator v in (3.1.10).

$$\frac{\Gamma}{\pi} = \mathrm{Im}\int \phi^*(\vec{k}),\delta h_k \cos\phi\, G_Q(\vec{k}\phi;\vec{d}'\phi')\cos\phi'\,\delta h_{k'}\,\phi(\vec{k}')\,d^3k\,d^3k'\,d\phi\,d\phi' \tag{4.2}$$

where G_Q is defined by (3.1.8) and $\delta h_k = \omega\alpha k_i$. We again expand it in powers of V but here we find that the first surviving term is second order in V.

$$G_Q()k\phi,k'\phi') = G_k^{(+)}(\phi,\phi')\ \delta(\vec{k}-\vec{k}')$$

$$+ \int_{-\pi}^{\pi} d\phi_1\ G_k^{(+)}(\phi,\phi_1)\tilde{V}(\vec{k}-\vec{k}')G_{k'}^{(+)}(\phi_1,\phi')$$

$$+ \int\ d\phi_1 d\phi_2 d^3k_1\ G_k^{(+)}(\phi,\phi_1)\tilde{V}(\vec{k}-\vec{k}_1)G_{k_1}^{(+)}(\phi_1\phi_2)\tilde{V}(\vec{k}_1-\vec{k}')G_{k'}^{(+)}(\phi_2,\phi')$$

$$+ \ldots \tag{4.3}$$

Consider first the contribution of the first term of (4.3) in Γ. We note that $G^{(+)}{}_k(\phi,\phi')$, (A.1.31), is an hermitian operator except for the appearance of $i\eta$ in the denominator $\Delta_k(\eta)$. Since the remaining operators in (A.1.31) are hermitian the imaginary part operation just extracts a term from G proportional to $(\Delta_k{}^{-1}(\eta) - \Delta_k{}^{-1}(-\eta))$ which is proportional to

$$\delta(J_{\nu_k}(x_k)J_{-\nu_k}(x_k)) \tag{4.4}$$

(We saw in (3.1.19) that the real part of this operator is proportional to the principal value of $((J_{\nu_k}J_{-\nu_k})^{-1})$. But returning to (3.1.24) and (3.1.30) we see that $\Phi(k)$ must have the same zeros as $J_{\nu_k}(x_k)J_{-\nu_k}(x_k)$ which guarantees that the contribution of the first term of (4.3) to (4.2) will vanish. The second term of (4.3) will also give a vanishing contribution to (4.2) for much the same reason. We see that the structures $\bar{G}_k(\phi)$, defined in (Appendix I) and the related

$$\tilde{G}(\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi'\ G_k^{(+)}(\phi,\phi')\cos\phi' \tag{4.5}$$

enter into this term of Γ as

$$\Gamma^{(2)} = \mathrm{Im}\int d^3k d^3k'\, \phi^*(\vec{k})\delta h_k \bar{G}_k(\phi)\tilde{V}(\vec{k}-\vec{k}')\tilde{G}_k(\phi)\delta h_{k'}\phi(\vec{k}')d\phi \tag{4.6}$$

A direct calculation using (3.1.19) yields

$$\bar{G}_k(\phi) = (\delta h_k)^{-1}\ (1 - \frac{\xi_k(\phi)}{\Delta_k(\eta)}\ ) \tag{4.7}$$

and

$$\tilde{G}_k(\phi) = (\delta h_k)^{-1}\ (1 - \frac{\xi_k{}^*(\phi)}{\Delta_k(\eta)}\ ) \tag{4.8}$$

where

$$\xi_k(\phi) = \gamma'_k(\phi)\ (\gamma^*{}_k(\phi)\ (\gamma'^*{}_k - \gamma'_k) + \gamma_k\gamma'_k \tag{4.9}$$

The further requirement $PG_k^{(+)} = G_k^{(+)}P = 0$ which is easily obtained from (3.1.18) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\ \xi_k(\phi) = \Delta_k(0) = \Delta^*{}_k(0). \tag{4.10}$$

Substitution of (4.7) and (4.8) into (4.6) with the use of (4.10) yields

$$\Gamma^{(2)} = \frac{1}{2}\int d^3k d^3k'\ \Phi(\vec{k})\delta h_k \ln\left(\frac{1}{\Delta_k(-\eta)}\ \Delta^*{}_k(-\eta)\right) \times$$

$$\int_{-\pi}^{\pi} d\phi(\xi_k(\phi)\xi^*{}_{k'}(\phi) + \xi^*{}_k(\phi)\xi_{k'}(\phi)))\ \delta h_{k'}\Phi(\vec{k}') \tag{4.11}$$

The ϕ integral is real so the operation of forming the imagionary part again yields a delta function, (4.4) and its symmetrization ($k \to k'$). Again the functions Φ in (4.11) will make this vanish.

The first non-vanishing term of Γ is $\Gamma^{(3)}$ arising from the third term of (4.3). It is

$$\frac{\Gamma^{(3)}}{2\pi} = \int d^3k d^3k' d^3k_1 \Phi(\vec{k})\ \delta h_k\ \tilde{V}(\vec{k}-\vec{k}_1)\tilde{V}(\vec{k}_1-\vec{k}')\ \delta h_{k'}\Phi(\vec{k})\ S(\vec{k}\vec{k}_1\vec{k}_1) \tag{4.12}$$

where, using (3.1.19) and (4.4)

$$S\ (\vec{k}\vec{k}_1\vec{k}') = \ln \int d\phi_1 d\phi_2 \bar{G}_k(\phi_1)G_{k_1}{}^{(+)}(\phi_1,\phi_2)G_{k'}{}^{(+)}(\phi_2). \tag{4.13}$$

The expressions (4.7) and (4.8) are inserted here and it is readily shown by using $PG^{+}=0$, that only the terms proportional ξ_k and $\xi_{k'}{}^*$ survive. Then using the second form of (A.1.32)

$$S(\vec{k}\vec{k}_1\vec{k}') = \ln(\delta h_k \delta h_{k'} \Delta_k(\eta)\Delta_{k'}(\eta))^{-1} \int d\phi_1 d\phi_2 \; x$$

$$\xi_k(\phi_1)\;(g_{k_1}{}^{(1)}(\phi_1,\phi_2) + \frac{g^{(2)}{}_{k_1}(\phi_1,\phi_2)}{\Delta_{k_1}(\eta)})\;\xi^*{}_{k'}(\phi_2). \tag{4.14}$$

The term proportional to $g_{k_1}{}^{(1)}$ can be shown to vanish since $g^{(1)}$ is hermitian and so the imaginary part operation again extracts the delta function, (4.4) (or $k \to k'$) which vanishes in the integral, (4.12) containing $\Phi(k)$. The remaining term in (4.13) is

$$S(\vec{k}\vec{k}_1\vec{k}') = (\delta h_k \delta h_{k'})^{-1}\, N(\vec{k}\vec{k}_1\vec{k}')\, \ln(\Delta_k(\eta)\Delta_{k_1}(\eta)\Delta_{k'}(\eta))^{-1} \tag{4.15}$$

where the real function N, is given by

$$N(\vec{k}\vec{k}\vec{k}') = \int d\phi_1 d\phi_2\; \xi_k(\phi_1) g^{(2)}{}_{k_1}(\phi_1,\phi_2)\, \xi^*{}_{k'}(\phi_2) \tag{4.16}$$

and in the context of the preceeding discussion, the factors proportional to the delta function (4.4) and ($k \to k'$) that arise from the imaginary part operation can be dropped. The function $\Delta_k(\eta)$ is given in (3.1.23), which using (3.1.24), (A.1.10) and the generating equation for Bessel functions can be written as

$$\Delta_k(\eta) = 2\sin\pi(\nu_k - i\eta) \sum_n \frac{J^2{}_n(x_k)}{n+\nu_k-i\eta}$$

$$= 2\pi\ J_{\nu_k-i\eta}(x_k)\ J_{-\nu_k+i\eta}(x_k) \qquad (4.17)$$

where we have used (3.1.28). A straight forward calculation yields (Appendix V)

$$\lim_{\eta\to 0} \Delta_k{}^{-1}(\eta) = \frac{p.p}{J_{\nu_k(x_k)}J^{-\nu_k(x_k)}} + i\pi\ \delta\ (J_{\nu_k}(x_k)J_{-\nu_k}(x_k))\ x$$

$$\text{syn}\ (L_{\nu_k}(x_k)J_{-\nu_k}(x_k) - L_{-\nu_k}(x_k)J_{\nu_k}(\nu_k)) \qquad (4.18)$$

where

$$L_\nu(x) = \frac{\partial}{\partial\nu}\ J_\nu(x) \qquad (4.19)$$

and it is easy to show from the series representations that

$$L_0(x) = \frac{\pi}{2}\ N_0(x) \qquad (4.20)$$

where N_0 is the Neuman function. Then the last factor of (4.15), is obtained as

$$\frac{\pi\delta(J_{\nu_{k_1}}(x_{k_1})J_{-\nu_{k_1}}(x_k))}{J_{\nu_k}(x_k)J_{-\nu_k}(x_k)J_{\nu_k}(x_k)J_{-\nu_{k'}}(x_k)} \text{ sgn } (L_{\nu_{k_1}}(x_{k_1})J_{-\nu_{k_1}}(s_{k_1}) -$$

$$- L_{-\nu_{k_1}}(x_{k_1})J_{\nu_{k_1}}(x_{k_1})) \tag{4.21}$$

where terms containing the non-contributing delta functions, (4.4), have been dropped. If we refer back to the discussion following (3.1.44) we see that the limit $\nu_k \to 0$ is taken, and justified by the result, (3.3.22). We follow a similar procedure in (4.21) where, with some care this limit yields for (4.21)

$$\frac{1}{J^2{}_0(x_k)J^2{}_0(x_{k_1})} \sum_i \frac{\delta(x_{k_1}-y_i(0))}{\left|y'_i(0)\right|(J'_0(y_i(0)))^2} \frac{\omega}{\nu_{k_1}{}^2} \text{ sgn } J'_0(y_i(0) \tag{4.22}$$

where the sum runs over the zeros of the J_0 Bessel function. We have defined the ι^{th} zero of J_ν $(y_i(\nu))$ and its derivative with respect to the order

$$\frac{d}{d\nu} y_i(\nu) = y_i'(\nu) \tag{4.23}$$

The evaluation of N, (4.16), with the aid of (3.1.19) and (4.9) is a straightforward but somewhat lengthy double integral. It is simplified by keeping only the leading terms in the expansion in powers of ν_k as was already done with reference to (3.1.14) and (4.22). We also drop terms which vanish because of the delta function in (4.22). The result is

$$N(\vec{k}\vec{k}_1\vec{k}') = \frac{(2\pi)^2}{\omega} \nu_{k_1} J_0(x_k) J_0(x'_k) J_0(x_{kk_1}) J_0(x_{k'k_1}) q(x_{k_1}) \tag{4.24}$$

where

$$q(x) = 2 \sum_{n=1}^{\infty} \frac{J_n^2(x)}{n^2} = {}_{\lim x \to \infty} \frac{\pi}{3|x|} \tag{4.25}$$

and $x_{kk_1} = x_k - x_{k_1}$.

If we assemble these the expression for $\Gamma^{(3)}$ becomes

$$\frac{\Gamma^{(3)}}{2\pi} = A^2 \sum_{i} \int d^3k d^3k_1 d^3k' U_1(\vec{k}) \tilde{V}(\vec{k}-\vec{k}_1) \tilde{V}(\vec{k}_1-\vec{k}') U_1(\vec{k}')$$

$$\times \; J_0(x_k) J_0(x_{k'}) J_0(x_{kk_1}) J_0(x_{k'k_1}) q(x_{k_1}) / (\omega \nu_{k_1} (J^1{}_0(y_i(o)))^2 \left| y^1{}_i(o) \right|) \tag{4.26}$$

where we have used (3.1.35), (3.2.1) and (4.20). The normalization factor A is obtained $(\Phi,\Phi) = 1$, which form (3.1.35) and (3.2.1) leads to

$$A^2 \int d^3k \; J^4{}_0(x_k) \; U^2{}_1(\vec{k}) = 1 \tag{4.27}$$

or for large $\lambda = \alpha\xi$

$$A^{-2} = \frac{2\pi}{3} \frac{\lambda^2}{\beta^6 \alpha^3} f \tag{4.28}$$

where

$$f = \int_0^{\infty} dt\ J^4{}_0(t) = 0.8996... \tag{4.30}$$

The evaluation of $\Gamma^{(3)}$ essentially requires the performance of a nine dimensional integral which is not for the faint hearted. In performing it we shall need to make one drastic approximation which, we believe, will preserve the correct behavior of Γ for large α but will probably change the coefficient. The approximation is that of replacing the zeros of the Bessel function by their large argument values. That is, we write

$$y_i{}^{(o)} = \pi(i + \tfrac{3}{4}) \approx \pi i, \quad \left|y_i{}^1(o)\right| = \frac{\pi}{2} \tag{4.31}$$

and

$$J^1{}_0(y_i{}^{(o)}) = \left(\frac{2}{\pi\left|y_i{}^{(o)}\right|}\right)^{1/2} (-1)^{i+1}$$

$$q(y_i{}^{(o)}) = \frac{\pi}{3\left|y_i{}^{(o)}\right|}. \tag{4.30}$$

If we scale all the $\vec{k}$ s in the integral by $\vec{k} = \xi\vec{\ell}$ then (4.26) becomes

$$\frac{\Gamma^{(3)}}{2\pi} = \frac{2mA^2}{\beta^6}\ \frac{\xi^3}{\lambda}\ \left(\frac{e^2}{2\pi^2}\right)^2 \sum_i \frac{\pi}{6}\ (-1)^{i+1}\ h(y_i{}^{(o)}) \tag{4.32}$$

where

$$h(y) = \int \int_{-\infty}^{\infty} \frac{d\ell_z d\ell'_z}{(\ell^2{}_z+1)(\ell'^2{}_z+1)} F(\frac{1}{\beta}(\ell_z - \frac{y_i}{\lambda}), \frac{1}{\beta}(\ell'_z - \frac{y_i}{\lambda})) \tag{4.33}$$

where

$$F(a_1 a_2) = \beta^6 \int \frac{d^2\ell \; d^2\ell_1 \; d^2\ell}{(\ell^{\;2}+\beta^2)(\ell'^2+\beta^2)^2} \left[\ell_1{}^2 + \frac{y_i{}^2}{\lambda^2} - \frac{2m\varepsilon}{\xi^2}\right] x$$

$$\frac{1}{\left|(\ell-\ell_1)^2+\beta^2 a_1{}^2\right|[(\ell'-\ell_1)^2+\beta^2 a^2{}_2]}. \tag{4.34}$$

It is easily seen that $h(y_i)$ is an even function of its argument so the sum in (4.32) can be taken as twice the sum over only positive values of i. This is then broken into even and odd values of i so that the net effect is

$$\sum_i (-1)^{i+1} h(y_i(o) = 2 \sum_{n=1}^{\infty} (h(2\pi n-\pi) - h(2\pi n)). \tag{4.35}$$

We shall see that large values of n contribute so that the sum can be converted to an integral and a Taylor series can be used

$$h(2\pi n-\pi) \approx h(2\pi n) - \pi \frac{\partial}{\partial(2\pi n)} h(2\pi n). \tag{4.36}$$

The integral is then simply done with the result

$$\sum_{i} (-1)^{i+1} h(y_i(o)) \approx h(2\pi) \qquad (4.37)$$

We use $2\pi/\lambda << 1$ and (4.28) to rewrite

$$\frac{\Gamma(3)}{2\pi} = R_y\ (4\pi^4 f)^{-1}\ h(o) \qquad (4.38)$$

Before proceding we note that the justification for this approximations comes from the fact that large $y_i(o)$ contribute to $\Gamma.^{(3)}$ The convergence of the sum in (4.32) comes from the third factor in (4.34) where it is seen that

$$y_i^1 \leq (-\frac{2m\varepsilon\lambda^2}{\xi^2}) = K^2 = \frac{8}{3\pi^2}\ \frac{\alpha}{a_o}(\ln\lambda)^2 >> 1 \qquad (4.39)$$

is the principal range of contribution of the sum.

For large λ, it is small ℓ_z which gives the major contribution to the integral in (4.33). (This is similar to the evaluation in (3.3.5). We may therefore perform the $\vec{\ell}$ and $\vec{\ell}'$ integrals in (4.33) for small values of ℓ_z and ℓ'_z. The result is

$$\int \frac{d^3\ell}{(\ell^{\ 2}+\beta^2)^2}\ \frac{1}{(\ell - \ell_{\ 1})^2+\ell^2{}_z} = \frac{\pi}{\ell_1{}^{\ 2}+\beta^2}\ (\ln \frac{\ell_1{}^{\ 2}+\beta^2}{\ell_z{}^2 - 1+O(\ell^2{}_z)} - 1 + O(\ell_z{}^2)) \qquad (4.40)$$

Then the $\vec{\ell}$ integral can be performed keeping only the dominant terms yielding

$$F\left(\frac{\ell_z}{\beta}, \frac{\ell'_z}{\beta}\right) = \frac{\pi^5}{3}\left(\ln\frac{\ell_z^{\,2}}{\beta^2}\right)\left(\ln\frac{\ell_z'^{\,2}}{\beta^2}\right)\left(\frac{\xi^2}{-2m\varepsilon}\right). \tag{4.41}$$

If this is substituted back into (4.33) we obtain

$$h(o) = \frac{\pi^2}{8}\,\frac{a_0}{\alpha}\left(\frac{\lambda}{\ln\lambda}\right)^2\left[\int_{-\infty}^{\infty}\frac{d\ell_z}{\ell_z^{\,2}+1}\,J^2{}_0(\lambda\ell_z)\,\ln\frac{\ell_z^{\,2}}{\beta^2}\right]^2. \tag{4.42}$$

The remaining integral can be evaluated in the limit of large λ by integration by parts.

$$\int_0^{\infty}\frac{d\ell}{\ell^2+1}\,J^2{}_0(\lambda\ell_z)\,\ln\frac{\ell}{\beta} \approx \frac{1}{\pi\lambda}\Big(-\frac{5}{6}(\ln\lambda)^2 - \frac{1}{2}\ln\lambda\,\Big(\frac{2}{3}(a_2-a_1)$$

$$+\frac{8}{3}\pi\mu\Big) + O(\ln\lambda)^0\Big) \tag{4.43}$$

where (3.3.19) has been used. Then assembling (4.33), (3.41), and (4.37) the result is

$$\Gamma(3) = \left(\frac{5\pi}{6}\right)^2\frac{1}{f}\,R_y\,\frac{a_0}{\alpha} \tag{4.44}$$

and using the first of (3.3.22)

$$\frac{\Gamma(3)}{|\varepsilon|} \approx \frac{1}{3f}\left(\frac{5\pi^2}{4}\right)^2\left[1 + O(\ln\lambda)^{-1}\right] \tag{4.45}$$

We see that the binding energy and width are the same order of magnitude as a function of α. As pointed out above, our calculation of the numerical coefficient of the width is not reliable so we can not conclude that this new kind of state which we have obtained is narrow. But we have shown that it does not disappear as the intensity becomes very high.

Finally we should point out that our assumption that the width is much less than the binding energy is not substantiated by our crude estimation of the width. We only find the two parameters to have the same behavior with large α. We do not expect this to change our basic result.

CONCLUSION

We should say at this point that our motivation in solving the problem of atomic structure and photoionization rate at a hydrogen atom in ultrastrong laser field is purely theoretical. There is no experimental results that could give more information concerning this problem.

However, rapid development of ultrastrong lasers encourages us to believe that experimental work on this subject will follow soon.

In the Introductory chapter we described previous methods and results that were obtained in calculations of the photoionization rate of hydrogen atom in ultrastrong laser .

Our approach have been to use variational calculation to find the upper bound of the energy minimum of a hydrogen atom for two different polarizations of ultrastrong laser field.

The result in Chapter III indicates the existence of the metastable state in the case of linearly polarized radiation field. For the circularly polarized radiation field the upper bound of the energy minimum is positive so that we are not able to find a bound state of the system.

In Chapter IV we calculated ionization rate as the width of the bound state obtained in Chapter III. Very lengthy analytical calculations gives a width as a function of laser intensity but numerous approximations made a numerical value somewhat uncertain. So we can see that there is some room for improvement left in obtaining better accuracy in ionization rate calculations.

We can also use more complex trial function (with some more variational parameters) which might give a different result for the circularly polarized case. The interpretation of the existance of a metastable state of hydrogen atom in ultrastrong laser fields needs a deeper understanding.

There are some indications (see Casati IV ICOMP Bolder 1987) that this unusual result could be placed in the context of quantum suppression of classical chaos. This all indicates that more research needs to be done on this very interesting subject.

APPENDIX I

Equation 3.8 could be rewritten as

$$(-i\varepsilon\frac{\partial}{\partial\phi} + 1 - i\eta\,\frac{\varepsilon}{\omega} + a\cos\phi)\ G_k{}^{(+)}(\phi,\phi') = -$$

$$- \frac{\varepsilon}{\omega}\ (\delta(\phi-\phi') - \frac{1}{2\pi}) + a\bar{G}_k{}^{(+)}(\phi') \tag{A.1.1}$$

where η is real and

$$P\bar{G}_k{}^{(+)} = 0,\ \bar{G}_k{}^{(+)} = P\cos\phi G_k{}^{(+)}(\phi,\phi') \tag{A.1.2}$$

Let us define

$$G_k{}^{(+)}(\phi,\phi') = e^{\frac{\eta}{\omega}(\phi-\phi')}\ G_{1k}{}^{(+)}(\phi,\phi') \tag{A.1.3}$$

so that

$$(-i\varepsilon\frac{\partial}{\partial\phi} + 1 + a\cos\phi)\ G_k{}^{(+)}(\phi,\phi') =$$

$$- (\delta(\phi-\phi') - \frac{1}{2\pi})\ \frac{\varepsilon}{\omega}\ e^{\frac{\eta}{\omega}(\phi-\phi')} + ae^{\frac{\eta}{\omega}(\phi-\phi')} + ae^{\frac{\eta}{\omega}(\phi-\phi')}\ \bar{G}^{(+)}{}_k\ (\phi') \tag{A.1.4}$$

Let us assume that the solution for the differential equation is given in this form

$$G^{(+)}{}_{1k}\ (\phi,\phi') = \gamma'^{*}(\phi)\gamma'(\phi')g(\phi,\phi') \tag{A.1.5}$$

where

$$\gamma'(\phi) = e^{\frac{i}{\varepsilon}\phi + i\frac{a}{\varepsilon}\sin\phi}, \quad \gamma'^{*}(\phi) = \gamma'^{-1}(\phi) \tag{A.1.6}$$

so that

$$\left(-i\varepsilon\frac{\partial}{\partial\phi} + 1 + a\cos\phi\right)\gamma'^{*} = 0$$

Substitution of (A.5) in (A.4) leads to

$$g(\phi,\phi') = -\frac{i}{2\omega}\,\theta\,(\phi-\phi') + C(\phi') + \frac{i}{2\pi\omega}\int_{-\pi}^{\phi} ds\; e^{\frac{\eta}{\omega}(s-\phi')}\,\gamma'(s)\gamma'^{*}(\phi')$$

$$+\, ia\nu_k \int_{-\pi}^{\phi} ds\; e^{\frac{\eta}{\omega}(s-\phi')}\,\gamma'(s)\gamma'^{*}(\phi')\,\bar{G}_k^{(+)}(\phi') \tag{A.1.7}$$

where

$$\nu_k = \frac{1}{\varepsilon(k)}, \quad \theta(\phi-\phi') = \mathrm{sgn}(\phi-\phi')$$

That means that

$$G_{1k}^{(+)} = -\frac{i}{2\omega}\,\gamma'^{*}(\phi)\gamma'(\phi')\theta(\phi-\phi') + C(\phi')\gamma'^{*}(\phi)\gamma'(\phi') + \frac{i}{2\pi\omega}\gamma'^{*}(\phi)\ \mathrm{x}$$

$$\mathrm{x}\int_{-\pi}^{\phi} ds\; e\,\frac{\eta}{\omega}\,(s-\phi')\,\gamma'(s) + ia\nu_k\gamma'^{*}(\phi)\ \mathrm{x}$$

$$\times \int_{-\pi}^{\phi} ds\ e^{\frac{\eta}{\omega}(s-\phi')}\ \gamma'(s)\ \bar{G}_k^{(+)}(\phi') \tag{A.1.8}$$

there are two unknown functions that should be determined $C(\phi')$ and $\bar{G}^{(+)}(\phi')$. Applying

$$PG_k^{(+)}(\phi,\phi') = 0 \tag{A.1.9}$$

Due to the fact that $G^{(+)}$ is completely in a Q space. We have

$$\int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\ e^{-\frac{\eta}{\omega}(\phi-\phi')}\ G_{1k}^{(+)}(\phi,\phi') = 0 \tag{A.1.10}$$

which gives

$$-\frac{i}{2\omega}\ e^{\frac{\eta}{\omega}\phi'}\gamma'(\phi') \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\ e^{-\frac{\eta}{\omega}\phi}\ \gamma'^{*}(\phi)\theta(\phi-\phi') + C(\phi')\gamma'(\phi')e^{\frac{\eta}{\omega}\phi'} \times$$

$$\times \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\ e^{-\frac{\eta}{\omega}\phi}\ \gamma'^{*}(\phi) + \frac{i}{2\pi\omega}\ e^{\frac{\eta}{\omega}\phi}$$

$$\int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\ e^{-\frac{\eta}{\omega}\phi}\ \gamma'^{*}(\phi) \int_{-\pi}^{\phi} ds\ e^{\frac{\eta}{\omega}(s-\phi')}\gamma'(s)$$

$$+ \, ia\nu_k\bar{G}_k^{(+)}(\phi') \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \, e^{-\frac{\eta}{\omega}\phi}\gamma'^*(\phi) \int_{-\pi}^{\phi} ds e^{\frac{\eta}{\omega}s}\gamma'(s) = 0 \tag{A.1.11}$$

Let us define

$$\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \, \gamma'^*(\phi)\theta(\phi-\phi') = \frac{1}{2\pi} \, (\gamma^*-2\gamma^*(\phi')) \tag{A.1.12}$$

where $\gamma = \gamma(\pi)$ and $\gamma'(\pi) = \gamma'$. Also

$$\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \, \gamma'^* = \frac{\gamma^*(\pi)}{2\pi} = \frac{\gamma*}{2\pi} = \frac{\gamma}{2\pi} \tag{A.1.13}$$

and γ is real.

We will need the integral

$$\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \, \gamma'^*(\phi) \, \gamma(\phi) \quad z \tag{A.1.14}$$

$\gamma'(\phi)$ is defined by (A.1.6) so that

$$Z = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \, e^{-\frac{i}{\varepsilon}(\phi+a\sin\phi)} \int_{-\pi}^{\phi} d\phi' e^{\frac{i}{\varepsilon}(\phi'+a\sin\phi')} \tag{A.1.15}$$

where

$$\varepsilon = \frac{\omega}{\varepsilon_k - W + U_p}, \; a = \frac{\omega \alpha k_i}{\varepsilon_k - W + U_p}, \; \frac{a_k}{\varepsilon_k} = \alpha k_i = x_k$$

To evaluate this integral we make use of Jacobi's expansion[7]

$$e^{\pm a \sin\phi} = \sum_{n=-\infty}^{\infty} (-1)^n \, e^{in\phi} J_n(a) \qquad (A.1.16)$$

so that the calculation of

$$Z = \sum_{n1} \sum_{n2} \frac{J_{n_1} J_{n_2}}{(\frac{1}{\varepsilon}+n_2)i} \left\{ \int_{-\pi}^{\pi} {}^{-i\phi(n_1 n_2)} \frac{d\phi}{2\pi} - e^{-i\,\frac{\pi}{\varepsilon}} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \, e^{-i\phi(\frac{1}{\varepsilon}\, n_1)} \right\} \qquad (A.1.17)$$

so that

$$Z = -i \sum_n \frac{J_n^2(x)}{n+\frac{1}{\varepsilon}} + ie^{\frac{i\pi}{\varepsilon}} \frac{\sin\frac{1}{\varepsilon}}{\pi} \left(\sum_n \frac{(-1)^n J_n(x)}{n+\frac{1}{\varepsilon}} \right)^2 \qquad (A.1.18)$$

Also

$$\mathrm{Re}(\Upsilon' Z) = \sin\frac{\pi}{\varepsilon} \sum_n \frac{J_n^2(\frac{a}{\varepsilon})}{n+\frac{1}{\varepsilon}} \qquad (A.1.19)$$

In order to simplify (A.1.11) let us define

$$I_1^* \, (\eta) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \, e^{\,\eta\nu_k\phi} \, \gamma'^*(\phi) \tag{A.1.20}$$

and

$$I_2(\eta) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \, e^{\eta\nu_k\phi} \, \gamma'^*(\phi) \int_{-\pi}^{\phi} ds e^{-\eta\nu_k s} \, \gamma'(s) \tag{A.1.21}$$

Also

$$I_1^*(\phi) = I_1^*(\phi_1\eta) = \int_{-\pi}^{\phi} \frac{d\phi'}{2\pi} \, e^{\eta\nu_k\phi'} \, \gamma'^*(\phi') \tag{A.1.22}$$

and

$$I_1^* \, (\pi) = I_1^*$$

Now we can write A.1.11 as

$$\frac{i\nu_k}{2} \, e^{-\eta\nu_k\phi'} \, [I_1^*(\eta) - 2I_1^*(\phi')] \, \gamma'(\phi') + \quad e^{-\eta\nu_k\phi'} C(\phi') \, x$$
$$x \, \gamma'(\phi') I_1^*(\eta) - i/2\pi \, \nu_k I_2(\eta) + ia\nu_k I_2(\eta) \bar{G}_k^{(+)}(\phi') = 0 \tag{A.1.23}$$

To find the other equation for evaluating $C(\phi')$ and $G^-{}_k{}^{(+)}(\phi')$ let us operate by P on the original integrodifferential equation. The result is easy to obtain

$$G_k^{(+)}(\pi,\phi) = -G_k^{(+)}(-\pi,\phi') \tag{A.1.24}$$

or

$$e^{2\pi\eta\nu_k} G_{1k}^{(+)}(\pi,\phi') = G_{1k}^{(+)}(-\pi,\phi') \tag{A.1.25}$$

Combining A.1.20 and A.1.8 one gets

$$C(\phi')\gamma'(\phi')(\gamma'^* e^{2\pi\eta\nu_k} - \gamma') + G^-{}_k{}^{(+)} e^{\eta\nu_k\phi'}(i2\pi a\nu_k e^{2\pi\eta\nu_k} - \gamma'^* I_1(-\eta)) = \frac{1}{2}\nu_k(\gamma'+\gamma'^* e^{2\pi\eta\nu_k})\gamma'(\phi') + + i\,\nu_k e^{2\pi\eta\nu_k}\gamma'^* I_1(-\eta) e^{\eta\nu_k\phi'} \tag{A.1.26}$$

Now we have two equations that can give us unknown functions $C(\phi')$ and $\bar{G}_k^{(+)}(\phi')$. After some algebraic manipulations one can get

$$\bar{G}_k^{(+)}(\phi') = \frac{I_1^*(\eta)}{ia\Delta(\eta)}\nu_k \{-\frac{i}{2}\nu_k(\gamma'+\gamma'^* e^{2\pi\eta\nu_k})\gamma'(\phi')e^{-\frac{\eta}{\varepsilon}\phi'} + i\nu_k\gamma'^* I_1(-\eta)e^{2\pi\eta\nu_k} + \frac{i\nu_k}{2I_1(\eta)} e^{-\eta\phi\nu_k}(I_1^*(\eta) - 2I_1(\phi')) (\gamma'^* e^{2\pi\eta\nu_k} - \gamma') - \frac{i}{2\pi}\nu_k \frac{I_2(\eta)}{I_1^*(\eta)}(\gamma'^* e^{2\pi\eta\nu_k} - \gamma')\} \tag{A.1.27}$$

where

$$\Delta_k(\eta) = I_2(\eta)(\gamma' - \gamma'^* e^{2\pi\eta\nu_k}) + e^{2\pi\eta\nu_k} 2\pi\gamma'^* I_1(-\eta)I_1(\eta) \tag{A.1.28}$$

The other function is

$$C(\phi') = \frac{I_2(\eta)}{\Delta(\eta)\gamma(\phi')} 2\gamma'(\phi') [I_1^*(\eta)\gamma' + I_1^*(\phi') (\gamma'^* e^{2\pi\eta\nu_k})]$$

$$- \frac{i\nu_k}{2I_1^*(\eta)} (I_1^*(\eta) - 2I_1^*(\phi') \gamma'(\phi') \tag{A.1.29}$$

Notice here tha for $\eta \to 0$

$$I_2 \to z \text{ and } I_1^*(\eta) \to \gamma \tag{A.1.30}$$

If we use that limit everywhere except in the denominator, where the limit will produce a principle value and delta function in the usual way.

$$G_k^{(+)}(\phi,\phi') = -\frac{i}{2\omega} \gamma_{k'}^*(\phi)\gamma_{k'}(\phi)\theta(\phi-\phi') -$$

$$- \frac{i}{2\omega\Delta_k(\eta)} \gamma_{k'}^*(\phi)\gamma_{k'}(\phi') \{-\gamma_k(\phi)\gamma_k^*(\phi')(\gamma_{k'*}-\gamma_{k'}) -$$

$$- (z_k^*\gamma_{k'}^* - z_k\gamma_{k'}) + \frac{1}{\pi} \gamma_k (\gamma_{k'}^*(\phi') - \gamma'_k\gamma_k(\phi)\} \tag{A.1.31}$$

or

$$G_k^{(+)}(\phi_1\phi') = g_k^{(1)}(\phi,\phi') + \frac{1}{\Delta_k(\eta)} g_k^{(2)}(\phi,\phi') \tag{A.1.32}$$

Thereby defining g_1 and g_2.

APPENDIX II

We defined the modified potential v Eq. (3.1.11) as

$$v = P\delta h G_Q \delta h P \tag{A.2.1}$$

After replacing G_Q by $G_k^{(+)}$ where the approximation $\delta h >> V$ has been introduced

$$v(k) = \frac{1}{(2\pi)^2} \frac{(\omega\alpha k_z)^2}{\varepsilon - \varepsilon_k} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos\phi G^{+}{}_k(\phi,\phi')\cos\phi' d\phi d\phi' \tag{A.2.2}$$

We defined (see Appendix I)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi' \cos\phi' G^{(+)}{}_k(\phi,\phi') = \bar{G}^{(+)}{}_k(\phi) \tag{A.2.3}$$

so that

$$v(k) = \frac{(\omega\alpha k_z)^2}{\varepsilon - \varepsilon_k} I \tag{A.2.4}$$

where

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\cos\phi\bar{G}^{(+)}{}_k(\phi) \tag{A.2.5}$$

From (Appendix I) in the limit of $\eta \to 0+$

$$\bar{G}^{(+)}{}_k = \frac{1}{2\pi a}\left[1 - \frac{\gamma'_k(\phi)[\gamma^*_k(\phi)(\gamma'^*_k-\gamma'_k)-\gamma_k\gamma'_k]}{Z\gamma'-Z^*\gamma^{*\prime}}\right] \tag{A.2.6}$$

so that

$$I = \frac{1}{2\pi a(Z\gamma'+Z^*\gamma'^*)}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\cos\phi\gamma'_k(\phi)[\gamma^*_k(\phi)(\gamma'^*_k-\gamma'_k) + \gamma_k\gamma'_k] \tag{A.2.7}$$

Let us integrate by parts

$$\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\cos\phi\gamma'(\phi) = -\frac{i\varepsilon}{2\pi a}\left[\gamma' - \gamma'^* - \frac{i\gamma}{\varepsilon}\right] \tag{A.2.8}$$

and

$$\int_{-\pi}^{\pi}\cos\phi\gamma'(\phi)\gamma^*(\phi) = -\frac{i\varepsilon}{2\pi a}\left[\gamma'\gamma - 2\pi - \frac{2\pi i}{\varepsilon}z^*\right] \tag{A.2.9}$$

Here we used the definition of γ and z given in Appendix I. Integral I is then equal to

$$I = \frac{1}{2\pi a^2} + \frac{i\varepsilon}{2\pi a^2}\,\frac{\gamma'-\gamma'^*}{Z\gamma'+Z^*\gamma'^*} \tag{A.2.10}$$

Notice that $\gamma' = e^{i\pi/\varepsilon}$ so

$$I = \frac{1}{2\pi a^2}\,(1 - \frac{2\varepsilon\sin\frac{\pi}{\varepsilon}}{R_e\Delta_k(0)}) \tag{A.2.11}$$

Here $Re\Delta_k(0) = 2Re\gamma' z$ where, again using the result (A.1.10) (Appendix I)

$$R_e\Delta_k(0) = 2\sin\frac{\pi}{\varepsilon} \sum_{n=\infty}^{\infty} \frac{J^2{}_n(\frac{a}{\varepsilon})}{n+\frac{1}{\varepsilon}} \tag{A.2.12}$$

so that

$$I = \frac{1}{2\pi a^2}\left[1 - \frac{\varepsilon}{P(\frac{a}{\varepsilon},\varepsilon)}\right] \tag{A.2.13}$$

At this point we have to evaluate the sum

$$P = \sum_{n=-\infty}^{\infty} \frac{J^2{}_n(x)}{Z+n} \tag{A.2.14}$$

where $z = 1/\varepsilon$ and $x = a/\varepsilon$.

Consider the sum

$$S = \sum_{n=-\infty}^{\infty} J^2{}_n(x) \left(\frac{1}{z} + \frac{1}{n-z}\right) \tag{A.2.15}$$

which can be written as

$$S = \frac{1}{z}\,(1 - J^2{}_0(x)) + \sum_{n \neq 0} \frac{J^2{}_n(x)}{n-z} \tag{A.2.16}$$

We see here that the second term has simple poles for $z = n$ (n = non zero integer) and that the first term has pole at $z = 0$.

Consider the function

$$-\frac{\pi}{\sin \pi Z}\, J_Z(x) J_{-Z}(x) = B(z) \tag{A.2.17}$$

that has the same poles as the second term. Then the expression $\frac{1}{z} - B(z)$ has the same poles as the sum S. Also we have to prove that this function is analytic for $z \to \infty$. From (G & R pp 960) one can find

$$J_Z(x) J_{-Z}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k} (2k)!}{(k!)^2 \Gamma(k+1-z)\Gamma(k+1+z)} \tag{A.2.18}$$

and (Abranowitz & Stegun pp.   )

$$\Gamma(k+1+z)\Gamma(k+1-z) = (1-z^2)(2^2-z^2)\ldots\frac{\pi z}{\sin\pi} \qquad (A.2.19)$$

So one can write that

$$J_z(x)J_{-z}(x) = \frac{\sin\pi z}{\pi z}\left[\sum_{k=1}^{\infty} \frac{(-1)^k(\frac{x}{2})^{2k}}{(k!)^2} \prod_{\ell=1}^{k} \frac{1}{\ell^2-z^2} + 1\right] \qquad (A.2.20)$$

which makes the expression

$$\frac{1}{z} - \frac{\pi}{\sin\pi z} J_z(x)J_{-z}(x) \sim 0\left(\frac{1}{z^3}\right) \qquad (A.2.21)$$

when $z \to \infty$.

So we see that we can write

$$S \sim \frac{1}{z} - \frac{\pi}{\sin\pi z} J_z(x)J_{-z}(x) \qquad A.2.22)$$

and that

$$\sum_n \frac{J^2{}_n(x)}{n-z} = -\frac{\pi}{\sin\pi z} J_z(x)J_{-z}(x) \qquad (A.2.23)$$

Finally we can write that

$$P = \sum_n \frac{J^2{}_{n}(a\ \varepsilon)}{n+1\ \ \varepsilon} = \frac{\pi}{\sin \frac{\pi}{\varepsilon}} J_{1\ \varepsilon}\left(\frac{a}{\varepsilon}\right) J_{-1\ \varepsilon}\left(\frac{a}{\varepsilon}\right) \qquad \text{(A.2.24)}$$

For v small and defining $\varepsilon = W - U_p\ \eta \to 0$, $P\psi = \phi$ we get (3.1.34).

APPENDIX III

The trial function

$$u_1(\vec{k}) = \frac{\xi^6}{(k_z^2+\xi^2)(k^2+\xi^2\beta^2)^2} \tag{A.3.1}$$

can be written as

$$u_1(\vec{k}) = \frac{1}{((\frac{k_z}{\xi})^2 + 1)\ ((\frac{k}{\xi})^2 + \beta^2)^2} \tag{A.3.2}$$

where $\beta = \eta/\xi$.

Let's calculate the normalization integral N, Eq. (3.27). With the substitution $\lambda = \alpha\xi$ and $\frac{\vec{k}}{\xi} = \vec{\ell}$, one gets

$$\langle N\rangle = 2\pi\xi^3 \int_0^\infty \frac{d\ell\ \ell}{[\ell^2+\beta^2]^4} \int_{-\infty}^\infty \frac{d\ell_z}{(\ell_z^2+1)^2}\ J^2{}_0(\lambda\ell_z) \tag{A.3.4}$$

or

$$\langle N\rangle = 2\pi\xi^3\ \frac{2}{6\beta^6}\ I_1 \tag{A.3.5}$$

Let us integrate I_1 by parts

$$I_1 = \int_{-\infty}^{\infty} \frac{d\ell_Z}{(\ell_Z{}^2+1)^2} J^2{}_0\,(\lambda \ell_Z) \tag{A.3.6}$$

If we integrate

$$I = \int_0^{\infty} \frac{d\ell}{\ell^2+1} J^2{}_0(\lambda\ell) \tag{A.3.7}$$

by parts and then take a derivative with respect to the parameter in the denominator one will get value of I_1 making $du = d\ell\ J^2{}_0(\lambda\ell)$, $v = \frac{1}{\ell^2+1}$, so that $u = \frac{1}{\lambda} f_1(\lambda\ell)$, $dv = v'd\ell$ where

$$f_1(x) = \int_0^x d_t J^2{}_0(t) = \int_0^x dt\left[J^2{}_0(t) - \frac{\theta(t-1)}{\pi t}\right] + \frac{1}{\pi}\ x\ \ell n x \theta(x-1) \tag{A.3.8}$$

or

$$f_1(x) = \bar{f}_1(x) + \frac{1}{\pi}\theta(x-1)\ell n x \tag{A.3.9}$$

Here we need an asymptotic form of $J^2{}_0(x) \to \frac{2}{\pi x}$ for $x \to \infty$.

$$I = -\frac{1}{\lambda}\int_0^{\infty} d\ell v'(f_1(\lambda\ell) + \frac{1}{\pi}\theta(\lambda\ell-1)\ell n\lambda\ell) \tag{A.3.10}$$

or

$$I = \frac{1}{\pi\lambda}\,\ell n\lambda - \frac{1}{\lambda}\int_0^\infty d\ell v'\bar{f}_1(\lambda\ell) \tag{A.3.11}$$

Integrating by parts once more we get

$$I = \frac{1}{\pi\lambda}\,\ell n\lambda + \frac{1}{\lambda^2}\int_0^\infty d\ell v''f_2(\lambda\ell) \tag{A.3.12}$$

where $f_2(x) = \int_0^x dt\bar{f}_1(t)$.

Now

$$f_2(x) = \int_0^x dt \int_0^t ds \left[J^2{}_0(s) - \frac{\theta(s-1)}{\pi s}\right] \tag{A.3.13}$$

and it could be written as

$$f_2(x) = \chi_1 x - \chi_2 + f_2(x) \tag{A.3.14}$$

where

$$\chi_1 = \int_0^\infty ds \; (J^2{}_0(s) - \frac{\theta(s-1)}{\pi s} \approx 0.8455 \tag{A.3.15}$$

The value for χ_1 is obtained numerically on VAX 780 computer.

$$\chi_2 = \int_0^\infty dx \int_x^\infty ds\ (J^2_0(s) - \frac{\theta(s-1)}{\pi s}) \qquad (A.3.16)$$

Integrating by parts one can get

$$\chi_2 = \int_0^\infty dx \left[xJ^2_0(x) - \frac{1}{\pi}(1 + \sin 2x)\right] + \frac{1}{2\pi} + \int_0^1 \frac{dx}{\pi} \qquad (A.3.17)$$

The first two terms vanish after numerical integration has been performed. So that

$$\chi_2 = \frac{1}{\pi} \qquad (A.3.18)$$

$$\bar{f}_2(x) = \int_x^\infty dt \int_t^\infty ds\ (J^2_0(s) - \frac{\theta(s-1)}{\pi s}) \qquad (A.3.19)$$

This leads to

$$I = \frac{1}{\pi\lambda}\ \ell n\lambda + \frac{\chi_1}{\lambda} + \frac{1}{\lambda^2}\int_0^\infty d\ell v''\bar{f}_2(\lambda\ell) \qquad (A.3.20)$$

The last term could be done again by parts which leads to term of order

$0(\frac{1}{\lambda^3})$

So

$$I = \frac{1}{\pi\lambda}\ \ell n\lambda + \frac{\chi_1}{\lambda} + 0\ (\frac{1}{\lambda^3}) \tag{A.3.21}$$

The final step in evaluating I_1 is

$$I_1 = \lim_{\beta^2\to 1} \left[\frac{\partial}{\partial\beta^2}\int_0^\infty \frac{d\ell}{\ell^2+\beta^2}\ J^2{}_0\ (\lambda\ell)\right] \tag{A.3.22}$$

or

$$I_1 = \frac{1}{\pi\lambda}\ \ell n\lambda + \frac{1}{2\lambda}\ (2\chi_1 - \frac{1}{\pi}) + 0(\frac{1}{\lambda^3}) \tag{A.3.23}$$

Substituting back in A.3.3

$$\langle N\rangle = \frac{2\lambda^2}{3\beta^6\alpha^3}\ (\ell n\lambda + \pi\chi_1 - \frac{1}{2} + 0(\frac{1}{\lambda^2})) \tag{A.3.24}$$

Kinetic energy part of Eq. (3.26)

$$\langle T\rangle = \int d^3k((u_1(k))^2 J^2{}_0(x_k)\varepsilon_k \tag{A.3.25}$$

can be done by using results obtained in calculating $\langle N\rangle$

$$\langle T\rangle = 2\pi \frac{\xi^5}{2m} \int_0^\infty \frac{d\ell\ \ell}{[\ell^2+\beta^2]^4} \int_{-\infty}^\infty d\ell_z \frac{J^2{}_0(\lambda\ell_z)}{(\ell_z{}^2+1)^2} (\ell^2+\ell_z{}^2) \tag{A.3.26}$$

or

$$\langle T\rangle \frac{\pi\xi^5}{m} 2 \int_0^\infty d\ell_z\ J^2{}_0(\frac{\lambda\ell_z}{(\ell_z{}^2+1)^2} \left[\frac{\ell_z{}^2}{6\beta^6} + \frac{1}{12\beta^4}\right]$$

$$= \frac{\pi\xi^5}{3m} \left[\frac{1}{\beta^6} I_2 + \frac{1}{2\beta^4} I_1\right] \tag{A.3.27}$$

where

$$I_2 = \int_0^\infty \frac{d\ell\ \ell^2}{(\ell^2+1)^2} J^2{}_0(\lambda\ell) = I - I_1 \tag{A.3.28}$$

so that

$$\langle T\rangle = \frac{\lambda^4}{6\beta^6\alpha^5 m} \left[\beta^2 \ell n\lambda + \beta^2(\pi\chi-\frac{1}{2}) + 1 + 0(\frac{1}{\lambda^2})\right] \tag{A.3.29}$$

The last part in the energy evaluation is the evaluation of the V integral

$$\langle V\rangle = -\frac{e^2}{2\pi^2} \xi^4 \int_{-\infty}^\infty \int_{-\infty}^\infty d\ell_z d\ell'_z J^2{}_0(\lambda\ell_z) G(\ell_z,\ell'_z)\ J^2{}_0(\lambda\ell'_z) \tag{A.3.30}$$

where

$$G(\ell_z,\ell'_z) = \int_0^\infty \int_0^\infty d\ell\, d\ell'\, \ell\, \ell' \int_0^{2\pi} \int_0^{2\pi}$$

$$\frac{d\phi d\phi'}{(\ell^2{}_z+1)(\ell^2 + \beta^2)^2[(\ell_z-\ell'_z)^2 + (\vec{\ell} - \vec{\ell}'\ )^2](\ell'_z+1)(\ell\ '^2+\beta^2)^2} \qquad \text{(A.3.31)}$$

or

$$G(\ell_z,\ell'_z) = \frac{1}{(\ell_z{}^2+1)(\ell'^2{}_z+1)}\, g((\ell_z-\ell'_z)^2) \qquad \text{(A.3.32)}$$

After performing ϕ, ϕ' integrations we get

$$g(x^2) = (2\pi)^2 \int_0^\infty \int_0^\infty \frac{d\ell\, d\ell'\, \ell\, \ell'}{(\ell\ ^2+\beta^2)(\ell\ '^2+\beta^2)^2}\, (x^4+2x^2(\ell\ ^2+\ell'^2\ ) + + (\ell\ ^2-\ell'^2))^{-1/2} \qquad \text{(A.3.33)}$$

V integral can be written as (3.32). $g(x^2)$ is the integral of the Coulomb potential over the components of the momenta perpendicular to the $\hat{z}$ axis. Let's rewrite $g(x^2)$ so that

$$g(x^2) = \frac{8\pi^2}{4\beta^4} \int_0^\infty \frac{ydy}{(y+1)^2} \int_0^1 \frac{ds}{(sy+1)^2}\, (x^4+2x^2\beta^2 y(1+s) + + r^2\beta^4(1-s)^2)^{-1/2} \qquad \text{(A.3.34)}$$

The dominant contribution is for $x \to 0$ near $s \to 1$. Setting $s = 1$ where possible and $x^4 << x^2\beta^2$.

$$g(x^2) = \frac{8\pi^2}{4\beta^4}\int_0^\infty \frac{ydy}{(y+1)^2})\int_0^1 \frac{ds}{(y+1)^2}\,(4x^2\beta^2y + y^2\beta^4(1-s)^2)^{-1/2} \qquad (A.3.35)$$

Now we can express $\langle V\rangle$ as a function of $g(x^2)$ in this form

$$\langle V\rangle = -\frac{e^2\xi^4}{\pi^2}\int_0^\infty\int_0^\infty \frac{d\ell d\ell' J^2{}_0(\lambda\ell)J^2{}_0(\lambda\ell')}{(\ell^2+1)\;(\ell'^2+1)}\,[g((\ell+\ell')^2) + $$
$$+ g\;(\ell-\ell'))^2] \qquad (A.3.36)$$

In order to perform this integration let's use the same approach as for evaluation of integral I in Appendix I.

Let

$$J^2{}_0(\lambda\ell) = \frac{\theta(\lambda\ell-1)}{\pi\lambda\ell} + (J^2{}_0(\lambda\ell) - \frac{\theta(\lambda\ell-1)}{\pi\lambda\ell}) \qquad (A.3.37)$$

and similarly for $J^2{}_0(\lambda\ell')$. We can denote integration region as one from 0 to $\frac{1}{\lambda}$ and $\frac{1}{\lambda} \to \infty$. That leaves us with three integrals to perform

$$\langle V\rangle = V_{11} + 2V_{12} + V_{22} \qquad (A.3.38)$$

where

$$V_{11} = \frac{e^2\xi^4}{\pi^4\lambda^2}\int_{\frac{1}{\lambda}}^{\infty}\int_{\frac{1}{\lambda}}^{\infty}\frac{d\ell d\ell'}{\ell\ell'(\ell^2+1)(\ell'^2+1)}[g((\ell+\ell')^2) + g((\ell-\ell')^2)] \quad (A.3.40)$$

$$V_{12} = -\frac{e^2\xi^4}{\pi\lambda}\frac{2\pi^2}{\beta^4}\int_{\frac{1}{\lambda}}^{\infty}\frac{d\ell}{\ell(\ell^2+1)}\int_0^{\infty}\frac{d\ell'}{(\ell'^2+1)}(J^2{}_0(\lambda\ell') - \frac{\theta(\lambda\ell'-1)}{\pi\lambda\ell'})$$

$$\times [\gamma(\ell+\ell')^2) + \gamma((\ell-\ell')^2)] \quad (A.3.41)$$

where we set

$$g(x^2) = \frac{2\pi^2}{\beta^4}\gamma(x^2) \quad (A.3.42)$$

and

$$V_{22} = -\frac{2e^2\xi^4}{\beta^4}\int\int^{\infty}\frac{d\ell d\ell'}{(\ell^2+1)(\ell'^2+1)}[J^2{}_0(\lambda\ell - \frac{\theta(\lambda\ell-1)}{\pi\lambda\ell}][J^2{}_0(\lambda\ell') - \theta(\lambda\ell')]$$

$$\times [\gamma(\ell+\ell')^2+\gamma(\ell-\ell')^2] \quad (A.3.43)$$

Evaluation of these integrals is done using approximations that $\lambda \to \infty$ which is justified for high intensity lasers and keeping terms in the expansion up to the order of $(\ell n x)^2$.

In order tò evaluate those integrals let us write

$$\gamma((\ell+\ell')^2) + \gamma(\ell-\ell')^2) = -\frac{1}{3\beta^2}\left[\frac{\ell n\ (\ell^2-\ell'^2)}{\beta^2} - \frac{1}{6}\right] + q(\ell,\ell') \quad (A.3.44)$$

where $q(\ell_1\ell')$ is defined by the difference between (3.3.7) and (A.3.44). $q(\ell_1\ell')$ has the property that it vanishes for ℓ or ℓ' equal zero. It therefore contributes a small correction to V_{11} when $\lambda = \alpha\xi$ is large.

Evaluating V_{11}

$$V_{11} = \frac{2e^2\xi^4}{\pi^2\lambda^2\beta^4} \int \int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell d\ell'}{\ell\ell'(\ell^2+1)(\ell'^2+1)} \{- \frac{1}{3\beta^2} \ln \frac{\ell^2-\ell'^2}{\beta^2} +$$

$$+ \frac{1}{18\beta^2} + \xi(\ell,\ell')\} \tag{A.3.45}$$

The integration of the first part

$$V_{11a} = - 2\frac{e^2\xi^4}{\pi^2\lambda^2\beta^4} \{- \frac{2}{3\beta^2} \int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell}{\ell(\ell^2+1)} \int_{\frac{1}{\lambda}}^{\ell} \frac{d\ell}{\ell'(\ell'^2+1)} \ln (\ell^2-\ell'^2) +$$

$$+ (\frac{1}{3\beta^2} \ln \beta^2 + \frac{1}{18\beta^2}) (\int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell}{\ell(\ell^2+1)})^2 \tag{A.3.46}$$

is performed in the approximation of large λ so that

$$V_{11a} = - \frac{2e^2\xi^4}{\pi^2\lambda^2\beta^4} \{- \frac{2}{3\beta^2} \int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell}{\ell(\ell^2+1)} \int_{(\ell\lambda)^{-1}}^{1} \frac{d\eta}{\eta(\eta^2\ell^2+1)}$$

$$\left[2\ln\ell + \ln(1-\eta^2) + (\ln\lambda)^2 (\frac{1}{3\beta^2} \ln\beta^2 + \frac{1}{18\beta^2})\right]\} \tag{A.3.47}$$

where $\ell' = \eta\ell$ and

$$\int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell}{\ell(\ell^2+1)} = \frac{1}{2}\,\ell n(\lambda^2+1) \qquad \text{(A.3.48)}$$

If we drop terms of order $\ell n\lambda$ and $(\ell n\lambda)^0$, after some integrations one gets that

$$V_{11a} = -\frac{2e^2\xi^4}{\pi^2\lambda^2\beta^6}\left[\frac{2}{9}\,(\ell n\lambda)^3 + \frac{1}{3}\,(\ell n\lambda)\,(2\ell n\beta + \frac{1}{6}) + 0\,(\ell n\lambda)\right] \qquad \text{(A.3.49)}$$

Integral V_{11b} that is of the form

$$V_{11} = -\frac{2e^2\xi^4}{\pi^2\lambda^2\beta^4}\int_{\frac{1}{\lambda}}^{\infty}\int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell d\ell'}{\ell\ell'(\ell^2+1)(\ell'^2+1)}\; q(\ell_1\ell') \qquad \text{(A.3.50)}$$

brings up terms that are $O(\ell)$ or $O(\ell')$ so the integral V_{11b} can bring up only terms of $O(\ell n\lambda)$ which could be neglected.

Evaluation of the integral V_{12} is done by making the same approximations

$$V_{12} = -\frac{2e^2\xi^4}{\pi\lambda\beta^4}\int_{\frac{1}{\lambda}}^{\infty}\frac{d\ell}{\ell(\ell^2+1)}\int_{0}^{\infty}\frac{d\ell'}{(\ell'^2+1)}\,(F^2{}_0(\lambda\ell) - \frac{0(\lambda\ell-1)}{\pi\lambda\ell'})$$

$$[\gamma((\ell+\ell')^2) + \gamma\,((\ell-\ell')^2)] \qquad \text{(A.3.51)}$$

We use asymptotic values for $([\gamma(\ell+\ell')^2) + \gamma(\ell-\ell')^2)]$. We also take the asymptotic form

$$J^2_0(\lambda\ell') - \frac{\theta(\lambda\ell'-1)}{\pi\lambda\ell'} \approx \frac{1}{\pi\lambda\ell'} \sin 2\eta\ell' \qquad \text{(A.3.52)}$$

and by scaling $\ell' = \frac{x}{\lambda}$

$$V_{12} \approx \frac{2}{3} \frac{e^2\xi^4}{\pi^2\beta^6\lambda^2} \int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell}{\ell(\ell^2+1)} \int_0^{\infty} \frac{dx}{x} \frac{\sin 2x}{(\frac{x^2}{\lambda^2} + 1)} \ln \frac{(\ell^2-(\frac{x}{\lambda})^2)}{\beta^2} \qquad \text{(A.3.53)}$$

for $\lambda \to \infty$

$$V_{12} \approx \frac{2e^2\xi^4}{3\beta^6\pi^2\lambda^2} \int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell}{\ell(\ell^2+1)} \ln \frac{\ell^2}{\beta^2} \int_0^{\infty} \frac{dx}{x} \sin 2x \qquad \text{(A.3.54)}$$

$$\int_0^{\infty} \frac{dx}{x} \sin 2x = \frac{\pi}{2} \qquad \text{(A.3.55)}$$

so

$$V_{12} \approx \frac{2e^2\xi^4}{3\beta^6\pi\lambda^2} \int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell}{\ell(\ell^2+1)} \ln \frac{\ell}{\beta} \qquad \text{(1.3.61)}$$

Again, neglecting the terms of $O(\ell n\lambda)$ and smaller

$$\int_{\frac{1}{\lambda}}^{\infty} \frac{d\ell}{\ell(\ell^2+1)} \ell n \frac{\ell}{\beta} \to = -\frac{1}{2} (\ell n\lambda)^2 - \ell n\beta \ell n\lambda \tag{A.3.56}$$

$$V_{12} \approx -\frac{2e^2\lambda^2}{\pi^2\alpha^4\beta^6} \left(\frac{\pi}{6} (\ell n\lambda)^2 + \frac{\pi}{3} \ell n\lambda \ell n\beta\right) \tag{A.3.57}$$

The final part of integral $\langle V \rangle$

$$V_{22} = -\frac{2e^2\xi^4}{\beta^4} \int_0^{\infty} \int_0^{\infty} \frac{d\ell d\ell'}{(\ell^2+1)(\ell'^2+1)} \left[J^2{}_0(\lambda\ell) - \frac{\theta(\lambda\ell-1)}{\pi\lambda\ell}\right] \left[J^2{}_0(\lambda\ell') - \frac{\theta(\lambda\ell'-1)}{\pi\lambda\ell'}\right] \times$$
$$\times \left[\Upsilon(\ell+\ell')^2 - \Upsilon(\ell-\ell')^2\right] \tag{A.3.58}$$

Taking asymptotic forms for the three expressions in the big brackets

$$V_{22} \approx -\frac{2e^2\xi^4}{\beta^4\pi^2\lambda^2} \int_0^{\infty} \int_0^{\infty} \frac{d\ell d\ell'}{\ell(\ell^2+1)\ell'(\ell^2+1)} \times$$
$$\times \sin 2\lambda\ell \sin 2\lambda\ell' \left(-\frac{1}{3\beta^2} \ell n \frac{\ell^2-\ell'}{\beta^2}\right) \tag{A.3.59}$$

If we scale $\lambda\ell = x$ and $\lambda\ell' = x'$

$$V_{22} = \frac{4}{3} \frac{e^2\xi^4}{\beta^2\pi^2\lambda^2} \int_0^{\infty} \frac{dx}{x(\frac{x^2}{\lambda^2} + 1)} \int_0^{x} \frac{dx'}{x'(\frac{x^2}{\lambda^2} + 1)}$$

$$\sin 2x \, \sin 2x' \ln \left(\frac{x^2 - x'^2}{\beta^2 \lambda^2}\right) \tag{A.3.60}$$

For $\lambda \to \infty$

$$V_{22} \approx \frac{4}{3} \frac{e^2 s^4}{\beta^2 \pi^2 \lambda^2} \int_0^\infty \frac{dx}{x} \sin 2x \int_0^x \frac{dx'}{x'} \sin 2x \left[\ln(x^2 - x'^2) - 2\ln\beta\lambda\right] \tag{A.3.61}$$

Those are convergent integrals and the leading term is order $O(\ln\lambda)$ so to follow our approximation we will neglect the whole V_{22} integral.

Combining results for V_{11}, V_{12} and V_{22} one gets Eq. (3.41)

$$V = -\frac{2e^2\lambda^2}{\pi^2\alpha^4\beta^6} \{(\frac{2}{9} (\ln\lambda)^3 + (\ln\lambda)^2 (\frac{2}{3} \ln\beta + \frac{1}{18} + \frac{\pi}{3}) + \\ + \frac{2\pi}{3} \ln\lambda \ln\beta + O(\ln\lambda)\}. \tag{A.3.62}$$

APPENDIX IV

Calculation of a binding energy of a hydrogen atom in ultrastrong circularly polarized laser field is done by expanding the $\varepsilon = \varepsilon(\lambda)$ Eq. (3.4.3) in power of λ^{-1} where λ is large parameter. Our prime task is to determine whether the system described above forms a bound state.

Again we start with Eq. (3.4.3) and use the form (3.4.4) of a trial function.

A.4.1 Evaluation of Normalization Integral <N>

Integral

$$\langle N\rangle = \int d^3k J^2{}_0(\alpha k\ )\left|U_1(\frac{k}{\xi},\beta)\right|^2 \tag{A.4.1}$$

is transformed by scaling $\frac{\vec{k}}{\xi} = \vec{\ell}$ and $\lambda = \alpha\xi$ into

$$\langle N\rangle = 2\pi\xi^3 \int_0^\infty \frac{d\ell\ \ell\ J^2{}_0(\lambda\ell\ )}{\ell^{\ 2}+1} \int_{-\infty}^\infty \frac{d\ell_z}{\ell_z{}^2+\beta^2} \tag{A.4.2}$$

the second integral being

$$\int_{-\infty}^\infty \frac{d\ell_z}{(\ell_z{}^2+\beta^2)^2} = \frac{\pi}{2\beta^3} \tag{A.4.3}$$

The first integral is see (Gradstain & Rizil pp. 678)

$$\int_0^\infty \frac{d\ell\ \ell\ J^2{}_0(\lambda\ell\ )}{(\ell\ ^2+\beta^2)^2} = (-\ \frac{\lambda}{2}\ \frac{\partial}{\partial\lambda}\ K_0\ (\lambda)\ I_0(\lambda) \tag{A.4.4}$$

where K_0 and I_0 are Bessel functions of the first kind.

Now, lets use the asymptotic expansion of $k_0(\lambda)$ and $I_0(\lambda)$ for large values of argument λ see (G & R pp. 962)

$$I_0(\lambda) \sim \frac{e^{\lambda}}{\sqrt{2\pi\lambda}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\lambda)^k} \frac{\Gamma(k+{}^1{}_2)}{k!\Gamma({}^1{}_2-k)} - \frac{e^{-\lambda}}{\sqrt{2\pi\lambda}} \sum_{k=0}^{\infty} \frac{1}{2(\lambda)^k} \frac{\Gamma(k+{}^1{}_2)}{k!\Gamma({}^1{}_2-k)}$$

$$K_0(\lambda) \sim \sqrt{\frac{\pi}{2\lambda}}\ e^{-\lambda} \left[\sum_{k=0}^{n-1} \frac{\Gamma(k+{}^1{}_2)}{k!\Gamma({}^1{}_2-k)} + \dots\right] \tag{A.4.5}$$

where Γ is the gamma function.

If we go up to $O(\frac{1}{\lambda^3})$, then

$$\langle N\rangle = \frac{2\pi^2\xi^3}{2\beta^3} \left[\frac{1}{2\lambda} + \frac{1}{16\lambda^3} + 0\ (\frac{1}{\lambda^5}) + ..\right] \tag{A.4.6}$$

A.4.2 <u>Evaluation of the Expectation Value of Kinetic Energy <T></u>

<T> integral has been evaluated in much the same way as integral <N>

$$\langle T\rangle = \frac{1}{2m} \int d^3k\ k^2J^2{}_0(\alpha k\ )U_1(\frac{k}{\xi},\beta)|^2 \tag{A.4.7}$$

The usual scaling yields

$$\langle T\rangle = \frac{\pi\xi^5}{m}\int_0^\infty \frac{d\ell\ \ell\ J^2_0(\lambda\ell)}{(\ell^2+1)^2}\left[\frac{\pi}{2\beta} + \ell^2\ \frac{\pi}{2\beta^3}\right] \qquad (A.4.8)$$

where the big bracket represent the result of $d\ell_z$ integration.

$$\langle T\rangle = \frac{\xi^2\pi^2}{2m\beta^3}\left[1 + (\beta^2-1)\ (-\frac{\lambda}{2}\ \frac{\partial}{\partial\lambda})\right] K_0(\lambda)I_0(\lambda) \qquad (A.4.9)$$

Asymptotic expansion in large argument λ yields

$$\langle T\rangle = \frac{\xi^5\pi^2}{2m\beta^3}\left[\frac{1}{2\lambda}\ \beta^2+\frac{1}{2} + \frac{1}{16\lambda^3}\ 3\beta^2-\frac{1}{2} + \theta(\frac{1}{\lambda^5}) + \ldots\right] \qquad (A.4.10)$$

A.4.3 <u>Evaluation of the <v> Integral</u>

Expectation value for the modified potential Eq. (3.4.12) is

$$\langle V\rangle = -\frac{e^2}{2\pi^2}\int_0^\infty\int_0^\infty d\ell\ d\ell'(f_0(\lambda\ell) + \frac{1}{\pi})\ g(\ell,\ell')(f_0(\lambda\ell) + \frac{1}{\pi} \qquad (A.4.11)$$

Here we use the same idea as in evaluating integral <v> for linearly polarized light.

$$f_0(\lambda\ell) = \lambda\ell\ J^2_0(\lambda\ell) - \frac{1}{\pi} \qquad (A.4.12)$$

Function $g(\ell, \ell')$ is

$$g(\ell\ ,\ell\ ') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\ell_Z d\ell_Z{}' \int_0^{2\pi}\int_0^{2\pi} \frac{d\phi d\phi' U_1(\ell,\beta)U_1(e',\beta)}{|\ell-\ell'|^2} \qquad (A.4.13)$$

We can brake <v> integral into three integrals

$$\langle v\rangle = V_{11} + V_{12} + V_{22} \qquad (A.4.14)$$

where the first one is

$$V_{11} = \frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty} d\ell\ d\ell\ 'g(\ell\ ,\ell\ ') \qquad (A.4.15)$$

We can see that V_{11} is independent of λ.

We will show later that it is a crucial information about V_{11} so

$$V_{11} = v,(\beta) \qquad (A.4.16)$$

In order to find value of V_{12} Eq. (3.4.16) we have to evaluate

$$\Upsilon(\ell\ ) = \int_0^{\infty} d\ell\ 'g(\ell\ ,\ell\ ') \qquad (A.4.17)$$

The expression for V_{12}

$$V_{12} = \frac{2}{\pi}\int_0^\infty d\ell\ f_0(\lambda\ell\ )\Upsilon(\ell\ ) \tag{A.4.18}$$

will be integrated by parts so that

$$d_u = d\ell\ f_0(\lambda\ell\ ) \quad \text{and} \quad v = \Upsilon(\ell\ )$$

$$u = \frac{1}{\lambda}\ f_1(\lambda\ell\ ) \qquad\qquad dv = d\ell\ \Upsilon'(\ell\ )$$

where

$$f_1(\lambda\ell\ ) = \int_0^{\lambda\ell} dt\ f_0(+) \tag{A.4.19}$$

We have to know asymptotic behavior of $\Upsilon(\ell\ )$ in order to see whether integral V_{12} converges or not.

Let us first evaluate $g(\ell\ ,\ell\ ')$. If we perform $d\ell_z$, $d\ell_z{}'$ integration first we have integral

$$I(\ell\ ,\ell\ ',\phi,\phi') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\ell_z d\ell_z{}'}{(\ell_z{}^2+\beta^2)\left[(\ell_z-\ell_z{}')^2+\left|\vec{\ell}\ -\vec{\ell}\ '\right|^2\right](\ell_z{}'^2+\beta^2)} \tag{A.4.20}$$

which we can evaluate by performing complex integration where the contour is Fig. 1 and contour integral vanishes for $|\ell_z|\to\infty$.

$$\oint \frac{d\ell_Z'}{\ell_Z^{2\prime}+\beta^2} \oint \frac{d\ell_Z}{(\ell_Z^2+\beta^2)[(\ell_Z-\ell_Z')^2+6^2]} \tag{A.4.21}$$

where $b^2 = |\underset{\sim}{\ell} - \underset{\sim}{\ell}\,'|^2$, and

$$\oint \frac{d\ell_Z}{(\ell_Z^2+\beta^2)[(\ell_Z-\ell_Z')^2+b^2]} = 2\pi i \ \Sigma \text{ residium} \tag{A.4.22}$$

where poles are $\ell_{Z1,2} = \pm\, i\beta$ and $\ell_{Z3,4} = \ell_Z' \pm i\beta$.

Similar procedure in contour integration over $d\ell_Z'$ yields to

$$I(\ell\ ,\ell\ ',\phi,\phi') = \frac{\pi^2}{\beta^2|\underset{\sim}{\ell} - \underset{\sim}{\ell}\,'|(2\beta)+|\underset{\sim}{\ell} - \underset{\sim}{\ell}\,'|} \tag{A.4.23}$$

so that

$$g(\ell\ ,\ell\ ') = \frac{\pi^2}{\beta^2}\int_0^{2\pi}\int_0^{2\pi} \frac{d\phi\, d\phi'}{(\ell\ ^2+1)(\ell\ '^2+1)|\underset{\sim}{\ell} - \underset{\sim}{\ell}\,'|\left[2\beta+|\underset{\sim}{\ell} - \underset{\sim}{\ell}\,'|\right]} \tag{A.4.24}$$

where $|\underset{\sim}{\ell} - \underset{\sim}{\ell}\,'| = (\ell\ ^2 + \ell\ '^2 - 2\ell\ \ell\ '\cos(\phi-\phi'))^{1/2}$

and

$$g(\ell\ ,\ell\ ') = \frac{2\pi^3}{\beta^2}\int_0^{2\pi} \frac{d\phi}{(\ell\ ^2+1)(\ell\ '^2+1)|\underset{\sim}{\ell} - \underset{\sim}{\ell}\,'|(2\beta+|\underset{\sim}{\ell} - \underset{\sim}{\ell}\,'|)} \tag{A.4.25}$$

At this point we have to find the value of $\Upsilon(\ell)$ in the limits of $\ell \to 0$ and $\ell \to \infty$ which is necessary for the result of (A.4.19)

$$\Upsilon(\ell) = \frac{2\pi^3}{\beta^2}\int_0^\infty d\ell' \int_0^{2\pi} \frac{d\phi}{|\boldsymbol{\ell}-\boldsymbol{\ell}'|\left[2\beta+|\boldsymbol{\ell}-\boldsymbol{\ell}'|\right](\ell^2+1)(\ell'^2+1)} \quad (A.4.26)$$

In the limit of $\ell \to 0$ we have to be careful about the first term in the denominator because there is a singularity of $\ell \to 0$.

$$\ell \to 0 \quad \Upsilon(\ell) = \frac{2\pi^2}{\beta^3}\int_0^\infty \frac{d\ell'}{(\ell'^2+1)(2\beta+\ell')}\int_0^{2\pi} \frac{d\phi}{|\boldsymbol{\ell}-\boldsymbol{\ell}'|} \quad (A.4.27)$$

Now integral

$$\int_0^{2\pi} \frac{d\phi}{|\boldsymbol{\ell}-\boldsymbol{\ell}'|} = 2\int_0^{\pi} \frac{d\phi}{\ell\,(1+x^2-2x\cos\phi)^{\frac{1}{2}}} = 2K(x) \quad (A.4.28)$$

where $x = \frac{\ell'}{\ell}$ and $x^2 < 1$. K(x) is complete eliptic integral of the first kind (see G & R pp.387).

$$\Upsilon(\ell) = \frac{2\pi^3}{\beta^2}\int_0^\infty \frac{d\ell}{2\beta+\ell'}\,\frac{K(\frac{\ell<}{\ell<})}{\ell>} \quad (A.4.29)$$

where $\ell<$ and $\ell>$, are smaller and larger of ℓ, ℓ.

$$\gamma(\ell)_{\ell \to 0} = \frac{4\pi^3}{\beta^2}\left[\frac{1}{\ell}\int_0^{\ell} \frac{d\ell' \frac{K(\ell')}{\ell}}{(\ell'+2\beta)(\ell'^2+1)}\right] + \int_{\ell}^{\infty} \frac{d\ell'}{\ell'} \frac{K(\frac{\ell}{\ell'})}{(\ell'^2+1)(\ell'+2\beta)} \quad (A.4.30)$$

The first integral is finite when $\ell \to 0$. The second integral yields

$$\gamma(\ell) = \frac{4\pi^3}{\beta^2}\left[\int_{\ell}^{1} \frac{d\ell' K(\frac{\ell}{\ell'})}{\ell'(\ell'+2\beta)(\ell'^2+1)} + \int_{1}^{\infty} \frac{d\ell'}{\ell} \frac{K(\frac{\ell}{\ell'})}{(\ell'+2\beta)(\ell'^2+1)}\right] \quad (A.4.31)$$

the second integral in (A.4.31) is finite so we can drop it from evaluating asymptotic behavior of $\gamma(\ell)$ when $\ell \to 0$.

The last remaining integral

$$\int_{\ell}^{1} \frac{d\ell' K(\frac{\ell}{\ell'})}{\ell'(\ell'+2\beta)(\ell'^2+1)} = \int_{\ell}^{1} \frac{d\ell'}{\ell}\left[\frac{\pi}{2\beta} + \left(\frac{K(\frac{\ell}{\ell'})}{(\ell'+2\beta)(\ell'^2+1)} - \frac{\pi}{2\beta}\right)\right] \quad (A.4.32)$$

where we used the fact that $K(o) = \pi$.

So that

$$\gamma(\ell)_{\ell \to 0} = -\frac{2\pi^4}{\beta^3}\ln\ell \quad (A.4.33)$$

and

$$\gamma'(\ell) = -\frac{2\pi^4}{\beta^3\ell}, \quad \ell\,\gamma'(\ell) = -\frac{2\pi^4}{\beta^3} \quad (A.4.34)$$

If we set a limit $\ell \to \infty$ in A.4.18 then

$$\Upsilon(\ell) \approx \frac{2\pi^5}{\beta^2 \ell^4} \tag{A.4.35}$$

after we integrated over ℓ .

Now we can go back to partial integration of V_{12} Eq. (A.4.19)

$$V_{12} = -\frac{2}{\pi}\frac{1}{\lambda}\int_0^\infty d\ell\ f_1(\lambda\ell)\Upsilon'(\ell) \tag{A.4.36}$$

If we define

$$\bar{f}_1(x) = \frac{1}{x} f_1(x)$$

then

$$V_{12} = -\frac{2}{\pi}\int_0^\infty d\ell\ \bar{f}_1(\lambda\ell)\,[\ell\ \Upsilon'(\ell)] \tag{A.4.37}$$

Now let's integrate by parts once more

$$du = \bar{f}_1(\lambda\ell)d\ell \qquad\qquad v = \ell\ \Upsilon'(\ell)$$

$$u = \frac{1}{\lambda} f_2(\lambda\ell) \qquad\qquad dv = d\ell\ (\ell\ \Upsilon'(\ell))'$$

where

$$f_2(x) = \int_0^x dt\ \bar{f}_1(\ ) = \int_0^x \frac{dt}{t} \int_o^t ds\ f_0(s) \qquad (A.4.38)$$

Note that $v(\ell\ ) \to \frac{1}{\ell}$, when $\ell\ \to \infty$ and $v(o) \to$ finite. So

$$V_{12} = \frac{2}{\pi\lambda} \int_0^\infty d\ell\ f_2(\lambda\ell\ )\ (\ell\ \gamma'(\ell\ ))' \qquad (A.4.39)$$

Let us do $f_2(\lambda\ell\ )$ first. It is

$$f_2(x) = \left(\int_0^1 \frac{dt}{t} + \int_1^x \frac{dt}{t}\right) \left\{\int_0^\infty ds\left[sJ^2_0(s) - \frac{1}{\pi}\ (1 + \sin 2s)\right] + \frac{1}{2\pi} - \right.$$

$$\left. - \frac{1}{2\pi} \cos 2t - \int_t^\infty \left(sJ^2_0(s) - \frac{1}{\pi}\ (1 + \sin 2s)\right)\right\} \qquad (A.4.40)$$

where $x \to \infty$.

The first two terms in the big bracket vanishes after numerical integration is performed (see Appendix 7).

$$f_2(x) = \int_0^1 \frac{dt}{t} \int_0^t ds\left(sJ^2_0(s) - \frac{1}{\pi}\right) + \int_1^x \frac{dt}{t} \left[- \frac{1}{2\pi} \cos 2t - \right.$$

$$\left. - \int_t^\infty ds\left(sJ^2_0(s) - \frac{1}{\pi}\ (1 + \sin 2s)\right)\right] \qquad (A.4.41)$$

We can still simplify this expression by doing partial integration on integrals

$$\int_0^1 \frac{dt}{t} \int_0^\infty ds(sJ^2_0(s) = - \int_0^1 dt\ t\ \ell n t J^2_0(t) \qquad (A.4.42)$$

and

$$\int_0^1 \frac{dt}{t} (sJ^2_0(s) - \frac{1}{\pi} (1 + \sin 2s) = \int_1^\infty dt\ \ell n t(tJ^2_0(t) - \frac{1}{\pi} (1 + \sin 2t)) \qquad (A.4.43)$$

So that $f_2(\infty) = k_{21}$

$$k_{21} = - \int_0^1 dt\ \ell n t J^2_0(t) - \frac{1}{\pi} - \frac{1}{2\pi} \int_0^1 \frac{dt}{t} \cos 2t -$$

$$- \int_1^\infty dt\ \ell n t(tJ^2_0(t) - \frac{1}{\pi} (1 + \sin 2t)) \qquad (A.4.44)$$

k_{21} is evaluated numerically (Appendix 7). Finally

$$V_{12} = - \frac{2}{\pi\lambda} \int_0^\infty d\ell\ [k_{21} + (f_2(\lambda\ell\ ) - k_{21})]\ (\ell\ \gamma'(\ell\ ))' \qquad (A.4.45)$$

with a definition

$$\bar{f}_2(x) = f_2(x) - k_{21} = -\int_x^\infty \frac{dt}{t} \int_0^t \frac{dt}{t} \, ds \, f_0(s) \qquad (A.4.46)$$

and doing one more partial integration

$$V_{12} = \frac{2k_{21}}{\pi\lambda} (\ell \; \gamma'(\ell \;))_{\ell \; = 0} - \frac{2}{\pi\lambda^2} \int_0^\infty d\ell \;\; f_3(\ell \;) \; (\ell \; \gamma'(\ell \;))' \quad (A.4.47)$$

the last integral converges and the whole term is of order $0(\frac{1}{\lambda^2})$ so we will neglect it.

$$V_{12} = -\frac{2k_{21}}{\pi\lambda} [\ell \;\; \gamma'(\ell \;)]_{\ell \;\; 0} = \frac{1}{\lambda} \frac{4\pi^3 k_{21}}{\beta^3} \qquad (A.4.48)$$

The last integral is

$$V_{22} = \int_0^\infty \int_0^\infty d\ell \; d\ell \; 'f_0(\lambda\ell \;)g(\ell \;,\ell \;')f_0(\lambda\ell \;') \qquad (A.4.49)$$

We will use the result of integration of $g(\ell \;,\ell \;')$ so that

$$g(\ell \;,\ell \;') = \frac{2\pi^3}{\beta^2(\ell \;^2+1)(\ell \;'^2+1)} \int_0^{2\pi} \frac{d\phi}{b[b+2\beta]} \qquad (A.4.50)$$

where $b = (\ell \;^2 + \ell \;'^2 - 2\ell \; \ell \;' \cos\phi)^1/_2$

The integral is partially done as

$$\int_0^{2\pi} \frac{d\phi}{b[b+2\beta]} = \frac{1}{\beta} [\frac{1}{\ell >} K (\frac{\ell <}{\ell >}) - \int_0^{\pi} \frac{d\phi}{b+2\beta} \qquad (A.4.51)$$

Here again K is the complete eliptic integral of the first kind and ℓ <, are smaller and larger of ℓ , ℓ .

So if ℓ ' = ℓ .x integral V_{22} is

$$V_{22} = 2 \int_0^{\infty} \ell \, d\ell \int_0^1 dx \, f_0(\lambda\ell \,) g(\ell \,, \ell \, x) f_0(\lambda\ell \, x) \qquad (A.4.52)$$

where

$$g(\ell \,, \ell \, x) = \frac{2\pi^3}{\beta^3(\ell \,^2+1)(\ell \,^2x^2+1)} [\frac{1}{\ell} K(x) - \int_0^{\pi} \frac{d\phi}{\ell \, a + 2\beta} \qquad (A.4.53)$$

and $a = (1 + x^2 - 2x\cos\phi)^{1/2}$.

Let

$$f_0(y) = \bar{f}_0(y) + \frac{1}{\pi} \sin 2y \qquad (A.4.54)$$

so that V_{22} brakes into three integrals

$$V_{22} = V_{22}(1) + V_{22}(2) + V_{22}(3) \qquad (A.4.55)$$

In the first of them let $\xi = \lambda\ell$, where λ is very large

$$V_{22}(1) = \frac{2}{\lambda} \int_0^\infty d\xi \int_0^1 dx(\frac{s}{\lambda}, \frac{\xi}{\lambda} x))\bar{f}_0(\xi)\bar{f}_0(\xi x) \qquad (A.4.56)$$

where for large λ

$$V_{22}(1) = \frac{2}{\lambda} \frac{2\pi^3}{\beta^3} \int_0^1 dxK(x) \int_0^\infty d\xi\bar{f}_0(\xi)\bar{f}_0(\xi,x) = \frac{4\pi^3}{\lambda\beta^3} k_{22}(1) \qquad (A.4.57)$$

where $k_{22}{}^1$ is a number independent of λ obtained by numerical integration

$$k_{22}{}^1 = \qquad (A.4.58)$$

The second integral can be written the same way as the first one

$$V_{22}(2) = \frac{4\pi^2}{\lambda\beta^3} k_{22}(2) \qquad (A.4.59)$$

where

$$k_2 2(2) = \int_0^1 dxK(x) \int_0^\infty d\xi\left[\bar{f}_0(\xi)\sin 2\xi x + \bar{f}_0(\xi x)\sin 2\xi\right] \qquad (A.4.60)$$

that after numerical integration yields (Appendix 7)

$$k_{22}(2) =$$

The last integral is

$$V_{22}(3) = \frac{2}{\pi 2}\int_0^\infty d\ell \int_0^1 dx \ell\ g(\ell\ ,\ell\ x)\sin 2\lambda\ell\ \sin 2\lambda\ell\ x \qquad (A.4.61)$$

where $\ell\ g(\ell\ ,\ell\ x)$ is given by ( ). Let us write

$$\sin 2\lambda\ell\ \sin 2\lambda\ell\ x = \frac{1}{2}\left[\cos 2\lambda\ell\ (1-x) - \cos 2\lambda\ell\ (1+x)\right] \qquad (A.4.62)$$

We will need integral

$$\int_0^\infty d\ell\ \frac{\cos\xi\ell}{(\ell^{\ 2}+1)(\ell^{\ 2}x^2+1)} = -\frac{1}{1-x^2}\int_0^\infty d\ell\ \cos\xi\ell\left[\frac{x^2}{\ell^{\ 2}x^2+1} - \frac{1}{\ell^{\ 2}+1}\right] \qquad (A.4.63)$$

where q is $2\lambda(1-x)$ or $2\lambda\ (1+x)$. Lets go back to (G&R pp. 378). There we find

$$\int_0^\infty \frac{d\ell\ \cos\xi\ell}{(\ell^{\ 2}+1)(\ell^{\ 2}x^2+1)} = -\frac{1}{1-x^2}\ \frac{\pi}{2}\ \{x\ e^{-2/x} - e^{-\xi}\} \qquad (A.4.64)$$

If we first take into account part of $\ell\ g(\ell\ ,\ell\ x)$ that has eliptic integral and setting the value for q we find that this first integral in $\ell\ g(\ell\ ,\ell\ x)$ is ($V_{22}(3) = V_{22a}(3) - V_{2222b}(3)$)

$$V_{22a}(3) = \frac{\pi}{\beta^3}\int_0^1 \frac{K(x)}{1}-x^2\ [x\ e^{-2\lambda\ \frac{1-x}{x}} - x\ e^{\frac{2\lambda}{x}(1+x)} - e^{-2\lambda(1-x)} + e^{-2\lambda(1+x)}$$

We can drop $e^{-2\lambda(1+x)}$ and $e^{-\frac{2\lambda}{x}(1+x)}$ for large λ.

So

$$V_{22a}(3) = -\frac{\pi}{\beta^3}\int_0^1 \frac{K(x)}{1}-x^2\ [x\ e^{-2\lambda(\frac{1-x}{x})} - e^{2\lambda(1-x)}] \qquad (A.4.66)$$

Let $y = 1-x$, then

$$V_{22a}(3) = -\frac{\pi}{\beta^3}\int_0^1 \frac{dy}{y(2-y)}\ K(1-y)\ [(1-y)e^{-2\lambda\ \frac{y}{1-y}} - e^{-2\lambda y}] \qquad (A.4.67)$$

the exponents peak around $y = 0$. So let us expand the integrant around $y = 0$, except for $K(1-y)$ since $K(1) \to \infty$

then

$$V_{22a}(3) = \frac{\pi}{2\beta^3}\int_0^1 dyK(1-y)\lambda y\ e^{-2\lambda y} \qquad (A.4.68)$$

Taking the limit of small y in $K(1-y)$

$$K(1-y) \approx \ell n \frac{4}{\sqrt{1-(1-y)^2}} \approx \ell n \frac{2\sqrt{2}}{\sqrt{y}} \tag{A.4.69}$$

so that with $t = 2\lambda y$ and set upper limit to $t \to \infty$ with an error $O(e^{-2\lambda})$

$$V_{22a}(3) \approx \frac{\pi}{8\lambda\beta^3}(4\ell n2 + \ell n\lambda) - \frac{\pi}{8\lambda\beta^3}\int_0^\infty dt\ \ell nt\ t\ e^{-t} \tag{A.4.70}$$

The second term is small so drop it.

So

$$V_{22a}(3) \sim \frac{\pi}{8\lambda\beta^3}(4\ell n2 + \ell n\lambda) \tag{A.4.71}$$

The second integral of $V_{22}(3) = V_{22a}(3) + V_{22b}(3)$ still get peak for small $\ell \sim \frac{1}{2\lambda}$. That means that $\ell\ a \sim \frac{a}{2\lambda} << 2\beta$ and we can do ϕ integration.

$$\ell\ g(\ell\ ,\ell\ x) \approx \frac{2\pi^3}{\beta^3(\ell^{\ 2}+1)(\ell^{\ 2}x^2+1)}\left(-\frac{\pi}{2\beta}\ell\right) \approx -\frac{\pi^4}{\beta^4}\frac{\ell}{(\ell^{\ 2}+1)(\ell^{\ 2}x^2+1)} \tag{A.4.72}$$

Setting this back into equation that gets its maximum contribution for $x \approx 1$

$$V_{22b}(3) = \frac{2}{\pi^2}\int_0^\infty d\ell \int_0^1 dx\left(-\frac{\pi^4}{\beta^4}\frac{\ell}{(\ell^{\ 2}+1)^2}\frac{1}{2}\cos 2\lambda\ell\ (1-x)\right. \tag{A.4.73}$$

which after x integration yields to

$$V_{22b}(3) = -\frac{\pi^2}{2\lambda\beta^4}\int_0^\infty d\ell \; \frac{\sin 2\lambda\ell}{(\ell^2+1)^2} = -\frac{\pi^2}{2\lambda\beta^4}\left(\frac{1}{2\lambda} - \frac{2}{\lambda}\int_0^\infty \frac{d\ell \; \ell \cos 2\lambda\ell}{(\ell^2+1)^3}\right) \quad (A.4.74)$$

We see that

$$V_{22b}(3) \approx -\frac{\pi^2}{4\lambda\beta^4}\left(1 + 0\left(\frac{1}{\lambda}\right)\right) \quad (A.4.75)$$

is small in comparison with the first term so we will neglect it, and have that

$$V_{22}(3) \sim \frac{\pi}{8\lambda\beta^3}\,(4\ell n2 + \ell n\lambda) \quad (A.4.76)$$

If we assemble all terms for the expectation value of $\langle V\rangle$

$$\langle V\rangle = -\frac{e^2}{2\pi^2}\,\frac{\lambda^2}{\alpha^4}\Big[v_1(\beta) + \frac{4\pi^3}{\lambda\beta^3}K_{21} + \frac{4\pi^3}{\lambda\beta^3}K_{22}(1) + \frac{4\pi^2}{\lambda\beta^3}K_{22}(2) + + \frac{\pi}{8\lambda\beta^3}(4\ell n2 + \ell n\lambda)\Big] \quad (A.4.77)$$

The last step is to variationally determine the minimum of energy

$$\varepsilon = \frac{\langle T\rangle + \langle V\rangle}{\langle N\rangle} \quad (A.4.78)$$

We see that only large negative value of V can make ε negative

$$\varepsilon = \frac{\lambda^2}{2m\alpha^2}\left[\beta^2+\frac{1}{2} - \frac{1}{4\lambda^2} + \ldots\right] - \frac{2e^2\beta^3}{\pi^4\alpha}\left[v_1(\beta) + \frac{A'}{\lambda\beta^3} + \frac{\pi}{\gamma\lambda\beta^3}\,\ln\lambda\right] \quad (A.4.79)$$

where

$$A' = 4\pi^3(k_{21} + k_{22}(1) + \frac{k_{22}(2)}{\pi} + \frac{2\ln 2}{\pi^2} = \quad (A.4.80)$$

Let us find the first derivative $\frac{\partial E}{\partial\lambda} = 0$

$$\frac{\partial E}{\partial\lambda} = \frac{2\lambda}{\alpha^2}\ \beta^2+\frac{1}{2} + \frac{2e^2A'}{\pi^4\alpha\lambda^2} + \frac{e^2\pi}{8\lambda^2}\ln\lambda - \frac{e^2\pi}{8\lambda^2} = 0 \quad (A.4.81)$$

From here

$$\lambda^3 = -\frac{2m\alpha e^2}{\pi^4(\beta^2+1)}\,A' - \frac{e^2\pi}{8}\,(\ln\lambda-1) \quad (A.4.82)$$

Since A' is A' > 0, we see that λ is negative which makes $\varepsilon > 0$ and lead us to the conclusion that we did not find a binding energy for a hydrogen atom in the ultraintense laser field that is circularly polarized.

APPENDIX V

This appendix is related to the rather lengthy calculations described in Chapter IV. We are not going to go into calculations of very single integral since many of then are already done in Appendix I, II, III & IV but rather to justify points that are crucial for the result of Chapter IV.

A.5.1 Calculation of $\Gamma^{(1)}$ and $\Gamma^{(2)}$ terms

Here we should prove that the imaginary part of the expectation value of $v = P\delta h G^{(+)}\delta h P$ vanishes.

$$\frac{\Gamma 1}{2\pi} = I_m \, \langle\phi(k)\delta h_k \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d\phi d\phi'}{2\pi} \cos\phi G^{(+)}{}_k(\phi,\phi'\eta)\cos\phi'\delta h_k \Phi(k)\rangle \quad (A.5.1)$$

Here we need

$$I_m \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d\phi d\phi'}{(2\pi)^2} \cos\phi G^{(+)}{}_k(\phi,\phi',\eta)\cos\phi' \quad (A.5.2)$$

which is

$$I_m \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi} \left[\frac{1}{2\pi a_k}\left(1 - \frac{\xi_k(\phi')}{\Delta_k(-\eta)}\right)\right]\cos\phi' \quad (A.5.3)$$

The first integral vanishes so that we have to calculate integral

$$- \frac{I_m}{2\pi a_k} \int_{-\pi}^{\pi} d\phi' \; \frac{\xi_k(\phi')\cos\phi'}{\Delta_k(-\eta)} \tag{A.5.4}$$

or we could also use

$$- \frac{I_m}{2\pi a_k} \int_{-\pi}^{\pi} d\phi \; \frac{\cos\phi\,\xi_k{}^*(\phi)}{\Delta_k(-\eta)}. \tag{A.5.5}$$

The imaginary part is then

$$\frac{\Gamma^{(1)}}{2\pi} = - \frac{1}{4\pi a_k} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \cos\phi \; [\xi_k(\phi) + \xi_k{}^*(\phi)] \; \frac{1}{2i} \left(\frac{1}{\Delta_k(-\eta)} - \frac{1}{\Delta_k(\eta)}\right) \tag{A.5.6}$$

The result in Appendix I

$$\Delta_k(-\eta) = - 2\sinh\nu_k(1 - i\eta) \sum_n \frac{J^2{}_n(x_k)}{\nu_k(1-i\eta)+n} =$$

$$- 2 \; J_{\nu k(1-i\eta)}(x_k) J_{-\nu k(1-i\eta)}(x_k) \tag{A.5.7}$$

or when $\eta \to 0$ one can expand (see Abranowitz & Stegun pp.   ) so that

$$\Delta_k(-\eta) = - 2 \; [J_{\nu_k} J_{\nu_k} - i\eta \; (L_{\nu_k} J_{-\nu_k} - L_{-\nu_k} \; J_{\nu_k}) ..] \tag{A.5.8}$$

where

$$L_\nu = \frac{\partial}{\partial\nu} J_\nu \qquad \text{(A.5.9)}$$

This makes the imaginary part of

$$\frac{1}{\Delta_k(-\eta)} - \frac{1}{\Delta_k(\eta)} = -\frac{1}{2}\{2\pi i\delta(J_{\nu_k}J_{-\nu_k}\text{sqn}(L_{\nu_k}J_{-\nu}-L_{-\nu}J_\nu))\} \qquad \text{(A.5.10)}$$

Let us write

$$\delta(J_{\nu_k}J_{-\nu_k}) = \frac{\delta(J_{\nu_k})}{|J_{-\nu_k}|} + \frac{\delta(J_{-\nu_k})}{|J_{\nu_k}|} \qquad \text{(A.5.11)}$$

and look at the $\delta(J_{\nu_k})$

If we define zeros of Bessel function as $J_{\nu_k}(Y_i(\nu_k) = 0$ where $i = 1,2,\ldots,\infty$ and then expand around $x_k - y_i(\nu k)$

so that

$$\delta(J_{\nu_k}) = [\Sigma\ (J'_{\nu_k}(y_i(\nu_k))(x_k-y_i(\nu_k))\ldots] =$$

$$\sum_{i=1}^{\infty} \delta(x_k-y_i(\nu_k))/|J'\nu_k(y_i(\nu_k)) + \ldots \qquad \text{(A.5.12}$$

The next step is to expand around

$$y_i(\nu_k) = y_i(0) + \nu_k y_i(0) + \ldots \qquad \text{(A.5.13)}$$

and

$$J'_{\nu_k}(y_i(\nu_k)) \approx J'_0(y_i(0)) + \nu_k \left[- \frac{y_i(0)}{y_i(0)} J'_0(y_i(0) + L_0(y_i(0))\right] \quad (A.5.14)$$

The crucial point here is that since we have δ functions Eq. (A.5.12) the expectation value integral that defines Γ_1 and Γ_2 vanishes became of the form of the wave function $\phi(k)$ Eq. (3.2.4)

A.5.2 Calculation of $\Gamma^{(3)}$ Term

The first nonvanishing term is

$$\frac{\Gamma^{(3)}}{2\pi} = I_m\langle \phi(k)\delta h_k V(k-k_1)V(k_1-k')\delta h_{k'}\Phi(k')S(k,k',k_1)\rangle \quad (A.5.15)$$

where

$$S = I_m = \int \frac{d\phi_1 d\phi_2}{(2\pi)^2} \bar{G}_k(\phi_1)G_{k_1}{}^{(+)}(\phi_1\phi_2,\eta)\bar{G}_{k'}(\phi_2) \quad (A.5.16)$$

Having use of $PG_k{}^{(+)} = G^{(+)k}P = 0$ one can write

$$S = \frac{I_m}{(2\pi)^2 a_k a_{k'} \Delta_k \Delta_{k'}} \int \frac{d\phi_, d\phi_2}{(2\pi)^2} \xi_k(\phi_1) \left[g^{(1)}{}_{k_1} + \frac{g_{k_1}{}^{(2)}}{\Delta_{k_1}}\right] \xi_{k'}{}^{*}(\phi_2) \quad (A.5.17)$$

or we can write it as

$$S = \frac{N^{(2)}(k,k',k_1)}{(2\pi)^2 a_k a_{k'}}) \, I_m \, \frac{1}{\Delta_k(-\eta)\Delta_{k_1}(-\eta)\Delta_{k'}(-\eta)} \tag{A.5.18}$$

Remember that we have $\phi(k)$ and $\phi(k')$ in the expression for $\Gamma^{(3)}$. We can write that

$$iI_m \frac{1}{\Delta_k \Delta_{k_1} \Delta_{k'}} = \frac{i\pi\delta(A)\mathrm{sgn}B}{A_1 A'} + i\pi \frac{\delta(A_1)\mathrm{sgn}B_1}{AA'} + \frac{i\pi\delta(A')\mathrm{sgn}}{AA_1} \tag{A.5.19}$$

where $\Delta_k = A + i\eta B$. Again the only term that survives is the second one and

$$S(k_1,k',k) = \frac{1}{(2\pi)^2 a_k a_{k'}} N^{(2)}(k,k_1,k')\delta(J_{\nu_k} J_{\nu_k})\mathrm{sgn}(L_{\nu_k} J_{-\nu_k} - L_\nu J_{\nu_k}) \tag{A.5.20}$$

where $N^{(2)}$ is a real function defined as

$$N^{(2)} = \int \frac{d\phi_1 d\phi_2}{(2\pi)^2} \xi_k(\phi_1) g_{k_1}{}^{(2)}(\phi_1,\phi_2)\xi_{k'}{}^*(\phi_2) \tag{A.5.21}$$

Final evaluation of $\Gamma^{(3)}$ is done by performing a ninedimensional integration

$$\Gamma^{(3)}/2\pi = \int d^3k\, d^3k'\, d^3k_1\, u_1(k)V(k-k_1)V(k_1-k')u(k')S(k,k_1,k') \tag{A.5.22}$$

The evaluation of $N^{(2)}$ is somewhat simplified due to the fact that all integrals that have Bessel functions with the argument x_{k_1} vanish because of the δ function. Following the procedure outlined in Appendix I where we make use of Jacobis expansion the general form of $N^{(2)}$

$$N^{(2)}(k,k',k_1) = \sum_{i,j}^{4} x_{kk_1}{}^{(i)} x_{k_1k'}{}^{(j)} A^{(ij)}(k,k_1,k') \qquad \text{(A.5.23)}$$

where x_{kk_1}, $x_{k_1x'}$ and $A(k,k_1k')$ are integrals defined as Eq.

$$x_{kk_1}{}^{(1)} = \int \frac{d\phi_1}{2\pi} \gamma'_k(\phi_1)\gamma_k{}^*(\phi_1)\gamma_{k_1}{}'^*(\phi_1)\gamma_{k_1}(\phi_1)$$

$$x_{kk_1}{}^{(2)} = \int \frac{d\phi_1}{2\pi} \gamma'_k(\phi_1)\gamma_{k_1}{}'^*(\phi_1)\gamma_{k_1}(\phi_1)$$

$$x_{kk_1}{}^{(3)} = \int \frac{d\phi_1}{2\pi} \gamma_k{}'(\phi_1)\gamma_k{}^*(\phi_1)\gamma_{k_1}{}'^*(\phi_1)$$

$$x_{kk_1}{}^{(4)} = \int \frac{d\phi_1}{2\pi} \gamma_k{}'(\phi_1)\gamma_{k_1}{}'^*(\phi_1) \qquad \text{(A.5.24)}$$

and

$$A_{11} = -\frac{4}{\pi\varepsilon_{k_1}} \sin\frac{\pi}{\varepsilon_k} \sin\frac{\pi}{\varepsilon_{k_1}} \sin\frac{\pi}{\varepsilon_{k_1}} \qquad \text{(A.5.25)}$$

$$A_{12} = - \frac{2i}{\pi\varepsilon_{k_1}} \gamma_{k_1}{}^2 \sin \frac{\pi}{\varepsilon_k} \sin \frac{\pi}{\varepsilon_{k'}} \tag{A.5.26}$$

$$A_{13} = \frac{2i}{\pi\varepsilon_{k_1}} \gamma_{k'} \sin \frac{\pi}{\varepsilon_k} \sin \frac{\pi}{\varepsilon_{k_1}} e^{-i\frac{\pi}{\varepsilon_{k'}}} \tag{A.5.27}$$

$$A_{14} = - \frac{1}{\pi\varepsilon_k} \gamma_{k_1}{}^2 \sin \frac{\pi}{\varepsilon_k} \gamma_{k'} e^{-i\frac{\pi}{\varepsilon_{k'}}} \tag{A.5.28}$$

$$A_{21} = - \frac{2i}{\pi\varepsilon_{k_1}} \gamma_k e^{i\frac{\pi}{\varepsilon_k}} \sin \frac{\pi}{\varepsilon_{k_1}} \sin \frac{\pi}{\varepsilon_{k'}} \tag{A.5.29}$$

$$A_{22} = \frac{\gamma_k}{\pi\varepsilon_k} e^{i\frac{\pi}{\varepsilon_k}} \gamma_{k_1}{}^2 \sin \frac{\pi}{\varepsilon_{k'}} \tag{A.5.30}$$

$$A_{23} = - \frac{1}{\pi\varepsilon_{k_1}} \gamma_k \gamma_{k'} e^{i\frac{\pi}{\varepsilon_k}} \sin \frac{\pi}{\varepsilon_{k_1}} e^{-i\frac{\pi}{\varepsilon_{k'}}} \tag{A.5.31}$$

$$A_{24} = - \frac{i\gamma_k}{2\pi\varepsilon_k} e^{i\frac{\pi}{\varepsilon_k}} \sin \frac{\pi}{\varepsilon_{k_1}} \gamma_{k'} e^{-i\frac{\pi}{\varepsilon_{k'}}} \tag{A.5.32}$$

$$A_{31} = \frac{1}{\pi\varepsilon_k} \gamma_{k_1} \gamma_{k_1}{}'^* \sin \frac{\pi}{\varepsilon_k} 2i \sin \frac{\pi}{\varepsilon_{k'}} \tag{A.5.33}$$

$$A_{32} = - \frac{\sin^{\pi} \varepsilon_k}{\varepsilon_k} (Z_{k_1}{}^* \gamma'_{k_1}{}^* - Z_{k_1} \gamma'_{k_1}) 2i \sin \frac{\pi}{\varepsilon_{k_1}} \tag{A.5.34}$$

$$A_{33} = \frac{1}{\pi\varepsilon_{k_1}} \gamma_{k_1} \gamma_{k_1}{}'^* \sin \frac{\pi}{\varepsilon_k} \gamma_{k'} e^{-i\frac{\pi}{\varepsilon_{k'}}} \tag{A.5.35}$$

$$A_{34} = -\frac{\gamma_{k'}}{\varepsilon_k} \sin \frac{\pi}{\varepsilon_k} (Z_{k_1}{}^* \gamma_{k_1}{}^* - Z_{k_1} \gamma_{k_1}{}') \gamma_{k'} \, e^{-i \frac{\pi}{\varepsilon_{k'}}} \tag{A.5.36}$$

$$A_{41} = \frac{i}{2\pi \varepsilon_{k_1}} \gamma_k \gamma_{k_1} \gamma_{k_1}{}'^* \, 2i \sin \frac{\pi}{\varepsilon_{k'}} \tag{A.5.37}$$

$$A_{42} = \frac{i}{\varepsilon_k} \gamma_k \, e^{-i \frac{\pi}{\varepsilon_k}} (Z_{k_1}{}^* \gamma_{k_1}{}'^* - Z_{k_1} \gamma_{k_1}{}') \sin \frac{\pi}{\varepsilon_{k'}} \tag{A.5.38}$$

$$A_{43} = \frac{1}{2\pi \varepsilon_k} \gamma_k \, e^{i \frac{\pi}{\varepsilon_k}} (Z_{k_1}{}^* \gamma_{k_1}{}'^* - Z_{k_1} \gamma_{k_1}{}') \, e^{-i \frac{\pi}{\varepsilon_{k'}}} \tag{A.5.39}$$

$$A_{44} = -\frac{1}{2\varepsilon_k} \gamma_k \, e^{i \frac{\pi}{\varepsilon_k}} \gamma_{k_1} \gamma_1{}'^* \gamma_{k'} \, e^{-i \frac{\pi}{\varepsilon_k}} \tag{A.5.40}$$

Here we have use of the fact that Z_{k_1} and γ_{k_1} have a Bessel function of the argument x_{k_1} so $X_{kk_1}{}^{(1)}$ and $X_{kk_1}{}^{(2)}$ do not contribute also among $A^{(ij)}$ coefficients the only one that survive are A_{11}, A_{13}, A_{21}, A_{23}, A_{43}, so the only survive them is $X_{kk_1}(3)\ X_{k_1k_1}{}^{(4)} A^{34}$ and the symmetric one. With all of it

$$N^{(2)}(k,k_1,k') = \frac{(2\pi)^2}{\omega} \nu_{k_1} J_0(x_k) J_0(x_{k_1}) J_0(x_{k-k_1} J_0(x_{k'} - x_k) q \tag{A.5.41}$$

where $q(x) = 2 \sum_{n=1}^{\infty} \frac{J^2{}_n(x)}{n2}$ The sum (A.5.41) can be evaluated if we take the asymptotic form for the Bessel function (G&R pp. 996) for the large argument and replace

$$J_y{}^2(x) \rightarrow \frac{2}{\pi x} \cos^2(x - \frac{\pi}{4} - \frac{\pi}{2} y) \tag{A.4.42}$$

so that

$$q(x) \rightarrow \frac{\pi}{3|x|} \qquad \text{(A.5.43)}$$

where we used

$$\int_0 \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2} \qquad \text{(A.5.44)}$$

In order to find $J'(\nu_k)$ that appears in S, let us expand

$$y_i(\nu_k) = y_i(0) + \nu_k y_i'(0) + \ldots \qquad \text{(A.5.45)}$$

and

$$J_{\nu_k}'(y_i(\nu_k) = J_0'(y_i(0) + \nu_k y_i'(0)) \qquad \text{(A.5.46)}$$

where

$$L_0(x) = \frac{\pi}{2} N_0(x) \qquad \text{(A.5.47)}$$

where N_0 is Newman function.

$$\Pi\, N_0(z) = 2\, J_0(z)(\ell n\, \frac{z}{2} + C) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} (\frac{z}{2})^{2k} \sum_{k=1}^{k} \frac{1}{m} \qquad (A.5.48)$$

And

$$J_{\nu_k}{}'(y_1(\nu_k)) \approx J_0{}'(y_1(0)) + \nu_k\, (-\frac{y_1(0)}{y_1(0)}\, J_0{}'(y_1(0) + L_0{}'(y_1(0)) \qquad (A.5.49)$$

Similarly we can expand

$$J_{-\nu_k}(x_k) \to J_{-\nu_k}(y_1(\nu_k)) \approx J_0(y_1(0) + \nu_k y'(0)) - \nu_k\, L_0(y_1(0)) \qquad (A.5.50)$$

and

$$J_{-\nu_k}(x_k) \approx 2\nu_k y_1{}'(0) J_0{}'(y_1(0)) \qquad (A.5.51)$$

where

$$y_1(0) J_0{}'(y_1(0)) + L_0(y_1(0)) = 0 \qquad (A.5.52)$$

Now we can write that

$$\frac{\Gamma(3)}{2\pi} = \int \frac{d^3k\, d^3k_1\, d^3k'}{(2\pi)^2 a_k a_{k'}}\, u_1(k) \delta h_k V(k-k_1) V(k_1-k') \delta h_{k'} u_1(k')\ x$$

$$\times \sum_{i=1}^{\infty} N(k,k_1,k')\varepsilon_{k'}/|J_o'(y_i(0)|^2|y_i'(0)|)\ \delta(x_{k_1}-y_i(0)\ L_o(y_i(0)) \quad (A.5.53)$$

the rest of the calculation is described in Chapter IV.

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