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NON-GAUSSIAN MODELS OF FINANCIAL MARKETS:
SIMULATION VIA SERIES REPRESENTATIONS

by

SERIGNE MODOU DIOP

A dissertation submitted to the Graduate Faculty in Mathematics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy, The City University of New York
2005

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*To the memory of my father,
who taught me that education is one
of the few risk-free investments.*

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Introduction

For almost three decades, research on contingent claim pricing in finance, have concentrated on refining and improving the seminal work of Black and Scholes [16]. Using the assumption of normality of the stock returns $(R_t)_{t>0}$, Black and Scholes modelled the evolution of the price of a basic financial asset as a geometric Brownian motion¹. In other words, the stock price process $(S_t)_{t>0}$ solves the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (0.0.1)$$

where $\sigma > 0$ and μ denote respectively the volatility and drift coefficients, and $(W_t)_{t>0}$ is a standard Brownian motion. The rate of return R_t is defined by

$$R_t = \log S_t - \log S_{t-1}. \quad (0.0.2)$$

Itô's lemma gives the solution of Equation 0.0.1:

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma W_t). \quad (0.0.3)$$

Furthermore, by using the principle of no arbitrage opportunities, Black and Scholes derived a unique price for a European option written on the asset.

¹It should be noted that the idea of modeling stock prices with Brownian motion originated with Bachelier [4] in his 1900 dissertation : *Théorie de la Spéculation* (written in Paris), and in subsequent work (see Bachelier [5] and Bachelier [6]). He introduced much of the standard framework in financial theory: random walk of financial market prices, Brownian motion, and martingales. Osborne [56] refined Bachelier's model, in 1959, by introducing the exponential $\exp(B_t)$ Brownian motion as a stock price model. Subsequently, Samuelson [67], in 1965, also used the exponential or geometric Brownian motion to model stock prices.

However, empirical studies examining their assumption that stock price returns followed a normal distribution, have shown a systematic mispricing.

Mandelbrot [49] showed that the returns of most financial assets have semi-heavy tails. In other words, the real kurtosis is higher than the kurtosis of the normal distribution. He had proposed a non-Gaussian Lévy process, the symmetric α -stable Lévy motion with index $\alpha < 2$, as a model. This process changes values only by jumps.

Exponential Lévy processes with non-stable distributions were used by Press [58] (1967). For a log-price process, he used a superposition of a Brownian motion and an independent compound Poisson process with normally distributed jumps.

Other well documented biases of the Gaussian models include volatility smiles and skewness premia suggesting that the Black-Scholes model overprices out-of-the-money calls and in-the-money puts while it underprices in-the-money calls and out-of-the-money puts. Rubinstein [65] showed that implied volatilities tended to rise for options that are deeply in or out of the money. And Bates [12] presented evidence that put options are underpriced relative to call options.

Those deficiencies have led researchers to develop alternative, more sophisticated models.

First, Merton [53] introduced the jump diffusion models by superimposing jumps upon the geometric Brownian motion. In other words, the stock price S_t follows the equation:

$$dS_t = (\mu - \lambda k)dt + \sigma dW_t + dN_t, \quad (0.0.4)$$

where μ is the expected return from the stock, λ the rate of occurrence of the jumps, k is the average jump size measured as a proportional increase in the stock price, and

N_t is a Poisson process generating the jumps. The standard Brownian motion W_t and the Poisson process are assumed to be independent.

However, Merton's introduction of a jump process led to another problem: the price of an option was no longer uniquely determined by the principle of absence of arbitrage. Since the market was no longer complete, there was no unique equivalent measure under which the discounted stock price process was a martingale.

Jarrow & Rosenfeld [39], and Schwert [71] provided some empirical evidence of the existence of jumps in the market portfolio. Kim, Oh, & Brooks [44] also documented empirically that jumps in stock returns are indeed systematic. Other empirical papers documenting jumps in asset returns include Ball & Torus [8]; Jorion [42]; Cao, Bakshi, & Chen [19]; Anderson, Benzoni & Lund [2], and Jiang [40].

Another class of models, the pure jump models, considered that stock prices followed a pure-jump process. This modelling technique has been implemented in various economic set-ups by Cox & Ross [21], and elaborated on in a later paper by Cox, Ross, and Rubinstein [22]. Jones [41], Shirakawa [72], Naik [54], Jarrow & Madan [38], and to a lesser extent, Madan, Carr & Chang [47] also made use of pure jump models. Such a class of models has the disadvantage that the absence of arbitrage ensures no unique price of the option. Hence, the jumps cannot be hedged completely. Moreover the jump amplitude has to fulfill certain requirements. Either it has to be a finite discrete random variable or it has to be a predictable process. In this way the jumps can be controlled to some extent. Moreover, if the jump amplitude is an independent absolutely continuous random variable, the market becomes incomplete. Hence, one has to be able to fix a probability measure under which pricing the derivatives and hedging against risk can be performed. Furthermore, the stochastic evolution of

more than just the underlying asset has to be specified in order to hedge the option completely.

A third family of models, the family of stochastic volatility models was introduced to correct the assumption that the volatility is constant. Hull and White consider the following stochastic volatility model for the risk-neutral behavior of a stock price:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{Y_t} S_t dW_t \\ dY_t &= (a + bY_t)dt + cY_t S_t dV_t \\ dV_t &= \rho dW_t + \sqrt{1 - \rho^2} dZ_t, \end{aligned} \tag{0.0.5}$$

where a , b , and c may depend on Y and t . The processes W and Z are uncorrelated standard Brownian motions. However, V and W are correlated standard Brownian motions. The variable Y_t in this model is the stock's volatility. Its square root is interpreted as the stock's variance rate. Wiggins [77] and Taylor [76] also documented that the inclusion of both jumps and stochastic volatility is required in order to give a reasonable fit of the data.

It is important to note, that with respect to option pricing, what these new models gain on Black-Scholes in terms of accuracy are arguably lost because they lead to intricate problems. On the one hand it is impossible to take care of all kinds of deficiencies within a particular model. On the other hand, the numerical problems which sometimes arise are often overwhelming.

Another key issue to note about the models cited above is the fact that the emphasis is laid on the probability densities of the distribution of the stochastic processes.

In this dissertation, we use another pure jump process, the Meixner process. It originated from the theory of orthogonal polynomials and was proposed to serve as a model of financial data by Grigelionis [34] and Schoutens [69].

To corroborate the semi-heavy nature of the tail of the distribution of financial assets, we simulate the path processes of the stock prices based on two types of distributions, Gaussian and non-Gaussian. As an example of a Gaussian distribution, we use the lognormal distribution (Black-Scholes). And for non-Gaussian distributions, we use several models: the Generalized Hyperbolic (GH) model (see Eberlein and Keller [26] or Rydberg [66]), the Variance-Gamma (VG) process (see Madan, Carr and Chang [47]), and the Meixner motion (MM) . Then we compare the performance of the different models.

The innovation in this thesis is the pricing of path-dependant exotic options. We consider single-barrier and double-barrier options. Their prices depend on whether the stock prices hit the predetermined barriers. The simulations of the paths of the stock price processes are done via series representations. Series representations of Lévy processes originated from the theory of series representations of infinitely divisible random variables. An extensive presentation of the theory can be found in Marcus [50].

The thesis is organized as follows: Chapter 1 is an exposé of general results on Lévy processes and the theory of series representation of non-Gaussian Lévy processes. Chapter 2 introduces the Meixner model for stock prices. In Chapter 3 we present results of statistical tests on stock price processes and compare the fit of the different models. Chapter 4 develops pricing formulas for plain vanilla European call options. Chapter 5 is devoted to a model for pricing exotic options.

Chapter 1

The Theory of Series Representation of Lévy Processes

1.1 Introduction

Lévy processes have been used for the modelling of financial assets since 1900, beginning with Bachelier, who introduced Brownian motion as a stock price model. In 1959, Osborne [56] improved Bachelier's model by substituting the Brownian motion by the exponential Brownian motion. Later, in 1963, Mandelbrot [49] observed that the logarithm of the relative price changes of cotton futures exhibited much heavier tails than the normal distribution. He suggested replacing the exponential Brownian motion by a symmetric α -stable Lévy motion with index $\alpha < 2$. That gave rise to the first pure-jump stock price process. We should note, that normal distributions are α -stable distributions with $\alpha = 2$. In his search for a model that better fits the empirically observed distribution of the changes of the log-prices, Press [58] suggested in 1967 an exponential Lévy process model with a non-stable distribution. He constructed such a process by a superposition of a Brownian motion and an independent compound Poisson process with normally distributed jumps. More recently (1990),

Madan and Seneta [48] proposed a pure jump process, the variance gamma process, as a model for stock prices. Their choice was justified by a statistical study of Australian stock price data. In 1995, Eberlein and Keller [26] introduced the exponential hyperbolic Lévy motion as a stock price model. The same year, Barndorff-Nielsen [9] proposed a model based on an exponential normal inverse Gaussian Lévy process. The last two models are subclasses of the generalized hyperbolic distributions introduced in 1977 by Barndorff-Nielsen and Halgreen [10]. In 1999, Grigelionis [34] presented the Meixner Lévy motion as a model for pricing financial securities. Since our primary interest in this thesis is to study the extent to which pure-jump Lévy processes, can be used as financial models, it is necessary that we develop a tool to analyze the basic structures of such processes. The series representation theory will constitute such a tool. To ease the presentation, from here on, when there is no possibility of confusion, we'll call a pure-jump Lévy process, simply a Lévy process.

This chapter develops the series representation of Lévy processes. The first section introduces the general notation that we will adopt throughout the rest of the thesis. It develops the procedure for modelling securities with Lévy processes. It also addresses the issue of the existence and the uniqueness of an equivalent martingale measure for pricing derivatives securities under an incomplete market setting. The idea is to use the so-called Esscher transform introduced by Gerber and Shui [32]. The second section develops the theory of series representations of Lévy processes. Sections 3, 4, and 5 are applications of series representations to the Generalized Hyperbolic Lévy motion, the Variance Gamma process and the α -stable process.

1.2 Lévy Processes as Log-Price Models

The following basic assumptions are made throughout the rest of the thesis.

1. The stochastic process $X = (X_t)_{t \geq 0}$ is adapted to the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.
2. The filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfies the usual conditions, i.e. $(\Omega, \mathcal{F}, P)$ is complete, all the null sets of $\mathcal{F}$ are contained in $\mathcal{F}_0$, and $\mathcal{F}_t$ is a right-continuous filtration: $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ are σ -algebras for $s, t \in \mathbb{R}, s \leq t$; and $\mathcal{F}_s = \bigcap_{t > s} \mathcal{F}_t$ for all $s \in \mathbb{R}$.
3. $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ also satisfies the saturatedness condition for null sets, i.e, given A and B, if $(A \subset B \in \mathcal{F})$ and if $P(B) = 0$, then $A \in \mathcal{F}$.
4. $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$.

Definition 1.2.1. The process X is called a Lévy process if it satisfies the following:

1. $\lim_{t \rightarrow s} X_t = X_s$, where the limit is in probability.
2. $X_t - X_s$ is independent of $\mathcal{F}_s$, $s < t < \infty$.
3. $X_t - X_s$ has same distribution as X_{t-s} , $s < t < \infty$.

Definition 1.2.2. The left limit at t , X_{t-} , is defined as $\lim_{s \uparrow t} X_s$. And the jump at t is $\Delta X_t = X_t - X_{t-}$. If $\sup_t |\Delta X_t| < k$, on bounded intervals for some $k \in \mathbb{R}$, we say that X has bounded jumps.

Moreover, if X_t is a Lévy process, there is a unique version of X which is càdlàg (continue à droite avec limites à gauche). From now on, we consider the càdlàg version of the Lévy processes.

Particular cases of Lévy processes include Brownian motion, stable processes, and compound Poisson processes. A detailed treatment on Lévy processes can be found in Rogers and Williams [63], Bertoin [14] [15], Protter and Talay [60], and Sato [68].

1.2.1 Jumps of a Lévy Process

The independence of the increments implies that all Lévy processes have the Markov property, i.e. that their future and past developments at every time t are conditionally independent, given the present value.

We say that a measure Q is locally equivalent to P , denoted $Q \stackrel{\text{loc}}{\sim} P$ if there exists a density process $(Y_t)_t$ such that $Y_t = dQ_t/dP_t$ for all $t \in \mathbb{R}_+$, where Q_t and P_t denote respectively the restrictions of Q and P to the σ -algebra $\mathcal{F}_t$.

Let A be a Borel set of $\mathbb{R}$ bounded away from 0. We define the random variables:

$$H_1 = \inf\{t > 0 : \Delta X_t \in A\}, \dots, H_{n+1} = \inf\{t > H_n, \Delta X_t \in A\}, \quad n \geq 1. \quad (1.2.1)$$

Also the process N is defined as:

$$N(A) = N_t(A) = \sum_{0 < s \leq t} 1_A(\Delta X_s) = \sum_{n \geq 1} 1_{\{H_n \leq t\}}. \quad (1.2.2)$$

Definition 1.2.3. The measure τ defined by:

$$\tau(A) = E(N_t) = E\left\{ \sum_{0 < s \leq t} 1_A(\Delta X_s) \right\} \quad (A \text{ as above}), \quad (1.2.3)$$

is called the Lévy measure of the Lévy process X .

1.2.2 The Lévy-Khintchine Formula

The following result, known as the Levy-Khintchine formula defines the Fourier transform of a Lévy process. Certain authors use the formula, along with the independence

of the increments as a standard definition of a Lévy process.

Definition 1.2.4. The characteristic function or Fourier transform of the Lévy process is

$$\phi_t(u) = E(e^{iuX_t}) = e^{-t\varphi(u)}, \quad (1.2.4)$$

where

$$\varphi(u) = \frac{a^2}{2}u^2 - iub + \int_{-\infty}^{\infty} (1 - e^{i\omega u} + i\omega h(\omega))\tau(d\omega), \quad (1.2.5)$$

for some numbers $a \geq 0$ and $b \in \mathbb{R}$, and a positive measure τ , the Lévy measure on $\mathbb{R}$, satisfying the integrability condition:

$$\left| \int_{\mathbb{R}/\{0\}} (1 \wedge \omega^2)\tau(d\omega) \right| < \infty. \quad (1.2.6)$$

And $h(\omega)$ denotes an arbitrary bounded continuous function satisfying

$$h(\omega) = \omega + O(\omega^2)$$

in a neighborhood of $\omega = 0$.

Definition 1.2.5. The triplet (a, b, τ) in Equation 1.2.5 is known as the Lévy-Khintchine triplet or the Lévy triplet.

In Equation 1.2.5, if $a = 0$, then X is said to be a Lévy jump process. And if also $b = 0$, the process is called a pure-jump Lévy process.

Lévy processes are increasing if and only if $a = 0$, $b \geq 0$, and $\tau(du)$ is concentrated on $(0, \infty)$ and satisfies the stricter integrability condition:

$$\left| \int_{0+}^{\infty} (1 \wedge \omega)\tau(d\omega) \right| < \infty. \quad (1.2.7)$$

An example of a increasing Lévy process is the Poisson process with rate λ , and Lévy measure $\tau(du) = \lambda\delta_1(du)$. If X has only non-negative increments, it is called a subordinator.

The next theorem is important from the standpoint of stochastic integration. See Protter [59], Theorem 40 for its proof.

Theorem 1.2.1. *If X is a Lévy process, then $X_t = Y_t + Z_t$, where Y and Z are Lévy processes, Y is a martingale with bounded jumps, $Y_t \in L^p$ for $p \geq 1$ and Z has paths of finite variation on compact sets.*

Therefore every Lévy process is a semimartingale (the sum of a martingale and a process with finite-variation paths). The last result ensures that basic stochastic integration theory for Lévy processes is routine.

In the rest of the thesis we assume that $a = b = 0$, so that Equation 1.2.5 can be written as :

$$\varphi(u) = \int_{-\infty}^{\infty} (1 - e^{i\omega u} + iuh(\omega))\tau(d\omega). \quad (1.2.8)$$

Remark 1.2.1. The Fourier transform satisfies:

1. $\phi_t(0) = 1$.
2. $\phi_{t+s}(u) = \phi_t(u)\phi_s(u)$.

1.2.3 Generating Lévy processes from infinitely divisible distributions.

Definition 1.2.6. A random variable X with distribution function F is said to be infinitely divisible if for any n , there exist independent random variables X_j , $1 \leq j \leq n$, such that $X = \sum_{j=1}^n X_j$. In terms of the distribution function F , this means that for each n , there is a distribution function F_n , such that F is the n -fold convolution $F_n * \cdots * F_n$ (n copies) of F_n .

Let $\Phi(u)$ denote the characteristic function of an infinitely divisible probability distribution. Recall that we can define for every such infinitely divisible distribution, a stochastic process $X = (X_t)_{t \geq 0}$, which starts at 0 and has independent and stationary increments such that the distribution of an increment $X_{t+s} - X_t$ over the time interval $[t, s + t]$, $t, s \geq 0$ has characteristic function $(\Phi(u))^t$. Such a stochastic process is a Lévy process. In other words, if

$$E(e^{i\lambda X_1}) = e^{-\psi(\lambda)}, \quad (1.2.9)$$

then we define

$$E(e^{i\lambda X_t}) \equiv e^{-t\psi(\lambda)}, \quad (1.2.10)$$

where ψ is given as in the Lévy-Khintchine formula.

This procedure enables the generation of Lévy processes from infinitely divisible probability distributions.

1.2.4 Lévy Processes as Financial Models.

Consider a financial market living on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We also assume that the market consists of two assets: a bond or cash account and a stock price process. The bond is a risk free asset $(B_t)_{0 \leq t \leq T}$ evolving according to the equation $B_t = e^{rt}$, where r denotes the risk-free rate of interest. The risky asset, the stock price process, denoted by $S = (S_t)_{0 \leq t \leq T}$, is given by the equation:

$$S_t = S_0 \exp(rt) \exp(X_t), \quad (1.2.11)$$

where $(X_t)_t$ is a purely discontinuous Lévy process. The valuation of options written on the stock requires finding a measure $\mathbf{Q}$ equivalent to the original probability measure $\mathbf{P}$ such that the process $e^{-rt}S_t$ is a martingale. This makes it possible to use the

rich machinery of martingale theory. Under the Black Scholes setting, the underlying process is continuous. The Cameron-Martin-Girsanov theorem ensures the existence and the uniqueness of the equivalent measure $\mathbf{Q}$. In this case, the market is said to be complete¹.

When the underlying process is a pure-jump Lévy process, the market is incomplete. That raises two questions: Does an equivalent martingale measure exist? And if it does, is it unique? The answer is, several martingale measures exist². In this thesis, we select, for its simplicity, an equivalent martingale measure, given by the Esscher transform, adopted by authors such as Eberlein [26], Barndorff-Nielsen [9], Raible [61], etc. The introduction of the Esscher transform is due to Gerber and Shui [32].

1.2.5 The Esscher Transform

Definition 1.2.7. For a probability density function $f(x)$, let θ be a real number such that

$$M(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f(x) dx \quad (1.2.12)$$

exists. As a function of x ,

$$f(x; \theta) = \frac{e^{\theta x} f(x)}{M(\theta)} \quad (1.2.13)$$

is a probability density, and is called an Esscher transform of the original one-dimensional distribution.

¹A complete market is a market where any derivative product can be dynamically replicated via cash and the underlying asset. The Black Scholes set of assumptions corresponds to a complete market.

²Whether the martingale measures are all equivalent, to resolve eventual arbitrage issues, is still an open problem. We refer interested readers to Delbaen and Haezendonck [23] for a study on equivalent martingale measures of compound Poisson processes.

Let $(X_t)_{t \geq 0}$ denote a stochastic process with stationary and independent increments such that $X_0 = 0$. Also, let the process $(S_t)_{t \geq 0}$ be defined by

$$S_t = S_0 e^{X_t} \quad t \geq 0. \quad (1.2.14)$$

For each t , let $F(x, t)$ denote the cumulative distribution function of X_t , i.e.,

$$F(x, t) = P(X_t \leq x). \quad (1.2.15)$$

The moment generating function is

$$M(z, t) = E[e^{zX_t}] \quad (1.2.16)$$

By assuming that $M(z, t)$ is continuous at $t = 0$, we have

$$M(z, t) = (M(z, 1))^t. \quad (1.2.17)$$

If X_t has density $f(x, t)$, then

$$M(z, t) = \int_{-\infty}^{\infty} e^{zx} f(x, t) dx. \quad (1.2.18)$$

Let θ be a real number such that $M(\theta, t)$ exists. Then the ratio,

$$\begin{aligned} f(x, t; \theta) &= \frac{e^{\theta x} f(x, t)}{\int_{-\infty}^{\infty} e^{\theta y} f(y, t) dy} \\ &= \frac{e^{\theta x} f(x, t)}{M(\theta, t)}, \end{aligned} \quad (1.2.19)$$

which is another probability density of X_t , $t > 0$, is called the Esscher transform of the original density. That is, the modified distribution of X_t is the Esscher transform of the original distribution. The corresponding moment-generating function is:

$$\begin{aligned} M(z, t; \theta) &= \int_{-\infty}^{\infty} e^{zx} f(x, t; \theta) dx \\ &= \frac{M(z + \theta, t)}{M(\theta, t)}. \end{aligned} \quad (1.2.20)$$

Since the exponential function is positive, the Esscher transform is equivalent to the original probability measure; that is, both probability measures have the same null sets.

To ensure that the stock prices are internally consistent, we have to find $\theta^* = \theta$ so that the discounted price process $(e^{-rt}S_t)_{t \geq 0}$ is a martingale with respect to the probability measure corresponding to θ^* . In particular,

$$\begin{aligned} S_0 &= E^*[e^{-rt}S_t] \\ &= e^{-rt}E^*[S_t], \end{aligned} \tag{1.2.21}$$

where r denotes the constant risk-free rate of interest and the expectation is taken with respect to the risk-neutral Esscher transform.

Proposition 1.2.2. *The parameter θ^* is the solution of the equation*

$$1 = e^{-rt}E^*[e^{X_t}] \tag{1.2.22}$$

or equivalently,

$$e^{rt} = M(1, t; \theta^*). \tag{1.2.23}$$

From Equation 1.2.17 we see that the solution does not depend on t , hence we may set $t = 1$. That leads to

$$e^r = M(1, 1; \theta^*), \tag{1.2.24}$$

or

$$r = \ln[M(1, 1; \theta^*)]. \tag{1.2.25}$$

Gerber and Shui [32] showed that the parameter θ^* is unique. The Esscher transform of parameter θ^* is called the risk-neutral Esscher transform, and the corresponding martingale measure, the risk-neutral Esscher measure.

The absence of arbitrage is essentially equivalent to the existence of an equivalent martingale measure, with respect to which the price of a contingent claim is the expected discounted payoff. This result known in the financial literature as the Fundamental Theorem of Asset Pricing is the foundation of the pricing formulas of Chapter 4.

1.2.6 Characterization of The Esscher Measure

Consider a Lévy process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. The Esscher transform of P can be defined by the relation

$$dQ = e^{\theta X_t - t \log M(\theta)} dP, \quad (1.2.26)$$

where $M(u) = E^P(e^{uX_1})$ is the moment generating function of X_1 and the parameter θ is defined as the solution of the following equation, where r denotes the risk-free rate of interest

$$r = \log \frac{M(\theta + 1)}{M(\theta)}. \quad (1.2.27)$$

In other words, the quantity $e^{\theta X_t - t \log M(\theta)}$ is the Radon-Nikodym derivative of Q with respect to P . In the Black-Scholes case, $M(\theta) = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}$. Solving Equation 1.2.27 yields $\theta = (r - \mu)/\sigma^2 - 1/2$. Hence

$$dQ = \exp \left[\left(\frac{r - \mu - \sigma^2}{\sigma} \right) W_t - \frac{1}{2} \left(\frac{r - \mu - \sigma^2/2}{\sigma} \right)^2 t \right] dP. \quad (1.2.28)$$

1.2.7 Pricing Call Stock Options.

As in Gerber and Shui [32], we consider a European call option (the right, but not the obligation to buy a stock) on a stock with time to maturity T and strike price K .

The value of this option at time $t = 0$ is given by:

$$C(T, K) = e^{-rT} E^Q[(S(T) - K)^+]. \quad (1.2.29)$$

If we let $k = \ln[K/S_0]$, Equation 1.2.29 becomes

$$\begin{aligned} C(T, K) &= e^{-rt} \int_k^\infty [S_0 e^x - K] f(x, T; \theta^*) dx \\ &= e^{-rT} S_0 \int_k^\infty e^x f(x, T; \theta^*) dx - e^{-rT} K [1 - F(k, T; \theta^*)]. \end{aligned} \quad (1.2.30)$$

From Equations 1.2.19, 1.2.20, and 1.2.23, we derive:

$$e^x f(x, T; \theta^*) = e^{rT} f(x, T, \theta^* + 1). \quad (1.2.31)$$

Thus the value of the European call option is obtained upon calculating the integral in Equation 1.2.30. We obtain

$$C(T, K) = S_0 [1 - F(k, T; \theta^* + 1)] - e^{\delta T} K [1 - F(k, T; \theta^*)]. \quad (1.2.32)$$

1.3 The Series Representation of Symmetric Lévy Processes

Series representations for real, independent increment stochastic processes without Gaussian components can be traced back to Ferguson and Klass [30]. Kallenberg [43] showed the uniform convergence in the Ferguson-Klass representation. Resnick [62] extended the Ferguson class representation to homogenous, independent increment processes with positive and negative jumps via the Itô-Lévy decomposition. A series representation of Hilbert-space-valued stable random vectors, which generalized and

improved the Ferguson-Klass representation of one dimensional stable random variables, was established by LePage, Woodroffe and Zine [45]. LePage [46] observed that symmetric stable random vectors could be represented as conditionally Gaussian processes. This important property was generalized and used by Marcus and Pisier [51] in their investigation of continuity of stable processes. Marcus and Pisier's work showed the significance of series decompositions in the study of stable probability measures on Banach spaces. Another generalization of the Ferguson-Klass representation (one-dimensional Lévy measure concentrated on positive reals) to the case of random vectors taking values in Banach spaces is also due to LePage [46]. Rosinski [64] gave a simple and general scheme of deriving series representations of arbitrary Banach Space valued infinitely divisible random vectors. For a current survey of series representations, see Rosinski [64].

Let $X = (X_t)_{0 \leq t \leq 1}$ be a Lévy process in $\mathbb{R}$ with Lévy measure τ and characteristic function given by

$$E[\exp(iuX_t)] = \exp t \left[\int_{\mathbb{R}/\{0\}} (e^{iux} - 1 - iux1_{\{|x|<1\}}) \tau(dx) \right]. \quad (1.3.1)$$

The case $\int_{\{|x| \leq 1\}} |x| \tau(dx) < \infty$ is referred to as the finite variation case, and the case $\int_{\{|x| \leq 1\}} |x| \tau(dx) = \infty$ is the compensated case. The term corresponding to $x1_{\{|x|<1\}}$ is used for centering purposes and is necessary for convergence in the compensated case.

The next section is adapted from Marcus [50]. We first develop series representations of a single infinitely divisible random variable.

1.3.1 Series Representation of a Random Variable

We adopt the following notation for the cumulant transform of the random variate x :

$$\Psi(\theta; x) = \log E(e^{-\theta x}), \quad (1.3.2)$$

and the Laplace transform

$$\bar{L}(\theta; x) = \exp(\Psi(\theta; x)). \quad (1.3.3)$$

Lemma 1.3.1. *Let Y be a non-negative random variable satisfying*

$$P(Y > \lambda) = e^{-\lambda}. \quad (1.3.4)$$

And let $\{Y_j\}_{j=1, \dots, \infty}$ be i.i.d. copies of Y . If we set

$$\Gamma_j = Y_1 + Y_2 + \dots + Y_j, \quad (1.3.5)$$

then

$$P(\Gamma_i < x) = \int_0^x \frac{u^{j-1} e^{-u}}{(j-1)!} du. \quad (1.3.6)$$

See Feller [29], p10, for a proof of the lemma.

Remark 1.3.1. The (Γ_j) can be considered to be the arrival times of a Poisson process with rate 1.

Now let X be a real-valued, symmetric, infinitely divisible random variable, with characteristic function given by:

$$E(e^{iuX}) = e^{-\varphi(u)}, \quad (1.3.7)$$

where τ satisfies

$$\varphi(u) = \int_{\mathbb{R}/\{0\}} (1 - \cos(ut)) d\tau(t) < \infty. \quad (1.3.8)$$

For convenience, we can extend the Lévy measure $\tau(dt)$ to $\mathbb{R}$ by setting $\tau(\{0\}) = 0$.

Also, let h be a random variable, independent of X such that

$$\tau_h[t, \infty) = E(\tau[t/|h|, \infty)) < \infty, \quad t > 0. \quad (1.3.9)$$

Theorem 1.3.2. *If the Lévy measure τ_h satisfies*

$$\left| \int_{\mathbb{R}/\{0\}} (1 \wedge |t|) d\tau_h[t, \infty) \right| < \infty, \quad (1.3.10)$$

then

$$X_t \stackrel{d}{=} \sum_{j=0}^{\infty} \tau^{-1}(\Gamma_j) h_j, \quad (1.3.11)$$

where $(h_j)_j$ are i.i.d. copies of h and

$$\tau^{-1}(t) = \sup\{u : \tau[u, \infty) > t\} \quad (1.3.12)$$

is the inverse of the Lévy measure. Furthermore, the series converges.

Proof. Let n be a positive number and define

$$K_n = \#\{j : \Gamma_j < n\}. \quad (1.3.13)$$

Then

$$P(K_n = k) = e^{-n} \frac{n^k}{k!}. \quad (1.3.14)$$

(See Feller [29], page 11.)

Now let $U = (U_1, U_2, \dots, U_n)$ be a sampling from a uniform $(0, 1)$ distribution, and $U_{(1)} < U_{(2)} < \dots < U_{(n)}$ be the first order statistics of U . Also let

$$S = \sum_{j=1}^{\infty} \tau^{-1}(\Gamma_j) h_j \quad (1.3.15)$$

and

$$S_n = \sum_{j=1}^{K_n} \tau^{-1}(\Gamma_j) h_j. \quad (1.3.16)$$

Conditional on K_n , the joint law of $(\Gamma_1/a, \Gamma_2/a, \dots, \Gamma_k/a)$ equals the joint law of $(U_{(1)}, U_{(2)}, \dots, U_{(k)})$, for any $a > 0$.

Conditioning on K_n and using product integration, the Laplace transform of S_n $\bar{L}(\theta; S_n) = E(e^{-\theta S_n})$ gives:

$$\begin{aligned} \bar{L}(\theta; S_n) &= E\left\{\prod_{j=1}^{K_n} \exp\{\Psi\{\theta; \tau^{-1}(\Gamma_j) h_j\}\}\right\} \\ &= E\left\{\prod_{j=1}^{K_n} \exp\{\Psi\{\theta; \tau^{-1}(nU_{(j)}) h_j\}\}\right\} \\ &= E\left\{\prod_{j=1}^{K_n} \exp\{\Psi\{\theta; \tau^{-1}(nU_j) h_j\}\}\right\} \\ &= E\{K_n \exp\{\Psi\{\theta; \tau^{-1}(nU) h\}\}\}. \end{aligned} \quad (1.3.17)$$

Since K_n follows a Poisson distribution, we obtain

$$\bar{L}(\theta; S_n) = E\left\{e^{-n} \sum_{j=0}^{\infty} \frac{n^j}{j!} \exp\{j\Psi\{\theta; \tau^{-1}(nU) h\}\}\right\} = \exp\{n(\bar{L}(\theta; \tau^{-1}(nU) h) - 1)\}. \quad (1.3.18)$$

But since

$$\exp\{n(\bar{L}(\theta; \tau^{-1}(nU) h) - 1)\} \rightarrow E\left[\int_{R^+} (1 - e^{-\theta x}) d\tau^{-1}(x)\right] \quad \text{as } n \rightarrow \infty$$

then

$$\bar{L}(\theta; \tau^{-1}(nU) h) = E\left[n^{-1} \int_{\tau^{-1}(U)}^{\infty} (1 - e^{-\theta x}) d\tau^{-1}(x)\right] + 1. \quad (1.3.19)$$

Hence

$$\bar{L}(\theta; S_n) = \lim_{n \rightarrow \infty} \bar{L}(\theta; S) \quad (1.3.20)$$

and the series S_n converges in distribution to S , which implies that S converges and is equal to X . $\square$

1.3.2 Series Representation of a Lévy Process

Corollary 1.3.3. *Let $X = (X_t)_t$ be a symmetric Lévy process. Also let $(\varepsilon_j)_j$ be a Rademacher sequence, i.e., a sequence of symmetric, independent random variables satisfying $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = 1/2$. Then*

$$X_t \stackrel{d}{=} \sum_{j=0}^{\infty} \varepsilon_j \tau^{-1}(\Gamma_j) 1_{[0,t]}(U_j), \quad 0 \leq t \leq 1 \quad (1.3.21)$$

where $1_{[0,t]}$ denotes the indicator function of the interval.

1.3.3 Series Representation of The Gamma Process

The density function of the $\text{gamma}(a, b)$ distribution with parameters $a > 0$ and $b > 0$ is given for $x > 0$ by

$$f_{\Gamma}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-xb}. \quad (1.3.22)$$

The characteristic function is

$$\phi_{\Gamma}(u; a, b) = \left(1 - \frac{iu}{b}\right)^{-a}. \quad (1.3.23)$$

The characteristic function is infinitely divisible. Therefore we can define the gamma process $(X_t)_{t \geq 0}$ with parameters $a, b > 0$ as the stochastic process which starts at zero and has stationary and independent gamma distributed increments. Precisely, time enters the first parameter, i.e., X_t follows a $\text{gamma}(at, b)$ distribution.

The Lévy triplet of the gamma process is given by

$$(a(1 - \exp(-b))/b, 0, a \exp(-bx) x^{-1} 1_{\{x>0\}} dx).$$

The Lévy measure τ of X_1 is concentrated on $(0, \infty)$. Its tail is given by:

$$\tau(x) = \tau([x, \infty)) = \int_x^{\infty} u^{-1} e^{-u/b} a du = a E_1(x/b), \quad (1.3.24)$$

where the exponential integral $E_p(x) = \int_x^\infty u^{-p} e^{-u} du$.

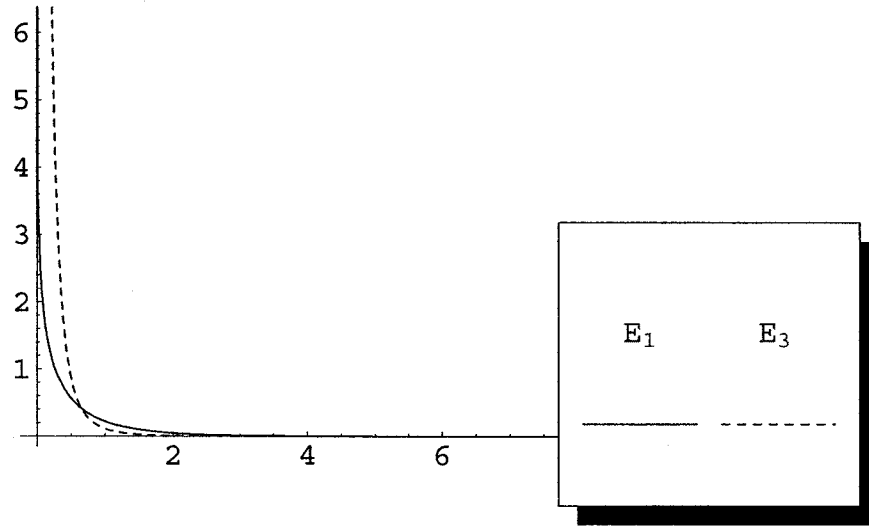


Figure 1.1: The graph of the exponential integral E_p , for $p = 1$ and $p = 3$.

The series representation of X is then given by:

$$X_t = \sum_{i=1}^{\infty} bE_1^{-1}(\Gamma_i/a)1_{\{h_i \leq t\}} \quad (1.3.25)$$

Figure 1.2 below, is a plot of the simulated jumps of a gamma process with parameters $a = 10$, $b = 2$. Figure 1.3 represents a simulated path of the gamma process for the same parameters. The jumps in Figure 1.2 are used in the series representation of the gamma process to generate Figure 1.3.

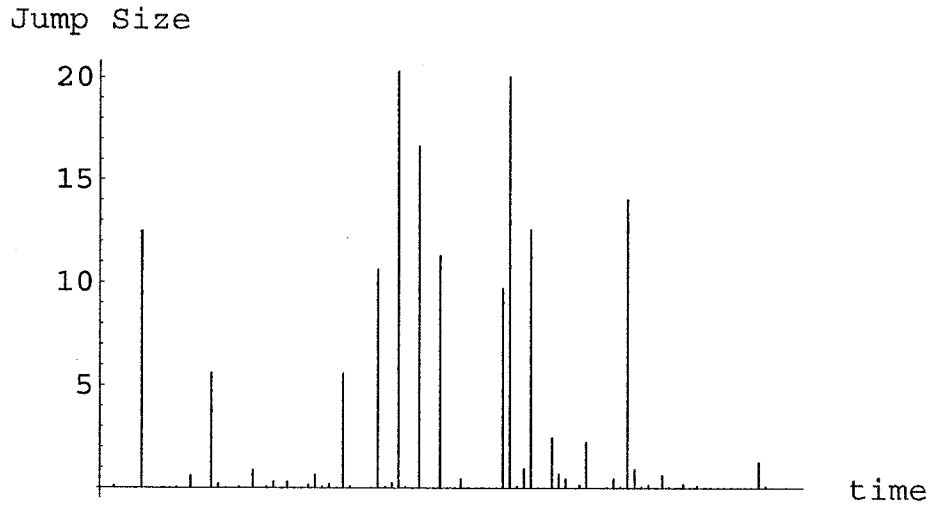


Figure 1.2: The jumps of a gamma process- $a = 10, b = 2$.

1.3.4 Series Representation of The Symmetric Stable Processes

Consider a stable distribution with characteristic function

$$\varphi(s) = e^{-|s|^\alpha} \quad 0 < \alpha < 1. \quad (1.3.26)$$

The Lévy measure is given by

$$\tau(y) = C_\alpha y^{-(1+\alpha)}, \quad (1.3.27)$$

where $C_\alpha = \alpha/\Gamma(1 - \alpha)$. Hence the inverse of the Lévy measure is given by :

$$\tau^{-1}(u) = \frac{C_\alpha^{1/\alpha}}{\alpha} u^{-1/\alpha}. \quad (1.3.28)$$

And the stable process can be represented by

$$X_t = \frac{C_\alpha^{1/\alpha}}{\alpha} \sum_{i=1}^{\infty} \varepsilon_j \Gamma_n^{-1/\alpha} 1_{\{U_j \leq t\}}. \quad (1.3.29)$$

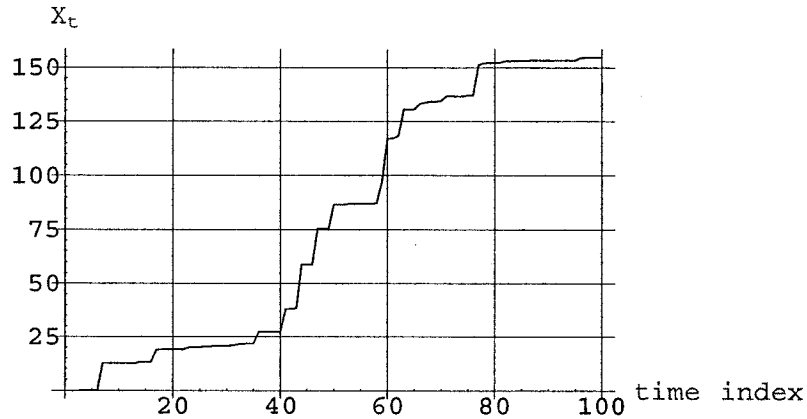


Figure 1.3: Simulated path of a gamma process- $a = 10, b = 2$.

Remark 1.3.2. Direct computation shows

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx \right)^{-1}. \quad (1.3.30)$$

There is a vast literature that focuses on proving that symmetric stable processes are in general a poor model for stock prices. See for example Barnea and Downes [11]; and Hagerman [35].

1.4 The Hyperbolic Lévy motion

Hyperbolic distributions were introduced by Barndorff-Nielsen [10](1977) in the context of the sand project as a variance-mean mixture of normal and generalized inverse Gaussian(GIG) distributions . They appear in various scientific fields: the modelling of the distribution of particle size from aeolian sand deposits, turbulence theory, and insurance theory, to name a few. Hyperbolic distributions are characterized by their

log-density being a hyperbola.

Definition 1.4.1. The one-dimensional hyperbolic distribution is defined by the following Lebesgue density, for any $x \in \mathbb{R}$,

$$\begin{aligned} gh(x; \lambda, \alpha, \delta, \mu) &= a(\lambda, \alpha, \beta, \delta) (\delta^2 + (x - \mu)^2)^{(\lambda - \frac{1}{2})/2} \\ &\quad \times K_{\lambda - 1/2} \left(\alpha \sqrt{\delta^2 + (x - \mu)^2} \exp(\beta(x - \mu)) \right), \quad (1.4.1) \\ a(\lambda, \alpha, \beta, \delta) &= \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi} \alpha^{\lambda - 1/2} \delta^\lambda K_\lambda(\delta \sqrt{\alpha^2 - \beta^2})}, \end{aligned}$$

where K_λ is a modified Bessel function of order λ . The parameters satisfy $\mu \in \mathbb{R}$ and

$$\begin{aligned} \delta &\geq 0, |\beta| < \alpha & \text{if } \lambda > 0 \\ \delta &> 0, |\beta| < \alpha & \text{if } \lambda = 0 \\ \delta &> 0, |\beta| \leq \alpha & \text{if } \lambda < 0. \end{aligned} \quad (1.4.2)$$

Also α and β determine the shape of the distribution, while δ and μ are scale and location parameters, respectively.

Hyperbolic distributions are infinitely divisible. This was shown by Barndorff-Nielsen and Halgreen [10] by proving infinite divisibility of the Generalized Inverse Gaussian distribution, which is used in the representation of the Hyperbolic distribution as a mixture of Normals. The infinite divisibility property implies that they generate a Lévy process, i.e. a process with stationary and independent increments such that the distribution of X_1 and thus of $(X_t - X_{t-1})$ is a Generalized Hyperbolic distribution. Such a Lévy process is known as the Hyperbolic Lévy motion. The density of the Lévy measure is

$$g(x; \zeta, \delta) = \frac{1}{\pi^2 |x|} \int_0^\infty \frac{\exp(-|x| \sqrt{2y + \frac{\zeta^2}{\delta}})}{y(J_1^2(\delta \sqrt{2y}) + Y_1^2(\delta \sqrt{2y}))} dy + \frac{\exp(-|x| \frac{\zeta}{\delta})}{|x|}. \quad (1.4.3)$$

Here J_1 and Y_1 are the Bessel functions of the first and second kinds respectively. Asymptotic relations about J_1 and Y_1 (formulae 9.1.7,9 and 9.2.1,2 in Abramowitz and Stegun [1]) show that the denominator of the integrand is asymptotically equivalent to a constant for $y \rightarrow 0$ and to $y^{-1/2}$ for $y \rightarrow \infty$. This remark is used when computing the inverse of the Lévy measure in the simulation process. It also implies that $g(x)$ behaves like $1/x^2$ at the origin, which means that there is an infinite number of small jumps in every finite time-interval. See Eberlein, Keller, & Prause [27] for a detailed development of the Generalized Hyperbolic motion.

Graph 1.4 represents a simulated path of the Hyperbolic process for $\delta = 0.03$, $\beta = -1.22$, $\alpha = 18.66$, $\mu = 0.15$, and $\lambda = -0.5$. The step in the time interval is one trading day, or $1/256$ years.

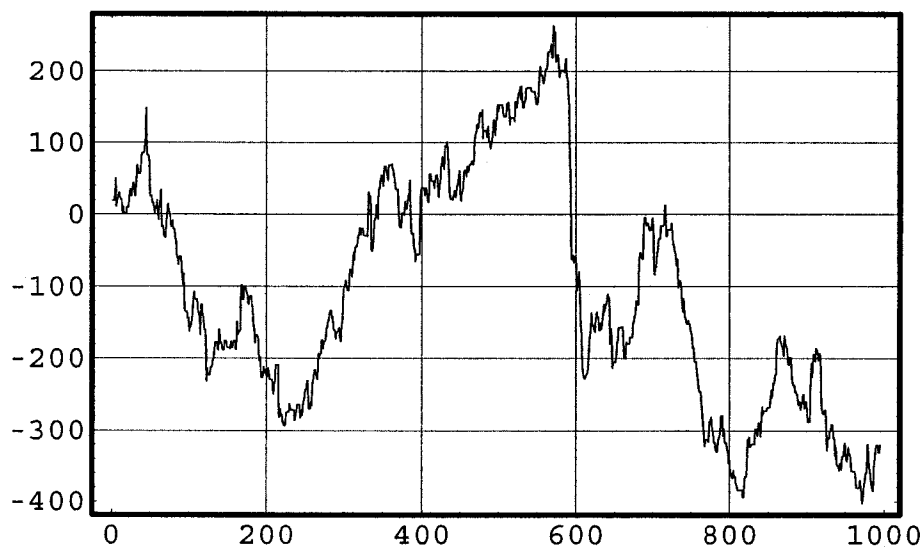


Figure 1.4: A simulated path of the hyperbolic process ($\delta = 0.03$, $\beta = -1.22$, $\alpha = 18.66$, $\mu = 0.15$, $\lambda = -0.5$.)

1.5 The Variance Gamma Process

The variance gamma (VG) process is obtained by evaluating a Brownian motion with constant drift and volatility at a random time change given by a gamma process. Consider the Brownian motion with drift θ and volatility σ

$$b(t; \theta, \sigma) = \theta t + \sigma W_t, \quad (1.5.1)$$

where W_t is a standard Brownian motion. The variance gamma process is the process $b(\gamma_t; \theta, \sigma)$. The process $\gamma_t = \gamma(t; \mu, \nu)$ has independent increments with gamma distribution over non-overlapping intervals of time, with mean rate $\mu > 0$ and variance rate ν . In other words, the density f_{γ_t} of the increment $I = \gamma_t(t+h; \mu, \nu) - \gamma_t(t; \mu, \nu)$ of γ_t is given by:

$$f_{\gamma}(I) = \frac{\mu}{\nu} \frac{\mu^{\frac{\mu^2 h}{\nu}} I^{\frac{\mu^2 h}{\nu} - 1} \exp(-\frac{\mu}{\nu} I)}{\Gamma(\frac{\mu^2 h}{\nu})} \quad I > 0, \quad (1.5.2)$$

where $\Gamma(x)$ is the gamma function. The VG process is in the class of subordinated processes introduced by Clark [20] in 1973. The density function for the VG process at time t is given by

$$f_{X_t}(\mu, \nu, \sigma, \theta) = \int_0^\infty \frac{1}{\sigma \sqrt{2\pi y}} \exp\left(-\frac{x - \theta y^2}{2\sigma^2 y}\right) \frac{y^{t/\nu} \exp(-y/\nu)}{\nu^{t/\nu} \Gamma(t/\nu)} dy. \quad (1.5.3)$$

The Lévy measure is given by:

$$g(x; \theta, \nu) dx = \frac{\exp(\theta x / \sigma^2)}{\nu |x|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}} |x|}{\sigma}\right) dx. \quad (1.5.4)$$

For a detailed development of the variance gamma process, see Madan, Carr & Chang [47].

Figure 1.5 is a simulated path of the variance gamma process for $\theta = -17.71$, $\nu = 13.22$, and $\sigma = 0.3$.

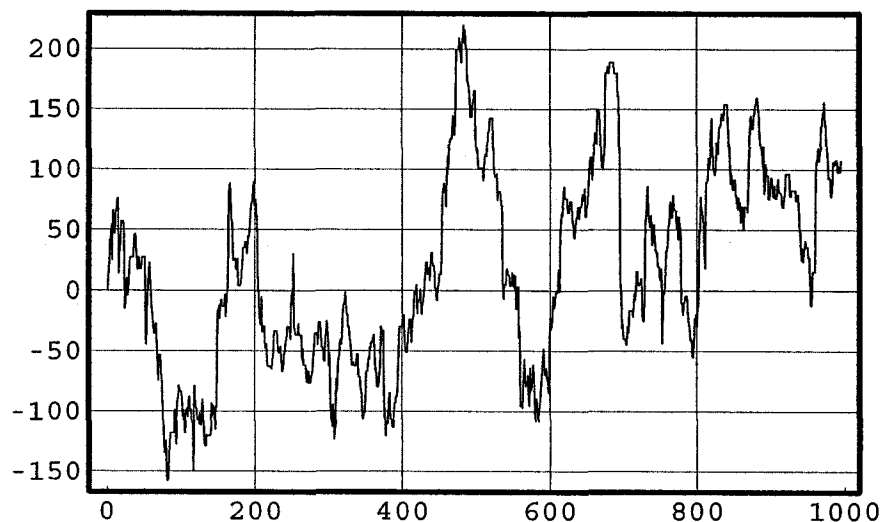


Figure 1.5: A simulated path of the variance gamma process- $\theta = -17.71$, $\nu = 13.22$, and $\sigma = 0.3$.

The simulation of the paths of the VG process, the generalized hyperbolic process, and the Meixner process are done via series representation. The only difficulty is the computation of the inverse of the Lévy measure. Since no closed-form formula is available, they were computed numerically. This is merely due to the relatively complicated forms of the density function of their distribution.

Chapter 2

The Meixner Model

2.1 Preliminaries

The use of non-Gaussian Lévy processes for modelling observational series in finance is growing at a fast pace. The desirable key features are semiheaviness of tails and self-decomposability of the one-dimensional marginal distribution. Self-decomposability is an important property because if markets are efficient and stable, then one expects price movements to be independent. In that case, since the price of a security at time $T + h$ is equal to the price at time T , plus all the price movements in between, the distribution of the price $(S_{T+h} - S_T)$ should be decomposable.

Definition 2.1.1. A probability distribution with density function f has semiheavy tails if the following holds:

$$\begin{aligned} f(x) &\sim C_- |x|^{\rho_-} e^{-\sigma_- |x|}, & \text{as } x \rightarrow -\infty, \\ f(x) &\sim C_+ |x|^{\rho_+} e^{-\sigma_+ |x|}, & \text{as } x \rightarrow +\infty, \end{aligned} \tag{2.1.1}$$

for some constants $\rho_+, \rho_- \in \mathbb{R}$, $C_+ > 0$, $C_- > 0$, $\sigma_+ > 0$, $\sigma_- > 0$.

Definition 2.1.2. A probability law of a random variable X with probability density function f is said to be self-decomposable if for any constant c , $0 < c < 1$, there exists

an independent random variable X_c such that:

$$X \stackrel{d}{=} cX + X_c. \quad (2.1.2)$$

In terms of the probability density function f , it means that there exists a characteristic function $\hat{f}_c(x)$ such that $\hat{f}(x) = \hat{f}(cx)\hat{f}_c(x)$, $x \in \mathbb{R}$ where

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{iux} f(x) dx. \quad (2.1.3)$$

Barndorff-Nielsen [10][9] introduced a famous class of self-decomposable distributions, the Generalized Hyperbolic distributions.

Another famous class of self-decomposable distributions, the z-distribution was defined by Prentice [57] in 1975. The Meixner distribution $M(\alpha, \beta, \delta, \mu)$ is in the class of z-distributions. It was introduced by Meixner [52] in 1934 and popularized by Grigelionis [34] in 1998 and Schoutens [70] in 2001.

The next section introduces the Meixner distribution. Section 3 is devoted to the Lévy measure of the Meixner distribution. Section 4 presents the loglikelihood function of the Meixner model.

2.2 The Meixner Distribution

The density of the Meixner¹ distribution $M(a, b, c, d, m)$ is given by:

$$f(x; a, b, c, d, m) = \frac{\cos^{2c}(b/2)}{2a\pi\Gamma(2c)} \exp\left(\frac{b(x-m)}{a}\right) \left|\Gamma\left(c + \frac{i(x-m)}{a}\right)\right|^{2d}, \quad (2.2.1)$$

where $a > 0$, $-\pi < b < \pi$, $c > 0$, $m \in \mathbb{R}$, and $d \geq 1$. To simplify the presentation, we set $d = 1$.

¹We have to mention, that we added the fifth parameter d to the original Meixner distribution, to create more flexibility when fitting empirical data. The original Meixner distribution uses only the parameters a , b , c , and m .

The Meixner distribution is a special case of the Generalized z -distribution. The Generalized z -distribution denoted by $GZD(a, b_1, b_2, d, m)$ can be defined through the characteristic function by

$$\phi_{GZD}(z; a, b_1, b_2, d, m) = \frac{B(b_1 + \frac{Iaz}{2\pi}, b_2 - \frac{Iaz}{2\pi})^{2d}}{B(b_1, b_2)} \exp(imz), \quad (2.2.2)$$

where $a, b_1, b_2, d > 0, m \in \mathbb{R}$. B denotes the Euler Beta function defined by:

$$B(b_1, b_2) = \Gamma(b_1)\Gamma(b_2)/\Gamma(b_1 + b_2) = \int_0^1 t^{b_1-1}(1-t)^{b_2-1}dt \quad (2.2.3)$$

The Generalized z -distribution is infinitely divisible. So it generates a Lévy process such that its distribution at time t has characteristic function $\phi(z, b_1, b_2, dt, mt)$.

The density of the Lévy measure for this distribution is given by

$$\begin{aligned} \tau_{GZ}(x) &= \frac{2d \exp(2\pi b_1 x/a)}{|x|(1 - \exp(-2\pi x/a))} & \text{if } x < 0; \\ \tau_{GZ}(x) &= \frac{2d \exp(2\pi b_2 x/a)}{x(1 - \exp(-2\pi x/a))} & \text{if } x \geq 0. \end{aligned} \quad (2.2.4)$$

Upon setting $b_1 = \frac{1}{2} + \frac{b}{2\pi}$ and $b_2 = \frac{1}{2} - \frac{b}{2\pi}$, we obtain the Meixner process.

The Meixner distribution $M(a, b, c, d, m)$ has semiheavy tails. This was shown by Grigelionis [34] by showing that

$$\begin{aligned} f(x; a, b, c, d, m) &\sim C_- |x|^{\rho_-} \exp(-\sigma_- |x|) & \text{as } x \rightarrow -\infty \\ f(x; a, b, c, d, m) &\sim C_+ |x|^{\rho_+} \exp(-\sigma_+ |x|) & \text{as } x \rightarrow +\infty, \end{aligned} \quad (2.2.5)$$

where

$$\rho_- = \rho_+ = 2c - 1, \sigma_- = (\pi - b)/a, \sigma_+ = (\pi + b)/a.$$

Grigelionis also proved that the moments of all order of the Meixner distribution exist. If a random variable X follows a Meixner distribution $M(a, b, c, m)$, then the

mean, the variance, the kurtosis Θ , and the skewness Υ are expressed respectively as:

$$\begin{aligned} E[X] &= m + \arctan(b/2) \\ E[(X - E[X])^2] &= \frac{a^2 c}{2} (\cos^{-2}(b/2)) \\ \Theta &= 3 + \frac{3 - 2 \cos^2(b/2)}{c} \\ \Upsilon &= \sqrt{(c/2) \sin(b/2)} \end{aligned} \quad (2.2.6)$$

It follows from the third equation that the kurtosis of the Meixner distribution is always greater than the kurtosis of the normal distribution.

The characteristic function of the Meixner distribution is given by:

$$\phi(u) = E[(\exp(iuX))] = \frac{\cos^{2c}(b/2)}{\cosh^{2c}(\frac{au - ib}{2})} \exp(imu). \quad (2.2.7)$$

It is obvious that $\phi(u)$ can be written as the n^{th} power of another characteristic function. It follows that the Meixner distribution is an infinitely divisible distribution. Hence it generates a process which starts at 0 and has stationary and independent increments. Such a Lévy process, the Meixner motion will be denoted by $M_t, t \geq 0$. The distribution of M_t is given in terms of the distribution of the Meixner distribution $M(a, b, ct, dt, mt)$. The parameter m determines the drift of the process.

2.2.1 The inverse of the Lévy Measure

The Lévy measure τ determines how the jumps occur. Jumps of sizes in a set A occur according to a Poisson process with parameter $\int_A \tau(dx)$. The Meixner Lévy motion is a pure jump process. It has no Brownian part. This was shown by Grigelionis [34]. The Lévy measure $\tau(dx)$ is given by

$$\tau(dx) = c \frac{\exp(bx/a)}{x \sinh(\pi x/a)} dx. \quad (2.2.8)$$

To simplify the notation, Equation 2.2.8 can be written as

$$\tau(u) = \tau([u, \infty)) = \int_u^\infty c \frac{\exp(bx/a)}{x \sinh(\pi x/a)} dx. \quad (2.2.9)$$

Since the function τ is a bijection, then it is invertible. We use what we denote as the computational formula 2.2.4 for all the computations.

Proposition 2.2.1. *The Lévy measure satisfies $\int_{-1}^1 |x| \tau(dx) = \infty$. It follows that the Meixner process is of infinite total variation.*

We refer the reader to Grigelionis [34], Page 37 for a proof of the proposition.

The difficulty in the simulation is that the inverse of the Lévy measure cannot be explicitly given ². Figure 2.1 and Figure 2.2 give the graphs of the Lévy measure and its inverse respectively.

Figure 2.3 represents a simulated path of the Meixner process for $a = 0.21$, $b = -0.38$, $c = 1.02$, and $d = 1$.

²Wolpert and Ickstadt [78] have developed a procedure to approximate the exponential integral function, using the S-Plus software.

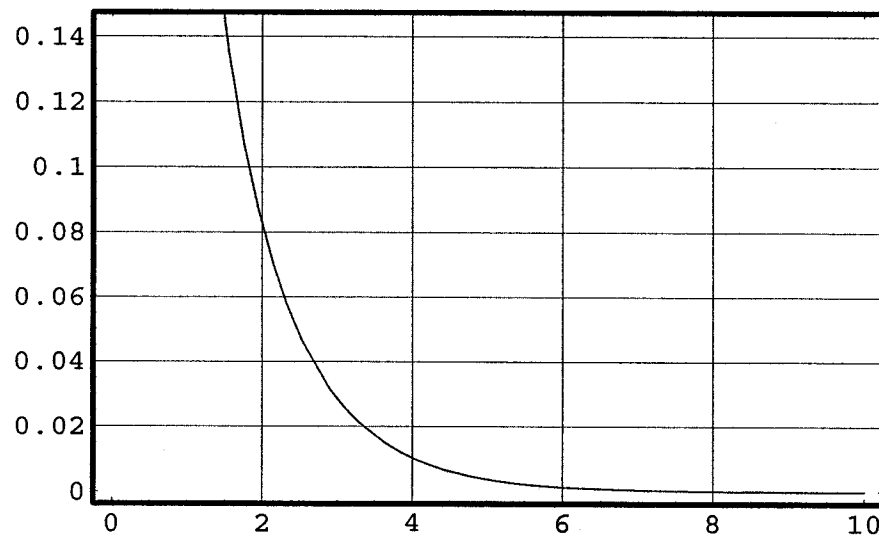


Figure 2.1: Graph of the Lévy measure $\tau(x)$ - $a = 0.021$, $b = -0.38$, $c = 1.02$, $d = 1$.

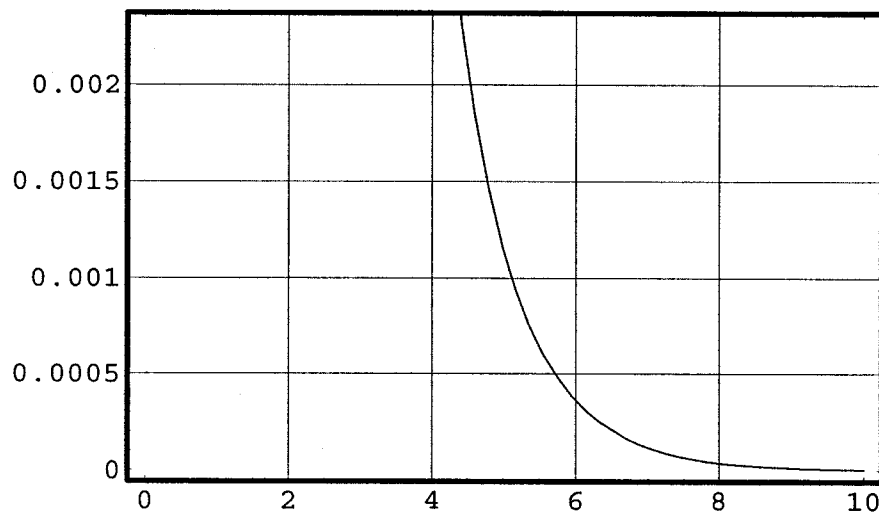


Figure 2.2: Graph of the inverse of the Lévy measure $\tau^{-1}(x)$ - $a = 0.021$, $b = -0.38$, $c = 1.02$, $d = 1$

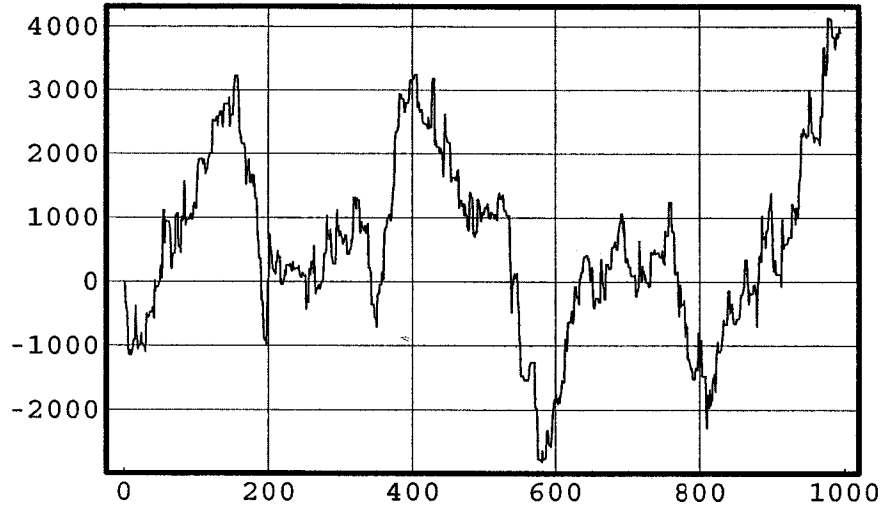


Figure 2.3: A simulated path of the Meixner process- $a = 0.021$, $b = -0.38$, $c = 1.02$, $d = 1$

2.2.2 Likelihood Function of The Meixner Model

Given the independent observations $x = (x_i), i = 1, \dots, n$, drawn from a Meixner(a, b, c, d, m) distribution, the likelihood function L is defined by:

$$L = L(x, \theta) = \prod_{i=1}^n f(x_i; a, b, c, d, m), \quad (2.2.10)$$

where f is the density of the Meixner distribution. Expanding the right hand side of the equation yields

$$L = \prod_{j=1}^n 2c \log(\cos(b/2)) - \log(2a\pi\Gamma(2c)) + \frac{b(x_j - m)}{a} + 2 \log \left| \Gamma\left(c + \frac{i(x_j - m)}{a}\right) \right|. \quad (2.2.11)$$

However, maximizing L is equivalent to maximizing the log-likelihood function $\log(L)$.

The power series expansion of $\log L$ is given by

$$\begin{aligned} \log L = & \frac{1}{a}(-n(bm + a \log(2) - ad \log(4) + a \log(a) + a \log(\pi) - 2ac \log(\cos(b/2))) \\ & + a \log(\Gamma(2d))) + 2a \sum_{j=1}^n \log(\Gamma(c + \frac{i(-m + x_j)}{a})) + b \sum_{j=1}^n x_j. \end{aligned} \quad (2.2.12)$$

Solving the equations ($\partial_a L = 0, \partial_b L = 0, \partial_c L = 0, \partial_d L = 0, \partial_m L = 0$) yields closed-form formulas for the estimates $\hat{a}$ and $\hat{b}$ of a and b respectively. The estimates of c , d , and m are obtained numerically. Also note that:

$$\begin{aligned} \hat{a} &= \frac{-1}{n}(-bmn + b \sum_{j=1}^n x_j) \\ \hat{b} &= 2 \arctan \frac{-mn + \sum_{j=1}^n x_j}{acn} \end{aligned} \quad (2.2.13)$$

2.3 Conclusion

We have introduced in this chapter a model for stock price processes, the Meixner Lévy motion. Although the Meixner process was introduced in 1975 by Prentice [57], it was only recently used (1998) to model financial securities by Grigelionis [34] and Schoutens [69]. To make the model more flexible, we extend the number of parameters. In contrast with Schoutens who estimated the parameters by the method of moments(MM), we obtained an estimation of the model parameters by the maximum likelihood (ML) method. The ML estimates are known to be more efficient and less biased than the MM estimates.

Chapter 3

Statistical Tests and Performances

3.1 Introduction

The evaluation of risk and the pricing of financial securities and their derivatives are based on statistical models describing the underlying assets. The quality of the evaluation of the risk and the validity of the prices are therefore strongly dependent on how well the statistical model captures the characteristics of the underlying asset prices. Therefore, a model that captures the most intrinsic features of the price process should produce better results. In this chapter, we study two key properties of financial time series: the distribution of the returns of the stock price process and the assumption of stationarity of the increments. The financial data consists of a discrete time series $S_1, S_2, S_3, \dots, S_n$, where S_i denotes the quoted price of a certain security at a fixed time interval $[i, i + 1]$.

The observed series $S = \{S_k, k \in [0, T]\}$ is a realization of a certain stochastic process S . Note that it is sufficient to restrict the index set of S to be the same as that of the realization. With that convention, the stochastic process can be described by a T -dimensional probability distribution, so that the relationship between a realization

and a stochastic process is analogous to that between the sample and the population in classical statistics.

We should however, emphasize that the procedure of using a single realization to infer the unknown parameters of a joint probability distribution is valid only if the process is ergodic, which roughly means that the sample moments for finite stretches of the realization approach their population counterparts as the length of the realization becomes infinite. However, since it is impossible to test for ergodicity using just a single realization, it will be assumed for the rest of this chapter, that all time series are ergodic.

In this chapter, we perform some statistical tests on stock price processes, to determine to what extent the different Lévy processes can be used to model financial securities. The statistical tests of stationarity and normality are performed with the SASTM statistical software¹. We also use maximum likelihood estimation to produce the parameters of the four models: the Black-Scholes model, the VG model, the Hyperbolic model, and the Meixner model. The model parameters are fitted against the empirical measures of central tendency and spread derived from the observed prices to compare the performances of the different models.

The second section of this chapter briefly introduces the data. The results of the normality test are presented in the third section. The fourth section is devoted to the test of stationarity. And Section 5 presents the parameter estimates of the four models. In Section 6, we present a comparative study of the models.

¹SASTM is a registered trademark of the SAS Institute.

3.2 The data

Since the evaluation of the prices of derivative securities are based on statistical models describing the underlying asset prices, the quality of the evaluation is strongly dependent on how well the statistical model captures the salient behavior of the asset. To get an insight into the microstructure of the asset, we collect stock price data and index price data. We then analyze the data to test if the general properties underlying Lévy processes are satisfied.

The first part of the data consists of the daily prices of three actively traded stocks on the NASDAQ : Dell Computer Corp.(DELL), Microsoft Corp.(MSFT), and International Business Machines Corp.(IBM). The second part of the data represents the daily values of the S&P-500 index² traded on the NASDAQ market and the values of the CAC-40³ index traded on the Paris Exchange. We consider the daily prices during the period from January 1998 to December 2000. This gives a time-series of 757 prices for each stock. We also consider the quoted tick prices of MSFT on November 3, 1999. The tick is the smallest allowable increment in price movement for a contract. Thus, the time lag for a tick price to be quoted is in general less than 10 seconds. That set consists of 1673 observations. We subdivided the tick prices into sets of 1-minute, 5- minutes, 10-minute, and 60-minute time lags to study the effect of the length of the time interval on the nature of the distribution.

The IBM daily prices from January 02, 1998 to December 29, 2000 are used to present the time series plot of Figure 3.1. The price dip on May 26, 1999 is due to

²The Standard & Poor's 500, or S&P500 stock Index, is considered to be the broadest measure of the U.S. stock market. It is a value-weighted index that consists of industrial stocks(400), financial stocks(40), transportation stocks(20), and public utilities(40)

³The CAC-40 "Cotation Assistée en Continu" index represents the 40 most quoted stocks on the Paris stock market exchange. It is valued at more than 50% of the French exchange.

a 2 for 1 stock split ⁴ . We also considered the IBM intraday prices from December 26 to December 29, 2000 to graph the quantile-quantile plots . These intraday prices are time-stamped quoted prices (bid and ask) collected from CRSP© ⁵.

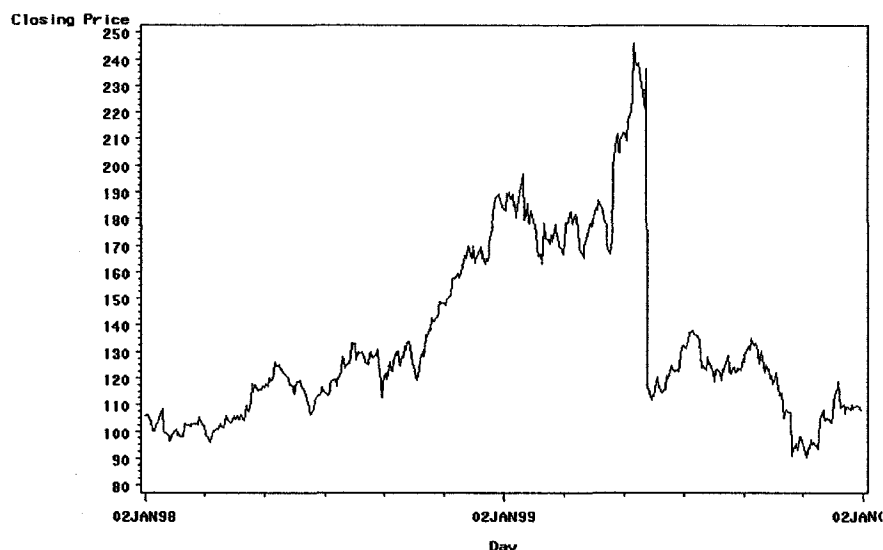


Figure 3.1: Daily IBM stock prices, Jan. 02, 1998- Dec. 29, 2000

The time series plots for the Dell and the Microsoft stock data for the same time horizon are presented below. We also obtain a time series plot for the S&P500 and the CAC40 indices.

⁴A 2 for 1 (2:1) stock split means that a stockholder will receive one additional share for each share of common stock owned. The Board of Directors declared stock split because the corporation believed that it will result in a market price that should be more attractive to a broader spectrum of investors, particularly individual investors.

⁵CRSP© is the Center for Research in Security Prices. It's a financial research center at the University of Chicago Graduate School of Business. CRSP creates and maintains premier historical US databases of stock (NASDAQ, AMEX, NYSE), indices, bond, and mutual fund securities. These databases are used by leaders in academic and corporate communities for financial, economic, and accounting research.

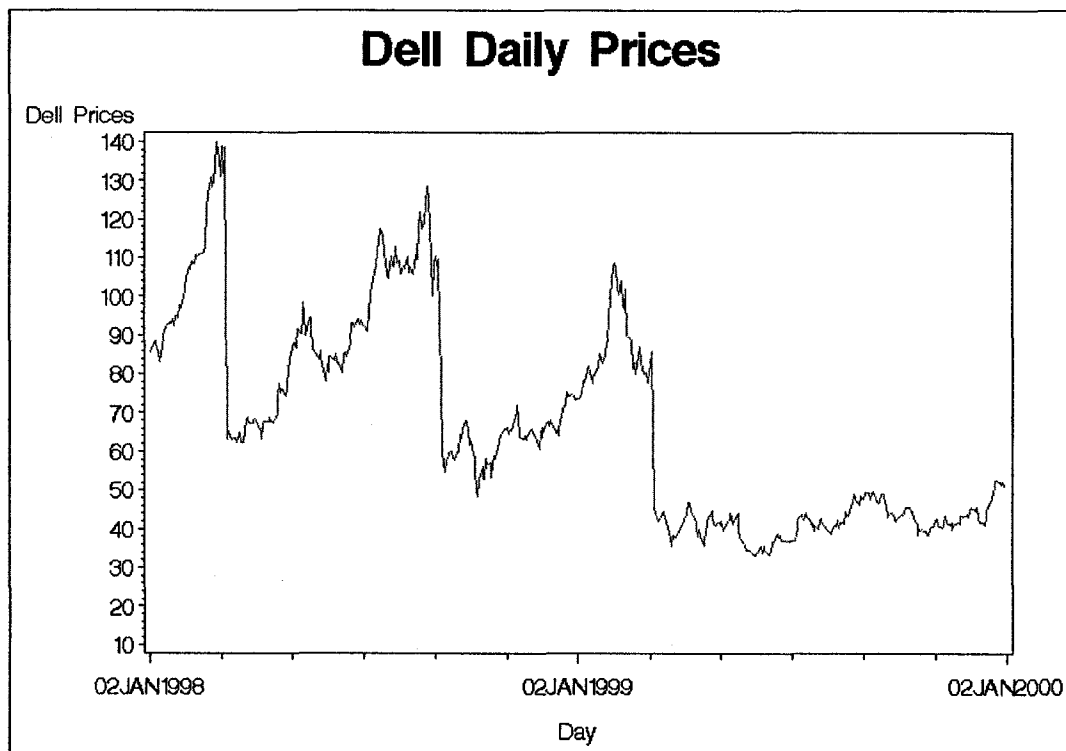


Figure 3.2: Daily Dell stock prices, Jan. 02, 1998- Dec. 29, 2000

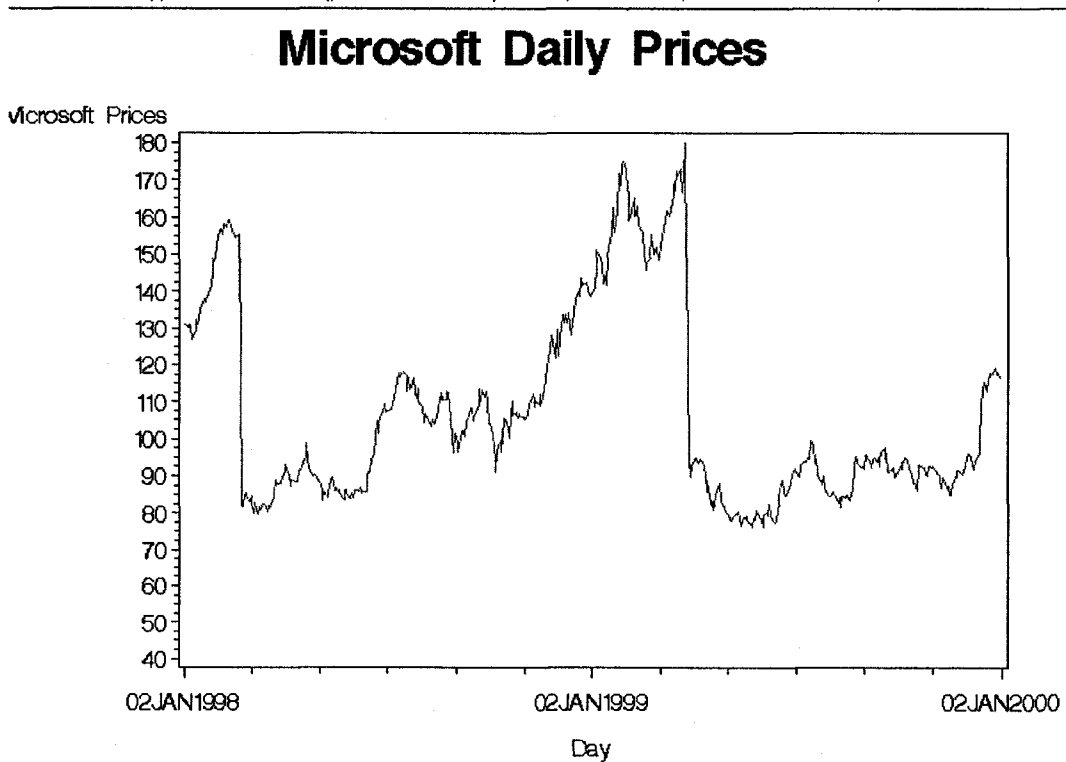


Figure 3.3: Daily Microsoft stock prices, Jan. 02, 1998- Dec. 29, 2000

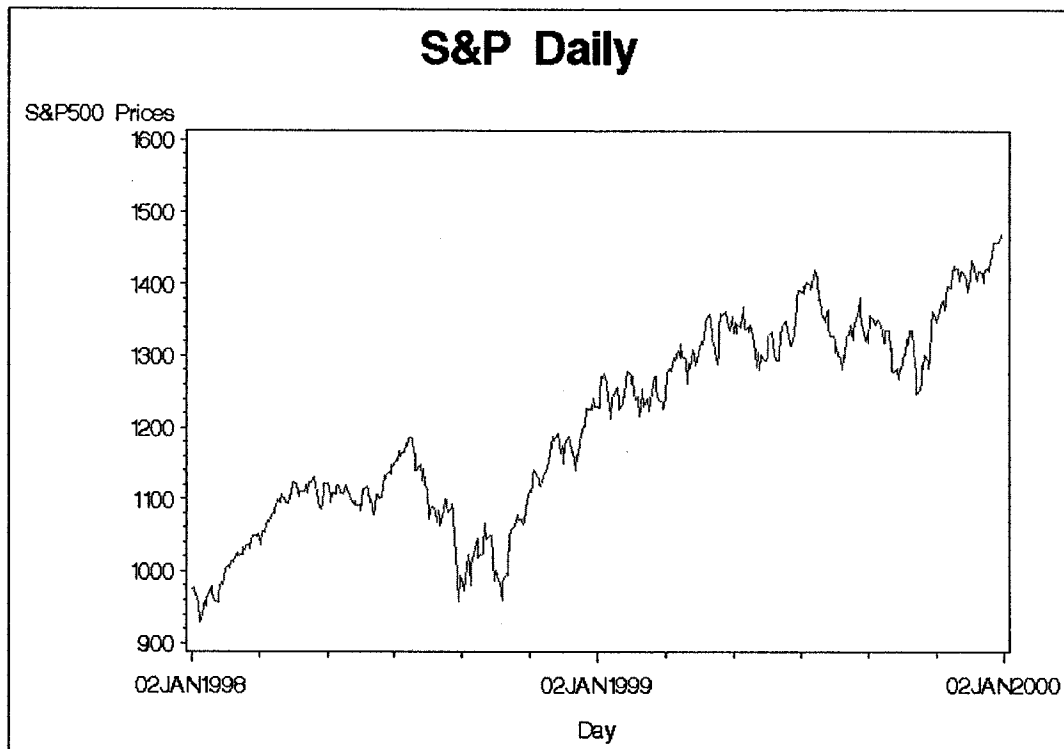


Figure 3.4: Daily S&P500 index prices, Jan. 02, 1998- Dec. 29, 2000

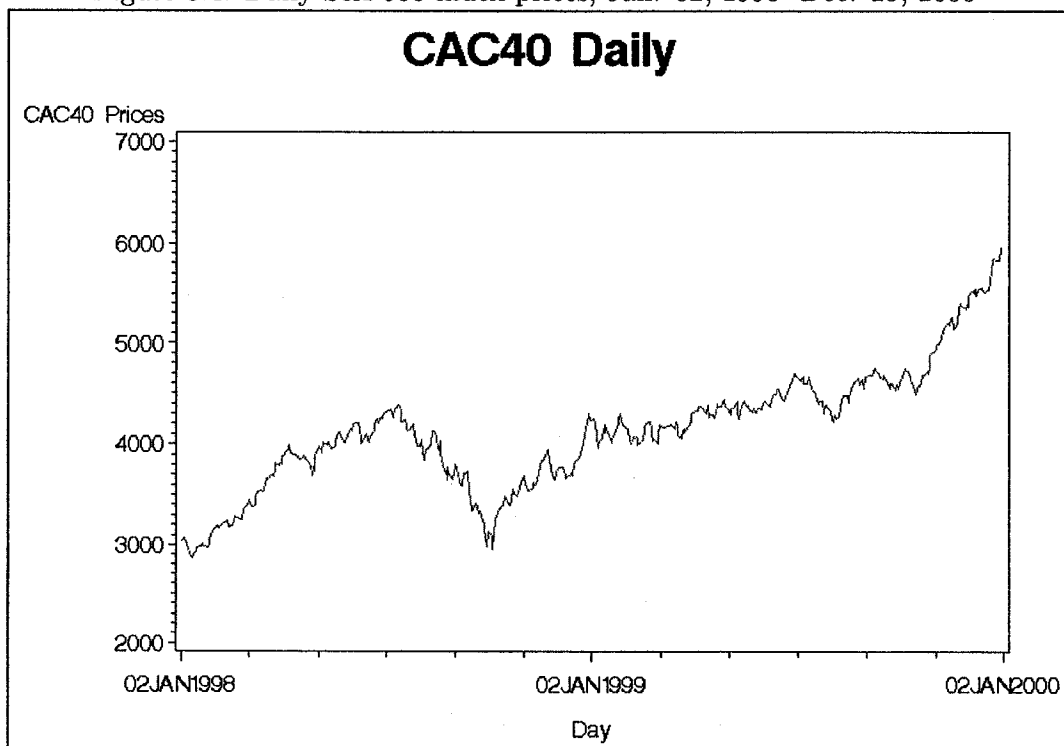


Figure 3.5: Daily CAC40 index price, Jan. 02, 1998- Dec. 29, 2000

Recall that the return on a stock from time t to $t + 1$ is defined as the sum of the dividend yield and the capital gain, i.e., as

$$R_{t+1} = \frac{S_{t+1} + D_t - S_t}{S_t}, \quad (3.2.1)$$

where D_t is the dividend paid during period t .

3.3 Test of Normality of The Distribution of The Returns

There are several methodologies to test for the normality of a time series data. In this section, we use two approaches, the test for skewness and kurtosis, and the quantile-quantile plot. Skewness is a measure of the tendency for values to spread out more on one side of the distribution than the other side, while kurtosis measures the thickness of the tails of the distribution. The skewness-kurtosis test is easily performed using SAS^{TM6}. Since the p-value is extensively used in this section, we give a brief overview of its interpretation in the next section.

3.3.1 What is the Interpretation of p-values?

In a hypothesis test, the p-value is defined as the smallest probability or significance level at which the null hypothesis can be rejected. To interpret this p-value, one must first know how the statistical test is structured. For instance, in the case of a two-sided t-test, about the equality of two population means, the hypotheses are:

⁶The Statistical Analysis Software (SAS) is an integrated system of software products for managing and analyzing data. It enables the user to perform data entry, retrieval and management; report writing and graphic design; statistical and mathematical analysis; business forecasting and decision support; operation research and project management; and application developments. It is distributed by the SAS Institute.

$H_o : \mu_1 = \mu_2$ (Null hypothesis: means of two groups are equal)

$H_a : \mu_1 \neq \mu_2$ (Alternative hypothesis: means of the two groups are not equal)

A low p-value for the statistical test points to rejection of the null hypothesis because it indicates how unlikely it is that a test statistic as extreme as, or more extreme than the one given by this data will be observed from this population, if the null hypothesis is true. If $p=0.015$, this means that if the population means were equal as hypothesized (under the null), there is a 15 in 1000 chance that a more extreme test statistic would be obtained using data from this population. If one agrees that there is enough evidence to reject the null hypothesis, that leads to the conclusion that there is significant evidence to support the alternative hypothesis.

3.3.2 Test of Skewness and Kurtosis

Consider a series $(x_t)_{t=1,2,\dots,T}$ with mean μ and standard deviation σ .

Let $\mu_r = E[(x - \mu)^r]$ be the r^{th} central moment of x_t . The coefficients of skewness and kurtosis are defined respectively as:

$$\begin{aligned} S &= \frac{\mu_3}{\sigma^3} = \frac{E[(x - \mu)^3]}{E[(x - \mu)^2]^{3/2}} \\ K &= \frac{\mu_4}{\sigma^4} = \frac{E[(x - \mu)^4]}{E[(x - \mu)^2]^2} \end{aligned} \quad (3.3.1)$$

Sample estimates of S and K can be obtained upon replacing population moments μ_r by the sample moments $\hat{\mu}_r = T^{-1} \sum_{t=1}^T (x_t - \bar{x})^r$.

For a Gaussian distribution, $S = 0$ and $K = 3$. We use the procedure PROC UNIVARIATE of the SAS software to conduct a skewness-kurtosis test.

PROC UNIVARIATE provides simple descriptive statistics such as the measures of central tendency and the measures of location for numerical variables. We perform

the test of normality of the increments on the tick prices, the 1 minute-lag prices, the hourly prices and the daily prices of the MSFT data. The results are presented in the table below.

	Time Interval			
	Tick	1 Min.	60 Min.	DAILY
KURTOSIS	55.68	2.86	45.95	111.13
SKEWNESS	430.10	0.0801	96.45	120.78

Table 3.1: Correlation Time Interval & Normality.

The table shows that for relatively small time intervals (1 minute-lag) , the returns tend to have a normal distributions. But for larger time intervals, returns are typically heavy-tailed and skewed. Also for high frequency data (tick prices), the kurtosis is much larger than 3, implying fatter tails of the distribution. This suggests that for large time scales or instantaneous data, assumptions underlying the Black-Scholes model are routinely rejected.

3.3.3 Normal Quantile-Quantile Plots

Another test of normality is the quantile-quantile plot. We use it with the skewness-kurtosis test because it presents a better visualization of the results of the skewness-kurtosis test. The quantile-quantile (Q-Q) test is a graphical technique for determining if a data set comes from a population with a given distribution. It is a plot of the quantiles of the first data set against the quantiles of the second data set. By a quantile, we mean the fraction (or percent) of points below a given data value. That is, the 30% quantile is the point at which 30% of the data fall below and 70% of the data fall above. A 45-degree reference line is also plotted. If the two sets come from

a population with the same distribution, the points should fall approximately along this reference line. The greater the departure from the reference line, the greater the evidence for the conclusion that the two data sets have come from populations with different distributions.

The results presented on the tables and the graphs to test the normality of the different numerical data suggest that financial price processes measured at very high frequencies are generated by processes with discontinuous sample paths. The plots also suggest that for returns computed over larger intervals of time, non-Gaussian Lévy processes are more appropriate for modelling purposes . This result corroborates studies of tick-by-tick data by Engle and Russell [28].

One possible response to the empirical rejection of the Black-Scholes model is to replace the role of the Brownian motion by a heavier-tailed Lévy process such as the VG process, the Generalized hyperbolic model, or the Meixner model.

In the next section, we perform a test of stationarity of the daily returns of IBM.

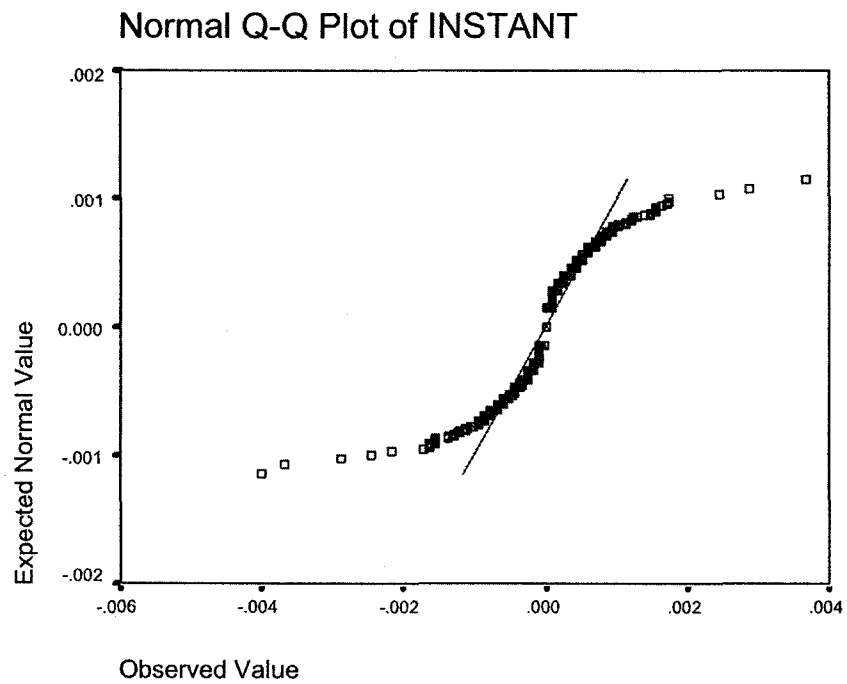


Figure 3.6: QQ-Plot- Lag < 1 minute.

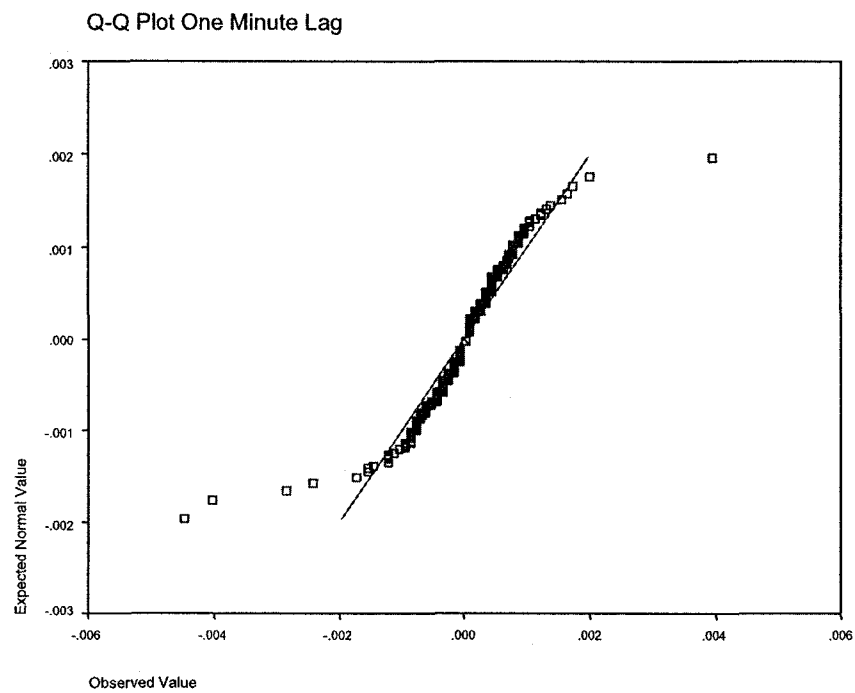


Figure 3.7: QQ-Plot- Lag = 1minute.

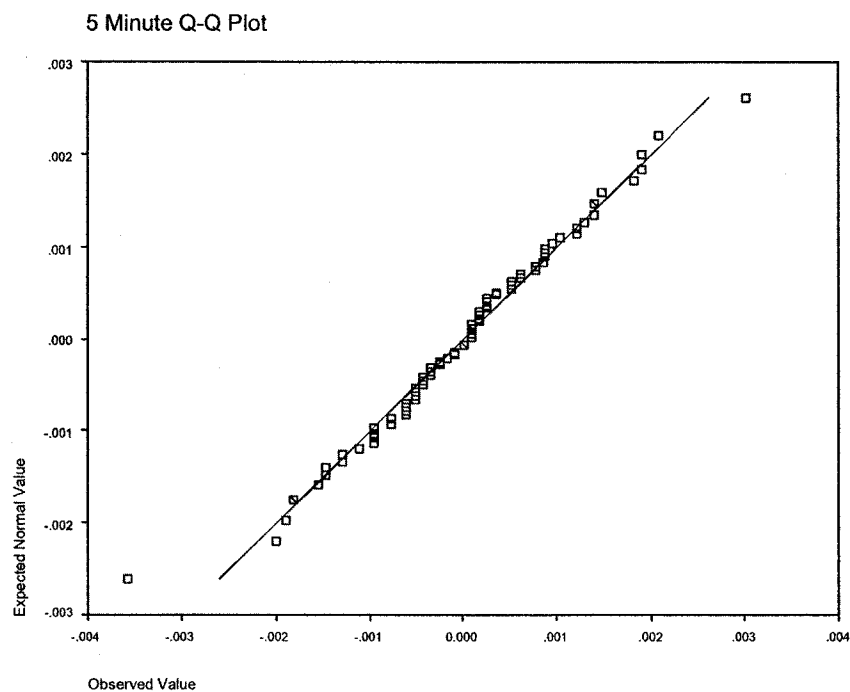


Figure 3.8: QQ-Plot- Lag = 5 minutes.

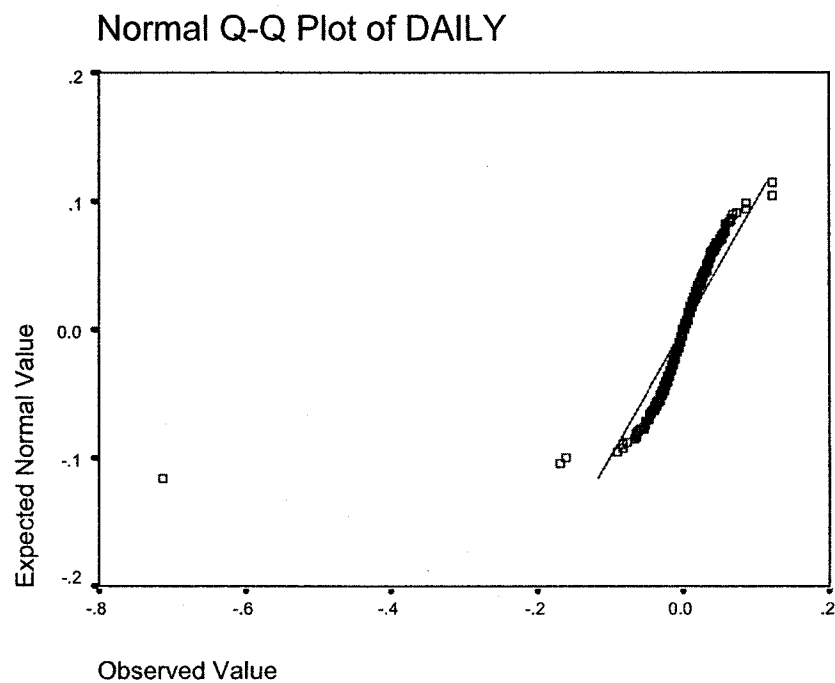


Figure 3.9: QQ-Plot- Lag = 1day.

3.4 Test of Stationarity of the Returns

The focus in this section is to test if the increments of the stochastic process $(X_t)_t$ underlying the stock price process $(S_t)_t$ form a stationary process. Let $X_{t1}, X_{t2}, \dots, X_{tn}$ represent a realization of the stochastic process $(X_t)_t$.

Definition 3.4.1. We define a stationary process as a stochastic process whose joint distribution is invariant with respect to displacement in time. In other words, if the series $(X_{ti})_{ti}$ is stationary, then the joint probability distribution function satisfies

$$f(X_{t1}, X_{t2}, \dots, X_{tn}) = f(X_{t1+h}, X_{t2+h}, \dots, X_{tn+h}), \quad (3.4.1)$$

for any $h > 0$.

The following notations will be used for the mean, the variance, the covariance, and the correlation function.

$$\mu_t = E[X_t] = E[X_{t+h}]. \quad (3.4.2)$$

The variance of the series is

$$\sigma^2(t) = \text{var}[X_t] = E[X_t - \mu]^2. \quad (3.4.3)$$

The covariance of the series is defined by

$$\tau(r, s) = \text{cov}[X_r, X_s] = E[(X_r - \mu_r)(X_s - \mu_s)]. \quad (3.4.4)$$

By dividing Equation 3.4.4 by the product of the standard deviations, we obtain the autocorrelation:

$$\rho(r, s) = \tau(r, s) / [\tau(r, r)\tau(s, s)]^{1/2}. \quad (3.4.5)$$

For a stationary process, the autocovariance can be expressed as:

$$\tau(k) = E[(X_t - \mu)(X_{t+k} - \mu)]. \quad (3.4.6)$$

This is known as the autocovariance coefficient at lag k . As a function of k , the function ρ is commonly referred to as the autocorrelation function (ACF) at lag k .

3.4.1 The Dickey-Fuller Test

The Dickey-Fuller (DF) test is a hypothesis test. The test uses the daily returns of the IBM stock data from January 02, 1997 to December 29, 1999. We test the null hypothesis that the returns time series is nonstationary against the alternative hypothesis that the time series is stationary. The test is carried through the DFTEST macro of the SASTM program. The test is based on tables of critical values provided in Dickey (1976) and Dickey, Hasza and Fuller (1984). Thus, if the test statistic is greater than the critical value provided in Dickey [24]; Dickey, Hasza and Fuller [25], we accept the null hypothesis and conclude that the time series is nonstationary. See Dickey, Hasza and Fuller [25] for a detailed description of the Dickey-Fuller test.

The results of the test are presented in Table 3.2.

Table 3.2: Dickey Fuller Test of Stationarity

DF Test Statistic	-29.27594	1% Critical Value	-2.5681
		5% Critical Value	-1.9412
		10% Critical Value	-1.6164

From the results of Table 3.2, we conclude that since the Dickey-Fuller test statistic (-29.27594) is smaller than the critical values (-2.5681, -1.9412, -1.6164) at the 1%,

5%, and 10% significant levels respectively, we reject the null hypothesis. Therefore, the series is stationary.

3.5 On Volatility Clustering

Volatility clustering refers to the fact that large price changes in the stock come in separated bursts. Figure 3.10 is an illustration of volatility clustering. Volatility has a key role to play in the determination of the risk and the valuation of options and other derivative securities. According to Black-Scholes,

$$d \ln S_t = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \quad (3.5.1)$$

where μ and σ respectively denote the rate of return and the stock volatility. It follows that the change in $\ln S$ between time t and T is normally distributed:

$$\ln S_T - \ln S_t \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)(T - t), \sigma\sqrt{T - t}\right] \quad (3.5.2)$$

where $N[m, s]$ denotes a normal distribution with mean m and standard deviation s . Thus, if we denote by τ the length of time interval in years and

$$u_i = \ln \left(\frac{S_i}{S_{i-1}} \right), \quad (3.5.3)$$

for $i = 1, 2, \dots, n$. Since $S_i = S_{i-1}e^{u_i}$, u_i is the continuously compounded return in the i^{th} interval. The usual estimate, s , of the standard deviation of the u_i 's is given by

$$s_n = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (u_i - \bar{u})^2} \quad (3.5.4)$$

where $\bar{u}$ is the mean of the u_i 's. From Equation 3.5.2, the standard deviation of the u_i 's is $\sigma_i\sqrt{\tau}$. The variable s_n is therefore an estimate of $\sigma_n\sqrt{\tau}$. It follows that σ_n itself can be estimated by $s_n/\sqrt{\tau}$.

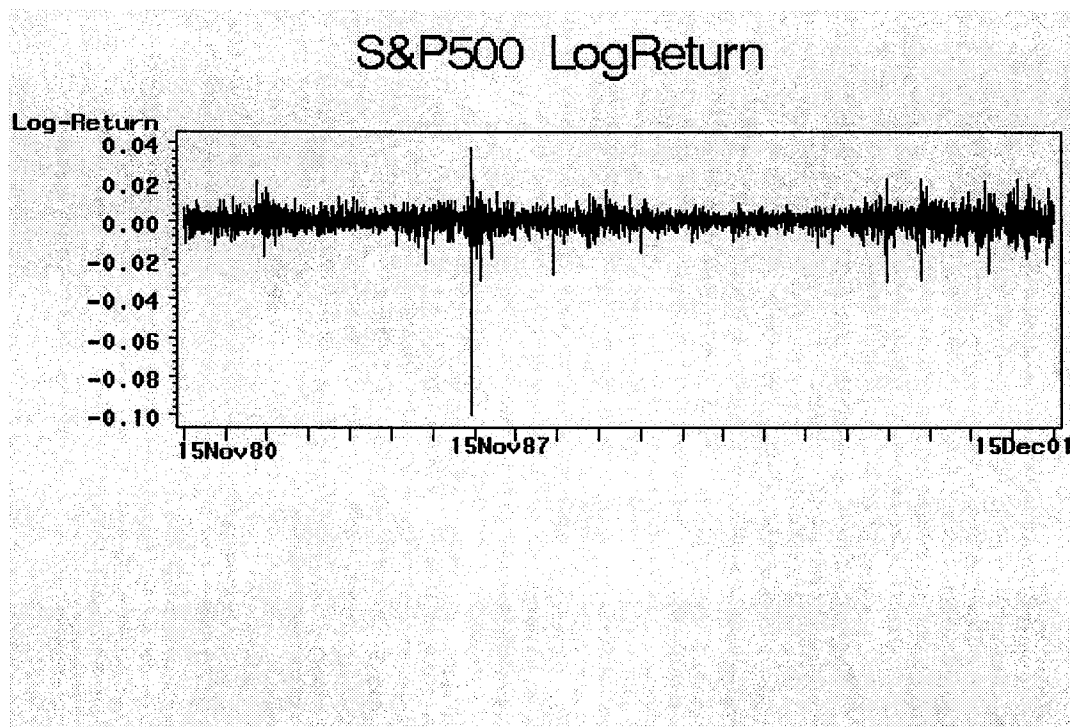


Figure 3.10: Volatility Clustering, Log-returns of the S&P 500, November 1980-November 2002

The main feature missing from the VG, the Hyperbolic, and the Meixner models is the incorporation of the fact that volatility changes over time. Practitioners find it necessary to frequently change the volatility parameter when using Black-Scholes to value options. In fact, the existence of stochastic volatility has been well documented. For example, Hull and White [36] compared the Black-Scholes model and their stochastic volatility model:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V} S_t dW_t \\ dV &= a(b - V)dt + cV S_t dW_V \end{aligned} \tag{3.5.5}$$

where a , b , and c may depend on V (the volatility) and t . And W_V and W_s are

standard Brownian motions. They found that the stochastic volatility model was more accurate than the Black-Scholes model.

The graph of Figure 3.10, is a plot of the daily log-returns of the *S&P500* between November 1980 and November 2002. It indicates that volatilities are clustered. It also clearly shows that there are periods with high absolute log-returns and periods with lower absolute log-returns. In other words, it indicates that volatility changes stochastically.

The next section is a comparative study of the Gaussian and non-Gaussian models for stock price processes.

3.6 Comparative Analysis of the Models

In this section, we test how well the Black-Scholes model, the variance gamma model, the hyperbolic model, and the Meixner model replicate the market stock prices. We consider the dataset that consists of daily stock prices of DELL, MSFT, IBM and daily index prices of the S&P 500 and the CAC 40 during the period from January 1998 to December 2000. Now we seek the models parameters that best fit these prices. Then we use those model parameters to simulate stock and index prices. The simulated prices are used to compare the models. The best model will be the model that produces prices that are closer to the empirical (market) prices. We use tests of goodness-of-fit to determine which model produces prices that are closer to the market prices. The Black-Scholes model, the hyperbolic model, the variance gamma model, and the Meixner model are given respectively by:

$$\begin{aligned}
 S_1(t) &= S_0 \exp((\mu - \sigma^2/2)t + \sigma B_t), \\
 S_2(t) &= S_0 \exp(rt + H_t), \\
 S_3(t) &= S_0 \exp(\mu t + V_t + \omega t), \\
 S_4(t) &= S_0 \exp(\mu t + \sigma M_t),
 \end{aligned} \tag{3.6.1}$$

where, $\sigma > 0$, $\mu \in \mathbb{R}$. Here $\omega = \frac{1}{\nu} \ln(1 - \theta\nu - \sigma^2\nu/2)$ where θ is the drift of the Brownian motion underlying the variance gamma process. The processes B_t , H_t and V_t are respectively a Brownian motion, a Hyperbolic Lévy motion, and a variance gamma process. M_t is the Meixner Lévy motion.

We estimate the parameters of the models by maximum likelihood methods using the densities of the lognormal, the variance gamma, the Hyperbolic, and the Meixner distributions by fitting the data. The parameters are α , β , μ , and δ for the hyperbolic model; θ , ν , and σ for the variance gamma model; μ and σ for the Black-Scholes

model; and m, a, b, c, d for the Meixner model. It should be noted that we estimate the parameters of the hyperbolic model by using a computer program written by *Michael Sørensen* and described in *Blaesild and Sørensen* [17]. The details of the estimation of the parameters by the maximum likelihood method are presented in the next section.

3.6.1 Maximum Likelihood Estimation

In this section, we develop the algorithm to estimate the parameters of the four models via the maximum likelihood (ML) method.

Suppose that one observes a random sample $x_1, x_2, \dots, x_n$ of values drawn independently from a distribution with density $f(x, \theta)$, governed by an unknown parameter θ . Because θ is usually unknown, it must be estimated from the data. The joint density of the x_i s is referred to as the likelihood function and denoted $L(x, \theta)$. For each θ this function tells how likely in some sense, we are to observe the sample $x = (x_1, x_2, \dots, x_n)$. As an estimate of θ , we select $\hat{\theta}$ such that when evaluated at $\hat{\theta}$, the expression for the likelihood of the samples is a maximum. Since the x_i 's are assumed to be independent, the likelihood function can be written as:

$$L(x, \theta) = \prod_{i=1}^n f(x_i, \theta). \quad (3.6.2)$$

When the sample sizes are too large, the right-hand side of Equation 3.6.2 can be an extremely large or an extremely small number, well beyond the range of a floating point number that digital computers can handle. For that reason, it's customary to maximize the logarithm of the likelihood function $l(x; \theta) = \log(L(x; \theta))$. Since it is a monotonically increasing function of $L(x; \theta)$, if $\hat{\theta}$ maximizes the loglikelihood

function, it must also maximize $L(x; \theta)$. The maximum is attained when the rate of change of $\log L$ with respect to θ equals zero. In other words

$$\frac{\partial \log L}{\partial \theta} = 0. \quad (3.6.3)$$

The solution of this equation yields the maximum likelihood estimate (MLE), $\hat{\theta}$. To ensure that $\log L$ is maximized when solving for θ , the slope of $\log L$ must be decreasing near the MLE. Therefore the following must hold:

$$\frac{\partial^2 \log L}{\partial \theta^2} < 0. \quad (3.6.4)$$

An example of the MLE method applied to a normal distribution is given next.

Example 3.6.1. Let $x_1, x_2, \dots, x_n$ be a random sample from a normal distribution $N(\mu, \sigma)$. Then

$$L(x; \mu, \sigma) = \frac{1}{(2\pi)^{n/2} \sigma^n} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right) \quad (3.6.5)$$

and

$$l(x; \mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log(\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2. \quad (3.6.6)$$

Then the partial derivatives of $\log L$ with respect to μ and σ give

$$\begin{aligned} \frac{\partial l}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = 0 \\ \frac{\partial l}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - \mu)^2 = 0. \end{aligned} \quad (3.6.7)$$

The MLE for μ and σ are respectively $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$.

3.6.2 Estimates of The Models Parameters

Using the maximum likelihood method, the parameters of the Black-Scholes model, the Meixner model, the hyperbolic model, and the variance gamma model are estimated for the daily stock prices of DELL, MSFT, IBM, and the daily index prices of S&P500 and CAC40. The results are presented in the tables below. Table 3.3 represents the parameters of the Black-Scholes model. In Table 3.4 we present the parameters of the hyperbolic model for the 3 stock price processes and the 2 index price processes. Table 3.5 contains the parameters of the variance gamma model. And Table 3.6 contains the estimated parameters of the Meixner model.

Stocks & Indices	$\hat{\mu}$	$\hat{\sigma}$
MSFT	-0.0014	0.044
DELL	-0.00211	0.0573
IBM	-0.00029	0.0366
S&P-500	0.000401	0.0128
CAC-40	0.00088	0.0147

Table 3.3: Estimated Parameters of The Black-Scholes Model.

Stocks & Indices	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$	$\hat{\delta}$
MSFT	207.45	2.54	-0.0002	0.0025
DELL	1.88.67	2.06	-0.0003	0.0012
IBM	95.07	7.20	-0.0034	0.0002
S&P-500	102.32	3.91	-0.00012	0.00043
CAC-40	305.83	1.02	-0.0007	0.0005

Table 3.4: Estimated Parameters of The hyperbolic Model.

Stocks & Indices	θ	ν	σ
MSFT	-0.0014	-573.839	0.0465801
DELL	-0.00211	-553.969	0.0586859
IBM	-0.00029	-637.513	0.037799
S&P-500	0.000401	-2.22599	0.0127812
CAC-40	0.00088	-1.44374	0.0146669

Table 3.5: Estimated Parameters of The Variance-Gamma Model.

Stocks & Indices	$\hat{m}$	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$
MSFT	0.00362866	0.874539	-1.47514	0.015214	1.000
DELL	0.0229738	0.629138	-1.90039	0.0284999	1.000
IBM	0.0022935	0.936066	-1.23778	0.00808006	1.000
S&P-500	0.0016011	0.0271472	-0.19824	0.444535	1.000
CAC-40	0.00489812	0.0206194	-0.375852	1.02472	1.000

Table 3.6: Estimated Parameters of The Meixner Model.

The three non-Gaussian models yield almost similar annualized mean returns μ and annual volatility σ for all the stocks and the indices. Also note that different time series data produce different model parameters.

The estimated parameters are used to simulate stock price processes and index price processes. Then the simulated prices are used to compare the models by performing some tests of goodness-of-fit. The different tests used and their results are presented in the next section.

3.7 Goodness-of-Fit Tests

Goodness-of-fits tests are a form of hypothesis testing to indicate whether or not it is reasonable to assume that a random sample comes from a specific distribution. The

null and alternative hypotheses can be stated as follows:

H_0 : The sample data come from the stated distribution.

H_a : The sample data do not come from the stated distribution.

There are three goodness-of-fit tests that we will use to compare the fit of the Black-Scholes model, the hyperbolic model, the variance gamma model, and the Meixner model: the χ^2 -test, the Kolmogorov-Smirnov test, and the Anderson-Darling test.

For the χ^2 -test, the datasets are divided into k bins and the test statistic is defined as:

$$\chi^2 = \sum_{i=1}^k (O_i - E_i)^2 / E_i^2, \quad (3.7.1)$$

where O_i is the observed frequency for bin i and E_i is its expected frequency. The expected frequency is calculated by:

$$E_i = N(F(x_u) - F(x_l)), \quad (3.7.2)$$

where F is the cumulative distribution function for the distribution being tested, x_u and x_l are respectively, the upper limit and the lower limit for class i , and N is the sample size. The test statistic follows approximately a χ^2 distribution with $N - m + 1$ degrees of freedom, where m is the number of parameters of the model. Therefore, the hypothesis that the data are from a population with the specified distribution is rejected if $\chi^2 > \chi_{\alpha, m}^2$, where $\chi_{\alpha, m}^2$ is the χ^2 -point function with $N - m + 1$ degrees of freedom and a significance level of α .

A disadvantage of the χ^2 -test is that the value of the statistic depends on how the data is binned.

The Kolmogorov-Smirnov test is more accurate than the χ^2 test because it does not depend on the size of the samples or the way that the data is partitioned. The

Kolmogorov-Smirnov distance is defined as

$$KS = \max_{1 \leq i \leq N} |F_{est}(x_i) - F_{emp}(x_i)|, \quad (3.7.3)$$

where F_{est} is the estimated cumulative distribution function (CDF) and F_{emp} is the empirical CDF. The best model is the model that minimizes the Kolmogorov-Smirnov distance. One disadvantage of the Kolmogorov-Smirnov test is related to the fact that it tends to be more sensitive near the center of the distribution than at the tails.

The Anderson-Darling test is an alternative of the χ^2 and the Kolmogorov-Smirnov tests. It corrects the Kolmogorov-Smirnov test by putting more weight to the tails. The Anderson-Darling test statistic is defined as:

$$AD = \max_{1 \leq i \leq N} \frac{|F_{emp}(x_i) - F_{est}(x_i)|}{\sqrt{(F_{est}(x_i) * (1 - F_{est}(x_i)))}}, \quad (3.7.4)$$

where F_{est} is the estimated cumulative distribution function (CDF) and F_{emp} is the empirical CDF. Again the null hypothesis will be rejected if the test statistics is greater than the critical value obtained from the table of the Anderson-Darling test, published by Stephens [73] [74] [75].

To perform the three goodness-of-fit tests, we consider the price process of the IBM, the MSFT, and the DELL stocks. We also consider the S&P500 and the CAC40 index prices. For the χ^2 test, we take 32 classes of equal width. And if necessary, we collapse outer cells such that the expected value of the observations becomes greater than 5. In the data set, we choose $-0.0210 + (j - 1) * (0.0015)$, $j = 1, \dots, N - 1$, as the boundary points of the classes. Because we estimate for the normal distribution two parameters, the numbers of degrees of freedom is $N - 3$. The numbers of degrees of freedom are respectively, $N - 5$, $N - 5$, and $N - 6$ for the hyperbolic distribution, the variance gamma distribution, and the Meixner distribution. The test results are

Stocks & Indices	p-value	Test Statistic	Critical Value
IBM	0.0611	121.66	30.03
DELL	0.0017	98.76	19.97
MICROSOFT	0.152	110.07	31.12
S & P 500	0.1801	321.54	78.00
CAC 40	0.003	276.09	88.65

Table 3.7: χ^2 -test Black-Scholes Model- Confidence level of 10%

Stocks & Indices	p-value	Test Statistic	Critical Value
IBM	0.4304	121.66	200.68
DELL	0.6732	98.76	188.34
MICROSOFT	0.720	110.07	232.12
S & P500	0.5843	321.54	378.00
CAC 40	0.673	276.09	398.65

Table 3.8: χ^2 -test Hyperbolic Model-Confidence level of 10%

presented below.

3.7.1 χ^2 Tests Results

The results of the χ^2 -tests are presented in Table 3.7 for the Black-Scholes model, Table 3.8 for the hyperbolic model, Table 3.9 for the variance gamma model, and Table 3.10 for the Meixner model.

The p-values in Table 3.7, Table 3.8, Table 3.9, and Table 3.10 suggest that the Black-Scholes model is strongly rejected in favor of the non-Gaussian models. It should be noted that the variance-gamma, the Meixner, and the hyperbolic models are not essentially different because even though Brownian motion has no jumps and

Stocks & Indices	p-value	Test Statistic	Critical Value
IBM	0.7200	121.66	324.03
DELL	0.6982	98.76	193.87
MICROSOFT	0.752	110.07	321.42
S & P 500	0.7211	321.54	378.00
CAC 40	0.5633	276.09	280.65

Table 3.9: χ^2 -test Variance Gamma Model-Confidence level of 10%

Stocks & Indices	p-value	Test Statistic	Critical Value
IBM	0.8711	121.66	212.04
DELL	0.7217	98.76	110.97
MICROSOFT	0.8150	110.07	431.12
S & P 500	0.8801	321.54	458.00
CAC 40	0.684	276.09	297.34

Table 3.10: χ^2 -test Meixner model- Confidence level of 10%

finite expectation at each time, a non-Gaussian Lévy process, with jumps at each time can be created by sampling it at random set of times. However, although both the variance gamma model and the hyperbolic model fare better than the geometric Brownian motion model, their fits are less accurate than the fit of the Meixner model. The Meixner model performs better than the other three models. This can be explained by the fact that the Meixner model uses more parameters, and therefore offers more flexibility for a better fit.

3.7.2 Kolmogorov-Smirnov Tests Results

We examine in this section the results of the Kolmogorov-Smirnov tests. Again the data is the same data as the one described in the χ^2 -test. The results are presented

in Table 3.11

Stocks & Indices	Black-Scholes	Variance Gamma	Hyperbolic	Meixner
IBM	0.3821	0.0810	0.1109	0.06623
DELL	0.5827	0.1862	0.2097	0.2013
MICROSOFT	0.09231	0.0212	0.0338	0.0281
S & P 500	0.1872	0.0231	0.0423	0.0222
CAC 40	0.6433	0.1982	0.0886	0.0234

Table 3.11: Kolmogorov-Smirnov Distances.

The results of the Kolmogorov-Smirnov test are consistent with the result of the χ^2 test. The Meixner model produces smaller Kolmogorov-Smirnov distances.

3.7.3 The Anderson-Darling Tests Results

The results of the Anderson-Darling tests are presented in Table 3.12. Again the tests were performed for all 4 models. We also use the DELL, IBM, and MSFT stock daily prices. For index prices, we use the S&P500 and the CAC40 values.

The Anderson-Darling tests results corroborate the conclusions drawn from the χ^2 -tests and the Kolmogorov-Smirnov tests. The pure-jump models perform better than the Black-Scholes model. This was predictable, since the normal distribution has much lighter tails than the Meixner, the variance gamma, and the hyperbolic distributions. In Table 3.12, we only present the value of the test statistics. The simple rule is, the lower the statistic, the better the fit.

Asset	Black-Scholes	Meixner	Hyperbolic	Variance Gamma
IBM	142.6421	0.12872	0.0828	1.2132
DELL	122.0076	2.8144	2.1121	4.2660
MICROSOFT	304.00	3.0076	0.2321	3.9000
S & P 500	98.2111	6.6362	0.9218	10.0987
CAC 40	231.09	2.0075	1.1121	8.0923

Table 3.12: Anderson-Darling Test Statistics.

3.8 Conclusion

This chapter contains a statistical analysis of time series data consisting of stock price processes. The analysis reveals the following properties:

1. The observations are heavy tailed.
2. As the sampling interval increases, the tails tend to become less heavy. It means that adding a certain number of consecutive observations may lead to lighter tails.
3. The stochastic process consisting of the log-returns has independent and stationary increments
4. The data has observable periods of high activity and low activity, often measured through volatility.
5. The aggregate returns over larger time intervals do exhibit volatility clustering.

The key feature of this chapter is the use of series representation to generate the paths of the pure jump Lévy processes to compare the performance of the underlying financial models. The simulation via series representation proves to be more accurate

and more efficient than a simple Monte-Carlo method because it captures better the nature of the jumps of the processes. Besides, to simulate the processes via series representation, one only needs the density of the Lévy measure, which in general, has a nicer expression than the density function of the underlying distribution of the processes. However, the expression of the density function is necessary to estimate the model parameters by the maximum likelihood method.

In light of the aforementioned findings, the log-returns of stock prices should be modelled by pure-jump Lévy processes. The necessity of the introduction of a stochastic volatility model is also strongly argued.

Chapter 4

The Pricing of Plain Vanilla Options

4.1 Introduction

An option¹ is a contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset (a stock, an index value, a foreign exchange rate, or an interest rate) at a specified price on or before a certain date (the time to maturity). The specified price is known as the strike price. When the investor is granted the right to buy the option, the option is said to be a call option. A put option gives the holder the right to sell the asset.

American options are options that may be exercised at any time between the date of purchase and the expiration. European options are options that can be exercised only at the time of expiration.

Standard options, traded on the exchanges, with no special or unusual features are called plain vanilla options. In contrast, exotic options are superficially created

¹Listed options have been available since 1973, when the Chicago Board of Options Exchange (CBOE) first opened.

by financial institutions for their corporate clients. They are options traded over-the-counter, namely outside exchanges; thus the name, OTC options. In this chapter, we develop a pricing scheme for plain vanilla options. In the next chapter we detail a pricing methodology for exotic options.

We now formally assemble the ingredients to calculate the price of a European call option under different models. We consider a model of a security market which consists of two assets: a bond or cash account and a stock price process. The bond is a risk-free asset $(B_t)_{0 \leq t \leq T}$ evolving according to the equation $B_t = e^{rt}$, where r denotes the risk-free rate of interest. The risky asset, the stock price process, is denoted by $S = (S_t)_{0 \leq t \leq T}$. The market is assumed to live in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Now we face the problem of finding a measure $\mathbf{Q}$, equivalent to the original probability measure $\mathbf{P}$, under which the process $(e^{-rt}S_t)_{0 \leq t \leq T}$ is a martingale. In the Black-Scholes case, $\mathbf{Q}$ is determined via the Cameron-Girsanov theorem. And in the incomplete market setting, $\mathbf{Q}$ is obtained by using the Esscher transform.

In this chapter, we present pricing formulas of a European call option following the different models. Section 1 defines the concept of options. In Section 2, we develop a pricing formula under the Meixner model. Section 3 revisits the Black-Scholes (BS) option pricing model. Sections 4 and 5 present pricing schemes under the hyperbolic, and the variance gamma (VG) models respectively. The last section is a comparative study of the different pricing formulas.

4.2 The Meixner Call Option Prices

Let the process for the stock price be given by

$$S_t = S_0 \exp(\mu t + \sigma X_t) \quad 0 \leq t \leq T, \quad (4.2.1)$$

where $\mu > 0$ is the mean rate of return on the stock, and σ is the stock volatility. $(X_t)_{(0 \leq t \leq T)}$ is the Meixner process with parameters² a, b, c, d, m . We define the Meixner process on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We denote by K the option strike price and T represents the time to maturity.

Recall that the density of the Meixner distribution $M(a, b, c, d, m)$ is given by:

$$f(x; a, b, c, m) = \frac{\cos^{2c}(b/2)}{2a\pi\Gamma(2c)} \exp\left(\frac{b(x-m)}{a}\right) \left| \Gamma\left(c + \frac{i(x-m)}{a}\right) \right|^{2d}, \quad (4.2.2)$$

where $a > 0$, $-\pi < b < \pi$, $c > 0$, $m \in \mathbb{R}$, and $d \geq 1$. The characteristic function of the one-dimensional Meixner distribution is given by:

$$\phi(u) = E^P[(\exp(iuX))] = \frac{\cos^{2c}(b/2)}{\cosh^{2c}\left(\frac{au-ib}{2}\right)} \exp(imu). \quad (4.2.3)$$

Since the Meixner process is a pure-jump process, the market is incomplete³. That is, there exist several measures equivalent to the original probability measure $\mathbf{P}$, such that under these measures, the discounted stock price process is a martingale. We face then the problem of selecting one equivalent martingale measure. Following Gerber and Shui [33], we select the Esscher transform⁴. We should note that they justify their choice of the Esscher transform by a utility maximizing argument.

²To simplify the presentation, we set $c=1$.

³An incomplete market may be thought of as a market where there are derivative securities that cannot be perfectly hedged; the price of such derivatives is not completely determined by the prices of other assets.

⁴Practitioners obtain the density of the Esscher transformed distribution by multiplying the original density by the function $\exp(\lambda x - \kappa(\lambda)t)$.

The Esscher transform Q of the probability measure P can be defined by the relation

$$dQ = e^{\theta X_t - t \log M(\theta)} dP, \quad (4.2.4)$$

where $M(u) = E^P(e^{uX_1})$ is the moment generating function of X_1 and the parameter θ is defined as the solution of the following equation, where r denotes the risk-free rate of interest

$$r = \log \frac{M(\theta + 1)}{M(\theta)}. \quad (4.2.5)$$

Grigelionis [34] showed that the characteristic function with respect to the Esscher probability measure Q is given for $\theta \in R$ by

$$E^Q(e^{iuX_1}) = \left(\frac{\cos(\frac{a\theta+b}{2})}{\cosh(\frac{au-i(a\theta+b)}{2})} \right)^{2d} \exp(imu). \quad (4.2.6)$$

Solving Equation 4.2.5 yields

$$\theta = -\frac{1}{a} \left(b + 2 \arctan \left(\frac{-\cos(a/2) + \exp(m-r)/(2c)}{\sin(a/2)} \right) \right). \quad (4.2.7)$$

Remark 4.2.1. The density of the Meixner distribution $f(x; a, b, c, d, m)$ is given by Equation 4.2.2. The right-hand side of Equation 4.2.6 is the characteristic function of a random variable with Meixner density $f(x; a, a\theta + b, c, d, m)$. That means that the change of measure only affects the second parameter of the distribution.

Proposition 4.2.1. *Let $C(T, K)$ denote the price of a European call stock option with strike price K and time to maturity T , at time 0. Then*

$$\begin{aligned} C(T, K) &= E^Q[(S(T) - K)^+] \\ &= S_0 \int_K^\infty f(x; a(\theta + 1) + b, dT, mT, cT) dx \\ &\quad - e^{-rT} K \int_K^\infty f(x; a\theta + b, dT, mT, cT) dx, \end{aligned} \quad (4.2.8)$$

where $k = \ln(K/S_0)$ and f , the density function of the Meixner distribution is given by:

$$f(x; a, b, c, d, m) = \frac{\cos^{2c}(b/2)}{2a\pi\Gamma(2c)} \exp\left(\frac{b(x-m)}{a}\right) \left|\Gamma\left(c + \frac{i(x-m)}{a}\right)\right|^{2d}, \quad (4.2.9)$$

with $a > 0$, $-\pi < b < \pi$, $c > 0$, $m \in \mathbb{R}$, and $d \geq 1$.

See Grigelionis [34] for a proof of the proposition.

4.2.1 The Price of a Put Option.

To see the relationship between a European call and a European put options, consider the following two portfolios at time t : one consisting of a European call option $C(t, T, K)$ and an amount of money $S_t e^{-r(T-t)}$; the second portfolio consists of a European put option $P(t, T, K)$ and one share S_0 of stock. At expiration, both portfolios are worth $\max(S_T, K)$. Therefore at the current time t , they are identical for arbitrage reason. Hence

$$C(t, T, K) + S_t e^{-r(T-t)} = P(t, T, K) + S_0. \quad (4.2.10)$$

The previous equation, known as the put-call parity, enables the evaluation of a European put option from its call counterpart.

4.3 The Black-Scholes European Call Option Prices

Assume that the stock price process is modelled by an exponential Brownian motion:

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma W_t), \quad (4.3.1)$$

with $\mu \in \mathbb{R}$, $\sigma > 0$, and $(W_t)_t$ a standard Brownian motion. Also assume that $B_t = \exp(rt)$ denotes the price of a tradable bond. $C(T, K)$, the price at time 0 of a European call option, with strike price K and expiration date T is given by the following expectation

$$C(T, K) = E^Q[(S(T) - K)^+], \quad (4.3.2)$$

where $\mathbf{Q}$ is the unique equivalent martingale measure for $B_t^{-1}S_t$. The existence and the unicity of $\mathbf{Q}$ are obtained via the Cameron-Martin-Girsanov theorem. We cite a version of the theorem from Baxter and Rennie [13].

Theorem 4.3.1. *If W_t is a Brownian motion with respect to a probability measure $\mathbf{P}$ and $(\gamma_t)_{t \geq 0}$ is an $\mathcal{F}$ -previsible process satisfying the boundedness condition $(E^P[\exp(\frac{1}{2} \int_0^T \gamma_t^2 dt)] < \infty)$, then there exists a measure $\mathbf{Q}$ such that*

1. $\mathbf{Q}$ is equivalent to $\mathbf{P}$.
2. $\frac{dQ}{dP} = \exp(-\int_0^T \gamma_t dW_t - \frac{1}{2} \int_0^T \gamma_t^2 dt)$.
3. $\tilde{W}_t := W_t + \int_0^T \gamma_s ds$ is a $\mathbf{Q}$ -Brownian motion.

The calculation of the integral ⁵ of Equation 4.3.2 yields the Black-Scholes formula given in the following proposition.

Proposition 4.3.2.

$$\begin{aligned} C(T, K) &= E^Q[(S(T) - K)^+] \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log(k/S_0) - rT}^{\infty} (S_0 e^x - k e^{-rT}) \exp\left(-\frac{(x + 1/2\sigma^2(T))^2}{2\sigma^2 T}\right) dx \\ &= S_0 \Phi\left(\frac{\log(S_0/k) + (r + 1/2\sigma^2)(T)}{\sigma\sqrt{T}}\right) - k e^{-rT} \Phi\left(\frac{\log(S_0/k) + (r - 1/2\sigma^2)T}{\sigma\sqrt{T}}\right), \end{aligned} \quad (4.3.3)$$

⁵We have developed in the appendix, a C++ function to calculate a European call option using the Black-Scholes formula.

where

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x \exp(-y^2/2) dy. \quad (4.3.4)$$

Remark 4.3.1. The simplicity of the Black-Scholes option pricing formula lies in the fact that it uses only one unknown parameter, the stock volatility σ .

Example 4.3.3. *If the stock has a volatility of 18% with a continuously compounded rate of interest of 6%, then the value of a European call option to buy a stock for \$25 in two years when the current price is \$20 is equal to \$1.22.*

4.4 The Variance Gamma European Call Option Prices

This section presents a pricing formula for a European call option using the variance-gamma model ⁶. See Madan, Carr, and Chang[47] for a detailed study of the model.

Consider the Brownian motion $b(t; \beta, \sigma)$ with constant drift μ and constant volatility σ ,

$$b(t; \beta, \sigma) = \beta t + \sigma W_t, \quad (4.4.1)$$

where W_t is a standard Brownian motion. The variance gamma (VG) process (X_t) is the process

$$X_t = b(\gamma_t; \beta, \sigma). \quad (4.4.2)$$

The process $\gamma_t = \gamma(t; \beta, \nu)$ has independent increments with gamma distribution over non-overlapping intervals of time, with mean rate $\mu > 0$ and variance rate ν . In other

⁶We have developed in the appendix a C++ program to calculate a European call option based on the Variance Gamma model.

words, the density f_{γ_t} of the increment $I = \gamma_t(t + h; \mu, \nu) - \gamma_t(t; \mu, \nu)$ of γ_t is given by:

$$f_{\gamma_t}(I) = \frac{\mu}{\nu} \frac{\mu^{\frac{\mu^2 h}{\nu}} I^{\frac{\mu^2 h}{\nu} - 1} \exp(-\frac{\mu}{\nu} I)}{\Gamma(\frac{\mu^2 h}{\nu})} \quad I > 0, \quad (4.4.3)$$

where $\Gamma(x)$ is the gamma function. Equation 4.4.2 implies that, conditional on the random time change γ_t , the VG random variate X_t over an interval of length t , is normally distributed with mean $\beta\gamma_t$ and variance $\sigma\sqrt{\gamma_t}$. So that we can write :

$$X_t = \beta\gamma_t + \sigma\sqrt{\gamma_t}W_t. \quad (4.4.4)$$

Let the process $(S_t)_{\{0 \leq t \leq T\}}$ denote the stock price process under the variance gamma model. As in Equation 3.6.1, S_t is given by

$$S_t = S_0 \exp(\mu t + X_t + \omega t), \quad (4.4.5)$$

where X_t denotes the variance gamma process, $\mu \in \mathbb{R}$, and $\omega = \frac{1}{\nu} \ln(1 - \zeta\nu - \sigma^2\nu/2)$. The parameter ζ is the drift of the Brownian motion underlying the variance gamma process.

The characteristic function of the VG process $\phi_{X_t}(u) = E[\exp(iuX_t)]$ is given by

$$\phi_{X_t}(u) = \left(\frac{1}{1 - i\zeta\nu u + (\sigma^2\nu/2)u^2} \right)^{t/\nu}. \quad (4.4.6)$$

The log-price relative z_t is defined by $z_t = \ln(S(t)/S(0))$. Madan, Carr, and Chang[47] showed in Theorem 1, Page 9, that if h denotes the density of z_t , i.e., $P(z_t \leq a) = \int_0^a h(z_t)dz_t$ then $h(z_t)$ is given by

$$h(z_t) = \frac{2 \exp(\theta x / \sigma^2)}{\nu^{t/\nu} \sqrt{2\pi\sigma} \Gamma(t/\nu)} \left(\frac{x^2}{2\sigma^2/\nu + \theta^2} \right)^{\frac{t}{2\nu} - \frac{1}{4}} \times J_{\frac{t}{\nu} - \frac{1}{2}} \left(\frac{1}{\sigma^2} \sqrt{x^2 (2\sigma^2/\nu + \theta^2)} \right), \quad (4.4.7)$$

where $J_{(\frac{t}{\nu}-\frac{1}{2})}$ is the modified Bessel function of the second kind with index $(\frac{t}{\nu}-\frac{1}{2})$.

The variable x in the equation is defined by

$$x = z_t - mt - \frac{t}{\nu} \ln(1 - \theta\nu - \sigma^2\nu/2). \quad (4.4.8)$$

The price of a European call option under the VG model is given by

$$C(T, K) = e^{-rt} E[(S(T) - K)^+], \quad (4.4.9)$$

Madan et al. [47] derived in Theorem 2, Page 10, a Black-Scholes type formula for $C(T, K)$. We present below their approach.

First the function $\Psi(a, b, \gamma)$ is defined as

$$\Psi(a, b, \gamma) = \int_0^\infty N\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) \frac{u^{\delta-1} \exp(-u)}{\Gamma(\gamma)} du. \quad (4.4.10)$$

Integrating Equation 4.4.10 gives

$$\begin{aligned} \Psi(a, b, \gamma) &= \frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^\gamma}{\sqrt{2\pi}\Gamma(\gamma)\gamma} \times \\ &K_{\gamma+\frac{1}{2}}(c)\Phi(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)) \\ &- \text{sign}(a) \frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^{\gamma+1}}{\sqrt{2\pi}\Gamma(\gamma)1+\gamma} \times \\ &K_{\gamma-\frac{1}{2}}(c)\Phi(1+\gamma, 1-\gamma, 2+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)) \\ &+ \text{sign}(a) \frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^{\gamma+1}}{\sqrt{2\pi}\Gamma(\gamma)\gamma} \times \\ &K_{\gamma-\frac{1}{2}}(c)\Phi(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)), \end{aligned} \quad (4.4.11)$$

where $c = |a|\sqrt{2+b^2}$, $u = \frac{b}{\sqrt{2+b^2}}$, and the degenerate hypergeometric function of two variables, $\hat{\Phi}$ has the integral representation

$$\hat{\Phi}(\alpha, \beta, \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} e^{uy} du. \quad (4.4.12)$$

The computation of the expectation in Equation 4.4.9 gives

$$C(T, K) = S_0 \Psi \left(d \sqrt{\left(\frac{1 - c_1}{\nu} \right)}, (\alpha + s) \sqrt{\frac{\nu}{1 - c_1}}, \frac{t}{\nu} \right) - K \exp(-rt) \Psi \left(d \sqrt{\frac{1 - c_2}{\nu}}, \alpha s \sqrt{\frac{\nu}{1 - c_2}}, \frac{t}{\nu} \right), \quad (4.4.13)$$

where

$$d = \frac{1}{s} \left[\log \left(\frac{S_0}{K} \right) + rt + \frac{t}{\nu} \log \left(\frac{1 - c_1}{1 - c_2} \right) \right] \quad (4.4.14)$$

$\alpha = \zeta s$, $\zeta = -\theta/\sigma^2$, and

$$\begin{aligned} s &= \frac{\sigma}{\sqrt{\left(1 - \frac{\theta^2 \nu}{\sigma^2} \right)}} \\ c_1 &= \frac{\nu(\alpha + s)^2}{2} \\ c_2 &= \frac{\nu\alpha^2}{2}. \end{aligned} \quad (4.4.15)$$

4.5 The Hyperbolic European Call Option Prices

The section presents a pricing formula for a European call option using the Hyperbolic model. See Keller & Eberlein [27] for a detailed study of the model.

Let the process $(X_t)_{t \geq 0}$ represent the Generalized Hyperbolic Lévy motion with parameters $(\lambda, \alpha, \beta, \delta, \mu)$. We assume that the process is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

The density function of the one-dimensional Hyperbolic distribution is given by:

$$f(x; \lambda, \alpha, \beta, \delta, \mu) = a(\alpha, \beta, \delta) (\delta^2 + (x - \mu)^2)^{(\lambda - 1/2)/2} \times J_{\lambda - 1/2}(\alpha \sqrt{\delta^2 + (x - \mu)^2}) \exp(\beta(x - \mu)), \quad (4.5.1)$$

where

$$a(\alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}. \quad (4.5.2)$$

J_ν denotes the modified Bessel function of the third kind with index ν . Also recall that the moment generating function M with respect to f is given by

$$M(u) = e^{\mu u} \frac{\sqrt{\alpha^2 - \beta^2}}{K_1(\delta\sqrt{\alpha^2 - \beta^2})} \frac{K_1(\sqrt{\alpha^2 - (\beta + u)^2})}{\sqrt{\alpha^2 - (\beta + u)^2}}, \quad |\beta + u| < \alpha. \quad (4.5.3)$$

The characteristic function $\phi(u)$ is given by

$$\phi(u) = e^{i\mu u} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\lambda/2} \times \frac{J_\lambda(\delta\sqrt{\alpha^2 - (\beta - iu)^2})}{J_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}. \quad (4.5.4)$$

The stock price process $(S_t)_{t \geq 0}$, modelled by the Generalized Hyperbolic model, is given by:

$$S_t = S_0 \exp(rt + X_t). \quad (4.5.5)$$

Keller & Eberlein [27] showed that there exists a unique value of θ that satisfies

$$r = \log M(\theta + 1) - \log M(\theta). \quad (4.5.6)$$

That unique θ gives the Esscher transform. Consider the probability measure P^θ defined by the Radon-Nikodym density

$$dP^\theta = \exp(\theta X_t - t \log M(\theta)) dP. \quad (4.5.7)$$

The price $C(T, K)$ of a European call option with strike price K is given by the discounted expectation of the payoff $(S_T - K)^+$ with respect to P^θ .

$$C(T, K) = E^{P^\theta} [e^{-rT} (S_T - K)^+]. \quad (4.5.8)$$

Keller & Eberlein [27], Page 15, showed that the computation of the expectation in Equation 4.5.8 gives

$$C(T, K) = S_0 \int_{\log(K/S_0)}^{\infty} f^{\theta}(x; \theta + 1) dx - e^{rT} K \int_{\log(K/S_0)}^{\infty} f^{\theta}(x; \theta) dx, \quad (4.5.9)$$

where

$$f^{\theta}(x; \theta) = \frac{e^{\theta x} f^{*t}(x)}{\int_{-\infty}^{\infty} e^{\theta y} f^{*t}(y) dy}, \quad (4.5.10)$$

f^{*t} is the t -fold convolution of the density of the Hyperbolic distribution. It is computed ⁷ by applying the Fourier inversion formula to the characteristic function.

4.6 Pricing Performances of the Different Models

4.6.1 The Models Prices

In this section, we compare the option prices produced by the Black-Scholes, the variance gamma, the Generalized Hyperbolic, and the Meixner models. We should note that, in this section, we merely present the call option prices obtained through the different models. The comparative study of the model prices is performed in the next section when we compare the model prices against the market call prices for options written on the S&P 500 index.

First, we estimate by the maximum likelihood method the models parameters. The data used consists of the Microsoft logprices from December 31, 2001 to January 23, 2003. The values of the parameters and the standard errors are given in Table 4.1.

⁷A hyperbolic option calculator can be found in Prof. Eberlein's website : <http://www.fdm.uni-freiburg.de/UK/calculator.html>

Parameters	Black-Scholes	Variance Gamma	Hyperbolic	Meixner
a				0.117576
b				-0.146326
c				1.01
d				0.234783
m				-0.00112
α			18.66	
β		0	-1.21788	
μ				-0.00163903
λ			-0.5	
σ	0.069	0.1213		0.0409286
ν		0.1686	0.09050	
θ		0.000153		
Mean	193.472	192.845	192.872	192.84
Std. Dev.	11.1875	10.6601	10.6453	10.6391

Table 4.1: Estimated Model Parameters and Summary Statistics- $S_0 = 1000$; $K = 800$.

Then we calculate the price of a plain vanilla call option to buy a share of stock, using Equations A.4.1, 4.3.3, A.3.1, and A.2.1 for different maturities and different stockprice/strike-price ratios (S_0/K). We present some of the results in Table 4.2.

Table 4.3 illustrates the difference between the prices obtained from the non-Gaussian models and the prices obtained from the Black-Scholes formula. We consider the same stock-price/strikeprice ratios as in Table 4.2. Then we calculate the ratios of the prices of the non-Gaussian models to the Black-Scholes prices. We note that the ratios are close to 1 when the strike-price/stockprice ratio is between 1 and 1.25.

S_0/K	BS	Meixner	Hyperbolic	VG
1.25	180.00	180.021	180.031	180.25
1.22	160.00	160.043	160.068	159.82
1.19	140.01	140.088	140.11	139.73
1.16	120.062	120.18	120.21	121.81
1.13	100.062	100.365	100.39	99.88
1.11	80.30	80.74	80.75	78.81
1.09	61.13	61.50	61.49	61.09
1.06	43.35	43.098	43.04	40.11
1.04	28.09	26.57	26.50	22.80
1.02	16.33	14.09	14.05	10.68
1.00	8.38	7.11	7.04	6.87
0.98	3.76	3.75	3.70	1.92
0.96	1.47	2.07	2.07	1.69
0.94	0.49	1.18	1.22	0.19
0.92	0.14	0.69	0.74	0.018
0.91	0.037	0.41	0.47	0.0087
0.89	0.008	0.25	0.30	0.0012
0.88	0.0015	0.15	0.20	0.0093
0.86	0.00026	0.097	0.13	0.00057
0.84	0.0000388	0.06	0.089	0.000032

Table 4.2: European call prices in the different models, for different stock-price/strikeprice ratio- $S_0 = 1000$, $T = 1$ year.

Figure 4.1, Figure 4.2, and Figure 4.3 are respectively plots of the difference between the Black-Scholes prices and the Meixner price, the Black-Scholes prices and the Hyperbolic prices, and the Black-Scholes prices and the variance gamma prices. The plots are taken for different strike prices.

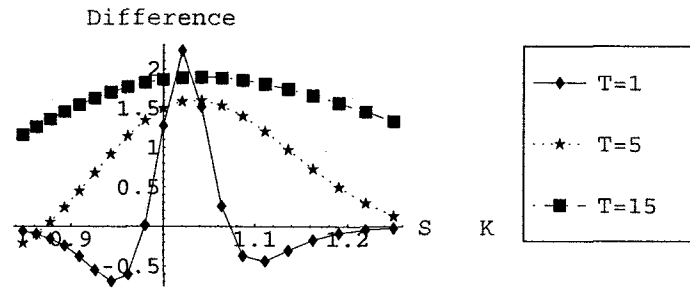


Figure 4.1: Difference between the Black-Scholes prices and the Meixner prices for $T = 1$, $T = 5$, and $T = 15$.

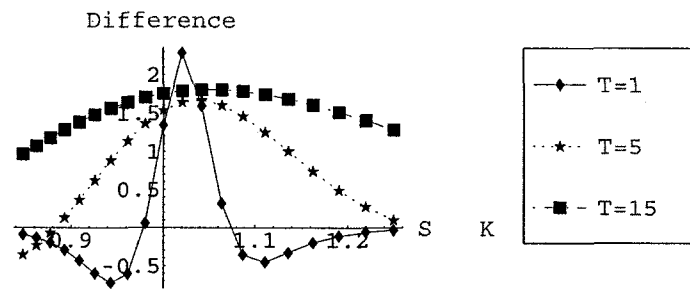


Figure 4.2: Difference between the Black-Scholes prices and the Hyperbolic prices for $T = 1$, $T = 5$, and $T = 15$.

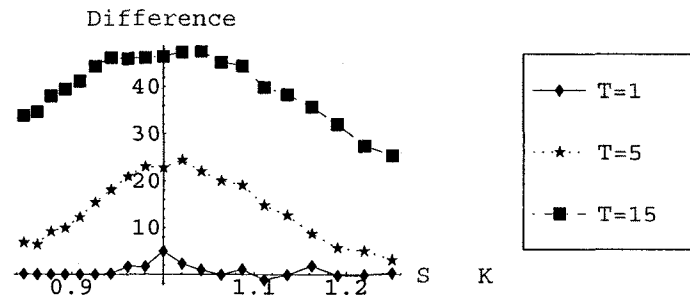


Figure 4.3: Difference between the Black-Scholes prices and the Variance Gamma prices for $T = 1$, $T = 5$, and $T = 15$.

$S(0)/K$	Meixner/BS	Hyperbolic/BS	Variance Gamma/BS
1.25	1.00	1.00	1.00
1.22	1.00	1.00	1.00
1.19	1.00	1.00	1.00
1.16	1.00	1.00	1.01
1.13	1.00	1.00	1.00
1.11	1.01	1.01	0.98
1.09	1.01	1.01	1.00
1.06	0.99	0.99	0.93
1.04	0.95	0.94	0.81
1.02	0.86	0.86	0.65
1.00	0.85	0.84	0.82
0.98	0.99	0.98	0.51
0.96	1.41	1.41	1.15
0.94	2.39	2.46	0.39
0.92	4.80	5.15	0.13
0.91	11.31	12.77	0.10
0.89	31.18	37.28	0.34
0.88	99.79	127.27	5.54
0.86	368.60	503.65	4.79
0.84	1562.03	2294.95	2414.72

Table 4.3: Ratios Meixner/BS, Hyperbolic/BS, and Variance Gamma/BS European call prices, for different strike-price/stockprice ratio- $S_0 = 1000$, $T = 1$ year .

4.6.2 Conclusion

As shown in Table 4.2, all models yield almost similar average price for a European call option written on the stock, when the stock-price/strikeprice is between 1 and 1.25. However the table also shows that the Black-Scholes model produces call option prices much smaller than the prices produced by the non-Gaussian models, when the strikeprice is less than 88% of the stock price. It is also remarkable that the Meixner model and the Hyperbolic model yield almost similar values for the different

strikeprices. The pricing formula under all the four models shows that the price of a call option increases as the time to maturity increases or as the current stock price increases. And it decreases as the exercise price increase. The graphs in Figure 4.1, Figure 4.2, and Figure 4.3 indicate that as the time to maturity increases, the curve becomes flatter.

4.6.3 Comparison of Model and Market Prices

The simple idea is to compare the four model prices with the market prices. First, the model parameters are estimated by maximum likelihood methods. Then, we compute for each model, the average absolute error with respect to the market prices. The best model, is the model that produces the smallest average absolute error.

4.6.4 The option data set

The data set consists of daily CBOE transaction records during the month of April 2002. Each transaction record includes the time at which the transaction takes place, the strike price, the bid and ask option prices, and the matching stock prices. The analysis is applied to the S&P 500 index . To avoid noisy information, all options with less than one week to expire are discarded, we only use call options maturing in July and August. There are around 1,000 transactions. The S&P 500 options are European options. Four maturity periods are selected: July, August, December, and March. Options with maturity less than one week are disregarded to avoid possible noises. There are typically 60 to 100 option data available everyday. Time to maturity are calculated for both options in terms of trading days. These prices are referred to as market prices.

To estimate the parameters we determine the model parameters that minimize the mean absolute error, as percentage of the mean option price. This quantity is known as the *ape*-statistic. It is given by

$$ape = \frac{1}{\text{mean option price}} \frac{\sum_{i=1}^n |C_{\text{market}}(i) - C_{\text{model}}(i)|}{\sum_{i=1}^n C_{\text{market}}(i)}, \quad (4.6.1)$$

where C_{market} is the observed market price, and C_{model} is the model price. The best

model is therefore the model that minimizes the value of the test statistic ape . The parameter estimates, the standard errors, and the values of the ape -statistic are presented in Table 4.4.

Parameters	Black-Scholes	VG	HYP	Meixner
a				0.8071
Std. Error				0.1202
b				0.7183
Std. Error				0.2011
c				1.01
Std. Error				0.02
d				0.2816
Std. Error				0.0032
m				0.0228
Std. Error				0.001
α			57.68	
Std. Error			0.811	
β			0.5842	
Std. Error			0.128	
μ			0.8551	
Std. Error			0.03581	
λ			-0.5	
Std. Error			0.001	
δ			11.52	
Std. Error			0.2856	
σ	0.12949	0.1805		
Std. Error	0.001	0.0021		
ν		-253.473		
Std. Error		0.0023		
θ		0.0012		
Std. Error		0.0001		
Ape	0.035	0.0259	0.0274	0.0278

Table 4.4: MLE Parameter Estimates and Mean Absolute Error on the S&P 500 Index.

The study shows that the pure-jump models yield a smaller value for the *ape* test statistic than the Black-Scholes model. The VG model outperforms the other non Gaussian models. The Meixner model and the generalized hyperbolic model yield almost similar prices. We also present in Graphs 4.4, 4.5, 4.6, and 4.7 a calibration of the model prices to the market prices.

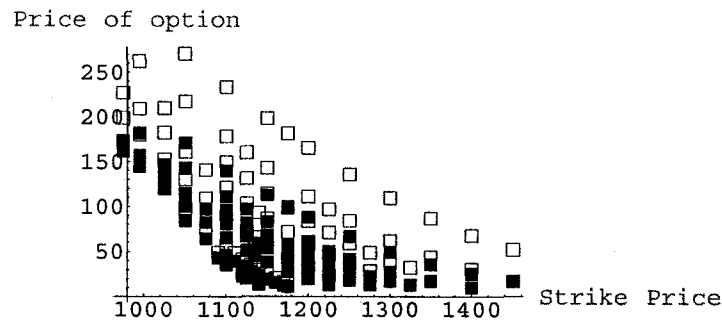


Figure 4.4: Calibration of the Black-Scholes call prices on S&P500. The Filled boxes are market prices. The clear boxes are BS prices.

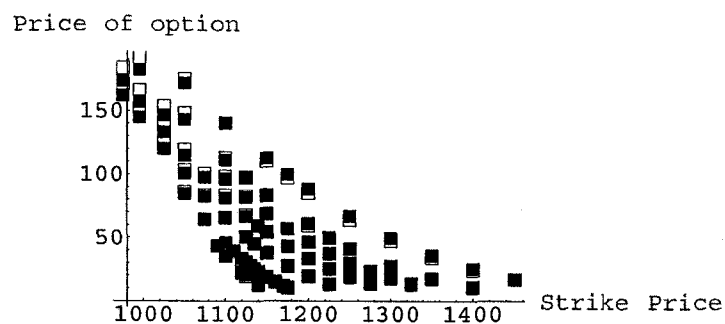


Figure 4.5: Calibration of the Meixner call prices on the S&P 500. The Filled boxes are market prices. The clear boxes are Meixner prices.

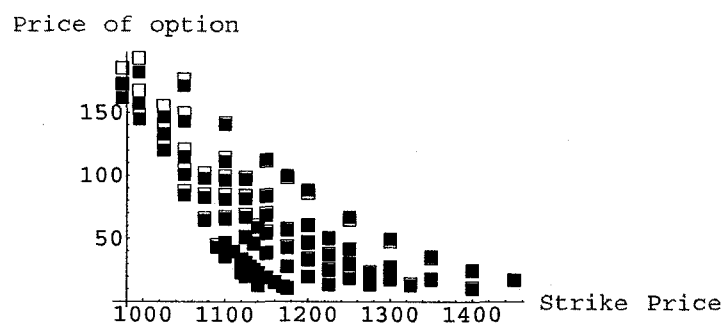


Figure 4.6: Calibration of the variance gamma call prices on S&P 500. The Filled boxes are market prices. The clear boxes are the variance gamma prices.

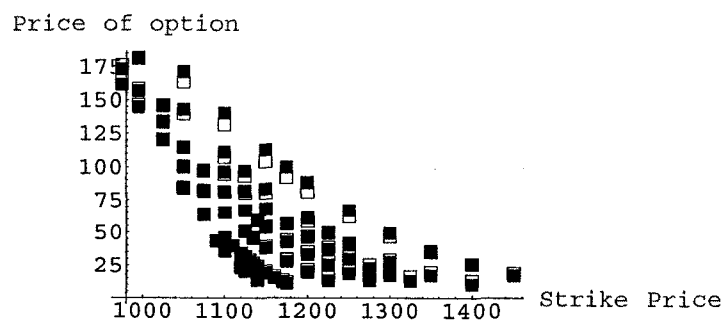


Figure 4.7: Calibration of the Hyperbolic call prices on S&P 500. The Filled boxes are market prices. The clear boxes are the Hyperbolic prices.

4.7 Conclusion

We have presented in this chapter, a general methodology for pricing non-path-dependent European options when the underlying stock price process is modelled by a pure-jump Lévy process. The method first derives an equivalent martingale measure under which the expectation of the discounted payoff of the option can be calculated. Next, a detailed description of the pricing formulas for the same European call option using the Black-Scholes model, the variance-gamma model, the Generalized hyperbolic model, and the Meixner model was provided. Then we performed a statistical test to compare the performances of the four models by computing the average absolute error produced by each model. And based on that test, we concluded that the variance gamma model appears to deliver more acceptable option prices than the Generalized hyperbolic and the Meixner models. We also concluded that the Black-Scholes model was outperformed by all the non-Gaussian models.

Chapter 5

The Pricing of Exotic Options

5.1 Introduction

The last few years have witnessed an increasing popularity of exotic options in financial markets. Exotic options are, unlike plain-vanilla options, nonstandard products created by financial engineers. There are different types of exotic options: forward start, compound, chooser, barrier, binary, lookback, shout, Parisian, Asian, and exchange options; to name a few. Everyday, new types are being developed. The popularity of exotic options can be explained by the fact that they are less expensive than the equivalent plain-vanilla options and more appropriate in hedging certain risks. For instance, a down-and-in-put with a low barrier offers an inexpensive protection against a big downward move of the underlying asset. Similarly, an up-and-out call on an index may enhance inexpensively an existing long position in the index. Most exotic options trade in the over-the-counter market and are designed by financial institutions to meet the requirements of their clients. For a complete survey on exotic options, see Hull [37].

The preceding chapter presented pricing formulas for plain-vanilla options based

on the Black-Scholes model, the variance-gamma model, the Generalized Hyperbolic model, and the Meixner model. In this chapter, we present a pricing methodology for barrier options based on the same models. However, since no closed-form formula is available in the pure-jump case for double barrier options, we generate the prices via simulation.

The first section introduces exotic options. Section 2 develops the closed form formula for single barrier options under the Black-Scholes setting. In the third section, we present another formula for pricing single barrier option in the non Gaussian setting (there is no closed-form formula for pricing double barrier options under the non Gaussian setting). The formula is due to Boyarchenko and Levendorskii. In Section 4, we develop a pricing methodology for barrier options (with both single and double barriers) using the simulation of the paths of the stock price processes. The methodology relies heavily on the series representation of Lévy processes to generate the stock prices sample paths. In Section 5, we present a comparative study of the Black-Scholes closed-form formula, the Boyarchenko-Levendorskii model, and the prices obtained via simulation under the Black-Scholes model, the Meixner model, the variance gamma model, and the Generalized Hyperbolic model.

5.2 Barrier Options

Barrier options are exotic options whose payoff depends on preset barriers. They can be classified in two groups, single barrier options and double barrier options. There are different types of single and double barrier options traded in the over-the-counter market. They can be classified as either knock-out barrier options or knock-in barrier options. A knock-out option is worth 0 if the underlying asset price reaches the

barrier(s). A knock-in option comes into existence only if the value of the underlying stock reaches the barrier(s). A down-and-out call is one type of knock-out option. It can be thought of as a regular call option whose value ceases to exist if the asset reaches a certain low barrier (less than the initial price of the asset). And a down-and-in call option is a call option that comes into existence only if the asset price reaches the barrier H .

5.2.1 Black-Scholes Prices of Single Barrier Options

In chapter 4, Equation 4.3.3 expressed the value of a plain-vanilla European call option with strike price K and time to expiration T , under the Black-Scholes model as:

$$C(T, K) = S_0 \Phi \left(\frac{\log(S_0/k) + (r + 1/2\sigma^2)(T+t)}{\sigma\sqrt{T+t}} \right) - ke^{-r(T+t)} \Phi \left(\frac{\log(S_0/k) + (r - 1/2\sigma^2)(T+t)}{\sigma\sqrt{T+t}} \right). \quad (5.2.1)$$

Using the same notation, the values of a down-and-in (C_{di}) call, a down-and-out (C_{do}) call, an up-and-in (C_{ui}) call, an up-and-out (C_{uo}) call, and a double-barrier (C_{db}) call options can be expressed as follows:

$$\begin{aligned} C_{di} &= e^{-rT} E^Q [(S_T - K)^+ 1_{\{m \leq H\}}] \\ C_{do} &= e^{-rT} E^Q [(S_T - K)^+ 1_{\{m > H\}}] \\ C_{ui} &= e^{-rT} E^Q [(S_T - K)^+ 1_{\{M \geq H\}}] \\ C_{uo} &= e^{-rT} E^Q [(S_T - K)^+ 1_{\{M < H\}}] \\ C_{db} &= e^{rT} E^Q [(S_T - K)^+ 1_{\{M < U\}} 1_{\{m > L\}}], \end{aligned} \quad (5.2.2)$$

where $M = \sup S_u$ and $m = \inf S_u$; $0 \leq u \leq t$; $0 \leq t \leq T$. The lower barrier and the upper barrier are respectively L and U .

Closed-form formulas for the single barrier options are obtained as follows:

If $H < K$

$$C_{di} = S_0 e^{-rT} (H/S_0)^{2\lambda} \Phi(y) - K e^{-rT} (H/S_0)^{2\lambda-2} \Phi(y - \sigma\sqrt{T}), \quad (5.2.3)$$

where

$$\lambda = \frac{r + \sigma^2/2}{\sigma^2}, \quad (5.2.4)$$

and

$$y = \frac{\ln(H^2/(S_0 K))}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}. \quad (5.2.5)$$

If $H \geq K$, then

$$\begin{aligned} C_{di} = & S_0 \Phi(x) e^{-rT} - K e^{-rT} \Phi(x - \sigma\sqrt{T}) \\ & - S_0 e^{-rT} (H/S_0)^{2\lambda} \Phi(y) + K e^{-rT} (H/S_0)^{2\lambda-2} \Phi(y - \sigma\sqrt{T}), \end{aligned} \quad (5.2.6)$$

where

$$\begin{aligned} x &= \frac{\ln(S_0/H)}{\sigma\sqrt{T}} + \lambda\sqrt{T} \\ y &= \frac{\ln(/HS_0)}{\sigma\sqrt{T}} + \lambda\sqrt{T}. \end{aligned} \quad (5.2.7)$$

To price the down-and-out counterpart option (C_{do}), it suffices to notice that if $H < K$, then $C_{do} = C - C_{di}$, where C is the price of the plain-vanilla option. However for $H \geq K$, then $C_{di} = C - C_{do}$. The values of up-and-out call options and up-and-in call options can be priced in a similar way. See Hull [37] for a thorough development of the formulas.

5.3 Closed-Form Formulas for Pricing Options under the Non Gaussian Models

Explicit formulas for pricing single barrier options using Regular Lévy Processes of Exponential Type¹ (RLPE) were derived in Boyarchenko-Levendorskii [18]. The formulas use the connection between the resolvent and the infinitesimal generator of the underlying Lévy process, the representation theorem for analytical semigroups, the Wiener-Hopf factorization method and the theory of pseudo differential operators. The following definitions are taken from Bertoin [14] and Nguyen-Ngoc [55].

We recall that if $X = (X_t)_{t \geq 0}$ is a Lévy process, the infinite divisibility of the law of X implies that its characteristic function can be expressed in the form

$$\phi_t(u) = E(e^{iuX_t}) = e^{-t\varphi(u)}, \quad (5.3.1)$$

where

$$\varphi(u) = \frac{a^2}{2}u^2 - iub + \int_{-\infty}^{\infty} (1 - e^{i\omega u} + i\omega h(\omega))\tau(d\omega), \quad (5.3.2)$$

for some numbers $a \geq 0$ and $b \in \mathbb{R}$, and a positive measure τ , the Lévy measure on $\mathbb{R}$.

We also recall that when one time-changes a Markov process M by an independent subordinator X , the resulting process $M \circ X$ is also a Markov process.

Also, if $T = T(q)$ designates an independent exponential time with parameter $q > 0$, the process X^q taking values in $[0, \infty)$ and given by $X_t^q = X_t$ if $t \in [0, T(q))$ and $X_t^q = \infty$ if $t \in [T(q), \infty)$, is called a subordinator killed at rate q .

¹The class of regular Lévy processes of exponential type contains in particular, Brownian motions, Hyperbolic processes, Meixner processes, truncated Lévy process, and variance gamma processes.

The stock price process $(S_t)_{t \geq 0}$ is expressed as

$$S_t = \exp(X_t), \quad (5.3.3)$$

where the Lévy process X is either a Meixner process, a Hyperbolic process, or a variance-gamma process.

We define the supremum process $P = (P_t)_{t \geq 0}$ of X as the process given by

$$P_t = \sup\{0 \vee X_s, 0 \leq s \leq t\}. \quad (5.3.4)$$

The process $(P - X)$ is called the reflected process at the supremum.

We also recall that a local time of X is any process $L = (L(t), t \geq 0)$ such that

$$dL(t) = \int_0^t 1_{\{X_s=0\}} ds \quad t \geq 0. \quad (5.3.5)$$

If $L = (L(t), t \geq 0)$ denotes the local time of $(P - X)$, we define

$$L^{-1}(\omega) = \inf \{s > 0 : L(s) > \omega\}, \quad (5.3.6)$$

as its right continuous inverse.

If we define the process $H = (H_t)_{t \geq 0}$, by $H_t = P_{\{L^{-1}(t)\}}$ if $L^{-1}(t) < \infty$, and $H(t) = \infty$ otherwise, then the pair (L^{-1}, H) is called the ladder process of X .

We denote by $T_k(X)$ the first passage time of X above k ($k \in R$), i.e.

$T_k(x) = \inf\{t : X_t > k\}$, and the first passage time below k is $\hat{T}_k(X)$, i.e.

$\hat{T}_k(X) = \inf\{t : X_t < k\}$. Note that in general, $X_{T_k(X)} > k$ and $X_{\hat{T}_k(X)} < k$ with positive probability.

In order to derive the price of single barrier options, one needs to know the law of $(T_k(X), X_{T_k(X)})$ for a fixed k . This law is characterized in terms of the Wiener-Hopf factorization.

We briefly present the essentials of the Wiener-Hopf factorization; for more details, see Nguyen-Ngoc [55]. It is well known that the ladder process $(T, X \circ T)$ is a two-dimensional Lévy process and that its Laplace exponent κ , defined by

$$e^{-t\kappa(\alpha, \beta)} = E[e^{-\alpha T(t) - \beta X \circ T(t)}] \quad (5.3.7)$$

is given by Fristedt's formula

$$\kappa(\alpha, \beta) = C \exp \left(\int_0^\infty \frac{dt}{t} \int_0^\infty (e^{-t} - e^{-\alpha t - \beta x}) P[X_t \in dx] \right), \quad (5.3.8)$$

where the constant $C > 0$ depends only on the normalization of the local time L . The Laplace exponent κ is called the ladder exponent of X . We recall the well-known Wiener-Hopf factorization for Lévy processes: for every $q > 0$ there exist two functions ϕ_q^+ and ϕ_q^- such that:

1. Both ϕ_q^+ and ϕ_q^- are characteristic functions. Besides, ϕ_q^+ is analytic in the half-plane $z \in \mathbb{C}, \Re(z) \leq 0$ and ϕ_q^- is analytic in the half-plane $z \in \mathbb{C}, \Re(z) \geq 0$.
2. For all $u \in \mathbb{R}$ the following identity holds (the Wiener-Hopf factorization)

$$\frac{q}{q + \phi(u)} = \phi_q^+(u) \phi_q^-(u). \quad (5.3.9)$$

The functions ϕ_q^+ and ϕ_q^- are called the Wiener-Hopf factors of X . They have the following probabilistic interpretation: If we let θ be the exponential random variable with parameter q , independent of X , then ϕ_q^+ is the characteristic function of M_θ , and ϕ_q^- is the characteristic function of $X_\theta - M_\theta$.

The Wiener-Hopf factors are related to the ladder exponent κ and the dual exponent of the dual ladder exponent $\tilde{\kappa}$ (i.e. the ladder exponent of the dual process

$\tilde{X} = -X$) by

$$\begin{aligned}\phi_q^+(u) &= \frac{\kappa(q, 0)}{\kappa(q, -iu)} & (u \leq 0) \\ \phi_q^-(u) &= \frac{\tilde{\kappa}(q, 0)}{\tilde{\kappa}(q, -iu)} & (u \geq 0)\end{aligned}\tag{5.3.10}$$

It turns out that the joint law of $(T_k(X), X_{T_k(X)})$ can be characterized entirely in terms of the function κ :

Theorem 5.3.1. (*The Pecherskii-Rogozin identity*)

Set $T_h = T_h(X)$. For every $\alpha, \beta, q > 0$, we have:

$$\int_0^\infty e^{-qh} E[e^{\alpha T_h - \beta(X_{T_h} - h)}] dh = \frac{1}{q - \beta} \left(1 - \frac{\kappa(\alpha, \beta)}{\kappa(\alpha, q)} \right).\tag{5.3.11}$$

Equation 5.3.11 can be written in terms of the Wiener-Hopf factor ϕ_α^+ :

$$\int_0^\infty e^{-qh} E[e^{\alpha T_h - \beta(X_{T_h} - h)}] dh = \frac{1}{q - \beta} \left(1 - \frac{\phi_q^+(iq)}{\kappa(\phi_q^+(i\beta))} \right).\tag{5.3.12}$$

Since $\hat{T}_k(X) = T_{-k}(-X)$, similar identities involving $\tilde{\kappa}$ or ϕ_q^- can be easily deduced for the joint law of $(\hat{T}_k(X), X_{\hat{T}_k(X)})$.

The valuation of barrier options is done in two steps.

1. Step 1:

The first step is to reduce the problem of the valuation of barrier options to the valuation of simpler options called digital options. The digital options are defined by their payoffs as follows:

Digital Up-and-In Call : $D_{uic}(H, K) = 1_{\{S_T > K\}} 1_{\{\sup_{t \leq T} S_t > K\}}$;

Digital Up-and-In Put: $D_{uip}(H, K) = 1_{\{S_T < K\}} 1_{\{\sup_{t \leq T} S_t > K\}}$.

The payoff from the barrier option in terms of the digital options is

$$C_{ui}(H, K) = \int_K^\infty D_{uic}(H, k) dk.\tag{5.3.13}$$

By Fubini's theorem, all is needed is to derive the price of a digital option.

2. Step 2: Pricing the digital options

Let us derive the price of a digital up-and-in call option, i.e. the probability $P_x[S_T > K, \{\sup_{t \leq T} S_t > K\}]$. The same method can be applied to calculate the price of the up-and-in put. Set

$$\hat{\pi}(T, h, k) = P_x[X_T, T(h) < T], \quad (5.3.14)$$

where $k = \ln(K)$ and $h = \ln(H)$. By the Markov property,

$$\hat{\pi}(T, h, k) = E_x[1_{\{T(h) < T\}} \hat{\beta}(T(h), X_{T(h)})] \quad (5.3.15)$$

where $\hat{\beta}$ is the function

$$\hat{\beta}(t, x) = P_x[X_{T-t} > k]. \quad (5.3.16)$$

Hence the price $\hat{\pi}(T, h, k)$ is characterized by Equation 5.3.11 and Equation 5.3.12.

It follows from straightforward computations that

$$\begin{aligned} \int_0^\infty dh \int_0^\infty dT \int_{-\infty}^\infty e^{(-\beta+i\lambda)h} e^{-qT} e^{i\lambda k \pi(h+x, k)} dk = \\ \frac{1}{i\lambda(q + \phi(\lambda))} \frac{1}{\beta - i\lambda} \left(1 - \frac{\phi_q^+(i\beta)}{\phi_q^+(-\lambda)} \right) \end{aligned} \quad (5.3.17)$$

See Nguyen-Ngoc [55] for a derivation of Equation 5.3.17

5.4 The Pricing of Barrier Options via Simulation

The other pricing method of barrier options under the non-Gaussian models relies on the theory of series representation of Lévy processes. The stock price process $(S_t)_{t \geq 0}$

is simulated according to the equation

$$S_t = \exp(\mu t + \sigma X_t), \quad (5.4.1)$$

where the parameters μ and σ represent respectively the drift coefficient and the stock's volatility; and $(X_t), t \geq 0$ is either the Meixner process, or the VG process, or the Hyperbolic Lévy motion. Recall that for the purpose of the simulation of the stock price process via the series representation theory, the Lévy measure is given by:

$$\tau(x) = \tau[x, \infty) = \int_x^\infty \tilde{\tau}(s) ds, \quad (5.4.2)$$

where the density of the Lévy measure $\tilde{\tau}(dx)$ is given respectively under the Meixner model, the Hyperbolic model, and the VG model by:

$$\begin{aligned} \tilde{\tau}_{Meix}(s) &= c \frac{\exp(bs/a)}{s \sinh(\pi s/a)} \\ \tilde{\tau}_{Hyp}(s) &= \frac{1}{\pi^2 |s|} \int_0^\infty \frac{\exp\left(-|s| \sqrt{2y + \frac{\zeta^2}{\delta}}\right)}{y(J_1^2(\delta \sqrt{2y}) + Y_1^2(\delta \sqrt{2y}))} dy + \frac{\exp(-|s| \frac{\zeta}{\delta})}{|s|} \\ \tilde{\tau}_{VG}(s) &= \frac{\exp(\theta s/\sigma^2)}{\nu |s|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}} |s|}{\sigma}\right), \end{aligned} \quad (5.4.3)$$

where J_1 and Y_1 are the Bessel functions of the first and second kinds respectively.

If we let ϵ represent the step of the simulation time, then we can generate the successive positions of $\log S_{t_i}$, at the time $t_i = i\epsilon$ according to

$$X_t \stackrel{d}{=} \sum_{j=0}^{\infty} \varepsilon_j \tau^{-1}(\Gamma_j) 1_{[0,t]}(U_j), \quad 0 \leq t \leq 1 \quad (5.4.4)$$

where the Lévy measure $\tau(x)$ is discretized. And its inverse is calculated numerically. Thus, we generate a number, say n , of paths of the stock price process. Then for each path i , we compute G_i , the value of the payoff function

$$G_i = (S_T - K)^+ 1_{\{T \leq H\}} \quad i = 1 \dots n, \quad (5.4.5)$$

where $\Upsilon = m$ or $\Upsilon = M$, depending on the nature of the single barrier option. Then we can estimate the value of the expected payoff:

$$G = \frac{1}{n} \sum_{i=1}^n G_i. \quad (5.4.6)$$

The final option price $e^{rT}G$, is then obtained by discounting the payoff G .

The standard error (SE) of the estimate of the payoff is given by:

$$SE = \sqrt{\frac{1}{(n-1)^2} \sum_{i=1}^n (G - G_i)^2}. \quad (5.4.7)$$

However, the estimator offered by this simple Monte Carlo simulation in the interval $[L, U]$ is biased. It's obvious that since the process is a pure-jump Lévy process, it's possible that the process exits and reenters $[L, U]$ without being detected. This error produces a systematic over estimation of the value of $\Sigma_{L,U}$, the first exit time outside of interval . Hence an error in the estimation of the probability of the event $A_{L,U} = \{\Sigma_{L,U} > T\}$ is possible. We should note also that a reduction of the simulation step ϵ produces negligible effects to reduce the error. The method developed in this thesis differs from a straightforward Monte Carlo method in that we handle the case where the stock price crosses the barriers without being detected. It is also an improvement upon the procedure suggested by Baldi [7] and Andersen and Brotherton-Ratcliffe [3]. If in the simulation of the two successive values of $\log S_{t_i}$, we find

$$\log S_{t_i} = x \quad \log S_{t_{i+1}} = y, \quad (5.4.8)$$

we calculate the probability $P = P(x, y, L, U)$ that $\log S$ crosses a barrier using the density function. With probability P , we can state that $\log(S_t)$ has crossed one barrier, and with probability $1 - P$, the simulation can continue. Therefore, it is evident that if the value of P can be specified, then the estimator is no longer biased.

It is important to note that in the Brownian motion case, $\log(S_t)$ in the interval $[t_i, t_{i+1}]$, conditioned on Equation (5.4.8) is a Brownian bridge. If there is only one constant barrier, say $U(t) \equiv b$, the probability $P(x, y, b)$ that the Brownian bridge crosses b under the condition that $x, y < b$ is given by

$$P(x, y, b) = \exp\left(-\frac{2}{\sigma^2 \epsilon}(b-x)(b-y)\right) \quad (5.4.9)$$

In the non Gaussian case, we can simply say that, in the interval $[t_i, t_{i+1}]$, conditioned on Equation 5.4.8, $\log(S_t)$ is the killed process Y^ϵ . So the probability p^ϵ that $\log S_t$ crosses one barrier $L(\epsilon t)$ or $U(\epsilon t)$, conditioned on $\log S_0 = x$ and $\log S_\epsilon = y$ in the interval $[0, \epsilon]$ is equal to the probability that Y^ϵ crosses the barriers $t \rightarrow L(\epsilon t)$ or $t \rightarrow U(\epsilon t)$ for $0 \leq t \leq 1$. And the theory of large deviations automatically implies:

$$\log p^\epsilon \sim \frac{I}{\epsilon} \quad (5.4.10)$$

where I is the infimum of the function:

$$\gamma \rightarrow \int_0^1 |\dot{\gamma}_s|^2 ds - \frac{1}{2}|y-x|^2 \quad (5.4.11)$$

taken on all trajectories such that $\gamma_0 = x, \gamma_1 = y$ and $\gamma_t \geq U(0)$ or $\gamma_t \leq L(0)$ for $0 \leq t \leq 1$. Computing the value of I is a simple calculus of variations, and we find $I = 0$ for $y > U(0)$, or $y < L(0)$, and

$$I = \frac{2}{\sigma^2} \min((x - U(0))(y - U(0)), (x - L(0))(y - L(0))). \quad (5.4.12)$$

In other words, for $L(0) < y < U(0)$,

$$I = \begin{cases} \frac{2}{\sigma^2}(x - U(0))(y - U(0)) & \text{for } x + y < L(0) + U(0) \\ \frac{2}{\sigma^2}(x - L(0))(y - L(0)) & \text{for } x + y > L(0) + U(0). \end{cases}$$

so that for $\epsilon \rightarrow 0$,

$$p^\epsilon = f(\epsilon)e^{-I/\epsilon} \quad (5.4.13)$$

where f is a function with sub-exponential behavior, for $\epsilon \rightarrow 0$. In our simulation, we consider for its simplicity $f(\epsilon) = e^{-\epsilon}$.

5.5 Numerical Results

In this section, we discuss the results obtained through the numerical simulation. We postpone the comparison of the model performances until the next section. We simply present the prices obtained using the different models. The option data used in this section ² is a set of European call options on the *S&P500* index at the close of the market on April 18, 2002. On this day, the *S&P500* closed at \$1124.47. The short term rate of interest was equal to $r = 0.019$. The data is used to estimate the models parameters. The estimates of the models parameters are obtained from calibration on the market vanilla prices of the *S&P500* index. The calibration is done by finding the model parameters that minimize the mean absolute error, the *ape*-statistics, given as in Chapter 4 by

$$ape = \frac{1}{\text{mean option price}} \frac{\sum_{i=1}^n |C_{\text{market}}(i) - C_{\text{model}}(i)|}{\sum_{i=1}^n C_{\text{market}}(i)}, \quad (5.5.1)$$

where C_{market} is the observed market price, and C_{model} is the model price. The parameters are presented in Table 5.1.

We take for all the single and double barrier options the time to maturity $T = 1$, the strike price $K = S_0 = 1124.47$. For all models, we make $n = 10,000$ simulations of paths covering the one year period. The time is discretized in 250 equally small time steps. And the barrier is set, depending on the type of the barrier option as follows: $H_{ui} = 1.10S_0$, $H_{uo} = 1.3S_0$, $H_{di} = 0.95S_0$, $H_{do} = 0.8S_0$. The barriers

²Special thanks Tykesson providing the data set.

Parameters	Black-Scholes	VG	HYP	Meixner
a				0.8071
b				0.7183
c				1.0001
d				82.3523
m				-0.6121
α			128.569	
β			0.001	
μ			0.8551	
λ			-0.5000	
δ			2.7546	
σ	0.1256	0.2030		
ν		-312.453		
θ		0.0021		
Ape	0.78%	0.62%	0.60%	0.63

Table 5.1: MLE Parameter Estimation on the S&P 500 Index option data .

for the double-barrier option are $U = 1.5S_0$ and $L = 3.5S_0$. It is necessary to point out that the estimator provided by this procedure is non-biased, at least in the case of a single constant barrier, since the value of p^ε is given by an explicit formula. Next, we present the prices obtained via the Black-Scholes-Monte-Carlo simulation method, the closed-form formula obtained via Black-Scholes, the closed-form formula obtained through Levendorskii-Boyarchenko under the variance-gamma model, the Generalized Hyperbolic model, and the Meixner model, and the prices obtained through the simulation by the series representation theory under the non Gaussian models. The results are presented in Table 5.2.

The price of a double barrier option is obtained via simulation. It's given in the second column of Table 5.2. For the Black-Scholes model, it is equal to 47.98. And for

Models	C_{db}	C_{ui}	C_{uo}	C_{di}	C_{do}
Black-Scholes	NA	75.30	78.63	17.97	85.98
B-S Monte Carlo	47.98	95.09	95.38	42.12	107.65
Standard Error.	0.003	0.003	0.001	0.002	0.003
Boy-Lev. VG	NA	78.98	86.30	20.43	88.76
Boy-Lev Meixner	NA	80.65	88.98	23.54	90.10
Boy-Lev - Hyperb.	NA	83.87	90.65	25.87	89.54
VG Simul Price	36.65	90.76	93.09	30.65	90.87
Standard Error.	0.003	0.003	0.001	0.002	0.003
Meix Simul Price	34.98	87.76	95.89	35.69	97.54
Standard Error.	0.003	0.003	0.001	0.002	0.003
Hyp Simul Price	34.76	93.98	91.76	34.85	93.23
Standard Error.	0.003	0.003	0.001	0.002	0.003

Table 5.2: Prices of Single and Double Barrier Options

the variance-gamma model, the Hyperbolic model and the Meixner model, it's equal respectively to 36.65, 34.76, and 34.98. Again the prices under the Meixner model and the hyperbolic model are very close. The prices of the single barrier options C_{di} , C_{do} , C_{ui} , and C_{uo} are obtained by both simulation and by closed-form formulas. The direct simulation under the Black-Scholes model (second row of Table 5.2) gives, for the double barrier option, the up-and-in option, the up-and-out option, the down-and-in option, and the down-and-out option, respectively 47.98, 95.09, 95.38, 42.12, and 107.65. These prices are at least 1.2 times higher than the corresponding prices using the Black-Scholes closed-form formula . That suggests that direct simulation without implementing a methodology that captures the event when the stock price exits the barriers is not reliable. Generally, simulation yields much higher prices than the closed-form formulas. However in the non Gaussian models, the difference in prices from simulation and from the closed-form formulas is much smaller than in

the Black-Scholes model. Hence, in the absence of a closed-form formula, it is more efficient to use a non Gaussian model to simulate the stock price process and to derive the price of barrier options. Also, in the simulation via series representation, one has to implement a methodology to avoid "hits-and-misses".

5.6 Models Performances

In this section we compare the pricing performance of the different models. The data consist of barrier options created by the "National Bank of Canada" for its corporate clients. We consider these empirical (market) prices during the month of January 2001. Again, we perform 10,000 simulations to generate the paths of the stock price process under the different models. On an AMD Athlon 2.2 GHz PC with 1GB of RAM, simulating the 10,000 paths of the stock price process for each model takes about 10 seconds. The computed average prices from the simulations and their standard deviations are then compared with the empirical (market) prices.

In general, the barriers are of the form:

$$L(t) = Ae^{\delta_1 t} \quad U(t) = Be^{\delta_2 t} \quad (5.6.1)$$

Here we set $\delta_1 = \delta_2 = 0$. Hence the barriers are constant. We look at the following 3 cases: $L = 1.5$ and $U = 2.5$; $L = 1.5$ and $U = 3$; $L = 1$ and $U = 3$.

Table 5.3 shows that the average market prices are 0.50, 0.10, and 0.06 for the three types of double-barrier options considered. One can see from the table that the Black-Scholes model and the Monte-Carlo Black-Scholes method appear strongly biased, producing prices far below or far beyond the empirical (market) prices. However,

	$\sigma = 0.2$ $r = 0.02$ $k = 2$ $L = 1.5, U = 2.5$	$\sigma = 0.5$ $r = 0.05$ $k = 2$ $L = 1.5, U = 3$	$\sigma = 0.5$ $r = 0.05$ $k = 1.75$ $L = 1, U = 3$
Black-Scholes	4.1230	6.5087	10.0802
Standard Error	0.001	0.001	0.001
B-S Monte Carlo	20.9225	32.0191	10.0772
Standard Error.	0.003	0.003	0.003
Var-Gamma	0.5087	0.0981	0.06702
Standard Error	0.001	0.001	0.001
Meixner	0.5658	0.0899	0.0700
Standard Error.	0.0004	0.0004	0.0004
Gen. Hyperb.	0.4521	0.0801	0.066
Standard Error	0.0002	0.0002	0.0002
Average Empirical Prices	0.5000	0.1000	0.0600

Table 5.3: Comparison of Models Prices and Market Prices .

the prices obtained through the pure-jump models appear to be very close to the empirical prices. The variance gamma model produces prices that are closer to the market prices. However, all three non-Gaussian prices are within an acceptable range of the market prices. They also outperform by far the Black-Scholes model. This can be explained by the fact that the distribution of the pure-jump models provide a better fit for the underlying stock price processes. Also, in the simulation of the Meixner , the Hyperbolic, and the variance gamma prices, we have implemented a mechanism that ensures that a barrier is not "hit and missed".

5.7 Conclusion

We have presented in this paper a simple but powerful procedure for pricing single-barrier and double-barrier options. The procedure is based on simulating the price process of the underlying stocks by using a series representation approach of Lévy processes.

The importance of the procedure comes from the absence of a closed-form formula for pricing double barrier options. For the pricing of single barrier exotic options, the simulation method has proven to be quite reliable, since it produces prices within an acceptable range of the prices obtained via the closed-form formula obtained by Boyarchenko-Levendorskii. However, more importantly, the method goes beyond the simple Monte Carlo model, since it handles the possibility that the price process crosses the barriers without being detected. This case is often overlooked by a straightforward approach.

Boyarchenko-Levendorskii's model derived explicit formulas by using the connection between resolvent and the infinitesimal generator of the underlying process, the representation theorem for analytical semigroups, the Wiener-Hopf factorization method and the theory of pseudo differential operators. One can argue that the gain in accuracy under the Boyarchenko-Levendorskii model is easily offset by the complexity of the implementation of the model.

We'll look into expanding that method to obtain a closed-form formula for pricing double barrier options, with or without rebates in future work.

Appendix A

Appendix

In the appendix, we develop the source code for the price of a European call option using the Black-Scholes formula. We also present an equivalent source code for the variance gamma model, the hyperbolic model, and the Meixner model. Each program begins with a model to simulate the stock price process under the different model, followed by an option pricing formula under the given model.

A.0.1 Appendix A

A.1 The Black-Scholes Model

The first part of each program presents the function header files necessary to execute the main program for each model.

file:///C:/Documents%20and%20Settings/guess/My%20Documents/black1.txt

```
// File: Normal.h
// Usage: Cumulative distribution function for a bivariate normal distribution,
//        see John Hull: Options, Futures and Other Derivatives
//        double ND2(double a, double b, double rho);
//        standard normal density function.
//        double ndf(double t);
//        standard normal cumulative distribution function
```

```
double nc(double x); double min(double, double);
double max(double, double);
```

```
//Black-Scholes Pricing Formula
```

```
double vanilla (double, double, double, double, double, double, int);
```

```

//*****
// Program : normal.cpp
// Author  : Serigne Diop
//
// Created : December 2001
//
// Usage   : auxiliary functions around the normal distribution
//*****
```

```
#include <math.h>
```

```
#define PI 3.141592654
```

file:///C:/Documents%20and%20Settings/guess/My%20Documents/black3.txt

```
double max(double a, double b) {
    if (a>b)
        return a;
    else
        return b;
}
double min(double a, double b) {
    if (a>b)
        return b;
    else
        return a;
}
```

// Replicates Sgn as in visual basic, the signum of a real number.

```
double sgn(double a) {
    if (a>0)
        return 1.;
    else if (a<0)
        return -1.;
    else
        return 0.;
}
```

// Standard normal density function

```
double ndf(double t) {
    return 0.398942280401433*exp(-t*t/2);
}
```

file:///C:/Documents%20and%20Settings/guess/My%20Documents/fblack5.txt

// Standard normal cumulative distribution function

```
double nc(double x) {
    double result;
    if (x<=-7.)
        result = ndf(x)/sqrt(1.+x*x);
    else if (x>7.)
        result = 1. - nc(-x);
    else {
        result = 0.2316419;

        static double a[5]={0.31938153,-0.356563782,
                            1.781477937,-1.821255978,
                            1.330274429};
        result=1./(1+result*fabs(x));
        result=1-ndf(x)*(result*(a[0]+result*(a[1]+result*(a[2]+result*(a[3]+result*a[4])))));

        if (x<=0.)
            result=1.-result;
    }
    return result;
}
```

//Function needed to calculate two dimensional cumulative distribution function, see Hull.

2

file:///C:/Documents%20and%20Settings/guess/My%20Documents/gblack6.txt

```
double fxy(double x, double y, double a, double b, double rho) {
    double a_s;
    double b_s;
    double result;
    a_s = a / sqrt(2 * (1 - rho * rho));
    b_s = b / sqrt(2 * (1 - rho * rho));
    result = exp(a_s * (2 * x - a_s) + b_s * (2 * y - b_s) +
        2 * rho * (x - a_s) * (y - b_s));
    return result;
}

// function needed to calculate two dimensional cumulative distribution function,
// see Hull.
// this equals ND2 if a and b and rho are all nonpositive,
// the generalization for the other cases is ND2 below.

double Ntwo(double a, double b, double rho) {
    static double aij[4]={0.325303,
        0.4211071,
        0.1334425,
        0.006374323};
    static double bij[4]={0.1337764,
        0.6243247,
        1.3425378,
        2.2626645};

    int i;
    int j;
    double result;
    result = 0;
```

file:///C:/Documents%20and%20Settings/guess/My%20Documents/gblack7.txt

```

for(i=0;i<=3;i++)
{
    for(j=0;j<=3;j++)
    {
        result+=aij[i] * aij[j] * fxy(bij[i], bij[j], a, b, rho);
    }
}
result = result * sqrt(1 - rho * rho) / PI;
return result;
}

//Calculates cumulative distribution function for a bivariate
//normal distribution
//see John Hull: Options, Futures and Other Derivatives
double ND2(double a, double b, double rho) {
    double rho1;
    double rho2;
    double denominator;
    double result;

    if (rho > 0.9999)
        result = nc(min(a, b));
    else if (rho < -0.9999)
        result = max(0, nc(a) - nc(-b));

```

file:///C:/Documents%20and%20Settings/guess/My%20Documents/gblack8.txt

```

else
{
  if (a * b * rho <= 0)
  {
    if (a <= 0 && b <= 0 && rho <= 0)
      result = Ntwo(a, b, rho);
    else if (a <= 0 && b * rho >= 0)
      result = nc(a) - Ntwo(a, -b, -rho);
    else if (b <= 0 && rho >= 0)
      result = nc(b) - Ntwo(-a, b, -rho);
    else
      result = nc(a) + nc(b) - 1 + Ntwo(-a, -b, rho);
  }
  else
  {
    denominator = sqrt(a * a - 2 * rho * a * b + b * b);
    rho1 = (rho * a - b) * sgn(a) / denominator;
    rho2 = (rho * b - a) * sgn(b) / denominator;
    result = ND2(a, 0, rho1) + ND2(b, 0, rho2) - (1 - sgn(a) * sgn(b)) / 4;
  }
  if (result < 0) result = 0;
}
return result;
}

```

file:///C:/Documents%20and%20Settings/guess/My%20Documents/gblack9.txt

double cndev(double u) { /* returns the inverse of cumulative
normal distribution function Reference> The Full Monte, by Boris
Moro, Union Bank of Switzerland
RISK 1995(2)*/

```
static double a[4]={2.50662823884,
                    -18.61500062529,
                    41.39119773534,
                    -25.44106049637};
static double b[4]={-8.47351093090,
                    23.08336743743,
                    -21.06224101826,
                    3.13082909833};
static double c[9]={0.3374754822726147,
                    0.9761690190917186,
                    0.1607979714918209,
                    0.0276438810333863,
                    0.0038405729373609,
                    0.0003951896511919,
                    0.0000321767881768,
                    0.0000002888167364,
                    0.0000003960315187};
double x,r; x=u-0.5; if (fabs(x)<0.42) { r=x*x;
r=x*(((a[3]*r+a[2])*r+a[1])*r+a[0])/
(((b[3]*r+b[2])*r+b[1])*r+b[0])*r+1.0); return(r); } r=u;
if(x>0.0) r=1.0-u;
    r=log(-log(r));
    r=c[0]+r*(c[1]+r*(c[2]+r*(c[3]+r*(c[4]+r*(c[5]+r*(c[6]+
        r*(c[7]+r*c[8])))))));
    if(x<0.0) r=-r;
return(r); }

}
```

A.1.1 Appendix B

A.2 The Hyperbolic Model

The following C code contains the program to perform the maximum likelihood estimation¹ of the parameters of the Hyperbolic model. It also contains the program to compute the value of a European call option under the Hyperbolic model. We recall that the price of a European call option under the Hyperbolic model is given by

$$C(T, K) = S_0 \int_{\log(K/S_0)}^{\infty} f^{\theta}(x; \theta + 1) dx - e^{rT} K \int_{\log(K/S_0)}^{\infty} f^{\theta}(x; \theta) dx, \quad (\text{A.2.1})$$

where

$$f^{\theta}(x; \theta) = \frac{e^{\theta x} f^{*t}(x)}{\int_{-\infty}^{\infty} e^{\theta y} f^{*t}(y) dy}, \quad (\text{A.2.2})$$

f^{*t} is the t -fold convolution of the density of the Hyperbolic distribution.

¹The code of the header file was provided by Sørensen.

```
file:///C:/Documents%20and%20Settings/guess/My%20Documents/black10.txt
```

```
// function : Black-Scholes formula for European put and call options (vanilla)
//          value and Greeks
```

```
double vanilla(double spot, double strike, double sigma, double
rf, double tau, int option)
```

```
{
```

```
double result; double d1, d2;
```

```
    d1 = (log(spot / strike) + (rf + sigma * sigma * 0.5) * tau) / (sigma * sqrt(tau));
    d2 = d1 - sigma * sqrt(tau);
```

```
    if (option == 1)
```

```
        result = exp(-rf * tau) * (spot * exp(rf * tau) * nc(d1) - strike * nc(d2));
```

```
    else
```

```
        result = strike * exp(-rf * tau) * nc(-d2) - spot * nc(-d1);
```

```
    return result; }
```

```
// Program : BlackScholes.cpp
```

```
// Author : Serigne Diop
```

```
//
```

```
// Created : December 2001
```

```
//
```

```
// Usage : Calculate the price of a vanilla put-call option
```

```
//          on a stock.
```

file:///C:/Documents%20and%20Settings/guess/My%20Documents/black12.txt

```
#include<iostream.h>
#include<math.h>
#include<fstream.h>
#include"normal.h"

void main() {

    double spotp;
    const double LOOPCOUNT = 760.0;
    const double strikep = 200.0;
    const double sigmap = 0.20 ; // Volatility of 20%
    const double rfp = 0.05;    // Risk free rate of interest of 5%
    const double taup = 2.0;    // The time is measured for two years.
    int option = 1;
    float count = 0.0;

    ifstream infile;
    ofstream outfile;

    infile.open("sprices.dat");
    outfile.open("oprices.txt");

    while (count < LOOPCOUNT && outfile.is_open())
    {

        infile >> spotp;
        outfile <<vanilla(spotp, strikep, sigmap,rfp, taup, option)<< endl;
        count = count + 1;
    }

}
```

```

/*
NAME
    hyp.h --- Header file for hyp
SYNOPSIS
    #include "hyp.h"
DESCRIPTION
    'hyp.h' contains constants, type definitions and function headers
    which are common for the modules in the program 'hyp'.

*/

/* constants to be changed by the user */
#define maxDim 3 /* maximal dimension to treat */
#define maxInterval 25 /* maximal length of intervalvector */

/* constants concerning plots */
/* maximal number of contour curves that possibly can be drawn */
#define maxNumberEllipse 30
/* the default number of contour curves that will be drawn */
/* #define defaultNumberEllipse 10 */
/* the number of points in a contour ellipse */
#define PointsInEllipse 300
/* the number of points in a log density graph */
#define PointsInLogDensity 100
/* the default minimal number of intervals with a positive number of
   observations when drawing conditional log histograms automatically */
/* #define CondLogHistDefaultMin 5 */
/* the default minimal number of groups with a positive number of
   observations when drawing conditional contour diagrams automatically */
/* #define CondContourDefaultMin 100 */

/* the precision for the integrations in multinomial likelihood */
/* #define IntegrationPrecision 1.0e-8 */

/* maximal height of the stack used in the integration routine */
#define MaxStackHeight 60

/* layout constants */
/* maximal number of columns to be printed when printing the observed
   and expected values */
#define ObsExpMaxCol 9
/* maximal number of columns to be printed when printing a matrix */
#define MatMaxCol 4
/* maximal number of columns to be printed when the log from the
   minimization procedure (bromg) is printed */
#define LogMaxCol 6

/* maximal length for a line on the data file */
#define MaxDataLineLength 200

/* maximal length of file names */
#define MaxFileNameLength 30

/* constants concerning start values */
/* default minimal and maximal values and number of log(delta), delta
   log(zeta), zeta and pi */
/*
define logDeltaDefaultMin -6
define logDeltaDefaultMax 2
define deltaDefaultMin 0.01
define deltaDefaultMax 3
define deltaDefaultNumber 20
define logZetaDefaultMin -6
define logZetaDefaultMax 2
define zetaDefaultMin 0.01
define zetaDefaultMax 3
define zetaDefaultNumber 20

```

```

define piDefaultMax 5
define piDefaultNumber 20
*/
/*=====DO NOT CHANGE BELOW THIS LINE=====*/

/* constants which are functions of the above constants */

/* maxObsIndex = maxInterval^maxDim */
/* maxObsIndex is the maximal number of groups in the grouped case */
#if maxDim == 1
#define maxObsIndex (maxInterval)
#else
#if maxDim == 2
#define maxObsIndex (maxInterval*maxInterval)
#else
#if maxDim == 3
#define maxObsIndex (maxInterval*maxInterval*maxInterval)
#endif
#endif
#endif

/* maxParamIndex is the maximal number of parameters */
#define maxParamIndex ((maxDim+1)*(maxDim+4) / 2 - 1)
/* maxMatDerIndex is the maximal number of derivatives of DELTA matrix */
#define maxMatDerIndex (maxDim*(maxDim+1)/2 - 1)

#define pi 3.141592653589793238462643

/*types*/
#define obsItype int /* obsIndex */
#define dimIndex int /* dimension index */
#define paramIndex int /* parameter index */
#define matDerIndex int /* index of derivative of matrix */

/* boolean type and constants */
#define bool int
#define true 1
#define false 0

typedef double vector[maxDim];
typedef double matrix[maxDim][maxDim];

/* data structure containing derivatives of DELTA-matrix w.r.t. FI */
typedef matrix dmatrix[maxMatDerIndex];

typedef struct
{
    double alfa;
    vector beta;
    double delta;
    double logdelta;
    vector my;
    matrix FI;
    matrix DELTA;
    matrix INVDELTA;
} parameters;
/* structure 'parameters' contains parameters of hyperbolic distribution */

typedef struct
{
    double alfa;
    vector beta;
    double delta;
    double logdelta;
    vector my;
    matrix FI;
    matrix DELTA;
    matrix INVDELTA;
    dmatrix dDELTA;

```

A.2.1 Appendix C

A.3 The Variance Gamma Model

The following code enables to calculate the price of a European call option under the variance gamma model. It follows the same idea described by Fiorani [31]. We recall that the price of a European call option under the variance gamma model is given by

$$C(T, K) = S_0 \Psi \left(d \sqrt{\frac{1-c_1}{\nu}}, (\alpha + s) \sqrt{\frac{\nu}{1-c_1}}, \frac{t}{\nu} \right) - K \exp(-rt) \Psi \left(d \sqrt{\frac{1-c_2}{\nu}}, \alpha s \sqrt{\frac{\nu}{1-c_2}}, \frac{t}{\nu} \right), \quad (\text{A.3.1})$$

where

$$d = \frac{1}{s} \left[\log \left(\frac{S_0}{K} \right) + rt + \frac{t}{\nu} \log \left(\frac{1-c_1}{1-c_2} \right) \right]. \quad (\text{A.3.2})$$

```

double deltaSqr;
int i; /* used for indexing derivatives wrt. beta_i, my_i and FI_ij */
int j; /* used for indexing derivatives wrt. FI_ij */
/* following is used when calculating precise versions of the
log likelihood functions and their derivatives */
vector x2; /* point in which the exponent or derivative is evaluated */
double c1; /* sqrt(delta^2 + (x2-my)%*(DELTA^1)%*(x2-my)) */
double c2; /* used as the derivative of the exponent in the point x2 */
} Dparam;
/* structure 'Dparam' contains parameters of hyperbolic distribution and
derivatives of DELTA-matrix w.r.t. the Cholesky factor FI of DELTA.
Note that 'parameters' is a "subtype" of 'Dparam', so
variables of type 'Dparam' can be used where variables of
type 'parameters' are used */

typedef struct
{
/* Observations and expected values of generalized hyperbolic distr.*/
int dim;
int dimVek[maxDim];
int origVar[maxDim];
double interval[maxDim][maxInterval+1];
double prob[maxObsIndex];
int obs[maxObsIndex];
/* The parameters of this generalized hyperbolic distribution */
int totalObs;
int twolambda;
parameters p;
} lowerDimDist;
/* structure 'lowerDimDist' contains parameters, observations and expected
values of the generalized hyperbolic distribution */

/* the parameter vector is stored as: alfa,log(delta),beta[],my[],FI[][] */
typedef double paramVek[maxParamIndex];
/* the parametermatrix (used for derivatives of parameter vector) */
typedef double paramMat[maxParamIndex][maxParamIndex];

/* structures for default values */
/* structure for numerical constants for procedure bromg */
typedef struct
{
int MaxIterations;
int MaxLineminltr;
int TimeLimit;
double Tolerance;
double LineminTol;
double FuncDiffLimit;
} BromgParameters;

typedef struct
{
BromgParameters group;
BromgParameters multi;
BromgParameters trunc;
BromgParameters ungroup;
double multi_IntegrationPrecision;
double trunc_IntegrationPrecision;

/* constants concerning start values */
/* default minimal and maximal values and number of log(delta), delta
log(zeta), zeta and pi */
double logDeltaDefaultMin;
double logDeltaDefaultMax;
double deltaDefaultMin;
double deltaDefaultMax;
int deltaDefaultNumber;
double logZetaDefaultMin;
double logZetaDefaultMax;
double zetaDefaultMin;

```

```

int zetaDefaultNumber;
double piDefaultMin;
double piDefaultMax;
int piDefaultNumber;

/* the default number of contour curves that will be drawn */
int defaultNumberEllipse;
/* the default minimal number of intervals with a positive number of
observations when drawing conditional log histograms automatically */
int CondLogHistDefaultMin;
/* the default minimal number of groups with a positive number of
observations when drawing conditional contour diagrams
automatically */
int CondContourDefaultMin;

/* number of groups to be made, when grouping ungrouped data */
int NoOfGroups;
} DefaultsStructure;

/* headers form miscellaneous files */
/* mathlib */
#include <math.h>

/* index.c */
#define alfaIndex 0
#define deltaIndex 1
paramIndex betaIndex();
paramIndex myIndex();
paramIndex FIIndex();
obsItype obsIndex();
obsItype margObsIndex();

/* integrals.c */
double adsim();

/* dfirst.c */
void DfirstInit();
void dloghyp();
void dmultinomial();
void dmultinomial2();
void dQuotient();
void dQuotient2();
void dtruncMult();
void dtruncMult2();
void dUnGroupLoghyp();

/* hypfun.c */
double normcon();
double lognormcon();
double exponent();
double Texponent();
double loghyp();
double multinomial();
double multinomial2();
double Quotient();
double Quotient2();
double truncMult();
double truncMult2();
double UnGroupLoghyp();

/* bessell.c */
void recursiveBessel();
double bessell();
double dLogBessell();

/* genhyp.c */
void generalHyp();
void generalTruncHyp();
double logGenHdist();

```

```
/* margcond.c */
void makeConditional();
void makeMarginal();
```

```
/* interio.c */
void readIn();
int scanint();
double scandoub();
bool yesanswar();
bool noanswar();
void readAnswer();
```

```
/*
NAME
```

```
    integral.c
```

```
SYNOPSIS
```

```
    #include "hyp.h"
    double adsim(n, a, b, ff, p, eps)
    int n;
    vector a,b;
    double (*ff)();
    int *p;
    double eps;
```

```
DESCRIPTION
```

adsim calculates the n-double integral, $n=0,1,2,\dots$, of the function ff the lower and the upper limit of the i'th integration variable being a[i] and b[i], respectively. That is adsim calculates the integral:

$$\int_{a[0]}^{b[0]} \int_{a[1]}^{b[1]} \dots \int_{a[n-1]}^{b[n-1]} ff(p,x) dx[0] dx[1] \dots dx[n-1]$$

The tolerance of the integral is eps in all dimensions.
The pointer *p is used for transferring additional information to the function ff. Function ff has to be declared as:

```
double ff(p,x)
vector x;
"something" *p;
{ ... }
```

The integration is performed using an adaptive version of Simphsons integration formula.

```
DIAGNOSTICS
```

If following is written on standard error
top exceeded MaxStackHeight in adsim --- MaxStackHeight = 30
then increase the value of the constant "#define MaxStackHeight 30"
and recompile the module, or decrease the integrationprecision.

```
*/
#include "hyp.h"
#include <stdio.h>
```

```
double fabs();
```

```
static vector xx; /* this is the integration variable */
```

```
double adsim(n, a, b, ff, p, eps)
int n;
vector a,b;
double (*ff)();
int *p;
double eps;
```

```

/* (integrates the rest over variable x[n-1]) */
{
    double f1, f2, f3, f4, f5, s0, sg, sn, x0, xe, xm, d, integ;
    int top;
    double s1[MaxStackHeight+1], s2[MaxStackHeight+1],
    s3[MaxStackHeight+1];
    /* s1,s2 and s3 are stacks for storing intermediate results */

    /* the 0-dimensional integral of a function is defined to be the
    function itself */
    if (n==0)
        return ff(p,xx);

    n--;
    xx[n]=a[n];
    f1 = n==0 ? ff(p,xx) : adsim(n,a,b,ff,p,eps);
    x0 = a[n] / 2;
    xx[n]=b[n];
    f5 = n==0 ? ff(p,xx) : adsim(n,a,b,ff,p,eps);
    xe = b[n] / 2;
    xx[n]=x0 + xe;
    f3 = n==0 ? ff(p,xx) : adsim(n,a,b,ff,p,eps);
    s0 = f1 + 4 * f3 + f5;
    top = 0;
    integ = 0;
    s1[0] = 0;
    s2[0] = 0;
    s3[0] = 0;
    while (top >= 0)
    {
        sg = s0;
        xm = (x0 + xe) / 2;
        xx[n]=x0 + xm;
        f2 = n==0 ? ff(p,xx) : adsim(n,a,b,ff,p,eps);
        xx[n]=xm + xe;
        f4 = n==0 ? ff(p,xx) : adsim(n,a,b,ff,p,eps);
        s0 = f1 + 4 * f2 + f3;
        d = xe - x0;
        sn = f3 + 4 * f4 + f5 + s0;
        if (fabs(2 * sg - sn) * d > eps)
        {
            top = top + 1;
            if (top > MaxStackHeight)
            {
                fprintf(stderr,"top exceeded MaxStackHeight \
in adsim --- MaxStackHeight = %d\n", MaxStackHeight);
                exit(1);
            }
            s1[top] = f4;
            s2[top] = f5;
            s3[top] = xe;
            xe = xm;
            f5 = f3;
            f3 = f2;
        }
        else
        {
            integ = integ + sn * d;
            f1 = f5;
            f3 = s1[top];
            f5 = s2[top];
            x0 = xe;
            xe = s3[top];
            top = top - 1;
            s0 = f1 + 4 * f3 + f5;
        }
    }
    return integ / 6;
}

```

```

/*
NAME
    hypfun.c --- Hyperbolic log likelihood functions

```

SYNOPSIS

```

#include <stdio.h>
#include "hyp.h"

double normcon(p)
parameters *p;

double exponent(p,x)
parameters *p;
vector x;

double Texponent(p,x)
Dparam *p;
vector x;

double loghyp(v)
paramVek v;

double multinomial(v)
paramVek v;

double multinomial2(v)
paramVek v;

double quotient(v)
paramVek v;

double quotient2(v)
paramVek v;

double truncMult(v)
paramVek v;

double truncMult2(v)
paramVek v;

double UnGroupLoghyp(v)
paramVek v;

```

DESCRIPTION

The value of normcon(p) is the norming constant of the hyperbolic distribution whose parameters are specified by the structure p of type 'parameters'.

The value of exponent(p,x) is the exponential part of the hyperbolic density (corresponding to the parameters in p) evaluated at x.

The value of Texponent(p,x) is simply exponent(p,x) divided by exponent(p,p->x2), where in applications p->x2 denotes a fixed value of the variable x.

This routine is used when calculating multinomial2, quotient2 and truncMult2 and their derivatives, see below.

loghyp(v) calculates minus the value of the log likelihood function corresponding to the parameter vector v, when the observations are grouped into boxes and it is assumed that all observations in the k'th box have a common value which is the centre of that box. That is: loghyp(v) is the value of the sum

$$- \sum_k N_k * \log\{h(\text{centre_of_}k)\}$$

where N_k is the number of observations in the k'th box and h is the density function of the hyperbolic distribution with parameters

multinomial(v) and multinomial2(v) calculates minus the value of the log likelihood function corresponding to the parameter vector v. The observations are grouped and the likelihood function is based on the multinomial distribution. If h is the density function of the hyperbolic distribution with parameters specified by v the value of multinomial(v) and multinomial2(v) is:

$$- \sum_k n_k * \log(\int_k h(x) dx)$$

where the sum is over all observation boxes and where n_k is the number of observations in box k and the integration is over the k'th box.

The integration procedure ADSIM, which is used throughout the program, is most precise if the integrand takes values around 1. In multinomial2(v) the integral is evaluated in the following way:

$$\int_k h(x) dx = h(x_c) \int_k h(x)/h(x_c) dx$$

where x_c is the centre of the k'th box. In multinomial(v) the integral is evaluated directly. Consequently, the difference between multinomial(v) and multinomial2(v) is that multinomial2(v) is more precise, but slower. The same applies for quotient(v) and quotient2(v) and for truncMult(v) and truncMult2(v), defined below.

quotient(v) and quotient2(v) calculates the quantity

$$\sum_k r_k * \log(\frac{r_k}{\int_k h(x) dx})$$

Here the sum is over all observation boxes and r_k is the relative number of observations in box k. Furthermore, h denotes the density function of the hyperbolic distribution with parameters specified by v and the integration is over the k'th box.

quotient(v) and quotient2(v) is minus the log likelihood ratio statistic for the hypothesis that the relative numbers of observations r_k are multinomial distributed with probabilities given by the hyperbolic distribution with parameters v. The one-dimensional version of this function was introduced by Barndorff-Nielsen (1977) in a study of the mass size distribution of sand sediments where only the relative weights are of relevance.

truncMult(v) and truncMult2(v) calculates the value of minus the truncated multinomial log likelihood function with probabilities specified by the hyperbolic distribution with parameter vector v. If e is the exponential term of the corresponding density function the value of truncMult(v) and truncMult2(v) is:

$$- \sum_k n_k * \log(\frac{\int_k e(x) dx}{\int_K e(x) dx})$$

where the sum is over all observation boxes and where n_k is the number of observations in box k, and K is the whole region, ie all the boxes (e is calculated by the routine exponent(p,x)).

UnGroupLoghyp(v) calculates minus the value of the log likelihood function corresponding to the parameter vector v in the case of ungrouped observations. If h denotes the density of the hyperbolic distribution with parameters specified by v the value of UnGroupLoghyp(v) is:

$$-\sum_x \log\{h(x)\}$$

where summation is performed over all observations.

```

*/
#include <stdio.h>
#include "hyp.h"

/*global variables*/
extern int dim;
extern int obs[maxObsIndex];
extern int totalObs;
extern double relobs[maxObsIndex];
extern double interval[maxDim][maxInterval+1];
extern double meaninterval[maxDim][maxInterval];
extern vector meanObsSum;
extern int dimVek[maxDim];
extern double huge *UnGroupedObs;
extern vector UnGroupedObsSum;
extern DefaultsStructure DEF;

double normcon(p)
parameters *p;
{
    double kappa,alfasqr,nom,denom,kapdel;
    dimIndex i,j;
    kappa=0.0;
    for (i=0; i<dim; i++)
        for (j=0; j<dim; j++)
            kappa+=p->beta[i]*p->beta[j]*p->DELTA[i][j];
    alfasqr=p->alfa*p->alfa;
    if (kappa>alfasqr)
    {
        fprintf(stderr,"alfa^2 < beta%%*%%*DELTA%%*%%*beta\n");
        exit(1);
    }
    kappa=sqrt(alfasqr-kappa);
    kapdel=kappa*p->delta;
    if (dim==2)
    {
        nom=kappa*kappa*kappa*exp(kapdel);
        denom=2*pi*p->alfa*(1+kapdel);
        return nom/denom;
    }
    else
    {
        nom=pow(kappa/p->delta,((double)(dim+1))/2);
        denom=pow(2*pi,((double)(dim-1))/2)*2*p->alfa;
        denom*=bessel(dim+1,kapdel);
        return nom/denom;
    }
}

static double lognormcon(p)
parameters *p;
/* returns the logarithm of the norming constant */
{
    double kappa,alfasqr,nom,denom,kapdel;
    dimIndex i,j;
    kappa=0.0;
    for (i=0; i<dim; i++)
        for (j=0; j<dim; j++)
            kappa+=p->beta[i]*p->beta[j]*p->DELTA[i][j];
    alfasqr=p->alfa*p->alfa;
    if (kappa>alfasqr)
    {
        fprintf(stderr,"alfa^2 < beta%%*%%*DELTA%%*%%*beta\n");

```

```

    }
    kappa=sqrt(alfasqr-kappa);
    kapdel=kappa*p->delta;
    if (dim==2)
    {
        nom=3*log(kappa)+kapdel;
        denom=log(2*pi*p->alfa*(1+kapdel));
        return nom-denom;
    }
    else
    {
        nom=log(kappa)*((double)(dim+1))/2;
        denom=log(2*pi)*((double)(dim-1))/2+log(2*p->alfa);
        denom+=p->logdelta*((double)(dim+1))/2;
        denom+=log(bessel(dim+1,kapdel));
        return nom-denom;
    }
}

double exponent(p,x)
parameters *p;
vector x;
{
    dimIndex i,j;
    vector myx;
    double xDx=0.0, bx=0.0;

    for (i=0; i<dim; i++)
        myx[i]=x[i]-p->my[i];

    for (i=0; i<dim; i++)
    {
        bx+=p->beta[i]*myx[i];
        for (j=0; j<dim; j++)
            xDx+=myx[i]*p->INVDELTA[i][j]*myx[j];
    }
    return exp(bx - p->alfa*sqrt(p->delta*p->delta + xDx));
}

double Texponent(p,x)
Dparam *p;
vector x;
/* p->c1 has to have the value
   { delta^2 + (x2-my)%*%(DELTA^-1)%*%(x2-my) }^1/2
*/
{
    dimIndex i,j;
    vector myx;
    double A=0,B=0;

    for (i=0; i<dim; i++)
    {
        myx[i]=x[i]-p->my[i];
        B+=p->beta[i]*(x[i]-p->x2[i]);
    }

    for (i=0; i<dim; i++)
        for (j=0; j<dim; j++)
            A+=myx[i]*myx[j]*p->INVDELTA[i][j];
    A=sqrt(p->deltaSqr+A);

    return exp(B + p->alfa*(p->c1 - A));
}

double loghyp(v)
paramVek v;
{
    parameters p;
    double expsum=0.0;

```

```

bool notDone=true;
vector myx;
double xDx, bxSum, exponent;

vek2param(v,&p);

for (i=0; i<dim; i++)
    o[i]=0;
while ( notDone )
{
    /*sum over the nonlinear part of the exponent*/
    if (obs[obsIndex(o)]!=0)
    {
        for (i=0; i<dim; i++)
            myx[i]=meaninterval[i][o[i]]-p.my[i];
        xDx=0.0;
        for (i=0; i<dim; i++)
        {
            for (j=0; j<dim; j++)
                xDx+=myx[i]*p.INVDELTA[i][j]*myx[j];
        }
        expsum+=obs[obsIndex(o)]
            *p.alfa*sqrt(p.delta*p.delta + xDx);
    }
    o[0]++;
    for (i=0; i<dim; i++)
        if (o[i]==dim Vek[i])
        {
            if (i<dim-1)
                o[i+1]++;
            else
                notDone=false;
            o[i]=0;
        }
        else
            break;
    }
    /*and add the linear part of the exponent*/
    bxSum=0.0;
    for (i=0; i<dim; i++)
        bxSum+=p.beta[i]*(meanObsSum[i] - totalObs*p.my[i]);
    exponent=bxSum-expsum;
    /*now we only need to calculate the norming constant*/
    return -(lognormcon(&p)*totalObs+exponent);
}

double multinomial(v)
param Vek v;
{
    parameters p;
    double intsum=0.0;
    int i,j,o[maxDim],ob;
    bool notDone=true;
    vector xmin,xmax;
    double expint;

    vek2param(v,&p);
    for (i=0; i<dim; i++)
        o[i]=0;
    while ( notDone )
    {
        /*sum over all boxes*/
        if ((ob=obs[obsIndex(o)])!=0)
        {
            for (i=0; i<dim; i++)
            {
                xmin[i]=interval[i][o[i]];
                xmax[i]=interval[i][o[i]+1];
            }

```

```

        expint=adsim(dim,xmin,xmax,exponent,&p,
            DEF.multi_IntegrationPrecision);
        /*and update the log likelihood function*/
        intsum+=(double)(ob)*log(expint);
    }
    o[0]++;
    for (i=0; i<dim; i++)
        if (o[i]==dimVek[i])
        {
            if (i<dim-1)
                o[i+1]++;
            else
                notDone=false;
            o[i]=0;
        }
        else
            break;
    }
    return -(lognormcon(&p)*(double)(totalObs)+intsum);
}

double multinomial2(v)
paramVek v;
{
    Dparam p;
    double intsum=0.0;
    int i,j,o[maxDim],ob;
    double relobso;
    bool notDone=true;
    vector xmin,xmax;
    double expint;
    vector xc; /* centre of integration area */
    vector myxc; /* xc-my */
    double xcDxc; /* (xc-my)%*(DELTA^-1)%*(xc-my) */

    vek2param(v,&p);
    p.deltaSqr=p.delta*p.delta;
    for (i=0; i<dim; i++)
        o[i]=0;
    while ( notDone )
    {
        /*sum over all boxes*/
        if ((ob=obs[obsIndex(o)])!=0)
        {
            relobso=relobs[obsIndex(o)];
            for (i=0; i<dim; i++)
            {
                xmin[i]=interval[i][o[i]];
                xmax[i]=interval[i][o[i]+1];
                p.x2[i]=xc[i]=meaninterval[i][o[i]];
            }
            /* we divide the integral with the value of the
            exponential term evaluated in the centre of the
            integration area so that the integrand takes
            values around 1. When the integration is
            done, the result is multiplied with the same
            number */

            for (i=0; i<dim; i++)
                myxc[i]=xc[i]-p.my[i];

            xcDxc=0;
            for (i=0; i<dim; i++)
                for (j=0; j<dim; j++)
                {
                    xcDxc+=myxc[i]*p.INVDELTA[i][j]*myxc[j];
                }
            p.c1=sqrt(p.deltaSqr + xcDxc);

```

```

        /*calculate integral of exponential term*/
        expint=adsim(dim,xmin,xmax,Texponent,&p,
            DEF.multi_IntegrationPrecision)*exponent(&p,xc);
        /*and update the log likelihood function*/
        intsum+=(double)(ob)*log(expint);
    }
    o[0]++;
    for (i=0; i<dim; i++)
        if (o[i]==dim Vek[i])
        {
            if (i<dim-1)
                o[i+1]++;
            else
                notDone=false;
            o[i]=0;
        }
        else
            break;
    }
    return -(lognormcon(&p)*(double)(totalObs)+intsum);
}

double quotient(v)
paramVek v;
{
    parameters p;
    double intsum=0.0;
    int i,j,o{maxDim};
    double relobso;
    bool notDone=true;
    vector xmin,xmax;
    double expint;

    vek2param(v,&p);
    for (i=0; i<dim; i++)
        o[i]=0;
    while ( notDone )
    {
        /*sum over all boxes*/
        if (obs[obsIndex(o)]!=0)
        {
            relobso=relobs[obsIndex(o)];
            for (i=0; i<dim; i++)
            {
                xmin[i]=interval[i][o[i]];
                xmax[i]=interval[i][o[i]+1];
            }
            /*calculate integral of exponential term*/
            expint=adsim(dim,xmin,xmax,exponent,&p,
                DEF.multi_IntegrationPrecision);
            /*and update the log likelihood function*/
            intsum+=relobso*log(relobso/expint);
        }
        o[0]++;
        for (i=0; i<dim; i++)
            if (o[i]==dim Vek[i])
            {
                if (i<dim-1)
                    o[i+1]++;
                else
                    notDone=false;
                o[i]=0;
            }
            else
                break;
    }
    return intsum - lognormcon(&p);
}

```

```

double quotient2(v)
param Vek v;
{
    Dparam p;
    double intsum=0.0;
    int i,j,o{maxDim};
    double relobso;
    bool notDone=true;
    vector xmin,xmax;
    double expint;
    vector xc; /* centre of integration area */
    vector myxc; /* xc-my */
    double xcDxc; /* (xc-my)%*(DELTA^1)%*(xc-my) */

    vek2param(v,&p);
    p.deltaSqr=p.delta*p.delta;
    for (i=0; i<dim; i++)
        o[i]=0;
    while ( notDone )
    {
        /*sum over all boxes*/
        if (obs[obsIndex(o)]!=0)
        {
            relobso=relobs[obsIndex(o)];
            for (i=0; i<dim; i++)
            {
                xmin[i]=interval[i][o[i]];
                xmax[i]=interval[i][o[i]+1];
                p.x2[i]=xc[i]=meaninterval[i][o[i]];
            }
            /* we divide the integral with the value of the
            exponential term evaluated in the centre of the
            integration area so that the integrant takes
            values around 1. When the integration is
            done, the result is multiplied with the same
            number */

            for (i=0; i<dim; i++)
                myxc[i]=xc[i]-p.my[i];

            xcDxc=0;
            for (i=0; i<dim; i++)
                for (j=0; j<dim; j++)
                {
                    xcDxc+=myxc[i]*p.INVDELTA[i][j]*myxc[j];
                }
            p.c1=sqrt(p.deltaSqr + xcDxc);

            /*calculate integral of exponential term*/
            expint=adsim(dim,xmin,xmax,Texponent,&p,
                DEF.multi_IntegrationPrecision)*exponent(&p,xc);
            /*and update the log likelihood function*/
            intsum+=relobso*log(relobso/expint);
        }
        o[0]++;
        for (i=0; i<dim; i++)
            if (o[i]==dimVek[i])
            {
                if (i<dim-1)
                    o[i+1]++;
                else
                    notDone=false;
                o[i]=0;
            }
            else
                break;
    }
}

```

```

    }

double truncMult(v)
param Vek v;
{
    parameters p;
    double intsum=0.0;
    int i,j,o[maxDim],ob;
    bool notDone=true;
    vector xmin,xmax;
    double expint,allint;

    vek2param(v,&p);
    allint=0.0;
    /* calculate the integral of the whole region as the sum of integrals
       over each box */
    for (i=0; i<dim; i++)
        o[i]=0;
    while ( notDone )
    {
        for (i=0; i<dim; i++)
        {
            xmin[i]=interval[i][o[i]];
            xmax[i]=interval[i][o[i]+1];
        }
        /* calculate integral of exponential term and add it
           to the overall integral */
        allint+=expint=adsim(dim,xmin,xmax,exponent,&p,
            DEF.trunc_IntegrationPrecision);
        /* and update the log likelihood function, if needed */
        if ((ob=obs[obsIndex(o)])!=0)
            intsum+=((double)(ob))*log(expint);

        o[0]++;
        for (i=0; i<dim; i++)
            if (o[i]==dimVek[i])
            {
                if (i<dim-1)
                    o[i+1]++;
                else
                    notDone=false;
                o[i]=0;
            }
            else
                break;
    }
    /* add integral over whole area */
    return intsum + ((double)(totalObs))*log(allint);
}

double truncMult2(v)
param Vek v;
{
    Dparam p;
    double intsum=0.0;
    int i,j,o[maxDim],ob;
    bool notDone=true;
    vector xmin,xmax;
    vector xc; /* centre of integration area */
    vector myxc; /* xc-my */
    double xcDxc; /* (xc-my)%*(DELTA^1)%*(xc-my) */
    double expint,allint;

    vek2param(v,&p);
    p.deltaSqr=p.delta*p.delta;
    allint=0.0;
    /* calculate the integral of the whole region as the sum of integrals
       over each box */
    for (i=0; i<dim; i++)

```

```

{
    my[i]=p.my[i];
    for (j=0; j<dim; j++)
        INVDELTA[i][j]=p.INVDELTA[i][j];
}

for (o=0; o<totalObs; o++)
{
    /*sum the nonlinear part of the exponent*/
    k=o*dim;
    for (i=0; i<dim; i++)
        myx[i]=UnGroupedObs[k+i]-my[i];
    xDx=0.0;
    for (i=0; i<dim; i++)
    {
        for (j=0; j<dim; j++)
            xDx+=myx[i]*INVDELTA[i][j]*myx[j];
    }
    expsum+=alfa*sqrt(deltasqr + xDx);
}
/*calculate the linear part of the exponent*/
bxSum=0.0;
for (i=0; i<dim; i++)
    bxSum+=p.beta[i]*(UnGroupedObsSum[i] - totalObs*p.my[i]);
/*calculate the exponent and the norming constant and put these
together
exponent=bxSum-expsum;
return -(log(normcon(&p))*totalObs+exponent);
*/
}

```

NAME

bessel.c --- module for Bessel functions

SYNOPSIS

```

void recursiveBessel(lambda,x,b0,b1)
int lambda;
double x, *b0, *b1;

double bessel(lambda,x)
int lambda;
double x;

double dLogBessel(lambda,x)
int lambda;
double x;

```

DESCRIPTION

The module calculates the modified Bessel function of third kind and with index $\lambda/2$ --- $K_{\lambda/2}$, where λ is a nonnegative integer.

The return values $b0$ and $b1$ for `recursiveBessel(lambda,x,b0,b1)` are:

$$b0 = K_{\lambda/2}(x)$$

$$b1 = K_{\lambda/2 - 1}(x)$$

The value of `bessel(lambda,x)` is $K_{\lambda/2}(x)$.

The value of `dLogBessel(lambda,x)` is the value of the first derivative of the logarithm of $K_{\lambda/2}(x)$, ie:

$$\frac{d \log(K_{\lambda/2}(x))}{dx}$$

The approximation formulas can be found in Abramowitz and Stegun: Handbook of Mathematical Functions (Dover publications 1972)

```

double k0(x)
double x;
/*Bessel function K_0 */
/* valid for all x > 0 */

{ /* k0 */
    if (x < 2.0)
        return k0low(x);
    else
        return k0high(x);
}

double k1high(x)
double x;
/*Bessel function K_1 */
/*formula 9.8.8 in Abramowitz & Stegun*/
/*valid for x >= 2 */
{
    static double a0=1.25331414,
        a1=.23498619,
        a2=-0.03655620,
        a3=.01504268,
        a4=-0.00780353,
        a5=.00325614,
        a6=-0.00068245;
    double r, t, t2, t3, t4, t5, t6;
        t = 2 / x;
        t2 = t*t;
        t3 = t * t2;
        t4 = t2*t2;
        t5 = t * t4;
        t6 = t3*t3;

    r = a0 + a1*t + a2*t2 + a3*t3 + a4*t4 + a5*t5 + a6*t6;
    return r / (exp(x) * sqrt(x));
}

double I1(x)
double x;
/*Bessel function I_1 */
/*formula 9.8.3 in Abramowitz & Stegun*/
/*valid for -3.75 <= x <= 3.75 */
{
    static double a0=.5,
        a1=.87890594,
        a2=.51498869,
        a3=.15084934,
        a4=.02658733,
        a5=.00301532,
        a6=.00032411;
    double t,r;
        t = x / 3.75;
        t=t*t;
    r = (((((a6*t+a5)*t+a4)*t+a3)*t+a2)*t+a1)*t+a0;
    return r * x;
}

double k1low(x)
double x;
/*Bessel function K_1 */
/*formula 9.8.7 in Abramowitz & Stegun*/
/*valid for 0 <= x < 2 */
{
    static double a0=1.0,
        a1=.15443144,
        a2=-0.67278579,
        a3=-0.18156897,

```

```

        *b1 = k1(x);
        break;
    }
    case 1:
    {
        *b0 = k12(x);
        *b1 = k12(x);
        break;
    }
    case 2:
    {
        *b0 = k1(x);
        *b1 = k0(x);
        break;
    }
    case 3:
    {
        *b0 = k32(x);
        *b1 = k12(x);
        break;
    }
    case 5:
    {
        *b0 = k52(x);
        *b1 = k32(x);
        break;
    }
    case 7:
    {
        *b0 = k72(x);
        *b1 = k52(x);
        break;
    }
    default :
    {
        double b2;
        /* calculate b1=K_(lambda/2 - 1)
           and b2=K_(lambda/2 - 2) */
        recursiveBessel(lambda - 2,x,b1,&b2);
        /*recursion formula cf. formula 2.17 in PB[1990]*/
        *b0 = b2 + ((double)(lambda - 2)) * *b1 / x;
    }
}

double bessel(lambda,x)
int lambda;
double x;
/*returns Bessel function K_(lambda/2) */
{
    switch (lambda)
    {
    case 0:
        return k0(x);
    case 1:
        return k12(x);
    case 2:
        return k1(x);
    case 3:
        return k32(x);
    case 5:
        return k52(x);
    case 7:
        return k72(x);
    default :
    {
        double b0,b1;
        recursiveBessel(lambda,x,&b0,&b1);
        return b0;
    }
    }
}

```

```
    }  
}  
  
double dLogBessel(lambda,x)  
int lambda;  
double x;  
/*returns first derivative of logarithm to Bessel function K_(lambda/2) */  
{  
    double kl,kl1;  
  
    recursiveBessel(lambda+2,x,&kl1,&kl);  
    return (double)(lambda)/(2*x) - kl1/kl;  
}
```

```

// The VG for option pricing formula
#include <stdlib.h>
#include <stdio.h>
#include <math.h>

//Global variables
double T, K, r, q;
double sigma, nu, theta;
double omega, lambdan,
lambdap; double ratioP, ratioN;

// Discretization
long N, M, exercise; double deltax, deltat; double xmin, xmax;
double *x_index; double *spotprice;

//Functions Declaration

// Exponential integral function
float expint(int n, float x);

// Computes final payoff of call

void finalPayoff(double*X);

// MAIN PROGRAM
int main() {
/* local variables */
int k,j;
FILE *fp; /* file pointer */

// initialization
T=1.0;
K=100.0;
r=0.066;
q=0.012;
sigma=0.3;
nu=0.25;
theta=-0.3;
N=250; /* stock */
M=300; /* time */
xmin=log(10.0);
xmax=log(260.0);
/* some resulting parameters */
omega=log(1.0-theta*nu-sigma*sigma*nu/2.0)/nu;
lambdan=sqrt((theta*theta)/(sigma*sigma*sigma*sigma)+2.0/
(sigma*sigma*nu))
+theta/(sigma*sigma);
lambdap=sqrt((theta*theta)/(sigma*sigma*sigma*sigma)+2.0/
(sigma*sigma*nu))
-theta/(sigma*sigma);
deltax= (xmax-xmin)/((double) N);
deltat=((double) T - 0.0)/((double)M);
x_index=(double *)malloc((N+1)*sizeof(double));
for (k=0; k<N+1; k++) {
x_index[k]=xmin + ((double) k)*deltax;
}
ratioP = 1.0/(nu*deltax*lambdap);
ratioN = 1.0/(nu*deltax*lambdan);
/* memory allocation */
X = (double *)malloc((N+1)*sizeof(double));
B = (double *)malloc((N+1)*sizeof(double));
eip = (double *)malloc((N+1)*sizeof(double));
ein = (double *)malloc((N+1)*sizeof(double));
expp = (double *)malloc((N+1)*sizeof(double));
expn = (double *)malloc((N+1)*sizeof(double));
einplusone = (double *)malloc((N+1)*sizeof(double));
eipminusone = (double *)malloc((N+1)*sizeof(double));
spotprice = (double *)malloc((N+1)*sizeof(double));

```

A.3.1 Appendix D

A.4 The Meixner Model

The following Mathematica code computes the price of a European call option under the Meixner model. We recall that the price of a European call option under the Meixner model is given by:

$$\begin{aligned} C(T, K) &= E^Q[(S(T) - K)^+] \\ &= S_0 \int_k^\infty f(x; a(\theta + 1) + b, dT, mT, cT) dx \\ &\quad - e^{-rT} K \int_k^\infty f(x; a\theta + b, dT, mT, cT) dx, \end{aligned} \quad (\text{A.4.1})$$

where $k = \ln(K/S_0)$ and f , the density function of the Meixner distribution is given by:

$$f(x; a, b, c, d, m) = \frac{\cos^{2c}(b/2)}{2a\pi\Gamma(2c)} \exp\left(\frac{b(x-m)}{a}\right) \left| \Gamma\left(c + \frac{i(x-m)}{a}\right) \right|^{2d}, \quad (\text{A.4.2})$$

with $a > 0$, $-\pi < b < \pi$, $c > 0$, $m \in \mathbb{R}$, and $d \geq 1$.

```

for (k=0; k<N+1; k++) {
eip[k] = expint(1, ((double)k + 1.0) * deltax * lambdap);
ein[k] = expint(1, ((double)k + 1.0) * deltax * lambdan);
expp[k] = exp(- ((double)k + 1.0) * deltax * lambdap);
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expn[k] = exp(- ((double)k + 1.0) * deltax * lambdan);
einplusone[k] = expint(1, ((double) k + 1.0) * deltax * (lambdan + 1.0));
eipminusone[k] = expint(1, ((double) k + 1.0) * deltax * (lambdap - 1.0));
spotprice[k] = exp(x_index[k]);
}
/* initialization of X by the terminal payoff. Step M.*/
finalPayoff(X,callput);
if(callput > 0) //call
{fp=fopen("Final_payoff_call.out","w");}
else //put
{fp=fopen("Final_payoff_put.out","w");}
for (k=0; k<N+1; k++) {
fprintf(fp,"%g %g\n",spotprice[k], X[k]);
}
fclose(fp);

/* set the boundary conditions */
if(callput > 0) { /* call */
X[N] = spotprice[N]*exp(-q*(M-j)*deltat) - K*exp(-r*(M-j)*deltat);
X[0] = 0.0;
}
X[0] = K*exp(-r*(M-j)*deltat) - spotprice[0]*exp(-q*(M-j)*deltat);
X[N] = 0.0;
}
}
fclose(fp);
return 0;
}
return X;
}

```

$\alpha = 18.66;$
 $\mu = -0.00163;$
 $\beta = -1.21788;$
 $\delta = 0.030372;$
 $a = 0.117576;$
 $b = -0.146;$
 $d = 0.23;$
 $m = -0.00161;$
 $r = 0.0002;$
 $S = 1000;$
 $K = 1000;$
 $w = 1.72;$
 $\sigma = 0.04;$

$$v = \frac{1}{a} (2Ar \cos[(\cosh[\frac{m-r}{4d}](\sin[\frac{b}{2}] + \sin[\frac{a+b}{2}]) - (\sin[\frac{b}{2}] + \sin[\frac{a+b}{2}])\sinh[\frac{m-r}{4d}])^2 (\cosh[\frac{m-r}{2d}] - \sinh[\frac{m-r}{2d}])) / (\sqrt{2}(-\cos[\frac{a}{2}] + \cosh[\frac{m-r}{2d}]) (\cosh[\frac{m-r}{2d}] - \sinh[\frac{m-r}{2d}]))$$

$bs = a v + b;$
 $t = 1;$

$$FindRoot[\mu + \delta(\sqrt{\alpha^2 - (\beta + w)^2} - \sqrt{\alpha^2 - (\beta + w + 1)^2}) == 0, \{w, \{-(\alpha + \beta), \alpha - (1 + \beta)\}$$

$T = \text{Range}[1, 41, 4]$

For [k=1; S=1000; K = 800; mxprice = { }, k< 21, k++, K = K+20; mxprice =
Append[mxprice ,

$$Oprice = N[e^{-rt} \int_{\log[K/S]}^{\infty} (Se^x - K) \frac{(2\cos[\frac{bs}{2}])^{2dt}}{2a\pi\Gamma[2dt]} x e^{\frac{bs(x-mt)}{s}} x \text{Abs}[\Gamma[dt + \frac{ix(x-mt)}{a}]^2 dx]]]$$

Bibliography

- [1] M. Abramowitz and I. A. Stegun. *Book of mathematical functions*. Dover, New York, 1968.
- [2] Benzoni L. Andersen, T. and J. Lund. *Estimating jump-diffusions for equity returns*. Unpublished Paper (1998), Kellog Graduate School of Management, Northwestern University.
- [3] L. Andersen and R. Brotherton-Ratcliffe. *Exact exotics*. Risk **9** (1996), 85–89.
- [4] L. Bachelier. *Théorie de la spéculation*. P.Cootner, 1964, ed., The Random Character of Stock Market Prices, M.I.T. Press, Cambridge, Massachusetts, 1900.
- [5] ———. *Théorie des probabilités continues*. Journal des Mathématiques Pures et Appliquées (1906).
- [6] ———. *Les probabilités cinématiques et dynamiques*. Annales de l'Ecole Normale Supérieure (1913).
- [7] P. Baldi. *Exact asymptotics for the probability of exit from a domain and application to simulation*. Ann. Probab. **23** (1995), 1644–1670.
- [8] C. A. Ball and N. Torous. *On jumps in common stock prices and their impact on call option pricing*. Journal of Finance **9** (1985), 155–173.

- [9] Barndorff-Nielsen. *Normal Inverse Gaussian distributions and the modelling of stock returns*. Technical Report, Research Report No. 300, Department of Theoretical Statistics, Aarhus University (1995).
- [10] O. E. Barndorff-Nielsen and O Halgreen. *Infinite divisibility of the Hyperbolic and Generalized Gaussian distributions*. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete **38** (1977), 309–319.
- [11] A. Barnea and D.H. Downes. *A reexamination of the empirical distribution of stock price changes*. Journal of The American Statistical Association **68** (1973), 348–350.
- [12] D. S. Bates. *Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options*. Review of Financial Studies **9** (1996), 69–107.
- [13] M Baxter and A Rennie. *Financial calculus: An introduction to derivative pricing*. Cambridge Press, Cambridge, 1996.
- [14] J. Bertoin. *Lévy processes*. Cambridge University Press, Cambridge, 1996.
- [15] J. Bertoin. *Subordinators: Examples and applications*. Ecole d'été de probabilités de Saint-Flour VVVII ed., Lecture Notes in Math., vol. 1727, Springer, 1999.
- [16] F. Black and M. S. Scholes. *The pricing of options and corporate liabilities*. Journal of Political Economy **81** (1973), 637–654.
- [17] P. Blæsild and M. K. Sørensen. *"hyp" - a computer program for analyzing data by means of the hyperbolic distribution*. Research Report 248, University of Aarhus, Department of Theoretical Statistics, Denmark, 1992.
- [18] S. Boyarchenko and S. Levendorskii. *Option pricing for truncated Lévy processes*. International Journal of Theoretical and Applied Finance **3.3** (2000), 549–552.

- [19] Bakshi G. Cao, C. Q. and Z. Chen. *Empirical performance of alternative option pricing models*. Journal of Finance **52** (1997), 2003–2049.
- [20] P. Clark. *A subordinated stochastic model with finite variance for speculative prices*. Econometrica **41** (1973), 135–156.
- [21] J. C. Cox and S. A. Ross. *The pricing of options for jump processes*. Working Paper 2-75, Rodney L. White Center for Financial Research, University of Pennsylvania, April 1975.
- [22] Ross S. A. Cox, J. C. and M. Rubinstein. *Option pricing: A simplified approach*. Journal of Financial Economics **7** (1979), 229–263.
- [23] F. Delbaen and J. Haezendonck. *A martingale approach to premium calculations principles in an arbitrage-free market*. Insurance: Mathematics and Economics **8** (1989), 269–77.
- [24] D. A. Dickey. *Estimation and hypothesis testing for nonstationary time series*. Ph-d dissertation, Iowa State University, Ames, IW, 1976.
- [25] Hasza D. P. Dickey, D. A. and W. A. Fuller. *Testing for unit roots in seasonal time series*. Journal of American Statistical Association **79** (1984), 355–367.
- [26] E. Eberlein and U. Keller. *Hyperbolic distributions in finance*. Bernoulli **1** (1995), 281–299.
- [27] Keller U. Eberlein, E. and K. Prause. *New insights into smile, mispricing, and value at risk: The hyperbolic model*. Journal of Business (1998), 371–405.
- [28] R. F. Engle and J. R. Russel. *Econometric analysis of discrete-valued, irregularly-spaced financial transactions data using a new autoregressive conditional multinomial model*. Unpublished Paper: Graduate School of Business, University of Chicago (1998).

- [29] W. Feller. *An introduction to probability theory and its applications*. First ed., vol. 2, Wiley & Sons, New-York, 1966.
- [30] T. S. Ferguson and M. S. Klass. *A representation of independent increment processes without Gaussian components*. *Annals of Mathematical Statistics* **43** (1972), 1634–1643.
- [31] H. U. Gerber and E. S. W. Shui. *Martingale approach to pricing perpetual American options*. In *Proceedings of the 4th AFIR International Colloquium April 20-22* (1994), 659–89.
- [32] ———. *Option pricing by Esscher transforms; with discussion*. *Transactions of The Society of Actuaries* **46** (1994), 99–191.
- [33] B. Grigelionis. *Processes of Meixner type*. *Lithuanian Math. Journal* **39(1)** (1999).
- [34] R. Hagerman. *More evidence on the distribution of security returns*. *Journal of Finance* **33** (1978), 1213–1221.
- [35] J. Hull and A. White. *The pricing of options on assets with stochastic volatilities*. *Journal of Finance* **42** (1987), 281–300.
- [36] J. C. Hull. *Options, futures, and other derivatives*. 4th edition ed., Prentice-Hall, 2000.
- [37] R. A. Jarrow and D. B. Madan. *Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns*. *Mathematical Finance* (1995), 311–336.
- [38] R. A. Jarrow and E. Rosenfeld. *Jump risks and the intertemporal capital*. *Journal of Business*. **57** (1984), 337–351.

- [39] J. Jiang, George. *Stochastic volatility and jump diffusion, implication on option pricing*. International Journal of Theoretical and Applied Finance **2** (1999), no. 4, 409–440.
- [40] J. Jones. *The yield curve: Cash and futures markets*. The Handbook of Financial Futures, New-York, U. A. (1984), 373–421.
- [41] P Jorrión. *On jump processes in the foreign exchange and in the stock markets*. The Review of Financial Studies (1988), 427–445.
- [42] O. Kallenberg. *Series of random processes without discontinuities of the second kind*. Annals of Probability **2** (1974), 729–737.
- [43] Oh Y. H. Kim, M. J. and R. Brooks. *Are jumps in stock returns diversifiable? Evidence and implication for option pricing*. Journal of Financial and Quantitative Analysis (1994), 609–632.
- [44] M. Woodroffe, J. Zine, and R. LePage. *Convergence to a stable distribution via order statistics*, Annals of Probability **9** (1981), 624–632.
- [45] R. LePage. *Multidimensional infinitely divisible random variables and processes, II*, Probability in Banach Spaces III, pp. 279–284, Springer, New-York, 1980.
- [46] Carr P. Madan, D. B. and E. C. Chang. *The variance gamma process and option pricing*. European Financial Review (1998).
- [47] D. B. Madan and E. Seneta. *The variance gamma model for share market returns*. Journal of Business (1990), no. 63, 511–524.
- [48] B. B. Mandelbrot. *The variation of certain speculative prices*. Journal of Business **36** (1963), 394–419.
- [49] M. B. Marcus. *ξ -Radial processes and random fourier series*. Memoirs of The American Mathematical Society, vol. 368, AMS, 1987.

- [50] M. B. Marcus and G. Pisier. *Characterizations of almost surely continuous p -stable random fourier series and strongly stationary processes*. Acta Mathematica **152**.
- [51] J. Meixner. *Orthogonale polynomsysteme mit einer besonderen gestalt der erzeugende funktion*. J. London Math. Soc. **9** (1934), 6–13.
- [52] R. C. Merton. *Option pricing when underlying stock returns are discontinuous*. Journal of Financial Economics **3** (1976).
- [53] V. Naik. *Option valuation and hedging strategies with jumps in the volatility of asset returns*. The Journal of Finance **5** (1993), 1969–1984.
- [54] Laurent Nguyen-Ngoc. *Exotic options in general Lévy models*. Prepublication 850, Univ. Paris 6, Laboratoire de Probabilites et Modeles Aleatoires. (2003).
- [55] M. F. M. Osbourne. *Brownian motion in the stock market*. Operation Research **7** (1959), 145–173.
- [56] R. L. Prentice. *Discrimination among some parametric models*. Biometrika **62**(3) (1975), 607–614.
- [57] S. J. Press. *A compound event model for security prices*. Journal of Business **40** (1967), 317–335.
- [58] P. Protter. *Stochastic integration and differential equations*. Springer-Verlag, New-York, 1992.
- [59] P. Protter and D. Tallay. *Convergence rate of the Euler scheme for stochastic differential equations driven by Lévy processes*. Annals of Probability **25** (1997), no. 1, 393–423.

- [60] S. I. Resnick. *An extremal decomposition of a process with stationary, independent increments*. Technical Report 79, Department of Statistics, Stanford University, 1976.
- [61] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales.. Itô Calculus*, vol. Volume 2, Wiley, Chichester, 1996.
- [62] J. Rosinski. *On series representations of infinitely divisible random vectors*. The Annals of Probability **18** (1990), no. 1, 405–430.
- [63] M. Rubinstein. *Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active cboe option classes from august 23, 1976 through august 31, 1978*. Journal of Finance **40** (1985), 455–480.
- [64] T. H. Rydberg. *Generalized Hyperbolic diffusions with applications toward finance*. Mathematical Finance **99** (0000), 183–201.
- [65] P. Samuelson. *Rational theory of warrant pricing*. Industrial Management Review **6** (1965), 13–32.
- [66] K. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, Cambridge, 1999.
- [67] W. Schoutens. *The Meixner process: Theory and applications in finance*. EURANDOM Report (2000).
- [68] ———. *The Meixner process in finance*. EURANDOM Report (2001-2002).
- [69] G. W. Schwert. *Stock volatility and the crash of '87*. Review of Financial Studies (1990), no. a, 77–102.
- [70] H. Shirakawa. *Interest rate option pricing with Poisson-Gaussian forward rate curve processes*. Mathematical Finance **1** (1991), 77–94.

- [71] M. A. Stephens. *Asymptotic results for goodness-of-fit statistics with unknown parameters*. Annals of Statistics **4** (1976), 357–369.
- [72] ———. *Goodness-of-fit for the extreme value distribution*. Biometrika **64** (1977), 583–588.
- [73] ———. *Goodness-of-fit with special reference to tests for exponentiality*. Technical Report No. 262, Department of Statistics, Stanford University, Stanford, CA (1977).
- [74] S. J. Taylor. *Modelling stochastic volatility*. Mathematical Finance **4** (1994), 183–204.
- [75] J. B. Wiggins. *Option values under stochastic volatility: Theory and empirical estimates*. Journal of Financial Economics **19** (1987), 351–377.
- [76] R. L. Wolpert and K. Ickstadt. *Simulation of Lévy random fields*. Müller, Dey, & Sinha, (eds.) ed., Practical Nonparametric and Semiparametric Bayesian Statistics, pp. 227–242, Springer-Verlag, 1998.