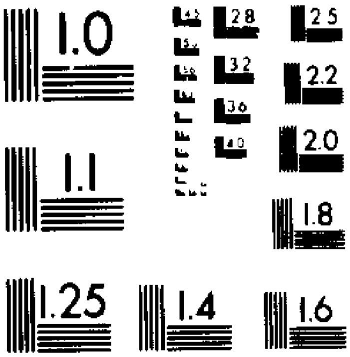

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STOCHASTIC INVENTORY CONTROL AND THE ASSUMPTION OF NON-INTERCHANGEABILITY

by

FARROKH NASRI

A dissertation submitted to the Graduate Faculty in Business in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University Of New York.

1986

This manuscript has been read and accepted by the Graduate Faculty in Business in satisfaction for the dissertation requirement for the degree of Doctor of Philosophy.

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ABSTRACT

Stochastic Inventory Control And The Assumption Of Non-Interchangeability

by

Farrokh Nasri

Advisor: Professor Georghios P. Sphicas

In less than half a century, literally thousands of articles have been published in the area of inventory management, each with its own set of objectives and assumptions. This research is another attempt to expand the body of knowledge in this field. The significance of this research is two-fold: (a) its application to some specific production problems and (b) its motivation of further studies investigating the significance of differences between inventory models that allow "cross over" and those that simply avoid it.

This dissertation deals with a (D,R) inventory model with deterministic demand and finite stochastic leadtimes. In contrast to the mainstream of inventory literature, we allow for crossing of orders and make a more restrictive assumption -- i.e., non-interchangeability (non-substitutability) of orders.

It is shown that, in general, the possibility of cross over arises if and only if the range of lead-time is larger than the optimal cycle time.

A stochastic generalization is given for basic EOQ's total optimal cost function when backorders are allowed. Some additional extensions are obtained for the case of uniformly distributed lead-times. After extensive algebraic manipulation, simplified cost expressions are obtained for other ranges of the model.

Lower and upper bounds of optimal costs are obtained for each model. Also, the shape and properties of the optimal cost as a function of the range of lead-time is given.

Finally, we obtain the probability of cross over of two consequent orders. We also extend the results to the case of uniform lead-time distribution.

It should be noted that this dissertation was initially simulation based. A large computer program with over five hundred lines of code was first developed to generate the solution to the problem through a search procedure and, then, to compare it with a simulated inventory model that relaxed the assumption of non-interchangeability. The numerical results obtained from printouts gave us insight to many of the problems addressed in this dissertation and eventually led to the analytical work that is described here.

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LIST OF SYMBOLS

K = Ordering cost per order

v = Cost per unit

h = Carrying cost per unit per unit of time

Ω = h/p

Ω_m = Maximum of h/p and p/h

$p = \pi$ = Backordering cost per unit per unit of time

R = Reorder level

Q = Quantity ordered per order (lot size)

q = Number of units of demand satisfied by each order

t = Time differential between placing an order and the start of q time units that will be satisfied by a given order

$K+vq$ = Ordering cost (including purchasing cost) for $q > 0$, 0 otherwise

D = Constant rate of demand per unit of time

$\tilde{D}$ = Expected demand per time period

$\bar{b}(r)$ = Expected number of shortages when an order arrives

r = Lead-time in units of time (stochastic)

$g(r)$ = Lead-time probability density function

$G(.)$ = Distribution function of lead-time

x = Demand during lead-time

$f(x)$ = Probability distribution of demand during lead-time

EAC = Expected annual cost per unit of time

ETC = Expected total cost per unit of time

μ = Mean of lead-time distribution

σ^2 = Variance of lead-time distribution

a = Lower bound of lead-time distribution

b = Upper bound of lead-time distribution

k = $2K/(h+p)D$

δ = $[2(b-a)/(1+\Omega_m)]^{1/2}$

$A(x)$ = A function of x (see figures 3.1.2 and 3.1.3)

$B(x)$ = A function of x (see figures 3.1.2 and 3.1.3)

α = The point in which both functions $A(x)$ and $B(x)$ reach a minimum

θ = A unique point to the right of α such that $A(\theta) = A(a)$

τ = A unique point to the left of α such that $A(\tau) = A(b)$

β = The point in which $A(x)$ crosses the horizontal axis

$*$ = Starred symbols represent optimal values

CHAPTER I

INTRODUCTION

1.1 OVERVIEW OF INVENTORY MANAGEMENT

Simply stated, inventory is some idle resource kept for future use. It is among the largest investments made by a firm and therefore it deserves considerable attention. In the United States alone, at any particular point in time hundreds of billions of dollars are invested in inventories at different types of organizations. Inventories are kept in companies that are small or large, manufacturing or service, profit or non-profit oriented, etc. Since inventory is a necessity to almost all organizations, and because an extremely large amount of money is invested in inventories, even a minor improvement in controlling inventories can bring about large savings. This research is an attempt to contribute to the state of art in the area of inventory control.

There are literally thousands of inventory models developed, each with different objectives and assumptions. This indicates the amount of attention being devoted by theoreticians to this area of management science. Yet, its importance is not always appreciated by top management. This gap between theory and practice may be due to a failure to recognize the impact that inventories have on costs and profits and/or the fact that inventory models with realistic assumptions are so abstract and hard for managers to digest.

As we will see in the following chapters, computing the values of decision variables, even for the simplest of stochastic inventory models, appears to be an extremely difficult problem. Due to this fact, some credit should be given to practitioners for not implementing inventory models in the real world. We believe there is hope. Computers and simulation soon will play the role of the most important gap-closer.

1.2 OBJECTIVES OF INVENTORY CONTROL

Inadequate control of inventories can result in both understocking and overstocking of items. Understocking results in lost sales, dissatisfied customers and production delays, while overstocking unnecessarily ties up funds that might be used more productively elsewhere.

There are two main objectives in inventory control. One is to maximize the level of customer services (i.e., have the right goods, in sufficient quantities, in the right place and at the right time). The other is to minimize the cost of providing the desired level of customer service.

These two objectives are generally in opposition: high levels of customer service lead to high costs, and low costs usually are accompanied by low levels of customer service. Consequently, inventory decisions (just as other decisions made by a decision-maker) are trade-offs involving a compromise between cost and customer service level.

Therefore, the decision-maker's problem is to achieve a balance with stocking decisions, avoiding both overstocking as well as understocking. The two fundamental decisions that must be made are:

(1) Timing (when to order or produce); and

(2) Size (how much to order or produce).

1.3 COSTS AND INVENTORY OPTIMIZATION

The aim in building an inventory model is to determine values for decision variables (such as timing and size) that are optimal (i.e., just yield the lowest cost). Therefore, an objective function must be formulated that relates the measure of performance of the inventory control policy to the decision variables. This measure of performance is the total cost per unit of time. Finding the optimal values of decision variables then requires minimizing the total cost per unit of time. The total cost per unit of time is composed of the following costs:

- Procurement cost (purchasing cost);
- Ordering cost (setup cost);
- Carrying cost (holding cost);
- Shortage cost (stockout cost).

The procurement cost (or purchasing cost) is the cost of obtaining (or acquiring) the materials. This cost, in most inventory models, is assumed to be constant.

Ordering costs are the costs associated with ordering and receiving inventory. These costs include determining how much is needed, preparing invoices, inspecting goods upon arrival for quality and quantity and moving the goods to temporary storage. Ordering costs are generally expressed as a fixed-dollar amount per order, regardless of order size. In the case in which a firm produces its own inventory instead of ordering it from a supplier, the costs of machine setup (*i.e.*, preparing equipment for the job at hand, adjusting the machine, wasted materials, and installing new fixtures) are analogous to ordering costs (i.e., they are expressed as a fixed charge per run, regardless of the size of the run).

Carrying or holding costs relate to physically holding items in storage. They include interest (or opportunity cost), insurance, taxes, depreciation, obsolescence, deterioration, spoilage, pilferage, breakage and warehousing costs (e.g., heat, electricity, rent and security). Frequently the most important cost is not a direct out-of-pocket cost but rather an opportunity cost. This is the cost incurred by having capital tied up in inventory rather than having it invested else where.

Shortage or stockout cost occur when demand exceeds the supply of inventory on-hand. In this situation one of the following may happen.

First, the unsatisfied demand (sale) may be lost entirely. The main cost here is "goodwill loss", which may include lost profits on sales of this and other items in the future due to the fact that the customer temporarily or permanently takes his business elsewhere or because he discourages other potential customers by telling them that he received unsatisfactory service. The cost of lost sales also includes the costs of any special procedures used to inform the customer that his demand can not be satisfied and the profit lost in not making the sale.

Second, at the other extreme the unsatisfied demand may be back-ordered. Backorder costs are extremely difficult to measure since they may include such factors as loss of customers' goodwill. Also, if the demand is for a spare part or raw material for an in-house production, the cost of the interruption or shutdown due to the lack of availability seems to be high and hard to measure. Other parts of the backorder cost are less serious and easier to detect. Such costs include the cost of notifying a customer that an item is not in stock and will be filled later, the cost of attempting to find out when the customer's order can be filled and giving him this information and, perhaps, the cost of buying from another supplier.

1.4 CONTINUOUS REVIEW (Q,R) MODELS

An ordering policy for a multi-period inventory model can be classfied as periodic or continuous. The real distinction is how frequently the inventory level must be observed (reviewed) in order to implement the policy.

In periodic review inventory models, the state of the inventory system is examined only at discrete, usually equally spaced, points in time. Decisions concerning the operation of the system, such as whether or not to place an order, are made only at these review times. In fact, the decision maker may know nothing about the state of the system at times other than the review times. This ordering policy consists of three parameters, R, r, and T, which are target inventory, reorder point, and review period length, respectively. Simply stated, we place an order of size R-I (difference between target inventory and observed inventory level at the review period) if observed inventory is less than or equal to the reorder point. Otherwise, no order is placed. In the literature, this is referred to as a periodic review R-r policy or (R,r,T) policy.

The second type of inventory policy, i.e., continuous review (R,r) policy (or transaction reporting), is the limiting form of (R,r,T) policy as T → 0. Continuous ordering policies assume that the state of the system is known at any particular point in time. An order is placed when the inventory level falls to a given level. The quantity to be ordered when an order is placed can either be fixed or variable.

A policy involving variable-order quantities usually specifies that the inventory level (on-hand plus on-order) be brought up to a level S whenever an order is placed. This policy is called (s,S) policy.

A policy involving fixed-order quantities usually specifies that an order of the same size, Q, be placed every time the inventory level drops to or below the reorder point. This policy is known in the literature as continuous-review (Q,R) policy, which is also called "fixed order quantity policy".

By the inventory level in the above ordering policies, we mean inventory position, _i.e._, net inventory plus the already on order quantity. This lets us prevent the difficulties when the lead-time is longer than the review period and when the order size is not sufficient to bring the on-hand inventory above the reorder point when the order is received.

Like all mathematical models, these inventory models are only approximations to reality. In an actual situation, then, the choice between a periodic and a continuous-review model depends on which is the better representation of reality.

The current research concentrates on continuous-review inventory models with fixed order size, _i.e._, continuous-review (Q,R) models.

1.5 VARIABILITY IN DEMAND AND LEAD-TIME

In the analysis of inventory models, one may face many sources of uncertainty. Perhaps the two most common are:

1. Demand timing; and
2. Procurement lead-time.

Usually in the real world, we are not sure when a demand for a certain commodity will occur; if it does occur, the number of units demanded remains uncertain. Thus, the total number of units demanded over a particular period of time will be a random variable, having a probability distribution that depends on the length of the period. When this probability distribution remains the same in every period of length t, we say that we have a stationary demand process. Otherwise, we have a dynamic demand process. Our analysis in this research is restricted to static demand processes.

In spite of great improvements in the means of transportation, communication and production, most procurement systems require some lead-time in supply which is often uncertain. This means that the lead-time is a random variable. These uncertainties make the solution to the inventory problem much more complex and also increase the cost of inventory for the same level of performance. Studies by Gross and Soriano [3] clearly demonstrate that lead-time variation has a major impact on inventory costs. However, before incorporating variability in lead-time into our model, we should make all reasonable efforts to eliminate such variability. The simplest of all is to negotiate with suppliers.

When lead-times are constant, all orders will arrive in the same sequence in which they were placed. If lead-times are independent random variables, there is a possibility that orders will cross over in time; that is, an order placed at time 2 might arrive before an earlier order placed at time 1.

However, orders do not usually cross in real-world situations, and it can easily be argued that in many cases lead-times are not strictly independent of themselves or the process generating demands. This dilemma has evoked various responses from inventory theorists, which will be discussed below.

1.6 LEAD-TIME DEMAND

The distribution of demand during lead-time is dependent on the distribution of lead-times and the distribution of demand during a fixed interval, assuming all distributions are stationary and independent.

When the lead-time variation is significant, it can be dangerous to use a model that assumes a constant lead-time. If one uses the mean of lead-time in the model, then this can lead to seriously underestimating the average period of time during which the system is out of stock. Consequently, the safety stock determined from the model may be too low. Therefore, adequate protection is not given against sales occurring before replenishments arrive. This means that the distribution of demand during lead-time must be found. The next section explains how the moments of this distribution can be found.

1.6.1 Moments Of The Lead-Time Demand Distribution

When the mean and variance of demand and lead-time are calculated, one can express the mean and variance of lead-time demand which is a random variable as:

$$E(L) = E(r).E(D)$$

$$Var(L) = E(r).Var(D)+[E(D)]^2.Var(r)$$

Where L is the lead-time demand, E(r) and E(D) and E(L) are the means of lead-time, demand, and lead-time demand, and Var(r), Var(D), Var(L) are the variances of lead-time, demand, and lead-time demand respectively.

These relationships are standard and the expressions for E(r), E(D), Var(r) can be calculated in most situations, so E(L) and Var(L) can be easily computed. But, of course, the moments alone are not enough; a distribution must be fitted to the moments and its success depends on how well the distribution can be supported in practice.

1.6.2 Distribution Of Demand During Lead-Time

In the analyses of inventory models under stochastic lead-times, the distribution of demand during lead-time is of more interest than the demand and the lead-time distributions, individually.

In Chapter II of this research, general and specific distributions of lead-time demand under probabilistic and deterministic demand will be discussed.

1.7 ANALYSIS OF THE ASSUMPTIONS AND SIGNIFICANCE OF THIS RESEARCH

Thousands of articles have been published in the area of inventory control in less than half a century. This in itself is evidence of the importance of this subject. Many of these articles are intended for practical applications, while others may have no practical application in the near future. The main reason for the lack of immediate applicability might be due to the nature of the restrictive assumptions that are made.

The significance of this research can be summarized in two major catagories:

(a) its application to some specific production control problems; and

(b) its motivation for further studies investigating the differences between models that allow orders to cross and those that just avoid cross over.

The inventory policy considered here is the well-known continuous review (Q,R) policy already discussed in Section 1.4 of this dissertation. In the era of computers, this modeling approach is quite applicable to real world situations.

Researchers and practitioners in the inventory theory have tended to emphasize only the variability of demand, usually neglecting that of lead-time. Recently, however, there has been an awakening of interest in the role of lead-time variability. This realistic assumption is made in this work. In reviewing the status of inventory management in 1980, for example, Harvey M. Wagner advocates further research into the influence of lead-time variability. He says: " Replenishment lead-times usually vary with uncertainty (to illustrate, for U.S. rail transporation, delivery times may be between 4 and 10 weeks, depending on the availability of the item from the vendor and the vagaries of the transportation network). Although, it is possible to adapt many replenishment formulas to handle the impact of lead-time variability (usually the approach is to inflate the value of variance of demand during lead-time), how to provide a numerical estimate of lead-time variability is still an art. It may be that this estimate always will involve more experienced judgment than scientific input, but a systematic exploration of what is at issue and what are the alternative approaches would be welcome." Gross and Soriano [3] note that an inventory system is more sensitive to lead-time variation than to demand variation. Vinson also finds lead-time variability to be more important in influencing inventory costs than mean lead-time or variability of period demand.

There are two restrictive assumptions made here that are different from the assumptions made by the mainstream of inventory literature:

(a) deterministic demand and finite stochastic lead-time; and

(b) non-interchangeability (or non-substitutability) of items.

One may reasonably argue that the probabilistic nature of the lead-time is well taken, but deterministic behavior of demand may not seem to be a realistic one. It is true that in a non-manufacturing context, when inventory is maintained for sales to arriving customers, demand is rarely constant. However, in the case of a manufacturer who stocks raw materials, parts or supplies for internal use (such as inputs to a production process), demand for inputs are rather constant for a given production level.

As far as theoretical interest in this assumption is concerned, by assuming deterministic demand, we can achieve two benefits:

(a) since the only source of variation is lead-time, its sensitivity can easily be analyzed and measured. This is in contrast to other stochastic inventory models, in which results are based on the compounding effect of both demand and lead-time variability and are mostly approximations;

(b) by excluding demand variation and making the assumption of non-interchangeability, we are able to more closely investigate the possibility of cross over of orders, cost differences, etc.

In order to make stochastic inventory analysis tractable, some authors assume that lead-times are exponentially distributed, making a Markovian analysis possible. Others assume that the possibility of cross-over is so small that they exclude it right from the start. For a more detailed analysis one can refer to [11]. The approach in this paper is the noble and unconventional one that was first presented by Washburn [18]. Here, we make the assumption of non-interchangeability, *i.e.*, each order is a special order and can only satisfy a particular unit of demand. This assumption makes the analysis of cross over, which is the main thrust of this research, possible. In this work, we show that the assumption of non-interchangeability is immaterial as long as a certain condition, which is based on parameters of the problem, is met. In such case, some explicit analytical results are obtained.

Washburn [18] states that the implication of the assumption of non-interchangeability in a manufacturing context is like assuming that parts are non-interchangeable. In a sales context, it is like assuming that each order is being "colored" to satisfy a particular customer's demand.

Here we show that the restrictive assumption of non-interchangeability becomes immaterial when orders do not cross or when a certain condition that is based on parameters of the problem is met.

In terms of cost measurement, an optimal cost function that is composed of two parts, is obtained -- the deterministic part, which is simply the same as basic EOQ's optimal cost function when backorders are allowed, and the stochastic part, which depends directly on the variance of the lead-time. As expected, the stochastic part drops out as lead-time variability approaches zero. Additional optimal cost expressions are also obtained for the cases in which the stochastic generalization of EOQ's optimal cost expression does not apply. Calculation of the above cost expressions requires extensive algebraic manipulations and simplifications. Therefore, as long as orders do not cross, our results apply to conventional inventory systems with interchangeable items. However, when cross over of orders is possible, our model provides higher costs to the extent that orders are likely to cross. Therefore, in general, our optimal total inventory cost is the upper-bound of conventional inventory systems, which is the result of cross over.

Optimal costs are obtained from different cost expressions, depending on the range of lead-time distribution. Lower and upper bounds of optimal cost for each range are obtained. Also, the shape and properties of the optimal cost as a function of the range of the lead-time is given.

The sensitivity of decision variables to the change in the lower bound of the lead-time, while its range remains constant, indicates that the optimal cost and the optimal order quantity are not sensitive to the change and therefore stay the same. The only change is in the time in which the order should be initiated.

This and many other results were observed by the author during his analysis of the results of a simulation study on the subject.

A general expression is obtained that allows us to calculate the probability of cross over of two subsequent orders. Also, explicit results are obtained for the case of uniform distribution of lead-time.

1.8 ORGANIZATION OF THE DISSERTATION

In Chapter One, we discussed the objectives and provide an overview of inventory control. We also discuss the type of assumptions that we are making, and the significance of this dissertation.

Chapter two attempts to review and integrate the existing relevant studies to this research, under two major catagories of "independent lead-time" and "dependent lead-time".

This dissertation is a continution of the study done by Sphicas and Nasri [13]. In Chapter three, we review this work in further detail, and provide additional results and proofs, and also fill some of the existing gaps.

Chapter four is basically on extension of Chapter three. It provides new and complementary analytical results mainly for the case of uniform distribution of lead-time. When possible, the results are generalized for the general lead-time distribution.

Chapter five is the conclusion of this dissertation. It includes :

a) a brief analysis of the assumptions made,

b) a discussion of preliminary empirical work, _i.e._, simulation, by the author prior to this analytical work,

c) a summary of the main results obtained, and

d) an insight for related future research.

CHAPTER II

ANALYSIS AND REVIEW OF LITERATURE

2.1 ANALYSIS

As we mentioned in the previous chapter, when lead-times are constant all orders will arrive in the same sequence in which they are placed. But, in most real world situations lead-times are stochastic and independent. In the case in which lead-times are independent random variables, there is a possibility that orders will cross each other in time.

The first part of section 2.2 reviews the main stream of inventory literature that assume lead-times are stochastic and independent random variables. These studies just igonre the possibility of cross over for the purpose of mathematical tractability.

In the second part of Section 2.2, we review the unorthodox inventory models that allow orders to cross each other, i.e., an order placed at time 1 arrives after another order placed at time 2. To make the analysis of these inventory models tractable, the assumption of non-interchangeability of items is imposed.

The last section of this chapter, briefly discusses the models with state-dependent lead-times.

2.2 INDEPENDENT LEAD-TIMES

Among the inventory models with stochastic independent lead-times, two different groups of models exist in the literature. The first group consists of those models that argue that orders do not usually cross in real-world situations. Of course, this seems to be conflicting. How could we assume independence of lead-times and also argue that orders do not cross? This dilemma has resulted in various responses from inventory theorists, which will be discussed in the next section. The second group consists of those recent models that allow the orders to cross and make the assumption of non-interchangeability of demand.

This work falls under the second group of models. We also assume that lead-times are independent random variables. When lead-times are independent random variables, we should allow for the possibility of cross over. More detailed analysis is given in section 2.2.2.

2.2.1 Review Of Models With The Assumption that "Orders Do Not Cross"

When lead-times are independent random variables, at any point in time there can be more than one order outstanding. There are two problems associated with this:

(1) orders may cross each other -- *i.e.*, an order placed at time 2 is being received before an order placed at time 1 -- where, in practice, orders are often received in the same sequence in which they are placed;

(2) this order crossing leads to errors in calculating costs.

To avoid these difficulties, the following assumptions, either explicitly or implicitly, are being made:

(1) the interval between successive orders is large, and therefore the probability of cross over is negligible and can be omitted [4];

(2) Wagner [17] assumes that the actual demand during lead-time does not exceed the order quantity.

(3) Sphicas [15], in his recent work, formally assumes a finite range for the lead-time demand distribution. Therefore, if a finite upper bound is placed on the lead-time demand distribution, one can assume that no orders

will cross (compare this with the previous assumption).

(Q,r) Heuristic Approximate Treatment Of Backorder Case

Even though in this formulation Hadley & Whitin [4] make a number of assumptions and approximations, the resulting models are especially useful for practical applications because of their relative simplicity. In this continuous review model, we order a lot of size Q when the inventory level drops to a reorder point of r.

Before starting the formal model, let us give a list of the notations used:

C = Cost per unit

K = Ordering cost/order

h = Carrying cost/unit/time period

π = Backordering cost/unit (independent of time of backorder)

Q = Quantity ordered (order size) per order

D = Units demanded/time period

$\bar{D}$ = Expected demand/time period

$\bar{b}(r)$ = Expected number of shortages when an order arrives

r = Reorder point

x = Demand during lead-time

$f(x)$ = Probability distribution of demand during lead-time

EAC = Expected annual cost

μ = Expected demand during lead-time

σ^2 = Variance of demand during lead-time

They [4] make the following assumptions:

- Demand is probabilistic.
- Lead-time is constant and also known with certainty.
- The unit cost C of the item is a constant and independent of Q.
- The backorder cost is π per unit backordered.
- There is never more than a single order outstanding.
- The cost of operating the information processing system is independent of Q and r.
- The reorder point r (based on the inventory position or net inventory) is positive.

The fifth assumption, as stated before, implies that at the time the reorder point is reached there are no orders outstanding, so that the inventory position (the amount on-hand plus on-order minus backorders) is equal to the net inventory (on-hand minus backorders).

Hadley-Whitin [4] argue that because of the seventh assumption there will be no backorders outstanding at the reorder point. To examine this model, any one of the inventory levels -- on-hand, net, or inventory position -- can be used to define the reorder point and the reorder point will have the same value for any one of them. Note that in order to use the on-hand level, we must assume that after an order arrives, it is sufficient to meet backorders and raise the on-hand inventory level above the reorder point. If this ever failed to happen, the reorder point would never be reached again and the system would proceed to accumulate backorders. When the reorder point is thought of in terms of the inventory position of the system, then the assumption guarantees the on-hand inventory will always be raised above the reorder point when an order arrives. Otherwise, it would not be possible to have only a single order outstanding.

Their approach finds the total cost per cycle and multiplies that by the number of cycles per unit of time. The only part of the cost function that deserves some discussion is the calculation of average on-hand inventory during the cycle. Obtaining an exact expression for average on-hand inventory during the cycle is difficult and therefore they use an approximation, which is good if the time the system is in a backorder condition during a cycle is small compared to the cycle length. The net inventory is at its minimum immediately before receipt of an order and at its maximum immediately after receipt of the order. The expected net inventory, therefore, is the average between $r-\mu$ and

$Q+r-\mu$ which is $(Q/2)+r-\mu$. The quantity $Q/2$ often is called the cycle stock and $r-\mu$ is referred to as the safety stock. Now we are in a position to express the total cost function per cycle:

$$K + CQ + (hQ/\bar{D})[Q/2 + r - \mu] + \pi\bar{b}(r) \tag{2.2.1.1}$$

Multiplying the total cost per cycle by the average number of cycles would provide us with the total annual cost per unit of time.

$$EAC = \bar{D}K/Q + C\bar{D} + h[Q/2 + r - \mu] + \pi\bar{D}\bar{b}(r)/Q \tag{2.2.1.2}$$

$$\text{where } \bar{b}(r) = \int_r^{\infty} (x-r)f(x)\,dx \tag{2.2.1.3}$$

To minimize EAC, we have to differentiate it with respect to Q and r.

$$\frac{\partial EAC}{\partial Q} = \frac{-K\bar{D}}{Q^2} + \frac{h}{2} - \frac{\pi\bar{D}\bar{b}(r)}{Q^2} = 0 \tag{2.2.1.4}$$

$$\frac{\partial EAC}{\partial r} = h + \frac{\pi\bar{D}}{Q} \cdot \frac{\partial\bar{b}(r)}{\partial r} = 0 \tag{2.2.1.5}$$

From the first equation we can find an expression for Q given reorder point r.

$$Q^* = \sqrt{\frac{2\bar{D}[K + \pi\bar{b}(r)]}{h}} \tag{2.2.1.6}$$

In the second equation,

$$\frac{\partial \bar{b}(r)}{\partial r} = \frac{\partial}{\partial r} \int_{r}^{\infty} (x - r) f(x) d(x)$$

$$= - \int_{r}^{\infty} f(x) d(x)$$

$$= - F^{c}(r) \qquad (2.2.1.7)$$

Where $F^{c}(r)$ is the complementary cumulative distribution of x evaluated at r. Therefore, solving the second equation for r in terms of Q gives:

$$F^{c}(r) = \frac{hQ}{\pi \bar{D}} \qquad (2.2.1.8)$$

The iterative procedure to find the optimal pair (Q^*, r^*):

(1) Let $\bar{b}(r) = 0$ and simply solve for Wilson Q^*_w. Call this value Q_1.

(2) Use equation (2.2.1.8) with $Q = Q_1$ to find the reorder point. Call this value r_1.

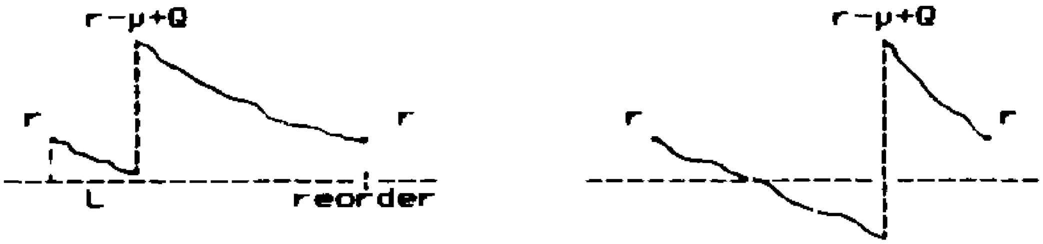
(3) Use equation (2.2.1.6) with $r = r_1$ to find a new lot size Q_2, having first found $\bar{b}(r_1)$ from equation (2.2.1.3).

(4) Repeat step 2 with $Q = Q_2$, etc. Convergence occurs when at iteration i, $Q_i = Q_{i-1}$ or $r_i = r_{i-1}$. Usually this is rapid.

Wagner [17] considers a discrete model basically with the same assumptions as Hadley and Whitin [4]:

- The probability distribution of demand during lead-time $P_x(x)$ does not depend on when the inventory reaches the reorder point r.
- The inventory level i can be treated as a continuous variable.
- After a replenishment order arrives, there exists a future moment in time when the inventory level i = r , and a reorder action occurs as a consequence.
- In an optimal policy, the reorder point $r > 0$ and during any lead-time, actual demand does not exceed the order quantity $(x \leq Q)$.
- Demand is probabilistic.

The only difference between the two models is the way in which the average on-hand inventory per unit of time is calculated. He distinguishes two cases:



Case 1: $\mu<r$

Case 2: $\mu>r$

Figure 2.2.1.1 Two possibilities for arrival time of an order.

Then he finds the expected average inventory level during lead-time and after replenishment until the next reorder as follows:

Average inventory level during lead-time =

$$\sum_{x=0}^{r}\frac{1}{2}[r+(r-x)]p_x(x) + \sum_{x>r}\frac{1}{2}[r]p_x(x) =$$

$$\frac{1}{2}[r + \sum_{x=0}^{r}(r-x)p_x(x)] \qquad (2.2.1.8)$$

Average inv. level after replenishment

until next reorder =

$$\frac{1}{2}[(r-\mu+Q) + r] = \frac{1}{2}(2r-\mu+Q) \qquad (2.2.1.9)$$

By weighting the above two quantities

Expected average inventory/unit time = (Fraction of time system is waiting for replenishment) X (Expected average inventory level during lead-time) + (Complement of the fraction of time system is waiting for replenishment) X (Expected average inventory level after replenishment until next reorder)

Where $\sum_{x=0}^{r} (r-x)p_x(x) = r - \mu - \sum_{x>r} (r-x)p_x(x)$ (2.2.1.10)

because $\sum_{x=0}^{\infty} (r-x)p_x(x) - \sum_{x>r}^{\infty} (r-x)p_x(x)$

$= r-\mu - \sum_{x>r}^{\infty} (r-x)p_x(x)$ (2.2.1.11)

Therefore:

Average inventory per unit of time =

$$\frac{\mu}{Q}\cdot\frac{1}{2}[-2r + \mu - Q + r + \sum_{x=0}^{r} (r-x)p_x(x)] + \frac{1}{2}(2r - \mu + Q) =$$

$$\frac{Q}{2} + r - \mu + \frac{\mu}{2Q} \cdot \sum_{x>r} (x-r)p_x(x) \qquad (2.2.1.12)$$

Now total expected average cost can be simply written as:

$$E[AC] = C\bar{D} + \frac{K\bar{D}}{Q} + h[\frac{Q}{2} + r - \mu + \frac{\mu}{2Q} . \sum_{x>r} (x-r)p_x(x)] +$$

$$+ \frac{\pi\bar{D}}{Q} . \sum_{x>r} (x-r)p_x(x)$$

or

$$EAC = C\bar{D} + \frac{K\bar{D}}{Q} + h(\frac{Q}{2} + r - \mu) +$$

$$+ (\frac{h\mu}{2Q} + \frac{\pi\bar{D}}{Q}) . \sum_{x>r} (x-r)p_x(x) \qquad (2.2.1.13)$$

Differentiating EAC with respect to Q and equating it to zero:

$$\frac{\partial EAC}{\partial Q} = - \frac{\bar{D}K}{Q^2} + \frac{h}{2} - (\frac{h\mu}{2Q^2} + \frac{\pi\bar{D}}{Q^2}) . \sum_{x>r} (x-r)p_x(x) = 0$$

$$(2.2.1.14)$$

Therefore:

$$Q^* = \sqrt{\frac{2\bar{D}K}{h} + (\mu + \frac{2\pi\bar{D}}{h}) \sum_{x>r} (x-r)p_x(x)} \qquad (2.2.1.15)$$

Differentiating with respect to r we have:

$$\frac{\partial EAC}{\partial r} = h - (\frac{h\mu}{2Q} + \frac{\pi\bar{D}}{Q}) \sum_{x>r} p_x(x) = 0 \qquad (2.2.1.16)$$

Therefore $1 - F^c(r)$ is:

$$F(r) = 1 - \frac{h}{\frac{h\mu}{2Q} + \frac{\pi\bar{D}}{Q}} \qquad (2.2.1.17)$$

which means, r is the smallest integer such that:

$$F(r) \geq 1 - \frac{h}{\frac{h\mu}{2Q} + \frac{\pi\bar{D}}{Q}} \qquad (2.2.1.18)$$

Finally EAC* can be simply developed as:

$$EAC^* = C\bar{D} + h(r-\mu) + \sqrt{2h[\bar{D}K + (\frac{h\mu}{2}+\pi\bar{D}) \sum_{x>r} (x-r)p_{\mu}(x)} \qquad (2.2.1.19)$$

Love [8] also developed a general continuous review stochastic model framework that is very similar to the previous two models by Hadley and Whitin [4] and Wagner [17]. He defines:

$\mu = \bar{D}L = \int_0^{\infty} x f(x) d(x)$ = average demand during lead-time

$r-\mu$ = average ending inventory (can be either positive or negative)

$\bar{y}(r) = \int_0^{r} (r-x) f(x) d(x) = r-\mu+\bar{b}(r)$

= average residual inventory when the initial inventory level r is depleted by demand during lead-time

$\bar{b}(r) = \int_r^{\infty} (x-r) f(x) d(x)$ = average shortage inventory when the initial inventory level r is depleted by demand during lead-time.

Then he approximates the average inventory level $\bar{y}(Q,r)$ as expected residual inventory $\bar{y}(r)$ plus one-half of the average amount added to the inventory when an order is received. Or, simply,

$$\bar{y}(Q,r) = \bar{y}(r) + \frac{1}{2}\,[Q - \bar{b}(r)] \qquad (2.2.1.20)$$

The expected total cost per unit of time may be written as:

$$EAC = C\bar{D} + \frac{K\bar{D}}{Q} + h\frac{Q}{2} + \left(\frac{h}{2} + \frac{\pi\bar{D}}{Q}\right)\bar{y}(r) + $$

$$+ \left(\frac{h}{2} - \frac{\pi\bar{D}}{Q}\right)(r-\mu) \qquad (2.2.1.21)$$

And after substituting for $\bar{y}(r)$ we get:

$$EAC = C\bar{D} + \frac{K\bar{D}}{Q} + h\left(\frac{Q}{2} + r - \mu\right) + $$

$$+ \left(\frac{\bar{D}\pi}{Q} + \frac{h}{2}\right)\bar{b}(r) \qquad (2.2.1.22)$$

The optimal values of Q and r can be determined by differentiating EAC with respect to Q and r, equating to zero, and solving for the two unknowns. Differentiating with respect to Q gives:

$$\frac{\partial EAC}{\partial Q} = \frac{-K\bar{D}}{Q^2} + \frac{h}{2} - \frac{\bar{D}\pi}{Q^2}.\bar{b}(r) = 0 \qquad (2.2.1.23)$$

which leads to

$$Q^* = \sqrt{\frac{2\bar{D}[K + \pi\bar{b}(r)]}{h}} \qquad (2.2.1.24)$$

Differentiating with respect to r gives:

$$\frac{\partial EAC}{\partial r} = h + (-\frac{\bar{D}\pi}{Q} + \frac{h}{2})[-\int_r^\infty f(x)dx] = 0 \qquad (2.2.1.25)$$

which leads to

$$F^c(r) = \frac{h}{\frac{\bar{D}\pi}{Q} + \frac{h}{2}} \qquad (2.2.1.26)$$

A closer look at the above three models by Hadley and Whitin [4], Wagner [17], and Love [8] indicates that the main difference among them is how the average inventory level during the cycle is approximated. The optimal order quantity is the same in the Hadley and Whitin [4] and Love [8] models but is slightly different in Wagner [17] model. The cumulative distribution of x evaluated at r is also slightly different in the aforementioned models.

One may argue the relevance of all these models since they make the assumption of constant lead-times. We have to point out that these models can be used as an approximation for the cases of stochastic demand and/or lead-times, especially when the time interval between placing orders is not very short.

These models are relatively easy to use, and simplify to EOQ where demand is constant [$\bar{b}(r)=0$]. Also, they converge fast when a solution exists. On the other hand, one may observe many limitations of these models. Among them are:

- There may be no solution or multiple solutions and there is no guarantee that the recursive procedure would indicate that .
- These are only heuristic treatments, therefore they are just approximations.
- Hadley and Whitin's model makes an implicit assumption that π is sufficiently large so that the term $hQ/\pi\bar{D}$ is always less than one.

Since it is possible that $F(r)=1-h/[h\pi/2Q+\pi\bar{D}/Q]$ in Wagner's model may be less than one while $F(r) = hQ/\pi\bar{D}$ in Hadley and Whitin's model may be greater than one, it can happen that for some small values of π, equations (2.2.1.15) and (2.2.1.18) may yield a solution while (2.2.1.6) and (2.2.1.8) will not. This is of course one advantage of Wagner's approach. However, a study by Gross and Ince [2] indicates that in general Hadley and Whitin's model is closer to the optimal than Wagner's a great portion of time. Nahmias [10] speculates that perhaps the Hadley-Whitin's model has compensating errors that allow it to perform better in many cases.

There are many other related studies. Each assumes that demand during lead-time follows a particular distribution such as, Gamma, Normal, Negative Binomial, Weibull, etc. For a complete review of these models, the reader can refer to Nasri [11].

Sphicas' [15] current research is also a continuous review inventory system with backorders. His model assumes that demand is deterministic and lead-time is stochastic, and it differs from the previously mentioned models in many ways. He [15] develops four models, models 1 and 2 which are complementary make the "average backorder assumption", i.e., the cost is proportional to the average backorders outstanding per unit of time. Models 3 and 4 which are more general, in addition to "average backorder assumption" include "the size of backorders assumption", i.e., the cost is proportional to the size of shortage when a replenishment arrives.

2.2.2 Review Of Models That Allow Orders To Cross

As mentioned at the beginning of the Section 2.2, the second group of inventory models that assume stochastic independent lead-times allow orders to cross each other. These models are more mathematically sound than the models in the previous Section. This is due to the fact that when lead-time is an independent random variable, there is always a possibility that an order placed at time 2 arrives before an order placed at time 1, i.e., orders may cross each other.

An unconventional solution to the lead-time modeling dilemma has been proposed by Washburn [18]. He [18] states that "each unit ordered can satisfy only one particular unit of demand. In a manufacturing context, this amounts to assuming that parts are not interchangeable. In a sales context, the corresponding assumption is that each item has been "colored" to suit the needs of a particular customer. The effect of the assumption is to decouple the orders so that it becomes immaterial whether they cross or not. If applied to conventional systems with interchangeable parts, errors will result to the extent that orders are likely to cross. The cost derived here will then constitute an upper bound on actual inventory cost."

Washburn [18] then makes the assumption that demand is constant per unit of time over an infinite horizon. He also assumes that a penalty cost per unit is incurred if the shipment never arrives ($\lim_{r \to \infty} G(r) \leq 1$). Then he formulates a cost expression and tries to minimize its present value. Finally, an algorithm is constructed to find the two decision variables.

In his article, Liberatore [7] states that Washburn's model formulation differs from the mainstream of traditional EOQ modeling in several respects. First, all costs are subject to continuous time discounting and the inventory holding costs are not stated explicitly but are merged into the time discount rate. Second, the lead-time probability distribution, $G(r)$, is permitted to have a "defect", $\text{Lim}_{r \dashrightarrow \infty} G(r) \leq 1$. Finally, although the model has a continuous time orientation, the inventory shortage cost is expressed as $/unit arrived late (or never arrived) in consonance with the previous assumption. This particular set of assumptions complicates the search for the optimal values of the decision variables and may yield non-unique solutions.

Liberatore [6] in his dissertation develops a finite time horizon, stochastic lead-time inventory model with deterministic demands. He develops and discusses his research in a freight transport selection context. As in Washburn, Liberatore makes the assumption of non-interchangeability of unit demands. Holding and backlogging costs are treated as fixed and ordering costs as variable and stationary over time. Then several results similar to Wagner-Whitin and Zabel are stated and proven and a recursive forward dynamic programming procedure is provided. In addition, several extensions of these models are examined. Three years later, Liberatore [7] published an article and considered a continuous deterministic demand, with stochastic lead-times in which he tried to formulate and solve a general EOQ model. Since Sphicas-Nasri's [13] paper (discussed in the next chapter) and other results and extensions in the following chapters are based on Liberatore's [7] article, we discuss his five page article in some detail.

To find the optimal values of the decision variables order size and timing, he explicitly makes the following assumptions:

- Continuous review (Q,R) policy is used.
- Demand is continuous and deterministic.
- Lead-time is general and stochastic and independent.
- Backlogging of demand is allowed.
- Unit demands are non-interchangeable (the impact of this assumption is that it allows orders to cross).

He then defines:

D = Constant demand rate (units/unit time demanded)

q = Number of time units of demand satisfied by each order

t = Time differential between placing an order and the start of the q time units that will be satisfied by the given order (See Figure 2.2.2.1)

h = Holding cost in $/unit/unit time

p = Backlogging cost in \$/unit/unit time

$g(r)$ = Lead-time probability mass or density function

$G(.)$ = Distribution function of lead-time

$K+vq$ = Ordering costs for $q>0$, 0 otherwise.

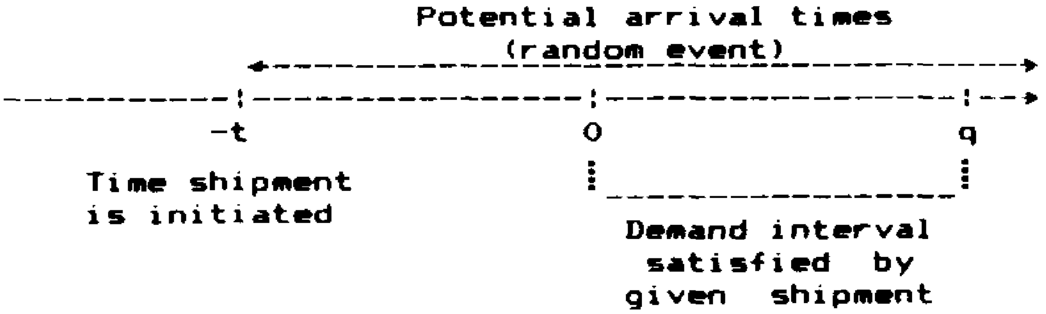


Figure 2.2.2.1 Time scale for one cycle in the steady-state continuous demand model.

Based upon the above assumptions, the expected total cost is expressed as:

$$\begin{aligned} ETC(t,q) = K+vq &+ \int_0^t \{hDq(t-r)+hDq^2/2\}g(r)dr \\ &+ \int_t^{t+q} \{pD(r-t)^2/2+hD(t+q-r)^2/2\}g(r)dr \\ &+ \int_{t+q}^{\infty} \{pDq^2/2+pDq(r-t-q)\}g(r)dr \end{aligned} \tag{2.2.2.1}$$

Note: For the graphical representation of ETC refer to chapter 4.

To find the optimal value of the decision variables, we differentiate the expected average cost $EAC(t,q) = ETC(t,q)/q$ with respect to q and then with respect to t and equate them to zero,

$$\frac{\partial EAC(t,q)}{\partial q} = \frac{-K}{q^2} + \frac{hD}{2} \int_0^t g(r)\,dr + \int_{t+q}^{t} \{ [\frac{-pD(r-t)^2}{2q^2}] + \frac{hD}{2}[1 - \frac{(r-t)^2}{q^2}] \} g(r)\,dr - \frac{pD}{2} \int_{t+q}^{\infty} g(r)\,dr \tag{2.2.2.2}$$

$$\frac{\partial EAC(t,q)}{\partial t} = hD \int_0^t g(r)\,dr + \int_t^{t+q} (\frac{-pD(r-t)}{q} + \frac{hD(t+q-r)}{q}) . g(r)\,dr - pD \int_{t+q}^{\infty} g(r)\,dr \tag{2.2.2.3}$$

After some algebraic manipulations we find two equations and two unknowns:

$$\frac{2K}{(h+p)D} + \int_{t^*}^{t^*+q^*} (r-t^*)^2 g(r)\,d(r) = (q^*)^2 \{ \int_0^{t^*+q^*} g(r)\,dr \,.\, \frac{p}{h+p} \} \tag{2.2.2.4}$$

and

$$\int_{t^*}^{t^*+q^*} (r-t^*) g(r)\,dr = q^* \{ \int_0^{t^*+q^*} g(r)\,dr - \frac{p}{h+p} \} \tag{2.2.2.5}$$

which are the unique global minima for t and q. Of course, t and q are global optimal if and only if the EAC function is strictly convex over the relevant domains. Therefore, Liberatore proves that diagonal elements and determinant of the Hessian matrix are positive (sufficient condition for strict convexity). Mathematically speaking, he proves that:

$$\frac{\partial^2 EAC(t,q)}{\partial q^2} > 0 \quad , \quad \frac{\partial^2 EAC(t,q)}{\partial t^2} > 0 \qquad (2.2.2.6)$$

$$\left[\frac{\partial^2 EAC(t,q)}{\partial q^2}\right]\left[\frac{\partial^2 EAC(t,q)}{\partial t^2}\right] - \left[\frac{\partial^2 EAC(t,q)}{\partial q . \partial t}\right]^2 > 0$$

(2.2.2.7)

In order to compare these results with the traditional EOQ model, Liberatore [7] assumed that lead-time to be deterministic. Therefore, he was able to represent the lead-time density function as a Dirac Delta function, that is, all probability concentrated at a single point. Equations (2.2.2.4) and (2.2.2.5) then become:

$$2K/(h+p)D + (r_o-t^*)^2 = (q^*)^2[h/(h+p)] \qquad (2.2.2.8)$$

and

$$(r_o-t^*) = q^* \; [h/(h+p)] \qquad (2.2.2.9)$$

From (2.2.2.9) we get:

$$q^* = (r_o - t^*)[(h + p)/h] \qquad (2.2.2.10)$$

By substituting q^* in (2.2.2.10) into (2.2.2.8) we get:

$$2K/(h+p)D + (r_o-t^*)^2 = (r_o-t^*)^2[(h+p)^2/h^2][h/(h+p)] \tag{2.2.2.11}$$

or simply

$$r_o-t^* = \sqrt{2Kh/p(h+p)D} \tag{2.2.2.12}$$

Substituting for $(r_o - t^*)$ in equation (2.2.2.10) and after some simplifications:

$$q^* = \sqrt{2K(h + p)/hpD} \tag{2.2.2.13}$$

Therefore, the optimal order quantity would be

$$Dq^* = D\sqrt{2K(h + p)/hpD}$$

$$= \sqrt{2KD(h + p)/hp} \tag{2.2.2.14}$$

Define $s^* = q^* - (r_o-t^*)$. Therefore, Ds^* is the amount of stock on-hand immediately after satisfying the backlogged demand.

$$Ds^* = \sqrt{2KDp/h(h + p)} \tag{2.2.2.15}$$

and the fraction of time no shortage exists is:

$$f^* = s^*/q^* = p/(h + p) \tag{2.2.2.16}$$

The above results are identical to those obtained for the classical EOQ model with backlogging of demand. Thus, Liberatore's [7] model is a stochastic lead-time generalization of the EOQ model with backlogging of demand.

Then he states some computational considerations. If the c.d.f. cannot be expressed in closed form, a numerical method such as Newton-Raphson must be used to iteratively solve the equations (2.2.2.4) & (2.2.2.5). Of course, the success of such methods depends largely on the starting value of the search. He showed that $G^{-1}[p/(h+p)]$ provides an upper bound, and, therefore a good initial guess for t^*.

At the end of his paper, Liberatore [7] gives an example where lead-times are uniformly distributed:

$$g(r:a,b) = \begin{cases} 1/(b-a) & , \text{ if } a<r<b \\ 0 & , \text{ otherwise} \end{cases} \qquad (2.2.2.17)$$

and by combining equations (2.2.2.4) and (2.2.2.5) and considering that $g(r)$ is uniformly distributed, we get:

$$\frac{2K}{(h+p)D} + \int_{t^*}^{t^*+q^*} \frac{(r-t^*)^2}{(b-a)}\,dr = q^* \int_{t^*}^{t^*+q^*} \frac{(r-t^*)^2}{(b-a)}\,dr$$

$$(2.2.2.18)$$

after integration and simplification, we will have:

$$q^* = \sqrt[3]{12K(b-a)/(h+p)D} \qquad (2.2.2.19)$$

Solving equation (2.2.2.5) for t^* when $g(r) = 1/(b-a)$ yields:

$$-t^* = (q^*/2) - [(ha+pb)/(h+p)] \qquad (2.2.2.20)$$

Liberatore stops at this point and fails to consider that this formulation is valid if and only if t* and (t* + q*) fall within the range of a and b, that is:

```
+-----+--------+-------------+------+

0     a        t*            t*+q*  b
```

Figure 2.2.2.2 The sequence in which Liberatore'a model applies.

This error was the motivation behind Sphicas-Nasri's [13] article, which will be discussed in the following chapter.

2.3 <u>DEPENDENT LEAD TIMES</u>

As long as there is never more than one order outstanding, no theoretical difficulties are involved. But, unfortunately, it is not always possible to specify that never more than a single order is outstanding. Therefore, in the case where more than a single order can be outstanding, difficulties are encountered in properly representing the lead-times as random variables. If we assume that the lead-time for a given order is independent of the lead-times of the orders which are outstanding, then we must allow orders to cross. As we have discussed before, we realistically allow for cross over of the orders in this work.

Some authors believe that, in the real world, almost always orders are received in the same sequence in which they are placed. Now, if this is true, then the lead-times can not be considered to be independent random variables, _i.e._, the time of arrival of an order placed at time t can depend on the times of arrival of the other orders on the books when the order is placed at time t. This is the case where the lead-times are dependent. This lead-time dependency, often causes serious analytical difficulties.

Due to the analytical difficulties in the treatment of lead-time state dependency, there has been very little work on this subject in the literature.

This brief section was included for completeness. Therefore review of the related literature is omitted.

CHAPTER III

ANALYSIS AND REVIEW OF AN INVENTORY MODEL WITH FINITE-RANGE STOCHASTIC LEAD-TIMES

In the real world, we are usually faced with stochastic lead-time, i.e., we are not totally sure about the arrival time of an order. This lead-time variability complicates the inventory model. One of the approaches to tackle this difficulty is given by Washburn [18]. This approach is extensively discussed in Section 2.2.2 (It is recommended that the reader review Section 2.2.2 of this dissertation at this point).

This research, which is based on Washburn's [18] approach, is originated from the recent work of Liberatore [7], which is also discussed in Section 2.2.2.

At the end of his paper, Liberatore [7] presents an example of uniform lead-time distribution, but he fails to consider that his formulation is not collectively exhaustive and is just valid for some particular range of decision variables.

In this chapter the main results of Sphicas-Nasri's [13] article are discussed. A generalized EOQ formula in conjunction with ranges for decision variables is obtained. Some additional results for the case of uniform distribution of lead-times is also provided.

3.1 GENERAL DISTRIBUTION OF LEAD-TIMES

The notations and assumptions used here are the same as the ones discussed in Section 2.2.2, except that lead-time distribution is considered to be finite and defined over the range (a,b).

At this point we need to define two functions A(x) and B(x). The difference between these functions used here and in Sphicas [14] is that in our paper g(r) is finite and defined between a and b.

The functions A(x) and B(x), which are defined below, only depend on g(r) and the cost ratio $\Omega = h/p$,

$$A(x) = (1 + \Omega) \int_0^x (x-r)g(r)dr - x \qquad (3.1.1a)$$

$$B(x) = [A(x) + x] \int_0^x (x-r)g(r)dr - \int_0^x (x-r)^2 g(r)dr \qquad (3.1.1b)$$

and must satisfy the following for optimum values of t and q:

$$A(t^*) = A(t^* + q^*) \tag{3.1.2}$$

$$B(t^* + q^*) - B(t^*) = k \quad \text{where } k = 2K/(h+p)D \tag{3.1.3}$$

The proof of the above two relationships is based on the shape and the properties of A(x) and B(x), which are given in [14].

One can summarize A(x) and B(x) for different ranges of x as follows:

$$A(x) = \begin{cases} -x & \text{if } x \le a \\ (1 + \Omega) \int_a^x (x-r)g(r)dr - x & \text{if } a \le x \le b \\ \Omega x - (1 + \Omega)\mu & \text{if } x \ge b \end{cases} \tag{3.1.4}$$

$$B(x) = \begin{cases} 0 & \text{if } x \le a \\ [A(x) + x] \int_a^x (x-r)g(r)dr - \int_a^x (x-r)^2 g(r)dr & \text{if } a \le x \le b \\ \Omega(x - \mu)^2 - \sigma^2 & \text{if } x \ge b \end{cases} \tag{3.1.5}$$

Where μ and σ^2 are the mean and variance of lead-time respectively. The proof of the above formula is given below:

When $X \le a$, the integral part of $A(x)$ and $B(x)$ drops out:

$$A(x) = -x \tag{3.1.6}$$

$$B(x) = 0 \tag{3.1.7}$$

When $a \le x \le b$

$$A(x) = (1 + \Omega) \int_a^x (x - r)g(r)dr - x \tag{3.1.8}$$

$$B(x) = [A(x) + x] \int_a^x (x-r)g(r)dr - \int_a^x (x-r)^2 g(r)dr \tag{3.1.9}$$

When $X \ge b$

$$\begin{aligned} A(x) &= (1 + \Omega) \int_a^b (x - r)g(r)dr - x \\ &= (1 + \Omega)(x - \mu) - x \\ &= \Omega x - (1 + \Omega)\mu \end{aligned} \tag{3.1.10}$$

$$\begin{aligned} B(x) &= [A(x) + x] \int_a^b (x - r)g(r)dr - \int_a^b (x - r)^2 g(r)dr \\ &= [(1 + \Omega)(x - \mu](x - \mu) - E(x - r)^2 \\ &= (1 + \Omega)(x - \mu)^2 - Ex^2 - Er^2 + 2EXr \\ &= x^2 + \mu^2 - 2x\mu + \Omega(x - \mu)^2 - x^2 - Er^2 + 2x\mu \\ &= \Omega(x - \mu)^2 - \sigma^2 \end{aligned} \tag{3.1.11}$$

It can also be shown that both functions are convex and reach the same minimum:

$$A(x) = (1 + \Omega) \int_a^x (x - r)g(r)dr - x \qquad (3.1.12)$$

$$A'(x) = (1 + \Omega)[\int_a^x g(r)dr] - 1$$

$$= (1 + \Omega)G(x) - 1 \qquad (3.1.13)$$

$$A''(x) = (1 + \Omega)g(x)$$

$$= (1 + \Omega)g(x) > 0 \qquad (3.1.14)$$

Therefore, $A(x)$ reaches minimum at $\alpha = G^{-1}(1/(1 + \Omega))$ Where $A'(x) = 0$.

$$B(x) = [A(x) + x] \int_a^x (x - r)g(r)dr - \int_a^x (x-r)^2 g(r)dr \qquad (3.1.15)$$

$$B(x) = (1 + \Omega)G(x) \int_a^x (x - r)g(r)dr + [A(x) + x)]$$

$$. \int_a^x g(r)dr - 2 \int_a^x (x - r)g(r)dr$$

$$= (1 + \Omega)G(x) \int_a^x (x - r)g(r)dr +$$

$$[(1+\Omega) \int_a^x (x-r)g(r)dr]G(x) - 2 \int_a^x (x-r)g(r)dr$$

$$= 2(1+\Omega) \int_a^x (x - r)g(r)dr . G(x) - 2 \int_a^x (x-r)g(r)dr$$

$$= 2 \int_a^x (x - r)g(r)dr[(1 + \Omega)G(x) - 1] = 0 \qquad (3.1.16)$$

Similarly it can be shown that $B''(x) > 0$ and $B(x)$ also reaches minimum when $\alpha = G^{-1}(1/(1 + \Omega))$.

It can easily be argued that $\Gamma = B(\beta)$ is positive.

To show this, rewrite (3.1.5)

$$B(x) = [A(x) + x]\int_a^x (x - r)g(r)dr - \int_a^x (x - r)^2 g(r)dr \qquad (3.1.18)$$

Letting $x = \beta$ and substituting for $A(\beta) = 0$, we have:

$$\Gamma = B(\beta) = \beta\int_a^\beta (\beta - r)g(r)dr - \int_a^\beta (\beta - r)^2 g(r)dr$$

$$= \int_a^\beta (\beta - r)[\beta - (\beta - r)]g(r)dr$$

$$= \int_a^\beta (\beta - r)rg(r)dr \geq 0 \qquad (3.1.19)$$

Functions $A(x)$ and $B(x)$ are shown in Figure 3.1.1.

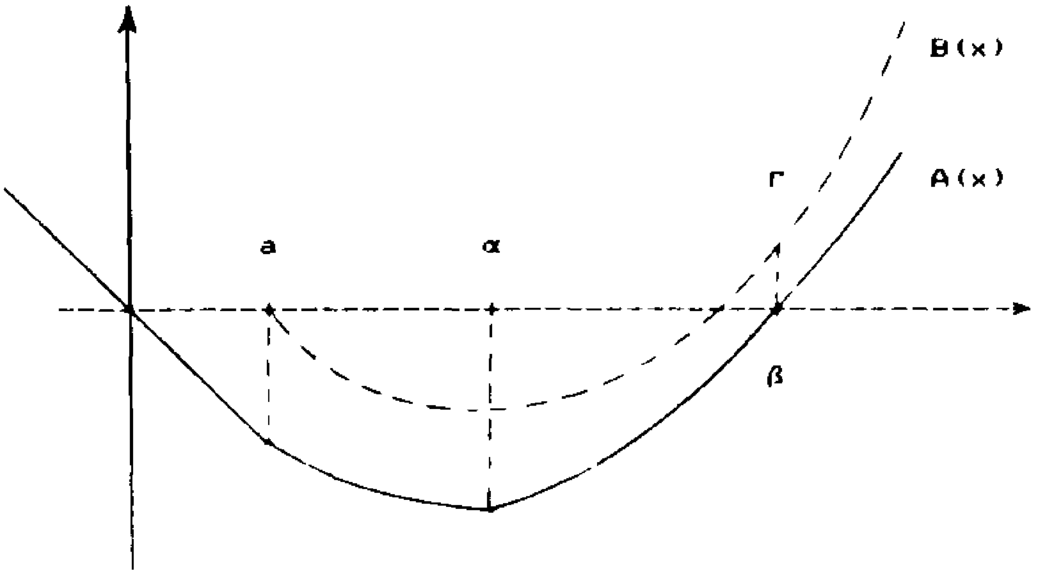


Figure 3.1.1 The functions A(x) and B(x).

Two quantities of interest that should be defined are θ and τ. θ is a unique point to the right of α such that $A(\theta) = A(a) = -a$. τ is a unique point to the left of α such that $A(\tau) = A(b) = \Omega b - (1 + \Omega)\mu$.

It can easily be argued that "a" is to the left of α and "b" is to its right. It is because at minimum $G(\alpha)=1/(1 + \Omega) \leq 0$, therefore, α is a value between "a" and "b". For $A(\theta)$ to be equal to $A(a)$, θ should be to the right of α , and for $A(\tau)$ to be equal to $A(b)$, τ should be to the left of α. Therefore, we conclude that "a" and τ fall to the left, and "b" and θ fall to the right of α.

The $b \leq \theta$ occurs if and only if $A(b) \leq A(a)$, i.e., $\Omega b-(1+\Omega)\mu \leq -a$. Therefore, the ordering $a \leq \tau \leq \alpha \leq b \leq \theta$ occurs if and only if $\Omega \leq (\mu - a)/(b - \mu)$. By the same reasoning $b \geq \theta$ happens if and only if $A(b) \geq A(a)$, i.e., $\Omega b - (1 + \Omega)\mu \geq - a$. This in turn dictates the ordering $\tau \leq a \leq \alpha \leq \theta \leq b$, iff $\Omega \geq (\mu-a)/(b-\mu)$.

We can summarize and graph the above arguments as follows:

Case 1 – If $\Omega \leq (\mu-a)/(b-\mu)$, the ordering is $a\leq\tau\leq\alpha<b\leq\theta$.

Case 2 – If $\Omega \geq (\mu-a)/(b-\mu)$, the ordering is $\tau\leq a\leq\alpha\leq\theta<b$.

The ranges for t^* and $t^* + q^*$ that satisfy equation (3.1.2) can easily be found if one uses Figures (3.1.2) and (3.1.3):

<u>Case 1</u> – If $\Omega \leq (\mu-a)/(b-\mu)$, t^* and $t^* + q^*$ are bounded as follows:

$$\begin{aligned} t^* &\leq a &&\text{iff}\quad \theta \leq t^* + q^* \\ a \leq t^* &\leq \tau &&\text{iff}\quad b \leq t^* + q^* \leq \theta \qquad (3.1.20a) \\ \tau \leq t^* &\leq \alpha &&\text{iff}\quad \alpha \leq t^* + q^* \leq b \end{aligned}$$

<u>Case 2</u> – If $\Omega \geq (\mu-a)/(b-\mu)$, t^* and t^*+q^* are bounded as follows:

$$\begin{aligned} t^* &\leq \tau &&\text{iff}\quad b \leq t^* + q^* \\ \tau \leq t^* &\leq a &&\text{iff}\quad \theta \leq t^* + q^* \leq b \qquad (3.1.20b) \\ a \leq t^* &\leq \alpha &&\text{iff}\quad \alpha \leq t^* + q^* \leq \theta \end{aligned}$$

Now, we are looking for a t^* and $w = t^*+q^*$ such that they satisfy the above bounds and also satisfy equation (3.1.3). If $w \geq \theta$ this is simple because to satisfy the bounds t^* should be less than or equal to a. For this range of t^*, $B(t^*)$ is zero. This in turn simplifies equation (3.1.3) to $B(t^*+q^*) = B(w) = k$. Therefore, to find the optimal set for the range $t^*+q^* \geq \theta$ we just have to search for a "w" such that $B(w)=k$ and then by mapping find t^*. Unfortunately, for the other two ranges the problem is not that simple because $B(t^*+q^*) - B(t^*)=k$ does not simplify to $B(t^*+q^*) = k$. In [14], Sphicas defines a function $H(w)$ that is increasing and convex over the range $w \geq \alpha$ (See Figures 3.1.2 and 3.1.3). Taking advantage of this function we can locate $(w = t^*+q^*)$ such that $H(w) = k$ and again by mapping, t^* can be found.

The above discussion and cases 1 and 2 can be summarized as follows:

If $k_2 \leq k$ then $t^* \leq t_2$ and $w_2 \leq t^*+q^*$

If $k_1 \leq k \leq k_2$ then $t_2 \leq t^* \leq t_1$ and $w_1 \leq t^*+q^* \leq w_2$

If $k \leq k_1$ then $t_1 \leq t^* \leq \alpha$ and $\alpha \leq t^*+q^* \leq w_1$

(3.1.21)

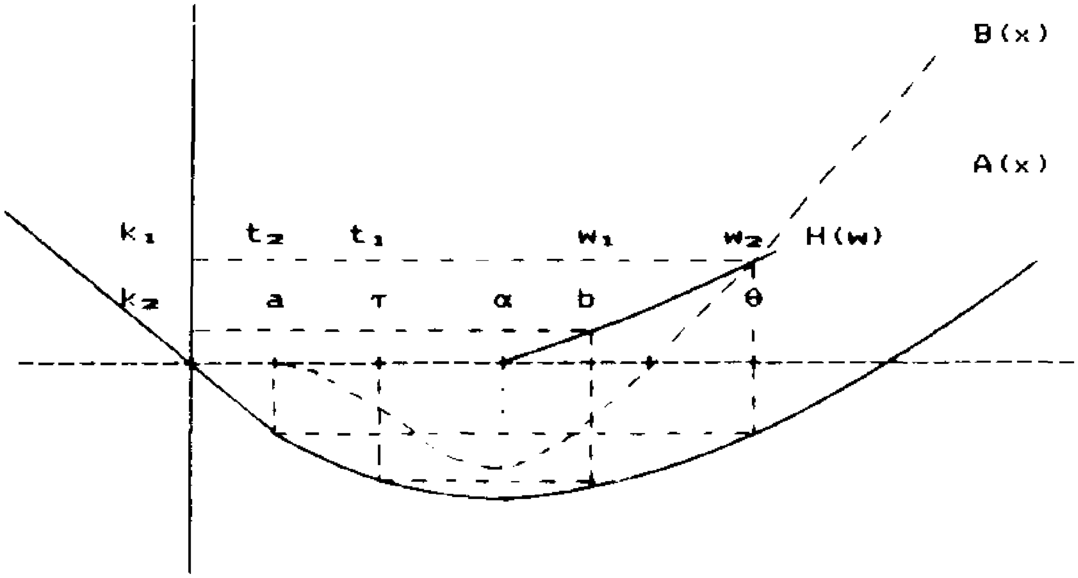


Figure 3.1.2 Case 1: $(b \le \theta)$ or $\Omega \le (\mu - a)/(b - \mu)$

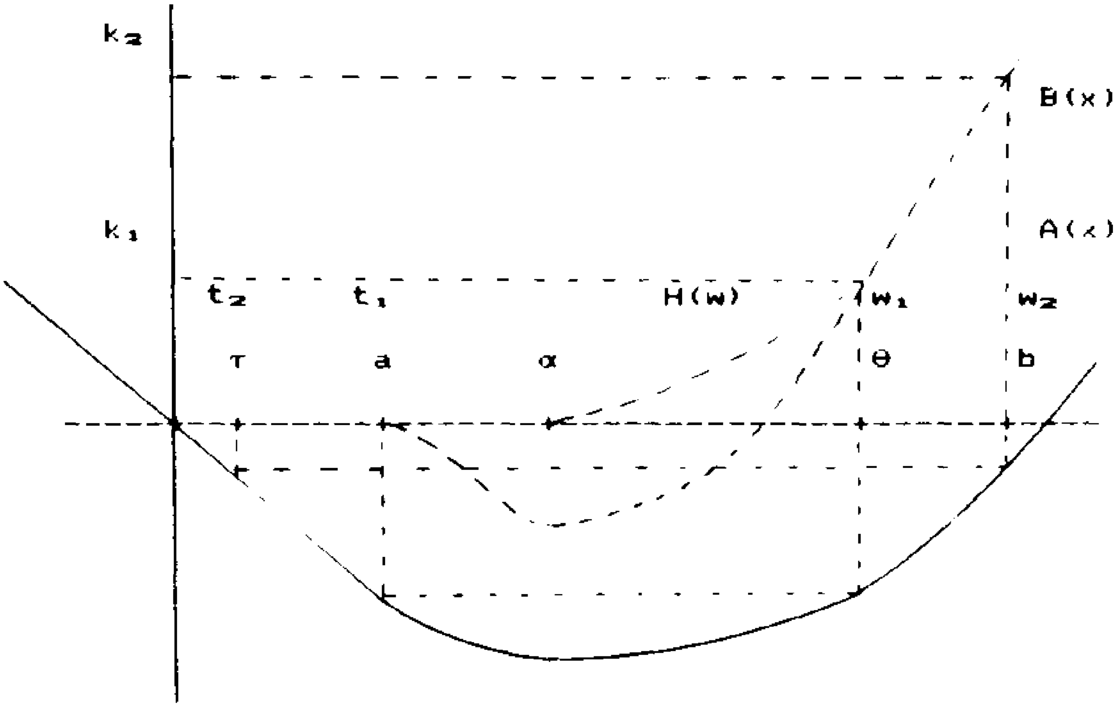


Figure 3.1.3 Case 2: $(b \ge \theta)$ or $\Omega \ge (\mu - a)/(b - \mu)$

Therefore, to find the optimal values in any given problem, the quantities t_2, t_1, w_1, w_2, k_1, and k_2 that are given in the following table should first be calculated based on the given parameters.

Table 3.1.1

Values of the critical points for general distribution

$\Omega \le (\mu-a)/(b-\mu)$	$\Omega \ge (\mu-a)/(b-\mu)$
$t_2 = a$	$t_2 = \mu - \Omega(b-\mu)$
$t_1 = \tau$, where $A(\tau)=\Omega(b-\mu) - \mu$	$t_1 = a$
$w_1 = b$	$w_1 = \theta$, where $A(\theta)=-a$
$w_2 = \mu + (\mu-a)/\Omega$	$w_2 = b$
$k_1 = \Omega(\mu-b)^2 - \sigma^2 - B(\tau)$	$k_1 = B(\tau)$
$k_2 = (\mu-a)^2/\Omega - \sigma^2$	$k_2 = \Omega(\mu-b)^2 - \sigma^2$

The proof of the above formulae is given below:

If $\Omega \le (\mu-a)/(b-\mu)$,

$$A(\tau) = A(b) \tag{3.1.22}$$

when $x \ge b$, $A(x) = \Omega x - (1+\Omega)\mu$, therefore,

$$A(b) = \Omega b - (1+\Omega)\mu \tag{3.1.23}$$

$$A(\tau) = A(b) = \Omega(b-\mu) - \mu \tag{3.1.24}$$

$$A(\theta) = A(a) = -a \tag{3.1.25}$$

$$\text{since } \theta > b,\ A(\theta) = \Omega\theta - (1+\Omega)\mu \qquad (3.1.26)$$

$$\text{so, } \Omega\theta - (1 + \Omega)\mu = - a \qquad (3.1.27)$$

$$\theta = \mu + (\mu-a)/\Omega \qquad (3.1.28)$$

$$k_1 = H(b) = B(b) - B(\tau)$$

$$= \Omega(b-\mu)^2 - \sigma^2 - B(\tau) \qquad (3.1.29)$$

$$k_2 = H(\theta) = B(\theta) \quad \text{where } \theta = \mu + (\mu-a)/\Omega \qquad (3.1.30)$$

$$\text{since } \theta > b, \text{ then } B(\theta) = \Omega(\theta-\mu)^2 - \sigma^2 \qquad (3.1.31)$$

$$k_2 = \Omega[\mu + (\mu-a)/\Omega - \mu]^2 - \sigma^2$$

$$= (\mu-a)^2/\Omega - \sigma^2 \qquad (3.1.32)$$

<u>If $\Omega \geq (\mu-a)/(b-\mu)$,</u>

$$t_2 \text{ is a point such that } A(t_2) = A(b) \qquad (3.1.33)$$

$$A(t_2) = A(\tau) = - t_2 \qquad (3.1.34)$$

$$A(b) = \Omega b - (1 + \Omega)\mu \qquad (3.1.35)$$

$$- t_2 = \Omega b - (1 + \Omega)\mu \qquad (3.1.36)$$

$$t_2 = \mu - \Omega(b - \mu) \qquad (3.1.37)$$

$$k_2 = B(b) = \Omega(b - \mu)^2 - \sigma^2 \qquad (3.1.38)$$

As mentioned before, to find the optimal solution one has to find all the critical values (from Table 3.1.1). One must obtain the range for t^* and t^*+q^*. Finally, through search we must find t^* and t^*+q^* such that (3.1.2) and (3.1.3) are satisfied.

It should be pointed out that the search is not necessary for all cases. If $k \geq k_2$, a general explicit analytical solution exists:

When $k \geq k_2$ then $t^* \leq a$ and $t^*+q^* \geq b$

$$B(t^*) = 0 \qquad (3.1.39)$$

$$B(t^* + q^*) = k \qquad (3.1.40)$$

Since $t^*+q^* \geq b$, $B(t^*+q^*) = \Omega(t^*+q^*-\mu)^2 - \sigma^2$

$$(3.1.41)$$

$$\Omega(t^*+q^*-\mu)^2 - \sigma^2 = k$$

$$t^*+q^* = \mu + \sqrt{(k+\sigma^2)/\Omega} \qquad (3.1.42)$$

$$A(x) = \Omega x - (1+\Omega)\mu \quad \text{when} \quad x \geq b \qquad (3.1.43)$$

$$A(t^*+q^*) = \Omega(t^*+q^*) - (1+\Omega)\mu \qquad (3.1.44)$$

$$A(t^*+q^*) = A(t^*) \quad \text{where } A(t^*) = -t^* \qquad (3.1.45)$$

$$\Omega(t^*+q^*) - (1 + \Omega)\mu = -t^*$$

$$t^* = \mu - \sqrt{\Omega(k + \sigma^2)} \qquad (3.1.46)$$

Solving (3.1.42) and (3.1.46) simultaneously,

$$q^* = (1 + \Omega) \sqrt{(k + \sigma^2)/\Omega} \qquad (3.1.47)$$

After some algebra and knowing that $Q^*=Dq^*$, $\Omega=h/p$, and $k = 2K/(h + p)D$,

$$Q^* = \sqrt{\frac{2DK\ (p + h) + \sigma^2 D^2 (p + h)^2}{ph}} \qquad (3.1.48)$$

which is a stochastic genralization of the Basic EOQ formula with backorders. This is due to the fact that in the deterministic case lead-time variance (σ^2) is zero. By substituting $\sigma^2 = 0$ in the above-mentioned formula, the basic EOQ formula with backlogging of demand can be obtained.

3.2 UNIFORM DISTRIBUTION OF LEAD-TIMES

The results obtained in Section 3.1 apply to any distribution of lead-times. In this section the previous results are applied to a uniform distribution of leadtimes. Obviously one expects more simplified results.

Let us begin by defining functions $A(x)$ and $B(x)$ when the lead-time distribution is uniform:

$$A(x) = \begin{cases} -x & \text{if } a \geq x \\ (1+\Omega)(x-a)^2/2(b-a) - x & \text{if } a \leq x \leq b \\ \Omega x - (1+\Omega)\mu & \text{if } x \geq b \end{cases} \tag{3.2.1}$$

$$B(x) = \begin{cases} 0 & \text{if } a \geq x \\ (1+\Omega)(x-a)^4/4(b-a)^2 - (x-a)^3/3(b-a) & \text{if } a \leq x \leq b \\ \Omega(x-\mu)^2 - \sigma^2 & \text{if } x \geq b \end{cases} \tag{3.2.2}$$

For the cases where $x \leq a$ and $x \geq b$, the formulas are the same as given for the general case in the previous section. The only two formulas which need to be proved are $A(x)$ and $B(x)$ for the range (a,b), which are given below:

$$A(x) = (1+\Omega)\int_a^x (x-r)g(r)dr - x \qquad \text{when} \qquad a \le x \le b$$

$$= [(1+\Omega)/(b-a)]\int_a^x (x-r)dr - x$$

$$= [(1+\Omega)/(b-a)][(x-r)^2/(-2)]_a^x - x$$

$$= (1+\Omega)(x-a)^2/2(b-a) - x \qquad (3.2.3)$$

$$B(x) = [A(x)+x]\int_a^x (x-r)g(r)dr - \int_a^x (x-r)^2 g(r)dr \quad \text{when } a \le x \le b$$

$$= [(1+\Omega)(x-a)^2/2(b-a)][1/(b-a)][(x-r)^2/(-2)]_a^x$$

$$- [1/(b-a)][(x-r)^3/(-3)]_a^x$$

$$= [(1+\Omega)(x-a)^2/2(b-a)^2][(x-a)^2/2]$$

$$- [1/(b-a)][(x-a)^3/3]$$

$$= (1+\Omega)(x-a)^4/4(b-a)^2 - (x-a)^3/3(b-a) \qquad (3.2.4)$$

From the table of critical values for general distributions of lead-time a new table can be constructed, which will include critical values for uniform distribution.

Table 3.2.1 Critical values for uniform distribution

$\Omega \le 1$	$\Omega \ge 1$
$t_2 = a$	$t_2 = a - [(\Omega-a)/2](b-a)$
$t_1 = a+[(1-\Omega)/(1+\Omega)](b-a)$	$t_1 = a$
$\alpha = a+[1/(1+\Omega)](b-a)$	$\alpha = a+[1/(1+\Omega)](b-a)$
$w_1 = b$	$w_1 = a+[2/(1+\Omega)](b-a)$
$w_2 = b+[(1-\Omega)/2\Omega](b-a)$	$w_2 = b$
$k_1 = [4\Omega^3/3(1+\Omega)^3](b-a)^2$	$k_1 = [4/3(1+\Omega)^3](b-a)^2$
$k_2 = [(3-\Omega)/12\Omega](b-a)^2$	$k_2 = [(3\Omega-1)/12](b-a)^2$

The proof of some of the critical values for the case of uniform distribution are as follows:

<u>**When $\Omega \le 1$,**</u>

From Table 3.2.1 we have,

$$A(b) = \Omega(b-\mu) - \mu \qquad (3.2.5)$$

From equation (3.2.1) we have,

$$A(\tau) = (1+\Omega)(\tau-a)^2/2(b-a) - \tau \quad \text{where } a \le \tau \le b \qquad (3.2.6)$$

We know that $A(\tau) = A(b)$, (3.2.7)

$$(1+\Omega)(\tau-a)^2/2(b-a) - \tau = \Omega(b-\mu) - \mu$$

$$(1+\Omega)(\tau-a)^2 - 2(b-a)\tau \pm (b-a)a = 2(b-a)[\Omega(b-\mu) - \mu]$$

$$(1+\Omega)(\tau-a)^2 - 2(b-a)(\tau-a) = 2(b-a)[(a-\mu)+\Omega(b-\mu)]$$

$$(1+\Omega)(\tau-a)^2 - 2(b-a)(\tau-a)$$
$$- 2(b-a)[(a-b)/2+\Omega(b-a)/2] = 0$$

$$(1+\Omega)(\tau-a)^2 - 2(b-a)(\tau-a) + (b-a)^2(1-\Omega) = 0$$

$$\tau-a = \frac{2(b-a) \pm \sqrt{4\Omega^2(b-a)^2}}{2(1+\Omega)} = \frac{2(b-a) \pm 2\Omega(b-a)}{2(1+\Omega)}$$

$$t_1 = \tau = a + [(b-a)(1-\Omega)/(1+\Omega)] \qquad (3.2.8)$$

$$G(\alpha) = \int_a^{\alpha} g(r)dr = [1/(b-a)] \int_a^{\alpha} dr = 1/(1+\Omega) \qquad (3.2.9)$$

$$(\alpha-a)/(b-a) = 1/(1+\Omega)$$

$$\alpha = a + (b-a)/(1+\Omega) \qquad (3.2.10)$$

$$A(\theta) = A(a) = -a \quad \text{where } \theta \geq b \qquad (3.2.11)$$

$$\Omega\theta - (1 + \Omega)\mu = -a \qquad (3.2.12)$$

$$w_2 = \theta = [(1+\Omega)(a+b)/2 - a]/\Omega$$

$$= [a + b + a\Omega + b\Omega - 2a]/2\Omega$$

$$= [a + b + a\Omega + b\Omega - 2a \pm b\Omega]/2\Omega$$

$$= b + [(b-a) - \Omega(b-a)]/2\Omega$$

$$= b + (1-\Omega)(b-a)/2\Omega \qquad (3.2.13)$$

$$k_1 = \Omega\,(\mu - b)^2 - \sigma^2 - b(\tau) \qquad (3.2.14)$$

$$\text{where} \quad \tau = a + [(1-\Omega)(b-a)/(1+\Omega)] \qquad (3.2.15)$$

$$\text{and} \quad B(\tau) = [(1+\Omega)(\tau-a)^4/4(b-a)^2-(\tau-a)^3/3(b-a)] \qquad (3.2.16)$$

substituting for τ in $B(\tau)$,

$$B(\tau) = [(1-\Omega)^4(b-a)^2/4(1+\Omega)^3-(1-\Omega)^3(b-a)^2/3(1+\Omega)^3]$$

$$= [(1-\Omega)^3(b-a)^2/(1+\Omega)^3][(-3\Omega -1)/12] \qquad (3.2.17)$$

Now substitute $B(\tau)$ in k_1,

$$k_1 = \Omega(\mu-b)^2 - \sigma^2 - [(1-\Omega)^3(b-a)^2/(1+\Omega)^3][(-3\Omega-1)/12]$$

$$= \Omega(b-a)^2/4 - (b-a)^2/12 - [(1-\Omega)^3(b-a)^2/(1+\Omega)^3]$$

$$[(-3\Omega-1)/12]$$

$$= [(b-a)^2/12(1+\Omega)^3][3\Omega(1+\Omega)^3-(1+\Omega)^3+(1-\Omega)^3(3\Omega+1)]$$

$$= [(b-a)^2/12(1+\Omega)^3][16\Omega^3]$$

$$= 4\Omega^3(b-a)^2/3(1+\Omega)^3 \qquad (3.2.18)$$

$$k_2 = B(\theta) \qquad \text{where } \theta \geq b \qquad (3.2.19)$$

$$k_2 = \Omega(\theta - \mu)^2 - \sigma^2 \qquad (3.2.20)$$

where $\theta = w_2 = b + [(1-\Omega)(b-a)/2\Omega]$ (3.2.21)

substituting for θ in k_2

$$k_2 = \Omega[b+(1-\Omega)(b-a)/2\Omega - \mu]^2 - (b-a)^2/12$$

$$= \Omega[(b-a)/2 + (1-\Omega)(b-a)/2\Omega]^2 - (b-a)^2/12$$

$$= \Omega[(b-a)^2/4][1+(1-\Omega)/\Omega]^2 - (b-a)^2/12$$

$$= (b-a)^2/4\Omega - (b-a)^2/12$$

$$= (3-\Omega)(b-a)^2/12\Omega \qquad (3.2.22)$$

<u>When $\Omega \geq 1$</u>,

$$A(\tau) = - \tau \qquad (3.2.23)$$

and $A(b) = \Omega b - (1+\Omega)\mu$ where $x \geq b$ (3.2.24)

knowing that $A(\tau) = A(b)$, (3.2.25)

$$- \tau = \Omega b - (1+\Omega)\mu$$

$$\tau = t_2 = (1+\Omega)(a+b)/2 - \Omega b + a/2$$

$$= a + b/2 + a\Omega/2 - b\Omega/2 - a/2$$

$$= a + (b-a)/2 - (b-a)\Omega/2$$

$$= a + (1-\Omega)(b-a)/2 \qquad (3.2.26)$$

α can be calculated in this case exactly the same way as was calculated in the case where $\Omega \leq 1$.

$$A(a) = - a \qquad (3.2.27)$$

$$A(\theta) = (1+\Omega)(\theta-a)^2/2(b-a)-\theta \qquad \text{where } a\leq\theta\leq b \qquad (3.2.28)$$

by definition $A(\theta) = A(a)$, (3.2.29)

$$(1+\Omega)(\theta-a)^2/2(b-a)-\theta = -a$$

$$(1+\Omega)(\theta-a)^2 - 2(b-a)(\theta-a) = 0 \qquad (3.2.30)$$

dividing both sides by $(\theta - a)$ and solving for $(\theta - a)$,

$$(\theta-a) = 2(b-a)/(1+\Omega)$$

$$\theta = w_1 = a + [2(b-a)/(1+\Omega)] \qquad (3.2.31)$$

$$k_1 = B(\theta) - B(a) = B(\theta), \text{ since } B(a) = 0$$

$$= (1+\Omega)(\theta-a)^4/4(b-a)^2 - (\theta-a)^3/3(b-a) \qquad (3.2.32)$$

where $\theta = w_1 = a + [2(b-a)/(1+\Omega)]$ (3.2.33)

substituting for θ in k_1,

$$k_1 = (1+\Omega)[2(b-a)/(1+\Omega)]^4/4(b-a)^2 - [2(b-a)/(1+\Omega)]^3/3(b-a)$$

$$= 4(b-a)^2/(1+\Omega)^3 - 8(b-a)^2/3(1+\Omega)^3$$

$$= 4(b-a)^2/3(1+\Omega)^3 \qquad (3.2.34)$$

$$k_2 = B(b) - B(\tau) \qquad (3.2.35)$$

where $B(b) = \Omega(b-\mu)^2 - \sigma^2$ (3.2.36)

and $B(\tau) = 0$ since $\tau \le a$ (3.2.37)

$$k_2 = \Omega[(b-a)/2]^2 - (b-a)^2/12$$

$$= (3\Omega-1)(b-a)^2/12 \qquad (3.2.38)$$

To find the optimal solution for t and q in the case of uniform distribution of lead-time is rather simple. The following table provides the formulas needed to solve for t^* and q^*. It can be simply observed that in the two cases $k \le k_1$, and $k \ge k_2$ we have been able to obtain explicit analytical expressions. In the case where $k_1 \le k \le k_2$, an analytical expression was not obtainaed. The solution of t^* and q^* can easily be obtained by a one dimensional search.

Table 3.2.2 Formulas for calculating decision variables

Uniform Distribution

$$\begin{cases} q^* = (1+\Omega)\sqrt{(k+\sigma^2)/\Omega} \\ t^* = \mu - \sqrt{\Omega(k+\sigma^2)} \end{cases} \qquad k_2 \le k$$

$$\begin{cases} (q^*)^2 - 2\delta(q^*)^{3/2}/3 = k(1+\Omega_m) \\ t^* = \begin{cases} a - q^* + \delta\sqrt{q^*} & \text{if } \Omega \ge 1 \\ b - \delta\sqrt{q^*} & \text{if } \Omega \ge 1 \end{cases} \end{cases} \qquad k_1 \le k \le k_2$$

$$\begin{cases} q^* = \sqrt[3]{6k(b-a)} \\ t^* = [(a\Omega + b)/(1+\Omega)] - q^*/2 \end{cases} \qquad k \le k_1$$

** Note $\delta = [2(b-a)/(1+\Omega)]^{1/2}$

The proof of the formulas given in Table 3.2.2. is given below:

When $k \ge k_2$, then $t^* \le a$ and $t^* + q^* \ge b$.

The proof of this case is already given in equations (3.1.42) and (3.1.46).

When $k \le k_1$, then $t^* \ge a$ and $t^* + q^* \le b$.

Using equation (3.2.1) for $A(x)$ and also equality (3.1.2), we have:

$$A(t^*+q^*) = (1+\Omega)(t^*+q^*-a)^2/2(b-a) - (t^*+q^*) \qquad (3.2.39)$$

$$A(t^*) = (1+\Omega)(t^*-a)^2/2(b-a) - t^* \qquad (3.2.40)$$

Knowing the relationship $A(t^*+q^*)=A(t^*)$, we have,

$$(1+\Omega)(t^*+q^*-a)^2/2(b-a) - (t^*+q^*) =$$

$$(1+\Omega)(t^*-a)^2/2(b-a) - t^*$$

$$[(1+\Omega)/2(b-a)][(t^*-a)^2+q^{*2}+2q^*(t^*-a)-(t^*-a)^2] = q^*$$

$$[(1+\Omega)/2(b-a)][q^* + 2t^* - 2a] = 1$$

$$t^* = (b-a)/(1+\Omega) + a - q^*/2$$

$$= (a\Omega+b)/(1+\Omega) - q^*/2 \qquad (3.2.41)$$

Using equation (3.2.2) for $B(x)$ and also equality (3.1.3), we have,

$$B(t^*+q^*) = [(1+\Omega)(t^*+q^*-a)^4/4(b-a)^2$$

$$-(t^*+q^*-a)^3/3(b-a)] \qquad (3.2.42)$$

$$B(t^*) = [(1+\Omega)(t^*-a)^4/4(b-a)^2-(t^*-a)^3/3(b-a)]$$

$$(3.2.43)$$

let $w = (t^*+q^*-a)/(b-a)$ (3.2.44)

and $z = (t^*-a)/(b-a)$ (3.2.45)

then $w+z = 2/(1+\Omega)$ (3.2.46)

and $w-z = q^*/(b-a)$ (3.2.47)

Substituting for $B(t^*+q^*)$ and $B(t^*)$ in (3.1.3), and using w and z,

$$(1+\Omega)w^4/4 - w^3/3 - (1+\Omega)z^4/4 + z^3/3 = k/(b-a)^2$$

$$(1+\Omega)(w^4-z^4)/4 - w^3/3 + z^3/3 = k/(b-a)^2$$

$$(1+\Omega)(w+z)(w^3-zw^2+z^2w-z^3)/4-w^3/3+z^3/3 = k/(b-a)^2$$

Substituting $2/(1+\Omega)$ for $(w+z)$ and multiplying both sides by 6,

$$3w^3 - 3zw^2 + 3z^2w - 3z^3 - 2w^3 + 2z^3 = 6k(b-a)$$

$$w^3 - z^3 + 3z^2w - 3zw^2 = 6k/(b-a)^2$$

$$(w-z)^3 = 6k/(b-a)^2$$

Substituting $q^*/(b-a)$ for $(w-z)$,

$$[q^*/(b-a)]^3 = 6k(b-a)^2$$

$$q^* = \sqrt[3]{6k(b-a)} \qquad (3.2.48)$$

When $k_1 \leq k \leq k_2$, then there are two possibilities,

i) $t^* \geq a$ and $t^*+q^* \geq b$ if $\Omega \leq 1$ (3.2.49)

ii) $t^* \geq a$ and $t^*+q^* \leq b$ if $\Omega \geq 1$ (3.2.50)

Again, let us define:

$$w = v+z = (t^*+q^*-a)/(b-a) \text{ and} \qquad (3.2.51)$$

$$z = 1-y = (t^*-a)/(b-a) \text{ and} \qquad (3.2.52)$$

$$v = w-z = q^*/(b-a) \text{ and} \qquad (3.2.53)$$

$$y = 1-z \qquad (3.2.54)$$

and as before use equations for $A(x)$ and $B(x)$ for the appropriate range of uniform distribution:

i) When $t^* \geq a$ and $t^*+q^* \geq b$,

$$A(t^*+q^*) = \Omega(t^*+q^*)-(1+\Omega)\mu \quad \text{since } t^*+q^* \geq b \qquad (3.2.55)$$

$$A(t^*) = (1+\Omega)(t^*-a)^2/2(b-a) - t^* \qquad (3.2.56)$$

Let $A(t^*+q^*) = A(t^*)$

$$\Omega(t^*+q^*)-(1+\Omega)\mu = (1+\Omega)(t^*-a)^2/2(b-a)-t^*$$

$$\Omega(t^*+q^*-a)+a\Omega-(a/2+b/2)-\Omega(a/2+b/2)=$$
$$(1+\Omega)(t^*-a)^2/2(b-a)-(t^*-a)-a$$

$$\Omega(t^*+q^*-a)+\Omega(a-b)/2+(a-b)/2=$$
$$(1+\Omega)(t^*-a)^2/2(b-a)-(t^*-a)$$

dividing both sides by $(b-a)$ and substituting,

$$\Omega w - (1+\Omega)/2 = (1+\Omega)z^2/2 - z$$

$$w = (1+\Omega)(1+z^2)/2\Omega - z/\Omega \tag{3.2.57}$$

$$v = w-z = [(1+\Omega)(1+z^2)/2\Omega - z/\Omega] - z$$

$$= (1+\Omega)(z-1)^2/2\Omega \tag{3.2.58}$$

$$B(t^*+q^*)= \Omega[(t^*+q^*) - \mu]^2 - \sigma^2 \tag{3.2.59}$$

$$B(t^*)=(1+\Omega)(t^*-a)^4/4(b-a)^2-(t^*-a)^3/3(b-a) \tag{3.2.60}$$

$$\text{Let } B(t^*)= B(t^*+q^*) - k \tag{3.2.61}$$

$$(1+\Omega)(t^*-a)^4/4(b-a)^2-(t^*-a)^3/3(b-a)$$

$$=\Omega[(t^*+q^*)-(a+b)/2]^2-(b-a)^2/12 - k$$

$$(1+\Omega)z^4/4-z^3/3$$

$$=\Omega[(t^*+q^*-a)+(a-b)/2]^2/(b-a)^2-1/12-k/(b-a)^2$$

$$=\Omega[(v+z)-1/2]^2 - 1/12 - k/(b-a)^2$$

$$=(1/\Omega)[(1+\Omega)(z^2+1-2z)/2+z\Omega-\Omega/2]^2-1/12-k/(b-a)^2$$

$$-(1+\Omega)z^4/4+z^3/3+(1/\Omega)[(1+\Omega)z^2/2-z+1/2]^2-1/12$$

$$= k/(b-a)^2$$

$$(1/\Omega)[-(1+\Omega)\Omega z^4/4+(1+\Omega)^2z^4/4+\Omega z^3/3-(1+\Omega)z^3+z^2$$

$$+(1+\Omega)z^2/2-z+1/4]-1/12 = k/(b-a)^2$$

$$(1/\Omega)[(1+\Omega)z^4/4-(3+2\Omega)z^3/3+(3+\Omega)z^2/2-z]+1/4\Omega-1/12$$

$$= k/(b-a)^2$$

Substitute y for (1−z),

$$(1/\Omega)[(1+\Omega)(1-4y+6y^2-4y^3+y^4)/4-(3+2\Omega)(1-3y+3y^2-y^3)/$$

$$3+(3+\Omega)(1-2y+y^2)/2-(1-y)]+ 1/4\Omega- 1/12 = k/(b-a)^2$$

$$(1+\Omega)\{(1+\Omega)y^4/4-[(1+\Omega)-1-2\Omega/3]y^3 +[3(1+\Omega)/2-(3+2\Omega)+$$

$$(3+\Omega)/2]y^2+[-(1+\Omega)+(3+2\Omega)-(3+\Omega)+1]y+$$

$$[(1+\Omega)/4-(3+2\Omega)/3+(3+\Omega)/2-1+1/4-\Omega/12]\}= k/(b-a)^2$$

$(1+\Omega)y^4/4\Omega - y^3/3 = k/(b-a)^2$

Substitute $\sqrt{2\Omega v/(1+\Omega)}$ for y,

$(1+\Omega)[2\Omega v/(1+\Omega)]^2 - 2\Omega v\sqrt{2\Omega v(1+\Omega)}/3(1+\Omega) = k/(b-a)^2$

$\Omega v^2/(1+\Omega) - 2\Omega v\sqrt{2\Omega v/(1+\Omega)}/3(1+\Omega)=k/(b-a)^2$

Substitute $q^*/(b-a)$ for v,

$\Omega q^{*2}/(1+\Omega)(b-a)^2$

$-2\Omega q^* \sqrt{2\Omega q^*/(1+\Omega)(b-a)}/3(1+\Omega)(b-a)$

$= k/(b-a)^2$

Multiply both sides by $(1+\Omega)(b-a)^2/\Omega$,

$q^{*2}-2(b-a)q^* \sqrt{2\Omega q^*/(1+\Omega)(b-a)}/3 = (1+\Omega)k/\Omega$

$q^{*2}-2[2\Omega(b-a)/(1+\Omega]^{1/2}q^{*3/2}/3=[(1+\Omega)/\Omega]k \quad (3.2.62)$

Let $\delta = [2(b-a)/(1+\Omega_m)]^{1/2} \quad (3.2.63)$

Where $\Omega_m = \max(\Omega, 1/\Omega) \quad (3.2.64)$

$q^{*2} - 2\delta q^{*3/2}/3 = k(1+\Omega_m) \quad (3.2.65)$

Now, we can easily obtain t^* in terms of q^* as follows:

From equation (3.2.58),

$v = (1+\Omega)(z-1)^2/2\Omega$ where $v = q^*/(b-a)$,

$z = (t^*-a)/(b-a) \quad (3.2.66)$

$q^*/(b-a) = (1+\Omega)[(t^*-a)/(b-a) - 1]^2$

$= [(1+\Omega)/2\Omega][(t^*-a)/(b-a)]^2 \quad (3.2.67)$

$[2(b-a)/(1+1/\Omega)]q^* = (t^*-b)^2$

$\delta^2 q^* = (t^*-b)^2$

$$t^* = b - \delta\sqrt{q^*} \tag{3.2.68}$$

Similarly, but more simply, similar results can be obtained for the following range:

ii) <u>When $t^* \leq a$ and $t^*+q^* \leq b$</u>

$$A(t^*) = -t^* \tag{3.2.69}$$

$$A(t^*+q^*)=(1+\Omega)(t^*+q^*-a)^2/2(b-a)-(t^*+q^*) \tag{3.2.70}$$

Equating $A(t^*+q^*)$ with $A(t^*)$, we get,

$$(1+\Omega)(t^*+q^*-a)^2/2(b-a)-(t^*+q^*) = -t^* \tag{3.2.71}$$

$$[(1+\Omega)/2]w^2 = q^*/(b-a) \tag{3.2.72}$$

$$[(1+\Omega)/2]w^2 = w - z = v \tag{3.2.73}$$

$$w = \sqrt{2v/(1+\Omega)} \tag{3.2.74}$$

$$B(t^*+q^*)=(1+\Omega)(t^*+q^*-a)^4/4(b-a)^2$$

$$-(t^*+q^*-a)^3/3(b-a) \tag{3.2.75}$$

$$B(t^*) = 0 \tag{3.2.76}$$

Using the relationship $B(t^*+q^*) - B(t^*) = k$

$$(1+\Omega)(t^*+q^*-a)^4/4(b-a)^2-(t^*+q^*-a)^3/3(b-a)=k \tag{3.2.77}$$

$$(1+\Omega)w^4/4 - w^3/3 = k/(b-a)^2 \tag{3.2.78}$$

Substituting $\sqrt{2v/(1+\Omega)}$ for w

$$(1+\Omega)[2v/(1+\Omega)]^2/4-2v(1+\Omega)\sqrt{2v/(1+\Omega)}/3$$

$$= k/(b-a)^2 \tag{3.2.79}$$

$$[v/(1+\Omega)][v-2\sqrt{2v/(1+\Omega)}/3] = k/(b-a)^2$$

$$v^2/(1+\Omega)-2v\sqrt{2v/(1+\Omega)/3}(1+\Omega) = k(b-a)^2 \tag{3.2.80}$$

Substituting $q^*/(b-a)$ for v.

$$q^{*2}/(1+\Omega)(b-a)^2-2q^*\sqrt{2q^*/(1+\Omega)(b-a)}/3(1+\Omega)(b-a)$$

$$= k/(b-a)^2$$

Multiplying both sides by $(1+\Omega)(b-a)^2$

$$q^{*2}-2[2(b-a)/(1+\Omega)]^{1/2}q^{*3/2}/3 = k(1+\Omega)$$

$$q^* - 2\delta q^{*3/2}/3 = k(1+\Omega_m) \qquad (3.2.81)$$

For this case t^* as a function of q^* can be obtained as follows:

From equation (3.2.71), we have

$$(1+\Omega)(t^*+q^*-a)^2/2(b-a) = q^*$$

$$t^*+q^* - a = [2(b-a)/(1+\Omega)]^{1/2}q^{*1/2}$$

$$t^* = a - q^* + \delta\sqrt{q^*} \qquad (3.2.82)$$

As can be seen, a single equation in terms of q^* can be obtained for both cases $\Omega \leq 1$ and $\Omega \geq 1$.

Table (3.2.3) indicates which model should be used to calculate t^* and q^*. Depending on the value of $k = 2K/(h+p)D$, we can select the right model. A similar table may be obtained that will guide us to determine the right model based on the range of lead-time (b-a).

The proof of the ranges in Table (3.2.3) is given below:

From Table (3.2.1), we have,

$$k_1 = 4\Omega^3(b-a)^2/3(1+\Omega)^3 \quad \text{when } \Omega \leq 1 \qquad (3.2.83)$$

and $k_1 = 4(b-a)^2/3(1+\Omega)^3$ when $\Omega \geq 1$ (3.2.84)

which can be combined to form one equation,

$$k_1 \leq 4(b-a)^2/3(1+\Omega_m)^3 \quad \text{where } \Omega_m = \max(\Omega, 1/\Omega) \tag{3.2.85}$$

when $k \leq k_1$, substituting for k_1

$$k \geq 4(b-a)^3/3(1+\Omega_m)^3$$

$$4(b-a)^3 \geq 3k(1+\Omega_m)^3$$

$$b - a \geq \sqrt[3]{3k(1+\Omega_m)^3/4} \tag{3.2.86}$$

Similarly, from Table (3.2.1), we have

$$k_2 = (3-\Omega)(b-a)^2/12\Omega \quad \text{when } \Omega \leq 1 \tag{3.2.87}$$

and $k_2 = (3\Omega-1)(b-a)^2/12$ when $\Omega \geq 1$ (3.2.88)

which can be combined to form one equation,

$$k_2 = (3\Omega_m-1)(b-a)^2/12 \tag{3.2.89}$$

when $k \geq k_2$, substituting for k_2,

$$k \geq (3\Omega_m-1)(b-a)^2/12$$

$$b - a \leq \sqrt{12k(3\Omega_m-1)} \tag{3.2.90}$$

Table 3.2.3 Formulas For Calculating t^* And q^* Based On The Range Of Uniform Lead-Time Distribution

MODEL 1	$k_2 \leq k$	$b - a \leq \sqrt{\dfrac{12k}{3\Omega_m - 1}}$	$q^* = (1 + \Omega)\sqrt{(k + \sigma^2)/\Omega}$ ** $t^* = \mu - \sqrt{\Omega(k + \sigma^2)}$
MODEL 2	$k_1 \leq k \leq k_2$	$\sqrt{\dfrac{12k}{3\Omega_m - 1}} \leq b - a \leq \sqrt{\dfrac{3k(1+\Omega_m)^3}{4}}$	$(q^*)^2 - \dfrac{2}{3}\delta(q^*)^{3/2} = k(1 + \Omega_m)$ $t^* = \begin{cases} a - q^* + \delta\sqrt{q^*} & \text{if } \Omega \geq 1 \\ b - \delta\sqrt{q^*} & \text{if } \Omega \leq 1 \end{cases}$ where $\delta = [2(b-a)/(1+\Omega)]^{1/2}$
MODEL 3	$k \leq k_1$	$\sqrt{\dfrac{3k(1+\Omega_m)^3}{4}} \leq b - a$	$q^* = \sqrt[3]{6k(b-a)}$ $t^* = [(a\Omega + b)/(1 + \Omega)] - \dfrac{q^*}{2}$

** This model applies to any lead-time distribution. It is not limited to uniform lead-time distribution like other models.

It is worth while to analyze the behavior of optimal cycle length q^* (or $Q^* = Dq^*$) as the range of lead-time distribution, $b - a$, changes. This analysis is based on results shown in Table (3.2.3).

Figure (3.2.1), which expresses the relationship between q^* and $(b-a)$, is given below. Within the range $0 - \sqrt{12k/(3\Omega_m - 1)}$, the function is increasing and convex. For the second range of lead-time, i.e., between $\sqrt{12k/(3\Omega_m - 1)}$ and $\sqrt{3k(1+\Omega_m)^3/4}$, the function is still increasing but concave. For the last range of lead-time, i.e., greater than $\sqrt{3k(1 + \Omega_m)^3/4}$, the function remains increasing and concave. It should also be noted that the Y-intercept is Basic EOQ with back-logging of demand.

The proof of the shape of $q^* = f(b-a)$ is given below. For all three ranges, it is proved that the first derivative is positive. It is also proved that in the range of model 1, the second derivative is positive and is negative for the ranges for models 2 and 3.

When model 1 applies,

$$q^* = (1+\Omega)\sqrt{(k+\sigma^2)/\Omega} \qquad (3.2.91)$$

$$= (1+\Omega)\{[12k+(b-a)^2]/12\Omega\}^{1/2} \qquad (3.2.92)$$

$$(dq^*)/d(b-a) = (1/2)(1+\Omega)\cdot$$

$$\{[12k+(b-a)^2]/12\Omega\}^{-1/2}(b-a)/6\Omega$$

$$= (1+\Omega)(b-a)\{12\Omega[12k+(b-a)^2]\}^{-1/2} \geq 0$$

(3.2.93)

$$(d^2q^*)/d(b-a)^2 = (-1/2)(1+\Omega)(b-a)\{12\Omega[12k + (b-a)^2]\}^{-3/2}[24\Omega(b-a)]$$

$$= -12\Omega(1+\Omega)(b-a)^2\{12\Omega[12k + (b-a)^2\}^{-3/2} \le 0 \qquad (3.2.94)$$

When model 2 applies, q^* can not be obtained explicitly as a function of b − a. So instead, (b−a) as a function of q^* is obtained and differentiated. Its inverse is then analyzed,

$$q^{*2} - 2\delta q^{*3/2}/3 = k(1+\Omega_m) \qquad (3.2.95)$$

where $\delta = [2(b-a)/(1+\Omega_m)]^{1/2}$ (3.2.96)

Let $A = 1+\Omega_m$ (3.2.97)

$$q^{*2} - 2[2(b-a)/A]^{1/2}q^{*3/2} = kA \qquad (3.2.98)$$

$$q^{*2} - kA = 2[2(b-a)/A]^{1/2}q^{*3/2}/3 \qquad (3.2.99)$$

$$(q^{*2}-kA)^2 = 8(b-a)q^{*3}/9A \qquad (3.2.100)$$

$$(b-a) = 9A(q^{*2}-kA)^2/8q^{*3} \qquad (3.2.101)$$

$$[d(b-a)]/dq^* = [18Aq^*(q^{*2}-kA)-27A(q^{*2}-ka)]/8q^{*4}$$

$$=[9A(q^{*2}-kA)/2q^{*2}]-[27A(q^{*2}-kA)^2/8q^{*4}]$$

$$= [9A(q^{*2}-kA)/2q^{*2}][1-3(q^{*2}-kA)/4q^{*2}]$$

$$= [9A(q^{*2}-kA)/2q^{*2}][(q^{*2}+3kA)/4q^{*2}]$$

$$= 9A(q^{*2}-kA)(q^{*2}+3kA)/8q^{*4} \ge 0$$

(3.2.102)

Note: It can easily be inferred from equation (3.2.99) that $(q^{*2}-kA) \ge 0$.

$$[d(b-a)/dq^*] = [9A(9q^{*2}+3kA)8q^{*4}](q^{*2}-kA) \qquad (3.2.103)$$

$$\begin{aligned} d^2(b-a)/dq^{*2} = {} & \{[(18Aq^*)(8q^{*4})-32q^{*3}(9A)(q^{*2} \\ & +3kA)]/64q^{*8}\}(q^{*2}-kA) \\ & +2q^*(9A)(q^{*2}+3kA))/8q^{*4} \\ = {} & (144Aq^{*3}/64q^{*8})[q^{*2}-2(q^{*2}+3kA)](q^{*2} \\ & -kA)+9A(q^{*2}+3kA)/4q^{*3} \\ = {} & -(9A/4q^{*5})(q^{*2}+6kA)(q^{*2}-kA) \\ & +(9A/4q^{*3})(q^*+3kA) \\ = {} & (9A/4q^{*3})[(q^*+3kA) \\ & -(1/q^{*2})(q^{*2}+6kA)(q^{*2}-kA)] \\ = {} & (9A/4q^{*3})[(q^*+3kA)-(1/q^{*2})(q^{*4}+5kAq^{*2} \\ & -6k^2A^2)] \\ = {} & (9A/4q^{*3})[-q^{*2}+q^*-2kA+(6k^2A^2/q^{*2})] \end{aligned} \tag{3.2.104}$$

To prove that the second part of equation (3.2.104) is positive, we use the relationship $(q^{*2}-kA) \geq 0$ that is implied from equation (3.2.99) and show that the following equation holds:

$$-q^{*2} + q^* - 2kA + (6k^2A^2/q^{*2}) \geq 0$$

Substitute kA for q^{*2},

$$-kA + \sqrt{kA} - 2kA + 6kA \geq 0$$

$$3kA + \sqrt{kA} \geq 0 \tag{3.2.105}$$

Now, since both parts of equation (3.2.104) are positive, therefore,

$$d^2(b-a)/dq^{*2} \geq 0$$

And its inverse.

$d^2q^*/d(b-a)^2 \leq 0$

Therefore (q^*) is a concave and increasing function of $(b-a)$.

When model 3 applies,

$q^* = [6k(b-a)]^{1/3}$

$dq^*/d(b-a) = 2k[6k(b-a)]^{-2/3} \geq 0$

$d^2q^*/d(b-a)^2 = -8k[6k(b-a)]^{-5/3} \leq 0$

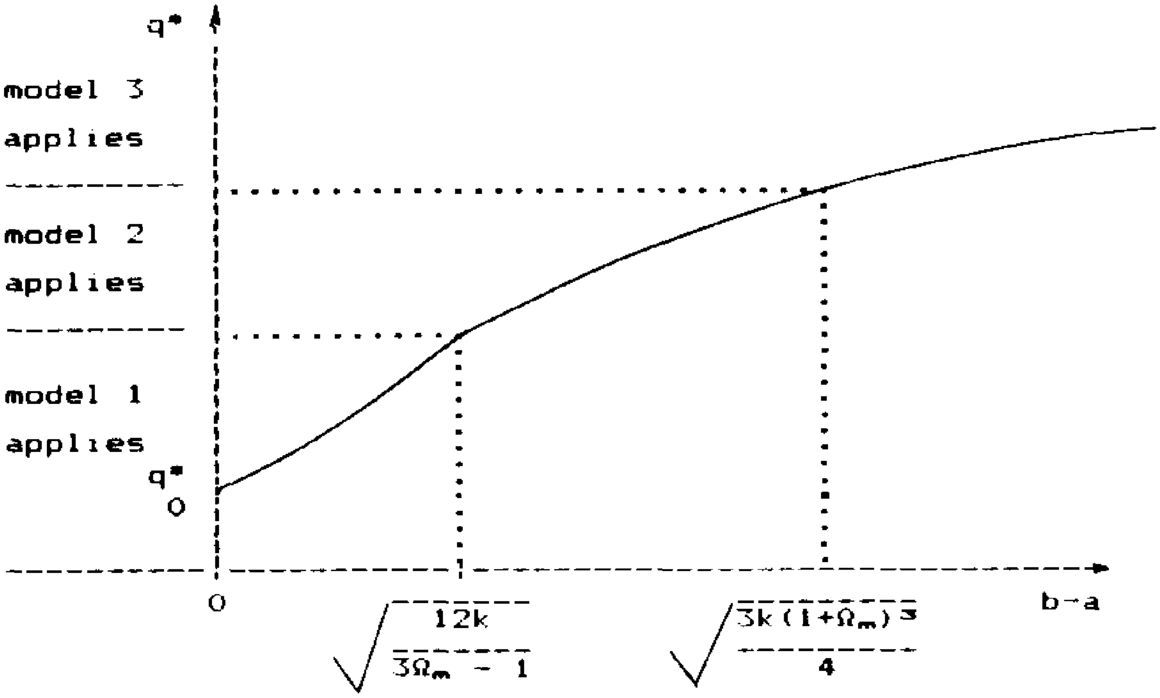


Figure 3.2.1 q^* as a function of b-a

CHAPTER IV

FURTHER ANALYTICAL RESULTS

In this chapter of the dissertation, additional very important analytical results are discussed.

In section 4.1, we obtain the ranges for which cross over may or may not occur.

In section 4.2, we develop explicit cost formulas for different ranges of lead-time distribution.

In sections 4.3 and 4.4, the behavior and shape of total cost and cycle time as a function of range of lead-time distribution is graphically and analytically analyzed.

In section 4.5, the sensitivity of decision variables and total optimal cost to lower bound of lead-time distribution is analyzed.

Finally, in section 4.6, we obtain an analytical expression for obtaining the probability of cross over of two subsequent orders. The results are illustrated for the case of uniform lead-time distribution.

4.1 CROSS OVER AND THE ASSUMPTION OF NON-INTERCHANGEABILITY

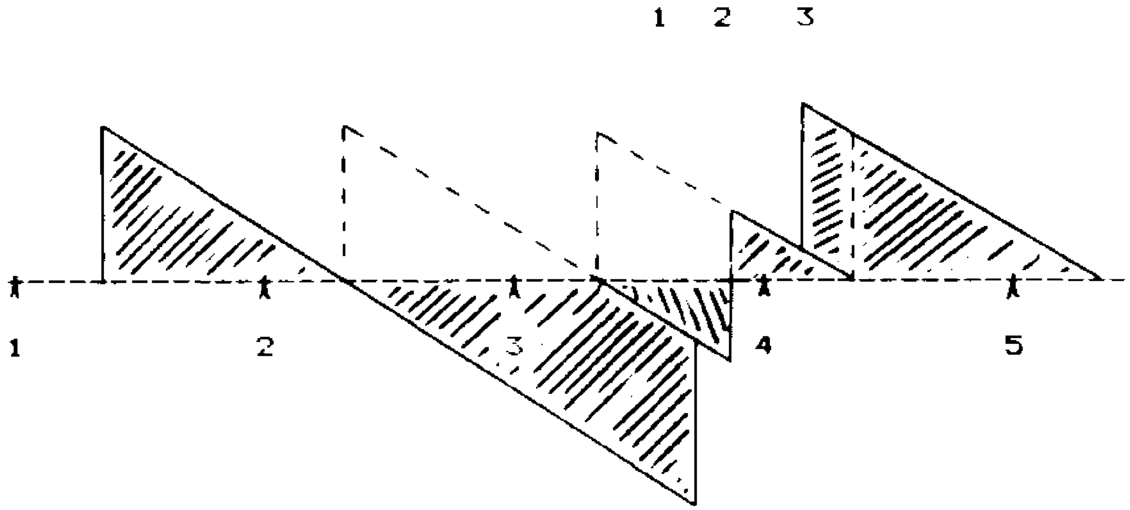
As we discussed in the introduction to this dissertation, cross over occurs if and only if an order placed later arrives before a previously placed order. In this paper, not only we allow for cross over but we also assume that unit demands are non-interchangeable. By non-interchangeability, we mean that each order is a special order, i.e., each order can satisfy a particular group of customers.

To understand the two concepts of cross over and non-interchangeability, the reader should refer to the following four figures.

The top figures indicate the inventory levels over time, when the assumption of non-interchangeability of demand is made. But, the bottom figures shows the inventory levels over time, when the conventional assumption of interchangeable items is made.

It should be noted that, in figures 4.1.1, and 4.1.2 the assumption of non-interchangeability is immaterial. This is due to the fact that, in these two figures orders arrive in sequence.

I. Unit demands are non-interchangeable



II. Unit demands are interchangeable

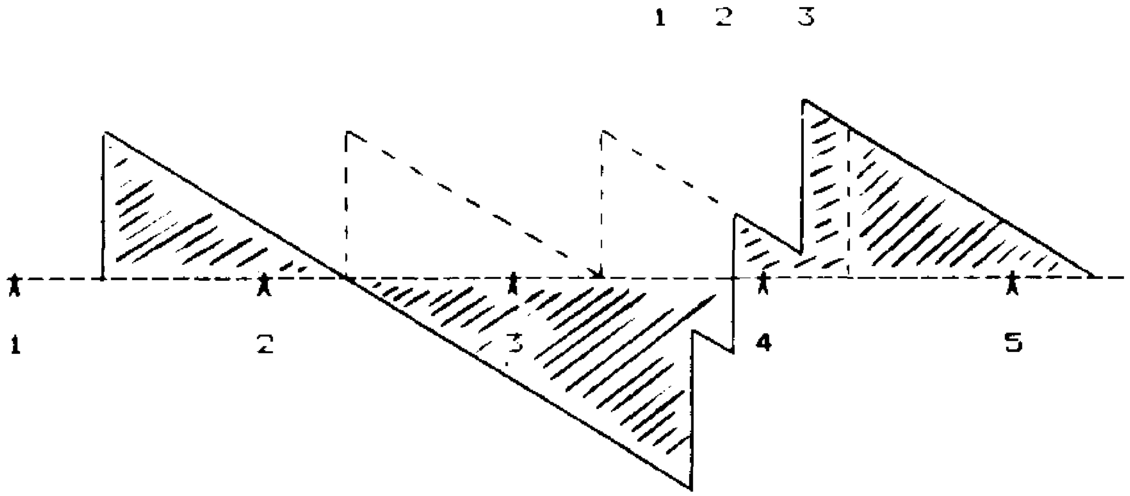
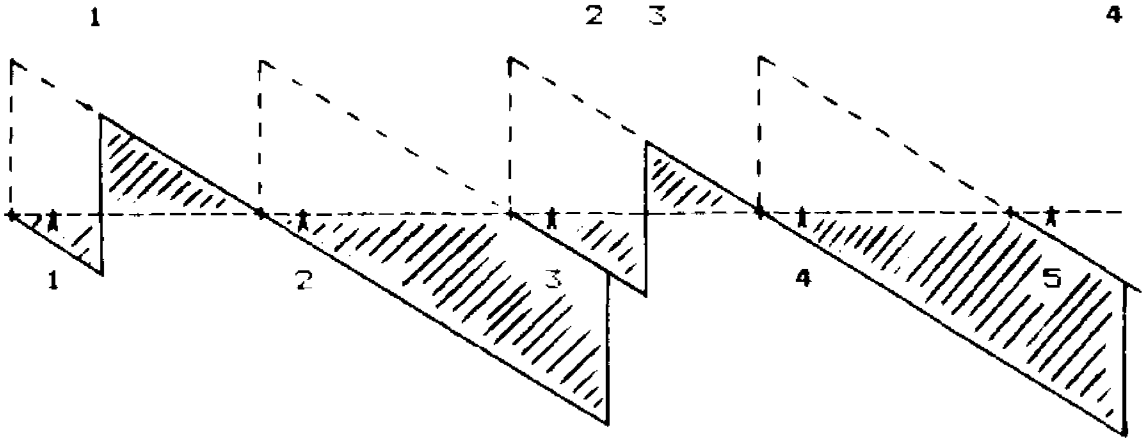


Figure 4.1.1 Comparison of the effect of the assump- of non-interchangeability when orders arrive in sequence and t is positive.

I. Unit demands are non-interchangeable



II. Unit demands are interchangeable

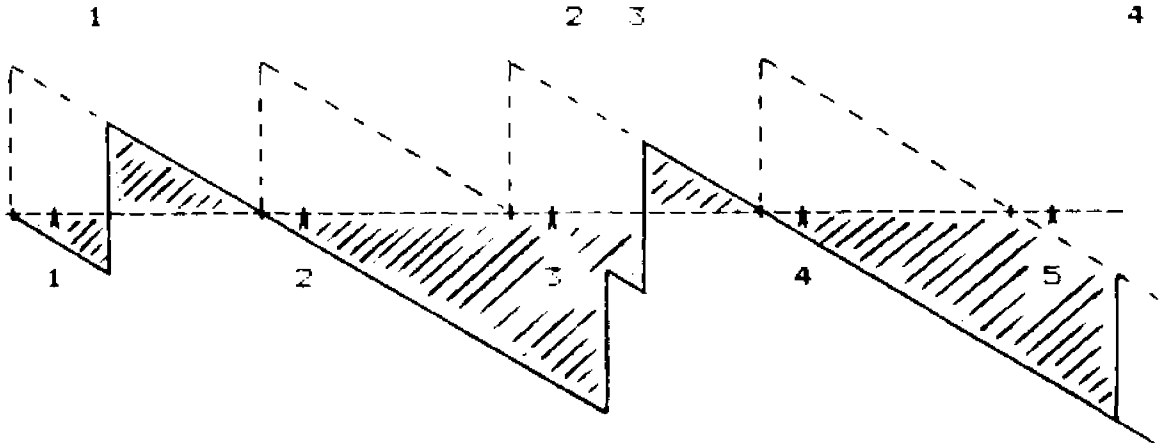
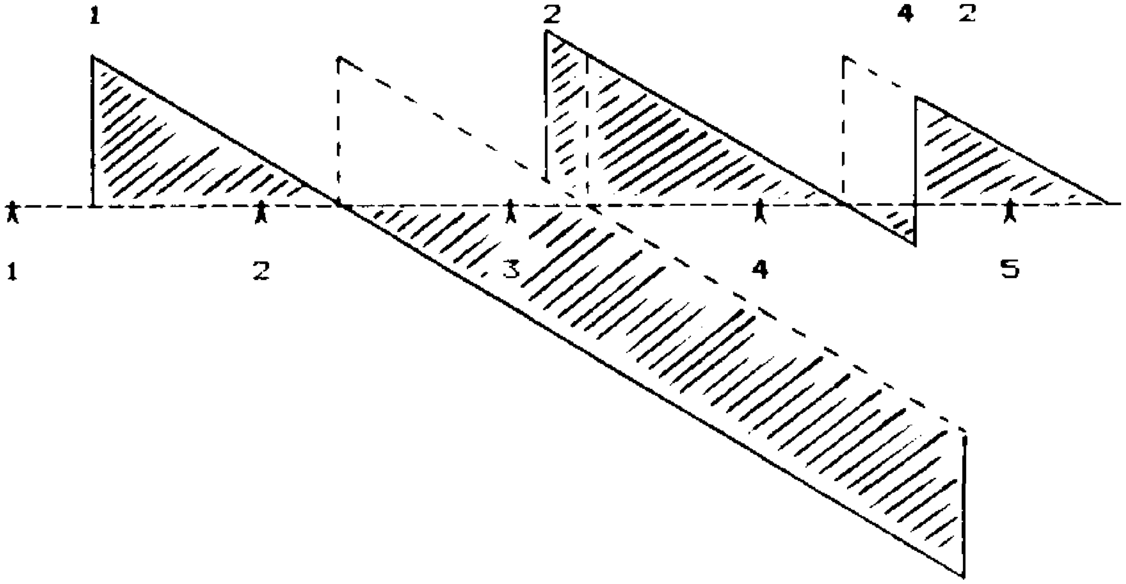


Figure 4.1.2 Comparison of the effect of the assumption of non-interchangeability when orders arrive in sequence and t is negative.

I. Unit demands are non-interchangeable



II. Unit demands are interchangeable

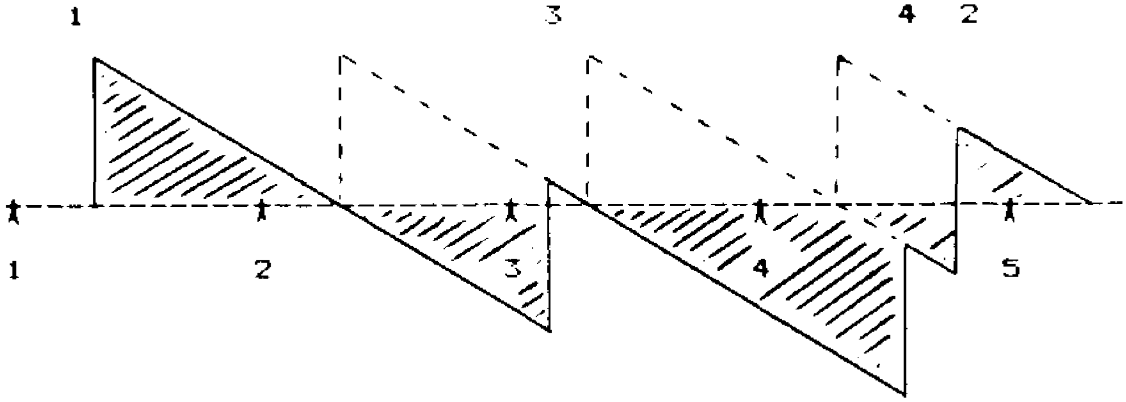
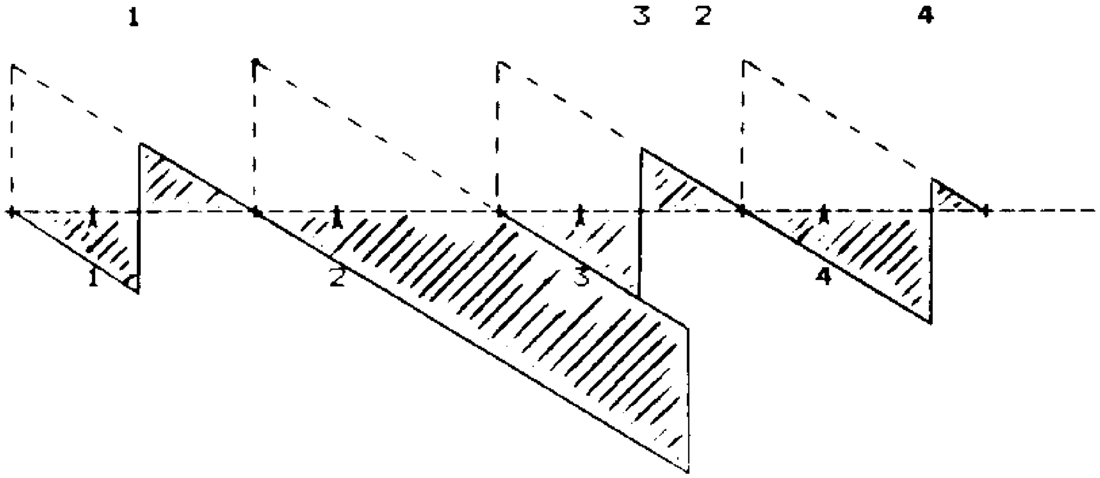


Figure 4.1.3 Comparison of the effect of the assumption of non-interchangeability when orders do *not* arrive in sequence (*i.e.*, orders may cross) and t is positive.

I. Unit demands are non-interchangeable



II. Unit demands are interchangeable

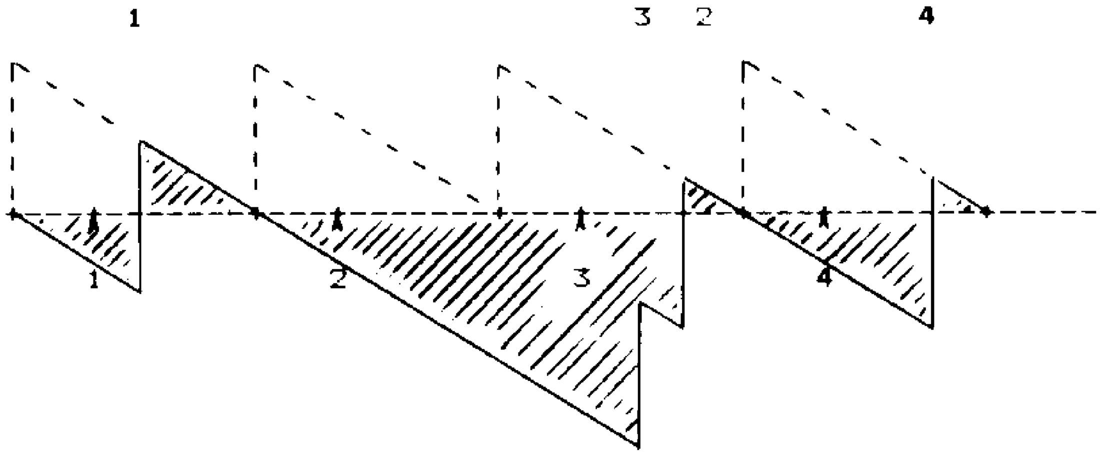


Figure 4.1.4 Comparison of the effect of the assumption of non-interchangeability when orders do not arrive in sequence (i.e., orders may cross) and t is negative.

Figures 4.1.1 and 4.1.2. are indicative of the fact that if orders do not cross, the assumption of non-interchangeability is immaterial. Therefore, our model is better than other existing models because it is exact rather than heuristic.

Knowing that there are only four possible ranges (shown below) in which a, b, t^*, t^*+q^* may occur we establish the following theorem.

THEOREM 4.1.1 The cross over may occur if and only if the range of the lead-time distribution, b-a, is greater than the optimal cycle time, q^*.

PROOF Four possible ranges exist for t^* and t^*+q^*. For each possible range we prove that b-a should be greater than q^* in order for cross over to occur:

Case 1 $t^* \leq a$ and $t^*+q^* \geq b$

Here we can distinguish two possible conditions:

i) t^* is positive and less than a,

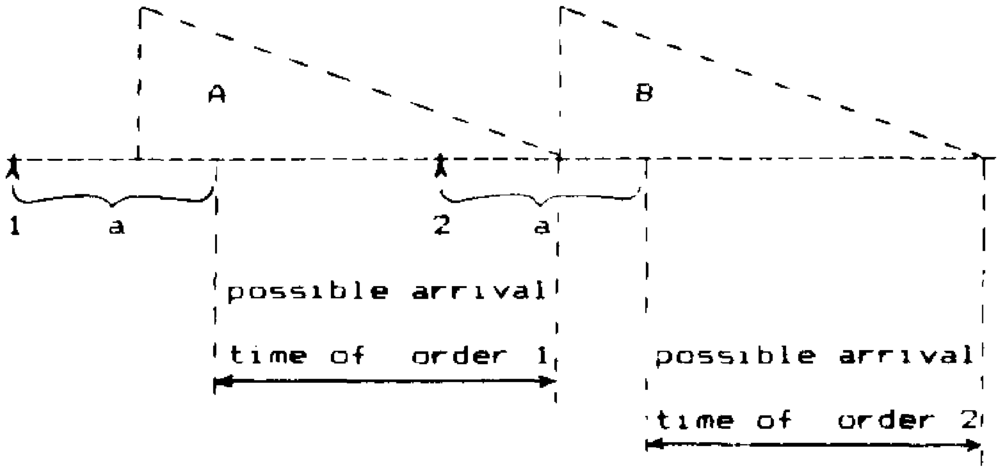


Figure 4.1.5 Possibility of cross over

when $a \geq t^* > 0$ and $t^*+q^* \geq b$.

For orders to cross, order 1 should arrive after point B, which is impossible because, $t^*+q^*\geq b$ is going to be violated. Therefore, in this case there is no possibility of cross over. The same argument is true when:

ii) t^* is negative and less than a,

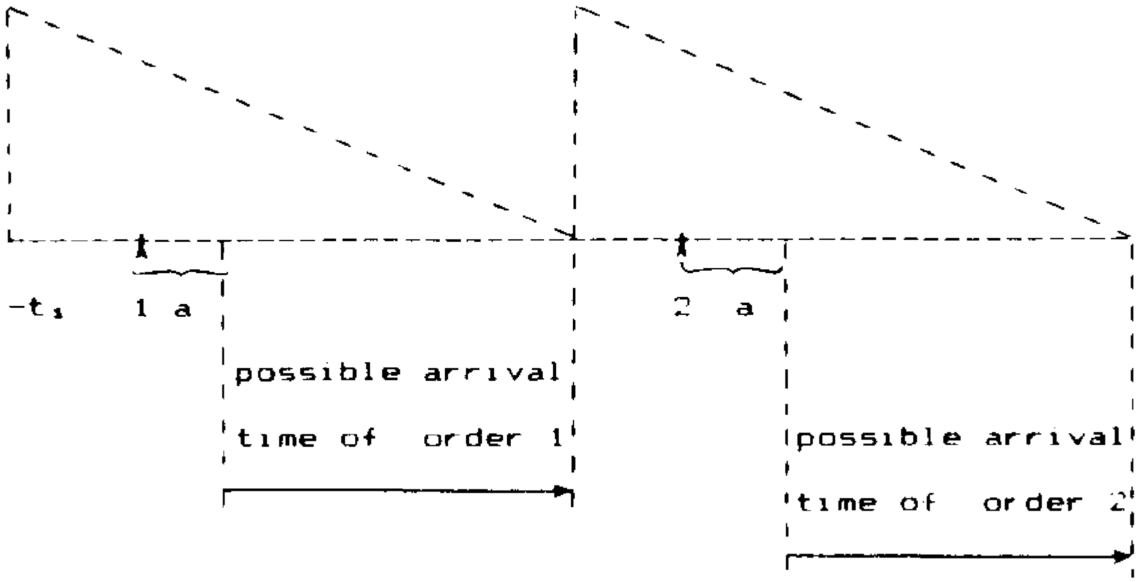


Figure 4.1.6 Possibility of cross over when $0\geq t^*\geq a$ and $t^*+q^*\geq b$.

Therefore, for the range to which model 1 applies there is no possibility of cross over.

<u>Case 2a</u> $t^* \leq a$ and $t^*+q^* \leq b$

Again, since "a" is both positive and greater than zero, we face two possibilities:

i) t^* is positive and less than a,

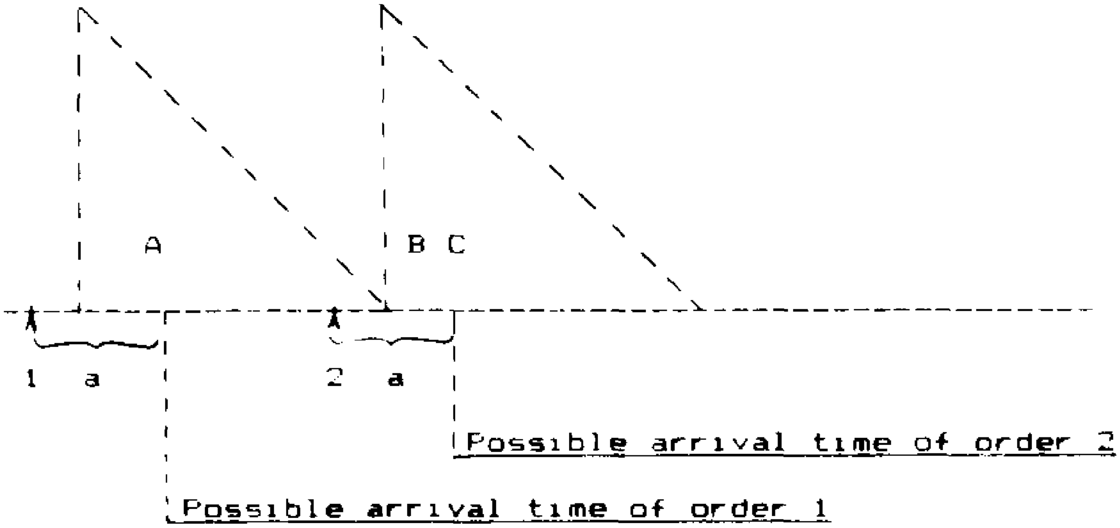


Figure 4.1.7 Possibility of cross over when $0 \leq t^* \leq a$ and $t^*+q^* \leq b$.

In this case possibility of cross over exists if the arrival of order 1 is after time C, mathematically,

$$a + q^* > b$$

or

$$b - a < q^*$$

ii) t^* is negative and less than a,

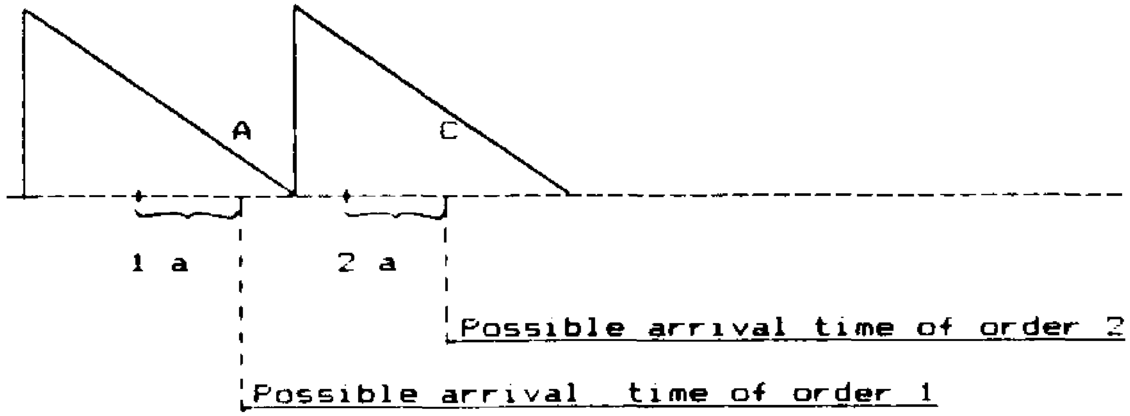


Figure 4.1.8 Possibility of cross over when $0 \geq t^* \leq a$ and $t^*+q^* \leq b$.

For orders to cross each other, the following should hold:

$$a + q^* < b$$

or

$$b - a > q^*$$

Case 2b $t^* \geq a$ and $t^*+q^* \geq b$

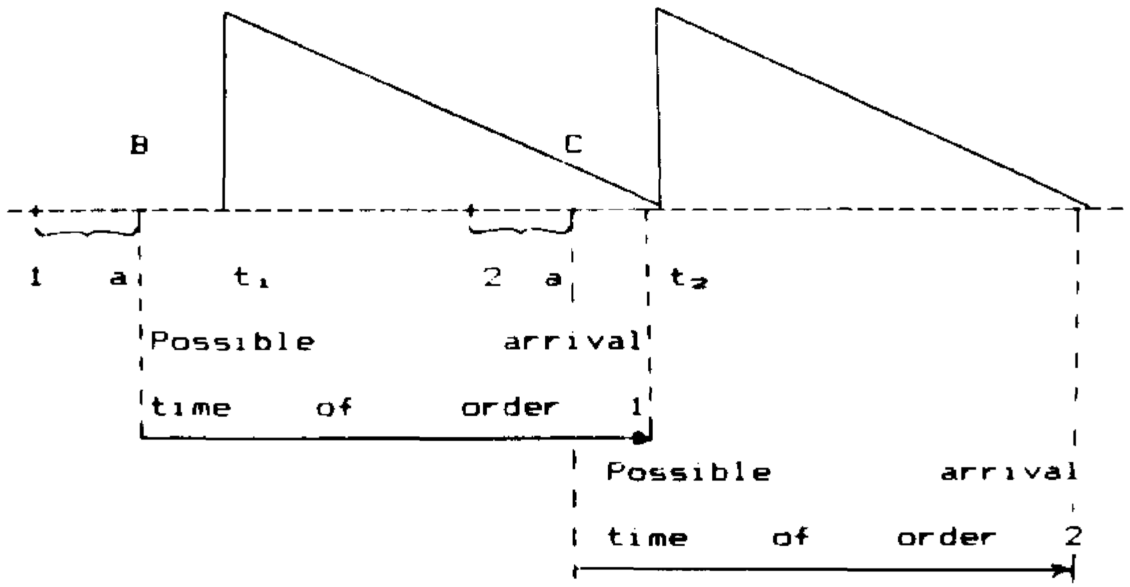


Figure 4.1.9 Possibility of cross over when $t^* \geq a$ and $t^*+q^* \geq b$.

In this case cross over also occurs if and only if $a+q^* < b$ or $b-a > q^*$.

Note: In case "b" is between B and C, $t^*+q^* \leq b$ is satisfied but orders do not cross. But if b is located after point C, then there is a possibility of cross-over.

And, finally:

Case 3 $t^* \geq a$ and $t^*+q^* \leq b$

Since "a" is always positive, this case applies only when t^* is positive, graphically:

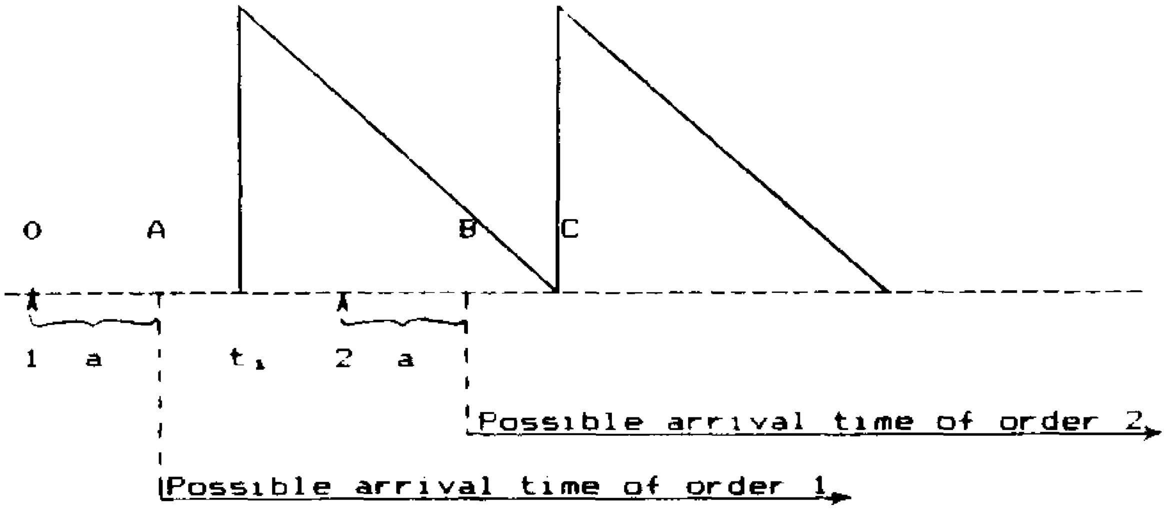


Figure 4.1.10 Possibility of cross over when $t^* \geq a$ and $t^*+q^* \leq b$.

According to Figure 4.1.10, there is a possibility of cross over as long as arrival time of order 1 is after point B. Since $t^*+q^* \le b$, i.e., b is located after point C, there is always a possibility of cross over in this case.

Therefore, the theorem is proved for all cases. The theorem also showed us that within the range in which model 1 applies, cross over can never occur. In the range in which model 2 applies (2a and 2b), if b is rather small there is never a possibility of cross-over. On the other hand, for large "b" there is a possibility of cross over. As described before, for the range in which model 3 applies there is always a possibility for cross over.

The above analysis leads us to the second theorem:

THEOREM 4.1.2 Given the parameters of the problem, Ω_m and $k = 2K/(h+p)D$, there is a possibility of cross-over if and only if:

$$q^* > \sqrt{\frac{k(1+\Omega_m)}{1 - \frac{2}{3}\sqrt{\frac{2}{1+\Omega_m}}}} \qquad (4.1.1)$$

PROOF: As we showed in Theorem 4.1.1, the possibility of cross over starts some where in the range in which model 2 applies and continues into the model 3 range.

It was proved in the previous chapter that the following applies for the middle range, <u>i.e.</u>, model 2:

$$q^{*2} - 2[2(b-a)/(1+\Omega_m)]^{1/2}\, q^{*3/2}/3 = k(1+\Omega_m) \qquad (4.1.2)$$

$$q^{*2} - (2/3)\sqrt{2/(1+\Omega_m)}\,\sqrt{b-a}\,.\,q^{*3/2} = k(1+\Omega_m)$$

$$\sqrt{b-a} = [q^{*2} - k(1+\Omega_m)]/[(2/3)\sqrt{2/(1+\Omega_m)}\,q^{*3/2}] \qquad (4.1.3)$$

From Theorem 4.1.1 , we know that cross over occurs if and only if:

$$b - a > q^{*} \qquad (4.1.4)$$

or

$$\sqrt{b - a} > \sqrt{q^{*}} \qquad (4.1.5)$$

Substituting the right hand side of the equation (4.1.3) for $\sqrt{(b - a)}$ in (4.1.5), we have,

$$[q^{*2} - k(1+\Omega_m)]/[(2/3)\sqrt{2/(1+\Omega_m)}\,.\,q^{*3/2}] > \sqrt{q^{*}} \qquad (4.1.6)$$

$$q^{*2} - k(1+\Omega_m) > (2/3)\sqrt{2/(1+\Omega_m)}\; q^{*2}$$

$$[1-(2/3)\sqrt{2/(1+\Omega_m)}]\; q^{*2} > k(1+\Omega_m)$$

$$q^{*} > \sqrt{\frac{k(1+\Omega_m)}{1-\frac{2}{3}\sqrt{\frac{2}{1+\Omega_m}}}} \qquad (4.1.7)$$

COROLLARY 4.1.1 There is a possibility for cross over when:

$$b-a > \sqrt{\frac{k(1+\Omega_m)}{1-\frac{2}{3}\sqrt{\frac{2}{1+\Omega_m}}}} \qquad (4.1.8)$$

From Theorem 4.1.1, we learned that the starting point for the possibility of cross over is when $(b-a)=q^{*}$.

Therefore, by changing q^* in equation (4.1.7) to (b-a) the corollary holds.

COROLLARY 4.1.2 The following relationships can be found between the lower and upper ranges of (b-a) and q^*.

$$[L_{b-a} = \sqrt{12k/(3\Omega_m-1)}] =$$

$$[2/(1+\Omega_m)][L_{q*} = (1+\Omega_m)\sqrt{3k/(3\Omega_m-1)}] \qquad (4.1.9)$$

and $[U_{b-a} = \sqrt{3k(1+\Omega_m)^3/4}] =$

$$[(1+\Omega_m)/2].[U_{q*} = \sqrt{3k(1+\Omega_m)}] \qquad (4.1.10)$$

It should be noted that $2/(1+\Omega_m)$ is always less than or equal to 1 and $(1+\Omega_m)/2$ is always greater than or equal to 1.

COROLLARY 4.1.3 When $\Omega_m = 1$, the lower and upper range for both b-a and q^*, as well as the starting point for the possibility of cross over, become $\sqrt{6k}$.

This simply means that only models 1 and 3 apply when $\Omega_m = 1$. Also, if b-a or q^* is less than $\sqrt{6k}$, there is no possibility for cross over. And, obviously, when b-a or q^* is greater than $\sqrt{6k}$, then the possibility of cross over exists.

Using Theorem 4.1.2 and corollary 4.1.1, we can give a more complete graph that expresses the relationship between (b-a) and q^*.

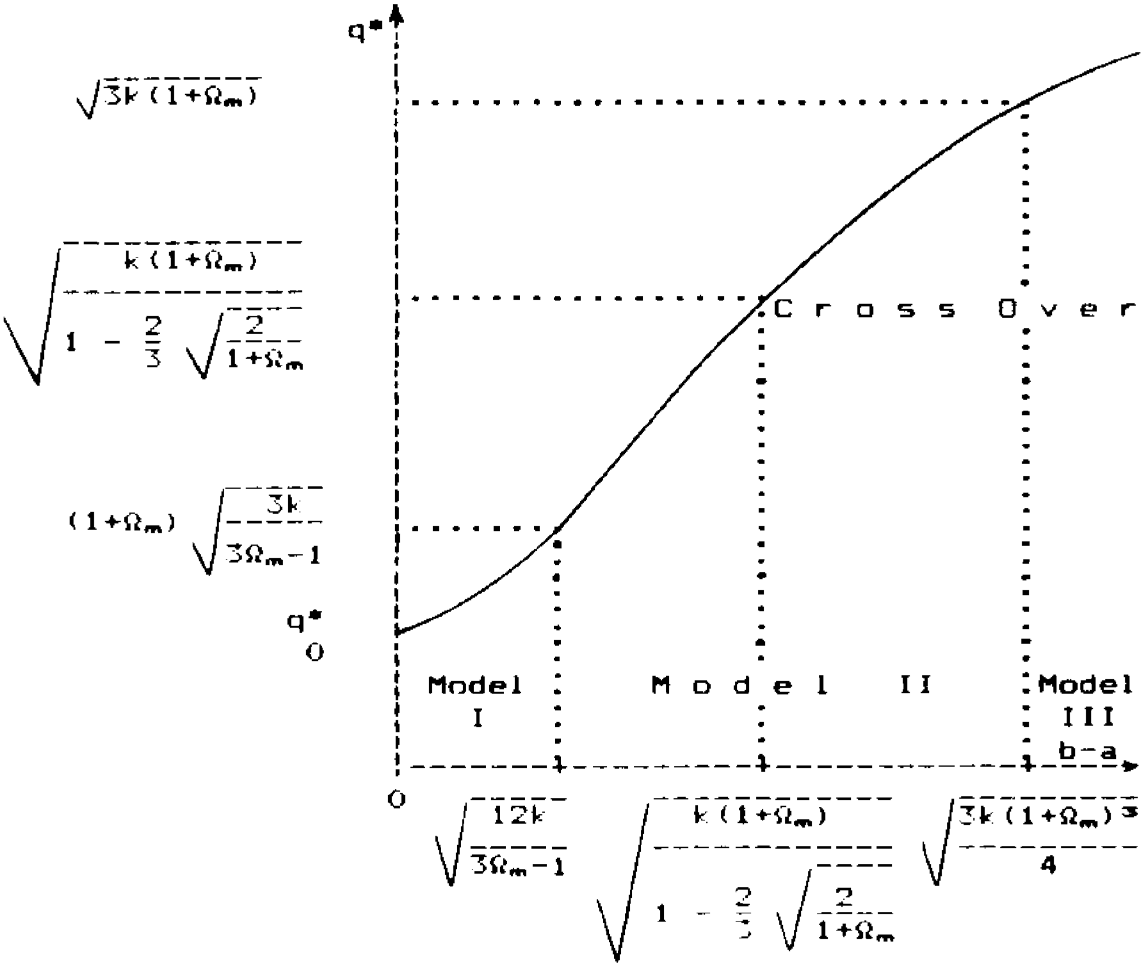


Figure 4.1.11 q* as a function of b-a and ranges for which each model applies when lead-time is uniformly distributed

4.2 CALCULATION OF TOTAL COST

This section of the chapter is devoted to calculation of total cost. As before, we distinguish different possible ranges for t^* and t^*+q^* and then obtain analytical results for total costs.

For a certain range in which model 1 applies, a total cost expression can be obtained. This expression, which applies to any distribution, is a stochastic generalization of basic EOQ's cost function.

Liberatore's [7] cost function is different from the mainstream of inventory cost functions. It is composed of three parts. To understand it fully, we will express it both algebraically and graphically.

$$ETC(t,q) = K + \int_0^t \{hDq(t-r)+hDq^2/2\}\, g(r)dr + \int_t^{t+q} \{pD(r-t)^2/2+hD(t+q-r)^2/2\}g(r)dr + \int_{t+q}^{\infty} \{pDq^2/2+pDq(r-t-q)\}g(r)dr \qquad (4.2.1)$$

Now we express each part of the cost function graphically. The arrival of the order, in the first integral, is assumed to occur before the start of the cycle for which the order is placed. Obviously, since the order has arrived before expected, this will increase the level of inventory. For this cycle only positive inventory is possible; therefore, the only inventory cost involved here is holding cost.

$$\int_0^t \{hDq(t-r) + hDq^2/2\}\, g(r)dr \qquad (4.2.2)$$

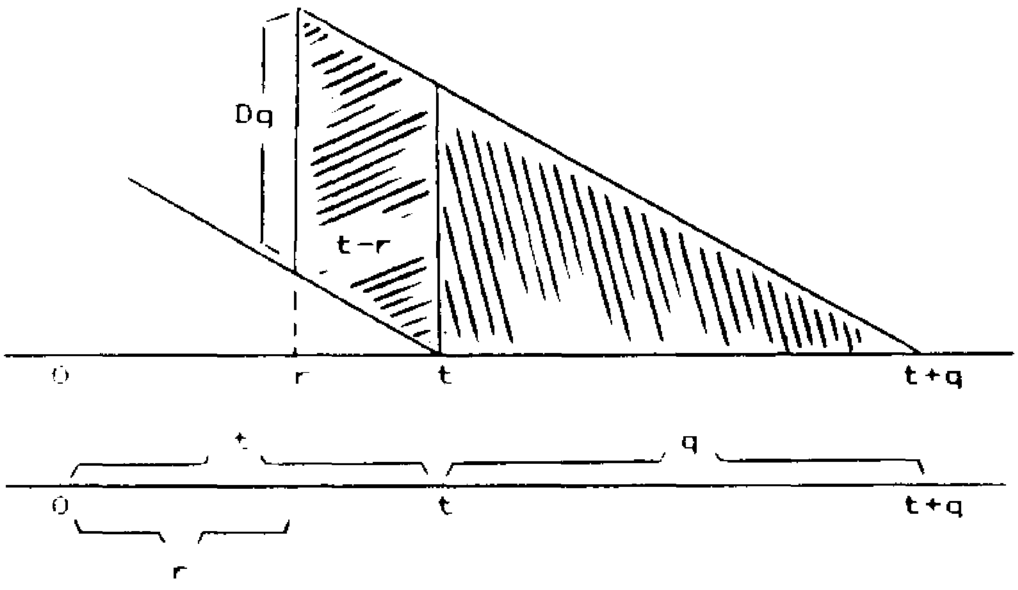


Figure 4.2.1 Cost, when the actual arrival of the order occurs before the start of the cycle for which the order is placed.

The first term in the above integration measures the area of parallelogram and the second term measures the area of the triangle. Both terms, multiplied by the holding cost per unit per unit of time, will then express the related holding cost.

The second integral in the total cost function is also composed of two terms. Here the arrival of the order is some where within the range of the cycle,

$$\int_{t}^{t+q} \{pD(r-t)^2/2 + hD(t+q-r)^2/2\}\, g(r)dr \qquad (4.2.3)$$

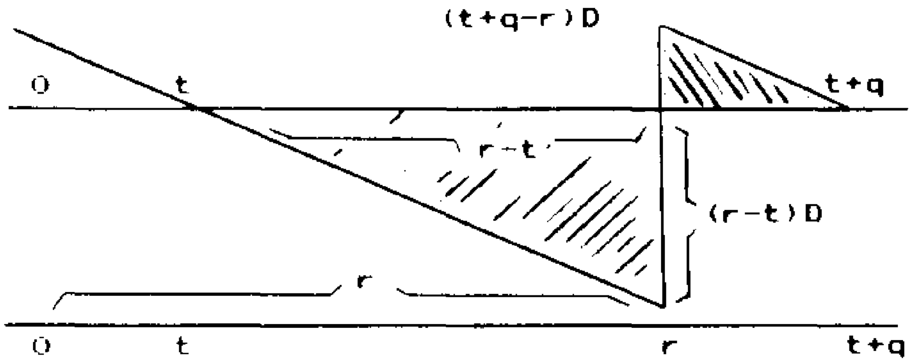


Figure 4.2.2 Cost, when the actual arrival of the order occurs some where within the range of its cycle for which the order is placed.

The area under the horizontal axis, $(r-t)[(r-t)D]/2$, multiplied by penalty cost per unit of time, represents total penalty cost involved. The area above the axis, $(t+q-r)[(t+q-r)D]/2$, multiplied by holding cost per unit of time, represents total holding cost involved here.

Finally, if the order arrives after the end of the cycle, the related cost is just penalty cost:

$$\int_{t+q}^{\infty} \{pDq^2/2 + pDq(r-t-q)\}\ g(r)dr \qquad (4.2.4)$$

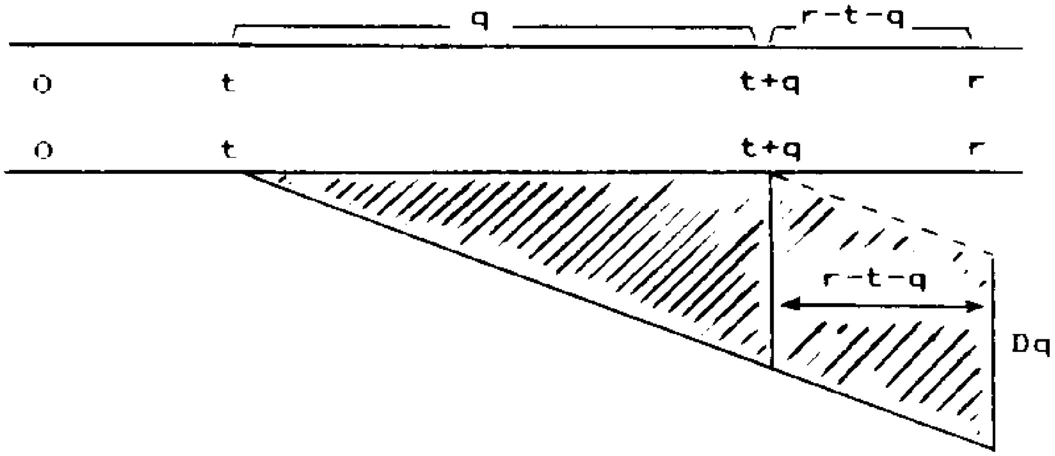


Figure 4.2.3 Cost, when the actual arrival of the order is after the end of the cycle for which the order is placed.

The total expected cost considered here is different from the above to the extent that here lead-time distribution is finite and is defined between a and b,

$$\begin{aligned} ETC(t,q) = K &+ \int_{a}^{t} \{hDq(t-r) + hDq^2/2\}\ g(r)dr \\ &+ \int_{t}^{t+q} \{pD(r-t)^2/2+hD(t+q-r)^2/2\}g(r)dr \\ &+ \int_{t+q}^{b} \{pDq^2/2+pDq(r-t-q)\}g(r)dr \qquad (4.2.5) \end{aligned}$$

And the expected average cost is ETC(t,q)/q,

$$EAC(t,q) = K/q + \int_{a}^{t}\{hD(t-r)+hDq/2\}g(r)\,dr$$
$$+\frac{1}{q}\int_{t}^{t+q}\{pD(r-t)^2/2+hD(t+q-r)^2/2\}g(r)\,dr$$
$$+\int_{t+q}^{b}\{pDq/2+pD(r-t-q)\}g(r)\,dr \quad (4.2.6)$$

As before, we distinguish three possible ranges and then for each range calculate the optimal expected average cost.

<u>THEOREM 4.2.1</u> If $k \le k_2$ or equivalently ($t^* \geq a$ and $t^*+q^* \geq b$), then the optimal expected average cost function is:

$$EAC(t^*,q^*) = \sqrt{2DKh[p/(h+p)]+hpD^2\sigma^2} \quad (4.2.7)$$

<u>Proof</u> Since a and b are between t^* and t^*+q^*, the first and the third integral in equation (4.2.6) drop out and we have:

$$EAC(t,q) = \frac{K}{q} + \left(\frac{1}{q}\right)\int_{a}^{b}\{pD(r-t)^2/2+hD(t+q-r)^2/2\}g(r)\,dr \quad (4.2.8)$$

$$EAC(t^*,q^*) = \frac{K}{q^*} + \left(\frac{1}{q^*}\right)\int_{a}^{b}\{pD(r-t)^2/2+hD(t^*+q^*-r)^2/2\}g(r)\,dr \quad (4.2.9)$$

Substituting for t^* and q^*.

$$EAC(t^*,q^*) = \frac{K}{q^*} + \frac{1}{q^*}\int_a^b \{\frac{pD}{2}[r-(\mu-\sqrt{\Omega(k+\sigma^2)})]^2 +$$

$$(hD/2)[(\mu+\sqrt{(k+\sigma^2)/\Omega}-r]^2\}g(r)\,dr$$

$$= \frac{K}{q^*} + \frac{1}{q^*}\int_a^b \{(\frac{pD}{2})[(r-\mu)^2+\Omega(k+\sigma^2)$$

$$+ 2(r-\mu)\sqrt{\Omega(k+\sigma^2)}] + (hD/2)[(r-\mu)^2$$

$$+ (k+\sigma^2)/\Omega - 2(r-\mu)\sqrt{(k+\sigma^2)/\Omega}]\}g(r)\,dr$$

After factoring and cancellation,

$$= \frac{K}{q^*} + \frac{1}{q^*}\{[(h+p)\frac{D}{2}]\int_a^b (r-\mu)^2 g(r)\,dr + (k+\sigma^2)(h+p)\frac{D}{2}\} \qquad (4.2.10)$$

Substituting σ^2 for $\int_a^b (r-\mu)^2 g(r)\,dr$ and getting the common denominator,

$$= \frac{k + (\frac{h+p}{2})D(\sigma^2+k+\sigma^2)}{\sqrt{\frac{h+p}{ph}[\frac{2K+(h+p)D\sigma^2}{D}]}}$$

$$= [K+(\frac{h+p}{2})D(\frac{2K+2(h+p)D\sigma^2}{(h+p)D})]$$

$$\cdot \sqrt{\frac{hp}{(h+p)}\,[\frac{D}{2K+(h+p)D\sigma^2}]}$$

$$= \sqrt{[2K+(h+p)D\sigma^2]^2(\frac{hp}{h+p})[\frac{D}{2K+(h+p)D\sigma^2}]}$$

$$= \sqrt{[2K+(h+p)D\sigma^2][\frac{hpD}{(h+p)}]}$$

$$= \sqrt{2DKh[\frac{p}{(h+p)}] + hpD^2\sigma^2} \qquad (4.2.11)$$

This explicit total cost function is a stochastic generalization of basic EOQ's total cost when backorders are allowed. The first term under the square root formula is basically the basic EOQ's cost function when backorders are allowed. The second term represents additional cost that is due to uncertainty in the lead-time and depends on the variance of lead-time distribution. This formula provides the optimal expected cost for any distribution of lead-time as long as its variance is known.

From the corollary in [13], we know that when $k_1 \leq k \leq k_2$ then there are two possibilities:

(a) $t^* \geq a$ and $t^*+q^* \geq b$, if $\Omega \leq (\mu-a)/(b-\mu)$ (4.2.12)

or (b) $t^* \geq a$ and $t^*+q^* \leq b$, if $\Omega \geq (\mu-a)/(b-\mu)$ (4.2.13)

Or in the case of uniform distribution of lead-time:

(a) $t^* \geq a$ and $t^*+q^* \geq b$, if $\Omega \leq 1$ (4.2.14)

(b) $t^* \geq a$ and $t^*+q^* \leq b$, if $\Omega \geq 1$ (4.2.15)

THEOREM 4.2.2 If $k_1 \leq k \leq k_2$ and $\Omega \leq 1$ equivalently $t^* \geq a$ and $t^*+q^* \geq b$ holds. For this case the optimal expected average cost is:

$$EAC(t^*,q^*) = Dh(t^*+q^*-\mu) \qquad (4.2.16)$$

Proof To find the expected total cost that applies in this case, we drop the third integration term and change the upper limit of the second integration term in equation (4.2.5) to "b" to fit the above ranges and we get:

$$ETC(t,q) = K + \int_a^t \{hDq(t-r) + \frac{hDq^2}{2}\}g(r)dr$$

$$+ \int_t^b \{pD\frac{(r-t)^2}{2} + hD\frac{(t+q-r)^2}{2}\}g(r)dr \quad (4.2.17)$$

Since we want to find the total average expected cost for uniform distribution, we have to replace g(r) with 1/(b-a) and then integrate:

$$ETC(t,q) = K + [hDq/(b-a)]\{[(t+q/2)r - r^2/2]_a^t\}$$

$$+ \{D/[6(b-a)]\}\{[p(r-t)^3 - h(t+q-r)^3]_b^t\}$$

$$= K + [hDq/(b-a)]\{(t+q/2)(t-a) - (t^2-a^2)/2\}$$

$$+ \{D/[6(b-a)]\}\{p(b-t)^3 - h(t+q-b)^3 + hq^3\}$$

$$= K + [hDq/(b-a)]\{(t-a)[t+ q/2 - t/2 - a/2]\}$$

$$+ \{D/[6(b-a)]\}\{p(b-t)^3$$

$$- h[q^3 - 3q^2(b-t) + 3q(b-t)^2 - (b-t)^3] + hq^3\}$$

$$= K + [hDq(t-a)/2(b-a)][(t-a)+q]$$

$$+ \{D/[6(b-a)]\}\{(h+p)(b-t)^3$$

$$+ 3h(b-t)q^2 - 3h(b-t)^2q\}$$

$$= K + [hD(t-a)^2/2(b-a)]q + [hD(t-a)/2(b-a)]q^2$$

$$+ [D(h+p)(b-t)^3/6(b-a)] + [3hD(b-t)/6(b-a)]q^2$$

$$- [3Dh(b-t)^2/6(b-a)]q$$

$$= K + \{3hD[(t-a)+(b-t)]/6(b-a)\}q^2$$

$$+ \{3hD[(t-a)^2 - (b-t)^2]/6(b-a)\}q$$

$$+ [D(h+p)(b-t)^3/6(b-a)]$$

$$= hDq^2/2+\{hD[(t-a)^2-(b-t)^2]/2(b-a)\}q$$

$$+ \{K+[D(h+p)(b-t)^3/6(b-a)]\} \qquad (4.2.18)$$

$$EAC(t,q) = ETC(t,q)/q=(hD/2)q+\{hD[(t-a)^2+(b-t)^2]/2(b-a)\}$$

$$+ \{K+[D(h+p)(b-t)^3/6(b-a)]\}/q \qquad (4.2.19)$$

We showed in the previous chapter that the optimal values of t^* and q^* are obtained from the following:

$$q^{*2} - 2\ [2(b-a)/(1+\Omega_m)]^{1/2}q^{*3/2}/3=k(1+\Omega_m) \qquad (4.2.20)$$

and $b-t^* = [2(b-a)/(1+\Omega_m)]^{1/2}q^{*1/2}$

$$= [2(b-a)h/(h+p)]^{1/2}q^{*1/2} \qquad (4.2.21)$$

From equations (4.2.20) and (4.2.21) we need to obtain $(a-t^*)^2$ and $(b-t^*)^2$ and then substitute in equation (4.2.19) , knowing that $(\Omega_m = p/h)$ in this case and

$k = 2K/(h+p)D$,

$$q^{*2}-2[2(b-a)h/(h+p)]^{1/2}q^{*3/2}/3=[2K/(h+p)D][(h+p)/h]$$

$$=2K/hD \qquad (4.2.22)$$

$$(a-t^*) =(a-b)+[2(b-a)h/(h+p)]^{1/2}q^{*1/2} \qquad (4.2.23)$$

$$(a-t^*)^2=(a-b)^2+[2(b-a)h/(h+p)]q^*$$

$$+2(a-b)[2(b-a)h/(h+p)]^{1/2}q^{*1/2} \qquad (4.2.24)$$

$$(b-t^*)^2=[2(b-a)h/(h+p)]q^* \qquad (4.2.25)$$

$$(t^*-a)^2-(b-t^*)^2=(a-b)^2+[2(b-a)h/(h+p)]q^*$$

$$+2(a-b)[2(b-a)h/(h+p)]^{1/2}q^{*1/2}$$

$$-[2(b-a)h/(h+p)]q^*$$

$$=(b-a)^2-2(b-a)[2(b-a)h/(h+p)]^{1/2}q^{*1/2}$$

$$(4.2.26)$$

$$EAC(t^*,q^*) = \frac{6(b-a)K+D(h+p)[2(b-a)h/(h+p)]^{3/2}q^{*3/2}}{6(b-a)} \cdot \frac{1}{q^*}$$

$$+ \frac{hD}{2} q^*$$

$$+ \frac{hD\{(b-a)^2-2(b-a)[2(b-a)h/(h+p)]^{1/2}q^{*1/2}\}}{2(b-a)}$$

$$= \frac{K}{q^*} + \frac{D(h+p)[2(b-a)h/(h+p)]^{3/2}}{6(b-a)} \cdot q^{*1/2}$$

$$+ \frac{hD}{2} q^* + \frac{hD}{2} (b-a) - hD[\frac{2(b-a)h}{(h+p)}]^{1/2} q^{*1/2}$$

$$= \frac{K}{q^*} + D\{\frac{(h+p)[2(b-a)h/(h+p)]}{6(b-a)} - h\}.$$

$$[\frac{2(b-a)h}{(h+p)}]^{1/2} q^{*1/2} + \frac{hD}{2}[q^*+(b-a)]$$

$$= \frac{K}{q^*} - \frac{2}{3} hD[\frac{2(b-a)h}{h+p}]^{1/2} q^{*1/2}$$

$$+ \frac{hD}{2} [q^*+(b-a)] \qquad (4.2.27)$$

Dividing both sides of equation (4.2.22) by q*, we get:

$$q^* - 2[\frac{2(b-a)}{1+\Omega_m}]^{1/2} q^{*1/2} = \frac{2}{hD} \cdot \frac{K}{q^*} \qquad (4.2.28)$$

Rewriting in terms of $\frac{K}{q^*}$,

$$\frac{K}{q^*} = \frac{hD}{2} \{q^* - \frac{2}{3}[\frac{2(b-a)}{(1+\Omega_m)}]^{1/2} q^{*1/2}\} \qquad (4.2.29)$$

Now substitute for $\frac{K}{q^*}$ in EAC(t*,q*),

$$\begin{aligned} EAC(t^*,q^*) = {} & (hD/2)\{q^*-(2/3)[2(b-a)/(1+\Omega_m)]^{1/2}q^{*1/2}\} \\ & -(2/3)hD[2(b-a)h/(h+p)]^{1/2}q^{*1/2} \\ & +(hD/2)[q^* + (b-a)] \qquad (4.2.30) \end{aligned}$$

Substituting for $b - t^*$ we get

$$\begin{aligned} EAC(t^*,q^*) &= (hD/2)[q^* - (2/3)(b-t^*)] - (2/3)hD(b-t^*) \\ &\quad + (hD/2)[q^*+(b-a)] \\ &= hDq^* - hD(b-t^*) + (hD/2)(b-a) \\ &= hD[(q^*-b+t^*+(b/2)-(a/2)] \qquad (4.2.31) \end{aligned}$$

Therefore, we will finally have

$$EAC(t^*,q^*) = Dh(t^*+q^*-\mu) \qquad (4.2.32)$$

THEOREM 4.2.3 If $k_1 \le k \le k_2$ and $\Omega \ge 1$ equivalently $t^* \le a$ and $t^*+q^* \ge b$ holds. For this case, the optimal expected average cost is:

$$EAC(t^*,q^*) = Dp(\mu-t^*)$$

Proof Given this range of decision variables, the expected total cost function, i.e., equation (4.2.5), changes to:

$$\begin{aligned} ETC(t,q) = {} & K + \int_a^{t+q} \{pD(r-t)^2/2+hD(t+q-r)^2/2\}g(r)\,dr \\ & + \int_{t+q}^{b} \{pDq^2/2+pDq(r-t-q)\}g(r)\,dr \qquad (4.2.33) \end{aligned}$$

Assuming uniform distribution of lead-time,

$$ETC(t,q) = K + \frac{D}{2(b-a)} \int_{a}^{t+q} [p(r-t)^2 + h(t+q-r)^2]dr$$

$$+ \frac{pDq}{(b-a)} \int_{t+q}^{b} (\frac{q}{2} + r - t - q)dr$$

$$= K + \frac{D}{6(b-a)} \{[p(r-t)^3 - h(t+q-r)^3]_{a}^{t+q}\}$$

$$+ \frac{pDq}{b-a} \{[(-\frac{q}{2} - t)r]_{t+q}^{b} + [\frac{r^2}{2}]_{t+q}^{b}\}$$

$$= K + \frac{D}{6(b-a)}\{p[q^3-(a-t)^3] + h(t+q-a)^3\}$$

$$+ \frac{pDq}{(b-a)}\{(-\frac{q}{2} - t)(b-t-q) + \frac{1}{2}[b^2-(t+q)^2]\}$$

$$= K + \frac{D}{6(b-a)}\{pq^3-p(a-t)^3 + h(t+q-a)^3\}$$

$$+ \frac{pDq}{(b-a)}\{(t+\frac{q}{2})(t+q-b) + \frac{1}{2}[b^2 - (t+q)^2]\}$$

$$=K+\frac{D}{6(b-a)}\{pq^3-p(a-t)^3+h[q^3-(a-t)^3+3q(a-t)(a-t-q)]\}$$

$$+ \frac{pDq}{(b-a)}[-b(t+\frac{q}{2})+(t+\frac{q}{2})(t+q) + \frac{b^2}{2} - \frac{(t+q)^2}{2}]$$

$$= K+ \frac{D}{6(b-a)}\{(h+p)q^3 - (h+p)(a-t)^3$$

$$+ 3hq(a-t)[(a-t)-q]\}$$

$$+ \frac{pDq}{2(b-a)}\{-2bt-bq+b^2+(t+q)[2t+q-(t+q)]\}$$

$$= K+ \frac{D}{6(b-a)}[(h+p)q^3-(h+p)(a-t)^3+3hq(a-t)^3$$

$$-3h(a-t)q^2] + \frac{pDq}{2(b-a)}[(b-t)^2-(b-t)q]$$

$$= \frac{D}{6(b-a)}\Big[\frac{6(b-a)K}{D} + (h+p)q^3-(h+p)(a-t)^3$$

$$+ 3h(a-t)^2q- 3h(a-t)q^2+3p(b-t)^2q-3p(b-t)q^2\Big]$$

$$= \frac{D}{6(b-a)}\{(h+p)q^3-3[h(a-t)+p(b-t)]q^2+3[h(a-t)^2$$

$$+p(b-t)^2]q+ \frac{6(b-a)K}{D} - (h+p)(a-t)^3\} \qquad (4.2.34)$$

$$EAC(t,q) = \frac{ETC(t,q)}{q} = \frac{D}{6(b-a)} \{(h+p)q^2-3[h(a-t)+p(b-t)]q$$

$$+3[h(a-t)^2+p(b-t)^2] + \frac{6(b-a)K}{Dq} - \frac{(h+p)(a-t)^3}{q}$$

$$= \frac{D}{2(b-a)}[h(a-t)^2+p(b-t)^2]+ \frac{K}{q} - \frac{D(h+p)}{6(b-a)}(a-t)^3.\frac{1}{q}$$

$$- \frac{D}{2(b-a)}[h(a-t)+p(b-t)]q+\frac{D(h+p)}{6(b-a)} q^2 \qquad (4.2.35)$$

To find the optimal expected cost, we need the optimal expression for q^* and $(a-t^*)$ and $(b-t^*)$. We have,

$$q^{*2}- \frac{2}{3}\Big[\frac{2(b-a)}{1+\Omega_m}\Big]^{1/2}q^{*3/2}=k(1+\Omega_m)=\frac{2K}{(h+p)D}\Big(\frac{h+p}{p}\Big)=\frac{2K}{Dp} \qquad (4.2.36)$$

and $a-t^*=q^*-\Big[\frac{2(b-a)}{1+\Omega_m}\Big]^{1/2}q^{*1/2}=q^*-\Big[\frac{2(b-a)p}{h+p}\Big]^{1/2}q^{*1/2}$ (4.2.37)

and $b-t^*=(b-a)+q^*-\Big[\frac{2(b-a)p}{h+p}\Big]^{1/2}q^{*1/2}$ (4.2.38)

Substituting for $(a-t)$ and $(b-t)$ in EAC, the optimal $(a-t^*)$ and $(b-t^*)$ we may obtain the $EAC(t^*,q^*)$,

$$EAC(t^*,q^*) = \frac{D}{2(b-a)} \{h[q^{*2} + \frac{2(b-a)p}{h+p} q^*$$

$$- 2(\frac{2(b-a)p}{h+p})^{1/2} q^{*3/2}] + p[(b-a)^2$$

$$+ q^{*2} + \frac{2(b-a)p}{h+p} q^* + 2(b-a)q^*$$

$$- 2(b-a)(\frac{2(b-a)p}{h+p})^{1/2} q^{*1/2}$$

$$- 2(\frac{2(b-a)p}{h+p})^{1/2} q^{*3/2}]\} + \frac{K}{q^*}$$

$$- \frac{D(h+p)}{6(b-a)} \cdot \frac{1}{q^*} \{q^{*3} - 3q^{*5/2}(\frac{2(b-a)p}{h+p})^{1/2}$$

$$+ 3q^{*2}(\frac{2(b-a)p}{h+p}) - (\frac{2(b-a)p}{h+p})^{3/2} q^{*3/2}\}$$

$$- \frac{Dq^*}{2(b-a)} \{h[q^* - (\frac{2(b-a)p}{h+p})^{1/2} q^{*1/2}]$$

$$+ p[(b-a) + q^* - (\frac{2(b-a)p}{h+p})^{1/2} q^{*1/2}]\}$$

$$+ \frac{D(h+p)}{6(b-a)} q^{*2}$$

$$= \frac{Dhp}{h+p} q^* - \frac{Dh}{b-a}[\frac{2(b-a)p}{h+p}]^{1/2} q^{*3/2} + \frac{Dp}{2}(b-a)$$

$$+ \frac{Dp^2}{h+p} q^* + Dpq^* - Dp[\frac{2(b-a)p}{h+p}]^{1/2} q^{*1/2}$$

$$- \frac{Dp}{b-a}[\frac{2(b-a)p}{h+p}]^{1/2} q^{*3/2} + \frac{K}{q^*}$$

$$+ \frac{D(h+p)}{2(b-a)}[\frac{2(b-a)p}{h+p}]^{1/2} q^{*3/2} - Dpq^*$$

$$+\frac{D(h+p)}{6(b-a)}\left[\frac{2(b-a)p}{h+p}\right]^{3/2}q^{*1/2}$$

$$+\frac{Dh}{2(b-a)}\left[\frac{2(b-a)p}{h+p}\right]^{1/2}q^{*3/2}-\frac{Dp}{2}q^{*}$$

$$+\frac{Dp}{2(b-a)}\left[\frac{2(b-a)p}{h+p}\right]^{1/2}q^{*3/2}$$

$$=\frac{Dp}{2}q^{*}-\frac{D(h+p)}{2(b-a)}\left[\frac{2(b-a)p}{h+p}\right]^{1/2}q^{*3/2}+\frac{Dp(b-a)}{2}$$

$$-Dp\left[\frac{2(b-a)p}{h+p}\right]^{1/2}q^{*1/2}+\frac{K}{q^{*}}+\frac{D(h+p)}{2(b-a)}.$$

$$.\left[\frac{2(b-a)p}{h+p}\right]^{1/2}q^{*3/2}+\frac{D(h+p)}{6(b-a)}\left[\frac{2(b-a)p}{h+p}\right]^{3/2}q^{*1/2}$$

$$=\frac{Dpq^{*}}{2}+\frac{Dp(b-a)}{2}+\frac{K}{q^{*}}-\frac{2}{3}Dp\left[\frac{2(b-a)p}{h+p}\right]^{1/2}q^{*1/2}$$

$$=\frac{K}{q^{*}}-\frac{2}{3}pD\left[\frac{2(b-a)p}{h+p}\right]^{1/2}q^{*1/2}+\frac{Dp}{2}[q^{*}+(b-a)] \tag{4.2.39}$$

Dividing both sides of equation (4.2.36) by q^{*},

$$q^{*}-\frac{2}{3}\left[\frac{2(b-a)}{1+\Omega_m}\right]^{1/2}q^{*1/2}=\frac{2K}{Dpq^{*}} \tag{4.2.40}$$

Rewriting in terms of $\frac{K}{q^{*}}$,

$$\frac{K}{q^{*}}=\frac{Dp}{2}q^{*}-\frac{Dp}{3}\left[\frac{2(b-a)}{1+\Omega_m}\right]^{1/2}q^{*1/2} \tag{4.2.41}$$

Substituting back in EAC(t*,q*) we have,

$$EAC(t^*,q^*) = \frac{Dp}{2}\,q^* - \frac{Dp}{3}\left[\frac{2(b-a)}{1+\Omega_m}\right]^{1/2} q^{*1/2}$$

$$- \frac{2}{3}\,Dp\left[\frac{2(b-a)p}{h+p}\right]^{1/2} q^{*1/2} + \frac{Dp}{2}\,q^*$$

$$+ \frac{Dp}{2}\,(b-a)$$

$$= Dpq^* - Dp\left[\frac{2(b-a)}{1+\Omega_m}\right]^{1/2} q^{*1/2} + \frac{Dp}{2}\,(b-a)$$

$$= Dp\left(q^* - \left[\frac{2(b-a)}{1+\Omega_m}\right]^{1/2} q^{*1/2}\right) + \frac{Dp}{2}\,(b-a)$$

$$= Dp\left[(a-t^*) + \frac{1}{2}\,b - \frac{1}{2}\,a\right]$$

$$= Dp\left[\frac{1}{2}\,(a+b) - t^*\right] \qquad (4.2.42)$$

And finally,

$$EAC(t^*,q^*) = Dp(\mu-t^*) \qquad (4.2.43)$$

<u>THEOREM 4.2.4</u> If $k \le k_1$ or equivalently $t^* \ge a$ and $t^*+q^* \le b$, the optimal expected average cost is

$$EAC(t^*,q^*) = \sqrt[3]{\frac{9K^2(h+p)D}{32(b-a)}} + \frac{hp(b-a)D}{2(h+p)} \qquad (4.2.44)$$

<u>Proof</u> Since t^* and t^*+q^* are within the range of lead-time distribution, none of the terms of equation (4.2.1), <u>i.e.</u>, the expected total cost function will drop out. The only changes are the lower bound of the first and the upper bound of the third integral,

$$\begin{aligned}
ETC(t,q) = K &+ \int_a^t \{hDq(t-r)+hDq^2/2\}[1/(b-a)]dr \\
&+ \int_t^{t+q} \{pD(r-t)^2/2 \\
&+ hD(t+q-r)^2/2\}[1/(b-a)]dr \\
&+ \int_{t+q}^b \{pDq^2/2+pDq(r-t-q)\}[1/(b-a)]dr \\
= K &+ [hDq/(b-a)]\{[(t+q/2)r]_a^t - [r^2/2]_a^t\} \\
&+ [D/6(b-a)]\{[p(r-t)^3-h(t+q-r)^3]_t^{t+q}\} \\
&+ [pDq/(b-a)]\{[(-q/2-t)r+r^2/2]_{t+q}^b\}
\end{aligned}$$

$$\begin{aligned}
&= K+[hDq/(b-a)][(t+q/2)(t-a)-(t^2-a^2)/2] \\
&\quad +[D/6(b-a)](pq^3 + hq^3) \\
&\quad +[pDq/b-a)]\{(-q/2-t)(b-t-q) \\
&\quad +[b^2-(t+q)^2]/2\}
\end{aligned}$$

$$\begin{aligned}
&= K+[hDq/(b-a)][(t+q/2)(t-a) \\
&\quad -(t+a)(t-a)/2] \\
&\quad +(h+p)Dq^3/6(b-a) \\
&\quad +[pDq/(b-a)][(b-t)(-t-q/2)+(t+q/2)q \\
&\quad +(b^2-t^2)/2 - q^2/2 - tq]
\end{aligned}$$

$$\begin{aligned}
&= K+[hD(t-a)q/(b-a)][t+q/2-(t+a)/2] \\
&\quad +D(h+p)q^3/6(b-a) \\
&\quad +[pD(b-t)q/(b-a)][-t-q/2+(b+t)/2]
\end{aligned}$$

$$\begin{aligned}
&= K+[hD(t-a)(t+q-a)q/2(b-a)] \\
&\quad +[(h+p)Dq^3/6(b-a)] \qquad (4.2.45)
\end{aligned}$$

$$\begin{aligned}
EAC(t,q) &= ETC(t,q)/q = K/q + hD(t-a)(t+q-a)/2(b-a) \\
&\quad + (h+p)Dq^2/6(b-a)+pD(b-t)(b-t-q)/2(b-a)
\end{aligned}$$

$$\begin{aligned}
&= \frac{hD(t-a)^2+pD(b-t)^2}{2(b-a)} + \frac{K}{q} + \frac{hD(t-a)-pD(b-t)}{2(b-a)}q \\
&\quad + \frac{(h+p)D}{6(b-a)}q^2 \qquad (4.2.46)
\end{aligned}$$

To find the optimal expected average cost, we have to substitute the optimal values of t^* and q^* in the above expected average cost function. For this case the optimal values are:

$$q^{*} = \sqrt[3]{6k(b-a)} = \sqrt[3]{\frac{12K(b-a)}{(h+p)D}} \qquad (4.2.47)$$

$$\text{and} \quad t^{*} = a + \frac{(b-a)}{(1+\Omega)} - \frac{q^{*}}{2}$$

$$= a + \frac{p(b-a)}{(h+p)} - \frac{q^{*}}{2} \qquad (4.2.48)$$

$$\text{and } b-t^{*} = \frac{\Omega(b-a)}{1+\Omega} + \frac{q^{*}}{2}$$

$$= \frac{h}{h+p}(b-a) + \frac{q^{*}}{2} \qquad (4.2.49)$$

Hence,

$$EAC(t^{*},q^{*}) = \frac{1}{2(b-a)}\{hD[\frac{p(b-a)}{h+p} - \frac{q^{*}}{2}]^{2}$$

$$+ pD[\frac{h}{h+p}(b-a) + \frac{q^{*}}{2}]^{2}\} + \frac{K}{q^{*}}$$

$$+ \{\frac{hD}{2(b-a)}[\frac{p(b-a)}{h+p} - \frac{q^{*}}{2}]$$

$$- \frac{pD}{2(b-a)}.[\frac{h}{h+p}(b-a) + \frac{q^{*}}{2}]\}q^{*} + \frac{(h+p)D}{6(b-a)}q^{*2}$$

$$= \frac{hD}{2(b-a)}.\frac{p^{2}(b-a)^{2}}{(h+p)^{2}} + \frac{hD(q^{*2})}{2(b-a)(4)} - \frac{hDp(b-a)(q^{*})}{(b-a)(h+p)(2)}$$

$$+ \frac{pD}{2(b-a)}.\frac{h^{2}}{(h+p)^{2}}(b-a)^{2} + \frac{pD}{2(b-a)}.\frac{q^{*2}}{4}$$

$$+ \frac{pD}{(b-a)} \cdot \frac{h}{h+p}(b-a) \cdot \frac{q^*}{2} + \frac{K}{q^*}$$

$$+ \frac{hDp}{2(b-a)} \cdot \frac{(b-a)}{(h+p)} q^* - \frac{hD}{4(b-a)} q^{*2}$$

$$- \frac{pDh(b-a)}{2(b-a)(h+p)} q^* - \frac{pD}{2(b-a)} \cdot \frac{q^*}{2}$$

$$+ \frac{(h+p)D}{6(b-a)} q^{*2}$$

$$= \frac{D}{2(b-a)}\left[\frac{h}{4} + \frac{p}{4} - \frac{h}{2} - \frac{p}{2} + \frac{h+p}{3}\right] q^{*2}$$

$$+ \frac{hp(b-a)D}{2(h+p)^2}(h+p) + \frac{K}{q^*}$$

$$= \frac{(h+p)D}{24(b-a)} q^{*2} + \frac{hp(b-a)D}{2(h+p)} + \frac{K}{q^*}$$

$$= \frac{(h+p)D[12K(b-a)/(h+p)D] + 24(b-a)K}{24(b-a)q^*}$$

$$+ \frac{hp(b-a)D}{2(h+p)}$$

$$= \frac{3K}{2q^*} + \frac{hp(b-a)D}{2(h+p)}$$

$$= \frac{3K}{2} \cdot \sqrt[3]{\frac{(h+p)D}{12(b-a)K}} + \frac{hp(b-a)D}{2(h+p)}$$

$$= \sqrt[3]{\frac{27K^3}{8} \cdot \frac{(h+p)D}{12K(b-a)}} + \frac{hp(b-a)D}{2(h+p)} \tag{4.2.50}$$

And finally,

$$EAC(t^*, q^*) = \sqrt[3]{\frac{9K^2(h+p)D}{32(b-a)} + \frac{hp(b-a)D}{2(h+p)}} \qquad (4.2.51)$$

Table 4.2.1 Formulas For Calculating Optimal Expected Average Cost For Uniform Lead-Time Distribution

MODEL 1	$k_2 \le k$	$b - a \le \sqrt{\dfrac{12k}{3Q_m - 1}}$	$EAC^* = \sqrt{2DKh \cdot \dfrac{p}{h+p} + hpD^2\sigma^2}$ **
MODEL 2	$k_1 \le k \le k_2$	$\sqrt{\dfrac{12k}{3Q_m - 1}} \le b - a \le \sqrt{\dfrac{3k(1+Q_m)^3}{4}}$	$EAC^* = Dh(t^* + q^* - \mu)$ if $Q \le 1$ $EAC^* = Dp(\mu - t^*)$ if $Q \ge 1$
MODEL 3	$k \le k_1$	$\sqrt{\dfrac{3k(1+Q_m)^3}{4}} \le b - a$	$EAC^* = \sqrt[3]{\dfrac{9k^2(h+p)D}{32(b-a)}} + \dfrac{hp(b-a)D}{2(h+p)}$

** This model applies to any lead-time distribution. It is not limited to uniform lead-time distribution like other models.

4.3 BOUNDS OF OPTIMAL EXPECTED AVERAGE COST

The purpose of this section of chapter is to find the bounds for which each of the optimal expected average costs calculated in section 4.2, apply. In simple words, the lower and upper bounds of EAC* for each of the three models is obtained.

We also show the cost turning point between crossing and not crossing of orders.

THEOREM 4.3.1 The upper bound of optimal expected average cost for model 1 is

$$EAC^* = \sqrt{2DKh \frac{p}{h+p} \left(\frac{3\Omega_m}{3\Omega_m - 1}\right)} \tag{4.3.1}$$

Or

$$EAC^* = \sqrt{hpD^2k\left(\frac{3\Omega_m}{3\Omega_m - 1}\right)} \quad \text{where } k = 2K/(h+p)D \tag{4.3.2}$$

Proof From the Table (3.2.3) we know that if

$$b \leq \sqrt{12k/(3\Omega_m - 1)}, \text{ then}$$

$$EAC^* = \sqrt{2DKh \frac{p}{h+p} + hpD^2\sigma^2} \tag{4.3.3}$$

Where $\sigma^2 = (b-a)^2/12$ for uniform lead-time distribution.

We can distinguish two cases for $\sqrt{12k/(3\Omega_m-1)}$. They are given below:

$$\sqrt{\frac{12k}{3\Omega_m-1}} = \begin{cases} \sqrt{\dfrac{12\Omega k}{3-\Omega}} & \text{when} \quad \Omega \le 1 \qquad (4.3.4) \\ \sqrt{\dfrac{12k}{(3\Omega-1)}} & \text{when} \quad \Omega \ge 1 \qquad (4.3.5) \end{cases}$$

When $\Omega \le 1$, substitute $\sqrt{12\Omega k/(3-\Omega)}$ for $(b-a)$ in EAC* and k for $2K/(h+p)D$:

$$\text{EAC*} = \sqrt{hpD^2k + hpD^2[12\Omega k/(3-\Omega)]/12} \qquad (4.3.6)$$

$$= \sqrt{hpD^2[1 + \Omega/(3 - \Omega)]}$$

$$= \sqrt{3hpD^2k/(3-\Omega)} \qquad (4.3.7)$$

or
$$= \sqrt{2DKh.\frac{p}{p+h}.\left(\frac{3}{3-\Omega}\right)} \qquad (4.3.8)$$

When $\Omega \ge 1$, substitute $\sqrt{12k/(3\Omega-1)}$ for $(b-a)$ in EAC* and k for $2k/(h+p)D$:

$$\text{EAC*} = \sqrt{hpD^2k + hpD^2\ [12k/(3\Omega-1)]/12}$$

$$= \sqrt{hpD^2k\ [1 + 1/(3\Omega - 1)]}$$

$$= \sqrt{hpD^2k\ [3\Omega/(3\Omega - 1)]} \qquad (4.3.9)$$

or
$$= \sqrt{2DKh.\frac{p}{h+p}.\left(\frac{3\Omega}{3\Omega-1}\right)} \qquad (4.3.10)$$

Combining both cases where $\Omega \le 1$ and $\Omega \ge 1$ we can express EAC* as:

$$EAC^* = \sqrt{2DKh.\frac{p}{h+p}\left(\frac{3\Omega_m}{3\Omega_m - 1}\right)} \qquad (4.3.11)$$

or $$EAC^* = \sqrt{hpD^2k.\left(\frac{3\Omega_m}{3\Omega_m - 1}\right)} \qquad (4.3.12)$$

<u>THEOREM 4.3.2</u> The upper bound of optimal expected average cost for model 2 is

$$EAC^* = \frac{k}{b-a} + hD\left[1 - \frac{2}{3}\sqrt{\frac{2}{1+\Omega_m}}\right](b-a) \text{ if } \Omega \le 1 \qquad (4.3.13)$$

$$EAC^* = \frac{k}{b-a} + pD\left[1 - \frac{2}{3}\sqrt{\frac{2}{1+\Omega_m}}\right](b-a) \text{ if } \Omega \ge 1 \qquad (4.3.14)$$

<u>Proof</u> From equation (4.2.27) we have

$$EAC^* = \frac{K}{q^*} - \frac{2}{3} hD \left[\frac{2(b-a)h}{h+p}\right]^{1/2} q^{*1/2}$$

$$+ \frac{Dh}{2}\left[q^* + (b-a)\right] \text{ when } \Omega \le 1 \qquad (4.3.15)$$

In section 4.1 of this dissertation we proved that the turning point between cross over and not cross over is at the point in which q*=b-a. Now we substitute (b-a) for q* in EAC* and we get

$$EAC^* = [K/(b-a)] - (2/3)hD.\sqrt{[2h/(h+p)]}\cdot(b-a)$$

$$+ (Dh/2)[2(b-a)] \qquad (4.3.16)$$

$$EAC^* = [K/(b-a)] + hD\ [1-(2/3)\ \sqrt{2\Omega/(1+\Omega)}\ ].(b-a)$$

$$(4.3.17)$$

or $$EAC^* = [K/(b-a)] + hD\ [1-(2/3).\ \sqrt{2/(1+\Omega_m)}\ ].(b-a)$$

$$(4.3.18)$$

From equation (4.2.39) we have

$$EAC^* = K/q^* - (2/3)pD\ [2(b-a)p/(h+p)]^{1/2}q^{*1/2}$$

$$+ (Dp/2)[q^*+(b-a)] \quad \text{when } \Omega\geq 1 \qquad (4.3.19)$$

Again substitute (b-a) for q^* in EAC^*,

$$EAC^* = [K/(b-a)] - (2/3)pD[2p/(h+p)]^{1/2}(b-a)$$

$$+ (Dp/2)[2(b-a)] \qquad (4.3.20)$$

$$EAC^* = [K/(b-a)] + pD[1-(2/3)\ \sqrt{2/(1+\Omega)}\ ].(b-a)$$

$$(4.3.21)$$

or $$EAC^* = [K/(b-a)] + pD[1-(2/3)\ \sqrt{2/(1+\Omega_m)}\ ].(b-a)$$

$$(4.3.22)$$

<u>**THEOREM 4.3.3**</u> The upper bound of optimal expected average cost for model 3 is

$$EAC^* = (h + p)\ \sqrt{(3/8)(DK/h)} \quad \text{when } \Omega\leq 1 \qquad (4.3.23)$$

$$EAC^* = (h + p)\ \sqrt{(3/8)(DK/p)} \quad \text{when } \Omega\geq 1 \qquad (4.3.24)$$

<u>**Proof**</u> From table (4.2.1) we have

$$EAC^* = hpD(b-a)/[2(h+p)] + \sqrt[3]{9K^2(h+p)D/32(b-a)}$$

$$(4.3.25)$$

This applies if $b-a \geq \sqrt{3k(1+\Omega_m)^3/4}$. To find the lower limit of EAC* for this range we substitute the following relationships in EAC*:

$$b-a = \sqrt{\frac{3k(1+\Omega_m)^3}{4}} = \begin{cases} \sqrt{\frac{3}{4}k\left(1+\frac{1}{\Omega}\right)^3} & \text{when } \Omega \leq 1 \quad (4.3.26) \\ \sqrt{\frac{3}{4}k(1+\Omega)^3} & \text{when } \Omega \leq 1 \quad (4.3.27) \end{cases}$$

When $\Omega \leq 1$, substitute $\sqrt{\frac{3}{4}\,k\,\left(1+\frac{1}{\Omega}\right)^3}$ for (b-a) in EAC*,

$$EAC^* = \left[\frac{h^2p^2D^2}{4(h+p)^2}\cdot\frac{3}{4}\cdot\frac{2k}{(h+p)D}\cdot\left(\frac{h+p}{h}\right)^3\right]^{1/2}$$

$$\left\{\frac{9k^2(h+p)D}{32\left[\frac{3}{4}\cdot\frac{2k}{(h+p)D}\cdot\left(\frac{h+p}{h}\right)^3\right]^{1/2}}\right\}^{1/3}$$

$$= \left[\frac{3}{8}\cdot\frac{p^2Dk}{h}\right]^{1/2} + \left[\frac{81k^4(h+p)^2D^2}{32^2\cdot\frac{3}{2}\cdot\frac{k}{D}\cdot\frac{(h+p)^2}{h^3}}\right]^{1/6}$$

$$= [(3/8)(p^2Dk/h)]^{1/2} + [27k^3D^3h^3/512]$$

$$= \sqrt{(3/8)Dkp^2/h} + \sqrt{(3/8)Dkh}$$

$$= \sqrt{(3/8)Dkh\ [1 + (p/h)]}$$

$$= (h+p)\sqrt{(3/8)(Dk/h)} \qquad (4.3.28)$$

When $\Omega \geq 1$, substitute $\sqrt{(3/4)k(1+\Omega)^3}$ for (b-a) in EAC*

$$EAC^* = \left[\frac{h^2p^2D^2}{4(h+p)^2} . \frac{3}{4} . \frac{2K}{(h+p)D} . \left(\frac{h+p}{p} \right)^3 \right]^{1/2}$$

$$\left\{ \frac{9K^2(h+p)D}{32 \left[\frac{3}{4} . \frac{2K}{(h+p)D} . \left(\frac{h+p}{p} \right)^3 \right]^{1/2}} \right\}^{1/3}$$

$$= \left[\frac{3}{8} . DK . \frac{h^2}{p} \right]^{1/2} + \left[\frac{9^2K^4(h+p)^2D^2}{32^2 . \frac{3}{2} . \frac{K}{D} . \frac{(h+p)^2}{p^3}} \right]^{1/6}$$

$$= (h/p)[(3/8)DKp]^{1/2} + [(3/8)DKp]^{1/2}$$

$$= \sqrt{(3/8)DKp \; [1 + (h/p)]}$$

$$= (h+p) \sqrt{\frac{3}{8} \frac{DK}{p}} \qquad (4.3.29)$$

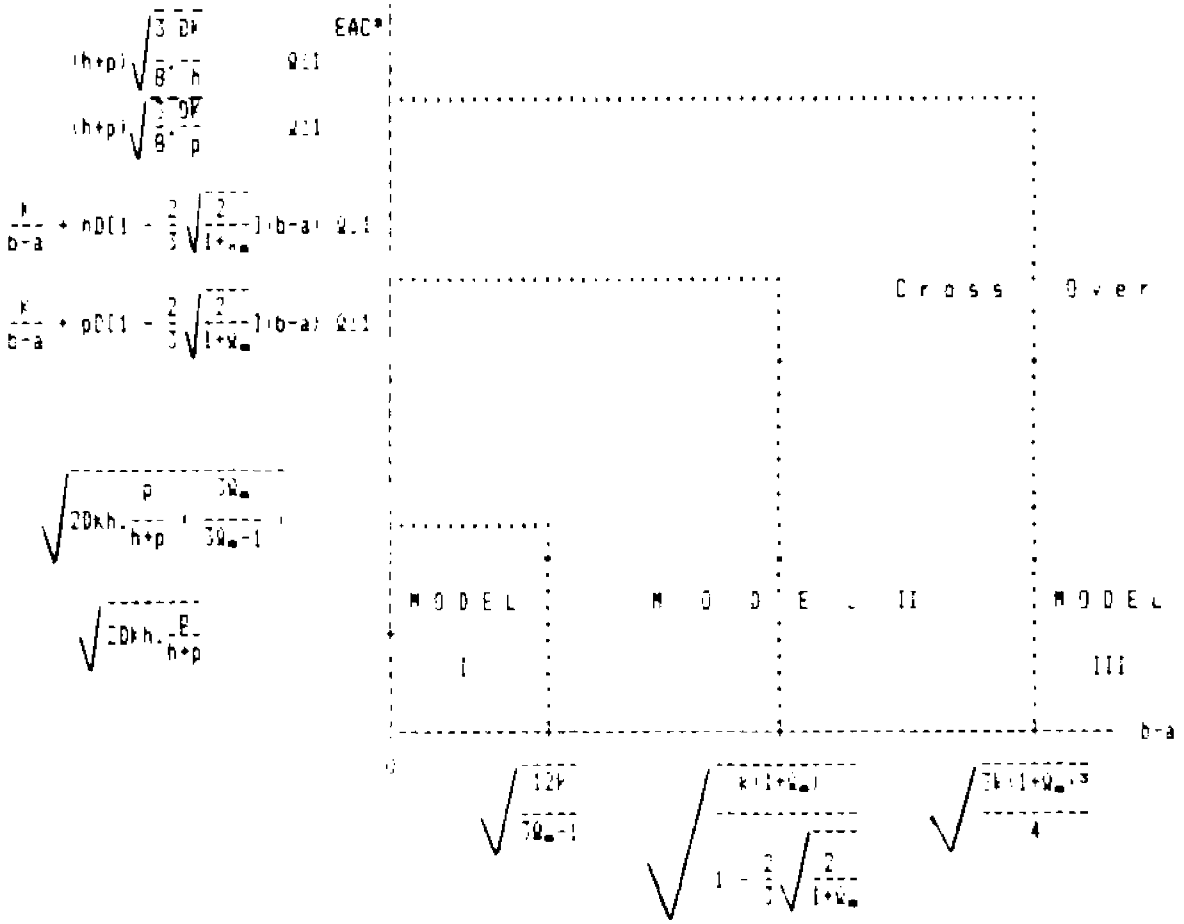


Figure 4.3.1 Uniform cost ranges for models I, II, and III

4.4 SHAPE AND PROPERTIES OF OPTIMAL EXPECTED AVERAGE COST

In section 4.2 we derived the optimal expressions for calculation of optimal cost for different ranges of lead-time distribution.

In this section, through differential calculus, we will try to find the shape of EAC* for different ranges of lead-time distribution.

THEOREM 4.4.1 When the range of uniform lead-time distribution is between $[0 - \sqrt{12k/(3\Omega_m-1)}]$, the EAC* is increasing and convex.

Proof In section 4.2 we proved if that the range of uniform lead-time distribution is between zero and $\sqrt{12k/(3\Omega_m-1)}$ then

$$EAC^* = \sqrt{2DKh.\frac{p}{h+p} + hpD\sigma^2} \qquad (4.4.1)$$

where $\sigma^2 = \frac{(b-a)^2}{12}$ for the case of uniform lead-time distribution. To prove that EAC^* is increasing and convex , it is sufficient to show that both the first and second derivatives are positive.

$$\frac{d\ EAC^*}{d(b-a)} = \frac{1}{2}\left[2DKh\frac{p}{p+h} + hpD^2\frac{(b-a)^2}{12}\right]^{-1/2}\left[\frac{hpD^2}{6}(b-a)\right]$$

(4.4.2)

$$= \frac{1}{2}\left[\frac{24DKhp + hpD^2(h+p)(b-a)^2}{12(h+p)}\right]^{-1/2}\left[\frac{hpD^2}{6}(b-a)\right]$$

$$= \frac{1}{2}\left[\frac{[12(h+p)]}{[24DKhp+hpD^2(h+p)(b-a)^2]}\,\frac{[hpD(hpD^3)(b-a)^2]}{[36]}\right]^{1/2}$$

$$= \frac{1}{2}\left[\frac{hpD^3(h+p)(b-a)^2}{72K + 3D(h+p)(b-a)^2}\right]^{1/2} \geq 0 \qquad (4.4.3)$$

Since each element in the first derivative is positive, then the first derivative is positive. Now let us show that the second derivative is also positive. If this is shown, then the theorem is proved.

$$\frac{d^2\ EAC^*}{d(b-a)^2} = \frac{1}{4}\left[\frac{72K+3D(h+p)(b-a)^2}{hpD^3(h+p)(b-a)^2}\right]^{1/2}\left\{\frac{[2hpD^3(h+p)(b-a)].}{1}\right.$$

$$\cdot\ \frac{[72K+3D(h+p)(b-a)^2]-6D(h+p)(b-a)].}{1}$$

$$\cdot\ \frac{[hpD^3(h+p)(b-a)^2]}{[72K+3D(h+p)(b-a)^2]^2}$$

$$= \frac{1}{4}\left[\frac{72K+3D(h+p)(b-a)^2}{hpD^3(h+p)(b-a)^2}\right]^{1/2} .$$

$$\cdot \left\{\frac{144hpD^3K(h+p)(b-a)}{[72K+3D(h+p)(b-a)^2]^2}\right\}$$

$$= \frac{1}{4}\left\{\frac{20736hpK^2D^2(h+p)}{[72K+3D(h+p)(b-a)^2]^3}\right\}^{1/2}$$

$$= \sqrt{\frac{1296hpK^2D^3(h+p)}{[72K+3D(h+p)(b-a)^2]^3}} \geq 0 \qquad (4.4.4)$$

Since both the first and second derivatives are positive, the theorem is proved, i.e, the optimal cost function is increasing and convex for this given range.

THEOREM 4.4.2 When the range of uniform lead-time distribution is greater than $\sqrt{3K(1+\Omega_m)^3/4}$, the EAC* is increasing and convex.

Proof For the above range of lead-time distribution the optimal expected average cost is:

$$EAC^* = \frac{hpD(b-a)}{2(h+p)} + \left[\frac{9K^2(h+p)D}{32(b-a)}\right]^{1/3} \qquad (4.4.5)$$

$$\frac{d\,EAC^*}{d(b-a)} = \frac{hpD}{2(h+p)} + \frac{1}{3}\left[\frac{9K^2(h+p)D}{32(b-a)}\right]^{-2/3}\left\{\frac{-32[9K^2(h+p)D]}{32^2(b-a)^2}\right\}$$

$$= \frac{hpD}{2(h+p)} + \frac{1}{3}\left[\frac{32(b-a)}{9k^2(h+p)D}\right]^{2/3}\left[\frac{9k^2(h+p)D}{32(b-a)}\right]\left[-\frac{1}{(b-a)}\right]$$

$$= \frac{hpD}{2(h+p)} - \frac{1}{3(b-a)}\left[\frac{9k^2(h+p)D}{32(b-a)}\right]^{1/3}$$

$$= \frac{hpD}{2(h+p)} - \left[\frac{k^2(h+p)D}{96(b-a)^4}\right]^{1/3} \qquad (4.4.6)$$

To prove that this first derivative is positive, we have to show that:

$$\frac{hpD}{2(h+p)} > \left[\frac{k^2(h+p)D}{96(b-a)^4}\right]^{1/3} \qquad (4.4.7)$$

or $$\frac{h^3p^3D^3}{8(h+p)^3} > \frac{k^2(h+p)D}{96(b-a)^4} \qquad (4.4.8)$$

or $$12h^3p^3D^2(b-a)^4 > k^2(h+p)^4 \qquad (4.4.9)$$

or $$(b-a)^4 > \frac{k^2(h+p)^4}{12h^3p^3D^2} \qquad (4.4.10)$$

To show that $(b-a)^4 > k^2(h+p)^4/12h^3p^3D^2$, we use the range relationship that this model applies. We know that the above cost expression holds if and only if:

$$b-a \geq \left[\frac{3}{4}(1+\Omega_m)^3 k\right]^{1/2}$$

$$\geq \left[\frac{3}{4}(1+\Omega_m)^3 \cdot \frac{2K}{(h+p)D}\right]^{1/2} \qquad (4.4.11)$$

When $\Omega = h/p \leq 1$ we have $\Omega_m = p/h$. Squaring both sides of equation (4.4.11), we get:

$$(b-a)^2 \geq \frac{3}{4}\left(\frac{h+p}{h}\right)^3 \cdot \frac{2K}{(h+p)D}$$

$$\geq \frac{3}{2} \cdot \frac{K(h+P)^2}{h^3 D} \tag{4.4.12}$$

$$(b-a)^4 \geq \frac{9}{4} \cdot \frac{K^2(h+p)^4}{h^4 D^2}$$

$$(b-a)^4 \geq 27\Omega_m^3 \cdot \left[\frac{K^2(h+p)^4}{12h^3p^3D^2}\right] \tag{4.4.13}$$

Comparing equation (4.4.10) with equation (4.4.13), knowing that $27\Omega_m^3 \geq 1$ leads to the conclusion that, equation (4.4.10) holds and as a result the first derivative is greater than zero.

Similarly we have to prove that the first derivative is positive when $\Omega_m = h/p \geq 1$, i.e., $\Omega_m = h/p$. Again from the range relationship for this case, we have:

$$b-a \geq \sqrt{\frac{3}{4}(1+\Omega_m)^2 k} \tag{4.4.14}$$

$$(b-a)^2 \geq \frac{3}{4}(1+\Omega_m)^3 \cdot \frac{2K}{(h+p)D} \tag{4.4.15}$$

Substituting h/p for Ω_m we have:

$$(b-a)^2 \geq \frac{3}{4}\left(\frac{h+p}{p}\right)^3 \cdot \frac{2k}{(h+p)D} \tag{4.4.16}$$

$$(b-a)^4 \geq \frac{9}{4} \frac{(h+p)^4 k^2}{p^4 D^2} \tag{4.4.17}$$

$$\geq 27\Omega_m^3 \left[\frac{k^2(h+p)^4}{12h^3p^3D^2}\right] \tag{4.4.18}$$

Comparing equation (4.4.10) with (4.4.18), knowing that $27\Omega_m^3 > 1$ leads to the conclusion that equation (4.4.10) holds and as a result the first derivative is greater than zero when $\Omega \geq 1$.

Now we show that the second derivative is also positive in this case.

$$\frac{d^2EAC}{d(b-a)^2} = -\frac{1}{3}\left[\frac{k^2(h+p)D}{96(b-a)^4}\right]^{-2/3}$$

$$\cdot\left(\frac{-4(96)(b-a)^3[k^2(h+p)D]}{(96)^2(b-a)^8}\right)$$

$$= \frac{1}{72}\left[\frac{96(b-a)^4}{k^2(h+p)D}\right]^{2/3}\left[\frac{k^2(h+p)D}{(b-a)^8}\right]$$

$$= \frac{1}{72}\left[\frac{96kD^{1/2}(h+p)^{1/2}}{(b-a)^{7/2}}\right]^{2/3}$$

$$= \frac{1}{9}\left[\frac{18k^2D(h+p)D}{(b-a)^7}\right]^{1/3} \geq 0$$

Hence the optimal cost function for this case has positive first and second derivatives. This means that in the range of $b-a \geq \sqrt{3k(1+\Omega_m)^3/4}$ the optimal cost function is increasing and convex.

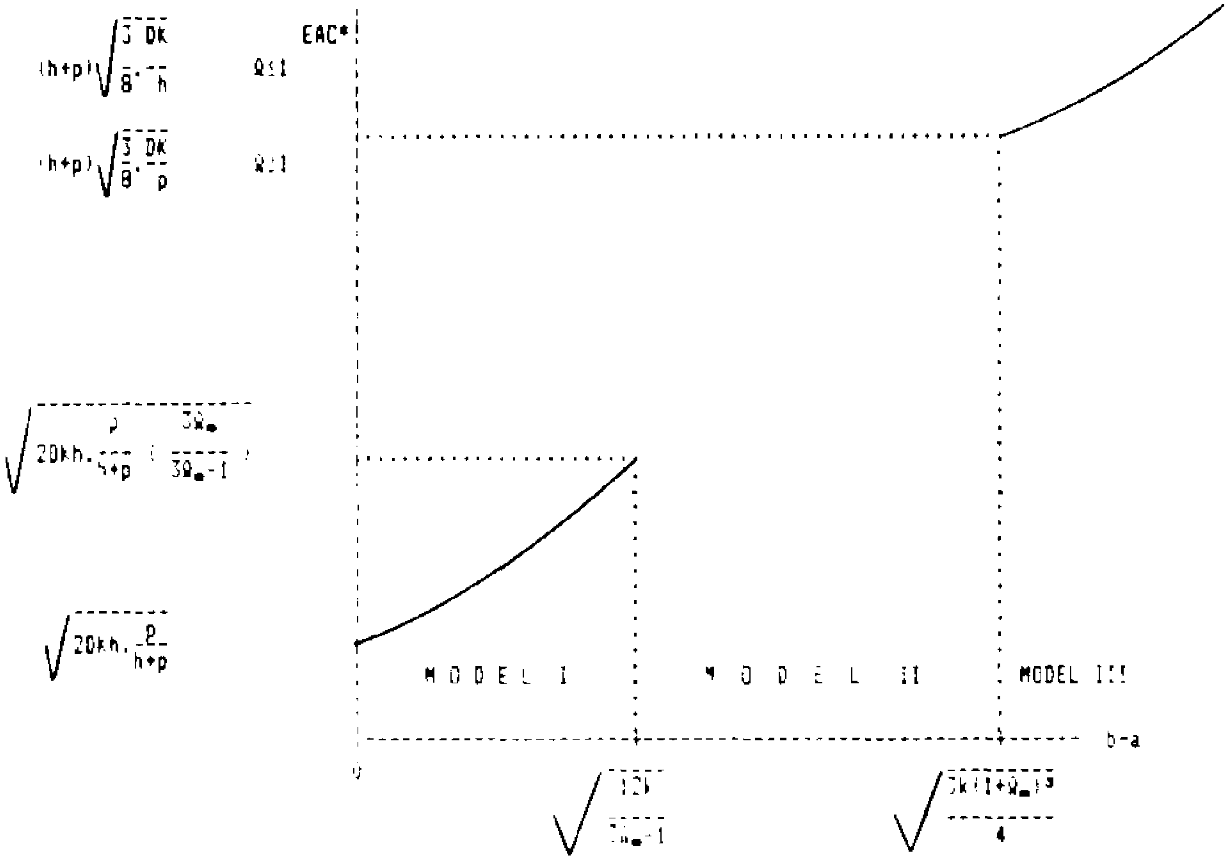


Figure 4.4.1 Shape of optimal cost function in the case of uniform lead-time.

4.5 SENSITIVITY OF DECISION VARIABLES AND TOTAL COST TO LOWER-BOUND OF LEAD-TIME DISTRIBUTION

A careful analysis of optimal solutions produced by computer search procedure, developed by the author, led to the following results. We observed that the optimal order quantity and optimal expected cost were insensitive to the changes in the lower bound of the uniform lead-time distribution, "a". The only decision variable that changed was "t*". Now, we state and prove the above observations in the following theorem.

THEOREM 4.5.1 If the lower bound of uniform lead-time distribution increases or decreases, given that the range of lead-time distribution remains the same, the optimal order quantity and the optimal expected cost will not change.

Proof To show that the optimal order quantity is insensitive to the changes in the lower bound of lead-time distribution, let us refer to Table (3.2.2). A close examination of the formulas for q* for all three ranges simply indicates that q* is a function of b-a and not "a" alone. Therefore, q^* is just sensitive to the range of the lead-time (b-a) and not to the lower bound of lead-time distribution "a".

A similar argument can be made for EAC^*, by just referring to Table (4.2.1). For the cases of $k \geq k_2$ and $k \leq k_1$ it is clear that q^* is a function of (b-a) and not "a" alone. In the following, we also show that the theorem also holds for the case in which $k_1 \leq k \leq k_2$. From table (4.2.1) when $k_1 \leq k \leq k_2$ and $\Omega \leq 1$, we have:

$$EAC^* = Dh\ (t^*+q^*-\mu) \qquad (4.5.1)$$

By substituting the appropriate t^* from Table 3.2.2 into equation (4.5.1), we get:

$$\begin{aligned} EAC^* &= Dh[(b - \delta\sqrt{q^*}) + q^* - \mu] \\ &= Dh[q^* - \delta\sqrt{q^*} + (b-a)/2] \qquad (4.5.2) \end{aligned}$$

Knowing that q^* and δ are both functions of (b-a) proves the theorem for the above-mentioned conditions. From Table 4.2.1, when $k_1 \leq k \leq k_2$ and $\Omega \geq 1$, we have:

$$EAC^* = Dp(\mu - t^*) \qquad (4.5.3)$$

By substituting the appropriate t^* from Table 3.2.2 into equation (4.5.3), we get:

$$\begin{aligned} EAC^* &= Dp[\mu - (a - q^* + \delta\sqrt{q^*})] \\ &= Dp[q^* - \delta\sqrt{q^*} + (b - a)\ /2] \qquad (4.5.4) \end{aligned}$$

Knowing that q^* and δ are both functions of $(b - a)$ completes the proof of the theorem.

THEOREM 4.5.2 If the lower bound of uniform lead-time distribution increases or decreases by a constant given that the range of the lead-time distribution remains the same, the optimal reorder level will increase or decrease by a quantity which is equal to the change in the lower range of lead-time multiplied by the demand rate.

Proof We give the proof for all three possible ranges. Our proof requires t^* values, for different ranges, which are given in Table 3.2.2.

$k \geq k_2$

$$t^* = \mu - \sqrt{\Omega(k+\sigma^2)} \tag{4.5.5}$$

$$= a + [(b-a)/2] - \sqrt{\Omega(k+\sigma^2)} \tag{4.5.6}$$

$k_1 \geq k \geq k_2$ and $\Omega \leq 1$

$$t^* = b - \delta\sqrt{q^*} \tag{4.5.7}$$

Since $(b-a)$ is constant, any change in "a" will change "b" by the same amount. In another words, if "a" increases or decreases by a constant, t^* will also change by the same amount.

$k_1 \geq k \geq k_2$ and $\Omega \geq 1$

$$t^* = a - q^* + \delta\sqrt{q^*} \tag{4.5.8}$$

$k \leq k_1$

$$t^* = [(a\Omega+b)/(1+\Omega)] - q^*/2 \tag{4.5.9}$$

$$= [a(1+\Omega) + (b-a)]/(1+\Omega) - q^*/2$$

Hence,

$$t^* = a + [(b-a)/(1+\Omega)] - q^*/2 \tag{4.5.10}$$

From equations (4.5.6), (4.5.7), (4.5.8), (4.5.10), it is clear that as "a" increases or decreases, t^* will also change accordingly.

We also know that $R^* = D.t^*$. Therefore the change in the optimal reorder level, R^*, is equal to the change in t^*, (i.e., "a") multiplied by the demand rate. This completes the theorem.

4.6 TWO CONSEQUENT ORDERS AND THE PROBABILITY OF CROSS OVER

In section 4.1 of this dissertation we showed that the possibility of cross over exists if and only if the range of lead-time distribution is greater than the cycle time (b-a > q). In the follwing theorem we assume that the range of lead-time distribution is between one and two cycle time lengths. The implication of this assumptiom is that, only two subsequent orders can possibly cross each other. Therefore at any point in time, only two orders can be outstanding.

THEOREM 4.6.1 the probability of cross over of orders when $2q^* \leq b-a \geq q^*$ is:

$$\int_{x_1=a+q}^{b} \int_{x_2=a}^{x_1-q} f(x_2).f(x_1)d(x_2)d(x_1) \qquad (4.6.1)$$

where

x_1 = time between placing and receiving order 1,

x_2 = time between placing and receiving order 2,

$f(x_1)$= probability density function of x_1,

$f(x_2)$= probability density function of x_2,

$f(x_1,x_2)$= joint probability density function of x_1, and x_2.

PROOF The following two figures show the range in which two subsequent orders can possibly cross each other,

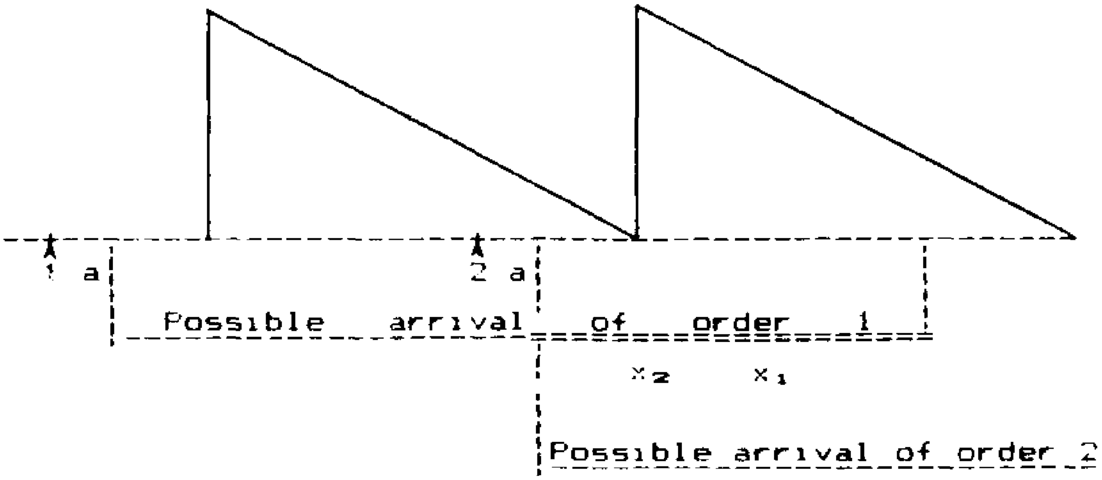


Figure 4.6.1 The range of cross over of two subsequent orders when t is positive.

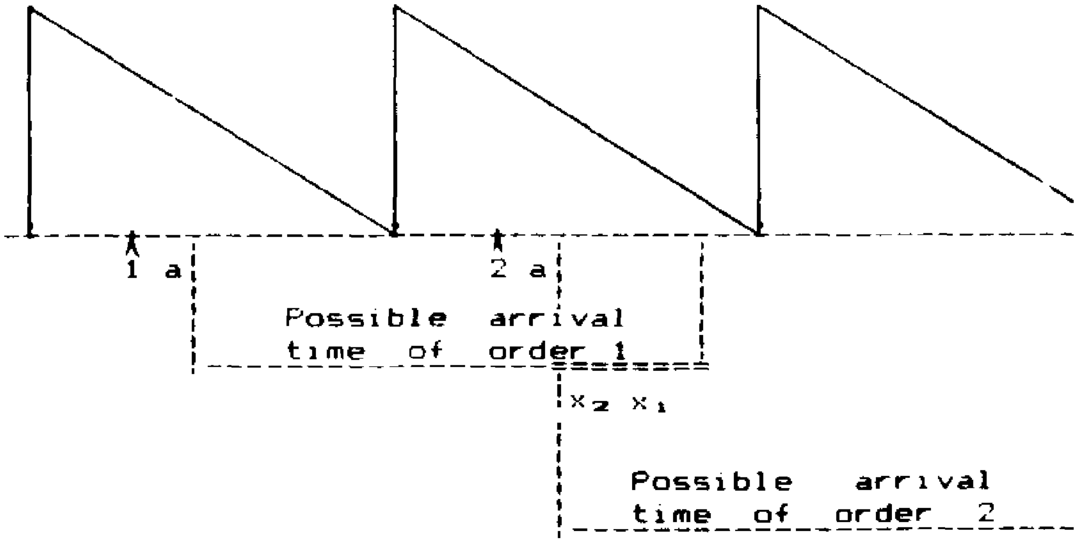


Figure 4.6.2 The range of cross over of two subsequent orders when t is negative.

Using figures 4.6.1 and 4.6.2 we can write

P(cross over of two subsequent orders)

$$= P(x_1 \geq q+x_2) \qquad (4.6.2)$$

$$= P(x_2 \leq x_1-q) \qquad (4.6.3)$$

$$= \int_{x_1=a+q}^{b} \int_{x_2=a}^{x_1-q} f(x_1,x_2)d(x_2)d(x_1) \qquad (4.6.4)$$

Since the lead-times are independent of each other, we get

$$= \int_{x_1=a+q}^{b} \int_{x_2=a}^{x_1-q} f(x_2)f(x_1)d(x_2)d(x_1) \qquad (4.6.5)$$

and the theorem is proved.

Now let us apply the results to the case of unifom distribution of lead-time.

P(cross over of two subsequent orders)

$$= \frac{1}{(b-a)^2} \int_{a+q}^{b} [(x_1-q)-a]d(x_1)$$

$$= \frac{1}{(b-a)^2} \{ \frac{1}{2} [x_1^2]_{a+q}^{b} - (a+q)[x_1]_{a+q}^{b} \}$$

$$= \frac{1}{(b-a)^2} \{ \frac{1}{2} [b^2 - (a+q)^2] - (a+q)[b - (a+q)] \}$$

$$= \frac{1}{(b-a)^2} \{ \frac{1}{2} [b^2 - (a+q)^2] - b(a+q) + (a+q)^2 \}$$

$$= \frac{1}{(b-a)^2} \left\{ \frac{1}{2} \left[b^2 + (a+q)^2 - 2b(a+q) \right] \right\}$$

$$= \frac{1}{(b-a)^2} \left\{ \frac{1}{2} \left[b - (a+q) \right]^2 \right\}$$

$$= \frac{1}{(b-a)^2} \left\{ \frac{1}{2} \left[(b-a) + q \right]^2 \right\}$$

$$= \frac{1}{2} \left(1 - \frac{q}{b-a} \right)^2 \qquad (4.6.6)$$

It should be noted that, when orders one and two cross each other, orders two and three can not possibly cross each other in this case.

Numerical example - Assuming that the lead-time distribution is uniformly distributed between 1 and 11 units of time, and the cycle time is between 5 and 10 units of time, we can construct the following table using equation (4.6.6):

TABLE 4.6.1 - Probability of cross over of two subsequent orders when lead-time distribution is uniform

q	Probability of cross over
10	0.0
9	0.005
8	0.020
7	0.045
6	0.080
5	0.125

CHAPTER V

SUMMARY, CONCLUSIONS, AND FUTURE RESEARCH

5.1 SUMMARY

The term inventory may not strike us in the same manner as the terms production and sales, but it is a fact that inventory usually constitutes the largest investment made by a firm. In economic terms, this means that even a small percentage of reduction in inventory level could translate into sizable savings in inventory costs.

The inventory model discussed here assumed that demand rate is deterministic and stochastic lead-time applies for a finite range. We also made the unorthodox assumption of non-interchangeability (non-substitutability) of items.

The constant demand pattern is mainly justified for manufacturers who require parts for their production processes. An inventory of parts is maintained to be used according to the requirements of production, thereby establishes a relatively constant demand pattern.

In actual practice, lead-time is not constant. In spite of great improvements in the means of transportation, communication and production, most procurement systems require some lead-time in supply, which is often uncertain. In mathematical terms, this means that the lead-time is a random variable. This assumption makes the analysis of inventory models much more difficult. In spite of its modeling complexity, we included this realistic assumption due to the fact that lead-time variation has a major impact on inventory costs.

To simplify the modeling, many authors assume that "orders can not possibly cross" right from the start. Here, we allowed the orders to cross each other, but imposed the assumption of non-interchangeability for mathematical tractability.

In this work we showed that this restrictive assumption becomes immaterial as long as orders do not cross, or when a certain condition, which is based on problem parameters, holds.

Washburn [18] provides the implications of the assumption of non-interchangeability in manufacturing and sales. In a manufacturing context, it is like assuming that parts are non-interchangeable. In a sales context, it is like assuming that each order is being "colored" to satisfy a particular customer's demand.

The objective of this dissertation was two-fold:

(1) To study the inventory models that allow for lead-time variability and at the same time

allow orders to cross each other due to the independent nature of lead-times.

(2) To examine the impact of the assumption of non-interchangeability.

Extensive related results were obtained by Sphicas-Nasri [13] prior to this work. Initially, it was felt that the only feasible continuation of that work is simulation. Therefore, a rather long computer program was designed. It was aimed at measuring the cost differences between the inventory models that assume non-interchangeability of items and those that do not.

First, a computer search procedure was designed to find the optimal solution to Sphicas-Nasri's [13] model, for different distributions of lead-time. Second, a simulated inventory model was developed, which allowed cross over without making the assumption of non-interchangeability. It was expected that as long as orders do not cross each other, the solution and costs of both models would be the same. And beyond that point, a higher cost was anticipated for our model due to the assumption of non-interchangeability. Simulation difficulties and some preliminary results were the moving force behind this analytical work.

5.2 CONCLUSIONS

Throughout this research, we have tried to keep a reasonable balance between the real-world application and mathematical tractability.

Like the majority of articles in the literature, this work may not have an immediate application in a real-world situation. We believe that this research, more than anything else, will smooth the way for future studies that will eventually lead to a better understanding of the behavior of inventory models that make the realistic assumption of variability in lead-time and allow for order crossing.

A computer search procedure was designed to find the optimal value of decision variables. Close review of computer results, which were tabulated for different parameters of the problem, indicated that the optimal cycle time, optimal order quantity and optimal cost do not change as long as the range of lead-time distribution, b-a, remains unchanged. But, the magnitude of change in the optimum reorder level, is equal to the change in "a" times demand rate.

We showed that cross over may occur if and only if a certain condition is met. Consequently, even before an attempt is made to solve the problem, we should be aware of whether or not the possibility for cross over exists. If orders do not cross each other the assumption of non-interchangeability becomes indifferent or redundant.

A cost expression, which is a stochastic generalization of basic EOQ's model, was developed. Its deterministic part is the same as the well-known deterministic basic EOQ model. Its stochastic part, which is composed of parameters of the problem is directly related to the variance of lead-time. As intuitively expected, the total optimal cost developed here approaches the traditional, well-known, basic EOQ model as the variance of lead-time approaches zero.

It was not possible to develop results that were applicable for all ranges of the model. Therefore, to obtain cost expressions, ranges, shape and properties of cost function, etc., our job was three times more tedious because each result had to be obtained for three different applicable ranges.

A general expression is obtained to calculate the probability of cross over of two subsequent orders. The results are illustrated for the case of uniform lead-time distribution.

5.3 SUGGESTIONS FOR FUTURE RESEARCH

Washburn [18] initially introduced the unorthodox assumption of non-interchangeability in 1979.

Liberatore [7] then developed a system of two equations that provides the optimal solution for t (time interval between the time an order is initiated and the start of the q time units of demand satisfied by that order) and q (cycle time).

Sphicas [14] presented a one dimensional search to solve the Liberatore's Continuous Review (Q,R) model.

Sphicas-Nasri [13] then obtained some analytical results for both general and uniform distributions of lead-time. They first indicated that one of Liberatore's results just applies for a given range and is not universal. For the range in which orders do not cross, some closed-form expressions were obtained for both t^* and q^*. Further results were obtained for the uniform lead-time distribution.

This work extended the previous studies just mentioned. Some general closed-form expressions were obtained. The range within which orders could possibly cross was determined. Some additional results for uniform distribution of lead-time were shown.

The natural extension of this and other previous works discussed is to allow for demand variability. Obviously, introducing one additional variable further complicates the model building.

We showed the range of cross over, based on the parameters of the problem. It applied to uniform lead-time distribution. Its extension to general distribution of lead-time based on problem parameters is valuable.

Many of the results obtained here applied to the uniform distribution of lead-time. An interested reader could investigate other distributions and then try to draw general conclusions.

Another possible area of future research is to extend the study of cross over discussed in this work to the situations in which more than two orders can possibly cross. The author has developed some rather large computer programs to study the effect of cross over on costs.

It is recommended by the author that the use of personal computers be avoided for further simulation studies in this subject. The personal computers are very slow in running rather large simulation programs.

There are also flaws in random number generators of IBM PC and many other PCs. Their random number generators have very short cycle lengths and should not actually be used for serious research. For further details about PCs and simulation the article by Modianos, Scott, and Cornwell is recommended.

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